Optimal Control

Using discrete time approach

The objective of this problem is to solve, via the **direct sequential approach**, the following scalar optimal control problem:

$$\begin{aligned}
\text{minimize:} \quad & \int_0^2 \tfrac{1}{2}[x_1(t)]^2 \, \mathrm{d}t && (1)\\
\text{subject to:} \quad & \dot{x}_1(t) = x_2(t) + u(t); \qquad x_1(0) = 1 && (2)\\
& \dot{x}_2(t) = -u(t); \qquad x_2(0) = 1 && (3)\\
& x_1(2) = x_2(2) = 0 && (4)\\
& -10 \leq u(t) \leq 10, \quad 0 \leq t \leq 2. && (5)
\end{aligned}$$

For simplicity, a **piecewise control parameterization** over n_s stages is considered for approximating,

$$u(t) = \omega^k, \quad t_{k-1} \leq t \leq t_k, \quad k = 1, \ldots, n_s,$$

with the time stages being equally spaced, $t_k = \frac{2k}{n_s}$.

Questions:

1. First, reformulate the optimal control problem into the Mayer form (denote the additional state variable by x_3).
2. In MatLab®, write a m-file calculating the values of the cost (1) and terminal constraints (4), for a given number n_s of stages and given values of the control parameters $\omega^1, \ldots, \omega^{n_s}$.
   - Use the function `ode15s` to integrate the differential equations; set both the absolute and relative integration tolerances to 10^{-8}.

   Application. Calculate the cost and constraint values for the following 2-stage control:

$$u^0(t) = \begin{cases} 10, & 0 \leq t < 1, \\ -10, & 1 \leq t \leq 2. \end{cases} \tag{6}$$

3. Solve the sequentially-discretized NLP problem using the `fmincon` function in MatLab's Optimization Toolbox:
   - You are to code a main program, as well as two separate m-files called by `fmincon` that calculate the values of the cost and of the constraints, respectively; these latter m-files should invoke the m-file developed in Question 2.
   - For simplicity, let `fmincon` calculate a finite-difference approximation of the cost and constraint derivatives; make sure that the minimum change in variable for finite differencing is consistent with the tolerances set previously for the ODE solver (10^{-8}): here, a value of 10^{-5} appears to be a reasonable choice.
   - Make sure to select the medium-scale SQP algorithm with quasi-Newton update and line-search, and set the solution point tolerance, the function tolerance and the constraint tolerance all to 10^{-9}; such tight tolerances are needed because the optimal control problem is singular.

- Set all the control coefficients equal to zero as the initial guess.

Application. Solve the optimal control problem for $n_s = 2$, 4, 8, 16 and 32 stages, then plot the results.

4. To increase the reliability and execution speed of the direct sequential procedure, you are to compute the derivatives of the cost (1) and terminal constraints (4) with respect to the control parameters $\omega^1, \ldots, \omega^{n_s}$ via the forward sensitivity method:

   (a) Write down the state sensitivity equations, as well as the cost and constraint derivatives per the forward sensitivity method.

   (b) Duplicate the m-file developed in Question 2 above and modify it so that it calculates the cost and constraint derivatives in addition to their values; run this m-file for the 2-stage control $u^0(t)$ given in (6).

   (c) Modify the two m-files passed to `fmincon` for cost/constraint function evaluation:
      - In the main program, tell `fmincon` to use the user-supplied cost/constraint derivatives (instead of finite-difference derivatives).
      - Use the DerivativeCheck feature of `fmincon` to detect inconsistencies in the computed derivatives!

   Application. Recalculate the solution to the optimal control problem for $n_s = 2$, 4, 8, 16 and 32 stages, and make sure that you get the same results as previously.