FVM: Theoretical Assignment 1

October 14, 2020

Deadline: 28/10/2020
Let $f \in L^2(0,1)$.

$$-u_{xx} = f(x) \quad \text{in } (0,1) \tag{0.1}$$

subject to a Neumann boundary condition:

$$u'(0) = g0, \quad -u'(1) = g1, \quad \int_0^1 f(x)dx = g0 + g1, \quad \int_0^1 u(x)dx = 0.$$

Let $(u_i)_{i\in[0,N+1]}$ with $g(u)_{1/2} = g0,\ g(u)_{N+1/2} = -g1$ statisfy

$$-d(g(u)) = f$$

where $f = \{f_i\}_{i\in[1,N]}$ and f_i is mean-value of f in T_i.
1. Using that $w(x) - w(y) = \int_y^x w'(s)ds$, prove the following trace inequality for all $w \in H^1(0,1)$:

$$|w(x) - \bar{w}| \le \|w'\|_{L^2(0,1)},$$

and the following Poincare inequality for all $w \in H^1(0,1)$:

$$\|w - \bar{w}\|_{L^2(0,1)} \le \|w'\|_{L^2(0,1)},$$

where we have used the notation

$$\bar{w} = \int_0^1 w(x)dx$$

2. Write the variational formulation of (0.1) and prove the following a priori estimate for a solution u of this variational formulation :

$$\|u'\|_{L^2(0,1)} \le \|f\|_{L^2(0,1)} + (|g0| + |g1|)$$

3. For $(w_i)_{i\in[0,N+1]}$, we define

$$\bar{w}_h = \sum_{i=1}^N |T_i| w_i.$$

(3.1) For all $i, j = 1, \cdots, N$, bound $|w_i - w_j|$ by a constant time $|w|_{1,D}$ and compute the constant.
(3.2) Deduce the following trace inequality for all $i = 1, \cdots, N$:

$$|w_i - \bar{w}_h| \le |w|_{1,D}.$$

(3.3) Deduce the following discrete Poincare inequality :

$$\|w - \bar{w}_h\|_{0,T} \leq |w|_{1,D}.$$

4. Given $\left\{v_{i+\frac{1}{2}}\right\}_{i=0}^{N}$, $\{w_i\}_{i=0}^{N+1}$. Prove that

$$(d(v), w)_T = -\left(v, g(w)\right)_D + v_{N+\frac{1}{2}} w_{N+1} - w_0 v_{\frac{1}{2}}$$

5. For any $(w_i)_{i\in[0,N+1]}$, prove that

$$(g(u), g(w))_D = (f, w)_T - g0 w_0 - g1 w_{N+1}$$

6. Using the previous Poincare and trace inequalities deduce that the discrete solution $(u_i)_{i\in[0,N+1]}$ satisfies the following a priori estimate

$$\|u\|_{1,D} \leq \|f\|_{0,T} + (|g0| + |g1|).$$

7. Since exact solution $u \in C^1(\bar{\Omega})$. We can define projection

$$\begin{aligned} \Pi : C^1(\bar{\Omega}) &\to \mathbb{R}^{N+2} \\ u &\mapsto (\Pi u)_i = u(x_i) \qquad \forall i \in [0, N+1] \end{aligned}$$

Since $u' \in C^0(\bar{\Omega})$. We can define projection

$$\begin{aligned} P : C^0(\bar{\Omega}) &\to \mathbb{R}^{N+1} \\ u' &\mapsto (Pu')_{i+1/2} = u'(x_{i+1/2}) \qquad \forall i \in [0, N] \end{aligned}$$

Let $(w_i)_{i\in[0,N+1]}$, we prove that

$$(gu, gw)_D = (Pu', gw)_D$$

8. We define

$$\varepsilon_{i+1/2} = u'(x_{i+1/2}) - \frac{u(x_{i+1}) - u(x_i)}{|D_{i+1/2}|}$$

We prove that

$$|D_{i+1/2}|\varepsilon_{i+1/2}^2 \leq \left(\frac{2}{3}\right)^2 4h^2 \int_{D_{i+1/2}} f^2(t)dt$$

If $x_{i+1/2}$ is midpoint of $D_{i+/12}$ then

$$|D_{i+1/2}|\varepsilon_{i+1/2}^2 \leq \left(\frac{4}{15}\right)^2 h^4 \|f'\|^2_{L^2(D_{i+1/2})}$$

where $h = \max\{|T_i| : i \in [1, N]\}$