18 Dense subspaces of L^p

Theorem 18.1. *Assume (X, d) is a metric space, μ is a Borel measure on X s.t. $\mu(B_R(x)) < \infty$, $\forall x \in X$ and $\forall R > 0$, $1 \le p \le \infty$. Then the bounded continuous functions on X with bounded support form a dense subspace of $L^p(X, d\mu)$. (Where by bounded support we mean that f is zero outside $B_R(x)$ for some x and $R > 0$.)*

If X is locally compact, then by $C_c(X)$ we denote the space of continuous functions on X with compact support.

Theorem 18.2. *Assume (X, d) is a separable, so it has a dense subset, locally compact metric space, μ is a Borel measure on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$, $| \le P < \infty$. Then $C_c(X)$ is dense in $L^p(X, d\mu)$.*

Remark. *Theorem 17.8 in the book is wrong.*

Remark. *These results do not extent to $p = \infty$.*

For $X = \mathbb{R}^n$, either theorem implies that if μ is a Borel measure on $\mathbb{R}^n$, s.t. $\mu(B_R(x)) < \infty$, $\forall x$, $\forall R > 0$, then $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$, $1 \le p < \infty$. Later we will prove that $C_c^\infty(\mathbb{R}^n)$ is still dense in $L^p(\mathbb{R}^n, d\mu)$. For $\mu = \lambda_n$ we write $L^p(\mathbb{R}^n)$ instead of $L^p(\mathbb{R}^n, d\lambda_n)$.

19 Modes of convergence

Theorem 19.1. ***(Egorov)***
Assume $(X, \mathscr{B}, \mu)$ is a measure space, $\mu(X) < \infty$. Assume $f_n, f : X \to \mathbb{C}$ are measurable functions and $f_n \to f$ a.e. Then $\forall \epsilon > 0$ there is $X_\epsilon \in \mathscr{B}$ s.t. $\mu(X_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash X_\epsilon$.

1. For measurable functions f_n, f, we write $f_n \to f$ in the $p - th$ mean, excluding $p = \infty$, if $\lim_{n\to\infty} ||f_n - f||_p = 0$.
2. For $p = 1$ we sat that $f_n \to f$ in mean and for $p = 2$ we say that $f_n \to f$ in quadratic mean.
3. We say that $f_n \to f$ in measure if $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \ge \epsilon\}) = 0$, $\forall \epsilon > 0$.

Theorem 19.2. *Assume $(X, \mathscr{B}, \mu)$ is a measure space, $1 \le p < \infty$, $f_n, f \to \mathbb{C}$ are measurable functions. Then*

1. *if $f_n \to f$ in the p-th mean, then $f_n \to f$ in measure*
2. *if $f_n \to f$ in measure, then there is a subsequence $(f_n)_{n=1}^\infty$ s.t. $f_{n_k} \to f$ a.e.*
3. *if $f_n \to f$ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.*

In particular, if $f_n \to f$ in the p-th mean, then $f_{n_k} \to f$ a.e. for a subsequence $(f_{n_k})_k$.

20 Hilbert spaces

Assume H is a vector space over $\mathbb{C}$.

Definition 20.1. A pre-inner product on H is a map $(\cdot, \cdot) : H \times H \to \mathbb{C}$ which is

1. sesquilinear: that is, linear in the first variable, and antilinear in the second, for $u, v, w, \in H$, $\alpha, \beta \in \mathbb{C}$
$$(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$$
$$(w, \alpha u + \beta v) = \bar{\alpha}(w, u) + \bar{\beta}(w, v)$$
2. Hermitian: $(u, v) = \overline{(v, u)}$
3. Positive semidefinite: $(u, u) \ge 0$ for $u \in H$.

It is called an inner product or a scalar product, if instead of (3) the map is

3. positive definite: $(u, u) > 0$ for $u \in H$, $u \ne 0$.

A similar definition makes sense for real vector spaces, but then $(\cdot, \cdot) : H \times H \to \mathbb{R}$ is

1. bilinear
2. Symmetric: $(u, v) = (v, u)$.

3. Positive semidefinite (for pre-inner products) or

3'. positive definite (for inner products)

We only consider complex vector spaces, but all the results will have analogues for real ones as well.

Theorem 20.2. ***Cauchy-Schwarz inequality*** *If $(\cdot,\cdot)$ is a pre-inner product, then*

$$|(u,v)| \leq (u,u)^{1/2}(v,v)^{1/2}$$

Corollary 20.3. *We have a seminorm $||u|| \equiv (u,u)^{1/2}$. It is a norm if and only if $(\cdot,\cdot)$ is an inner product.*

Definition 20.4. A Hilbert space is a complex vector space H with an inner product $(\cdot,\cdot)$ s.t. H is complete with respect to the norm $||u|| = (u,u)^{1/2}$. The norm of the Hilbert space is determined by the inner product, but the inner product can also be recovered from the norm by the polarisations identity

$$(u,v) = \frac{||u+v||^2 - ||u-v||^2}{4} + i\frac{||u+iv||^2 - ||u-iv||^2}{4},$$

which is proven by direct computation. A similar computation also proves the parallelogram law:

$$||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2.$$

Remark. *A norm in a vector space is given by an inner product if and only if it satisfies the parallelogram law, and then the scalar product is uniquely determined by the polarisation identity.*

Remark. *Assume $(X,\mathscr{B},\mu)$ is a measure space. Then $L^2(X,d\mu)$ is a Hilbert space with inner product*

$$(f,g) = \int_X f\bar{g}d\mu.$$

If $\mathscr{B} = \mathcal{P}(X)$ and μ is the counting measure, then we denote $L^2(X,d\mu)$ by $l^2(X)$. If $X = \mathbb{N}$ then we simply write l^2, this is called sequence space. A more formal definition is

$$l^2(X) = \left\{ f : X \to \mathbb{C} \ : \ \sum_x |f(x)|^2 < \infty \right\}$$

which is a Hilbert space with inner product $(f,g) = \sum_{x\in X} f(x)\bar{g}(x)$.

We recall that a subspace C of a vector space is called convex if $\forall u,v \in C$, $\forall t \in [0,1]$ we have $tu+(1-t)v \in C$. The following is one of the key properties of Hilbert spaces.

Theorem 20.5. *Assume H is a Hilbert space, $C \subset H$ is a closed convex subset, $u \in H$. Then there is a unique $u_0 \in C$ s.t.*

$$||u-u_0|| = d(u,C) = \inf_{v\in C} ||u-v||.$$