Practical Assignment 3: Diffusion Advection Equations on 1D

Deadline:06/01/2020

Consider the diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < 1$$

where ϵ is the diffusion coefficient. The initial condition is

$$u(x,0) = u_0(x)$$

The boundary condition is

$$u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t)$$

or

$$\frac{\partial u}{\partial x}(a,t) = \psi_a(t), \quad \frac{\partial u}{\partial x}(b,t) = \psi_b(t)$$

Example: Let $\alpha = \frac{1}{16}$ and $a = 2$ and
Initial condition: $u_0(x) = sin(2\pi x)$
Boundary condition: $u(0,t) = u(1,t) = -\sin(4\pi t)$
Exact solution: $u(x,t) = e^{-\frac{1}{4}\pi^2 t} sin(2\pi(x - at))$
We implement and compare the following methods
1.1 Explicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^n - u_{i-1/2}^n}{|T_i|} = \alpha\left(\frac{u_{i+1}^n - u_i^n}{|D_{i+1/2}||T_i|} - \frac{u_i^n - u_{i-1}^n}{|D_{i-1/2}||T_i|}\right)$$

where $u_{i+1/2}^n$ is upwind scheme for advection term.
Accuracy: $O(\Delta t, h^2)$.
Fourier condition: $k(\frac{2\epsilon}{h^2} + \frac{|a|}{h}) \leq 1$
1.2 Implicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^n - u_{i-1/2}^n}{|T_i|} = \alpha\left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{|D_{i+1/2}||T_i|} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{|D_{i-1/2}||T_i|}\right)$$

Fourier condition: $\frac{|a|k}{h} \leq 1$
1.3 Crank-Nicolson method (1947)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^n - u_{i-1/2}^n}{|T_i|} = \frac{\alpha}{2}\left(\frac{u_{i+1}^n - u_i^n}{|D_{i+1/2}||T_i|} - \frac{u_i^n - u_{i-1}^n}{|D_{i-1/2}||T_i|} + \frac{u_{i+1}^{n+1} - u_i^{n+1}}{|D_{i+1/2}||T_i|} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{|D_{i-1/2}||T_i|}\right)$$

Accuracy: $O((\Delta t)^2, h^2)$.
Fourier condition: $\frac{|a|k}{h} \leq 1$
1.4 Generalization

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^n - u_{i-1/2}^n}{|T_i|} = \alpha\left(\theta\left[\frac{u_{i+1}^n - u_i^n}{|D_{i+1/2}||T_i|} - \frac{u_i^n - u_{i-1}^n}{|D_{i-1/2}||T_i|}\right] + (1-\theta)\left[\frac{u_{i+1}^{n+1} - u_i^{n+1}}{|D_{i+1/2}||T_i|} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{|D_{i-1/2}||T_i|}\right.\right.$$

1.5 **Question**: Compare previous method? (accuracy, stability,computing time).