IEOR 4735 project

Zhengrui Fu zf2181 / Guoan Wang gw2355

November 2018

1 Notation

1. S_t, time t price of stock(STOXX50E)in Euro
2. X_t, time t exchange rate, number of USD per Euro (e.g. 1.13)
3. $r_t^{\$}$, time t US short rate
4. r_t^e, time t EUR short rate
5. $L(t, T_1, T_2)$, US Libor forward rate contracted at t for the period $[T_1, T_2]$
6. $P^{\$,*}(0, T)$, the observed US treasury bond price (now) maturing at time T
7. $P^{\$}(t, T)$, the time t price of US treasury bond maturing at time T
8. $f^{\$}(t, T)$, the instantaneous forward rate prevailing at time t for maturity $T > t$.
9. $p^{\$}(t, T, K, S)$, the time t price of a European put option with strike price K and exercise date T, on an underlying US S-bond

2 Objective

We would like to buy a derivative contract paying the amount $max(0, S_T - K)$ in USD at a pre-specified expiration date T , with S_T the price of STOXX50E denominated in EUR and K a given strike price in USD.

Additionally if the level of 3-month USD LIBOR observed on a specified date $T_1 < T$ is above a known barrier level L then the contract will "knock out" and terminate immediately without any payments.

Mathematically speaking, we want to calculate

$$E^Q[e^{-\int_0^T r_t^{\$} dt}(S_T - K)^+ \mathbb{1}_{L(T_1,T_1,T_1+\delta)\leq L}]$$

where δ is 3 months, Q is the USD money market risk neutral measure. We will calculate this using Monte Carlo.

3 Models

3.1 US short rate

Assumes the Ho-Lee dynamics of r_t

$$dr_t = \theta(t)dt + \sigma dW_t^Q$$

σ will be calibrated using the Interest Cap prices in the market. We observe the price of a cap with term T (e.g. $T = 1$ year), strike K^{cap} written on a δ-Libor (e.g. δ = 3-Month). A cap is essentially a bunch of caplets. Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$, where $t_{i+1} - t_i = \delta$. Then, for each $i \in 1, 2, ..., n$, we get a payoff of $(L(t_{i-1}, t_{i-1}, t_i) - K^{cap})^+$ at time t_i.

Each caplet, according to section 26.8 in Bjork, can be viewed as a put on bond, because:

$$\begin{aligned}
&(L(t_i, t_i, t_{i+1}) - K^{cap})^+ \cdot P^\$(t_i, t_{i+1}) \\
&= (\frac{1 - P^\$(t_i, t_{i+1})}{\delta P^\$(t_i, t_{i+1})} - K^{cap})^+ \cdot P^\$(t_i, t_{i+1}) \\
&= \frac{1}{\delta}(1 - (1 + K^{cap})P^\$(t_i, t_{i+1})) \\
&= \frac{1 + K^{cap}}{\delta} \cdot (\frac{1}{1 + K^{cap}} - P^\$(t_i, t_{i+1}))^+
\end{aligned}$$

Thus, the price of a cap if the sum of a series of European put options on bonds. A European call on a bond can be evaluated using the following formula(Proposition 24.5 and 24.9 in Bjork):

$$c^\$(0, T, K, S) = P^{\$,*}(0, S)N(d) - P^{\$,*}(0, T) * K * N(d - \sigma_p)$$

$$d = \frac{1}{\sigma_p} log(\frac{P^{\$,*}(0, S)}{P^{\$,*}(0, T) * K}) + \frac{1}{2}\sigma_p$$

$$\sigma_p = \sigma * (S - T) * \sqrt{T} \text{ (For Ho-Lee)}$$

$$\sigma_p = \frac{1}{a}(1 - e^{-a(S-T)}) \cdot \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(S-T)})} \text{ (For Hull-White)}$$

By put-call parity, the price of an European put on a bond is:

$$p^\$(0, T, K, S) = -P^{\$,*}(0, S)N(-d) + P^{\$,*}(0, T) * K * N(-d + \sigma_p)$$

Then, σ can be calibrated using market cap prices[1]. The prices we use are: For Ho-Lee, $\sigma \approx 0.7\%$, and for Hull-White, assuming $a = 3\%$, $\sigma \approx 0.8\%$.

In Ho-Lee model, $\theta(t)$ will be calibrated as follows (section 24.4.2 in Bjork)

$$\theta(t) = \frac{\partial f^*(0, t)}{\partial T} + \sigma^2 t$$

[1] We are using caps rather than caplets because we did not find caplets prices on Bloomberg

Term	Price(bps)	Strike
1	0.6553	2%
1	0.0542	3%

.

In Hull-White model, $\theta(t)$ will be calibrated as follows (section 24.4.4 in Bjork)

$$f_T^*(0,t) + \dot{g}(t) + a(f^*(0,t) + g(t))$$

where

$$g(t) = \frac{\sigma^2}{2} B^2(0,t)$$

$$B(t_1,t_2) = \frac{1}{a}(1 - e^{-a(t_2-t_1)})$$

3.2 Euro short rate

We will assume the Euro short rate is deterministic over time. Now (when we downloaded the data) the Euro short rate is -0.37%. We also use Ho-Lee and Hull-White methods to simulate Euro short rate, but since we assume the rate is deterministic, thus $\sigma = 0$.

3.3 US Libor

By definition,

$$L(T_1,T_1,T_1+\delta) = \frac{1 - P^{\$}(T_1,T_1+\delta)}{\delta * P^{\$}(T_1,T_1+\delta)}$$

How do we get $P^{\$}(T_1,T_1+\delta)$? According to Bjork Proposition 24.4 (The Ho-Lee Structure)

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} exp((T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t))$$

Or, using Hull-White Structure, according to Bjork Proposition 24.8

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} exp(B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t))$$

3.4 Stock

Under US Risk Neutral measure, the dynamic of stock price in Euro, are as follows:

$$dS_t = (r_t^e - \rho\sigma_X\sigma_S(S_t,t))S_t dt + \sigma_S(S_t,t)S_t dW_t$$

σ_X, the vol for FX, can be estimated using two methods, historical estimation or calibration using option prices. Using historical estimation, $\sigma_X \approx 6.5\%$; Bloomberg terminal gives option-implied $\sigma_X \approx 7.6\%$.

ρ, the correlation between FX brownian motion and Stock Brownian motion, is estimated from historical time series. $\rho \approx -0.248$.

Also, we should notice the stock Brownian motion and the US short rate Brownian motion might also have some correlation. Using historical estimation, this correlation is around 0.011.

The local vol surface is calibrated from the call price surface (strike from 2000 to 4400, maturity from 20 days to 10 years). We use a variation of Dupire Formula [2], because the Euro short rate is not constant over time:

$$\sigma_S(S,t)^2 = 2\frac{\frac{dC}{dT} + qC + (r_t - q)K\frac{dC}{dK}}{K^2\frac{d^2C}{dK^2}}\bigg|_{K=S,T=t}$$

4 Data and Interpolation

Historical stock prices and historical foreign exchange time series are found on yahoo finance. Historical US short rate data is found on iborate[3]. Other data (stock call prices, Euro short rate, stock dividend yield, US Libor cap prices, US treasury yields, etc.) are found on Bloomberg terminal as of Nov.28th, 2018.

For 1-d discrete data (such as prices of bonds of different maturities), we use scipy.interpolate.interp1d[4]. For 2-d descrete data (such as stock calls prices of different maturities and strike), we use scipy.interpolate.interp2d[5]. In both cases we use cubic interpolation.

5 Simulation

Simulation is done using Runge-Kutta Method rather than Euler Method.

$$\tilde{X}_{t_j} = X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\sqrt{\Delta t}$$

$$\begin{aligned} X_{t_{j+1}} &= X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\sqrt{\Delta t} Z_j \\ &+ \frac{1}{2}((\sigma(\tilde{X}_{t_j}, t_j) - \sigma(X_{t_j}, t_j))\sqrt{\Delta t}(Z_j^2 - 1) \end{aligned}$$

[2]The original Dupire Formula is from Bergomi equation 2.3
[3]http://iborate.com/usd-libor/
[4]https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.interp1d.html
[5]https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.interp2d.html