Lecture 2 : Posterior distribution, Bayes estimators

Exercice 1 (Sampling from the Gaussian model).
Let $X|\theta \sim \mathcal{N}(\theta, 1)$ with $\theta \sim \mathcal{N}(0, 10)$. Let's consider one observation $X = x$.

1. Show that the posterior distribution of θ is also a normal distribution whose parameters will be given.
2. Give the Bayes estimator of θ under the quadratic loss.

Exercice 2 (Sampling from the Bernouilli distribution).
We consider a vector of observations $\boldsymbol{x} = (x_1, \ldots, x_n)$ from a random sample of a Bernoulli distribution with parameter $\theta \in [0, 1]$. The prior on θ is a Beta with parameters a and b. This can be written under the following hierarchical model :

$$\begin{aligned} X_i|\theta &\sim \mathcal{B}(\theta) \text{ for } i = 1, \ldots, n \\ \theta &\sim \text{Beta}(a, b) \end{aligned}$$

Give the posterior $\pi(\theta|\boldsymbol{x})$.

Exercice 3 (Proportion of defective objects).
We are interested in the unknwon proportion θ of defective objects in a large batch. The prior on θ is the uniform distribution on the unit interval $[0, 1]$. Let's suppose that we take a random sample of n objects from this batch, $X_1, \ldots, X_n$.

1. Compute the density of $\boldsymbol{X} = (X_1, \ldots, X_n)$.
2. Give the posterior of θ and determine its nature.

Exercice 4 (Parameter of an exponential distribution).
We observe the lifetimes of fluorescent bulbs. We model the lifetime as an exponential distribution with parameter $\beta > 0$. We consider as prior on β a gamma distribution with the following density :

$$\pi(\beta) = \frac{20000^4}{3!} \beta^3 e^{-20000\beta}.$$

Let us suppose that we take a random sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ of n bulbs.
Give the posterior of θ given $(X_1 = x_1, \ldots, X_n = x_n)$.

Exercice 5 (Sampling from the Poisson distribution).
Let's consider the following hierarchical model :

$$\begin{aligned} X_i|\theta &\sim \mathcal{P}(\theta) \text{ for } i = 1, \ldots, n \\ \theta &\sim \text{Gamma}(a, b) \end{aligned}$$

wheree $\mathcal{P}(\theta)$ denotes the Poisson distribution with parameter $\theta > 0$; and $a, b > 0$.

Compute $\pi(\theta|\boldsymbol{x})$ and the Bayes estimator of θ under quadratic loss.

Exercice 6 (Gaussian sampling).
Let's consider a sample from the Normal distribution with mean μ and variance σ^2. The likelihood is denoted $f(\boldsymbol{x}|\mu, \sigma^2)$, for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

1. We first consider that only the mean is unknown : $\theta = \mu$ and σ^2 is known. We endow μ with a Normal prior, $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$. The parameters μ_0 and σ_0^2 are called hyperparameters. They are considered fixed here. In practice, when we don't know much about these hyperparameters, we choose μ_0 close to 0 and σ_0^2 large enough.
   (a) Compute the likelihood and the maximum likelihood estimator of μ.
   (b) Compute the posterior distribution of μ as well as the Bayes estimator of μ for a quadratic loss. Show that the Bayes estimator is a linear combination of the empirical mean and the prior expectation. What happens when the sample size is small ? Large ?
   (c) We are now going to illustrate this on simulations. First generate $n = 1$ observation from a Gaussian $\mathcal{N}(0, 2^2)$. Represent on the same graph the prior and the posterior of μ.
   Do the same by varying the sample size : $n = 10, 100, 1000$.
2. We now turn our attention in the case where the mean μ is known and the parameter θ is the variance σ^2. We choose as a prior for σ^2 an inverse-gamma with parameters α and β. Show that the posterior of σ^2 is also an inverse-gamma whose parameters will be specified.

   Recall : if $U \sim IG(\alpha, \beta)$ with $\alpha > 0, \beta > 0$, the p.d.f is :

$$f(u|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(1/u)^{\alpha+1} \exp\left(-\beta/u\right) \quad \text{for all } u > 0.$$