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Finite Volume Method.

Xét bài toán: Cho f $\in L^2(0,1)$:

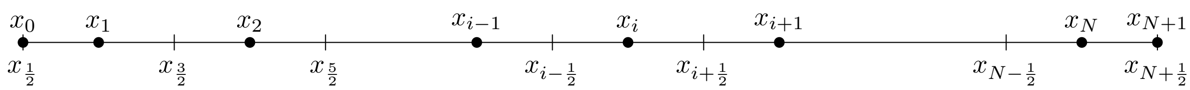
$$-u_{xx} = f(x) \qquad ,\forall x \in (0,1) \tag{1}$$

thỏa điều kiện biên Neumann:

$$\begin{cases} u'(0) = g0, \\ -u'(1) = g1. \end{cases}$$

điều kiện tồn tại nghiệm:

$$\begin{cases} \displaystyle\int_0^1 f(x)\mathrm{d}x &= g1 + g0, \\ \displaystyle\int_0^1 u(x)\mathrm{d}x &= 0. \end{cases}$$



Ta chọn (N+1) điểm $\{x_{i+\frac{1}{2}}\}_{i=\overline{0,N}}$ sao cho: $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1$.
Ta đặt $T_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$, $|T_i| = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad \forall i = \overline{1,N}$.
$x_0 = 0,\ x_{N+1} = 1,\ x_i \in T_i,\ \forall i = \overline{1,N}$.
$h = \max\limits_{i=\overline{1,N}}\{|T_i|\}$.
Ta gọi $(T_i)_{i=\overline{1,N}}$: là "control volume" và $(x_i)_{i=\overline{0,N+1}}$: là "control point".
(*) **Discrete divergence operator**

$$\begin{array}{rccc} d & : & \mathbb{R}^{N+1} & \longrightarrow & \mathbb{R}^N \\ & & \{v_{i+\frac{1}{2}}\}_{i=0}^N & \longmapsto & \{(dv)_i\}_{i=1}^N \end{array}$$

với $(dv)_i = \dfrac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{|T_i|}$ và $T_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$
Scalar product:
Cho $\{u_i\}_{i=1}^N, \{w_i\}_{i=1}^N$

$$(u,w)_T = \sum_{i=1}^N u_i w_i |T_i|$$
$$\|u\|_{0,T}^2 = (u,u)_T$$

(*) **Discrete gradient operator**

$$\begin{array}{rccc} g & : & \mathbb{R}^{N+2} & \longrightarrow & \mathbb{R}^{N+1} \\ & & \{u_i\}_{i=0}^{N+1} & \longmapsto & \{(gu)_{i+\frac{1}{2}}\}_{i=0}^N \end{array}$$

với $(gu)_{i+\frac{1}{2}} = \dfrac{u_{i+1} - u_i}{|D_{i+\frac{1}{2}}|}$ và $D_{i+\frac{1}{2}} = [x_i, x_{i+1}]$
<u>*Scalar product*</u>:
Cho $\{v_{i+\frac{1}{2}}\}_{i=0}^N, \{r_{i+\frac{1}{2}}\}_{i=0}^N$

$$(v, r)_D = \sum_{i=0}^{N} v_{i+\frac{1}{2}} r_{i+\frac{1}{2}} |D_{i+\frac{1}{2}}|$$
$$\|v\|_{0,D}^2 = (v, v)_D$$
$$|u|_{1,D}^2 = \|g(u)\|_{0,D}^2 = (g(u), g(u))_D$$

Câu 1:
Sử dụng tính chất: $w(x) - w(y) = \displaystyle\int_y^x w'(s)\mathrm{d}s$,
chứng minh bất đẳng thức "*Trace*" đúng với mọi $w \in H^1(0, 1)$

$$|w(x) - \overline{w}| \leq \|w'\|_{L^2(0,1)}$$

và bất đẳng thức "*Poincaré*" với mọi $w \in H^1(0, 1)$

$$\|w - \overline{w}\|_{L^2(0,1)} \leq \|w'\|_{L^2(0,1)}$$

trong đó $\overline{w} = \displaystyle\int_0^1 w(x)\mathrm{d}x$

Trả lời 1:
(*) Chứng minh bất đẳng thức "*Trace*".
Ta có:

$$\begin{aligned}
|w(x) - \overline{w}| &= \left| w(x) - \int_0^1 w(s)\mathrm{d}s \right| = \left| \int_0^1 w(x)\mathrm{d}s - \int_0^1 w(s)\mathrm{d}s \right| = \left| \int_0^1 w(x) - w(s)\mathrm{d}s \right| \\
&\leq \int_0^1 \big|w(x) - w(s)\big|\,\mathrm{d}s = \int_0^1 \left| \int_s^x w'(r)\mathrm{d}r \right| \mathrm{d}s \quad \text{(sử dụng bđt Holder)} \\
&\leq \int_0^1 \left\|w'\right\|_{L^2(0,1)} .|x - s|^{\frac{1}{2}}\mathrm{d}s \leq \left\|w'\right\|_{L^2(0,1)} .1 \qquad \text{(vì } 0 \leq s, x \leq 1 \text{ nên } |x - s| \leq 1)
\end{aligned}$$

(*) Chứng minh bất đẳng thức "*Poincaré*".
Ta có:

$$\begin{aligned}
&|w(x) - \overline{w}| \leq \|w'\|_{L^2(0,1)} \qquad \text{(Chứng minh trên)} \\
\Leftrightarrow &\int_0^1 |w(x) - \overline{w}|^2\mathrm{d}x \leq \int_0^1 \|w'\|_{L^2(0,1)}^2\mathrm{d}x \\
\Leftrightarrow &\int_0^1 |w - \overline{w}|^2(x)\mathrm{d}x \leq \|w'\|_{L^2(0,1)}^2 . \int_0^1 \mathrm{d}x \\
\Leftrightarrow &\|w - \overline{w}\|_{L^2(0,1)}^2 \leq \|w'\|_{L^2(0,1)}^2 \\
\Leftrightarrow &\|w - \overline{w}\|_{L^2(0,1)} \leq \|w'\|_{L^2(0,1)}
\end{aligned}$$

Câu 2:
Biến đổi phương trình (1) và chứng minh hệ thức sau cho nghiệm u của phương trình (1):

$$\|u'\|_{L^2(0,1)} \leq \|f\|_{L^2(0,1)} + |g1| + |g0|$$

Trả lời 2:
Ta có:

$$-u_{xx} = f(x) \qquad ,\forall x \in (0,1) \tag{1}$$

Nhân 2 vế với hàm *test function* (φ), với $\varphi \in H^1(0,1)$, và lấy tích phân trên miền [0,1].
Khi đó ta được:

$$\begin{aligned}
-\int_0^1 u_{xx}(x)\varphi(x)\mathrm{d}x &= \int_0^1 f(x)\varphi(x)\mathrm{d}x \qquad ,\forall \varphi \in H^1(0,1)\\
-u'(x)\varphi(x)|_0^1 + \int_0^1 u'(x)\varphi'(x)\mathrm{d}x &= \int_0^1 f(x)\varphi(x)\mathrm{d}x \qquad ,\forall \varphi \in H^1(0,1)\\
g1\varphi(1) + g0\varphi(0) + \int_0^1 u'(x)\varphi'(x)\mathrm{d}x &= \int_0^1 f(x)\varphi(x)\mathrm{d}x \qquad ,\forall \varphi \in H^1(0,1)
\end{aligned}$$

Thay $\varphi = u$, ta được:

$$\int_0^1 (u'(x))^2\mathrm{d}x = \int_0^1 f(x)u(x)\mathrm{d}x - g1u(1) - g0u(0)$$

Khi đó ta có:

$$\|u'\|^2_{L^2(0,1)} \leq \|f\|_{L^2(0,1)}\|u\|_{L^2(0,1)} + |g1u(1)| + |g0u(0)|$$

Theo câu 1, "*Trace*" có

$$|u(x) - \overline{u}| \leq \|u'\|_{L^2(0,1)} \qquad ,\forall x \in [0,1]$$

Với $\overline{u} = 0$, ta có:

$$\begin{aligned}
|u(1)| &\leq \|u'\|_{L^2(0,1)}\\
|u(0)| &\leq \|u'\|_{L^2(0,1)}
\end{aligned}$$

Suy ra,

$$\|u'\|^2_{L^2(0,1)} \leq \|f\|_{L^2(0,1)}\|u'\|_{L^2(0,1)} + |g1|\|u'\|_{L^2(0,1)} + |g0|\|u'\|_{L^2(0,1)}$$

Vậy,

$$\|u'\|_{L^2(0,1)} \leq \|f\|_{L^2(0,1)} + |g1| + |g0|$$

Câu 3:
Với $(w_i)_{i=\overline{0,N+1}}$ ta định nghĩa

$$\overline{w}_h = \sum_{i=1}^{N} |T_i| w_i$$

a) Chứng minh rằng với mọi $i, j = 1, \cdots, N$, ta có $|w_i - w_j| \leq |w|_{1,D}$.

b) Chứng minh bât đẳng thức "*Trace*" ở dạng rời rạc, với mọi $i = \overline{1, N}$:

$$|w_i - \overline{w}_h| \leq |w|_{1,D}$$

c) Chứng minh bât đẳng thức "*Poincaré*" ở dạng rời rạc:

$$\|w - \overline{w}_h\|_{0,T} \leq |w|_{1,D}$$

Trả lời 3:

a) Ta có:

$$\begin{aligned}|w|^2_{1,D} &= (g(w), g(w))_D = \sum_{i=0}^{N} \left(\frac{w_{i+1} - w_i}{|D_{i+\frac{1}{2}}|}\right)^2 |D_{i+\frac{1}{2}}| = \sum_{i=0}^{N} \frac{(w_{i+1} - w_i)^2}{|D_{i+\frac{1}{2}}|} \\ &\geq \sum_{i=0}^{j-1} \frac{(w_j - w_i)^2}{|D_{i+\frac{1}{2}}|} \geq \frac{(w_j - w_i)^2}{|x_j - x_i|} \geq (w_j - w_i)^2\end{aligned}$$

$$\Rightarrow |w_i - w_j| \leq |w|_{1,D}$$

b) Ta có:

$$\begin{aligned}|w_i - \overline{w}_h| &= \left|w_i \sum_{j=1}^{N} |T_j| - \sum_{j=1}^{N} |T_j| w_j\right| \leq \sum_{j=1}^{N} |T_j||w_i - w_j| \leq \sum_{j=1}^{N} |T_j||w|_{1,D} \quad \text{(câu a)} \\ &\leq |w|_{1,D} \sum_{j=1}^{N} |T_j| = |w|_{1,D}\end{aligned}$$

c) Ta có:

$$\begin{aligned}&\|w - \overline{w}_h\|^2_{0,T} = \sum_{j=1}^{N} (w_i - \overline{w}_h)^2 |T_j| \leq |w|^2_{1,D} \sum_{j=1}^{N} |T_j| = |w|^2_{1,D} \\ &\Rightarrow \|w - \overline{w}_h\|_{0,T} \leq |w|_{1,D}\end{aligned}$$

Câu 4:
Cho $\{v_{i+\frac{1}{2}}\}_{i=0}^{N}$, $\{w_i\}_{i=0}^{N+1}$. Chứng minh:

$$(d(v), w)_T = -(v, g(w))_D + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}}$$

Trả lời 4:

Với các toán tử **d** và **g**, các *Scalar product* $(\cdot,\cdot)_T$ và $(\cdot,\cdot)_D$ xác định ở phần đầu. Ta có:

$$\begin{aligned}
(d(v),w)_T+(v,g(w))_D &= \sum_{i=1}^{N}(dv)_i w_i|T_i| + \sum_{i=0}^{N} v_{i+\frac{1}{2}}(gw)_{i+\frac{1}{2}}|D_{i+\frac{1}{2}}| \\
&= \sum_{i=1}^{N}(v_{i+\frac{1}{2}}-v_{i-\frac{1}{2}})w_i + \sum_{i=0}^{N} v_{i+\frac{1}{2}}(w_{i+1}-w_i) \\
&= \sum_{i=1}^{N} v_{i+\frac{1}{2}}w_i - \sum_{i=1}^{N} v_{i-\frac{1}{2}}w_i + \sum_{i=0}^{N} v_{i+\frac{1}{2}}w_{i+1} - \sum_{i=0}^{N} v_{i+\frac{1}{2}}w_i \\
&= \sum_{i=1}^{N} v_{i+\frac{1}{2}}w_i - \sum_{i=1}^{N} v_{i-\frac{1}{2}}w_i + \sum_{i=0}^{N-1} v_{i+\frac{1}{2}}w_{i+1} + v_{N+\frac{1}{2}}w_{N+1} - \left(\sum_{i=1}^{N} v_{i+\frac{1}{2}}w_i + v_{\frac{1}{2}}w_0\right) \\
&= \sum_{i=1}^{N} v_{i+\frac{1}{2}}w_i - \sum_{i=1}^{N} v_{i-\frac{1}{2}}w_i + \sum_{i=1}^{N} v_{i-\frac{1}{2}}w_i + v_{N+\frac{1}{2}}w_{N+1} - \sum_{i=1}^{N} v_{i+\frac{1}{2}}w_i - v_{\frac{1}{2}}w_0 \\
&= v_{N+\frac{1}{2}}w_{N+1} - v_{\frac{1}{2}}w_0 \qquad \text{(điều cần chứng minh)}
\end{aligned}$$

Câu 5:
Với mọi $(w_i)_{i\in[0,N+1]}$, chứng minh rằng:

$$(g(u),g(w))_D = (f,w)_T - g0w_0 - g1w_{N+1}$$

Trả lời 5:
Với $(u_i)_{i\in[0,N+1]}$, ta có phương trình:

$$\begin{cases} -d(g(u))_i = f_i \\ (gu)_{\frac{1}{2}} = g0 \\ (gu)_{N+\frac{1}{2}} = -g1 \end{cases} \tag{5.1}$$

Nhân 2 vế phương trình (5.1) cho $|T_i|(w_i)_{i\in[0,N+1]}$ rồi lấy tổng theo i, ta được:

$$\begin{aligned}
-\sum_{i=1}^{N} d(g(u))_i w_i|T_i| &= \sum_{i=1}^{N} f_i w_i |T_i| \\
-(d(gu),w)_T &= (f,w)_T
\end{aligned}$$

Ta có ở **câu 4**:

$$(d(v),w)_T = -(v,g(w))_D + v_{N+\frac{1}{2}}w_{N+1} - w_0 v_{\frac{1}{2}}$$

Đặt $\{v_{i+\frac{1}{2}}\}_{i=0}^{N} = \{(gu)_{i+\frac{1}{2}}\}_{i=0}^{N}$, khi đó:

$$\begin{aligned}
-(d(gu),w)_T &= (f,w)_T \\
(g(u),g(w))_D - (gu)_{N+\frac{1}{2}}w_{N+1} + w_0(gu)_{\frac{1}{2}} &= (f,w)_T \\
(g(u),g(w))_D + g1w_{N+1} + g0w_0 &= (f,w)_T \\
(g(u),g(w))_D &= (f,w)_T - g1w_{N+1} - g0w_0
\end{aligned}$$

Câu 6:
Sử dụng bất đẳng thức *"Poincare"* và *"Trace"* đã chứng minh ở trước, chứng minh rằng nghiệm rời rạc $(u_i)_{i\in[0,N+1]}$ thỏa bất thức sau:

$$\|u\|_{1.D} \leq \|f\|_{0,T} + (|g0| + |g1|)$$

Trả lời 6:
Theo **câu 5**, ta có:

$$(g(u), g(w))_D = (f, w)_T - g0w_0 - g1w_{N+1}$$

Thay w = u, ta được:

$$\|u\|_{1,D}^2 = (f, u)_T - g0u_0 - g1u_{N+1}$$

Khi đó,

$$\|u\|_{1,D}^2 \leq \|f\|_{0,T}\|u\|_{0,T} + |g0||u_0| + |g1||u_{N+1}|$$

Theo **câu 3**, ta có:

$$\begin{aligned} |u_0| &\leq \|u\|_{1,D} \\ |u_{N+1}| &\leq \|u\|_{1,D} \\ \|u\|_{0,T} &\leq \|u\|_{1,D} \end{aligned}$$

Vậy

$$\|u\|_{1,D} \leq \|f\|_{0,T} + |g0| + |g1|$$

Câu 7:
Với $u \in C^1(\overline{\Omega})$. Ta xác định toán tử:

$$\begin{array}{cccll} \prod & : & C^1(\overline{\Omega}) & \longrightarrow & \mathbb{R}^{N+2} \\ & & u & \longmapsto & (\prod u)_i = u(x_i) \end{array} \quad \forall i \in [0, N+1]$$

Với $u' \in C^0(\overline{\Omega})$. Ta xác định toán tử:

$$\begin{array}{cccll} P & : & C^0(\overline{\Omega}) & \longrightarrow & \mathbb{R}^{N+1} \\ & & u' & \longmapsto & (Pu')_{i+\frac{1}{2}} = u'(x_{i+\frac{1}{2}}) \end{array} \quad \forall i \in [0, N]$$

Với mọi $(w_i)_{i\in[0,N+1]}$, chứng minh rằng:

$$(gu, gw)_D = (Pu', gw)_D$$

Trả lời 7:
Ta có phương trình:

$$-u_{xx} = f$$

Nguyên hàm 2 vế trên từng miền T_i ta được:

$$
\begin{aligned}
-\frac{1}{|T_i|}[u'(x_{i+\frac{1}{2}}) - u'(x_{i-\frac{1}{2}})] &= f_i \\
-\frac{1}{|T_i|}\left[(Pu')_{i+\frac{1}{2}} + (Pu')_{i-\frac{1}{2}}\right] &= f_i \\
-(d(Pu'))_i &= f_i
\end{aligned}
$$

Nhân 2 vế cho $(w_i)_{i\in[0,N+1]}$, với $w_0 = w_{N+1} = 0$, ta được:

$$
\begin{aligned}
-(d(Pu'))_i w_i &= f_i w_i \\
-\sum_{i=1}^{N}(d(Pu'))_i w_i |T_i| &= \sum_{i=1}^{N} f_i w_i |T_i| \\
-(d(Pu'), w)_T &= (f, w)_T \\
(Pu', g(w))_D - Pu'_{N+\frac{1}{2}} w_{N+1} + Pu'_{\frac{1}{2}} w_0 &= (f, w)_T \qquad \text{(theo câu 4)} \\
(Pu', g(w))_D &= (f, w)_T
\end{aligned}
$$

Mặt khác, theo câu 5, với $(w_i)_{i\in[0,N+1]}$, $(w_0 = w_{N+1} = 0)$:

$$(g(u), g(w))_D = (f, w)_T$$

Suy ra

$$(g(u), g(w))_D = (Pu', g(w))_D$$

Câu 8:
Ta đặt:

$$\epsilon_{i+\frac{1}{2}} = u'(x_{i+\frac{1}{2}}) - \frac{u(x_{i+1}) - u(x_i)}{|D_{i+\frac{1}{2}}|}$$

Ta chứng minh rằng:

$$|D_{i+\frac{1}{2}}|\epsilon^2_{i+\frac{1}{2}} \leq \left(\frac{2}{3}\right)^2 4h^2 \int_{D_{i+\frac{1}{2}}} f^2(t)\mathrm{d}t$$

Nếu $x_{i+\frac{1}{2}}$ là điểm chính giữa $D_{i+\frac{1}{2}}$ thì:

$$|D_{i+\frac{1}{2}}|\epsilon^2_{i+\frac{1}{2}} \leq \left(\frac{4}{15}\right)^2 h^4 \|f'\|_{L^2(D_{i+\frac{1}{2}})}$$

trong đó $h = \max\{|T_i| : i \in [1, N]\}$

Trả lời 8:

Ta có:

$$
\begin{aligned}
&|u'(x_{i+\frac{1}{2}})(x_{i+1}-x_{i+\frac{1}{2}})-u(x_{i+1})+u(x_{i+\frac{1}{2}})| = \left| u'(x_{i+\frac{1}{2}})\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \mathrm{d}s - \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} u'(s)\mathrm{d}s \right| \\
&= \left| \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} u'(x_{i+\frac{1}{2}}) - u'(s)\mathrm{d}s \right| = \left| \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{s}^{x_{i+\frac{1}{2}}} u''(r)\mathrm{d}r\mathrm{d}s \right| = \left| \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{s}^{x_{i+\frac{1}{2}}} f(r)\mathrm{d}r\mathrm{d}s \right| \\
&\leq \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left| \int_{s}^{x_{i+\frac{1}{2}}} 1^2\mathrm{d}r \right|^{1/2} \mathrm{d}s \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f(r)^2\mathrm{d}r \right)^{1/2} \\
&\leq \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left| x_{i+\frac{1}{2}} - s \right|^{1/2} \mathrm{d}s \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f(r)^2\mathrm{d}r \right)^{1/2} \\
&\leq \frac{2}{3}(x_{i+1}-x_{i+\frac{1}{2}})^{3/2} \left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} f(r)^2\mathrm{d}r \right)^{1/2}
\end{aligned}
$$

Tương tự ta có,

$$
|u'(x_{i+\frac{1}{2}})(x_{i+\frac{1}{2}}-x_i)-u(x_{i+\frac{1}{2}})+u(x_i)| \leq \frac{2}{3}(x_{i+\frac{1}{2}}-x_i)^{3/2} \left(\int_{x_i}^{x_{i+\frac{1}{2}}} f(r)^2\mathrm{d}r \right)^{1/2}
$$

Khi đó, ta có

$$
\begin{aligned}
|D_{i+\frac{1}{2}}\epsilon_{i+\frac{1}{2}}| &= |u'(x_{i+\frac{1}{2}})(x_{i+1}-x_i)-u(x_{i+1})+u(x_i)| \\
&\leq \frac{2}{3}(x_{i+1}-x_i)^{3/2} \left(\int_{x_i}^{x_{i+1}} f(r)^2\mathrm{d}r \right)^{1/2}
\end{aligned}
$$

Suy ra:

$$
|D_{i+\frac{1}{2}}\epsilon_{i+\frac{1}{2}}|^2 \leq \left(\frac{2}{3}\right)^2 (x_{i+1}-x_i)^3 \left(\int_{x_i}^{x_{i+1}} f(r)^2\mathrm{d}r \right) = \left(\frac{2}{3}\right)^2 D_{i+\frac{1}{2}}^3 \int_{D_{i+\frac{1}{2}}} f(r)^2\mathrm{d}r
$$

với $|D_{i+\frac{1}{2}}| = |x_{i+1}-x_i| \leq |x_{i+1}-x_{i+\frac{1}{2}}| + |x_{i+\frac{1}{2}}-x_i| = |T_{i+1}| + |T_i| \leq 2h$
Vậy

$$
|D_{i+\frac{1}{2}}|\epsilon_{i+\frac{1}{2}}^2 \leq \left(\frac{2}{3}\right)^2 (2h)^2 \int_{D_{i+\frac{1}{2}}} f(r)^2\mathrm{d}r
$$

(*) Nếu $x_{i+\frac{1}{2}}$ là điểm chính giữa $D_{i+\frac{1}{2}}$

Ta có:

$$
\begin{aligned}
&\left|u''(x_{i+\frac{1}{2}})\frac{(x_{i+1}-x_{i+\frac{1}{2}})^2}{2}+u'(x_{i+\frac{1}{2}})(x_{i+1}-x_{i+\frac{1}{2}})-u(x_{i+1})+u(x_{i+\frac{1}{2}})\right|\\
&=\left|u''(x_{i+\frac{1}{2}})\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}(s-x_{i+\frac{1}{2}})\mathrm{d}s+u'(x_{i+\frac{1}{2}})\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\mathrm{d}s-\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}u'(s)\mathrm{d}s\right|\\
&=\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}u''(x_{i+\frac{1}{2}})(s-x_{i+\frac{1}{2}})\mathrm{d}s+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}u'(x_{i+\frac{1}{2}})-u'(s)\mathrm{d}s\right|\\
&=\left|-\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}u''(x_{i+\frac{1}{2}})\mathrm{d}r\mathrm{d}s+\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}u''(r)\mathrm{d}r\mathrm{d}s\right|\\
&=\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}u''(r)-u''(x_{i+\frac{1}{2}})\mathrm{d}r\mathrm{d}s\right|=\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}\int_{x_{i+\frac{1}{2}}}^{r}u'''(t)\mathrm{d}t\mathrm{d}r\mathrm{d}s\right|\\
&=\left|\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}\int_{x_{i+\frac{1}{2}}}^{r}f'(t)\mathrm{d}t\mathrm{d}r\mathrm{d}s\right|\\
&\le\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}\left(\int_{x_{i+\frac{1}{2}}}^{r}\mathrm{d}t\right)^{1/2}\left(\int_{x_{i+\frac{1}{2}}}^{r}(f'(t))^2\mathrm{d}t\right)^{1/2}\mathrm{d}r\mathrm{d}s\\
&\le\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}\int_{s}^{x_{i+\frac{1}{2}}}\left(r-x_{i+\frac{1}{2}}\right)^{1/2}\mathrm{d}r\mathrm{d}s\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}(f'(t))^2\mathrm{d}t\right)^{1/2}\\
&=\frac{4}{15}\left(x_{i+1}-x_{i+\frac{1}{2}}\right)^{5/2}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}(f'(t))^2\mathrm{d}t\right)^{1/2}\\
&=\frac{4}{15}\left(\frac{D_{i+\frac{1}{2}}}{2}\right)^{5/2}\left(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}(f'(t))^2\mathrm{d}t\right)^{1/2}
\end{aligned}
$$

Tương tự ta có:

$$
\begin{aligned}
&\left|u''(x_{i+\frac{1}{2}})\frac{(x_{i+\frac{1}{2}}-x_i)^2}{2}+u'(x_{i+\frac{1}{2}})(x_{i+\frac{1}{2}}-x_i)-u(x_{i+\frac{1}{2}})+u(x_i)\right|\\
&\le\frac{4}{15}\left(\frac{D_{i+\frac{1}{2}}}{2}\right)^{5/2}\left(\int_{x_i}^{x_{i+\frac{1}{2}}}(f'(t))^2\mathrm{d}t\right)^{1/2}
\end{aligned}
$$

Cộng hai vế suy ra:

$$|D_{i+\frac{1}{2}}\epsilon_{i+\frac{1}{2}}|^2\le\left(\frac{4}{15}\right)^2\left(\frac{D_{i+\frac{1}{2}}}{2}\right)^5\left(\int_{x_i}^{x_{i+1}}f(t)^2\mathrm{d}t\right)$$

Suy ra:

$$|D_{i+\frac{1}{2}}|\epsilon_{i+\frac{1}{2}}^2\le\left(\frac{4}{15}\right)^2\left(\frac{D_{i+\frac{1}{2}}}{2}\right)^4\|f'\|_{L^2(D_{i+\frac{1}{2}})}^2\le\left(\frac{4}{15}\right)^2h^4\|f'\|_{L^2(D_{i+\frac{1}{2}})}^2$$