5 Hahn-Banach theorem

Theorem 5.1. *(<u>**Hahn-Banach theorem**</u>) Assume V is a real vector space, $V_0 \subset V$ a subspace, $\phi : V \to \mathbb{R}$ a convex function and $f : V_0 \to \mathbb{R}$ a linear functional s.t. $f \leq \phi$ on V_0. Then f can be extended to a linear functional F on V s.t. $F \leq \phi$.*

Theorem 5.2. *(<u>**Hahn-Banach theorem 2**</u>) Assume V is a real or complex vector space, p is a seminorm on V, $V_0 \subset V$ a subspace and f is a linear functional on V_0 s.t. $|f(x)| \leq p(x)$ $\forall x \in V_0$. Then f can be extended to a linear functional F on V s.t. $|F(x)| \leq p(x)$ $\forall x \in V$.*

Corollary 5.3. *Assume V is a normed vector space (real or complex), $V_0 \subset V$ a subspace and $F \in V_0^*$. Then there is $F \in V^*$ s.t. $F|_{V_0} = f$ and $||F|| = ||f||$.*

Corollary 5.4. *Assume V is a normed space and $x \in V, x \neq 0$. Then there is $F \in V^*$ s.t. $||F|| = 1$ and $F(x) = ||x||$.*

Such an F is called a supporting functional of x.

Definition 5.5. A normed space V is called <u>reflexive</u> if $V^{**} = V$.

Remark. *This is stronger than requiring $V \cong V^{**}$.*

Example 5.6.

1. Every f.d. normed vector space V is reflexive for dimensional reasons $\dim V^{**} = \dim V^* = \dim V$.

2. Every Hilbert space H is reflexive. By Riesz' theorem every bounded linear functional f on $\overline{H}$ has the form
$$f(\bar{x}) = (\bar{x}, \bar{y}) = (x, y) \quad y \in H,$$
which means that $f = y$ in H^{**}. As we will later see, the spaces $L^p(X, d\mu)$, with μ σ-finite and $1 < p < \infty$, are reflexive. The spaces $L^1(X, d\mu)$ and $L^\infty(X, d\mu)$ are usually not reflexive.