Finite Volume Method

(Stokes Equation)

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Finite Volume Method

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Introduction Stokes equation

Introduction to Stokes Equation

we use the finite volume method to solve the following equation in $\Omega = [0,1] \times [0,1]$

$$
\begin{aligned}
-\Delta U(x) + \triangledown p(x) = f(x) \quad \forall x \in \Omega \\
\triangledown \cdot U(x) = 0 \quad in \quad \Omega \\
U = 0 \quad on \quad \partial\Omega \\
\int_{\Omega} p(x)dx = 0
\end{aligned}
$$

Introduction Stokes equation

Introduction to Stokes Equation

It can be rewrite as follow :

$$-\Delta u(x,y)+\frac{\partial p}{\partial x}(x,y) = f_1(x) \quad \forall x \in \Omega \tag{1.1}$$

$$-\Delta v(x,y)+\frac{\partial p}{\partial y}(x,y) = f_2(x) \quad \forall x \in \Omega \tag{1.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad in \quad \Omega \tag{1.3}$$

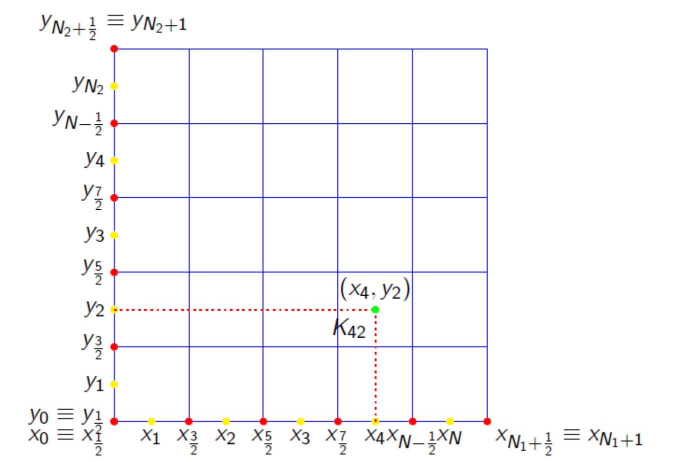
$$u = 0 \quad on \quad \partial\Omega \tag{1.4}$$

$$v = 0 \quad on \quad \partial\Omega \tag{1.5}$$

$$\int_\Omega p(x)dx = 0 \tag{1.6}$$

Mesh

Mesh


$y_{N_2+\frac{1}{2}} \equiv y_{N_2+1}$
y_{N_2}
$y_{N-\frac{1}{2}}$
y_4
$y_{\frac{7}{2}}$
y_3
$y_{\frac{5}{2}}$
y_2
$y_{\frac{3}{2}}$
y_1
$y_0 \equiv y_{\frac{1}{2}}$
$x_0 \equiv x_{\frac{1}{2}}$
x_1 $x_{\frac{3}{2}}$ x_2 $x_{\frac{5}{2}}$ x_3 $x_{\frac{7}{2}}$ x_4 $x_{N-\frac{1}{2}}$ x_N
$x_{N_1+\frac{1}{2}} \equiv x_{N_1+1}$
(x_4, y_2)
K_{42}

Mesh

Mesh

Consider $\Omega = (0,1) \times (0,1)$. On interval [0,1] we make two partition $(x_{i+\frac{1}{2}})_{i \in \overline{0,N_1}}$,$(y_{j+\frac{1}{2}})_{j \in \overline{0,N_2}}$ s.t

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N_1-\frac{1}{2}} < x_{N_1+\frac{1}{2}} = 1$$

$$0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \ldots < y_{N_1-\frac{1}{2}} < y_{N_1+\frac{1}{2}} = 1$$

Let $\tau = (T_{ij})_{i \in \overline{1,N_1}, j \in \overline{1,N_2}}$ be an admissible mesh of $(0,1) \times (0,1)$ s.t

$$T_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}]$$

T_{ij} is called a control volume of τ . The point $(x_{i+\frac{1}{2}}), y_{j+\frac{1}{2}})$ are called mesh points.

Mesh

Mesh

Choosing the sequences $(x_i)_{i\in\overline{0,N_1+1}}, (y_j)_{j\in\overline{0,N_2+1}}$ s.t

$$x_0 = x_{\frac{1}{2}}, x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), x_{N_1+1} = x_{N_1+\frac{1}{2}} = 1$$

$$y_0 = y_{\frac{1}{2}}, y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}), y_{N_2+1} = y_{N_2+\frac{1}{2}} = 1$$

The point (x_i, y_J) is called a control point of control volume T_{ij}
Let

$$h_i = |x_{i+\frac{1}{2})} - x_{i-\frac{1}{2})}|, \quad k_j = |y_{j+\frac{1}{2})} - y_{j-\frac{1}{2})}| \quad \forall i \in \overline{1, N_1}, j \in \overline{1, N_2}$$

$$h_{i+\frac{1}{2}} = |x_{i+1)} - x_i|, \quad k_{j+\frac{1}{2}} = |y_{j+1} - y_j| \quad \forall i \in \overline{0, N_1}, j \in \overline{0, N_2}$$

Then the area of control volume T_{ij}=$h_i k_j$, $h = max\{h_i, k_j\}$ is the mesh size

Preliminary

Formula

throughout this presentation we use the following formula :

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) + error \tag{2.1}$$

and

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} f = \frac{1}{4}\left(f(x_i, y_{j-\frac{1}{2}}) + f(x_i, y_{j+\frac{1}{2}}) + f(x_{i-\frac{1}{2}}, y_j) + f(x_{i+\frac{1}{2}}, y_j)\right) + er$$

Scheme

Scheme

The finite volume scheme is found by integrating 1.1 over each control volume T_{ij} , which gives

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} -\Delta u dxdy + \frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x} dxdy = \frac{1}{|T_{ij}|}\int_{T_{ij}} f dxdy \quad (2.2)$$

The first term in (2.3) is exactly what we have learn in our course so we just need consider the second term in LHS

Scheme

Using formula 2.1 we get :

$$\frac{1}{|T_{ij}|}\int_{T_{ij}}\frac{\partial p}{\partial x}dxdy = \frac{1}{|T_{ij}|}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\frac{\partial p}{\partial x}dxdy$$

$$= \frac{1}{|T_{ij}|}\left(\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} p(x_{i+\frac{1}{2}}, y) - p(x_{i-\frac{1}{2}}, y)dy\right)$$

$$= \frac{1}{h_i}\left(p(x_{i+\frac{1}{2}}, y_j) - p(x_{i-\frac{1}{2}}, y_j)\right)$$

Scheme

We will consider three following cases:

Case 1: if $i \neq 1$ and $i \neq N$

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x} dxdy = \frac{1}{h_i}\left(\frac{p_{i+1,j}+p_{i,j}}{2} - \frac{p_{i,j}+p_{i-1,j}}{2}\right) = \frac{1}{2h_i}(p_{i+1,j}-p_{i-1,j})$$

Case 2: if $i = 1$

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x} dxdy = \frac{1}{h_i}\left(\frac{p_{i+1,j}+p_{i,j}}{2} - p_{i,j}\right) = \frac{1}{2h_i}(p_{i+1,j}-p_{i,j})$$

Case 3: if $i = N$

$$\frac{1}{|T_{ij}|}\int_{T_{ij}} \frac{\partial p}{\partial x} dxdy = \frac{1}{h_i}\left(p_{i,j} - \frac{p_{i,j}+p_{i-1,j}}{2}\right)$$

Scheme

Scheme

At cell (i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$

$$-a_i u_{i-1,j} - b_i u_{i+1,j} - c_j u_{i,j-1} - d_j u_{i,j+1} + s_{i,j} u_{i,j} + \frac{1}{2h_i}(\epsilon_i p_{i-1,j} + \xi_i p_{i,j} + \eta_i p_{i+1,j}) = f_{i,j}$$

where

$$a_i = \frac{1}{h_i h_{i-\frac{1}{2}}}, b_i = \frac{1}{h_i h_{i+\frac{1}{2}}}$$

$$c_j = \frac{1}{k_j k_{i-\frac{1}{2}}}, d_j = \frac{1}{k_j k_{j+\frac{1}{2}}}$$

$$s_{i,j} = a_i + b_i + c_j + d_j$$

Scheme

$$\epsilon(i) = \begin{cases} -1 & \textbf{if} i \neq 1 \\ 0 & \textbf{if} i = 1 \end{cases}$$

$$\xi_i = \begin{cases} -1 & \textbf{if} i = 1 \\ 1 & \textbf{if} i = N \\ 0 & \textbf{if} i \neq 1, N \end{cases}$$

$$\eta(i) = \begin{cases} 1 & \textbf{if} i \neq N \\ 0 & \textbf{if} i = N \end{cases}$$

Scheme

absolutely similar we get :

Scheme

At cell (i,j) for $i \in [1, N_1]$ and $j \in [1, N_2]$

$$-a_i v_{i-1,j} - b_i v_{i+1,j} - c_j v_{i,j-1} - d_j v_{i,j+1} + s_{i,j} v_{i,j}$$
$$+\frac{1}{2h_i}\left(\epsilon_j p_{i,j-1} + \xi_j p_{i,j} + \eta_j p_{i,j+1}\right) = f_{i,j}$$

Scheme

we now consider the equations :

Scheme

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad in \quad \Omega$$

Intergrate over each control volume T_{ij} which gives :

$$\frac{1}{|T_{ij}|}\int_{T_{ij}}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Scheme

similarly when we doing with p we have :

Scheme

$$\frac{u_{i+1,j} + u_{i,j}}{2h_i} - \frac{u_{i,j} + u_{i-1,j}}{2h_i} + \frac{v_{i,j+1} + v_{i,j}}{2k_j} - \frac{v_{i,j} + v_{i,j-1}}{2k_j} = 0$$

Scheme

In other word,

Scheme

$$\frac{1}{2h_i}(\epsilon_i u_{i-1,j} + \xi_i u_{i,j} + \eta_i u_{i+1,j}) + \frac{1}{2k_j}(\epsilon_j v_{i,j-1} + \xi_j v_{i,j} + \eta_j v_{i,j+1}) = 0$$

Matrix form

$$K = \begin{bmatrix} A & 0 & B \\ 0 & A & D \\ B^T & D^T & 0 \end{bmatrix}$$

where A is exactly in our course

$$B = \frac{-1}{2h} \begin{bmatrix} -1 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & & \dots & & & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 1 \end{bmatrix}$$

Matrix form

$$D = \frac{-1}{2h}\begin{bmatrix} -I & I & & & & \\ -I & 0 & I & & & \\ & -I & 0 & I & & \\ & & & \cdots & & \\ & & & -I & 0 & I \\ & & & & -I & I \end{bmatrix}$$

Matrix form

$$KU = F$$

where

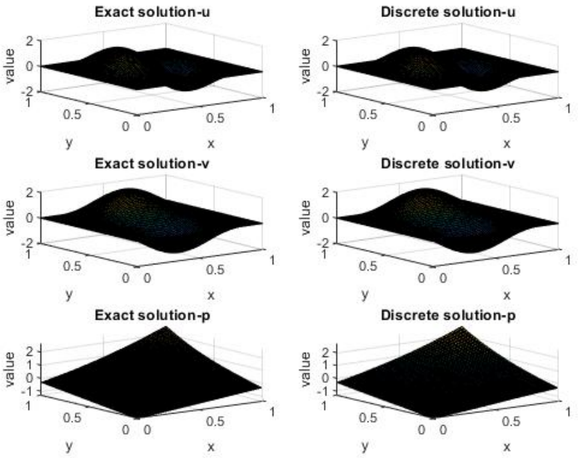
$$F = [F_1, F_2, 0], U = [u, v, p]$$

Experiment test

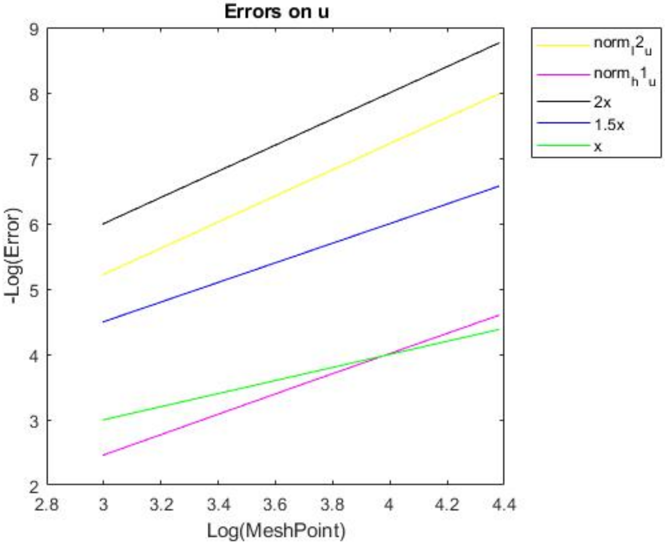
Experiment Test we set up with the following system of equation

$$
\begin{aligned}
u &= (1 - cos(2\pi x))sin(2\pi y) \\
v &= -(1 - cos(2\pi y))sin(2\pi x) \\
p &= xy + x + y + x^3y^2 - \frac{4}{3}
\end{aligned}
$$

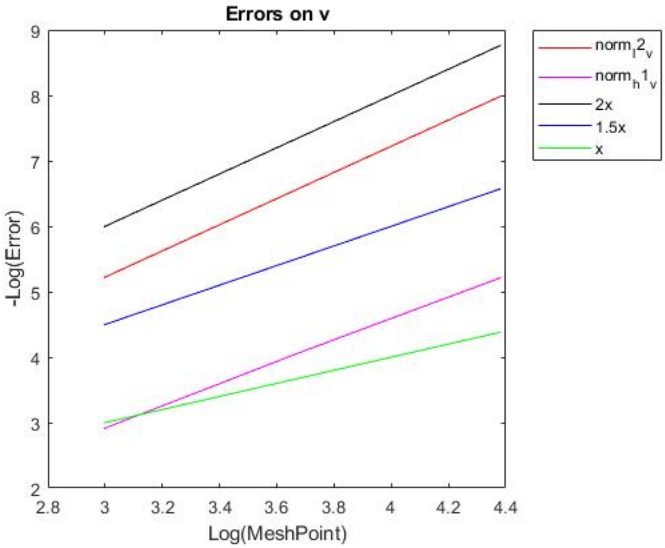
Experiment test



Experiment test



Experiment test



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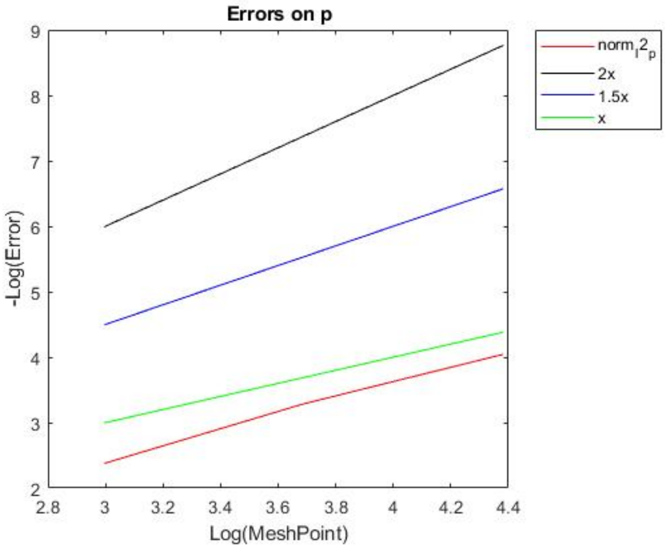


Error

Error

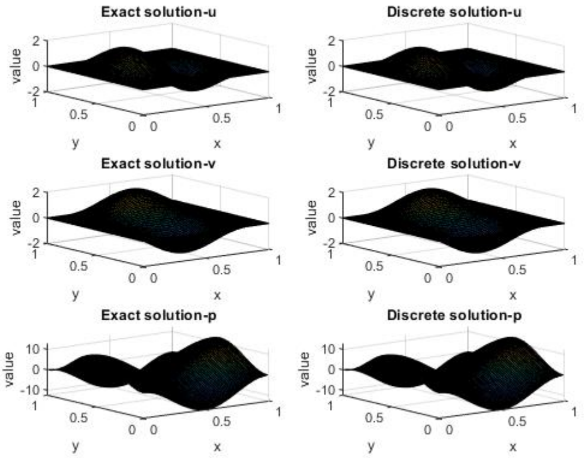
number of elements	20	40	80
Error of L_2 norm on u	0.0054	0.0014	0.0003
Error of H_0^1 norm on u	0.0859	0.0292	0.0101
Error of L_2 norm on v	0.0054	0.0014	0.0003
Error of H_0^1 norm on v	0.0545	0.0168	0.0054
Error of L_2 norm on p	0.0929	0.0374	0.0175

Experiment test

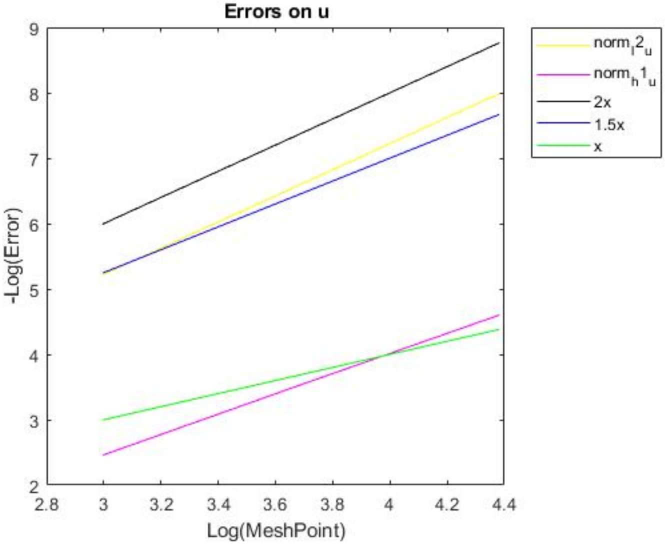
One more example !

$$u = (1 - cos(2\pi x))sin(2\pi y)$$
$$v = -(1 - cos(2\pi y))sin(2\pi x)$$
$$p = -2\pi(cos(2\pi x) - cos(2\pi y))$$

Experiment test



Experiment test



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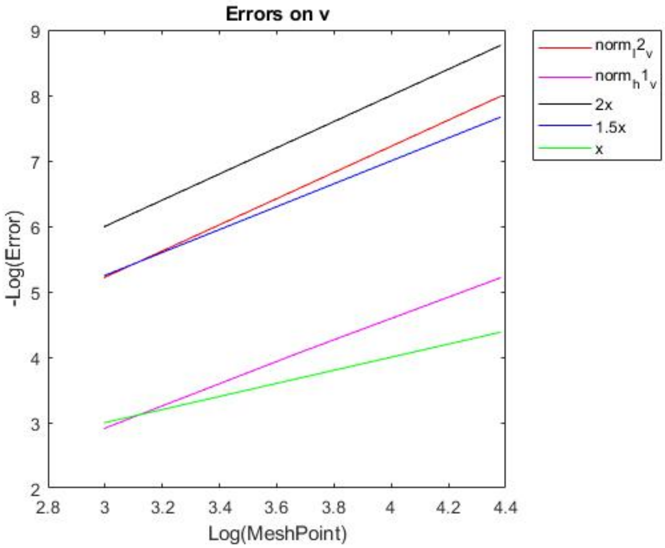
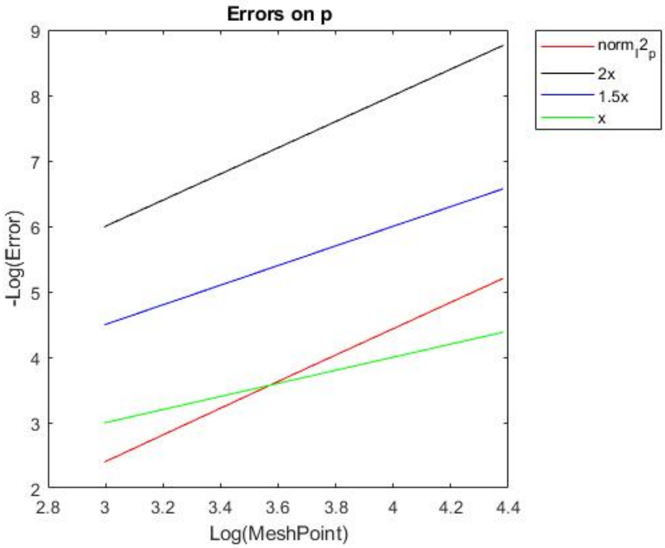


Experiment test



Error

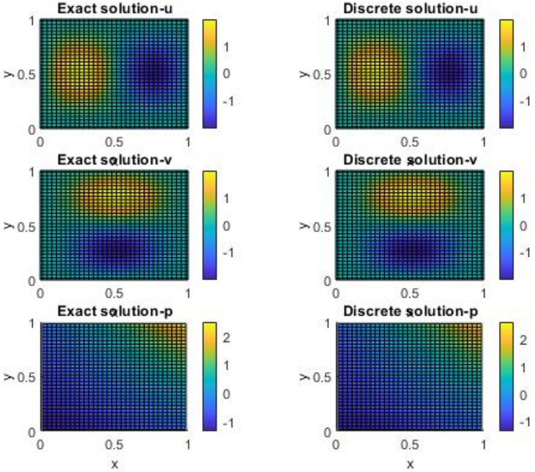
Error

number of elements	20	40	80
Error of L_2 norm on u	0.0054	0.0014	0.0003
Error of H_0^1 norm on u	0.0857	0.0292	0.0100
Error of L_2 norm on v	0.0054	0.0014	0.0003
Error of H_0^1 norm on v	0.0545	0.0168	0.0054
Error of L_2 norm on p	0.0054	0.0014	0.0003

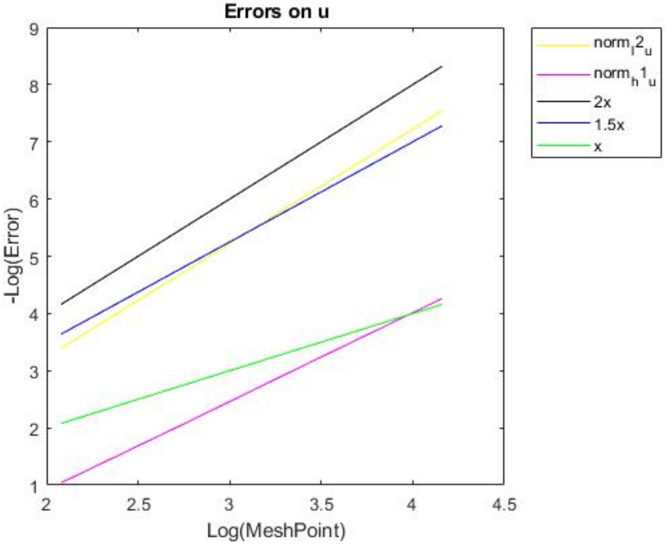
Solved by Uzawa method

$$
\begin{aligned}
u &= (1 - cos(2\pi x))sin(2\pi y) \\
v &= -(1 - cos(2\pi y))sin(2\pi x) \\
p &= xy + x + y + x^3y^2 - \frac{4}{3}
\end{aligned}
$$

Solved by Uzawa method



Solved by Uzawa method



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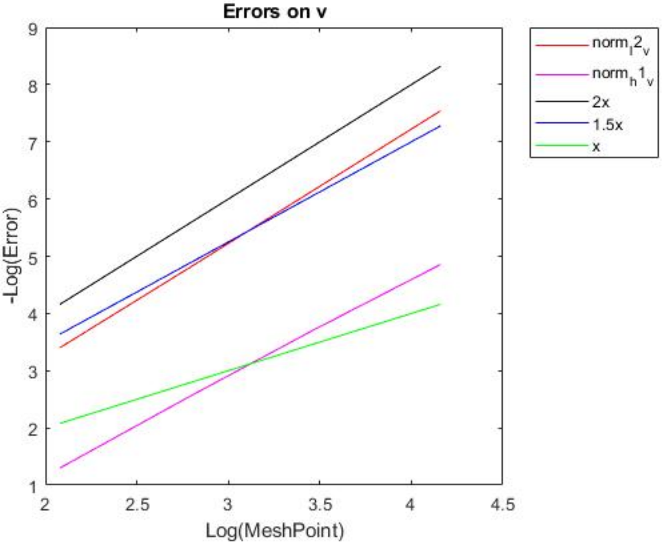

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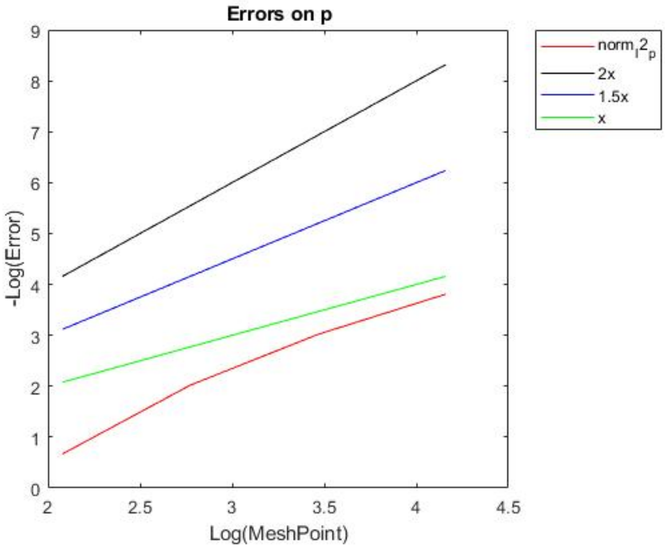


Error

Error

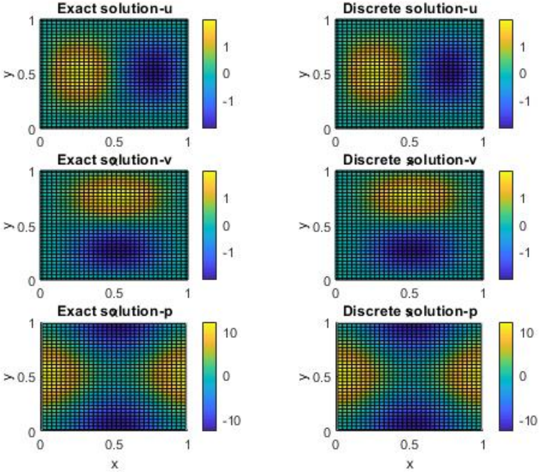
number of elements	8	16	32	64
Error of L_2 norm on u	0.0336	0.0085	0.0021	0.0005
Error of H_0^1 norm on u	0.3536	0.1218	0.0413	0.0141
Error of L_2 norm on v	0.0333	0.0085	0.0021	0.0005
Error of H_0^1 norm on v	0.2729	0.0803	0.0244	0.0078
Error of L_2 norm on p	0.5112	0.1325	0.0489	0.0222

Solved by Uzawa method

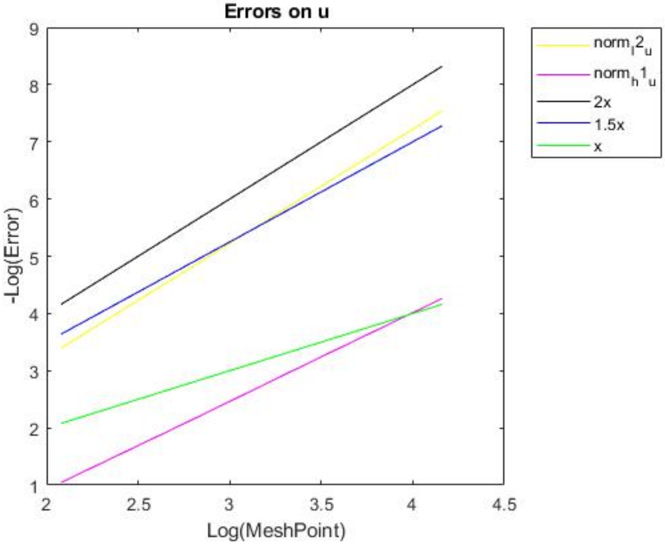
One more example !

$$
\begin{aligned}
u &= (1 - cos(2\pi x))sin(2\pi y) \\
v &= -(1 - cos(2\pi y))sin(2\pi x) \\
p &= -2\pi(cos(2\pi x) - cos(2\pi y))
\end{aligned}
$$

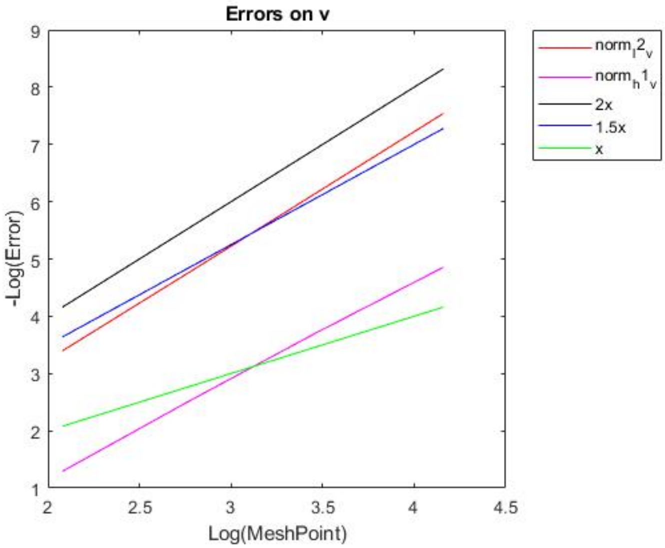
Solved by Uzawa method



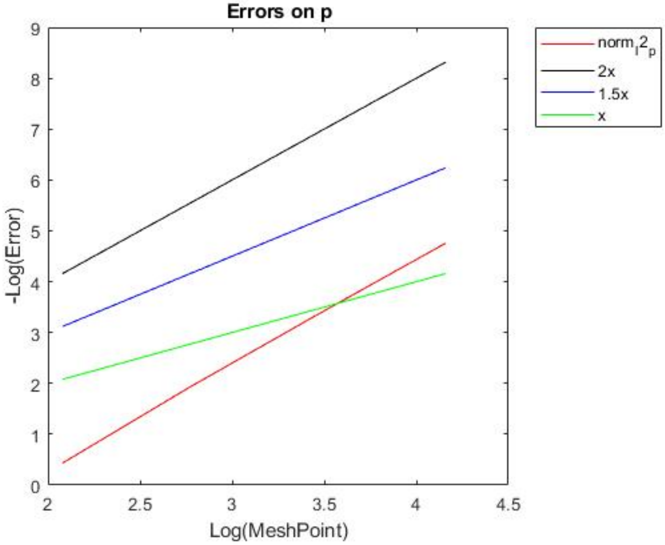
Solved by Uzawa method



Solved by Uzawa method



Solved by Uzawa method



Error

Error

number of elements	8	16	32	64
Error of L_2 norm on u	0.0335	0.0085	0.0021	0.0005
Error of H_0^1 norm on u	0.3510	0.1214	0.0412	0.0141
Error of L_2 norm on v	0.0335	0.0085	0.0021	0.0005
Error of H_0^1 norm on v	0.2745	0.0804	0.0244	0.0078
Error of L_2 norm on p	0.6482	0.1449	0.0348	0.0086

we can see that when we solved by uzawa , order convergence of p is increase !

Uniqueness of solution

we have that

$$\begin{pmatrix} A & B \\ B^T & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

in order to prove that it is suffice to prove if $f_1, f_2 = 0$ then $x_1, x_2 = 0$, indeed

$$\begin{cases} Ax_1 + Bx_2 = 0 \\ B^T x_2 = 0 \end{cases}$$

Uniqueness of solution

Or Multiplying the first row by $B^T A^{-1}$ and subtracting from the second row yields

$$-B^T A^{-1} B x_2 = 0$$

we have that $-B^T A^{-1} B$ is symmetric positive-definite we conclude that $x_2 = 0$ or $Ax_1 = 0$ since A is invertible we conclude that $x_1 = 0$

Thanks For Watching