27 Hilbert spaces

Assume H is a vector space over $\mathbb{C}$.

Definition 27.1. A pre-inner product on H is a map $(\cdot,\cdot) : H \times H \to \mathbb{C}$ which is

1. sesquilinear: that is, linear in the first variable, and antilinear in the second, for $u, v, w, \in H$, $\alpha, \beta \in \mathbb{C}$

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$$

$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v)$$

2. Hermitian: $(u, v) = \overline{(v, u)}$

3. Positive semidefinite: $(u, u) \geq 0$ for $u \in H$.

It is called an inner product or a scalar product, if instead of (3) the map is

3. positive definite: $(u, u) > 0$ for $u \in H$, $u \neq 0$.

A similar definition makes sense for real vector spaces, but then $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is

1. bilinear
2. Symmetric: $(u, v) = (v, u)$.
3. Positive semidefinite (for pre-inner products) or

3'. positive definite (for inner products)

We only consider complex vector spaces, but all the results will have analogues for real ones as well.

Theorem 27.2. ***Cauchy-Schwarz inequality*** *If $(\cdot,\cdot)$ is a pre-inner product, then*

$$|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$$

Corollary 27.3. *We have a seminorm $||u|| \equiv (u, u)^{1/2}$. It is a norm if and only if $(\cdot,\cdot)$ is an inner product.*

Definition 27.4. A Hilbert space is a complex vector space H with an inner product $(\cdot,\cdot)$ s.t. H is complete with respect to the norm $||u|| = (u, u)^{1/2}$. The norm of the Hilbert space is determined by the inner product, but the inner product can also be recovered from the norm by the polarisations identity

$$(u, v) = \frac{||u + v||^2 - ||u - v||^2}{4} + i\frac{||u + iv||^2 - ||u - iv||^2}{4},$$

which is proven by direct computation. A similar computation also proves the parallelogram law:

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2.$$

Remark. *A norm in a vector space is given by an inner product if and only if it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarisation identity.*

Remark. *Assume $(X, \mathscr{B}, \mu)$ is a measure space. Then $L^2(X, d\mu)$ is a Hilbert space with inner product*

$$(f, g) = \int_X f\bar{g}d\mu.$$

If $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, then we denote $L^2(X, d\mu)$ by $l^2(X)$. If $X = \mathbb{N}$ then we simply write l^2, this is called sequence space. A more formal definition is

$$l^2(X) = \left\{ f : X \to \mathbb{C} \ : \ \sum_x |f(x)|^2 < \infty \right\}$$

which is a Hilbert space with inner product $(f, g) = \sum_{x \in X} f(x)\bar{g}(x)$.

We recall that a subspace C of a vector space is called convex if $\forall u, v \in C$, $\forall t \in [0, 1]$ we have $tu + (1-t)v \in C$. The following is one of the key properties of Hilbert spaces.

Theorem 27.5. *Assume H is a Hilbert space, $C \subset H$ is a closed convex subset, $u \in H$. Then there is a unique $u_0 \in C$ s.t.*

$$||u - u_0|| = d(u, C) = \inf_{v \in C} ||u - v||.$$

28 Orthogonal projections

For a Hilbert space H and a subset $A \subset H$, let

$$A^{\perp} = \{x \in H \ : \ x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. $A^{\perp}$ is a closed subset of H.

Proposition 28.1. *Assume H_0 is a closed subspace of a Hilbert space H. Then every $u \in H$ uniquely decomposes as $u = u_0 + u_1$ with $u_0 \in H_0$ and $u_1 \in H_0^{\perp}$.*

For a closed subspace $H_0 \subset H$, consider the map $P : H \to H_0$ s.t. $pu \in H_0$ is a unique element satisfying $u - Pu \in H_0^{\perp}$. The operator $P : H \to H_0$ is linear. It is also contractive, meaning that $||Pu|| \leq ||u||$. It is called the orthogonal projection onto H_0. If H_0 is finite dimensional with an orthonormal basis $u_1, \ldots, u_n$, then

$$Pu = \sum_{k=1}^{n} (u, u_k) u_k.$$

Orthogonal basis can be defined for arbitrary Hilbert spaces.

Definition 28.2. An orthogonal system in H is a collection of vectors $u_i \in H$ $(i \in I)$ s.t.

$$(u_i, u_j) = \delta_{ij} \ \ \forall i, j \in I.$$

It is called an orthogonal basis if $span\{u_i\}_{i \in I}$ is dense in H.

Here $span\{u_i\}_{i \in I}$ denotes the linear span of $\{u_i\}_{i \in I}$, the space of finite linear combinations of the vectors u_i.

Theorem 28.3. *Every Hilbert space H has an orthonormal basis. If H is separable, then there is a countable orthonormal basis.*

Proposition 28.4. *Assume $\{u_i\}_{i \in I}$ is an orthogonal system in the Hilbert space H. Take $u \in H$. Then*

1. *(Bessel's inequality) $\sum_{i \in I} |(u, u_i)|^2 \leq ||u||^2$*
2. *(Parseval's identity) if $\{u_i\}_{i \in I}$ is an orthogonal basis then $\sum_{i \in I} |(u, u_i)|^2 = ||u||^2$.*

If $(u_i)_{i \in I}$ is an orthogonal basis, then the numbers (u, u_i) $(i \in I)$ are called the Fourier coefficients of u with respect to $(u_i)_{i \in I}$. The Parseval identity suggests that u is determined by its Fourier coefficients. This is true, and even more we have

Proposition 28.5. *Assume $(u_i)_{i \in I}$ is an orthogonal basis in a Hilbert space H. Then for every vector $(c_i)_{i \in I} \in l^2(I)$ there is a unique vector $u \in H$ with Fourier coefficients c_i. We write*

$$u = \sum_{i \in I} c_i u_i.$$

Corollary 28.6. *We have a linear isomorphism $U : l^2(I) \tilde{\to} H$, $U((c_i)_{i \in I}) = \sum_{i \in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is $||Ux|| = ||x|| \ \forall x \in l^2(I)$. By the polarisation identity this is equivalent to*

$$(Ux, Uy) = (x, y), \ \ \forall x, y \in l^2(I).$$

Hence U is unitary.

Corollary 28.7. *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2.*

Given two orthogonal bases $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$ in a Hilbert space H, we can decompose $u_i = \sum_{j \in J} (u_i, v_j) v_j$ and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable proves the following.

Remark. *Any two orthonormal bases in a Hilbert space have the same cardinality.*

29 Dual spaces

Lemma 29.1. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. TFAE:*

1. *f is continuous*
2. *f is continuous at 0*
3. *there is $c \geq 0$ s.t. $|f(x)| \leq c||x||$ $\forall x \in V$.*

If $(1) - (3)$ are satisfies, then f is called a <u>bounded linear functional</u>. The smallest such $c \geq 0$ in (3) is called the <u>norm</u> of f and is denoted by $||f||$. We have

$$||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||} = \sup_{||x||=1} |f(x)| = \sup_{||x|| \leq 1} |f(x)|.$$

Proposition 29.2. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V forms a Banach space V^*.*

Theorem 29.3. *(**<u>Riesz</u>**) Assume H is a Hilbert space. Then every $f \in H^*$ has the form*

$$f(x) = (x, y)$$

for a uniquely determined $y \in H$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space H we can define the <u>conjugate Hilbert space</u> $\overline{H}$. Its elements are the symbols $\bar{x}$ for $x \in H$. The linear structure and inner products are defined by

$$\bar{x} + \bar{y} = \overline{x + y}, \quad c\bar{x} = \overline{\bar{c}x}, \quad (\bar{x}, \bar{y}) = \overline{(x, y)} = (y, x).$$

Corollary 29.4. *For every Hilbert space H, we have an isometric isomorphism $\overline{H} \tilde{\to} H^*$, $\bar{x} \mapsto (\cdot, x)$.*