MAT4400: Notes on Linear analysis

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3 σ-Algebras

Definition 3.1 (σ-Algebra)**.** A family $\mathscr{A}$ of subsets of X with:

(i) $X \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$,

(iii) $(A_n)_{n\in\mathbb{N}} \in \mathscr{A} \Rightarrow \bigcup\limits_{n\in\mathbb{N}} A_n \in \mathscr{A}$

Remark. *It follows that,*

(i) $\emptyset \in \mathscr{A}$

(ii) $A, B \in \mathscr{A} \Rightarrow A \cup B \in \mathscr{A}$

(iii) $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A} \Rightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathscr{A}$

Theorem 3.2 (and Definition)**.**

(i) The intersection of arbitrarily many σ-algebras in X is againg a σ-algebra in X.

(ii) For every system of sets $p \subset \mathscr{P}(X)$ there exists a smallestσ-algebra containing$p$. This is the σ-algebra generated by p, denoted $\sigma(p)$, and $\sigma(p)$ is called its generator.

Definition 3.3 (Borel)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called **Borel σ-algebra**, and its members are called **Borel sets** or **Borel measurable sets**.

4 Measures

Definition 4.1. A (positive) measure μ on X is a map $\mu : \mathscr{A} \to [0, \infty]$ satisfying

1. $\mathscr{A}$ is a σ-algebra in X,
2. $\mu(\emptyset) = 0$,
3. $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ pairwise disjoint $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

If μ satisfies point 2. and 3., but $\mathscr{A}$ is not a σ-algebra, then μ is called a pre-measure.

Definition 4.2. Let X be a set and $\mathscr{A}$ a σ-algebra on X. The pair $(X, \mathscr{A})$ is called a measurable space. If μ is a measure on X, then $(X, \mathscr{A}, \mu)$ is called a measure space. A finite measure is a measure with $\mu(X) < \infty$, and a probability measure is a measure with $\mu(X) = 1$. The corresponding measure spaces are called finite measure spaces and probability spaces respectively.

Proposition 4.3. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $A, B, A_n, B_n \in \mathscr{A}$, $n \in \mathbb{N}$. Then*

1. $A \cap B = \emptyset \to \mu(A \cup B) = \mu(A) + \mu(B)$,
2. $A \subset B \to \mu(A) \leq \mu(B)$,
3. $A \subset B,\ \mu(A) < \infty \Rightarrow \mu(B \backslash A) = \mu(B) - \mu(A)$,
4. $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,
5. $\mu(A \cup B) \leq \mu(A) + \mu(B)$,
6. $A_n \uparrow A \Rightarrow \mu(A) = \sup_{n\in\mathbb{N}} \mu(A_n) = \lim_{n\to\infty} \mu(A_n)$,
7. $B_n \downarrow B, \mu(B_1) < \infty \Rightarrow \mu(B) = \inf_{n\in\mathbb{N}} \mu(B_n) = \lim_{n\to\infty} \mu(B_n)$,
8. $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$.

Definition 4.4. The set function λ^n on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ that assigns every half-open rectangle $[a, b) = [a_1, b_1) \times \ldots \times [a_n, b_b)$ the value

$$\lambda^n([a, b)) = \prod_{i=1}^{n} (b_i - a_i)$$

is called the n-dimensional Lebesgue measure.

5 Uniqueness of Measures

Lemma 5.1. *A Dynkin system D is a σ-algebra iff it is stable under finite intersections, i.e. $A, B \in D \Rightarrow A \cap B \in D$.*

Theorem 5.2 (Dynkin)**.** *Assume X is a set, S is a collection of subsets of X closed under finite intersections, that is, if $A, B \in S \Rightarrow A \cap B \in S$. Then $D(S) = \sigma(S)$.*

Theorem 5.3 (uniqueness of measures)**.** *Let (X, B) be a measurable space, and $S \subset P(X)$ be the generator of B, i.e. $B = \sigma(S)$. If S satisfies the following conditions:*

1. *S is stable under finite intersections ($\cap$-stable), i.e. $A, C \in S \Rightarrow A \cap C \in S$.*
2. *There exists an exhausting sequence $(G_n)_{N \in \mathbb{N}} \subset$ with $G_N \uparrow X$. Assume also that there are two measures μ, ν satisfying:*
3. *$\mu(A) = \nu(A),\ \forall A \in S$.*
4. *$\mu(G_n) = \nu(G_n) < \infty$.*

Then $\mu = \nu$.

6 Existence of Measures

Assume X is a set, $S \subset \mathcal{P}(X)$, $\emptyset \in S$, and $\mu : S \to [0, +\infty]$ is a map s.t. $\mu(\emptyset) = 0$. Can we extend μ to a measure on $(X, \sigma(S))$?

We define $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ by

$$\mu^* = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in S,\ A \in \bigcup_{n=1}^{\infty} A_n \right\}$$

In general μ^* is neither a measure nor it extends μ. But we have,

Lemma 6.1. *The map μ^* has the following properties:*

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B$ *then* $u^*(A) \leq \mu^*(B)$,
3. $\mu^* \left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$,

Any map $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ with properties (1.) – (3.) is called an <u>outer measure</u> on X. It turns out, every outer measure defines a measure on a certain σ-algebra.

Theorem 6.2 (Carathéodory's theorem)**.** *Assume X is a set and $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ is an outer measure. Let $\mathscr{B} \subset \mathcal{P}(X)$ be the collection of sets A s.t.*

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B), \ \ \forall B \in X.$$

Such sets A are called <u>Carathéodory measurable</u> with respect to μ^*. Then $\mathscr{B}$ is a σ-algebra and $\mu \overset{def}{=} \mu^*|_{\mathscr{B}}$ is a measure on $(X, \mathscr{B})$.

7 Measurable Mappings

We consider maps $T : X \to X'$ between two measurable spaces $(X, \mathscr{A})$ and $(X', \mathscr{A}')$ which respects the measurable structures and the σ-algbras on X and X'. These maps are useful as we can transport a measure μ, defined on $(X, \mathscr{A})$, to $(X', \mathscr{A}')$.

Definition 7.1. Let $(X, \mathscr{A})$, $(X', \mathscr{A}')$ be measurable spaces. A map $T : X \to X'$ is called $\mathscr{A}/\mathscr{A}'$-measurable if the pre-image of every measurable set is a measurable set:

$$T^{-1}(A') \in \mathscr{A}, \quad \forall A' \in \mathscr{A}'. \tag{1}$$

- A $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^m)$ measurable map is often called a Borel map.
- The notation $T : (X, \mathscr{A}) \to (X', \mathscr{A}')$ is often used to indicate measurability of the map T.

Lemma 7.2. *Let $(X, \mathscr{A})$, $(x', \mathscr{A}')$ be measurable spaces and let $\mathscr{A}' = \sigma(\mathscr{G}')$. Then $T : X \to X'$ is $\mathscr{A}/\mathscr{A}'$-measurable iff $T^{-1}(\mathscr{G}') \subset \mathscr{A}$, i.e. if*

$$T^{-1}(G') \in \mathscr{A}, \ \ \forall G' \in \mathscr{G}'. \tag{2}$$

Theorem 7.3. *Let $(X_i, \mathscr{A}_i)$, $\ i = 1, 2, 3$, be measurable spaces and $T : X_1 \to X_2$, $S : X_2 \to X_3$ be $\mathscr{A}_1/\mathscr{A}_2$ and $\mathscr{A}_2/\mathscr{A}_3$-measurable maps respectivly. Then $S \circ T : X_1 \to X_3$ is $\mathscr{A}_1/\mathscr{A}_3$-measurable.*

Corollary 7.4. *Every continuous map betwen metric spaces is a Borel map.*

Definition 7.5. (and lemma) Let $(T_i)_{i\in I}$, $T_I : X \to X_i$, be arbitrarily many mappings from the same space X into measurable spaces $(X_i, \mathscr{A}_i)$. The smallest σ-algebra on X that makes all T_i simultanously measurable is

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathscr{A}_i)\right) \tag{3}$$

Corollary 7.6. *A function $f : (X, \mathscr{B}) \to \mathbb{R}$ is measurable if $f((a, +\infty)) \in \mathscr{B}$, $\ \forall a \in \mathbb{R}$.*

Corollary 7.7. *Assume $(X, \mathscr{B})$ is a measurable space, (Y, d) is a metric space, $(f_n : (X, \mathscr{B}) \to Y)_{n=1}^{\infty}$ is a sequence of measurable maps. Assume this sequence of images $(f_n(x))_{n=1}^{\infty}$ is convergent in Y $\forall x \in X$. Define*

$$f : X \to Y, \ \ \text{by } f(x) = \lim_{n\to\infty} f_n(x). \tag{4}$$

Then f is measurable.

Theorem 7.8. *Let $(X, \mathscr{A})$, $(X', \mathscr{A}')$ be measurable spaces and $T : X \to X'$ be an $\mathscr{A}/\mathscr{A}'$-measurable map. For every measurable μ on $(X, \mathscr{A})$,*

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathscr{A}', \tag{5}$$

defines a measure on $(X', \mathscr{A}')$.

Definition 7.9. The measure $\mu'(\cdot)$ in the above theorem is called the push forward or image measure of μ under T and it is denoted as $T(\mu)(\cdot)$, $T_{*\mu}(\cdot)$ or $\mu \circ T^{-1}(\cdot)$.

Theorem 7.10. *If $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\lambda^n = T(\lambda^n)$.*

Theorem 7.11. *Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$S(\lambda^n) = |\det S^{-1}|\lambda^n = |\det S|^{-1}\lambda^n. \tag{6}$$

Corollary 7.12. *Lebesgue measure is invariant under motions: $\lambda^n = M(\lambda^n)$ for all motions M in $\mathbb{R}^n$. In particular, congruent sets have the same measure. Two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry*

8 Measurable Functions

A *measurable function* is a measurable map $u : X \to \mathbb{R}$ from some measurable space $(X, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}^1))$. They play central roles in the theory of integration.

We recall that $u : X \to \mathbb{R}$ is $\mathscr{A}/\mathscr{B}(\mathbb{R}^1)$-measurable if

$$u^{-1}(B) \in \mathscr{A}, \ \ \forall B \in \mathscr{B}(\mathbb{R}^1). \tag{7}$$

Moreover from a lemma from chapter 7, we actually only need to show that

$$u^{-1}(G) \in \mathscr{A}, \ \ \forall G \in \mathscr{G} \text{ where } \mathscr{G} \text{ generates } \mathscr{B}(\mathbb{R}^1). \tag{8}$$

Proposition 8.1.

1 *If $f, g : (X, \mathscr{B}) \to \mathbb{C}$ are measurable, then the function $f+g$, $f \cdot g$, cf, $(c \in \mathbb{C})$ are measurable.*

2 *If $b : \mathbb{C} \to \mathbb{C}$ is Borel and $b : (\mathbb{C}, \mathscr{B}) \to \mathbb{C}$ is measurable, then $b \circ f$ is measurable.*

3 *If $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$ and f_n are measurable, then f is measurable.*

4 *If $X = \bigcup_{n=1}^{\infty} A_n$, $(A_n \in \mathscr{B})$, $f|_{A_n} : (A_n, \mathscr{B}_{A_n}) \to \mathbb{C}$ is measurable $\forall n$, then f is measurable.*

Definition 8.2. Given a measurable space $(X, \mathscr{B})$, a measurable function $f : (X, \mathscr{B}) \to \mathbb{C}$ is called simple if

$$f(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}(x), \tag{9}$$

for some $c_k \in \mathbb{C}$, $A_k \in \mathscr{B}$, where $\mathbb{1}$ is the characteristic function,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{10}$$

The representation of simple function is **not** unique. We denote the standard representation of f by

$$f(x) = \sum_{n=0}^{N} z_n \mathbb{1}_{B_n}(x), \quad N \in \mathbb{N}, \ z_n \in \mathbb{R}, \ B_n \in \mathscr{A}, \ X = \bigcup_{n=1}^{N} B_n, \ \text{ for } B_n \cap B_m = \emptyset, \ \ n \neq m. \tag{11}$$

The set of simple functions is denoted $\mathcal{E}(\mathscr{A})$ or $\mathcal{E}$.

Definition 8.3. Assume μ is a measure on $(X, \mathscr{B})$. Given a *positive* simple function

$$f = \sum_{k=1}^{N} c_k \mathbb{1}_{A_k}, \quad (c_k \geq 0). \tag{12}$$

We define

$$\int_X f d\mu = \sum_{k=1}^{n} c_k \mu(A_k) \in [0, +\infty]. \tag{13}$$

We also denote this by $I_\mu(f)$.

Lemma 8.4. *This is well defined, that is, $\int_x f d\mu$ does not depend on the presentation of the simple function f.*

Properties 8.5. *For every positive simple function*

1 $\int_X cf d\mu = c \int_X f d\mu$, *for only* $c \geq 0$

2 $\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu$.

Corollary 8.6. *If $f \geq g \geq 0$ are simple functions, then*

$$\int_X f d\mu \geq \int_X g d\mu. \tag{14}$$

Definition 8.7. If $f : X \to [0, +\infty)$ is measurable, then we define

$$\int_X f d\mu = \sup\left\{ \int_X g d\mu \ : \ f \geq g \geq 0, \ \ g \text{ is simple} \right\} \tag{15}$$

Remark. *This means that any measurable function can be approximated by simple functions.*

Properties 8.8. *Measurable functions like this have the following properties*

1 $\int_X cf d\mu = c \int_X f d\mu, \quad \forall c \geq 0.$

2 If $f \geq g \geq 0$, then $\int_X f d\mu \geq \int_X g d\mu$ for any measurable g, f.

3 If $f \geq 0$ is simple, then $\int_X f d\mu$ is the same value as obtained before.

To advance in measure theory we consider measurable functions

$$f : X \to [0, +\infty].$$

Measurability is understood w.r.t the σ-algebra $\mathscr{B}([0, +\infty])$ generated by $\mathscr{B}([0, +\infty))$ and $\{+\infty\}$. In other words, $A \subset [0, +\infty] \in B([0, +\infty])$ iff $A \cap [0, +\infty) \in \mathscr{B}([0, +\infty)$.

Remark. *Hence* $f : X \to [0, +\infty]$ *is measurable iff* $f^{-1}(A)$ *is measurable* $\forall A \in \mathscr{B}([0, +\infty))$.

Definition 8.9. For measurable functions $f : X \to [0, +\infty]$, we define

$$\int_X f d\mu = \sup \left\{ \int_x g d\mu \ : \ f \geq g \geq 0 \ : \ g \text{ is simple} \right\} \in [0, +\infty]. \tag{16}$$

2

Theorem 8.10. *(**Monotone convergence theorem**) Assume* $(X, \mathscr{B}, \mu)$ *is a measure space,* $(f)_{n=1}^{\infty}$ *is an increasing sequence of measurable positive functions* $f_n : X \to [0, +\infty]$. *Define* $f(x) = \lim_{n\to\infty} f_n(x)$. *Then* f *is measurable and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu. \tag{17}$$

Theorem 8.11. *Assume* $(X, \mathscr{B})$ *is a measurable space and* $f : X \to [0, +\infty]$ *is measurable. Then there are simple functions* g_n*, s.t.*

$$0 \leq g_1 \leq g_2 \leq \dots \ , \ g_n(x) \to f(x), \ \forall x \in X.$$

Moreover, if f *is bounded, we can choose* g_n *s.t. the convergence is uniform, that is,*

$$\lim_{n\to\infty} \sup_{x\in X} |g_n(x) - f(x)| = 0. \tag{18}$$

9 Integration of measurable functions

Through this chapter $(X, \mathscr{A}, \mu)$ will be some measure space. Recall that $\mathcal{M}^+(\mathscr{A})$ $[\mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})]$ are the $\mathscr{A}$-measurable positive functions and $\mathcal{E}(\mathscr{A})$ $[\mathcal{E}^+_{\bar{\mathbb{R}}}(\mathscr{A})]$ are the positive and simple functions.

The fundamental idea of *Integration* is to measure the area between the graph of the function and the abscissa. For positive simple functions $f \in \mathcal{E}^+(\mathscr{A})$ in standard representation, this is done easily

$$\text{if } f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A}) \quad \text{then } \sum_{i=0}^{M} y_i \mu(A_i) \tag{19}$$

would be the μ-area enclosed by the graph and the abscissa. We note that the representation of f should not impact the integral of f.

Lemma 9.1. *Let $\sum_{i=0}^{M} y_i \mathbb{1}_{A_i} = \sum_{k=0}^{N} z_k \mathbb{1}_{B_k}$ be two standard representations of the same function $f \in \mathcal{E}^+(\mathscr{A})$. Then*

$$\sum_{i=0}^{M} y_i \mu(A_i) = \sum_{k=0}^{N} z_k \mu(B_k). \tag{20}$$

Definition 9.2. Let $f = \sum_{i=0}^{M} y_i \mathbb{1}_{A_i} \in \mathcal{E}^+(\mathscr{A})$ be a simple function in standard representation. Then the number

$$I_\mu(f) = \sum_{i=0}^{M} y_i \mu(A_i) \in [0, \infty] \tag{21}$$

(which is independent of the representation of f) is called the μ-integral of f.

Proposition 9.3. *Let $f, g \in \mathcal{E}^+(\mathscr{A})$. Then*

(i) $I_\mu(\mathbb{1}_A) = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $I_\mu(\lambda f) = \lambda I_\mu(f) \quad \forall \lambda \geq 0$.

(iii) $I_\mu(f + g) = I_\mu(f) + I_\mu(g)$.

(iv) $f \leq g \Rightarrow I_\mu(f) \leq I_\mu(g)$.

In theorem 8.8 we saw that we could for every $u \in \mathcal{M}^+(\mathscr{A})$ write it as an increasing limit of simple functions. By corollary 8.10, the suprema of simple functions are again measurable, so that

$$u \in \mathcal{M}^+(\mathscr{A}) \Leftrightarrow u = \sup_{n \in \mathbb{N}} f_n, \quad f \in \mathcal{E}^+(\mathscr{A}), \quad f_n \leq f_{n+1} \leq \dots .$$

We will use this to "inscribe" simple functions (which we know how to integrate) below the graph of a positive measurable function u and exhaust the μ-area below u.

Definition 9.4. Let $(X, \mathscr{A}, \mu)$ be a measure space. The (μ)-integral of a positive function $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ is given by

$$\int u d\mu = \sup \left\{ I_\mu(g) : g \leq u,\ g \in \mathcal{E}^+(\mathscr{A}) \right\} \in [0, +\infty]. \tag{22}$$

If we need to emphasize the *integration variable*, we write $\int u(x)\mu(dx)$. The key observation is that the integral $\int \ldots d\mu$ extends I_μ.

Lemma 9.5. *For all $f \in \mathcal{E}^+(\mathscr{A})$ we have $\int f d\mu = I_\mu(f)$.*

The next theorem is one of many convergence theorems. It shows that we could have defined 22 using any increasing sequence $f_n \uparrow u$ of simple functions $f_n \in \mathcal{E}^+(\mathscr{A})$.

Theorem 9.6. *(**Beppo Levi**) Let $(X, \mathscr{A}, \mu)$ be a measure space. For an increasing sequence of functions $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$, $0 \leq u_n \leq u_{n+1} \leq \ldots$, we have for the supremum $u = \sup_{n\in\mathbb{N}} u_n \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$ and*

$$\int \sup_{n\in\mathbb{N}} u_n d\mu = \sup_{n\in\mathbb{N}} \int u_n d\mu. \tag{23}$$

Note we can write $\lim_{n\to\infty}$ instead of $\sup_{n\in\mathbb{N}}$ as the supremum of an increasing sequence is its limit. Moreover, this theorem holds in $[0, +\infty]$, so the case $+\infty = +\infty$ is possible.

Corollary 9.7. *Let $u \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

$$\int u d\mu = \lim_{n\to\infty} \int f_n d\mu$$

holds for every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{E}^+(\mathscr{A})$ with $\lim_{n\to\infty} f_n = u$.

Proposition 9.8. *(of integral) Let $u, v \in \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then*

(i) $\int \mathbb{1}_A d\mu = \mu(A) \quad \forall A \in \mathscr{A}$.

(ii) $\int \alpha u d\mu = \alpha \int u d\mu \quad \forall \alpha \geq 0$.

(iii) $\int u + v d\mu = \int u d\mu + \int v d\mu$.

(iv) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$.

Corollary 9.9. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\bar{\mathbb{R}}}(\mathscr{A})$. Then $\sum_{n=1}^\infty u_n$ is measurable and we have*

$$\int \sum_{n=1}^\infty u_n d\mu = \sum_{n=1}^\infty \int u_n d\mu$$

(including the possibility $+\infty = +\infty$.)

Theorem 9.10. *(**Fatou**) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathscr{A})$ be a sequence of positive measurable functions. Then $u = \liminf_{n\to\infty} u_n$ is measurable and*

$$\int \liminf_{n\to\infty} u_n d\mu \leq \liminf_{n\to\infty} \int u_n d\mu \tag{24}$$

10 Integrals of Measurable Functions

We have defined our integral for positive measurable functions, i.e. functions in $\mathcal{M}^+(\mathscr{A})$. To extend our integral to not only functions in $\mathcal{M}^+(\mathscr{A})$ we first notice that

$$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) \Leftrightarrow u = u^+ - u^-,\ u^+, u^- \in \mathcal{M}^+_{\overline{\mathbb{R}}}, \tag{25}$$

i.e. that every measurable function can be written as a sum of **positive** measurable functions.

Definition 10.1 (μ-integrable)**.** A function $u : X \to \overline{\mathbb{R}}$ on $(X, \mathscr{A}, \mu)$ is μ-*integrable*, if it is $\mathscr{A}/\mathscr{B}(\overline{\mathbb{R}})$-measurable and if $\int u^+ d\mu, \int u^- d\mu < \infty$ (recall the definition for the integral of positive measurable functions). Then

$$\int u d\mu := \int u^+ d\mu - \int u^- d\mu \in (-\infty, \infty) \tag{26}$$

is the (μ-)*integral* of u. We write $\mathcal{L}^1(\mu)$ for the set of all real-valued μ-integrable functions [1].

Theorem 10.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then the following conditions are equivalent:*

(i) $u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(ii) $u^+, u^- \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iii) $|u| \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$.

(iv) $\exists w \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *with* $w \geq 0$ *s.t.* $|u| \leq w$.

Theorem 10.3 (Properties of the μ-integral)**.** *The μ-integral is:* ***homogeneous, additive****, and:*

(i) $\min\{u, v\}, \max\{u, v\} \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ *(lattice property)*

(ii) $u \leq v \Rightarrow \int u d\mu \leq \int v d\mu$ *(monotone)*

(iii) $\left|\int u d\mu\right| \leq \int |u| d\mu$ *(triangle inequality)*

Remark. *If $u(x) \pm v(x)$ is defined in $\overline{\mathbb{R}}$ for all $x \in X$ then we can exclude $\infty - \infty$ and the theorem above just says that the integral is linear:*

$$\int (au + bv) d\mu = a \int u d\mu + b \int v d\mu. \tag{27}$$

[1] In words, we extend our integral to ~~positive~~ measurable functions by noticing that we can write every measurable function as a sum of positive measurable functions, something that we do know how to integrate. We don't want to run into the problem of $\infty - \infty$, thus we require the integral of the positive and negative parts to both (separately) be less than infinity.

This is always true for real-valued $u, v \in \mathcal{L}^1(\mu) = \mathcal{L}^1_{\mathbb{R}}(\mu)$, making $\mathcal{L}^1(\mu)$ a vector space with addition and scalar multiplication defined by

$$(u+v)(x) := u(x) + v(x), \ \ (a \cdot u)(x) := a \cdot u(x), \tag{28}$$

and

$$\int ...d\mu : \mathcal{L}^1(\mu) \to \mathbb{R}, \ \ u \mapsto \int u d\mu, \tag{29}$$

is a ***positive linear functional****.*

Integration of complex functions

Assume $(X, \mathscr{B}, \mu)$ is a measure space.

Definition 10.4. A measurable function $f : X \to \mathbb{C}$ is called integrable (or μ-integrable) if

$$\int_X |f| d\mu < \infty.$$

Denote by $\mathcal{L}^1(X, \mathscr{B}, d\mu)$ or $\mathcal{L}^1(X, d\mu)$ the set of integrable functions. This is a vector space over $\mathbb{C}$, since

$$|f+g| \leq |f| + |g|, \quad \text{and} \quad |cf| = |c||f|, \ \ (c \in \mathbb{C}).$$

It is also spanned by positive functions since

$$f = (\operatorname{Re} f)_+ - (\operatorname{Re} f)_- + i(\operatorname{Im} f)_+ - i(\operatorname{Im} f)_-,$$

where for a function h we let $h_+ = max\{0, h\}$, $h_- = -\min\{h, 0\}$. If $f \in \mathcal{L}^1(X, d\mu)$, then

$$(\operatorname{Re} f)_\pm, (\operatorname{Im} f)_\pm \in \mathcal{L}^1(X, d\mu), \quad \text{as} \quad |(\operatorname{Re} f)_\pm|, |(\operatorname{Im} f)_\pm| \leq |f|.$$

Proposition 10.5. *The integral extends uniquely for the positive integrable functions to linear functionals $\mathcal{L}^1(X, d\mu) \to \mathbb{C}$, that is, to a map such that*

$$\int_X (f+g) d\mu = \int_x f d\mu + \int_X g d\mu \ \ , \ \ \int_X cf d\mu = c \int_X f d\mu, \ \ (c \in \mathbb{C}).$$

Proposition 10.6. *(**Triangle inequality**)*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

11 Null sets and the Almost Everywhere (lecture 08, 05. Feb.)

Definition 11.1. A (μ-)null set $N \in \mathcal{N}_\mu$ is a measurable set $N \in \mathscr{A}$ satisfying

$$N \in \mathcal{N}_\mu \iff N \in \mathscr{A} \text{ and } \mu(N) = 0. \tag{30}$$

This can be used generally about a 'statement' or 'property', but we will be interested in questions like 'when is $u(x)$ equal to $v(x)$', and we answer this by saying

$$u = v \text{ } a.e. \Leftrightarrow \{x : u(x) \neq v(x)\} \text{ is (contained in) a } \mu\text{-null set.}, \tag{31}$$

i.e.

$$u = v \quad \mu\text{-a.e.} \Leftrightarrow \mu\left(\{x : u(x) \neq v(x)\}\right) = 0. \tag{32}$$

The last phrasing should of course include that the set $\{x : u(x) \neq v(x)\}$ is in $\mathscr{A}$.

Theorem 11.2. *Let $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, then:*

(i) $\int |u| d\mu = 0 \Leftrightarrow |u| = 0$ *a.e.* $\Leftrightarrow \mu\{u \neq 0\} = 0$,

(ii) $\mathbb{1}_N u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \quad \forall\ N \in \mathcal{N}_\mu$,

(iii) $\int_N u d\mu = 0$.

(i) is really useful, later we will define $\mathcal{L}^p$ and the $||\cdot||_p$-(semi)norm. Then (i) means that if we have a sequence u_n converging to u in the $||\cdot||_p$-norm then $u_n(x) = u(x)$ a.e.

Corollary 11.3. *Let $u = v$ μ-a.e. Then*

(i) $u, v \geq 0 \Rightarrow \int u d\mu = \int v d\mu$,

(ii) $u \in \mathcal{L}^1_{\mathbb{R}}(\mu) \Rightarrow v \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ *and* $\int u d\mu = \int v d\mu$.

Corollary 11.4. *If $u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A})$, $v \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ and $v \geq 0$ then*

$$|u| \leq v \text{ } a.e. \ \Rightarrow u \in \mathcal{L}^1_{\mathbb{R}}(\mu). \tag{33}$$

Proposition 11.5 (Markow inequality)**.** *For all $u \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, $A \in \mathscr{A}$ and $c > 0$*

$$u\left(\{|u| \geq c\} \cap A\right) \leq \frac{1}{c} \int_A |u| d\mu, \tag{34}$$

if $A = X$, then (obviosly)

$$u\{|u| \geq c\} \leq \frac{1}{c} \int |u| d\mu. \tag{35}$$

Completions of measure spaces

Definition 11.6. A measure space $(X, \mathscr{B}, \mu)$ is called **complete** if whenever $A \in \mathscr{B}$ and $\mu(A) = 0$, we have $B \in \mathscr{B}, \ \forall B \subset A$.

Remark. *Any measure space can be completed as follows:*
Let $\bar{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathscr{B}$ with $B \subset A$ and $\mu(A) = 0$.

Thus, the completed σ-algebra is of the form

$$\bar{\mathscr{B}} = \{B \cup N \ : \ B \in \mathscr{B}, \ \ N \subset A, \ \ \text{for all } A \in \mathscr{B} \text{ with } \ \mu(A) = 0\}$$

Proposition 11.7. *The σ-algebra $\bar{\mathscr{B}}$ can also be described as follows:*

$$\bar{\mathscr{B}} := \{B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathscr{B} \text{ with } \mu(A_2 \backslash A_1) = 0\}, \tag{36}$$

with B, A_1, A_2 as above, we define

$$\bar{\mu} := \mu(A_1) = \mu(A_2) \tag{37}$$

Then $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$ is a complete measure space.

Definition 11.8. If μ is a Borel measure on a **metric** space (X, d), then the completion $\bar{\mathscr{B}}(X)$ of the Borel σ-algebra with respect to μ is called the σ-algebra of μ-measurable sets.

Remark. *For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ-algebra of **Lebesgue measurable sets**. Instead of $\bar{\lambda_n}$ we still write λ_n and call it the **Lebesgue measure**. A function $f : \mathbb{R}^n \to \mathbb{C}$, measurable w.r.t. the σ-algebra of Lebesgue measurable sets is called the **Lebesgue measurable**.*

The following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition 11.9. *Assume $(X, \mathscr{B}, \mu)$ is a measure space and consider its completion $\left(X, \bar{\mathscr{B}}, \bar{\mu}\right)$. Assume $f : X \to \mathbb{C}$ is $\bar{\mathscr{B}}$-measurable. Then there is a $\mathscr{B}$-measurable function $g : X \to \mathbb{C}$ s.t. $f = g$ $\bar{\mu}$-a.e.*

12 Convergence Theorems and Their Applications (lecture 9, 8. Feb.)

- To interchange limits and integrals in **Riemann integrals** one typically has to assume uniform convergence. ¡- The set of Riemann integrable functions is somewhat limited, see theorem 12.7

Theorem 12.1 (**Generalization of Beppo Levi, monotone convergence**).

*(i) Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be s.t. $u_1 \leq u_2 \leq ...$ with limit $u := \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$. Then $u \in \mathcal{L}^1(\mu)$ **iff***

$$\sup_{n\in\mathbb{N}} \int u_n d\mu < +\infty,$$

in which case

$$\sup_{n\in\mathbb{N}} \int u_n d\mu = \int \sup_{n\in\mathbb{N}} u_n d\mu.$$

(ii) Same thing only with a decreasing sequence ...$> -\infty$ in which case

$$\inf_{n\in\mathbb{N}} \int u_n d\mu = \int \inf_{n\in\mathbb{N}} u_n d\mu.$$

Theorem 12.2 (**Lebesgue; dominated convergence**). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ s.t.*

(a) $|u_n|(x) \leq w(x)$, $w \in \mathcal{L}^1(\mu)$,

(b) $u(x) = \lim_{n\to\infty} u_n(x)$ exists in $\bar{\mathbb{R}}$,

then $u \in \mathcal{L}^1(\mu)$ and we have

(i) $\lim\limits_{n\to\infty} \int |u_n - u| d\mu = 0$;

(ii) $\lim\limits_{n\to\infty} \int u_n d\mu = \int \lim\limits_{n\to\infty} u_n d\mu = \int u d\mu$;

Application 1: Parameter-Dependent Integrals

- We are interested in questions of the sort, when is

$$U(t) := \int u(t,x)\mu(dx),\ t \in (a,b),$$

again a smooth function of t? The answer involves interchange of limits and integration. Also, it turns out to better understand Riemann integrability, we need the Lebesgue integral.

Theorem 12.3 (continuity lemma)**.** *Let $\emptyset \neq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ satisfy*

(a) $x \mapsto u(t,x)$ *is in* $\mathcal{L}^1(\mu)$ *for every fixed* $t \in (a,b)$*;*

(b) $t \mapsto u(t,x)$ *is continuous for every fixed* $x \in X$*;*

(c) $|u(t,x)| \leq w(x)$ *for all* $(t,x) \in (a,b) \times X$ *and some* $w \in \mathcal{L}^1(\mu)$*.*

Then the function $U : (a,b) \to \mathbb{R}$ given by

$$t \mapsto U(t) := \int u(t,x)\mu(dx) \tag{38}$$

is continuous.

Theorem 12.4 (differentiability lemma)**.** *Let $\emptyset \leq (a,b) \subset \mathbb{R}$ be a non-degenerate open interval and $u : (a,b) \times X \to \mathbb{R}$ satisfy*

(a) $x \mapsto u(t,x)$ *is in* $\mathcal{L}^1(\mu)$ *for every fixed* $t \in (a,b)$*;*

(b) $t \mapsto u(t,x)$ *is continuous for every fixed* $x \in X$*;*

(c) $|\partial_t u(t,x)| \leq w(x)$ *for all* $(t,x) \in (a,b) \times X$ *and some* $w \in \mathcal{L}^1(\mu)$*.*

Then the function

$$t \mapsto U(t) := \int u(t,x)\mu(dx) \tag{39}$$

is differentiable and its derivative is

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int u(t,x)\mu(dx) = \int \frac{\partial}{\partial t}u(t,x)\mu(dx). \tag{40}$$

Application 2: Riemann vs Lebesgue Integration

Consider only $(X, \mathscr{A}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

Definition 12.5 (The Riemann Integral)**.** Consider on the finite interval $[a,b] \subset \mathbb{R}$ the partition

$$\Pi := \{a = t_0 < t_1 < ... < t_k < b\}, k = k(\Pi), \tag{41}$$

and introduce

$$S_\Pi[u] := \sum_{i=1}^{k(\Pi)} m_i(t_i - t_{i-1}), \qquad m_i := \inf_{x \in [t_{i-1}, t_i]} u(x), \tag{42}$$

$$S^\Pi[u] := \sum_{i=1}^{k(\Pi)} M_i(t_i - t_{i-1}), \qquad M_i := \sup_{x \in [t_{i-1}, t_i]} u(x). \tag{43}$$

$$\tag{44}$$

A bounded function $u : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if the values

$$\underline{\int} u := \sup_{\Pi} S_{\Pi}[u] = \inf_{\Pi} S^{\Pi}[u] =: \overline{\int} u \tag{45}$$

coincide and are finite. Their common value is called the **Riemann integral** of u and denoted by $(R) \int\limits_a^b u(x)dx$ or $\int\limits_a^b u(x)dx$.

Theorem 12.6. *Let* $u : [a, b] \to \mathbb{R}$ *be a **measurable** and **Riemann integrable** function. Then*

$$u \in \mathcal{L}^1(\lambda) \ \textit{and} \int\limits_{[a,b]} u d\lambda = \int\limits_a^b u(x)dx. \tag{46}$$

Theorem 12.7. *Let* $u : [a, b] \to \mathbb{R}$ *be a bounded function, it is Riemann integrable **iff** the points in* (a, b) *where* u *is discontinuous are a (subset of) Borel measurable null set.*

Improper Riemann Integrals

- The Lebesgue integral extends the (*proper*) Riemann integral. However, there is a further extension of the Riemann integral which cannot be captured by Lebesgue's theory. u is Lebesgue integrable *iff* $|u|$ ha finite Lebesgue integral. ¡- The Lebesgue integral does not respect sign-changes and cancellations. However, the following *improper Riemann integral* does:

$$(R) \int\limits_0^\infty u(x)dx := \lim_{n\to\infty} (R) \int\limits_0^a u(x)dx. \tag{47}$$

Corollary 12.8. *Let* $u : [0, \infty) \to \mathbb{R}$ *be a measurable, Riemann integrable function for every interval* $[0, N]$, $N \in \mathbb{N}$. *Then* $u \in \mathcal{L}^1[0, \infty)$ ***iff***

$$\lim_{N\to\infty} (R) \int\limits_0^N |u(x)|dx < \infty. \tag{48}$$

In this case, $(R) \int_0^\infty u(x)dx = \int_{[0,\infty)} u d\lambda$

Example of a function which is *improperly Riemann integrable* but **not** *Lebesgue integrable*:

$$f(x) = \frac{\sin(x)}{x}. \tag{49}$$

Proposition 12.9 (appearing as example 12.13 in Schilling)**.** *Let* $f_\alpha(x) := x^\alpha, x > 0$ *and* $\alpha \in \mathbb{R}$. *Then*

(i) $f_\alpha \in \mathcal{L}^1(0,1) \Leftrightarrow \alpha > -1.$

(ii) $f_\alpha \in \mathcal{L}^1[1,\infty) \Leftrightarrow \alpha < -1.$

13 The Function Spaces $\mathcal{L}^p$ (lecture 11, 15. Feb.)

Assume V is a vector space over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Definition 13.1. A seminorn on V is a map $p : V \to [0, +\infty)$ s.t.

(1) $p(cx) = |c|p(x) \ \ \forall x \in V, \forall c \in \mathbb{K}$.

(2) $p(x+y) \leq p(x) + p(y) \ \ \forall x, y \in V$. **triangle inequality**.

A seminorm is called a norm if we also have

$$p(x) = 0 \iff x = 0.$$

A norm is commonly denoted $||x||$, and a vectorspace equipped with a norm is called a **normed space**.

Definition 13.2. Assume (X, d) is a measure space. Fix $1 \leq p \leq \infty$. For every measurable function $f : X \to \mathbb{C}$ we define the following

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p} \in [0, +\infty]. \tag{50}$$

We can see that $||cf||_p = |c|||f||_p \ \ \forall c \in \mathbb{C}$.

Notice that by Theorem 11.2(i) we have that $||f||_p = 0 \Rightarrow f = 0$ a.e. Consider for example $\lim_{n\to\infty} ||f_n - f||_p = 0$, then we can find a subsequence s.t. $\lim_{k\to\infty} |f_{n(k)} - f| = 0$ a.e., i.e. $\lim_{k\to\infty} f_{n(k)} = f$ a.e.

Lemma 13.3.

$$||f + g||_p \leq ||f||_p + ||g||_p. \tag{51}$$

Definition 13.4. We define

$$\mathcal{L}^p(X, d\mu) = \{ f : X \to \mathbb{C} \ \mid \ f \text{ is measurable and } ||f||_p < \infty \}.$$

This is a vectorspace with seminorm $f \mapsto ||f||_p$. And in general this is not a normed space, since $||f||_p = 0 \iff f = 0$ a.e.

Generally, if p is a seminorm on a vectorspace V, then

$$V_0 = \{ x \in V \ \mid \ p(x) = 0 \} \tag{52}$$

which is a subspace of V. Then we consider the quotient/factor space V/V_0.

Definition 13.5. For $x, y \in V$, define

$$x \sim y \iff x - y \in V_0. \tag{53}$$

This is an equivalence relation on V. The representation class of V is defined by $[x]$ or $x + V_0$.

Then V/V_0 is equals the set of equivalence classes. We can show that it is a normed space.

$$[x] + [y] = [x + y] \ \ , \ \ c[x] = [cx] \ \ , \ \ ||[x]|| = p(x).$$

Applying this to $\mathcal{L}^p(X, d\mu)$ we get the normed space

$$L^p(X, d\mu) := \mathcal{L}^p(X, d\mu)/\mathcal{N} = \mathcal{L}^p(X, d\mu)/_\sim. \tag{54}$$

Where $\mathcal{N}$ is the space of measurable functions f s.t. $f = 0$ a.e. The equivalence relation $\sim$ is defined by

$$u \sim v \iff \{u \neq v\} \in \mathcal{N}_\mu \iff \mu\{u \neq v\} = 0,$$

and so $L^p(X, d\mu)$ consists of *all equivalence classes* $[u]_p = \{v \in \mathcal{L}^p | u \sim v\}$. So for every $u \in L^p$ there is no $v \in L^p$ such that $\mu\{u \neq v\} \neq 0$.

We will further continue to denote the norm by $|| \cdot ||_p$, and we will normally **not** distinguish between $f \in \mathcal{L}^p(X, d\mu)$ and the vector in $L^p(X, d\mu)$ that f defines.

Definition 13.6. A normed space $(X, || \cdot ||)$ is called a Banach space if V is complete w.r.t the metric $d(x, y) = ||x - y||$.

Theorem 13.7. *If $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p \leq \infty$, then $L^p(X, d\mu)$ is a Banach space.*

Definition 13.8. A measurable function $f : X \to \mathbb{C}$ is called **essentially bounded** if there is $c \geq 0$ s.t.

$$\mu(\{x \ : \ |f(x)| > c\}) = 0. \tag{55}$$

That is $|f| \leq c$ a.e. The smallest such c is called the essential supremum of f and is denoted by $||f||_\infty$.

Definition 13.9.

$$\mathcal{L}^\infty(X, d\mu) = \{f : X \to \mathbb{C} \ | \ f \text{ is measurable and } ||f||_\infty < \infty\}.$$

$$L^\infty(X, d\mu) = \mathcal{L}^\infty(X, d\mu)/\mathcal{N}.$$

Where by the previous definiton these spaces become the spaces of all essentially bounded functions.

Theorem 13.10. *If $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, then $L^\infty(X, d\mu)$ is a Banach space.*

Convergence in $\mathcal{L}^p$ and completeness

Lemma 13.11. *For any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, of positive functions $u_n \geq 0$ we have*

$$\Big|\Big|\sum_{n=1}^{\infty} u_n\Big|\Big|_p \leq \sum_{n=1}^{\infty} ||u_n||_p.$$

Theorem 13.12 (**Riesz-Fischer**). *The spaces $\mathcal{L}^p(\mu), p \in [1,\infty)$, are complete, i.e. every Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$ converges to some limit $u \in \mathcal{L}^p(\mu)$*

Corollary 13.13. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$ with $\mathcal{L}^p - \lim_{n\to\infty} u_n = u$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ s.t. $\lim_{k\to\infty} u_{n_k}(x) = u(x)$ holds for almost every $x \in X$.*

Theorem 13.14. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence of functions s.t. $|u_n| \leq w$ $\forall n \in \mathbb{N}$ and some $w \in \mathcal{L}^p(\mu)$. If $u(x) = \lim_{n\to\infty} u_n(x)$ exists for (almost) every $x \in X$, then*

$$u \in \mathcal{L}^p \text{ and } \lim_{n\to\infty} ||u - u_n||_p = 0.$$

Theorem 13.15 (**F. Riesz convergence theorem**). *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu), p \in [1,\infty)$, be a sequence s.t. $\lim_{n\to\infty} u_n(x) = u(x)$ for almost every $x \in X$ and some $u \in \mathcal{L}^p(\mu)$. Then*

$$\lim_{n\to\infty} ||u_n - u||_p = 0 \Longleftrightarrow \lim_{n\to\infty} ||u_n||_p = ||u||_p.$$

14 Product measures

Assume $(X_1, \mathscr{B}_1, \mu_1)$ and $(X_2, \mathscr{B}_2, \mu_2)$ are measure spaces. Consider $X_1 \times X_2$. Consider the σ-algebra $\mathscr{B}_1 \times \mathscr{B}_2$ (sometimes denoted by $\mathscr{B}_1 \otimes \mathscr{B}_2$) of subsets of $X_1 \times X_2$, generated by all the sets of form $A_1 \times A_2$, with $A_1 \in \mathscr{B}_1$ and $A_2 \in \mathscr{B}_2$.

We want to define a measure $\mu = \mu_1 \times \mu_2$ such that $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.

Theorem 14.1. *Assume $(X_1, \mathscr{B}_1, \mu_1)$ and $(X_2, \mathscr{B}_2, \mu_2)$ are σ-finite measure spaces, $f : X_1 \times X_2 \to [0, +\infty]$ is $(\mathscr{B}_1 \otimes \mathscr{B}_2)$-measurable. Then the following functions are measurable,*

1. $(X_1, \mathscr{B}_1) \to [0, +\infty]$, $\ x_1 \mapsto f(x_1, x_2)\ \forall x_2 \in X_2$,
2. $(X_2, \mathscr{B}_2) \to [0, +\infty]$, $\ x_2 \mapsto f(x_1, x_2)\ \forall x_1 \in X_1$,
3. $(X_1, \mathscr{B}_1) \to [0, +\infty]$, $\ x_1 \mapsto \int_{X_2} f(x_1, x_2) d\mu(x_2)$,
4. $(X_2, \mathscr{B}_2) \to [0, +\infty]$, $\ x_2 \mapsto \int_{X_1} f(x_1, x_2) d\mu(x_1)$,

and we have

$$\int_{X_1} \left(\int_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = \int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2)$$

Theorem 14.2. *Assume $(X_1, \mathscr{B}_1, \mu_1)$, $(X_2, \mathscr{B}_2, \mu_2)$ are σ-finite measure spaces. Then there is a unique measure $\mu = \mu_1 \times \mu_2$ in $(X_1 \times X_2, \mathscr{B}_1 \times \mathscr{B}_2)$ s.t.*

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \qquad \text{for all } A_1 \in \mathscr{B}_1,\ A_2 \in \mathscr{B}_2.$$

Theorem 14.3. ***Tonelli and Fubini's theorem*** *Assume $(X_1, \mathscr{B}_1, \mu_1)$, $(X_2, \mathscr{B}_2, \mu_2)$ are σ-finite measure spaces.*

1. *Tonelli's theorem: If $f : X_1 \times X_2 \to [0, +\infty]$ is $(\mathscr{B}_1 \times \mathscr{B}_2)$-measurable, then*
$$\begin{aligned}\int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \times \mu_2)(x_1, x_2) &= \int_{X_1} \left(\int_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2)\end{aligned}$$

2. *Fubini's theorem: If $f : X_1 \times X_2 \to \mathbb{C}$ is $(\mathscr{B}_1 \times \mathscr{B}_2)$-measurable and any of the integrals*
$$\begin{gathered}\int_{X_1 \times X_2} |f(x_1, x_2)| d(\mu_1 \times \mu_2)(x_1, x_2), \\ \int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| d\mu_2(x_2) \right) d\mu_1(x_1), \\ \int_{X_2} \left(\int_{X_1} |f(x_1, x_2)| d\mu_1(x_1) \right) d\mu_2(x_2).\end{gathered}$$

are finite, then the two other are finite as well, f is integrable w.r.t. $\mu_1 \times \mu_2$ and

$$\begin{aligned}\int_{X_1\times X_2} |f(x_1,x_2)| d(\mu_1\times\mu_2)(x_1,x_2) &= \int_{X_1}\left(\int_{X_2} |f(x_1,x_2)| d\mu_2(x_2)\right) d\mu_1(x_1)\\ &= \int_{X_2}\left(\int_{X_1} |f(x_1,x_2)| d\mu_1(x_1)\right) d\mu_2(x_2)\end{aligned}$$

Remark.

1. *σ-finiteness is important.*
2. *In the last equality in the Fubini's theorem it is important that f is integrable. Without it is possible that both of the iterative integrals are defined, but not equal!*

Corollary 14.4. *In the setting of the theorem, consider $A \in \mathscr{B}_1 \times \mathscr{B}_2$. Consider for $x_1 \in X_1$, the set*

$$A_{x_1} = \{x_2 \in X_2 \ : \ (x_1,x_2) \in A\}, \quad \textit{thus } A = \bigcup_{x_1\in X_1} (\{x_1\}\times A_{x_1}).$$

Thus, for each $x_1 \in X_1$, we have $A_{x_1} \in \mathscr{B}_2$ and

$$(\mu_1\times\mu_2)(A) = 0 \Leftrightarrow \mu_2(A_{x_1}) = 0 \quad \textit{for } \mu_1\textit{-a.e.} \quad x_1 \in X_1$$

Lebesgue measures on $\mathbb{R}^{\mathbf{n}}$

Theorem 14.5. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets,*

$$F : \Omega_1 \to \Omega_2 \textit{ be a } C^1\textit{-diffeomorphism, (invertible, has cont. first derivative.)}$$

Consider the restriction μ_1, μ_2 of λ_n to Ω_1, Ω_2. Then F is Lebesgue measurable, $F_*\mu_1 << \mu_2$ and

$$\frac{d(F_*\mu_1)}{d\mu_2} = |\det J_{F^{-1}}| \circ F^{-1},$$

where $(J_F)_{ij} = \partial F_i/\partial x_j$ is the Jacobi matrix of F.

15 Integrals with respect to image measures

Definition 15.1. For functions $f, g : \mathbb{R}^n \to \mathbb{C}$, the <u>convolution</u> is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

A good question is when this is well-defined.

Lemma 15.2. *If $f, g \in L^1(\mathbb{R}^n)$, then the function $y \mapsto f(y)g(x-y)$ is integrable for a.e. $x \in \mathbb{R}^n$, $f * g \in L^1(\mathbb{R}^n)$, and*

$$||f * g||_1 \leq ||f||_1 ||g||_1.$$

Lemma 15.3. *Assume $\phi : (a, b) \to [0, +\infty)$ is a convex function. Then ϕ is continuous and*

$$\phi(x) = \sup\{l(x) : \phi \geq l, \ \ l(t) = \alpha t + \beta\}, \quad \forall x \in (a, b).$$

Theorem 15.4. *(<u>**Jensen's inequality**</u>) Assume $(X, \mathscr{B}, \mu)$ is a probability space ($\mu(X) = 1$), $\phi : [0, +\infty) \to [0, +\infty)$ is convex. Then for every integrable $f : X \to [0, +\infty]$, we have*

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi \circ f d\mu.$$

If in addition $\lim_{x \to \infty} \phi(x) = +\infty$, then for every measurable function $f : X \to [0, +\infty]$ we have

$$\phi\left(\int f d\mu\right) \leq \int_X \phi \circ f d\mu,$$

where we put $\phi(+\infty) = +\infty$.

Lemma 15.5. *Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$. Then the function $y \mapsto f(y)g(x-y)$ is integrable for a.e. x, $f * g \in L^p(\mathbb{R}^n)$ and*

$$||f * g||_p \leq ||f||_1 ||g||_p.$$

Note that

$$\int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

so $f * g = g * f$.

Remark. *More generally, for $\mu \in M(\mathbb{R}^n)$ ($M(X)$ is the space of complex Borel measures on X) and $g \in L^p(\mathbb{R}^n)$, we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by*

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y).$$

Then $||\mu * g||_p \leq |\mu|(\mathbb{R}^n)||g||_p$.

One might ask what convolutions are good for? The following example sheds some light on it,

Example 15.6. Consider

$$f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}.$$

Then

$$(f * g)(x) = \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y)dy = \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y)dy.$$

Where we note that $\lambda_n(B_r(0)) = \lambda_n(B_r(x))$.

For a multi-index variable $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, write ∂^α for $\partial^{\alpha_1+\dots\alpha^n}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. Denote by $L^1_{loc}(\mathbb{R}^n)$ the space of Lebesgue measurable functions that are integrable on every ball. We identify functions that coincide a.e. (so, more pedantically, $L^1_{loc}(\mathbb{R}^n)$ is a space of equivalence classes of functions). We have $L^p(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$ for all $1 \le p \le \infty$.

Lemma 15.7. *If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then $\phi * f \in C^\infty(\mathbb{R}^n)$ and*

$$\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f.$$

By choosing suitable ϕ we can make $(\phi * f)$ close to f, as we will see shortly.

Definition 15.8. A <u>positive mollifier</u> is a function $\phi \in C_c(\mathbb{R}^n)$ s.t. $\phi \ge 0$ and

$$\int_{\mathbb{R}^n} \phi(x)dx = 1.$$

For a function ϕ on $\mathbb{R}^n$ and $\epsilon > 0$, define

$$\phi^\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon}).$$

Also note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{R^n} \phi dx = 1$, then $\int_{\mathbb{R}^n} \phi^\epsilon dx = 1$.

Example 15.9. Consider the function h on $\mathbb{R}$ defined by

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & , |t| < 1, \\ 0 & t \ge 1. \end{cases}$$

Then $h \in C_c^\infty(R^n)$. Hence $\phi(x) = c_n h(|x|)$ is a mollifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|)dx\right)^{-1}$.

Proposition 15.10. *Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \ge 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. Then we have:*

1. *if $f \in C_0(\mathbb{R}^n)$, then $\phi^\epsilon * f \in C_0(\mathbb{R}^n)$ and*

$$\lim_{\epsilon \to 0^+} ||\phi^\epsilon * f - f|| = 0 \quad \textit{(uniform norm)},$$

2. if $f \in L^p(\mathbb{R}^n)$*,* $1 \le p < \infty$*, then*

$$\lim_{\epsilon \to 0^+} ||\phi^\epsilon * f - f||_p = 0.$$

Corollary 15.11. *For any Radon measure* μ *on* $\mathbb{R}^n$*,* $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n, d\mu)$ *for* $1 \le p < \infty$*.*

17 Dense subspaces of L^p

Theorem 17.1. *Assume (X, d) is a metric space, μ is a Borel measure on X s.t. $\mu(B_R(x)) < \infty$, $\forall x \in X$ and $\forall R > 0$, $1 \leq p \leq \infty$. Then the bounded continuous functions on X with bounded support form a dense subspace of $L^p(X, d\mu)$. (Where by bounded support we mean that f is zero outside $B_R(x)$ for some x and $R > 0$.)*

If X is locally compact, then by $C_c(X)$ we denote the space of continuous functions on X with compact support.

Theorem 17.2. *Assume (X, d) is a separable, so it has a dense subset, locally compact metric space, μ is a Borel measure on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$, $1 \leq p < \infty$. Then $C_c(X)$ is dense in $L^p(X, d\mu)$.*

Remark. *Theorem 17.8 in the book is wrong.*

Remark. *These results do not extent to $p = \infty$.*

For $X = \mathbb{R}^n$, either theorem implies that if μ is a Borel measure on $\mathbb{R}^n$, s.t. $\mu(B_R(x)) < \infty$, $\forall x$, $\forall R > 0$, then $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$, $1 \leq p < \infty$. Later we will prove that $C_c^\infty(\mathbb{R}^n)$ is still dense in $L^p(\mathbb{R}^n, d\mu)$. For $\mu = \lambda_n$ we write $L^p(\mathbb{R}^n)$ instead of $L^p(\mathbb{R}^n, d\lambda_n)$.

Modes of convergence

Theorem 17.3. *(**Egorov**)*

Assume $(X, \mathscr{B}, \mu)$ is a measure space, $\mu(X) < \infty$. Assume $f_n, f : X \to \mathbb{C}$ are measurable functions and $f_n \to f$ a.e. Then $\forall \epsilon > 0$ there is $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.

1. For measurable functions f_n, f, we write $f_n \to f$ in the $p - th$ mean, excluding $p = \infty$, if $\lim_{n\to\infty} ||f_n - f||_p = 0$.
2. For $p = 1$ we sat that $f_n \to f$ in <u>mean</u> and for $p = 2$ we say that $f_n \to f$ in <u>quadratic mean</u>.
3. We say that $f_n \to f$ in <u>measure</u> if $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$, $\forall \epsilon > 0$.

Theorem 17.4. *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p < \infty$, $f_n, f \to \mathbb{C}$ are measurable functions. Then*

1. *if $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure*
2. *if $f_n \to f$ in measure, then there is a subsequence $(f_n)_{n=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.*
3. *if $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.*

In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.

19 Fourier transform

Write $L^1(\mathbb{R}^n)$ for $L^1(\mathbb{R}^n, d\lambda_n)$. We also write

$$\int_{\mathbb{R}^n} f(x)dx \quad \text{for} \quad \int_{\mathbb{R}^n} f(x)d\lambda_n(x).$$

The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(y)e^{-i\langle x,y\rangle}dy, \quad \text{where } \langle x,y\rangle = x_1y_1 + \ldots + x_ny_n.$$

More generally, if μ is a complex measure on $\mathbb{R}^n$, then its Fourier transform is the function $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle}d\mu(y).$$

If $f \in L^1(\mathbb{R}^n)$ and μ_f is defined by $d\mu_f = fd\lambda_n$ then $\hat{\mu}_f = \hat{f}$.

Remark. *(**and warning**)*
There are many conventions:

1. *Instead of the normalisation* $1/(2\pi)^n$*, one has* 1*,* $1/(2\pi)^{n/2}$*.*
2. *Instead of* $e^{-i\langle x,y\rangle}$*, one has* $e^{i\langle x,y\rangle}$*,* $e^{\pm i2\pi\langle x,y\rangle}$*.*

Lemma 19.1. *If μ is a complex Borel measure (and hence finite) on $\mathbb{R}^n$, then $\hat{\mu}$ is a continuous function on $\mathbb{R}^n$ and*

$$|\hat{\mu}(x)| \le \frac{|\mu|(\mathbb{R}^n)}{(2\pi)^n}.$$

(In particular, if $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is continuous and $|\hat{f}(x)| \le ||f||_1/(2\pi)^n$)

Lemma 19.2.

1. *If $f_t(x) = f(x-t)$, then*
$$\hat{f}_t(x) = e^{-i\langle t,x\rangle}\hat{f}(x).$$
2. *Uf $l_t(x) = e^{i\langle t,x\rangle}$ $(x, t \in \mathbb{R}^n)$, then $\widehat{l_tf}(x) = \hat{f}(x-t)$.*
3. *If $T \in GL_n(\mathbb{R})$, then $\widehat{f \circ T} = \frac{1}{|\det T|}\hat{f} \circ (T^T)^{-1}$.*
4. *$\hat{\bar{f}}(x) = \bar{\hat{f}}(-x)$.*

Example 19.3. (*With omitted proof, good exercise tho*). Let $f(x) = e^{-\frac{|x|^2}{2}}$, where $|x| = \langle x, x\rangle^{1/2}$. Then

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$$

. More, generally, if $f(x) = e^{-\frac{C|x|^2}{2}}$ $(C > 0)$, then

$$\hat{f}(x) = \frac{1}{(2\pi C)^{n/2}} e^{-\frac{|x|^2}{2C}}.$$

Proposition 19.4. *If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = (2\pi)^n \hat{f}\hat{g}$.*

Lemma 19.5. *(**Rieemann-Lebesgue lemma**)*

If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Theorem 19.6. *(**Fourier inversion theorem**)*

Assume $f \in L^1(\mathbb{R}^n)$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$, we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y\rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x), \quad \text{for a.e. } x.$$

Lemma 19.7. *For any $f, g \in L^1(\mathbb{R}^n)$, we have*

$$\int_{R^n} \hat{f}(x) g(x) dx = \int_{R^n} f(x) \hat{g}(x) dx$$

Corollary 19.8. *If $f \in L^1(\mathbb{R}^n)$ is s.t. $\hat{f} = 0$, then $f = 0$ a.e.*

A linear map $U : H \to K$ between Hilbert spaces is called an isometry if $||Ux|| = ||x||$ $\forall x \in H$. By the polarisation identity, this is equivalent to

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \quad \forall x, y, \in H.$$

If U is in addition surjective, then U is called a unitary operator.

Theorem 19.9. *(**Plancherel**)*

There is a unique unitary operation $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ s.t.

$$Uf = (2\pi)^{\frac{n}{2}} \hat{f}, \quad \text{for } f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Definition 19.10. We define the Fourier transform of $f \in L^2(\mathbb{R}^n)$ by

$$\hat{f} = (2\pi)^{-\frac{n}{2}} Uf.$$

If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\hat{f}(X) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y\rangle} dy.$$

Schwartz space

Proposition 19.11. *Assume $f \in L^1(\mathbb{R}^n)$ and $x_j f \in L^1(\mathbb{R}^n)$ for some $1 \le j \le n$. Then*

$$\partial_j \hat{f} = -i\widehat{x_j f}, \qquad \left(\partial_j = \frac{\partial}{\partial x_j}\right).$$

(Here by $x_j f$ we mean the function $x_j \mapsto x_j f(x)$.)

Proposition 19.12. *Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is such that $\partial_j f \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{\partial_j f} = i x_j \hat{f}.$$

Corollary 19.13. *If $f, \partial_j f \in L^1(\mathbb{R}^n)$, then*

$$\lim_{x\to\infty} x_j \hat{f}(x) = 0.$$

Corollary 19.14.

1. *If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \le N$, then $\hat{f} \in C^N(\mathbb{R}^n)$ and $\partial^\alpha \hat{f} = (-i)^{|\alpha|}\widehat{x^\alpha f}$.*
2. *If $f \in C^N(\mathbb{R}^n)$ and $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \le N$, then $\widehat{\partial^\alpha f} = i^{|\alpha|} x^\alpha \hat{f}$ and hence $(1+|x|)^N \hat{f}(x) \xrightarrow[x\to\infty]{} 0$.*

(Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_+$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x_1^\alpha = x^{\alpha_1} \ldots x_n^{\alpha_n}$.)

Definition 19.15. A function f on $\mathbb{R}^n$ is called a Schwartz function if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f$ is bounded for all multi-indices α, β. The space $S(\mathbb{R}^n)$ of Schwartz functions is called a Schwartz space.

Note that for every $f \in C^\infty(\mathbb{R}^n)$ the following conditions are equivalent:

1. $x^\alpha \partial^\beta f$ is bounded for all α, β,
2. $x^\alpha (\partial^\beta f)(x) \xrightarrow[x\to\infty]{} 0$ for all α, β,
3. $(1+|x|)^N \partial^\beta f$ is bounded for all $N \ge 1$ and all β.

Example 19.16. $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $e^{-a|x|^2} \in S(\mathbb{R}^n)$ for all $a > 0$. If $f \in S(\mathbb{R}^n)$, then $x^\alpha \partial^\beta f \in S(\mathbb{R}^n)$. The product of two Schwartz functions is a Schwartz function.

Clearly, $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$. From the Corollary above we conclude that if $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$. By the Fourier inversion theorem we then get

Theorem 19.17. *The Fourier transform maps $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$.*

Remark. *This gives another proof to the fact that the image of the Fourier transform $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is dense, which is needed to prove Plancherel's theorem*

Remark. *If $f \in \mathbb{C}_c^\infty(\mathbb{R}^n)$, $f \neq 0$, then $\hat{f} \in S(\mathbb{R}^n)$, but $\hat{f}$ is <u>never</u> compactly supported, since it extends to an analytic function on $\mathbb{C}^n$*

$$\hat{f}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x,y\rangle} dy.$$

20 Radon-Nikodym theorem

Assume $(X, \mathscr{B}, \mu)$ is a measure space. Are there other measures on $(X, \mathscr{B})$?

Example 20.1. Take a measurable function $f : X \to [0, +\infty]$ and define

$$\nu(A) = \int_A f d\mu \quad \text{for } \; A \in \mathscr{B}.$$

This is a measure by the monotone convergence theorem. We write $d\nu = f d\mu$.

Proposition 20.2. *Assume $(X, \mathscr{B})$ is a measurable space, μ and ν are σ-finite measures on $(X, \mathscr{B})$. Then there exist $N \in \mathscr{B}$ and a measurable $f : X \to [0, +\infty)$ s.t. $\mu(N) = 0$ and*

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathscr{B}.$$

We can discard the term $\nu(A \cap N)$ as follows.

Definition 20.3. Given measures μ and ν on $(X, \mathscr{B})$, we say that ν is absolutely continuous with respect to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}$ and $\mu(A) = 0$.

One can justify the name with the following lemma

Lemma 20.4. *Assume μ and ν are measures on $(X, \mathscr{B})$, $\nu(X) < \infty$. Then $\nu << \mu$ if and only if $\forall \epsilon > 0$ there exists $\delta > 0$ s.t. if $A \in \mathscr{B}$ and $\mu(A) < \delta$, then $\nu(A) < \epsilon$.*

Remark. *The result is not true for infinite ν.*

Theorem 20.5. *(**Radon-Nikodym**) Assume μ and ν are σ-finite measures on a measurable space $(X, \mathscr{B})$, $\nu << \mu$. Then there is a measurable function $f : X \to [0, +\infty)$ s.t. $d\nu = f d\mu$. If $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ μ-a.e.*

The function f is called the Radon-Nikodym derivative of ν with regard to μ and is denoted by $\frac{d\nu}{d\mu}$.

$$\nu(A) = \int_A \frac{d\nu}{d\mu} d\mu$$

Example 20.6. Consider a real-valued function $f \in C^1[a, b]$ s.t. $f'(t) > 0$ $\forall t \in [a, b]$, let $c = f(a)$ and $d = f(b)$. We know that for any Riemann integralbe function g on $[c, d]$ we have

$$\int_c^d g(t)dt = \int_a^b g(f(t))f'(t)dt.$$

Equivalently, for any Riemann integrable function g on $[a, b]$, we have

$$\int_c^d g \circ f^{-1} dt = \int_a^b g f' dt.$$

Denote by $\lambda_{[a,b]}$, $\lambda_{[c,d]}$ the Lebesgue measure restricted to the Borel subsets of $[a,b]$ and $[c,d]$ respectivly. Then the integral above implies that

$$d((f^{-1})_*\lambda_{[c,d]}) = f' d\lambda_{[a,b]},$$

since the integration of $\mathbb{1}_{[\alpha,\beta]}$ gives the same result for any interval $[\alpha,\beta] \subset [a,b]$ and since a finite Borel measure on $[a,b]$ is determined by its value on such intervals. Thus $(f^{-1})_*\lambda_{[c,d]} << \lambda_{[a,b]}$ and

$$\frac{d((f^{-1})_*\lambda_{[c,d]})}{d\lambda_{[a,b]}} = f'.$$

Decomposition theorems

Definition 20.7. Two measures ν and μ on a measurable space $(X, \mathscr{B})$ are called mutually singular, or we say that ν is singular with regard to μ, if there is $N \in \mathscr{B}$ such that $\nu(N^c) = 0$, $\mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem 20.8. *(**Lebesgue decomposition theorem**) Assume ν, μ are σ-finit measures on $(X, \mathscr{B})$. Then there exists unique measures ν_a and ν_s s.t.*

$$\nu = \nu_a + \nu_s, \quad \nu_a << \mu, \quad \nu_s \perp \mu$$

Theorem 20.9. *(**Polar decomposition of complex measures**) Assume ν is a complex measure on $(X, \mathscr{B})$. Then there exists a finite measure μ on $(X, \mathscr{B})$ and a measurable function $f : X \to \Pi$ such that $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ-a.e.*

For signed measures we have the following decomposition

Theorem 20.10. *(**Hahn decomposition theorem**) Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exists $P, N \in \mathscr{B}$ such that $X = P \cup N$, $P \cap N = \emptyset$, $\nu(A \cap P) \geq 0$, $\nu(A \cap N) \leq 0$ $\forall A \in \mathscr{B}$. Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then*

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

Corollary 20.11. *(**Jordan's decomposition theorem**) Assume ν is a finite signed measure on $(X, \mathscr{B})$. Then there exists unique finite measures ν_+, ν_- on $(X, \mathscr{B})$ such that*

$$\nu = \nu_+ - \nu_- \quad \textit{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

21 Duals of L^p-spaces

Assume $(X, \mathscr{B}, \mu)$ is a measure space, $1 \leq p < \infty$. What is the dual of $L^p(X, d\mu)$? When does a measurable function $g : X \to \mathbb{C}$ define a bounded linear functional on $L^p(X, d\mu)$ by

$$\phi(f) = \int_X fg d\mu?$$

Theorem 21.1. *(**Young's inequality**) Assume $f : [0, a] \to [0, b]$ is a strictly increasing continuous functions, $f(0) = 0$, $f(a) = b$. Then for all $s \in [0, a]$ and $t \in [0, b]$ we have*

$$st \leq \int_0^s f(x)dx + \int_0^t f^{-1} dy dy$$

and the equality holds if and only if $t = f(s)$.

If we apply this to $f(s) = s^{p-1}$. Then $f^{-1}(t) = t^{q-1}$, where q is the Hölder conjugate of p. $(p-1)(q-1) = 1$, that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We get

$$st \leq \int_0^s x^{p-1} dx + \int_0^t y^{q-1} dy = \frac{s^p}{p} + \frac{t^q}{q}.$$

Theorem 21.2. *(**Hölder's inequality**) If $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$, $1 < p < \infty$ and $1/p + 1/q = 1$. Then*

$$fg \in L^1(X, d\mu) \quad \textit{and} \quad ||fg||_1 \leq ||f||_p ||g||_q.$$

It follows that every $g \in L^q(X, d\mu)$ defines a bounded linear functional

$$l_g : L^p(X, d\mu) \to \mathbb{C}, \quad l_g(f) = \int_X fg d\mu, \quad \text{and} \quad ||l_g|| \leq ||g||_q.$$

The same makes sense for $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$, when μ is σ-finite as

$$\int_X |fg| d\mu \leq \int_X |f| d\mu ||g||_\infty = ||f||_1 ||g||_\infty.$$

Lemma 21.3. *Assume $1 \leq p \leq \infty$, $1/p + 1/q = 1$ and μ is σ-finite if $p = 1$ or $p = \infty$. For $g \in L^q(X, d\mu)$ consider $l_g \in L^p(X, d\mu)^*$. Then*

$$||l_g|| = ||g||_q.$$

Therefore we can view $L^q(X, d\mu)$ as a subspace of $L^p(X, d\mu)^*$ using the isometric embedding

$$L^q(X, d\mu) \hookrightarrow L^p(X, d\mu), \quad g \mapsto l_g.$$

Theorem 21.4. *Assume $(X, \mathscr{B}, \mu)$ is a σ-finite measure space, $1 \leq p < \infty$, $1/p + 1/q = 1$. Then*

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark. *This is usually not true for $p = \infty$.*

Dual of $C(X)$ spaces (lecture 4.april)

When μ is sigma finite we know that

$$L^p(X, d\mu)^* = L^q(X, d\mu)$$

when $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$. However, what is the dual space of $C(X)$? Assume (X, d) is a locally compact metric space. Consider the space $C_c(X)$ of continuous functions with compact support on X. A linear functional

$$\phi : C_c(X) \to \mathbb{C}$$

is called positive if $\phi(f) \geq 0$ for $f \geq 0$.

Example 21.5. Assume μ is a Borel measure on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$. Then

$$\phi(f) = \int_X f d\mu$$

defines a positive linear functional on $C_c(X)$.

Theorem 21.6. *(**Riesz-Markov**) Assume (X, d) is a locally compact metric space and $\phi : C_c(X) \to \mathbb{C}$ is a positive linear functional. Then there is a Borel measure μ such that $\mu(K) < \infty$ for every compact $K \subset X$ and*

$$\phi(f) = \int_X f d\mu, \quad \forall f \in C_c(X).$$

Such a measure is unique if X is separable.

To prove this we need some other results

Lemma 21.7. *(**Urysohn's lemma**) Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there is a continuous function $f : X \to [0, 1]$ s.t. $f|_A = 1$, $f|_B = 0$.*

Lemma 21.8. *Assume (X, d) is a compact metric space, assume we have a finite open cover $U = (U_i)_{i=1}^n$ of X. Then there exists continuous functions $\rho_1, \ldots, \rho_n$ such that*

$$supp(\rho_i) \subset U_i, \quad \rho_i \geq 0, \quad \sum_{i=1}^n \rho_i = 1.$$

Every such function is called a partition of unity subordinate to U.

Remark. *Without separability of X we can assume that the uniqueness holds within the class of Borel measures μ s.t.*

1. *$\mu(K) < \infty$ for every compact K*
2. *μ is outer regular $\mu(A) = \inf_{A \subset U,\ U \text{ open}} \mu(U)$*

3. μ is inner regular on open sets, $\mu(U) = \sup_{K \subset U,\ K\ compact} \mu(K)$

Such measures are called Radon measures.

As an application we will describe $C(X)^*$ for compact metric spaces X in terms of measures. Denote by $M(X)$ the space of complex Borel measures on X. For every $\nu \in M(X)$ we want to make sense of

$$\int_X f d\nu \;\text{ for }\; f \in C(X).$$

It is enough to consider finite signed measures, as we can then define

$$\int_X f d\nu = \int_X f d(Re\nu) + i \int_X f d(Im\nu).$$

So assume that ν is a finite signed measure. Then $\nu = \mu_1 - \mu_2$ for positive measures and we can define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \int M(X)$ defines a linear functional $\phi_\nu : C(X) \to \mathbb{C}$ by

$$\phi_\nu(f) = \int_X f d\nu$$

and the map $\nu \mapsto \phi_\nu$ is linear.

Lemma 21.9. *If $\nu \in M(X)$ and $d\nu = gd|\nu|$ is its polar decomposition, then*

$$\int_X f d\nu = \int_X f g d|\nu|, \;\; \forall f \in C(X).$$

Lemma 21.10. *For every $\nu \in M(X)$, the linear functional ϕ_ν is bounded and*

$$||\phi_\nu|| = |\nu|(X).$$

Recall that the norm on $C(X)$ is

$$||f|| = \sup_{x \in X} |f(x)|.$$

Now consider (X, d) to be a compact metric space. Denote $M(X)$ to be the set of complex Borel measures on X.

$$M(X) \to C(X)^*, \;\; \nu \mapsto \phi_\nu, \;\; \phi_\nu(f) = \int_X f d\nu, \;\; ||\phi_\nu|| = |\nu|(X).$$

Lemma 21.11.

1. *If $\phi : C(X) \to \mathbb{C}$ is positive, then it is bounded and $||\phi|| = \phi(1)$.*
2. *The positive linear functional on $C(X)$ span $C(X)^*$.*

Theorem 21.12. *Assume (X, d) is a compact metric space. Then the map*

$$M(X) \to C(X)^*, \;\; \nu \mapsto \phi_\nu, \;\; \phi_\nu(f) = \int_X f d\mu,$$

is a linear isomorphism.

Remark.

1. *As a byproduct we see that $M(X)$ is a Banach space with norm $||\nu|| = |\nu|(X)$.*
2. *Assume (X, d) is a locally compact metric space*

$$C_0(X) \equiv \textit{the space of all continuous functions vanishing at infinity}$$

$$= \{f : X \to \mathbb{C} \;:\; f \textit{ is continuous}, \;\; f(x) \to 0, \;\; \textit{as } x \to 0\}.$$

Where "$f(x) \to 0$, as $x \to 0$" means that $\forall \epsilon > 0 \exists$ compact $K \subset X$ s.t. $|f(K)| < \epsilon \; \forall x \in K^c$. This is a Banach space with norm $||f|| = \sup_{x \in X} |f(x)|$ and $C_c(X)$ is dense in $C_0(X)$. One can prove that if X is separable, then we again have a linear isomorphism $M(X) \xrightarrow{\sim} C_0(X)^*$, $\nu \mapsto \phi_\nu$, $\phi_\nu(f) = \int_X f d\nu$. For general X we simply have $M_{reg}(X) \xrightarrow{\sim} C_0(X)^*$, where $M_{reg}(X)$ is the space of regular complex Borel measures on X. Here regularity of ν means that $|\nu|$ is regular.

26 Hilbert spaces

Assume H is a vector space over $\mathbb{C}$.

Definition 26.1. A pre-inner product on H is a map $(\cdot,\cdot) : H \times H \to \mathbb{C}$ which is

1. sesquilinear: that is, linear in the first variable, and antilinear in the second, for $u, v, w, \in H, \quad \alpha, \beta \in \mathbb{C}$

$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$$

$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v)$$

2. Hermitian: $(u, v) = \overline{(v, u)}$

3. Positive semidefinite: $(u, u) \geq 0$ for $u \in H$.

It is called an inner product or a scalar product, if instead of (3) the map is

3. positive definite: $(u, u) > 0$ for $u \in H$, $u \neq 0$.

A similar definition makes sense for real vector spaces, but then $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is

1. bilinear

2. Symmetric: $(u, v) = (v, u)$.

3. Positive semidefinite (for pre-inner products) or

3'. positive definite (for inner products)

We only consider complex vector spaces, but all the results will have analogues for real ones as well.

Theorem 26.2. *(**Cauchy-Schwarz inequality**) If $(\cdot,\cdot)$ is a pre-inner product, then*

$$|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}$$

Corollary 26.3. *We have a seminorm $||u|| \equiv (u, u)^{1/2}$. It is a norm if and only if $(\cdot,\cdot)$ is an inner product.*

Definition 26.4. A Hilbert space is a complex vector space H with an inner product $(\cdot,\cdot)$ s.t. H is complete with respect to the norm $||u|| = (u, u)^{1/2}$. The norm of the Hilbert space is determined by the inner product, but the inner product can also be recovered from the norm by the polarisations identity

$$(u, v) = \frac{||u + v||^2 - ||u - v||^2}{4} + i\frac{||u + iv||^2 - ||u - iv||^2}{4},$$

which is proven by direct computation. A similar computation also proves the parallelogram law:

$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2.$$

Remark. *A norm in a vector space is given by an inner product if and only if it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarisation identity.*

Remark. *Assume $(X, \mathscr{B}, \mu)$ is a measure space. Then $L^2(X, d\mu)$ is a Hilbert space with inner product*

$$(f, g) = \int_X f\bar{g}d\mu.$$

If $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, then we denote $L^2(X, d\mu)$ by $l^2(X)$. If $X = \mathbb{N}$ then we simply write l^2, this is called sequence space. A more formal definition is

$$l^2(X) = \left\{ f : X \to \mathbb{C} \ : \ \sum_x |f(x)|^2 < \infty \right\}$$

which is a Hilbert space with inner product $(f, g) = \sum_{x \in X} f(x)\bar{g}(x)$.

We recall that a subspace C of a vector space is called <u>convex</u> if $\forall u, v \in C$, $\forall t \in [0, 1]$ we have $tu + (1 - t)v \in C$. The following is one of the key properties of Hilbert spaces.

Theorem 26.5. *Assume H is a Hilbert space, $C \subset H$ is a closed convex subset, $u \in H$. Then there is a unique $u_0 \in C$ s.t.*

$$||u - u_0|| = d(u, C) = \inf_{v \in C} ||u - v||.$$

Orthogonal projections

For a Hilbert space H and a subset $A \subset H$, let

$$A^\perp = \{x \in H \ : \ x \perp y \ \forall y \in A\},$$

where $x \perp y$ means that $(x, y) = 0$. $A^\perp$ is a closed subset of H.

Proposition 26.6. *Assume H_0 is a closed subspace of a Hilbert space H. Then every $u \in H$ uniquely decomposes as $u = u_0 + u_1$ with $u_0 \in H_0$ and $u_1 \in H_0^\perp$.*

For a closed subspace $H_0 \subset H$, consider the map $P : H \to H_0$ s.t. $Pu \in H_0$ is a unique element satisfying $u - Pu \in H_0^\perp$. The operator $P : H \to H_0$ is linear. It is also <u>contractive</u>, meaning that $||Pu|| \leq ||u||$. It is called the <u>orthogonal projection</u> onto H_0. If H_0 is finite dimensional with an orthonormal basis $u_1, \ldots, u_n$, then

$$Pu = \sum_{k=1}^{n} (u, u_k)u_k.$$

Orthogonal basis can be defined for arbitrary Hilbert spaces.

Definition 26.7. (**Projection operators**) A projection on a vector space V is a linear operator
$$P : V \to V \text{ such that } P^2 = P.$$
If $V = H$ is a Hilbert space, then we can use the notion of orthogonality, then a projection P on H is called orthogonal if it satisfies
$$(Px, y) = (x, Py) \text{ for all } x, y \in H$$
.

Definition 26.8. An orthogonal system in H is a collection of vectors $u_i \in H$ $(i \in I)$ s.t.
$$(u_i, u_j) = \delta_{ij} \;\; \forall i, j \in I.$$
It is called an orthogonal basis if $span\{u_i\}_{i\in I}$ is dense in H.

Here $span\{u_i\}_{i\in I}$ denotes the linear span of $\{u_i\}_{i\in I}$, the space of finite linear combinations of the vectors u_i.

Theorem 26.9. *Every Hilbert space H has an orthonormal basis. If H is separable, then there is a countable orthonormal basis.*

Proposition 26.10. *Assume $\{u_i\}_{i\in I}$ is an orthogonal system in the Hilbert space H. Take $u \in H$. Then*

1. *(Bessel's inequality)* $\sum_{i\in I} |(u, u_i)|^2 \le ||u||^2$
2. *(Parseval's identity) if $\{u_i\}_{i\in I}$ is an orthogonal basis then* $\sum_{i\in I} |(u, u_i)|^2 = ||u||^2$.

If $(u_i)_{i\in I}$ is an orthogonal basis, then the numbers (u, u_i) $(i \in I)$ are called the Fourier coefficients of u with respect to $(u_i)_{i\in I}$. The Parseval identity suggests that u is determined by its Fourier coefficients. This is true, and even more we have

Proposition 26.11. *Assume $(u_i)_{i\in I}$ is an orthogonal basis in a Hilbert space H. Then for every vector $(c_i)_{i\in I} \in l^2(I)$ there is a unique vector $u \in H$ with Fourier coefficients c_i. We write*
$$u = \sum_{i\in I} c_i u_i.$$

Corollary 26.12. *We have a linear isomorphism $U : l^2(I) \tilde{\to} H$, $U((c_i)_{i\in I}) = \sum_{i\in I} c_i u_i$. By Parseval's identity this isomorphism is isometric, that is $||Ux|| = ||x|| \; \forall x \in l^2(I)$. By the polarisation identity this is equivalent to*
$$(Ux, Uy) = (x, y), \;\; \forall x, y \in l^2(I).$$
Hence U is unitary.

Corollary 26.13. *Up to a unitary isomorphism, there is only one infinite dimensional separable Hilbert space, namely, l^2.*

Given two orthogonal bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ in a Hilbert space H, we can decompose $u_i = \sum_{j\in J}(u_i, v_j)v_j$ and using that the sets $\{j : (u_i, v_j) \neq 0\}$ are countable proves the following.

Remark. *Any two orthonormal bases in a Hilbert space have the same cardinality.*

Dual spaces

Lemma 26.14. *Assume V is a normed space over $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider a linear functional $f : V \to K$. TFAE:*

1. *f is continuous*
2. *f is continuous at 0*
3. *there is $c \geq 0$ s.t. $|f(x)| \leq c||x||\ \forall x \in V$.*

If $(1)-(3)$ are satisfies, then f is called a <u>bounded linear functional</u>. The smallest such $c \geq 0$ in (3) is called the <u>norm</u> of f and is denoted by $||f||$. We have

$$||f|| = \sup_{x\neq 0}\frac{|f(x)|}{||x||} = \sup_{||x||=1}|f(x)| = \sup_{||x||\leq 1}|f(x)|.$$

Proposition 26.15. *For every normed vector space V over $K = \mathbb{R}$ or $K = \mathbb{C}$, the bounded linear functionals on V forms a Banach space V^*.*

Theorem 26.16. *(<u>**Riesz representation theorem**</u>) Assume H is a Hilbert space. Then every $f \in H^*$, where H^* is the dual-space of H, containing continuous, linear functionals, has the form*

$$f(x) = (x, y)$$

for a uniquely determined $y \in H$. Moreover, we have $||f|| = ||y||$.

For every Hilbert space H we can define the <u>conjugate Hilbert space</u> $\overline{H}$. Its elements are the symbols $\bar{x}$ for $x \in H$. The linear structure and inner products are defined by

$$\bar{x} + \bar{y} = \overline{x+y},\ \ c\bar{x} = \overline{\bar{c}x},\ \ (\bar{x}, \bar{y}) = \overline{(x,y)} = (y, x).$$

Corollary 26.17. *For every Hilbert space H, we have an isometric isomorphism $\overline{H}\tilde{\to}H^*$, $\bar{x} \mapsto (\cdot, x)$.*

Appendix

H Regularity of measures (lecture 10, 12. Feb.)

We let (X,d) be a metric space and denote by $\mathcal{O}$ the open, by $\mathcal{C}$ the closed and $\mathscr{B}(X) = \sigma(\mathcal{O})$ the Borel set of X.

Definition H.1. A measure μ on $(X,d,\mathscr{B}(X))$ is called outer regular, if

$$\mu(B) = \inf\{\mu(U) \mid B \subset U,\ U \text{ open}\} \tag{56}$$

and inner regular, if $\mu(K) < \infty$ for all compact sets $K \subset X$ and

$$\mu(U) = \sup\{\mu(K) \mid K \subset U,\ K \text{ compact}\}\,. \tag{57}$$

A measure which is both inner and outer regular is called **regular**. We write $\mathfrak{m}_r^+(X)$ for the family of regular measures on $(X,\mathscr{B}(X))$.

Remark. *The space X is called σ-compact if there is a sequence of compact sets $K_n \uparrow X$. A typical example of such a space is a locally compact, separable metric space.*

Theorem H.2. *Let (X,d) be a metric space. Every finite measure μ on $(X,\mathscr{B}(X))$ is outer regular. If X is σ-compact, then μ is also inner regular, hence regular.*

Theorem H.3. *Let (X,d) be a metric space and μ be a measure on $(X,B(X))$ such that $\mu(K) < \infty$ for all compact sets $K \subset X$.*

1 If X is σ-compact, then μ is inner regular.

2 If there exists a sequence $G_n \in \mathcal{O},\ G_n \uparrow X$ such that $\mu(G_n) < \infty$, then μ is outer regular.

I Hahn-Banach theorem

Theorem I.1. *(**Hahn-Banach theorem**) Assume V is a real vector space, $V_0 \subset V$ a subspace, $\phi : V \to \mathbb{R}$ a convex function and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \leq \phi$ on V_0. Then f can be extended to a linear functional F on V s.t. $F \leq \phi$.*

Theorem I.2. *(**Hahn-Banach theorem 2**) Assume V is a real or complex vector space, p is a seminorm on V, $V_0 \subset V$ a subspace and f is a linear functional on V_0 s.t. $|f(x)| \leq p(x)\ \forall x \in V_0$. Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x)\ \forall x \in V$.*

Corollary I.3. *Assume V is a normed vector space (real or complex), $V_0 \subset V$ a subspace and $F \in V_0^*$. Then there is $F \in V^*$ s.t. $F|_{V_0} = f$ and $||F|| = ||f||$.*

Corollary I.4. *Assume V is a normed space and $x \in V, x \neq 0$. Then there is $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

Such an F is called a supporting functional of x.

Definition I.5. A normed space V is called reflexive if $V^{**} = V$.

Remark. *This is stronger than requiring $V \cong V^{**}$.*

Example I.6.

1. Every f.d. normed vector space V is reflexive for dimensional reasons $\dim V^{**} = \dim V^* = \dim V$.

2. Every Hilbert space H is reflexive. By Riesz' theorem every bounded linear functional f on $\overline{H}$ has the form

$$f(\bar{x}) = (\bar{x}, \bar{y}) = (y, x) \quad y \in H,$$

which means that $f = y$ in H^{**}. As we will later see, the spaces $L^p(X, d\mu)$, with μ σ-finite and $1 < p < \infty$, are reflexive. The spaces $L^1(X, d\mu)$ and $L^\infty(X, d\mu)$ are usually not reflexive.

J Complex and signed measures

Assume $(X, \mathscr{B})$ is a measurable space.

Definition J.1. A complex measure on $(X, \mathscr{B})$ is a map $\nu : \mathscr{B} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and $\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ for any disjoint $A_n \in \mathscr{B}$ where the series is assumed to bee absolutely convergent. If ν takes values in $\mathbb{R}$, then ν is called a finitely signed measure.

Remark. *More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$. Given a complex measure ν on $(X, \mathscr{B})$, its total variation is the map*

$$|\nu| : \mathscr{B} \to [0, +\infty]$$

defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^{N} |\nu(A_n)| \ : \ A = \bigcup_{n=1}^{N} A_n, \ \ A_n \in \mathscr{B}, \ \ A_n \cap A_m = \emptyset \ \ \text{for } n \neq m \right\}$$

Proposition J.2. *$|\nu|$ is a finite measure on $(X, \mathscr{B})$.*

Definition J.3. If $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, then we say that ν is absolutely continuous with regards to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}$ and $\mu(A) = 0$. Equivalently $|\nu| << \mu$.

Theorem J.4. ***Radon-Nikodym theorem for complex measures***
Assume $(X, \mathscr{B}, \mu)$ is a measurable space, ν is a complex measure on $(X, \mathscr{B})$, $\nu << \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.

K Kolmogorov extension theorem

Assume X is a set and $(\mathscr{B}_n)_{n=1}^\infty$ is an increasing sequence of σ-algebras of subsets of X. Assume μ_n is a measure on $(X, \mathscr{B}_n)$ and

$$\mu_{n+1}|_{\mathscr{B}_n} = \mu_n, \quad \forall n.$$

Can we define a measure μ on $(X, \mathscr{B})$, where $\mathscr{B} = \sigma\left(\bigcup_{n=1}^\infty \mathscr{B}_n\right)$ s.t. $\mu|_{\mathscr{B}_n} = \mu_n$, $\forall n$? In general <u>no</u>. But we have the following:

Theorem K.1. *In the above setting, assume in addition that $\mu_n(X) = 1$ $\forall n$ and there is a collection of subsets $\mathscr{C} \subset \mathscr{B}$ s.t.:*

1. $\mu_n(A) = \sup\{\mu_n(C) \ : \ C \subset A, C \in \mathscr{C} \cap \mathscr{B}_n\}$ $\forall A \in \mathscr{B}_n$,
2. *if $(C_n)_{n=1}^\infty$ is a sequence in $\mathscr{C}$ and $\cap_{n=1}^\infty C_n = \emptyset$, then $\cap_{n=1}^N C_n = \emptyset$ for some N.*

Then there is a unique measure μ on $(X, \mathscr{B})$ s.t. $\mu|_{\mathscr{B}_n} = \mu_n$.

Remark. *If in addition $\mathscr{C}$ is closed under finite unions and countable intersections, then*

$$\mu(A) = \sup\{\mu(C) \ : \ C \subset A, C \in \mathscr{C}\} \quad \forall A \in \mathscr{B}.$$

Indeed, consider the collection $\tilde{\mathscr{B}}$ of sets $A \in \mathscr{B}$ s.t.

$$\begin{aligned}
\mu(A) &= \sup\{\mu(C) \ : \ C \subset A, C \in \mathscr{C}\}, \\
\mu(A^c) &= \sup\{\mu(C) \ : \ C \subset A^c, C \in \mathscr{C}\}.
\end{aligned}$$

Then $\tilde{\mathscr{B}}$ is a σ-algebra by the same argument as in the lecture on regularity of measures, and $A \subset \tilde{\mathscr{B}}$. Hence, $\tilde{\mathscr{B}} = \mathscr{B}$.

Assume now we have a collection $((X_i, \mathscr{B}_i))_{i \in I}$ of measurable spaces, (It can be infinite and uncountable). Consider $X = \prod_{i \in I} X_i$. Denote by $\prod_{i \in I} \mathscr{B}_i$ the σ-algebra generated by all sets of the form

$$\prod_{i \in I} A - i \times \prod_{j \in F^c} X_j,$$

where $F \subset I$ is finite, $A_i \subset \mathscr{B}_i$, $(i \in F)$.

Example K.2. Consider a sequence $((X_n, d_n))_{n=1}^\infty$ of separable metric spaces. We assume $d_n(x, y) \leq 1$, $\forall x, y$. Note that any metric can be replaced by definition $\tilde{d}(x, y) = d(x, y)/(1 + d(x, y))$. Then $\prod_{n \in I} X_n$ is a metric space with metric

$$d(\underline{x}, \underline{y}) = \sum_{n=1}^\infty \frac{1}{2^n} d_n(x_n, y_n),$$

where $\underline{x} = (x_n)_{n=1}^\infty \in X$. Given a sequence $(\underline{x}(n))_{n=1}^\infty$ in X, we have $\underline{x}(n) \xrightarrow[n \to \infty]{} \underline{x}$ if and only if $x(n)_k \xrightarrow[n \to \infty]{} x_k$, $\forall k$. Consider the Borel σ-algebra $\mathscr{B}(X_n)$. Then $\prod_{n=1}^\infty \mathscr{B}(X_n) = \mathscr{B}(X)$.

To see this, for every n, choose open sets, $U_{nk} \subset X_n$ $(k = 1, 2, ...)$ s.t. every open set in X_n is the union of some of U_{nk}'s. This is possible by separability. Take a dense countable subset of X_n and then all balls of rational radii with centers at points of this subset. Then every open subset of X is the union of sets of the form

$$U_{1,k_1} \times U_{2,k_2} \times \ldots \times U_{n,k_n} \times \prod_{m=n+1}^{\infty} X_m.$$

Therefore, such sets generate the σ-algebra $\mathscr{B}(X)$, and as $U_{n,k}$ $(n \in \mathbb{N})$ generate $\mathscr{B}(X_n)$, we conclude that $\cap_{n=1}^{\infty} \mathscr{B}(X_n) = \mathscr{B}(X)$.

Remark. *In relation to this example, we will need the following:*

Theorem K.3. *(**Tychonoff**)*
Assume $((X_n, d_n))_{n=1}^{\infty}$ is a sequence of compact metric spaces. Then $\prod_{n=1}^{\infty} X_n$ (with metrics as in the example) is compact.

Theorem K.4. *(**Kolmogorov extension theorem**)*
Assume $(X_i)_{i \in I}$ is a collection of metric spaces. Consider $X = \prod_{i \in I} X_i$, $\mathscr{B} = \prod_{i \in I} \mathscr{B}(X_i)$. Assume for every finite $F \subset I$ we are given a regular Borel probability measure μ_F on X_F s.t.

$$(\pi_{G,F})_* \mu_G = \mu_F$$

for all finite $F \subset G \subset I$. Then there is a unique probability measure μ on $(X, \mathscr{B})$ s.t.

$$(\pi_{I,F})_* \mu = \mu_F$$

for all finite $F \subset I$.

Remark. *If in addition the spaces X_i are separable, then we can also conclude that for every $A \in \mathscr{B} = \prod_{i \in I} \mathscr{B}(X_i)$ we have*

$$\mu(A) = \sup \mu(C),$$

where the supremum is taken over the sets $C \subset A$ of the form

$$C = K \times \prod_{i \in I \setminus J} X_i,$$

where $J \subset I$ is countable and $K \subset X_J$ is compact.

L Random variables and stochastic processes

Assume $(X, \mathscr{B}, \mathbb{P})$ is a probability measure space. Assume $(Y, \mathscr{C})$ is a measurable space.

A measurable map $f : X \to Y$ is called a random variable. For $A \in \mathscr{C}$, define

$$\mathbb{P}(f \in A) \stackrel{\text{def}}{=} \mathbb{P}(f^{-1}(A)) = (f_*\mathbb{P})(A),$$

as the probability that f takes values in A. The measure $f_*\mathbb{P}$ is called the probability distribution of f.

Definition L.1. A stochastic process is a collection of random variables $(f_t : X \to Y)_{t\in T}$, T stands for "time". Typically, $T = \mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$.

Given different $t_1, \dots, t_n \in T$, we can consider the disjoint distribution of $f_{t_1}, \dots, f_{t_n}$ as the measures

$$\mu_{t_1,\dots,t_n} = (f_{t_1} \times \dots \times f_{t_n})_* \mathbb{P}, \quad \text{on } (Y^n, \mathscr{C}^n).$$

Thus,

$$\mu_{t_1,\dots,t_n}(A_1 \times \dots \times A_n) = \mathbb{P}(f_{t_1} \in A_1, \dots, f_{t_n} \in A_n) \stackrel{\text{def}}{=} \mathbb{P}\left(\bigcap_{k=1}^{n} f_{t_k}^{-1}(A_k)\right).$$

Theorem L.2. *Assume T is a set, and for all different $t_1, \dots t_n$ we are given a Borel probability measure $\mu_{t_1,\dots,t_n}$ on $\mathbb{R}^n$ s.t.*

1. *if $\delta \in S_n$ (S_n is the permutation group) and $A_1, \dots, A_n \in \mathscr{B}(\mathbb{R})$, then*

$$\mu_{t_1,\dots,t_n}(A_1 \times \dots \times A_n) = \mu_{t_{\delta(1)},\dots,t_{\delta(n)}}(A_{\delta(1)} \times \dots \times A_{\delta(n)})$$

2. *Additionally,*

$$\mu_{t_1,\dots,t_n,s_1,\dots,s_m}\left(A_1 \times \dots \times A_n \bigtimes_{k=1}^{m} \mathbb{R}\right) = \mu_{t_1,\dots,t_n}(A_1 \times \dots \times A_n)$$

for all different $t_1, \dots, t_n, s_1, \dots, s_m \in T$ and $A_1, \dots, A_n \in \mathscr{B}(\mathbb{R})$.

Then there is a probability measure space $(X, \mathscr{B}, \mathbb{P})$ and a stochastic process $(f_t : X \to \mathbb{R})_{t\in T}$ s.t. $\mu_{t_1,\dots,t_n}$ is the joint distribution of $f_{t_1}, \dots f_{t_n}$ for all different $t_1, \dots, t_n \in T$.

Remark. *Instead of $\mathbb{R}$ we could have taken any complete separable metric space. Or we could have taken any metric space, but we would have to require that $\mu_{t_1,\dots,t_n}$ are regular measures.*

Definition L.3. Random variables $f_1, \dots f_n : X \to Y$ are called <u>independent</u> if

$$\mathbb{P}(f_1 \in A_1, \dots, f_n \in A_n) = \mathbb{P}\left(\bigcap_{k=1}^{n} f_k^{-1}(A_k)\right) = \mathbb{P}(f_1 \in A_1) \dots \mathbb{P}(f_n \in A_n).$$

In other words, this means that if the joint distribution of $f_1, \dots f_n$ is the product of distributions of $f_1, \dots f_n$. For such measures the theorem gives the following results:

Assume we are given a Borel probability measure μ_t on $\mathbb{R}$ or any other complete separable metric space for $t \in T$. Then there is a unique measure

$$\mathbb{P} = \prod_{t \in T} \mu_t \quad \text{s.t.} \quad (\pi_{T,F})_* \mathbb{P} = \prod_{t \in T} \mu_t \ \text{ for all finite } \ F \subset T.$$

Example L.4. Consider the process of tossing a coin. Write 0 for tail and 1 for head. We can model the process as follows:

$$X = \prod_{n=1}^{\infty} \{0,1\}, \ \ \mathbb{P} = \prod_{n=1}^{\infty} \nu, \quad \text{where} \quad \nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1.$$

With $\mathscr{B} = \prod_{n=1}^{\infty} \mathcal{P}(\{0,1\})$.

Then if we define $f_n : X \to \{0,1\}$, $f_n(\underline{x}) = x_n$, for $\underline{x} = (x_m)_{m=1}^{\infty}$. For a.e. $\underline{x} \in X$ we have

$$\frac{1}{n} \sum_{k=1}^{n} f_n(\underline{x}) \xrightarrow[n \to \infty]{} \frac{1}{2} \qquad (\textbf{Law of large numbers}).$$

If Kolmogorov's theorem requires some regularity assumption, infinite products of probability spaces always exists:

Theorem L.5. *Assume $((X_i, \mathscr{B}_i, \mu_i))_{i \in I}$ is a collection of probability measure spaces. Consider $X = \prod_{i \in I} X_i$, $\mathscr{B} = \prod_{i \in I} \mathscr{B}_i$. Then there is a unique measure μ on $(X, \mathscr{B})$ s.t. $(\pi_{I,F})_* \mu = \prod_{i \in F} \mu_i$ for all finite $F \in I$.*

In other words,

$$\mu\left(\prod_{i \in F} A_i \times \prod_{j \in J \setminus F} X_j\right) = \prod_{i \in F} \mu_i(A_i).$$

Remark. *One nice property worth minding used in the proof of this theorem is that when proving σ-additivity for $B = \bigcap_{n=1}^{\infty} B_n$, $B_n \cap B_m = \emptyset$ for $n \neq m$. Proving that,*

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n) \Leftrightarrow \mu(A_n) \xrightarrow[n \to \infty]{} 0 \ , \quad A_n = B \setminus \left(\bigcup_{m=1}^{n-1} B_m\right), \ \ A_n \downarrow \emptyset,$$

for a pre-measure on an <u>algebra</u> of sets. Hence, this gives us an alternative method of proving σ-additivity.