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Assignment 2: Finite Difference Method.

Question :
We consider the convection diffusion equations:

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} = \kappa\frac{\partial^2 u}{\partial x^2}. \quad t > 0 \quad ,0 < x < 1 \tag{1}$$

where κ is the diffusion coefficient.
The initial condition is:

$$u(x, 0) = u_0(x)$$

The boundary condition is:

$$u(0, t) = \phi_0(t), \quad u(1, t) = \phi_1(t)$$

Write Forward Euler method with the Lax-Friedrich, Lax-Wendroff, and Upwind schemes for convection term and find stable condition:

1. Using Forward Euler method with $\kappa = 0$.
2. Using Forward Euler method with $\kappa \neq 0$.
3. Using Crank-Nicolson method with $\kappa = 0$.
4. Using Crank-Nicolson method with $\kappa \neq 0$.

Answer :
Sử dụng Phương pháp Sai phân hữu hạn cho bài toán (1) với lưới đều trên khoảng [0,1] được chia thành $N_x + 1$ điểm x_i với khoảng cách ("Space step") là: h = $\frac{1}{N_x}$; Và lưới đều trên khoảng [0,T] được chia thành $N_t + 1$ điểm t_j với khoảng cách ("Time step") là: k = $\frac{T}{N_x}$
Đặt $U_i^n = u(x_i, t_n)$ biểu diễn cho giá trị xấp xỉ tại điểm lưới (x_i, t_n)
1, Using Forward Euler method with $\kappa = 0$.
Với $\kappa = 0$, phương trình (1) trở thành:

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} = 0 \tag{2}$$

($\star$) Lax-Friedrich schemes:
Rời rạc hóa phương trình (2) ta được:

$$\frac{U_j^{n+1} - U_j^n}{k} + \alpha\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$
$$U_j^{n+1} = U_j^n - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n)$$

Trên thực tế thì phương pháp xấp xỉ trên "instability" nên thay thế bằng việc cho $U_j^n = \dfrac{1}{2}(U_{j+1}^n + U_{j+1}^n)$ khi đó ta được:

$$U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j+1}^n) - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n)$$

Sử dụng Von-Newmann analysis:
Đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi} = \frac{1}{2}(e^{i(j+1)h\xi} + e^{i(j-1)h\xi}) - \frac{\alpha k}{2h}(e^{i(j+1)h\xi} - e^{i(j-1)h\xi})$$

Chia 2 vế cho $e^{ijh\xi}$ ta được:

$$G(\xi) = \frac{1}{2}(e^{ih\xi} + e^{-ih\xi}) - \frac{\alpha k}{2h}(e^{ih\xi} - e^{-ih\xi})$$

Sử dụng tính chất sau

$$\begin{cases} \dfrac{1}{2}(e^{ih\xi} - e^{-ih\xi}) = isin(h\xi) \\ \\ \dfrac{1}{2}(e^{ih\xi} + e^{-ih\xi}) = cos(h\xi) \end{cases}$$

Khi đó ta được:

$$G(\xi) = cos(h\xi) - \frac{\alpha k}{h}isin(h\xi)$$

Lấy chuẩn (chuẩn số phức) 2 vế ta được

$$|G(\xi)| = \sqrt{[cos(h\xi)]^2 + \left[\frac{\alpha k}{h}sin(h\xi)\right]^2}$$

ta có điều kiện stable:

$$\begin{aligned} |G(\xi)| &\leq 1 \\ \sqrt{[cos(h\xi)]^2 + \left[\frac{\alpha k}{h}sin(h\xi)\right]^2} &\leq 1 \\ cos^2(h\xi) + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi) &\leq 1 \end{aligned}$$

Đánh giá :

$$cos^2(h\xi) + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi) \leq max\left[1, \left(\frac{\alpha k}{h}\right)^2\right](cos^2(h\xi) + sin^2(h\xi)) = max\left[1, \left(\frac{\alpha k}{h}\right)^2\right]$$

Khi đó nếu điều dưới đây xảy ra thì ta có điều kiện stable:

$$
\begin{aligned}
max\left[1, \left(\frac{\alpha k}{h}\right)^2\right] &\leq 1 \\
\Rightarrow \quad \left(\frac{\alpha k}{h}\right)^2 &\leq 1 \\
\Rightarrow \quad \frac{\alpha k}{h} &\leq 1
\end{aligned}
$$

($\star$) Lax-Wendroff schemes:
Using ODE system:

$$
\begin{aligned}
U'(t) &= AU(t) \\
\Rightarrow \quad U"(t) &= A^2U(t)
\end{aligned}
$$

Khi đó ta có thể viết lại phương trình (2):

$$U_j^{n+1} = U_j^n - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}\left(\frac{\alpha k}{2h}\right)^2 (U_{j+2}^n - 2U_j^n + U_{j-2}^n)$$

Trên thực tế phương pháp xấp xỉ với 5 điểm này không stable nên ta lấy 3 điểm:

$$U_j^{n+1} = U_j^n - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

Tương tự sử dụng Von-Neumann analysis đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi} = e^{ijh\xi} - \frac{\alpha k}{2h}(e^{i(j+1)h\xi} - e^{i(j-1)h\xi}) + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (e^{i(j+1)h\xi} - 2e^{ijh\xi} + e^{i(j-1)h\xi})$$

Rút gọn 2 vế cho $e^{ijh\xi}$

$$
\begin{aligned}
G(\xi) &= 1 - \frac{\alpha k}{2h}(e^{ih\xi} - e^{-ih\xi}) + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (e^{ih\xi} - 2 + e^{-ih\xi}) \\
&= 1 - \frac{\alpha k}{h} isin(h\xi) + \left(\frac{\alpha k}{h}\right)^2 (cos(h\xi) - 1) \\
&= 1 - \frac{\alpha k}{h} 2isin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right) + \left(\frac{\alpha k}{h}\right)^2 \left[-2sin^2\left(\frac{h\xi}{2}\right)\right]
\end{aligned}
$$

Lấy chuẩn 2 vế ta được

$$|G(\xi)| = \sqrt{\left[1 - \left(\frac{\alpha k}{h}\right)^2 2sin^2\left(\frac{h\xi}{2}\right)\right]^2 + \left[\frac{\alpha k}{h} 2sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right)\right]^2}$$

Điều kiện stable:

$$
\begin{aligned}
|G(\xi)| &\leq 1 \\
\sqrt{\left[1-\left(\frac{\alpha k}{h}\right)^2 2sin^2\left(\frac{h\xi}{2}\right)\right]^2+\left[\frac{\alpha k}{h}2sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2} &\leq 1 \\
\left[1-\left(\frac{\alpha k}{h}\right)^2 2sin^2\left(\frac{h\xi}{2}\right)\right]^2+\left[\frac{\alpha k}{h}2sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2 &\leq 1 \\
1-4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^4 sin^4\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right) &\leq 1 \\
-4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^4 sin^4\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right) &\leq 0
\end{aligned}
$$

Chia 2 vế cho $4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) \geq 0$ ta được:

$$
\begin{aligned}
-1+\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+cos^2\left(\frac{h\xi}{2}\right) &\leq 0 \\
\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+cos^2\left(\frac{h\xi}{2}\right) &\leq 1
\end{aligned}
$$

Tới đây hoàn toàn tương tự như chứng minh của Lax-Friedrich ở trước đó, ta suy ra được điều kiện Stable:

$$\frac{\alpha k}{h} \leq 1$$

($\star$) Upwind schemes:
Xét với trường hợp $\alpha \geq 0$
Một xấp xỉ cho phương trình (2) là:

$$
\begin{aligned}
\frac{U_j^{n+1}-U_j^n}{k}+\alpha\frac{U_j^n-U_{j-1}^n}{h} &= 0 \\
U_j^{n+1} &= U_j^n-\frac{\alpha k}{h}(U_j^n-U_{j-1}^n)
\end{aligned}
$$

Sử dụng Von-Neumann analysis đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi} = e^{ijh\xi}-\frac{\alpha k}{h}(e^{ijh\xi}-e^{i(j-1)h\xi})$$

Chia 2 vế cho $e^{ijh\xi}$, ta được:

$$\begin{aligned} G(\xi) &= 1 - \frac{\alpha k}{h}(1 - e^{-ih\xi}) \\ &= 1 - \frac{\alpha k}{h}\left[1 - (cos(h\xi) - isin(h\xi))\right] \\ &= \left[1 - \frac{\alpha k}{h}(1 - cos(h\xi))\right] - \frac{\alpha k}{h} isin(h\xi) \\ &= \left[1 - 2\left(\frac{\alpha k}{h}\right) sin^2\left(\frac{h\xi}{2}\right)\right] - \frac{\alpha k}{h} 2isin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right) \end{aligned}$$

Lấy chuẩn 2 vế ta được

$$|G(\xi)| = \sqrt{\left[1 - 2\left(\frac{\alpha k}{h}\right) sin^2\left(\frac{h\xi}{2}\right)\right]^2 + \left[2\frac{\alpha k}{h} sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right)\right]^2}$$

Điều kiện stable:

$$\begin{aligned} |G(\xi)| &\leq 1 \\ \sqrt{\left[1 - 2\left(\frac{\alpha k}{h}\right) sin^2\left(\frac{h\xi}{2}\right)\right]^2 + \left[2\frac{\alpha k}{h} sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right)\right]^2} &\leq 1 \\ 1 - 4\left(\frac{\alpha k}{h}\right) sin^2\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha k}{h}\right)^2 sin^4\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) cos^2\left(\frac{h\xi}{2}\right) &\leq 1 \\ -4\left(\frac{\alpha k}{h}\right) sin^2\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha k}{h}\right)^2 sin^4\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) cos^2\left(\frac{h\xi}{2}\right) &\leq 0 \end{aligned}$$

Chia 2 vế cho $4sin^2\left(\frac{h\xi}{2}\right) \geq 0$ ta được:

$$\begin{aligned} -\left(\frac{\alpha k}{h}\right) + \left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) + \left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\leq 0 \\ -\left(\frac{\alpha k}{h}\right) + \left(\frac{\alpha k}{h}\right)^2 \left[sin^2\left(\frac{h\xi}{2}\right) + cos^2\left(\frac{h\xi}{2}\right)\right] &\leq 0 \\ \left(\frac{\alpha k}{h}\right)^2 &\leq \left(\frac{\alpha k}{h}\right) \\ \left(\frac{\alpha k}{h}\right) &\leq 1 \end{aligned}$$

Hoàn toàn tương tự cho trường hợp $\alpha < 0$.

2, Using Forward Euler method with $\kappa \neq 0$.

Những chứng minh sau tại 1 số bước được lược bỏ nhanh:

($\star$) Lax-Friedrich schemes:

Rời rạc hóa phương trình (1) ta được:

$$\begin{aligned}\frac{U_j^{n+1}-U_j^n}{k}+\alpha\frac{U_{j+1}^n-U_{j-1}^n}{2h}&=\kappa\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2}\\ U_j^{n+1}&=U_j^n-\frac{\alpha k}{2h}(U_{j+1}^n-U_{j-1}^n)+\frac{\kappa k}{h^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)\end{aligned}$$

Sử dụng Von-Newmann analysis đặt: $U_j^n=e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1}=G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$\begin{aligned}G(\xi)e^{ijh\xi}&=e^{ijh\xi}-\frac{\alpha k}{2h}(e^{i(j+1)h\xi}-e^{i(j-1)h\xi})+\frac{\kappa k}{h^2}(e^{i(j+1)h\xi}-2e^{ijh\xi}+e^{i(j-1)h\xi})\\ G(\xi)&=1-\frac{\alpha k}{2h}(e^{ih\xi}-e^{-ih\xi})+\frac{\kappa k}{h^2}(e^{ih\xi}-2+e^{-ih\xi})\end{aligned}$$

Khi đó ta được:

$$\begin{aligned}G(\xi)&=1-\frac{\alpha k}{h}isin(h\xi)+2\frac{\kappa k}{h^2}(cos(h\xi)-1)\\ &=1-i\frac{\alpha k}{h}2sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)+2\frac{\kappa k}{h^2}\left(-2sin^2\left(\frac{h\xi}{2}\right)\right)\end{aligned}$$

Lấy chuẩn 2 vế ta được

$$|G(\xi)|=\sqrt{\left[2\frac{\alpha k}{h}sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2+\left[1-4\frac{\kappa k}{h^2}sin^2\left(\frac{h\xi}{2}\right)\right]^2}$$

Điều kiện stable:

$$\begin{aligned}|G(\xi)|&\le 1\\ \sqrt{\left[2\frac{\alpha k}{h}sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2+\left[1-4\frac{\kappa k}{h^2}sin^2\left(\frac{h\xi}{2}\right)\right]^2}&\le 1\\ 4\left(\frac{\alpha k}{h}\right)^2sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right)+\left[1-8\frac{\kappa k}{h^2}sin^2\left(\frac{h\xi}{2}\right)+16\left(\frac{\kappa k}{h^2}\right)^2sin^4\left(\frac{h\xi}{2}\right)\right]&\le 1\\ -8\frac{\kappa k}{h^2}sin^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^2sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right)+16\left(\frac{\kappa k}{h^2}\right)^2sin^4\left(\frac{h\xi}{2}\right)&\le 0\end{aligned}$$

Chia 2 vế cho $4sin^2\left(\frac{h\xi}{2}\right)\ge 0$ ta được:

$$\begin{aligned}-2\frac{\kappa k}{h^2}+\left(\frac{\alpha k}{h}\right)^2cos^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\kappa k}{h^2}\right)^2sin^2\left(\frac{h\xi}{2}\right)&\le 0\\ \left(\frac{\alpha k}{h}\right)^2cos^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\kappa k}{h^2}\right)^2sin^2\left(\frac{h\xi}{2}\right)&\le 2\frac{\kappa k}{h^2}\end{aligned}$$

Ta có đánh giá:

$$\begin{aligned}\left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) + 4\left(\frac{\kappa k}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right) &\leq max\left[\left(\frac{\alpha k}{h}\right)^2, 4\left(\frac{\kappa k}{h^2}\right)^2\right]\left[cos^2\left(\frac{h\xi}{2}\right) + sin^2\left(\frac{h\xi}{2}\right)\right] \\ &= max\left[\left(\frac{\alpha k}{h}\right)^2, 4\left(\frac{\kappa k}{h^2}\right)^2\right]\end{aligned}$$

Vậy ta chỉ cần chứng minh:

$$max\left[\left(\frac{\alpha k}{h}\right)^2, \left(2\frac{\kappa k}{h^2}\right)^2\right] \leq 2\frac{\kappa k}{h^2}$$

Với $\left(\frac{2\kappa k}{h^2} \geq \frac{\alpha k}{h}\right) \Rightarrow \kappa \geq \frac{\alpha k}{h}\frac{h^2}{2k} = \frac{\alpha h}{2}$ thì

$$\begin{aligned}\left(\frac{2\kappa k}{h^2}\right)^2 &\leq \frac{2\kappa k}{h^2} \\ \frac{2\kappa k}{h^2} &\leq 1 \\ k &\leq \frac{h^2}{2\kappa}\end{aligned}$$

Với $\left(\frac{2\kappa k}{h^2} < \frac{\alpha k}{h}\right) \Rightarrow \kappa < \frac{\alpha h}{2}$ thì

$$\begin{aligned}\left(\frac{\alpha k}{h}\right)^2 &\leq \frac{2\kappa k}{h^2} \\ \frac{\alpha^2 k^2}{h^2}\frac{h^2}{2\kappa k} &\leq 1 \\ \frac{\alpha^2 k}{2\kappa} &\leq 1 \\ k &\leq \frac{2\kappa}{\alpha^2}\end{aligned}$$

Vậy điều kiện kiện stable cho Lax-Friedrich schemes sử dụng phương pháp Forward Euler trong trường hợp $\kappa \neq 0$ là:

$$\begin{cases} k & \leq \frac{h^2}{2\kappa}, \quad \forall \kappa \geq \frac{\alpha h}{2} \\ k & \leq \frac{2\kappa}{\alpha^2}, \quad \forall \kappa < \frac{\alpha h}{2} \end{cases}$$

($\star$) Lax-Wendroff schemes:
Using ODE system:

$$\begin{aligned} & U'(t) = AU(t) \\ \Rightarrow \quad & U"(t) = A^2U(t)\end{aligned}$$

$$
\begin{aligned}
U_j^{n+1} &= U_j^n - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (U_{j+1}^n - 2U_j^n + U_{j-1}^n) + \left(\frac{\kappa}{h^2}\right)(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \\
&= U_j^n - \frac{\alpha k}{2h}(U_{j+1}^n - U_{j-1}^n) + \left(\frac{\alpha^2 k^2 + 2\kappa}{2h^2}\right)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)
\end{aligned}
$$

Tương tự đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$
G(\xi)e^{ijh\xi} = e^{ijh\xi} - \frac{\alpha k}{2h}(e^{i(j+1)h\xi} - e^{i(j-1)h\xi}) + \left(\frac{\alpha^2 k^2 + 2\kappa}{2h^2}\right)(e^{i(j+1)h\xi} - 2e^{ijh\xi} + e^{i(j-1)h\xi})
$$

Rút gọn 2 vế cho $e^{ijh\xi}$

$$
\begin{aligned}
G(\xi) &= 1 - \frac{\alpha k}{2h}(e^{ih\xi} - e^{-ih\xi}) + \left(\frac{\alpha^2 k^2 + 2\kappa}{2h^2}\right)(e^{ih\xi} - 2 + e^{-ih\xi}) \\
&= 1 - \frac{\alpha k}{h} i sin(h\xi) + \left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)(cos(h\xi) - 1) \\
&= 1 - i\frac{\alpha k}{h} 2 sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right) + \left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)\left[-2 sin^2\left(\frac{h\xi}{2}\right)\right]
\end{aligned}
$$

Lấy chuẩn 2 vế ta được

$$
|G(\xi)| = \sqrt{\left[1 - 2\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right) sin^2\left(\frac{h\xi}{2}\right)\right]^2 + \left[2\frac{\alpha k}{h} sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right)\right]^2}
$$

Điều kiện stable:

$$
\begin{aligned}
|G(\xi)| &\le 1 \\
\sqrt{\left[1 - 2\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right) sin^2\left(\frac{h\xi}{2}\right)\right]^2 + \left[2\frac{\alpha k}{h} sin\left(\frac{h\xi}{2}\right) cos\left(\frac{h\xi}{2}\right)\right]^2} &\le 1 \\
\left[1 - 4\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right) sin^2\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)^2 sin^4\left(\frac{h\xi}{2}\right)\right] + 4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) cos^2\left(\frac{h\xi}{2}\right) &\le 1 \\
-4\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right) sin^2\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)^2 sin^4\left(\frac{h\xi}{2}\right) + 4\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) cos^2\left(\frac{h\xi}{2}\right) &\le 0
\end{aligned}
$$

Chia 2 vế cho $4 sin^2\left(\frac{h\xi}{2}\right) \ge 0$ ta được:

$$
\begin{aligned}
-\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right) + \left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right) + \left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\le 0 \\
\left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right) + \left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\le \left(\frac{\alpha^2 k^2 + 2\kappa}{h^2}\right)
\end{aligned}
$$

Ta có đánh giá :

$$\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+\left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) \leq max\left[\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right]\left[cos^2\left(\frac{h\xi}{2}\right)+sin^2\left(\frac{h\xi}{2}\right)\right]$$
$$= max\left[\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right]$$

Vậy ta chỉ cần chứng minh:

$$max\left[\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right] \leq \left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)$$

Với $\left(\frac{\alpha^2k^2+2\kappa}{h^2} \geq \frac{\alpha k}{h}\right) \Rightarrow \kappa \geq \frac{1}{2}\left(\frac{\alpha k}{h}-\frac{\alpha^2k^2}{h^2}\right)=\frac{\alpha k}{2h}\left(1-\frac{\alpha k}{h}\right)$ thì

$$\begin{aligned}\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)^2 &\leq \frac{\alpha^2k^2+2\kappa}{h^2}\\ \frac{\alpha^2k^2+2\kappa}{h^2} &\leq 1\\ k^2 &\leq \frac{h^2-2\kappa}{\alpha^2}\end{aligned}$$

Đẳng thức trên đúng và có nghĩa khi: $h^2-2\kappa>0 \Rightarrow \kappa<\frac{h^2}{2}$

Khi đó với k>0,

$$k \leq \sqrt{\frac{h^2-2\kappa}{\alpha^2}}$$

Với $\left(\frac{\alpha^2k^2+2\kappa}{h^2} < \frac{\alpha k}{h}\right) \Rightarrow \kappa < \frac{\alpha k}{2h}\left(1-\frac{\alpha k}{h}\right)$ thì

$$\begin{aligned}\left(\frac{\alpha k}{h}\right)^2 &\leq \frac{\alpha^2k^2+2\kappa}{h^2}\\ 0 &\leq \frac{2\kappa}{h^2}\\ 0 &\leq \frac{1}{h^2}\frac{\alpha k}{2h}\left(1-\frac{\alpha k}{h}\right)\\ 0 &\leq 1-\frac{\alpha k}{h}\\ \frac{\alpha k}{h} &\leq 1\end{aligned}$$

Vậy điều kiện kiện stable cho Lax-Wendroff schemes sử dụng phương pháp Forward Euler trong trường hợp $\kappa \neq 0$ là:

$$\begin{cases} k \leq \sqrt{\frac{h^2-2\kappa}{\alpha^2}}, & \forall\kappa: \ \frac{h^2}{2}>\kappa \geq \frac{\alpha k}{2h}\left(1-\frac{\alpha k}{h}\right)\\ k \leq \frac{h}{\alpha}, & \forall\kappa < \frac{\alpha k}{2h}\left(1-\frac{\alpha k}{h}\right)\end{cases}$$

($\star$) Upwind schemes:
Xét với trường hợp $\alpha \geq 0$ta có xấp xỉ

$$\frac{U_j^{n+1}-U_j^n}{k}+\alpha\frac{U_j^n-U_{j-1}^n}{h}=\kappa\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2}$$
$$U_j^{n+1}=U_j^n-\frac{\alpha k}{h}(U_j^n-U_{j-1}^n)+\frac{\kappa k}{h^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)$$

Đặt: $U_j^n=e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1}=G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi}=e^{ijh\xi}-\frac{\alpha k}{h}(e^{ijh\xi}-e^{i(j-1)h\xi})+\frac{\kappa k}{h^2}(e^{i(j+1)h\xi}-2e^{ijh\xi}+e^{i(j-1)h\xi})$$

Chia 2 vế cho $e^{ijh\xi}$, ta được:

$$\begin{aligned}G(\xi)&=1-\frac{\alpha k}{h}(1-e^{-ih\xi})+\frac{\kappa k}{h^2}(e^{ih\xi}-2+e^{-ih\xi})\\&=1-\frac{\alpha k}{h}\left[1-(cos(h\xi)-isin(h\xi))\right]+2\frac{\kappa k}{h^2}(cos(h\xi)-1)\\&=\left[1-\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)(1-cos(h\xi))\right]-\frac{\alpha k}{h}isin(h\xi)\\&=\left[1-2\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)\right]-\frac{\alpha k}{h}2isin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\end{aligned}$$

Lấy chuẩn 2 vế ta được

$$|G(\xi)|=\sqrt{\left[1-2\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)\right]^2+\left[2\frac{\alpha k}{h}sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2}$$

Điều kiện stable:

$$|G(\xi)|\leq 1$$
$$\sqrt{\left[1-2\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)\right]^2+\left[2\frac{\alpha k}{h}sin\left(\frac{h\xi}{2}\right)cos\left(\frac{h\xi}{2}\right)\right]^2}\leq 1$$
$$1-4\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2sin^4\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^2sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right)\leq 1$$
$$-4\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2sin^4\left(\frac{h\xi}{2}\right)+4\left(\frac{\alpha k}{h}\right)^2sin^2\left(\frac{h\xi}{2}\right)cos^2\left(\frac{h\xi}{2}\right)\leq 0$$

Chia 2 vế cho $4sin^2\left(\frac{h\xi}{2}\right) \geq 0$ ta được:

$$
\begin{aligned}
-\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)+\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+\left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\leq 0\\
\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+\left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\leq \left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)
\end{aligned}
$$

Ta có đánh giá:

$$
\begin{aligned}
\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2 sin^2\left(\frac{h\xi}{2}\right)+\left(\frac{\alpha k}{h}\right)^2 cos^2\left(\frac{h\xi}{2}\right) &\leq max\left[\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right]\left[cos^2\left(\frac{h\xi}{2}\right)+sin^2\left(\frac{h\xi}{2}\right)\right]\\
&= max\left[\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right]
\end{aligned}
$$

Vậy ta chỉ cần chứng minh:

$$
max\left[\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2,\left(\frac{\alpha k}{h}\right)^2\right] \leq \left(\frac{\alpha^2 k^2+2\kappa}{h^2}\right)
$$

Với $\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2} \geq \frac{\alpha k}{h}\right) \Rightarrow \kappa \geq 0$ thì

$$
\begin{aligned}
\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\right)^2 &\leq \frac{\alpha k}{h}+\frac{2\kappa k}{h^2}\\
\frac{\alpha k}{h}+\frac{2\kappa k}{h^2} &\leq 1\\
k &\leq \frac{h^2}{\alpha h+2\kappa}
\end{aligned}
$$

Với $\left(\frac{\alpha k}{h}+\frac{2\kappa k}{h^2} < \frac{\alpha k}{h}\right) \Rightarrow \kappa < 0$ thì

$$
\begin{aligned}
\left(\frac{\alpha k}{h}\right)^2 &\leq \frac{\alpha k}{h}+\frac{2\kappa k}{h^2} < \frac{\alpha k}{h} \quad (do\ \kappa < 0)\\
\left(\frac{\alpha k}{h}\right)^2 &< \frac{\alpha k}{h}\\
\frac{\alpha k}{h} &< 1
\end{aligned}
$$

Vậy điều kiện kiện stable cho Upwwind schemes sử dụng phương pháp Forward Euler trong trường hợp $\kappa \neq 0$ là:

$$
\begin{cases}
k & \leq \dfrac{h^2}{\alpha h+2\kappa}\ , \qquad \forall\kappa \geq 0\\[2ex]
k & \leq \dfrac{h}{\alpha}\ , \qquad \forall\kappa < 0
\end{cases}
$$

3, Using Crank – Nicolson Euler method with $\kappa = 0$.

($\star$) Lax-Friedrich schemes:

Rời rạc hóa phương trình (2) ta được:

$$\frac{U_j^{n+1} - U_j^n}{k} + \alpha \left(\frac{1}{2} \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \frac{1}{2} \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2h} \right) = 0$$

$$U_j^{n+1} + \frac{\alpha k}{4h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) = U_j^n - \frac{\alpha k}{4h}(U_{j+1}^n - U_{j-1}^n)$$

Thay $U_j^n = \dfrac{1}{2}(U_{j+1}^n + U_{j+1}^n)$ khi đó ta được xấp xỉ theo phương pháp Lax-Friedrich:

$$U_j^{n+1} + \frac{\alpha k}{4h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) = \frac{1}{2}(U_{j+1}^n + U_{j+1}^n) - \frac{\alpha k}{4h}(U_{j+1}^n - U_{j-1}^n)$$

Sử dụng Von-Neumann analysis:

Đặt: $U_j^n = e^{ijh\xi}$

Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$

Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left[1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi})\right] = e^{ijh\xi}\left[\frac{1}{2}(e^{ih\xi} + e^{-ih\xi}) - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi})\right]$$

Chia 2 vế cho $e^{ijh\xi}$ ta được:

$$G(\xi) = \frac{\frac{1}{2}(e^{ih\xi} + e^{-ih\xi}) - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi})}{1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi})}$$

$$G(\xi) = \frac{\frac{1}{2}cos(h\xi) - i\frac{\alpha k}{2h}sin(h\xi)}{1 + i\frac{\alpha k}{2h}sin(h\xi)}$$

Ta có điều kiện stable:

$$|G(\xi)| \le 1$$

$$\left|\frac{1}{2}cos(h\xi) - i\frac{\alpha k}{2h}sin(h\xi)\right| \le \left|1 + i\frac{\alpha k}{2h}sin(h\xi)\right|$$

$$\sqrt{\frac{1}{4}cos^2(h\xi) + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi)} \le \sqrt{1 + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi)}$$

$$\frac{1}{4}cos^2(h\xi) + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi) \le 1 + \left(\frac{\alpha k}{h}\right)^2 sin^2(h\xi)$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\alpha, k, h > 0$.

Vậy "Lax-Friedrich schemes using Crank-Nicolson with $\kappa = 0$" luôn stable.

($\star$) Lax-Wendroff schemes:
Ta có:

$$U_j^{n+1} + \frac{1}{2}\left(\frac{\alpha k}{2h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) - \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1})\right)$$
$$= U_j^n - \frac{1}{2}\left(\frac{\alpha k}{2h}(U_{j+1}^{n} - U_{j-1}^{n}) - \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n})\right)$$

Tương tự sử dụng Von-Neumann analysis đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left(1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) - \frac{1}{4}\left(\frac{\alpha k}{h}\right)^2 (e^{ih\xi} - 2 + e^{-ih\xi})\right)$$
$$= e^{ijh\xi}\left(1 - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) + \frac{1}{4}\left(\frac{\alpha k}{h}\right)^2 (e^{ih\xi} - 2 + e^{-ih\xi})\right)$$

Rút gọn 2 vế cho $e^{ijh\xi}$

$$G(\xi) = \frac{1 - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) + \frac{1}{4}\left(\frac{\alpha k}{h}\right)^2 (e^{ih\xi} - 2 + e^{-ih\xi})}{1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) - \frac{1}{4}\left(\frac{\alpha k}{h}\right)^2 (e^{ih\xi} - 2 + e^{-ih\xi})}$$
$$= \frac{1 - i\frac{\alpha k}{2h}sin(h\xi) - \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi))}{1 + i\frac{\alpha k}{2h}sin(h\xi) + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi))}$$

Điều kiện stable:

$$|G(\xi)| \leq 1$$
$$\sqrt{\left(1 - \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)} \leq \sqrt{\left(1 + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)}$$
$$1 - \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi)) \leq 1 + \frac{1}{2}\left(\frac{\alpha k}{h}\right)^2 (1 - cos(h\xi))$$
$$-\left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right) \leq \left(\frac{\alpha k}{h}\right)^2 sin^2\left(\frac{h\xi}{2}\right)$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\alpha, k, h > 0$.
Vậy "Lax-Wendroff schemes using Crank-Nicolson with $\kappa = 0$" luôn stable.
($\star$) Upwind schemes:
Xét với trường hợp $\alpha \geq 0$

Một xấp xỉ cho phương trình (2) là:

$$\frac{U_j^{n+1}-U_j^n}{k}+\alpha\left(\frac{1}{2}\frac{U_j^n-U_{j-1}^n}{h}+\frac{1}{2}\frac{U_j^{n+1}-U_{j-1}^{n+1}}{h}\right)=0$$

$$U_j^{n+1}+\frac{\alpha k}{2h}(U_j^{n+1}-U_{j-1}^{n+1})=U_j^n-\frac{\alpha k}{2h}(U_j^n-U_{j-1}^n)$$

Sử dụng Von-Neumann analysis đặt: $U_j^n=e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1}=G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left(1+\frac{\alpha k}{2h}(1-e^{-ih\xi})\right)=e^{ijh\xi}\left(1-\frac{\alpha k}{2h}(1-e^{-ih\xi})\right)$$

Chia 2 vế cho $e^{ijh\xi}$, ta được:

$$\begin{aligned}G(\xi)&=\frac{1-\frac{\alpha k}{2h}(1-e^{-ih\xi})}{1+\frac{\alpha k}{2h}(1-e^{-ih\xi})}\\&=\frac{1-\frac{\alpha k}{2h}\left(1-cos(h\xi)+isin(h\xi)\right)}{1+\frac{\alpha k}{2h}\left(1-cos(h\xi)+isin(h\xi)\right)}\\&=\frac{\left(1-\frac{\alpha k}{2h}(1-cos(h\xi))\right)-\frac{\alpha k}{2h}isin(h\xi)}{\left(1+\frac{\alpha k}{2h}(1-cos(h\xi))\right)+\frac{\alpha k}{2h}isin(h\xi)}\end{aligned}$$

Điều kiện stable:

$$|G(\xi)|\leq 1$$

$$\sqrt{\left(1-\frac{\alpha k}{2h}(1-cos(h\xi))\right)^2+\left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)}\leq\sqrt{\left(1+\frac{\alpha k}{2h}(1-cos(h\xi))\right)^2+\left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)}$$

$$-\frac{\alpha k}{2h}(1-cos(h\xi))\leq\frac{\alpha k}{2h}(1-cos(h\xi))$$

$$-\frac{\alpha k}{h}sin^2\left(\frac{h\xi}{2}\right)\leq\frac{\alpha k}{h}sin^2\left(\frac{h\xi}{2}\right)$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\alpha, k, h>0$.
Vậy "UpWind schemes using Crank-Nicolson with $\kappa=0$" luôn stable.
Hoàn toàn tương tự cho trường hợp $\alpha<0$.

4, Using Crank − Nicolson method with $\kappa \neq 0$.
$(\star)$ Lax-Friedrich schemes:

Rời rạc hóa phương trình (1) ta được:

$$U_j^{n+1} + \frac{\alpha k}{4h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) - \frac{\kappa k}{2h^2}(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) = U_j^n - \frac{\alpha k}{4h}(U_{j+1}^n - U_{j-1}^n) + \frac{\kappa k}{2h^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

Sử dụng Von-Newmann analysis đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left(1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) - \frac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})\right) = e^{ijh\xi}\left(1 - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) + \frac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})\right)$$

Rút gọn 2 vế cho $e^{ijh\xi}$ ta được:

$$\begin{aligned} G(\xi) &= \frac{1 - \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) + \frac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})}{1 + \frac{\alpha k}{4h}(e^{ih\xi} - e^{-ih\xi}) - \frac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})} \\ &= \frac{1 - i\frac{\alpha k}{2h}sin(h\xi) - \frac{\kappa k}{h^2}(1 - cos(h\xi))}{1 + i\frac{\alpha k}{2h}sin(h\xi) + \frac{\kappa k}{h^2}(1 - cos(h\xi))} \end{aligned}$$

Điều kiện stable:

$$\begin{aligned} |G(\xi)| &\leq 1 \\ \sqrt{\left(1 - \frac{\kappa k}{h^2}(1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)} &\leq \sqrt{\left(1 + \frac{\kappa k}{h^2}(1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)} \\ -\frac{\kappa k}{h^2}(1 - cos(h\xi)) &\leq \frac{\kappa k}{h^2}(1 - cos(h\xi)) \\ -\frac{\kappa k}{h^2}\left(2sin^2\left(\frac{h\xi}{2}\right)\right) &\leq \frac{\kappa k}{h^2}\left(2sin^2\left(\frac{h\xi}{2}\right)\right) \end{aligned}$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\kappa, k, h > 0$.
Vậy "Lax-Friedrich schemes using Crank-Nicolson with $\kappa \neq 0$" luôn stable.

($\star$) Lax-Wendroff schemes:

$$\begin{aligned} U_j^{n+1} + \frac{\alpha k}{4h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) - \left(\frac{\alpha^2 k^2 + 2\kappa}{4h^2}\right)(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) \\ = U_j^n - \frac{\alpha k}{4h}(U_{j+1}^n - U_{j-1}^n) + \left(\frac{\alpha^2 k^2 + 2\kappa}{4h^2}\right)(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \end{aligned}$$

Tương tự đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$

Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left(1+\frac{\alpha k}{4h}(e^{ih\xi}-e^{-ih\xi})-\left(\frac{\alpha^2k^2+2\kappa}{4h^2}\right)(e^{ih\xi}-2+e^{-ih\xi})\right)$$
$$=e^{ijh\xi}\left(1-\frac{\alpha k}{4h}(e^{ih\xi}-e^{-ih\xi})+\left(\frac{\alpha^2k^2+2\kappa}{4h^2}\right)(e^{ih\xi}-2+e^{-ih\xi})\right)$$

Rút gọn 2 vế cho $e^{ijh\xi}$

$$G(\xi)=\frac{1-\dfrac{\alpha k}{4h}(e^{ih\xi}-e^{-ih\xi})+\left(\dfrac{\alpha^2k^2+2\kappa}{4h^2}\right)(e^{ih\xi}-2+e^{-ih\xi})}{1+\dfrac{\alpha k}{4h}(e^{ih\xi}-e^{-ih\xi})-\left(\dfrac{\alpha^2k^2+2\kappa}{4h^2}\right)(e^{ih\xi}-2+e^{-ih\xi})}$$
$$=\frac{1-i\dfrac{\alpha k}{2h}sin(h\xi)-\left(\dfrac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))}{1+i\dfrac{\alpha k}{2h}sin(h\xi)+\left(\dfrac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))}$$

Điều kiện stable:

$$|G(\xi)|\leq 1$$
$$\sqrt{\left(1-\left(\frac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))\right)^2+\left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)}\leq\sqrt{\left(1+\left(\frac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))\right)^2+\left(\frac{\alpha k}{2h}\right)^2 si}$$
$$-\left(\frac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))\leq\left(\frac{\alpha^2k^2+2\kappa}{2h^2}\right)(1-cos(h\xi))$$
$$-\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)\leq\left(\frac{\alpha^2k^2+2\kappa}{h^2}\right)sin^2\left(\frac{h\xi}{2}\right)$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\kappa, k, h > 0$.
Vậy "Lax-Wendroff schemes using Crank-Nicolson with $\kappa \neq 0$" luôn stable.

($\star$) Upwind schemes:
Xét với trường hợp $\alpha \geq 0$ ta có xấp xỉ

$$U_j^{n+1}+\frac{\alpha k}{2h}(U_j^{n+1}-U_{j-1}^{n+1})-\frac{\kappa k}{2h^2}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})=U_j^n-\frac{\alpha k}{2h}(U_j^n-U_{j-1}^n)+\frac{\kappa k}{2h^2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)$$

Đặt: $U_j^n = e^{ijh\xi}$
Khi đó ta có: $U_j^{n+1} = G(\xi)e^{ijh\xi}$
Thế vào ta được:

$$G(\xi)e^{ijh\xi}\left(1+\frac{\alpha k}{2h}(1-e^{-ih\xi})-\frac{\kappa k}{2h^2}(e^{ih\xi}-2+e^{-ih\xi})\right)=e^{ijh\xi}\left(1-\frac{\alpha k}{2h}(1-e^{-ih\xi})+\frac{\kappa k}{2h^2}(e^{ih\xi}-2+e^{-ih\xi})\right)$$

Chia 2 vế cho $e^{ijh\xi}$, ta được:

$$
\begin{aligned}
G(\xi) &= \frac{1 - \dfrac{\alpha k}{2h}(1 - e^{-ih\xi}) + \dfrac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})}{1 + \dfrac{\alpha k}{2h}(1 - e^{-ih\xi}) - \dfrac{\kappa k}{2h^2}(e^{ih\xi} - 2 + e^{-ih\xi})} \\
&= \frac{1 - \dfrac{\alpha k}{2h}\left(1 - cos(h\xi) + isin(h\xi)\right) + \dfrac{\kappa k}{h^2}(cos(h\xi) - 1)}{1 + \dfrac{\alpha k}{2h}\left(1 - cos(h\xi) + isin(h\xi)\right) - \dfrac{\kappa k}{h^2}(cos(h\xi) - 1)} \\
&= \frac{\left(1 - \left(\dfrac{\alpha k}{2h} + \dfrac{\kappa k}{h^2}\right)(1 - cos(h\xi))\right) - \dfrac{\alpha k}{2h} isin(h\xi)}{\left(1 - \left(\dfrac{\alpha k}{2h} - \dfrac{\kappa k}{h^2}\right)(1 - cos(h\xi))\right) + \dfrac{\alpha k}{2h} isin(h\xi)}
\end{aligned}
$$

Điều kiện stable:

$$
\begin{aligned}
|G(\xi)| &\leq 1 \\
\sqrt{\left(1 - \left(\frac{\alpha k}{2h} + \frac{\kappa k}{h^2}\right)(1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2(h\xi)} &\leq \sqrt{\left(1 - \left(\frac{\alpha k}{2h} - \frac{\kappa k}{h^2}\right)(1 - cos(h\xi))\right)^2 + \left(\frac{\alpha k}{2h}\right)^2 sin^2
} \\
-\left(\frac{\alpha k}{2h} + \frac{\kappa k}{h^2}\right)(1 - cos(h\xi)) &\leq -\left(\frac{\alpha k}{2h} - \frac{\kappa k}{h^2}\right)(1 - cos(h\xi)) \\
-\frac{\kappa k}{h^2} sin^2\left(\frac{h\xi}{2}\right) &\leq \frac{\kappa k}{h^2} sin^2\left(\frac{h\xi}{2}\right)
\end{aligned}
$$

Ta thấy đẳng thức cuối luôn đúng với mọi $\kappa, k, h > 0$.
Vậy "Upwind schemes using Crank-Nicolson with $\kappa \neq 0$" luôn stable.