5 Radon-Nikodym theorem

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

Example 5.1. Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) = \int_A f d\mu \quad \text{for } A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$.

Proposition 5.2. *Assume $(X, \mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty)$ s.t. $\mu(N) = 0$ and*

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathscr{B}.$$

We can discard the term $\nu(A \cap N)$ as follows.

Definition 5.3. Given measures μ and ν on $(X, \mathscr{B})$, we say that ν is absolutely continuous with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}$ and $\mu(A) = 0$.

One can justify the name with the following lemma

Lemma 5.4. *Assume μ and ν are measures on $(X, \mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ if and only if $\forall \epsilon > 0$ there exists $\delta > 0$ s.t. if $A \in \mathscr{B}$ and $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*

Remark. *The result is not true for infinite ν.*

Theorem 5.5. *(**Radon-Nikodym**) Assume μ and ν are σ-finite measures on a measurable space $(X, \mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0, +\infty)$ s.t. $d\nu = f d\mu$. If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ μ-a.e.*

The function f is called the Radon-Nikodym derivative of ν with regard to μ and is denoted bt $\frac{d\nu}{d\mu}$.

Example 5.6. Consider a real-valued function $f \in C^1[a, b]$ s.t. $f'(t) > 0$ $\forall t \in [a, b]$, let $c = f(a)$ and $d = f(b)$. We know that for any Riemann integralbe function g on $[c, d]$ we have

$$\int_c^d g(t)dt = \int_a^b g(f(t))f'(t)dt.$$

Equivalently, for any Riemann integrable function g on $[a, b]$, we have

$$\int_c^d g \circ f^{-1} dt = \int_a^b g f' dt.$$

Denote by $\lambda_{[a,b]}$, $\lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a, b]$ and $[c, d]$ respectivly. Then the integral above implies that

$$d((f^{-1})_* \lambda_{[c,d]}) = f' d\lambda_{[a,b]},$$

since the integration of $\mathbb{1}_{[\alpha,\beta]}$ gives the same result for any interval $[\alpha, \beta] \subset [a, b]$ and since a finite Borel measure on $[a, b]$ is determined by its value on such intervals. Thus $(f^{-1})_* \lambda_{[c,d]} << \lambda_{[a,b]}$ and

$$\frac{d((f^{-1})_* \lambda_{[c,d]})}{d\lambda_{[a,b]}} = f'.$$

6 Complex and signed measures

Assume $(X, \mathscr{B})$ is a measurable space.

Definition 6.1. A complex measure on $(X, \mathscr{B})$ is a map $\nu : \mathscr{B} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and $\nu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \nu(A_n)$ for any disjoint $A_n \in \mathscr{B}$ where the series is assumed to bee absolutely convergent. If ν takes values in $\mathbb{R}$, then ν is called a finitely signed measure.

Remark. *More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$. Given a complex measure ν on $(X, \mathscr{B})$, its total variation is the map*

$$|\nu| : \mathscr{B} \to [0, +\infty]$$

defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^N |\nu(A_n)| \ : \ A = \bigcup_{n=1}^N A_n, \ \ A_n \in \mathscr{B}, \ \ A_n \cap A_m = \emptyset \ \text{ for } n \neq m \right\}$$

Proposition 6.2. $|\nu|$ *is a finite measure on* $(X, \mathscr{B})$.

Definition 6.3. If $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, then we say that ν is absolutely continuous with regards to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}$ and $\mu(A) = 0$. Equivalently $|\nu| << \mu$.

Theorem 6.4. ***Radon-Nikodym theorem for complex measures***

Assume $(X, \mathscr{B}, \mu)$ *is a measurable space,* ν *is a complex measure on* $(X, \mathscr{B})$, $\nu << \mu$. *Then there is a unique* $f \in L^1(X, d\mu)$ *s.t.* $d\nu = f d\mu$.