# Finite Volume Method
## (Stokes Equation)

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Finite Volume Method

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# Introduction Stokes equation

## Introduction to Stokes Equation

we use the finite volume method to solve the following equation in $\Omega = [0,1] \times [0,1]$

$$
\begin{gathered}
-\Delta U(x) + \bigtriangledown p(x) = f(x) \quad \forall x \in \Omega \\
\bigtriangledown \cdot U(x) = 0 \quad in \quad \Omega \\
U = 0 \quad on \quad \partial\Omega \\
\int_{\Omega} p(x) dx = 0
\end{gathered}
$$

# Introduction Stokes equation

## Introduction to Stokes Equation

It can be rewrite as follow :

$$-\Delta u(x,y)+\frac{\partial p}{\partial x}(x,y)=f_1(x) \quad \forall x\in\Omega \tag{1.1}$$

$$-\Delta v(x,y)+\frac{\partial p}{\partial y}(x,y)=f_2(x) \quad \forall x\in\Omega \tag{1.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad in \quad \Omega \tag{1.3}$$

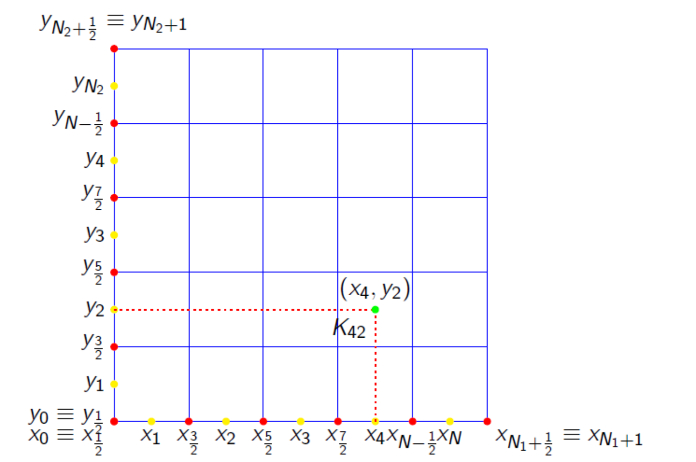
$$u=0 \quad on \quad \partial\Omega \tag{1.4}$$

$$v=0 \quad on \quad \partial\Omega \tag{1.5}$$

$$\int_\Omega p(x)dx=0 \tag{1.6}$$

# Mesh

## Mesh


$y_{N_2+\frac{1}{2}} \equiv y_{N_2+1}$
$y_{N_2}$
$y_{N-\frac{1}{2}}$
$y_4$
$y_{\frac{7}{2}}$
$y_3$
$y_{\frac{5}{2}}$
$y_2$
$y_{\frac{3}{2}}$
$y_1$
$y_0 \equiv y_{\frac{1}{2}}$
$x_0 \equiv x_{\frac{1}{2}}$
$x_1$ $x_{\frac{3}{2}}$ $x_2$ $x_{\frac{5}{2}}$ $x_3$ $x_{\frac{7}{2}}$ $x_4$ $x_{N-\frac{1}{2}}$ $x_N$
$x_{N_1+\frac{1}{2}} \equiv x_{N_1+1}$
$(x_4, y_2)$
$K_{42}$

# Mesh

## Mesh

Consider $\Omega = (0,1) \times (0,1)$ . On interval [0,1] we make two partition $(x_{i+\frac{1}{2}})_{i\in\overline{0,N_1}}$,$(y_{j+\frac{1}{2}})_{j\in\overline{0,N_2}}$ s.t

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N_1-\frac{1}{2}} < x_{N_1+\frac{1}{2}} = 1$$

$$0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < ... < y_{N_1-\frac{1}{2}} < y_{N_1+\frac{1}{2}} = 1$$

Let $\tau = (T_{ij})_{i\in\overline{1,N_1},j\in\overline{1,N_2}}$ be an admissible mesh of $(0,1) \times (0,1)$ s.t

$$T_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}]$$

$T_{ij}$ is called a control volume of $\tau$ . The point $(x_{i+\frac{1}{2})}, y_{j+\frac{1}{2}})$ are called mesh points.

# Mesh

## Mesh

Choosing the sequences $(x_i)_{i\in\overline{0,N_1+1}}, (y_j)_{j\in\overline{0,N_2+1}}$ s.t

$$x_0 = x_{\frac{1}{2}}, x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), x_{N_1+1} = x_{N_1+\frac{1}{2}} = 1$$

$$y_0 = y_{\frac{1}{2}}, y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}), y_{N_2+1} = y_{N_2+\frac{1}{2}} = 1$$

The point $(x_i, y_J)$ is called a control point of control volume $T_{ij}$ Let

$$h_i = |x_{i+\frac{1}{2})} - x_{i-\frac{1}{2})}|, \quad k_j = |y_{j+\frac{1}{2})} - y_{j-\frac{1}{2})}| \quad \forall i \in \overline{1, N_1}, j \in \overline{1, N_2}$$

$$h_{i+\frac{1}{2}} = |x_{i+1)} - x_i|, \quad k_{j+\frac{1}{2}} = |y_{j+1} - y_j| \quad \forall i \in \overline{0, N_1}, j \in \overline{0, N_2}$$

Then the area of control volume $T_{ij}{=}h_i k_j$, $h = max\left\{h_i, k_j\right\}$ is the mesh size

# Preliminary

## Formula

throughout this presentation we use the following formula :

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) + error \qquad (2.1)$$

and

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} f = \frac{1}{4}\left(f(x_i, y_{j-\frac{1}{2}}) + f(x_i, y_{j+\frac{1}{2}}) + f(x_{i-\frac{1}{2}}, y_j) + f(x_{i+\frac{1}{2}}, y_j)\right) + er$$

# Scheme

## Scheme

The finite volume scheme is found by integrating 1.1 over each control volume $T_{ij}$ , which gives

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} -\Delta u dxdy + \frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x} dxdy = \frac{1}{|T_{ij}|}\int_{T_{ij}} f dxdy \quad (2.2)$$

The first term in (2.3) is exactly what we have learn in our course so we just need consider the second term in LHS

# Scheme

Using formula 2.1 we get :

$$\frac{1}{|T_{ij}|}\int_{T_{ij}}\frac{\partial p}{\partial x}dxdy = \frac{1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial p}{\partial x}dxdy$$

$$= \frac{1}{2k_j}\left(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial p}{\partial x}(x_{i+\frac{1}{2}},y) + \frac{\partial p}{\partial x}(x_{i-\frac{1}{2}},y)dy\right)$$

$$= \frac{1}{2k_j}\left(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial p}{\partial x}(x_{i+\frac{1}{2}},y)dy + \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\frac{\partial p}{\partial x}(x_{i-\frac{1}{2}},y)d\right.$$

# Scheme

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \frac{\partial p}{\partial x}(x_{i+\frac{1}{2}}, y)dy = k_j \frac{\partial p}{\partial x}(x_{i+\frac{1}{2}}, y_j)$$

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \frac{\partial p}{\partial x}(x_{i-\frac{1}{2}}, y)dy = k_j \frac{\partial p}{\partial x}(x_{i+\frac{1}{2}}, y_j)$$

$$\begin{aligned}\frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x}dxdy &= \frac{1}{2}\left(\frac{\partial p}{\partial x}(x_{i+\frac{1}{2}}, y_j) + \frac{\partial p}{\partial x}(x_{i+\frac{1}{2}}, y_j)\right) \\ &= \frac{1}{2}\left(\frac{p_{i+1,j} - p_{i,j}}{h_{i+\frac{1}{2}}} + \frac{p_{i,j} - p_{i-1,j}}{h_{i-\frac{1}{2}}}\right) \\ &= \frac{1}{2}\left(\frac{1}{h_{i+\frac{1}{2}}}p_{i+1,j} + (\frac{1}{h_{i-\frac{1}{2}}} - \frac{1}{h_{i+\frac{1}{2}}})p_{i,j} - \frac{1}{h_{i-\frac{1}{2}}}p_{i-1,j}\right.\end{aligned}$$

# Scheme

## Scheme

At cell (i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$

$$-a_i u_{i-1,j} - b_i u_{i+1,j} - c_j u_{i,j-1} - d_j u_{i,j+1} + s_{i,j} u_{i,j}$$
$$+\frac{1}{2}\left(\frac{1}{h_{i+\frac{1}{2}}} p_{i+1,j} + \left(\frac{1}{h_{i-\frac{1}{2}}} - \frac{1}{h_{i+\frac{1}{2}}}\right) p_{i,j} - \frac{1}{h_{i-\frac{1}{2}}} p_{i-1,j}\right) = f_{i,j}$$

where

$$a_i = \frac{1}{h_i h_{i-\frac{1}{2}}}, b_i = \frac{1}{h_i h_{i+\frac{1}{2}}}$$

$$c_j = \frac{1}{k_j k_{i-\frac{1}{2}}}, d_j = \frac{1}{k_j k_{j+\frac{1}{2}}}$$

$$s_{i,j} = a_i + b_i + c_j + d_j$$

# Scheme

absolutely similar we get :

## Scheme

At cell (i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$

$$-a_i v_{i-1,j} - b_i v_{i+1,j} - c_j v_{i,j-1} - d_j v_{i,j+1} + s_{i,j} v_{i,j}$$
$$+\frac{1}{2}\left(\frac{1}{k_{j+\frac{1}{2}}} p_{i,j+1} + (\frac{1}{k_{j-\frac{1}{2}}} - \frac{1}{k_{j+\frac{1}{2}}}) p_{i,j} - \frac{1}{k_{j-\frac{1}{2}}} p_{i,j-1}\right) = f_{i,j}$$

# Scheme

we now consider the equations :

## Scheme

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad in \quad \Omega$$

Intergrate over each control volume $T_{ij}$ which gives :

$$\int_{T_{ij}} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0$$

# Scheme

similarly when we doing with p we have :

## Scheme

$$\frac{u_{i+1,j} - u_{i,j}}{h_{i+\frac{1}{2}}} + \frac{u_{i,j} - u_{i-1,j}}{h_{i-\frac{1}{2}}} + \frac{v_{i,j+1} - v_{i,j}}{k_{j+\frac{1}{2}}} + \frac{v_{i,j} - v_{i,j-1}}{k_{j+\frac{1}{2}}} = 0$$

# Matrix form

$$K = \begin{bmatrix} A & 0 & B \\ 0 & A & D \\ \frac{-1}{2h}B & \frac{-1}{2h}D & 0 \end{bmatrix}$$

where A is exactly in our course

$$B = \frac{-1}{2h} \begin{bmatrix} -1 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & & \dots & & & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 1 \end{bmatrix}$$

# Matrix form

$$D = \frac{-1}{2h}\begin{bmatrix} -I & I & & & & \\ -I & 0 & I & & & \\ & -I & 0 & I & & \\ & & & \dots & & \\ & & & -I & 0 & I \\ & & & & -I & I \end{bmatrix}$$

# Matrix form

$$KU = F$$

where

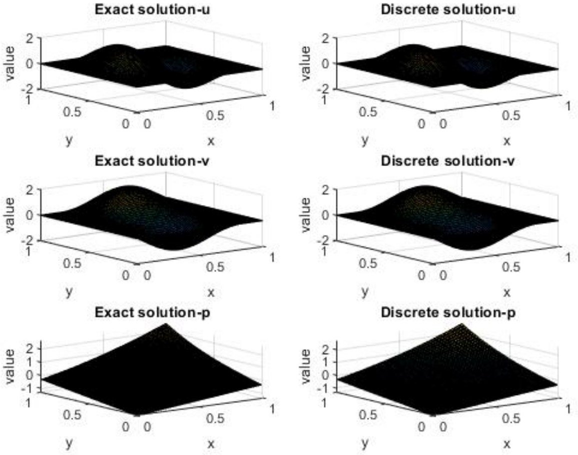
$$F = [F_1, F_2, 0], U = [u, v, p]$$

# Experiment test

Experiment Test we set up with the following system of equation

$$
\begin{aligned}
u &= (1 - cos(2\pi x))sin(2\pi y) \\
v &= -(1 - cos(2\pi y))sin(2\pi x) \\
p &= xy + x + y + x^3y^2 - \frac{4}{3}
\end{aligned}
$$

# Experiment test



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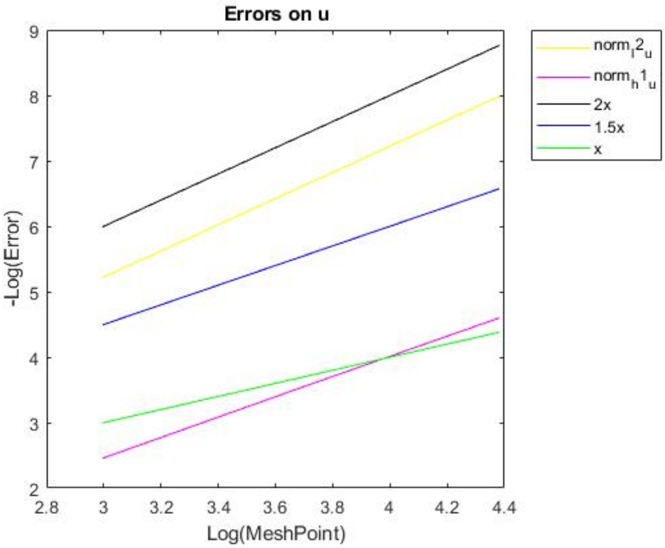

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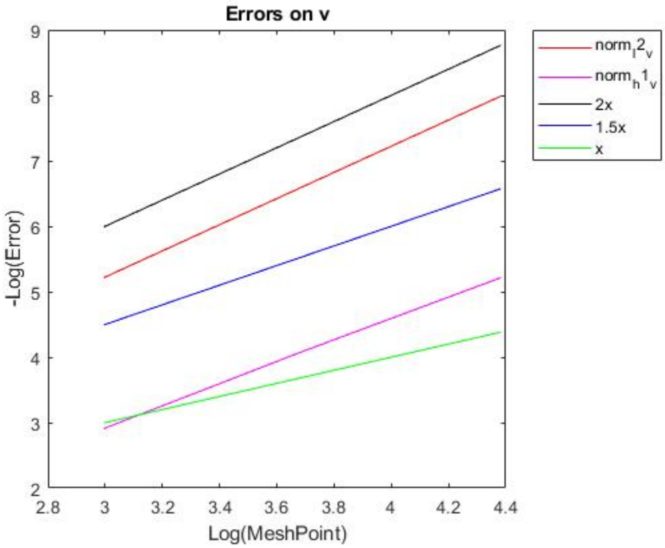

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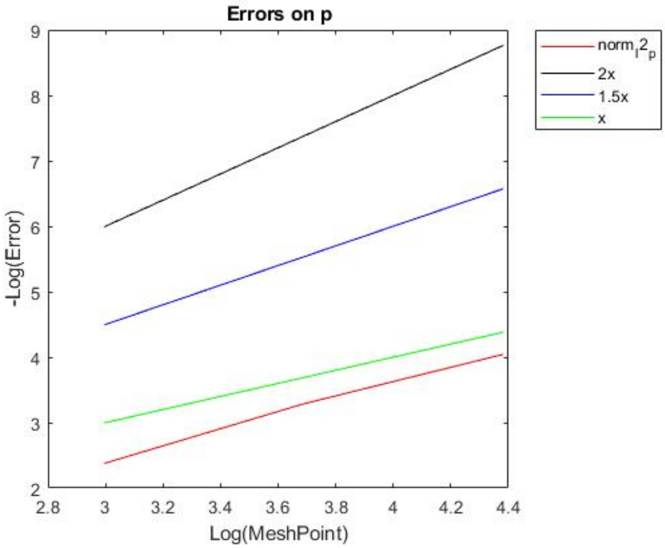

# Error

## Error

| number of elements | 20 | 40 | 80 |
|---|---|---|---|
| Error of $L_2$ norm on u | 0.0054 | 0.0014 | 0.0003 |
| Error of $H_0^1$ norm on u | 0.0859 | 0.0292 | 0.0101 |
| Error of $L_2$ norm on v | 0.0054 | 0.0014 | 0.0003 |
| Error of $H_0^1$ norm on v | 0.0545 | 0.0168 | 0.0054 |
| Error of $L_2$ norm on p | 0.0929 | 0.0374 | 0.0175 |

# Thanks For Watching