5 Complex and signed measures

Assume $(X, \mathscr{B})$ is a measurable space.

Definition 5.1. A complex measure on $(X, \mathscr{B})$ is a map $\nu : \mathscr{B} \to \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and $\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ for any disjoint $A_n \in \mathscr{B}$ where the series is assumed to bee absolutely convergent. If ν takes values in $\mathbb{R}$, then ν is called a finitely signed measure.

Remark. *More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$. Given a complex measure ν on $(X, \mathscr{B})$, its total variation is the map*

$$|\nu| : \mathscr{B} \to [0, +\infty]$$

defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^{N} |\nu(A_n)| \ : \ A = \bigcup_{n=1}^{N} A_n, \ \ A_n \in \mathscr{B}, \ \ A_n \cap A_m = \emptyset \ \text{ for } n \neq m \right\}$$

Proposition 5.2. $|\nu|$ *is a finite measure on* $(X, \mathscr{B})$.

Definition 5.3. If $(X, \mathscr{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathscr{B})$, then we say that ν is absolutely continuous with regards to μ and write $\nu << \mu$, if $\nu(A) = 0$ whenever $A \in \mathscr{B}$ and $\mu(A) = 0$. Equivalently $|\nu| << \mu$.

Theorem 5.4. ***Radon-Nikodym theorem for complex measures***
Assume $(X, \mathscr{B}, \mu)$ is a measurable space, ν is a complex measure on $(X, \mathscr{B})$, $\nu << \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t. $d\nu = f d\mu$.