IEOR 4735 project

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1 Notation

1. S_t, time t price of stock(STOXX50E)in Euro
2. X_t, time t exchange rate, number of USD per Euro (e.g. 1.13)
3. $r_t^{\$}$, time t US short rate
4. r_t^e, time t EUR short rate
5. $L(t, T_1, T_2)$, US Libor forward rate contracted at t for the period $[T_1, T_2]$
6. $P^{\$,*}(0, T)$, the observed US treasury bond price (now) maturing at time T
7. $P^{\$}(t, T)$, the time t price of US treasury bond maturing at time T
8. $f^{\$}(t, T)$, the instantaneous forward rate prevailing at time t for maturity $T > t$.
9. $p^{\$}(t, T, K, S)$, the time t price of a European put option with strike price K and exercise date T, on an underlying US S-bond

2 Objective

We would like to buy a derivative contract paying the amount $max(0, S_T - K)$ in USD at a pre-specified expiration date T , with S_T the price of STOXX50E denominated in EUR and K a given strike price in USD.

Additionally if the level of 3-month USD LIBOR observed on a specified date $T_1 < T$ is above a known barrier level L then the contract will "knock out" and terminate immediately without any payments.

Mathematically speaking, we want to calculate

$$E^Q[e^{-\int_0^T r_t^{\$} dt}(S_T - K)^+ \mathbb{1}_{L(T_1,T_1,T_1+\delta)\leq L}]$$

where δ is 3 months, Q is the USD money market risk neutral measure. We will calculate this using Monte Carlo.

3 Models

3.1 US short rate

Assumes the Ho-Lee dynamics of r_t

$$dr_t = \theta(t)dt + \sigma dW_t^Q$$

σ will be calibrated using the Interest Cap prices in the market. We observe the price of a cap with term T (e.g. $T = 1$ year), strike K^{cap} written on a δ-Libor (e.g. δ = 3-Month). A cap is essentially a bunch of caplets. Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$, where $t_{i+1} - t_i = \delta$. Then, for each $i \in 1, 2, ..., n$, we get a payoff of $(L(t_{i-1}, t_{i-1}, t_i) - K^{cap})^+$ at time t_i.

Each caplet, according to section 26.8 in Bjork, can be viewed as a put on bond, because:

$$\begin{aligned}
&(L(t_i, t_i, t_{i+1}) - K^{cap})^+ \cdot P^{\$}(t_i, t_{i+1}) \\
&= (\frac{1 - P^{\$}(t_i, t_{i+1})}{\delta P^{\$}(t_i, t_{i+1})} - K^{cap})^+ \cdot P^{\$}(t_i, t_{i+1}) \\
&= \frac{1}{\delta}(1 - (1 + \delta K^{cap})P^{\$}(t_i, t_{i+1})) \\
&= \frac{1 + \delta K^{cap}}{\delta} \cdot (\frac{1}{1 + \delta K^{cap}} - P^{\$}(t_i, t_{i+1}))^+
\end{aligned}$$

Thus, the price of a cap if the sum of a series of European put options on bonds. A European call on a bond can be evaluated using the following formula(Proposition 24.5 and 24.9 in Bjork):

$$c^{\$}(0, T, K, S) = P^{\$,*}(0, S)N(d) - P^{\$,*}(0, T) * K * N(d - \sigma_p)$$

$$d = \frac{1}{\sigma_p} log(\frac{P^{\$,*}(0, S)}{P^{\$,*}(0, T) * K}) + \frac{1}{2}\sigma_p$$

$$\sigma_p = \sigma * (S - T) * \sqrt{T} \text{ (For Ho-Lee)}$$

$$\sigma_p = \frac{1}{a}(1 - e^{-a(S-T)}) \cdot \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(S-T)})} \text{ (For Hull-White)}$$

By put-call parity, the price of an European put on a bond is:

$$p^{\$}(0, T, K, S) = -P^{\$,*}(0, S)N(-d) + P^{\$,*}(0, T) * K * N(-d + \sigma_p)$$

Then, σ can be calibrated using market cap or caplets prices prices. We found cap prices of term 1Y, 2Y, etc. on Bloomberg; or we can use artificial caplet prices as input.

In Ho-Lee model, $\theta(t)$ will be calibrated as follows (section 24.4.2 in Bjork)

$$\theta(t) = \frac{\partial f^*(0, t)}{\partial T} + \sigma^2 t$$

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In Hull-White model, $\theta(t)$ will be calibrated as follows (section 24.4.4 in Bjork)

$$f_T^*(0,t) + \dot{g}(t) + a(f^*(0,t) + g(t))$$

where

$$g(t) = \frac{\sigma^2}{2} B^2(0,t)$$

$$B(t_1, t_2) = \frac{1}{a}(1 - e^{-a(t_2 - t_1)})$$

3.2 Euro short rate

We will assume the Euro short rate is deterministic over time. Now (when we downloaded the data) the Euro short rate is -0.37%. We also use Ho-Lee and Hull-White methods to simulate Euro short rate, but since we assume the rate is deterministic, thus $\sigma = 0$.

3.3 US Libor

By definition,

$$L(T_1, T_1, T_1 + \delta) = \frac{1 - P^{\$}(T_1, T_1 + \delta)}{\delta * P^{\$}(T_1, T_1 + \delta)}$$

How do we get $P^{\$}(T_1, T_1 + \delta)$? According to Bjork Proposition 24.4 (The Ho-Lee Structure)

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} exp((T-t)f^*(0,t) - \frac{\sigma^2}{2} t(T-t)^2 - (T-t)r(t))$$

Or, using Hull-White Structure, according to Bjork Proposition 24.8

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} exp(B(t,T)f^*(0,t) - \frac{\sigma^2}{4a} B^2(t,T)(1 - e^{-2at}) - B(t,T)r(t))$$

3.4 Stock

Under US Risk Neutral measure, the dynamic of stock price in Euro, are as follows:

$$dS_t = (r_t^e - \rho \sigma_X \sigma_S(S_t, t))S_t dt + \sigma_S(S_t, t) S_t dW_t$$

σ_X, the vol for FX, can be estimated using two methods, historical estimation or calibration using option prices. Using historical estimation, $\sigma_X \approx 6.5\%$; Bloomberg terminal gives option-implied $\sigma_X \approx 7.6\%$.

ρ, the correlation between FX brownian motion and Stock Brownian motion, is estimated from historical time series. $\rho \approx -0.248$.

Also, we should notice the stock Brownian motion and the US short rate Brownian motion might also have some correlation. Using historical estimation, this correlation is around 0.011.

For the vol of the stock, $\sigma_S(S_t, t)$, we considered two different methods, local vol (not implemented) and Black Scholes implied vol (in which case $\sigma_S(S_t, t)$ is a constant σ_S^{imp}.

3.4.1 Black Scholes

In this case we assume the dynamics of stock price is log normal, whose drift is just EUR short rate.

We can use Black Scholes to back-out σ_S by minimizing the mean square error over observed vanilla Euro call prices.

$$C^{BS}(K,T) = P(0,T)[F_0(T)N(d_1) - KN(d_2)]$$
$$F_0(T) = S_0 * e^{-qT}/P(0,T)$$
$$d_1 = \frac{ln\frac{F_0}{K} + \frac{1}{2}\sigma_S^2 T}{\sigma_S\sqrt{T}}$$
$$d_2 = d_1 - \sigma_S\sqrt{T}$$

We solve for σ_S by minimizing the mean square error as below:

$$\min_{\sigma_S} \Sigma_{K,T}(C^{OBS}(K,T) - C^{BS}(K,T))^2$$

3.4.2 Loval Vol (very slow)

The local vol surface is calibrated from the call price surface (strike from 2000 to 4400, maturity from 20 days to 10 years). The original Dupire formula [1] assumes constant short rate, but here we have deterministic yet not constant short rate. According to Hu[2], the corresponding dupire formula is

$$\sigma_S^{LV}(S,T)^2 = 2\frac{\frac{dC}{dT} + q_T C + (r_T - q_T)K\frac{dC}{dK}}{K^2\frac{d^2C}{dK^2}}\Bigg|_{K=S,T=T}$$

The formula above agrees with Dupire formula with stochastic interest rate, as in Joshi[3], equation 2.2, or in madan[4], as shown in class.

$$\sigma_S^{LV}(S,T)^2 = 2\frac{\frac{dC}{dT} - KE[exp(-\int_0^T r_t dt)r_T \mathbb{1}_{S_T>K}] + q_T(C - KC_k)}{K^2\frac{d^2C}{dK^2}}\Bigg|_{K=S,T=T}$$

[1] The original Dupire Formula is from Bergomi equation 2.3

[2] "LOCAL VOLATILITY MODEL WITH STOCHASTIC INTEREST RATE", equation 2.5.8, https://yorkspace.library.yorku.ca/xmlui/bitstream/handle/10315/30721/Hu_Bing_HB_2015_Masters.pdf

[3] "Joshi, Mark and Ranasinghe, Navin, Local Volatility Under Stochastic Interest Rates Using Mixture Models (May 15, 2016). Available at SSRN: https://ssrn.com/abstract=2780072 or http://dx.doi.org/10.2139/ssrn.2780072"

[4] https://staff.fnwi.uva. nl/p.j.c.spreij/winterschool/RenMadanQian.pdf

Since Euro short rate is deterministic, the expectation can be simplified, and we get the same equation as in Hu.

For the local vol surface, we could use SVI parametrization. For each slice of maturity T, given parameter set $\{a, b, \rho, m, \sigma\}$, the total variance is assumed to be:

$$w(k; \{a, b, \rho, m, \sigma\}) = a + b\{\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\}$$

where

$$w(k, t) = \sigma_{BS}^2(k, t) * t$$

$$K(\text{strike price}) = F_t(\text{forward price})e^k$$

For each time slice T, we find the parameter set $\{a, b, \rho, m, \sigma\}$ by minimizing the mean square difference between fitted price and market price.

In time dimension, we use linear interpolation in total implied variance $(w(k, t))$.

In terms of implied volatility, the local volatility can be written as[5]:

$$\sigma_S^{LV}(S, T)^2 = \frac{\frac{df}{dt}}{(\frac{y}{2f}\frac{df}{dy} - 1)^2 + \frac{1}{2}\frac{d^2f}{dy^2} - \frac{1}{4}(\frac{1}{4} + \frac{1}{f})(\frac{df}{dy})^2}\Bigg|_{y=ln(\frac{S}{F_t})}$$

$$y = ln(\frac{K}{F_t})$$

$$f(t, y) = (t - t_0)\sigma_{BS}(K, t)^2$$

Equivalently,

$$\sigma_S^{LV}(S, T)^2 = \frac{\frac{1}{2}\frac{\sigma_{BS}}{T} + \frac{\partial\sigma_{BS}}{\partial T} + K(r_T - q)\frac{\partial\sigma_{BS}}{\partial K}}{\frac{1}{2}K^2\left[\frac{1}{\sigma_{BS}K^2T} + 2\frac{\partial\sigma_{BS}}{\partial K}\frac{d_1}{\sigma_{BS}K\sqrt{T}} + \frac{\partial^2\sigma_{BS}}{\partial K^2} + \left(\frac{\partial\sigma_{BS}}{\partial K}\right)^2\frac{d_1d_2}{\sigma_{BS}}\right]}\Bigg|_{K=S,T=T}$$

4 Data and Interpolation

Historical stock prices and historical foreign exchange time series are found on yahoo finance. Historical US short rate data is found on iborate[6]. Other data (stock call prices, Euro short rate, stock dividend yield, US Libor cap prices, US treasury yields, etc.) are found on Bloomberg terminal as of Nov.28th, 2018.

For 1-d discrete data (such as prices of bonds of different maturities), we use scipy.interpolate.interp1d[7], with cubic interpolation.

For 2-d discrete data (i.e. stock calls prices of different maturities and strike), we use SVI parametrization as described above.

[5]Bergomi equation 2.19

[6]http://iborate.com/usd-libor/

[7]https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.interp1d.html

5 Simulation

Simulation is done using Runge-Kutta Method rather than Euler Method.

$$\tilde{X}_{t_j} = X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\sqrt{\Delta t}$$

$$\begin{aligned} X_{t_{j+1}} &= X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\sqrt{\Delta t} Z_j \\ &+ \frac{1}{2}((\sigma(\tilde{X}_{t_j}, t_j) - \sigma(X_{t_j}, t_j))\sqrt{\Delta t}(Z_j^2 - 1) \end{aligned}$$