On interior point methods for LCPs

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Why complementarity is so important?

1. Linear and Nonlinear programming.
2. Optimization Problems with Equilibrium Constraints.
3. Relaxation of binary variables.
4. ...

For given $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, the $LCP(M, q)$
is the following problem

$$\text{find } z \in \mathbb{R}^n \quad : \qquad 0 \leqslant z \perp w := Mz + q \geqslant 0$$

- $LCP(M, q)$ is an NP-hard problem for general matices M
- Fortunately, for different large classes of matrices M, there exist polynomial or at least efficient numerical methods.

Classes of matrices

There are some usefull classes of matrices when studying LCPs.

- SS (Skew symmetric) matrices:

$$\forall x \in \mathbb{R}^n, \quad x^T M x = 0.$$

- SDP matrices:

$$M =^t M \ : \ \forall x \in \mathbb{R}^n, \quad x^T M x \geqslant 0.$$

- P matrices :

$$\forall x \in \mathbb{R}^n, \exists i \in \{1, ..., n\} \ : \quad x_i (Mx)_i > 0.$$

- P_0 matrices :

$$\forall x \in \mathbb{R}^n, \exists i \in \{1, ..., n\} \ : \quad x_i (Mx)_i >= 0.$$

- SU : CSU (or RSU) :

$$\forall x \in \mathbb{R}^n, \forall i \in \{1, ..., n\} \ : \quad x_i (Mx)_i \leqslant 0 \Rightarrow x_i (Mx)_i = 0.$$

(resp.)

$$\forall x \in \mathbb{R}^n, \forall i \in \{1, ..., n\} \ : \quad x_i (M^T x)_i \leqslant 0 \Rightarrow x_i (M^T x)_i = 0.$$

Other classes of matrices

- $P_*(\kappa)$ matrices ($\kappa > 0$) such that

$$\forall x \in \mathbb{R}^n, (1+4\kappa) \sum_{i \in I^+(x)} x_i(Mx)_i + \sum_{i \in I^-(x)} x_i(Mx)_i >\geqslant 0.$$

  where $I^+(x) := \{i\ : x_i(Mx)_i > 0$ et $I^-(x) := \{i\ : x_i(Mx)_i < 0$

- P_* matrices : Union of $P_*(\kappa)$ matrices ($\kappa > 0$).

  We have in fact

$$SS \subset SDP \subset P_* = SU \subset P_0,$$

$$P \subset P_* \text{ and } P \cap SS = \emptyset.$$

Principle of IPMs

We suppose M to be symmetric and positive definite.
Construct a sequence $(z(\varepsilon), w(\varepsilon))$ satisfying

$$w(\varepsilon) \geqslant 0,\ z(\varepsilon) \geqslant 0,\ \|w(\varepsilon) - Mz(\varepsilon) - q\| \leqslant \varepsilon \quad \text{et} \quad z(\varepsilon)^T w(\varepsilon) \leqslant \varepsilon$$

To produce such sequence, perturbe the equations $LCP(M, q)$, in the following scheme

$$\begin{cases} w = Mz + q \\ w.z = \mu e \end{cases} \tag{2.1}$$

for a sequence of positive parameters μ going to 0
Since M is positive definite, the perturbed system of equations has a unique solution for each μ if $F_0 \neq \emptyset$.

Principle of IPMs (2)

Remark: In general, we don't have (for free) any solution $(\mu_0, z(\mu_0), w(\mu_0))$ of the perturbed system. If it is the case, the algorithm is said to be feasible.

- consider $\mu_0 > 0$ and an initial point $(z^0, w^0) > 0$ such that $w^0.z^0 = \mu_0 e$.
- Compute r^0 such that $\mu_0 r^0$ is the initial feasibility error:

$$\mu_0 r^0 = w^0 - Mz^0 - q$$

The infeasible central path

- $(\mu, z(\mu), w(\mu)) > 0$ belongs to the Central path if

$$\begin{cases} w(\mu) - Mz(\mu) - q = \mu r^0 \\ w(\mu).z(\mu) = \mu e \end{cases} \tag{2.2}$$

  Remark: we can use another parameter ($\neq \mu$) to model the infeasibility.

- To measure the distance or proximity to the central path, one can use

$$\delta(\mu, z, w) = \frac{1}{2}\|v - v^{-1}\| \qquad \text{where } v = \sqrt{\frac{z.w}{\mu}}.$$

  Remark:

$$\delta(\mu, z, w) = 0 \Leftrightarrow v = e \Leftrightarrow (\mu, z, w) \in TC$$

Generic Algorithm

1. **Initialization:** Choose a proximity tolerance τ, a global precision ε and a reduction parameterθ.Choose an initial point (μ_0, z^0, w^0) and let $(\mu, z, w) := (\mu_0, z^0, w^0)$.
2. **Stopping criterion** While $\max(\|w - Mz - q\|, z^T w) > \varepsilon$, repeat
   - **Prediction step:**
     Compute a direction $(\Delta z, \Delta w)$ such that
     $$\begin{cases} M\Delta z - \Delta w = \theta\mu r^0 \\ w.\Delta z + z\Delta w = (1-\theta)\mu e - z.w \end{cases} \tag{2.3}$$
     $z = z + \Delta z, \quad w = w + \Delta w$, and $v := \sqrt{\frac{z.w}{\mu}}$
     Compute $\delta(v)$.
     Reduce the central path parameter $\mu = (1-\theta)\mu$
   - **Correction step:** While $\delta(v) > \tau$, repeat
     Compute a direction $(\Delta z, \Delta w)$ using
     $$\begin{cases} M\Delta z - \Delta w = 0 \\ w.\Delta z + z\Delta w = \mu e - z.w \end{cases} \tag{2.4}$$
     $z = z + \Delta z, \quad w = w + \Delta w$, and$v := \sqrt{\frac{z.w}{\mu}}$
     Compute $\delta(v)$.

Convergence

I present here a simple convergence result without proof when θ and τ are constant and using complete steps.

Theorem

If $\theta = \frac{1}{12n}$ and $\tau = \frac{1}{4}$ then the generic algorithm needs at most $12n \log \frac{33 z0^T w0}{32\varepsilon}$ iterations to get an ε solution for the LCP(M, q) $\quad$ ($O(n \log \frac{n}{\varepsilon})$ iterations).

The best known complexity for IPM is $O(\sqrt{n} \log \frac{n}{\varepsilon})$

- Write a matlab function IPMLCP with the following calling sequence

$$[z, w] = IPMLCP(M, q, \varepsilon, \theta, \tau);$$

- Generate a reduction parameter, a precision and a proximity tolerance

$$\varepsilon = \ldots; \theta = \ldots; \tau = \ldots$$

- Generate an LCP with symmetric definite matrix M and some vector q
- Solve the corresponding LCP using your matlab function

$$[z, w] = IPMLCP(M, q, \varepsilon, \theta, \tau);$$

- Compute the feasibility and the complementarity errors
- Make the size of the problem higher and higher, compute the outer iterations number and comment
- What can happen if M is not symmetric definite ?