20 Decomposition theorems

Definition 20.1. Two measures ν and μ on a measurable space $(X, \mathscr{B})$ are called mutually singular, or we say that ν is singular with regard to μ, if there is $N \in \mathscr{B}$ such that $\nu(N^c) = 0$, $\mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 20.2. *(**Lebesgue decomposition theorem**) Assume ν, μ are σ-finit measures on $(X, \mathscr{B})$. Then there exists unique measures ν_a and ν_s s.t.*

$$\nu = \nu_a + \nu_s, \quad \nu_a << \mu, \quad \nu_s \perp \mu$$

Theorem 20.3. *(**Polar decomposition of complex measures**) Assume ν is a complex measure on $(X, \mathscr{B})$. Then there exists a finite measure μ on $(X, \mathscr{B})$ and a measurable function $f : X \to \Pi$ such that $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ-a.e.*

For signed measures we have the following decomposition

Theorem 20.4. *(**Hahn decomposition theorem**) Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exists $P, N \in \mathscr{B}$ such that $X = P \cup N$, $P \cap N = \emptyset$, $\nu(A \cap P) \geq 0$, $\nu(A \cap N) \leq 0$ $\forall A \in \mathscr{B}$. Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then*

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

Corollary 20.5. *(**Jordan's decomposition theorem**) Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exists unique finite measures ν_+, ν_- on $(X, \mathscr{B})$ such that*

$$\nu = \nu_+ - \nu_- \quad \textit{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

21 Duals of L^{p}-spaces

Assume $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p < \infty$. What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \to \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\phi(f) = \int_X f g d\mu?$$

Theorem 21.1. *(**Young's inequality**) Assume $f : [0, a] \to [0, b]$ is a strictly increasing continuous functions, $f(0) = 0$, $f(a) = b$. Then for all $s \in [0, a]$ and $t \in [0, b]$ we have*

$$st \leq \int_0^s f(x)dx + \int_0^t f^{-1} dy dy$$

and the equality holds if and only if $t = f(s)$.

If we apply this to $f(s) = s^{p-1}$. Then $f^{-1}(t) = t^{q-1}$, where q is the Hölder conjugate of p. $(p-1)(q-1) = 1$, that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We get

$$st \leq \int_0^s x^{p-1} dx + \int_0^t y^{q-1} dy = \frac{s^p}{p} + \frac{t^q}{q}.$$

Theorem 21.2. *(**Hölder's inequality**) If $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$, $1 < p < \infty$ and $1/p + 1/q = 1$. Then*

$$fg \in L^1(X, d\mu) \quad \textit{and} \quad ||fg||_1 \leq ||f||_p ||g||_q.$$

It follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \to \mathbb{C}, \quad l_g(f) = \int_X f g d\mu, \quad \text{and} \quad ||l_g|| \leq ||g||_q.$$

The same makes sense for $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$, when μ is σ-finite as

$$\int_X |fg| d\mu \leq \int_X |f| d\mu ||g||_\infty = ||f||_1 ||g||_\infty.$$

Lemma 21.3. *Assume $1 \leq p \leq \infty$, $1/p + 1/q = 1$ and μ is σ-finite if $p = 1$ or $p = \infty$. For $g \in L^q(X, d\mu)$ consider $l_g \in L^p(X, d\mu)^*$. Then*

$$||l_g|| = ||g||_q.$$

Therefore we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \hookrightarrow L^p(X, d\mu), \quad g \mapsto l_g.$$

Theorem 21.4. *Assume $(X, \mathscr{B}, \mu)$ is a σ-finit measure space, $1 \leq p < \infty$, $1/p + 1/q = 1$. Then*

$$L^P p(X, d\mu)^* = L^q(X, d\mu).$$

Remark. *This is usually not true for $p = \infty$.*