Chapitre 4

Multigrid methods

Contents

Multigrid methods are a prime source of important advances in algorithmic efficiency , finding a rapidly increasing number of users. Unlike other known methods, multigrid offers the possibility of solving problems with N unknowns with O(N) work and storage, not just for special cases, but for large classes of problems. It relies on the use of several nested grids.

4.1 Geometric multigrid

For the modal presentation of the method, we refer to [7],[3], [6]. For the finite element part, we refer to [2].

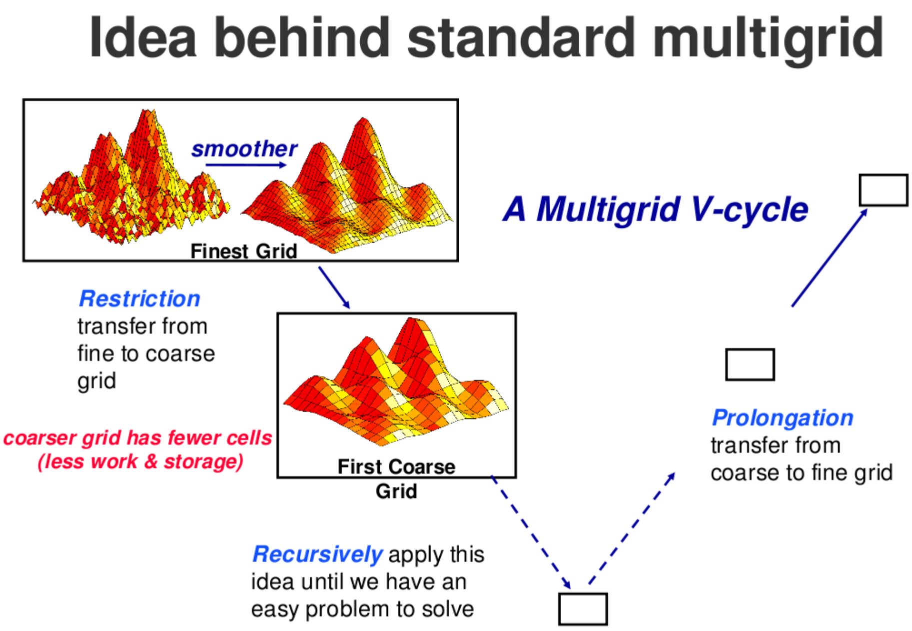


FIGURE 4.1 – scheme for a V-cycle, courtesy of David Keyes, Columbia University

4.1.1 The V- cycle process

One cycle of the multigrid method is given as follows. Suppose we want to solve $A^h \bar{U}^h = b^h$. We take an initial guess U^h, and define $MG(A^h, b^h, U^h)$ to be

Step 1 : smoothing N_1 iterations of the smoother, with initial guess U^h.

$$U^{h,1} = \mathcal{S}^h(A^h, b, U^h, N_1), \quad e^{h,1} = \bar{U}^h - U^{h,1}.$$

The residual is $r^{h,1} = b^h - A^h U^{h,1} = A^h e^{h,1}$.
It is projected on the coarse grid

$$r^{2h} = P_h^{2h} r^{h,1}$$

Step 2 : Coarse resolution The system $A^{2h} \tilde{U}^{2h} = r^{2h}$ is solved approximately by p iterations of the multigrid solver on the coarse grid

$$U^{2h,r} = MG(A^{2h}, r^{2h}, U^{2h,r-1}), \quad U^{2h,0} = 0, 1 \leq r \leq p.$$

It is projected on the fine grid

$$U^{h,2} = U^{h,1} + P_{2h}^h U^{2h,r}, \quad e^{h,2} = e^{h,1} - P_{2h}^h U^{2h,r}$$

Step 3 : Smoothing again N_2 iterations of the smoother

$$U^{h,3} = \mathcal{S}^h(A^h, b^h, U^{h,2}, N_2).$$

We will describe the process in the simple case where the coarse problem is solved exactly, *i.e.*

$$U^{h,2} = U^{h,1} - P^h_{2h}\tilde{U}^{2h}$$

$$h = \frac{1}{n+1}, \quad n = 2^\ell - 1, \quad A^h\bar{U}^h = f^h, \quad A^h \in \mathcal{M}_n(\mathbb{R})$$

Example : $n = 7$

$$A^h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \bar{U}^h = \begin{pmatrix} \bar{U}^h_1 \\ \bar{U}^h_2 \\ \bar{U}^h_3 \\ \bar{U}^h_4 \\ \bar{U}^h_5 \\ \bar{U}^h_6 \\ \bar{U}^h_7 \end{pmatrix}$$

$$A^{2h} = \frac{1}{4h^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad U^{2h} = \begin{pmatrix} U^{2h}_1 \\ U^{2h}_2 \\ U^{2h}_3 \end{pmatrix}$$

The Smoother

We will use one of the stationary methods , Gauss-Seidel or relaxed Jacobi, that is

$$U - \frac{1}{\omega}D^{-1}(AU - b) \quad \text{or } U - (D - E)^{-1}(AU - b).$$

The matrix of the iteration is

$$S = I - \frac{1}{\omega}D^{-1} \quad \text{or } I - (D - E)^{-1}A.$$

See chapter 1. The result of the smoothing is

$$e^{h,1} = S^{N_1}e^0, \quad r^{h,1} = A^h e^{h,1}. \tag{4.1}$$

Projection on the coarse grid

The fine grid is $(kh) = (\frac{k}{n+1})$ for $1 \leq k \leq n$. The coarse grid is $(k2h) = (\frac{2k}{n+1})$ for $1 \leq k \leq (n-1)/2$.

$$P^{2h}_h : \mathbb{R}^n \to R^{(n-1)/2}, \quad (P^{2h}_h U^h)_j = \frac{1}{4}(U^h_{2j-1} + 2U^h_{2j} + U^h_{2j+1}).$$

The matrix of P^{2h}_h is

$$P^{2h}_h = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad \mathbb{R}^7 \to \mathbb{R}^3$$

$$P_h^{2h}V = \begin{pmatrix} \frac{1}{4}V_1 + \frac{1}{2}V_2 + \frac{1}{4}V_3 \\ \frac{1}{4}V_3 + \frac{1}{2}V_4 + \frac{1}{4}V_5 \\ \frac{1}{4}V_5 + \frac{1}{2}V_6 + \frac{1}{4}V_7 \end{pmatrix} \quad \mathbb{R}^7 \to \mathbb{R}^3$$

Define now

$$r^{2h} := P_h^{2h} r^h = P_h^{2h} A^h e^{h,1}.$$

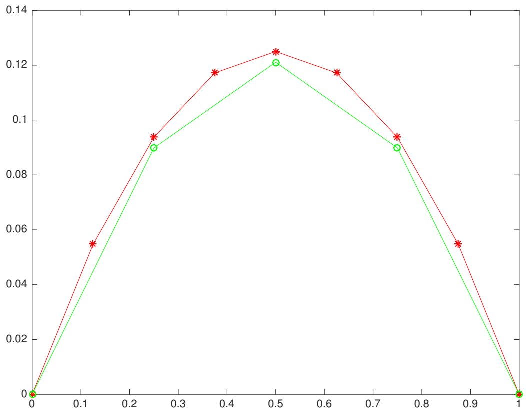


FIGURE 4.2 – Projection from fine to coarse grid

Coarse resolution

Suppose the coarse grid problem is solved exactly.

$$A^{2h}\tilde{U}^{2h} = r^{2h}$$

.

Projection on the fine grid

We define the projection operator as :

$$P_{2h}^h : \mathbb{R}^{(n-1)/2} \to R^n, \quad \begin{cases} (P_{2h}^h U^{2h})_{2j} = U_j^{2h} \\ (P_{2h}^h U^{2h})_{2j+1} = \frac{1}{2}(U_j^{2h} + U_{j+1}^{2h}) \end{cases}$$

The matrix is

$$P_{2h}^h = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = 2P_h^{2h} \quad \mathbb{R}^3 \to \mathbb{R}^7 \quad P_{2h}^h \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}U_1 \\ U_1 \\ \frac{1}{2}(U_1 + U_2) \\ U_2 \\ \frac{1}{2}(U_2 + U_3) \\ U_3 \\ \frac{1}{2}U_3 \end{pmatrix}$$

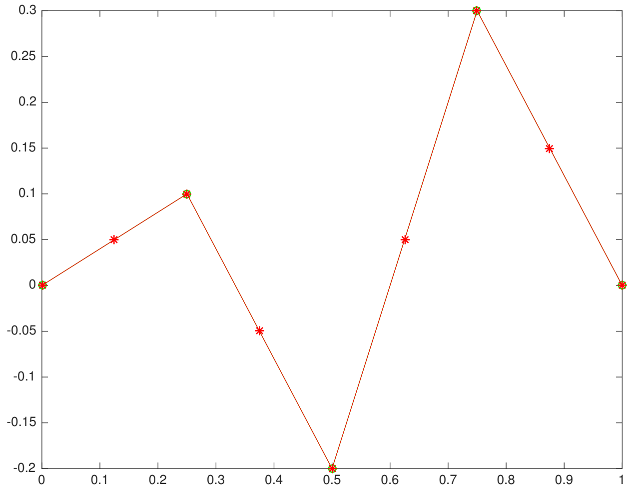


FIGURE 4.3 – Projection from coarse to fine grid

Result of the coarse walk

$$e^{h,2} = (I - P^h_{2h}(A^{2h})^{-1}P^{2h}_h A^h)e^{h,1}$$

Lemma 4.1

$$\mathcal{K}er\, P^{2h}_h A^h = \{V \in \mathbb{R}^n,\, V_{2j} = 0,\, j = 1 \cdots, (n-1)/2\}, \tag{4.2}$$

$$\mathcal{K}er\, P^{2h}_h A^h \oplus \mathcal{I}m P^h_{2h} = \mathbb{R}^n, \tag{4.3}$$

$$\forall V \in \mathbb{R}^{(n-1)/2}, \forall j, (A^h P^h_{2h} V)_{2j+1} = 0, \tag{4.4}$$

$$P^{2h}_h A^h P^h_{2h} = A^{2h}. \tag{4.5}$$

Proof It is easy to compute for $n = 7$,

$$P^{2h}_h A^h U = \frac{1}{h^2}\begin{pmatrix} \frac{1}{2}U_2 - \frac{1}{4}U_4 \\ -\frac{1}{4}U_2 + \frac{1}{2}U_4 - \frac{1}{4}U_6 \\ -\frac{1}{4}4_4 + \frac{1}{2}U_6 \end{pmatrix} = \frac{1}{4h^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_4 \\ U_6 \end{pmatrix} = A^{2h}\begin{pmatrix} U_2 \\ U_4 \\ U_6 \end{pmatrix}$$

Denoting by U^e the vector of the even coordinates of U, we have proved that for any vector $U \in \mathbb{R}^n$,

$$P^{2h}_h A^h U = A^{2h} U^e. \tag{4.6}$$

Therefore the kernel of $P^{2h}_h A^h$ is equal to the space of U such that $U^e = 0$, which proves (4.2).

Now by the rank theorem,

$$\dim \mathcal{K}\mathrm{er} P^{2h}_h + \dim \mathcal{I}\mathrm{m} P^{2h}_h = n.$$

Since A^h is an isomorphism in $\mathbb{R}^n$, $\dim \mathcal{K}\mathrm{er} P^{2h}_h = \dim \mathcal{K}\mathrm{er} P^{2h}_h A^h$. Then

$$\dim \mathcal{K}\mathrm{er} P^{2h}_h A^h + \mathrm{rg} P^{2h}_h = n.$$

Since $P^{2h}_h = \frac{1}{2}(P^h_{2h})^T$, they have the same rank, and therefore

$$\dim \mathcal{K}\mathrm{er} P^{2h}_h A^h + \mathrm{rg} P^h_{2h} = n.$$

Furthermore, any U in $\mathcal{K}\mathrm{er} P^{2h}_h A^h \cap \mathcal{I}\mathrm{m} P^h_{2h}$ is equal to $P^h_{2h} w$, and $U_{2j} = 0$. Since $(P^h_{2h} w)_{2j} = w_j$, this proves that $w = 0$. Hence (4.3) is proved.

We now can prove in the same way, first that for V in $\mathbb{R}^{(n-1)/2}$ (coarse),

$$(A^h P^h_{2h} V)_{2j+1} = 0, \quad (A^h P^h_{2h} V)_{2j} = \frac{1}{2h^2}(-V_{j-1} + 2V_j - V_{j+1}) = 2(A^{2h} v)_j.$$

Then using (4.6), V in $\mathbb{R}^{(n-1)/2}$ (coarse),

$$P^{2h}_h A^h P^h_{2h} V = A^{2h}(P^h_{2h} V)^e = A^{2h} V$$

which finally gives (4.5). ∎

Lemma 4.2

$$e^{h,1} = d^h + P^h_{2h} e^{2h},$$

with

$$d^h_{2j} = 0, \quad d^h_{2j+1} = \frac{h^2}{2}(A^h e^{h,1})_{2j+1}, \quad e^{2h}_j = e^{h,1}_{2j}$$

Proof By (4.3), we can expand $e^{h,1}$ as

$$e^{h,1} = d^h + P^h_{2h} e^{2h},$$

with $d^h \in Ker P^{2h}_h A^h$. By (4.2), $d^h_{2j} = 0$, and

$$e^{h,1}_{2j} = (P^h_{2h} e^{2h})_{2j} = e^{2h}_j,$$

which determines the components of e^{2h}. Compute now the odd components,

$$e^{h,1}_{2j+1} = d^h_{2j+1} + (P^h_{2h} e^{2h})_{2j+1} = d^h_{2j+1} + \frac{1}{2}(e^{2h}_j + e^{2h}_{j+1}) = d^h_{2j+1} + \frac{1}{2}(e^{h,1}_{2j} + e^{h,1}_{2j+2})$$

Therefore

$$d^h_{2j+1} = \frac{1}{2}(2e^{h,1}_{2j+1} - e^{h,1}_{2j} - e^h_{2j+2}) = \frac{h^2}{2}(A^h e^{h,1})_{2j+1}.$$

∎

Apply the lemma to compute $e^{h,2}$.

$$P^h_{2h}(A^{2h})^{-1} P^{2h}_h A^h e^{h,1} = P^h_{2h}(A^{2h})^{-1} P^{2h}_h A^h (d^h + P^h_{2h} e^{2h}) = P^h_{2h}(A^{2h})^{-1} \underbrace{P^{2h}_h A^h P^h_{2h}}_{A^{2h}} e^{2h} = P^h_{2h} e^{2h}.$$

Therefore

$$e^{h,2} = e^{h,1} - P^h_{2h} e^{2h} = d^h,$$

which implies the elegant formula

$$e^{h,2}_{2j} = 0, \quad e^{h,2}_{2j+1} = \frac{h^2}{2}(A^h e^{h,1})_{2j+1} = \frac{h^2}{2} r^{h,1}_{2j+1}.$$

the even components have disappeared.

Postsmoothing

$$e^{h,3} = S^{N_2} e^{h,2}.$$

$$\boxed{e^{h,3} = S^{N_2} \Pi_o \frac{h^2}{2} A^h S^{N_1} e^h}$$

Spectral analysis

The smoothing matrix S has eigenvalues λ_k, and eigenvectors $\Phi^{(k)}$. For relaxed Jacobi or the Gauss-Seidel algorithm, the eigenvalues are

$$\begin{aligned} \lambda_k^J(\omega) &= 1 - 2\omega \sin^2(\frac{k\pi h}{2}) && \text{for } 1 \le k \le n, \\ \lambda_k^{GS} &= \cos^2 k\pi h && \text{for } 1 \le k \le n, \end{aligned}$$

Figure 4.4 shows the eigenvalues as a function of k for $n = 2^5 - 1$.

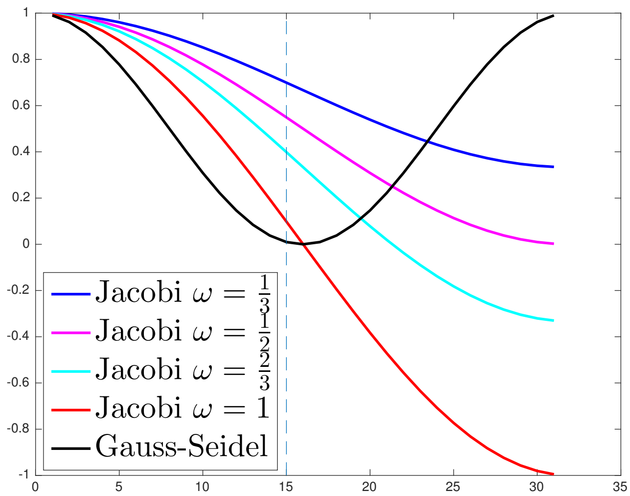


FIGURE 4.4 – Eigenvalues of the relaxed Jacobi iteration matrix as a function of k for several values of ω together with Gauss-Seidel

∗ For small k, $\lambda_k^J(\omega) \sim 1 - \omega \frac{k^2\pi^2h^2}{2}$.

∗ For $\omega = 2/3$, $(n-1)/2 \le k \le n \Rightarrow |\lambda_k^J(\omega)| \le \underbrace{1/3}_{\text{smoothing factor}}$

∗ For other modes. $|\lambda_k^J(\omega)| \in (1/3, 1 - \frac{4}{3}\sin^2(\frac{\pi h}{2}))$

When using Gauss-Seidel as a smoother, one can observe that the eigenvalues are small in the neighbourhood of $k \sim (n-1)/2$.

For an initial error $e^h = \Phi^{(k)}$, eigenmode of A, the error and residual after N_1 iterations is

$$e^{h,1} = \lambda_k^{N_1}\Phi^{(k)}, \quad r^{h,1} = \mu_k \lambda_k^{N_1}\Phi^{(k)}.$$

From

$$e_{2j}^{h,2} = 0, \quad e_{2j+1}^{h,2} = \frac{h^2}{2} r_{2j+1}^{h,1}$$

we obtain

$$e^{h,2} = \frac{h^2}{2}\mu_k \lambda_k^{N_1}\Phi_{2j+1}^k.$$

If the same smoother is applied in postprocessing,

$$e^{h,3} = \lambda_k^{N_2} e^{h,2},$$

and finally,

$$e_{2j}^{h,3} = 0, \quad e_{2j+1}^{h,3} = \frac{h^2}{2}\mu_k \lambda_k^{N_1+N_2}\Phi_{2j+1}^k.$$

We can see now that even the low frequencies are damped. Choose relaxed Jacobi with $\omega = 2/3$. For $(n-1)/2 \leq k \leq n$, we have, with $N = N_1 + N_2$,

$$|e^{h,3}_{2j+1}| \leq (\frac{1}{3})^N |\Phi^k_{2j+1}|,$$

and for $1 \leq k \leq (n-1)/2$,

$$|e^{h,3}_{2j+1}| \leq \sup_{x\in(0,1)} (x(1-\omega x)^N)|\Phi^k_{2j+1}| \leq \frac{1}{\omega(N+1)}\left(\frac{N}{N+1}\right)^N |\Phi^k_{2j+1}|$$

For three iterations of the smoother (N=3), the low frequencies have been damped by a factor 0.1582**, and the high frequencies by a factor 0.2963 ! !** The figures below show the result of one cycle of the above described algorithm, compared to three iterations of relaxed Jacobi, or Gauss-Seidel, for several inital guesses. $n = 10$.

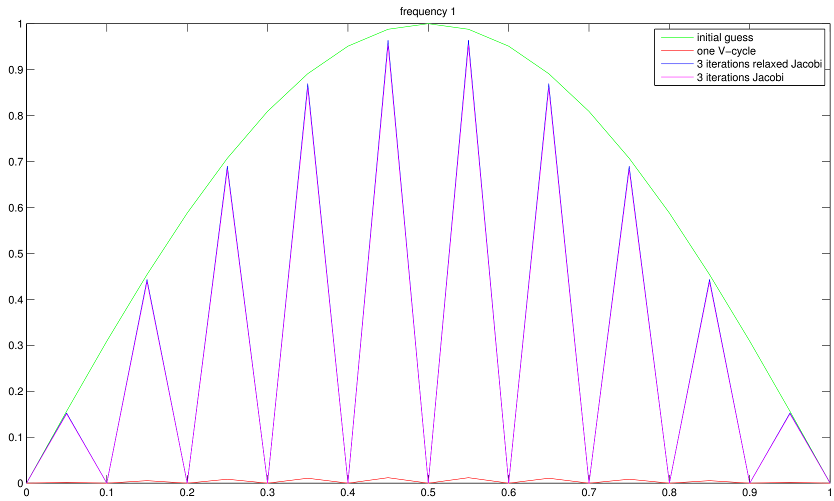


FIGURE 4.5 – Comparison of the iterative methods. Initial guess $\sin \pi x$

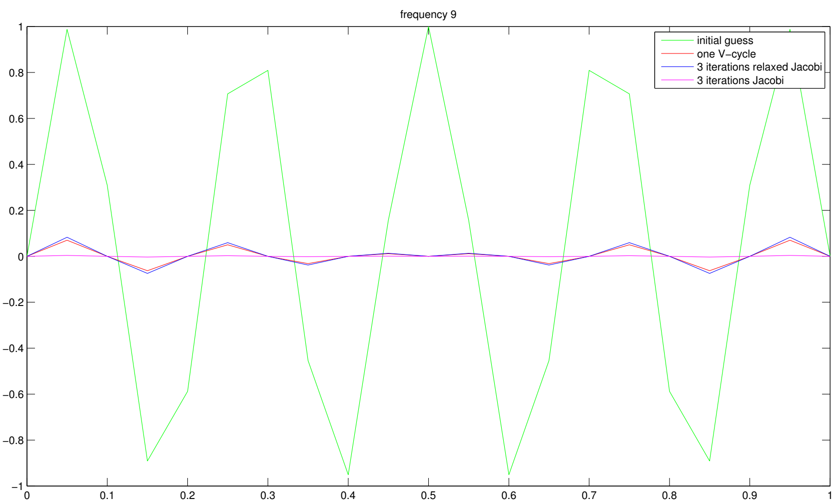


FIGURE 4.6 – Comparison of the iterative methods. Initial guess $\sin(n-1)/2\pi x$.

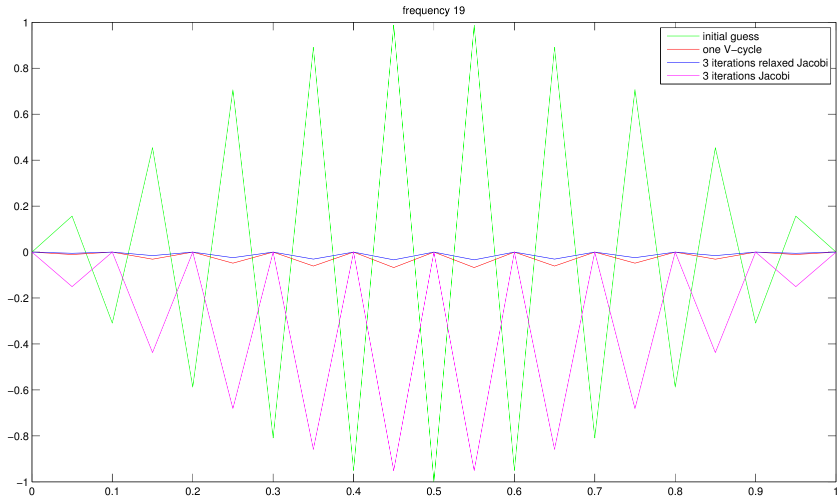
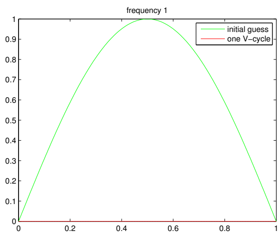
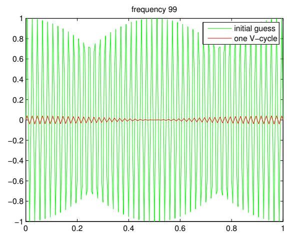


FIGURE 4.7 – Comparison of the iterative methods. Initial guess $\sin(n-1)\pi x$.

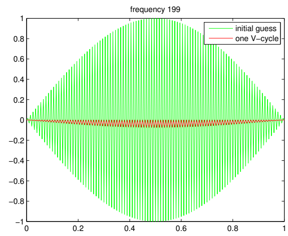
THE EFFECT OF ONE V-CYCLE ON ONE SINGLE MODE FOR $n = 201$.



Initial guess $\sin \pi x$



Initial guess $\sin(n-1)/2\pi x$



Initial guess $\sin(n-1)\pi x$.

4.1.2 L^∞*estimates*

Suppose the computation on the coarse grid can be done exactly, and use relaxed Jacobi with $\omega = 2/3$. Then $S = I - h^2A^h/3 = 1/3B$ where

$$B = \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & 0 & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & 1 & 1 \\ & & & 1 & 1 \end{pmatrix}.$$

Compute

$$(SU)_j = \frac{1}{3}(U_{j-1} + U_j + U_{j+1}), \Rightarrow \|SU\|_\infty \leq \|U\|_\infty.$$

Furthermore

$$h^2(SA^he^h)_{2j+1} = \frac{1}{6}(-e^h_{2j-2} + e^h_{2j-1} + e^h_{2j+1} - e^h_{2j+2}),$$

from which we deduce that

$$max_j|h^2(SA^he^h)_{2j+1}| \leq \frac{2}{3}\|e^h\|_\infty.$$

Therefore we obtain

$$\|e^{h,2}\|_\infty = \sup_j |\frac{h^2}{2} A^h r^{h,1}_{2j+1}|$$

from formula (??), we deduce for $\omega = 2/3$:

$$e^{h,2}_{2j+1} = -\frac{1}{6} e_{2j-1} + \frac{1}{6} e_{2j} + \frac{1}{6} e_{2j+2} - \frac{1}{6} e_{2j+3}.$$

and therefore we have the error estimate

$$e^{h,2}_{2j} = 0, \quad |e^{h,2}_{2j+1}| = \frac{2}{3} \|e_h\|_\infty.$$

Since the smoothing produces for any $U \in \mathbb{R}^{2n-1}$

$$(SU)_j = \frac{1}{3}(U_{j-1} + U_j + U_{j+1}),$$

we have

$$\|SU\|_\infty \leq \|U\|_\infty,$$

and

$$\|e^{h,3}\|_\infty \leq \frac{2}{3} \|e_h\|_\infty.$$

THE CONVERGENCE IS INDEPENDENT OF THE SIZE OF THE MATRIX

Number of elementary operations

method	number of operations
Gauss elimination	n^2
optimal overrelaxation	$n^{3/2}$
preconditionned conjugate gradient	$n^{5/4}$
FFT	$n \ln_2(n)$
multigrid	n

TABLE 4.1 – Asymptotic order of the number of elementary operations as a function of the number of grid points in one dimension for the Laplace equation (sparse matrix)

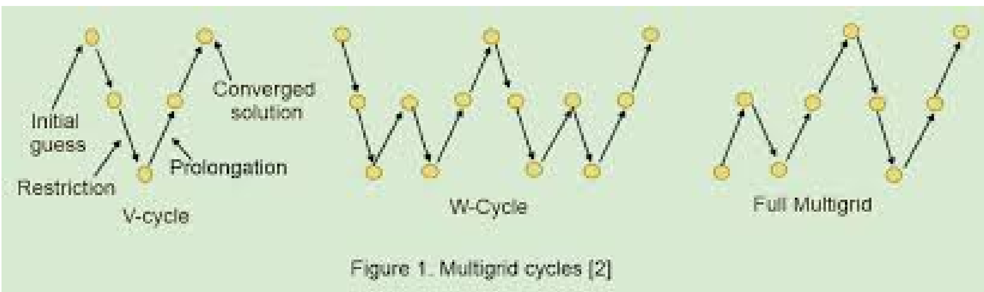


FIGURE 4.8 – full multigrid