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Department of Economics
University of Copenhagen

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Emil Rode

Pricing of Callable Mortgage Bonds

Supervisor: Rolf Poulsen

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Abstract

This thesis studies the structural model of Stanton (1995) for pricing of callable mortgage bonds. The model is adjusted to suit the Danish mortgage market by using the short rate dynamics suggested by Hull & White (1990) and by incorporating the delivery option embedded in Danish mortgage bonds. Object oriented programming is utilised for an efficient implementation of the numerical solution to the model. The numerical solution applies a Crank-Nicolson scheme in order to approximate the pricing PDEs obeyed by the callable mortgage bond. Estimation of the model is performed by means of the Generalised Method of Moments using observed prepayment rates. The final model is used to calculate option adjusted key figures, and by applying a backtest the model key figures are found to accurately describe the price movements observed in the market.

Contents

Preface

Supposedly, the title of this thesis will not raise many questions about what the pages to follow will deal with. However, I would like to emphasize that the aim of the thesis has not been to reinvent modelling of callable mortgage bonds. Pricing models have been developed many times before suggesting all sorts of ways to model the prepayment behaviour of borrowers and the evolution of the term structure of interest rates. Regardless of the fact that many models already exist, there will always be room for new ones. The reason for this is that it will always be possible to point out new aspects to incorporate in the model since markets are constantly changing. Instead, it has been the aim of the thesis to introduce a well known model and perform only the most necessary changes to make it consistent with the current market environment. In this way the model has been kept simple while the implementation has been in focus. The implementation of the model has been the most demanding part of the thesis, as it has required an extensive piece of objected oriented programming in order to numerically solve the model in an efficient way. I believe that the theoretical foundation of the model as well as a clean, efficient and reliable implementation should be assigned equal attention. Ultimately, the intended user of the model is a portfolio manager, a trader or a risk manager, so there is clearly little room for error.

In the empirical parts of this thesis, it has been of fundamental importance to have prepayment data available. This data can, with the right amount of patience, be gathered and structured from the market notices that are publicly available through the exchange. However, I have had the privilege of having this data available in structured form through my employer, SEB Investment Management, for which I am grateful.

Finally, I would like to thank my supervisor, Rolf Poulsen, for his valuable comments through out the process of writing this thesis.

1 Introduction

Mortgage bonds have a long history in Denmark dating back to 1797, where the first mortgage bank was established [4]. The overall idea of how the mortgage system should work has only been subject to minor changes proving a high degree of stability. Instead of having a one to one relation between a borrower's mortgage loan and the investor's mortgage bond, the Danish system pools the borrowers and issue mortgage bonds on behalf of the pool as a whole. An investor buying a bond issued from the pool is effectively buying a share of the pool entitling him to receive interest and repayments proportional to his owner share. What distinguishes the Danish mortgage system from that of other countries is the so called balance principal. The balance principal ensures that there is a near-perfect match between the interest and repayments paid and received by borrowers and investors respectively. Figure 1 illustrates the cash flows occurring between the borrower, the mortgage bank and the investor. When a loan is granted to a borrower, the mortgage bank will issue bonds in the market accordingly. Investors will then buy the bonds, and the proceeds from the trades will go to the borrower. The borrower will then pay interest, repayments and fees to the mortgage bank who will then pass through the interest and repayments. Finally, and importantly, the mortgage bank takes on the credit risk against the borrower. Since the borrower will pledge his dwelling against the loan, the credit risk is lowered from the viewpoint of the mortgage bank. The default risk held by the investor is even further reduced since the mortgage bank has to go into default before the investor will be exposed to a credit event. In a potential credit event of the mortgage bank, the investors will have the right to the cover pool. The cover pool will consist of collateral in terms of the claims against the borrowers as well as additional securities posed by the mortgage bank to protect the investor from losses. These securities constitute what is known as overcollateralisation and should be of very high credit quality [7]. Since the modelling will be performed from the investors point of view, and the default risk held by the investor has been brought to a minimum, we will not introduce default risk in our model.

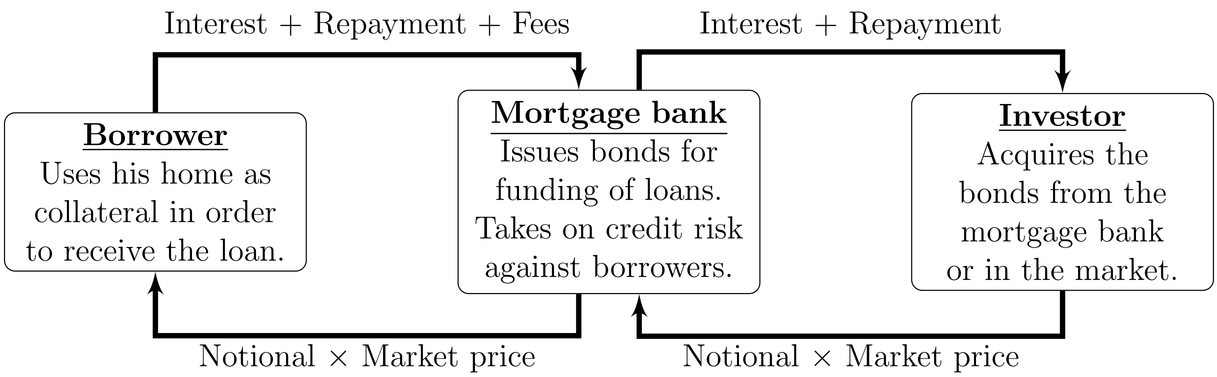

Figure 1: Simplified illustration of the relationships and payment streams between the homeowner, the mortgage bank and the investor in the Danish mortgage system.

The reason why it is interesting to be able to understand the Danish mortgage system in detail is due to its significant size and impact on the Danish economy. In 2016 the Danish mortgage system grew in size to EUR 386 billion making it the largest of its kind in Europe [6]. According to the Danish central bank, Nationalbanken, the foreign ownership of Danish mortgage bonds totalled DKK 679 billion as of July 2017 or equivalently 24% of the total outstanding bonds [19]. The market for Danish mortgage bonds consists mainly of fixed coupon callable bonds, adjustable rate bonds and capped floating rate bonds. The focus of this thesis will be on the fixed coupon callable bonds due it being the most complex of the aforementioned.

In order to understand the different aspects of the callable bond, we will first have to introduce the term structure theory and build a framework for describing the behaviour of the yield curve. This is done in section 2. Next we will have to come up with a model for the prepayment behaviour of borrowers in the mortgage pool. Section 3 goes through one of many methods of describing this behaviour and combines it with the term structure theory in order to price the callable mortgage bond. Section 4 builds an estimation procedure that utilises the Generalised Method of Moments in order to estimate the model. With an estimated pricing module in place, section 5 goes through the risk and return aspect of investing in a callable mortgage bond. The most common key figures as duration, convexity, option adjusted spread and expected returns are calculated and a back test is performed to measure the reliability of the risk figures. Finally, section 6 searches for structural changes in the mortgage market around the financial crisis in order to identify possible shortcomings of the model.

2 The Term Structure

In the classic Black-Scholes-Merton model (1973) [9] the variable of interest was the price of a stock; a tangible and measurable quantity. Pricing and hedging was an easy task, since the model had only one source of risk. When introducing stochastic volatility, as done for example in the models by Heston (1993) [28] or Hagan et. al. (2002) [20], a side effect was the incomplete market arising as a consequence of volatility itself not being a traded asset. By assuming the existence of a market for derivatives, it was possible to hedge the volatility risk of one derivative by use of another. The term structure theory will seem somewhat similar. Assuming a market for fixed income securities depending on a stochastic short rate, we will likewise be able to hedge one fixed income security with another. We will start the section out by defining the short rate.

2.1 The Short Rate

We will assume the existence of a locally risk-free short rate, $r(t)$, which follows a one-factor diffusion model. That is, given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a state space, $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a probability measure, the short rate follows a diffusion of the form:

$$\mathrm{d}r_t = \alpha(t, r_t)\mathrm{d}t + \beta(t, r_t)\mathrm{d}W_t^{\mathbb{P}} \tag{2.1}$$

$$r_0 = \bar{r}_0, \tag{2.2}$$

where $W^{\mathbb{P}} = (W_t^{\mathbb{P}})_{t \geq 0}$ is a standard Brownian motion under the probability measure $\mathbb{P}$. We are explicit about the measure $\mathbb{P}$ since we will be changing it later. We will let $(\mathcal{F}_t)_{t \geq 0}$ be the information-filtration generated by $(W_t^{\mathbb{P}})_{t \geq 0}$. That the short rate is locally risk-free means that during an infinitesimal time period $[t, t + \mathrm{d}t]$, net deposits of size A_t will earn $\mathrm{d}A_t = r_t A_t \mathrm{d}t$ when placed in the risk-free rate. Hence, over a time interval $[t, T]$ we will have that A_t will grow into A_T given by

$$A_T = A_t \mathrm{e}^{\int_t^T r_s \mathrm{d}s}. \tag{2.3}$$

From this general representation of the risk-free rate, we will now turn to the most simple fixed income product; the zero coupon bond.

2.2 Zero Coupon Bonds

A zero coupon bond is a bond that promises the holder one unit of currency at maturity. We will make the assumption that these zero coupon bonds trade in the market. This assumption is not that hard to justify since we will often be able to replicate the zero coupon bond's payoff by static arbitrage arguments. We will further assume that the time-t price, P_t^T, of a zero coupon bond maturing at time T will at most depend on the current time, the maturity and the short rate, i.e. P_t^T is given by some function $P(t, r_t, T)$. If we interpret the bond as a derivative written on the short rate, then by use of standard arbitrage arguments we have the following theorem:

Theorem 2.1. *(The Term Structure PDE) The price function, P_t^T, for a zero coupon bond maturing at time T must satisfy the following partial differential equation:*

$$\frac{\partial P_t^T}{\partial t} + (\alpha(t, r_t) - q_t \beta(t, r_t)) \frac{\partial P_t^T}{\partial r} + \frac{1}{2}\beta^2(t, r_t) \frac{\partial^2 P_t^T}{\partial r^2} = r_t P_t^T, \quad (t, r_t) \in [0; T) \times \mathbb{R} \tag{2.4}$$

with terminal condition $P_T^T = 1$.

Proof. If we are willing to assume that $P_t^T = P(t, r_t, T)$ is twice continuously differentiable in r_t and once continuously differentiable in t then we may apply Ito's lemma to get

$$\begin{aligned} \mathrm{d}P^T(t, r_t) &= \left(\frac{\partial P^T}{\partial t} + \alpha(t, r_t) \frac{\partial P^T}{\partial r} + \frac{1}{2}\beta^2(t, r_t) \frac{\partial^2 P^T}{\partial r^2} \right) \mathrm{d}t + \beta(t, r_t) \frac{\partial P^T}{\partial r} \mathrm{d}W_t^{\mathbb{P}} &(2.5) \\ &= \alpha^T(t, r_t) P_t^T \mathrm{d}t + \beta^T(t, r_t) P_t^T \mathrm{d}W_t^{\mathbb{P}}, &(2.6) \end{aligned}$$

where

$$\alpha^T(t, r_t) = \frac{\frac{\partial P^T}{\partial t} + \alpha(t, r_t)\frac{\partial P^T}{\partial r} + \frac{1}{2}\beta^2(t, r_t)\frac{\partial^2 P^T}{\partial r^2}}{P_t^T} \quad \text{and} \quad \beta^T(t, r_t) = \frac{\beta(t, r_t)\frac{\partial P^T}{\partial r}}{P_t^T}. \tag{2.7}$$

We will now establish a self-financing portfolio V_t of h_t^T zero coupon bonds maturing at time T and h_t^S zero coupon bonds maturing at time S, i.e. V_t satisfies

$$\begin{aligned} V_t &= h_t^T P_t^T + h_t^S P_t^S &(2.8) \\ \mathrm{d}V_t &= h_t^T \mathrm{d}P_t^T + h_t^S \mathrm{d}P_t^S. &(2.9) \end{aligned}$$

The portfolio will therefore have the following dynamics

$$\begin{aligned} \mathrm{d}V_t = &\left(h_t^T \alpha^T(t, r_t) P_t^T + h_t^S \alpha^S(t, r_t) P_t^S \right) \mathrm{d}t \\ &+ \left(h_t^T \beta^T(t, r_t) P_t^T + h_t^S \beta^S(t, r_t) P_t^S \right) \mathrm{d}W_t^{\mathbb{P}}. \end{aligned} \tag{2.10}$$

The idea is now to choose the portfolio in such a way that the stochastic term vanishes. Doing so leaves us with two equations in two unknowns:

$$\begin{aligned} h_t^T \beta^T(t, r_t) P_t^T + h_t^S \beta^S(t, r_t) P_t^S &= 0 &(2.11) \\ h_t^T P_t^T + h_t^S P_t^S &= V_t, &(2.12) \end{aligned}$$

where the second equations stems from the self-financing condition. By inserting the second equation in the first yields

$$h_t^T \beta^T(t, r_t) P_t^T + \frac{V_t - h_t^T P_t^T}{P_t^S} \beta^S(t, r_t) P_t^S = 0. \tag{2.13}$$

Solving for h_t^T gives

$$h_t^T = \frac{V_t \beta^S(t, r_t)}{(\beta^S(r, r_t) - \beta^T(r, r_t)) P_t^T}, \tag{2.14}$$

and by using (2.12) we also have

$$h_t^S = -\frac{V_t \beta^T(t, r_t)}{(\beta^S(r, r_t) - \beta^T(r, r_t)) P_t^S}. \tag{2.15}$$

By selecting exactly this portfolio composition, we have made the stochastic part of (2.10) vanish. Hence, the portfolio has become locally risk free, and for there to be no arbitrage, the portfolio must earn the locally risk free rate. We therefore have

$$\left(h_t^T \alpha^T(t, r_t) P_t^T + h_t^S \alpha^S(t, r_t) P_t^S \right) \mathrm{d}t = r_t V_t \mathrm{d}t. \tag{2.16}$$

Inserting the portfolio holdings and simplifying we get

$$\left(\frac{\beta^S(t, r_t) \alpha^T(t, r_t)}{\beta^S(t, r_t) - \beta^T(t, r_t)} - \frac{\beta^T(t, r_t) \alpha^S(t, r_t)}{\beta^S(t, r_t) - \beta^T(t, r_t)} \right) \mathrm{d}t = r_t \mathrm{d}t. \tag{2.17}$$

Dropping the dt-terms and rearranging we obtain

$$\frac{\alpha^T(t, r_t) - r_t}{\beta^T(t, r_t)} = \frac{\alpha^S(t, r_t) - r_t}{\beta^S(t, r_t)}. \tag{2.18}$$

The special thing about the relation (2.18) is that the left hand side does not depend on S and the right hand side does not depend on T. Hence, the ratios must be maturity independent and we may define the ratio as a function $q_t = q(t, r_t)$ of time and the short rate:

$$q_t = \frac{\alpha^T(t, r_t) - r_t}{\beta^T(t, r_t)} \quad \forall \quad T > 0. \tag{2.19}$$

q_t is often referred to as the market price of risk or the Sharpe ratio, as it is the excess return over the risk free rate per unit of risk. We will, nevertheless, simply regard q_t as a consistency relation that has to hold in the market for zero coupon bonds. By inserting the expressions for $\alpha^T(t, r_t)$ and $\beta^T(t, r_t)$ from (2.7) in equation (2.19) and rearranging, we arrive at the following PDE:

$$\frac{\partial P_t^T}{\partial t} + (\alpha(t,r_t) - q_t\beta(t,r_t))\frac{\partial P_t^T}{\partial r} + \frac{1}{2}\beta^2(t,r_t)\frac{\partial^2 P_t^T}{\partial r^2} = r_t P_t^T, \quad (t,r_t) \in [0;T) \times \mathbb{R}$$

Since the zero coupon bond matures at time T with value 1, we may add the terminal condition $P_T^T = 1$. □

Theorem 2.1 gives us the price of any zero coupon bond in terms of a PDE. However, we cannot simply solve the PDE since we do not know the explicit formulation of q_t. We can deal with this problem in two ways. The first way is to assume some structure for q_t in which case we are implicitly making an assumption about the aggregate risk profile in the market for bonds. The second way is to assume that a market already exist and then imply q_t from the market prices. As we will see later, this corresponds to stating the short rate dynamics under a market consistent probability measure.

The solution to the PDE in theorem 2.1 can be found in different ways, but we will now introduce one particular way. Due to the Feynman-Kac theorem, we can give the price function P_t^T a stochastic representation, as presented in the following theorem.

Theorem 2.2. *(Feynman-Kac) Let P_t^T be a solution to the term structure PDE in* (2.4)*, then P_t^T will have the representation*

$$P_t^T = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s}\right], \tag{2.20}$$

where $\mathbb{Q}$ is an alternative probability measure under which the short rate follows the dynamics

$$\mathrm{d}r_t = (\alpha(t,r_t) - q_t\beta(t,r_t))\mathrm{d}t + \beta(t,r_t)\mathrm{d}W_t^{\mathbb{Q}}, \tag{2.21}$$

and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.

Proof. Let r_t follow the $\mathbb{Q}$ dynamics in (2.21). Now we will apply Ito's product rule to the function $g(P_t^T, A_t) = \frac{P_t^T}{A_t}$ with $A_t = e^{\int_0^t r_s \mathrm{d}s}$ as usual.

$$\begin{aligned}
\mathrm{d}g(P_t^T, A_t) &= P_t^T \mathrm{d}\frac{1}{A_t} + \frac{1}{A_t}\mathrm{d}P_t^T + \mathrm{d}P_t^T \mathrm{d}\frac{1}{A_t} \\
&= -r_t\frac{P_t^T}{A_t}\mathrm{d}t + \frac{1}{A_t}\mathrm{d}P_t^T,
\end{aligned}$$

where $\mathrm{d}P_t^T \mathrm{d}\frac{1}{A_t} = 0$ since $\mathrm{d}\frac{1}{A_t}$ only contains $\mathrm{d}t$ terms. The dynamics of P_t^T are as follows

$$\begin{aligned}
\mathrm{d}P_t^T &= \left(\frac{\partial P_t^T}{\partial t} + (\alpha(t,r_t) - q_t\beta(t,r_t))\frac{\partial P_t^T}{\partial r} + \frac{1}{2}\beta(t,r_t)\frac{\partial^2 P_t^T}{\partial r^2}\right)\mathrm{d}t + \beta(t,r_t)\frac{\partial P_t^T}{\partial r}\mathrm{d}W_t^{\mathbb{Q}} \\
&= r_t P_t^T \mathrm{d}t + \beta(t,r_t)\frac{\partial P_t^T}{\partial r}\mathrm{d}W_t^{\mathbb{Q}},
\end{aligned}$$

which follows by using the term structure PDE. The dynamics of $g(P_t^T, A_t)$ become

$$\mathrm{d}g(P_t^T, A_t) = \frac{\beta(t, r_t)}{A_t} \frac{\partial P_t^T}{\partial r} \mathrm{d}W_t^{\mathbb{Q}}.$$

Integrating from t to T yields

$$g(P_T^T, A_T) = g(P_t^T, A_t) + \int_t^T \frac{\beta(s, r_s)}{A_s} \frac{\partial P_s^T}{\partial r} \mathrm{d}W_s^{\mathbb{Q}}.$$

Taking conditional expectations and assuming the integrand of the stochastic integral to be in L^2 yields the result[1]:

$$P_t^T = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \right].$$

□

Theorem 2.2 provides a very important linkage between the solution to the term structure PDE and the expected value of a random variable. If the PDE is difficult to solve, then it might be more convenient to compute the expected value analytically or by simulations.

2.3 Interest Rate Swaps

A heavily traded interest rate instrument is the interest rate swap. An interest rate swap is an agreement to exchange a stream of fixed rate payments against a stream of floating rate payments. The counterparty receiving the fixed rate payment is said to have entered a receiver swap while the counterparty paying the fixed rate is said to have entered a payer swap. We will denote the payment dates of the swap by $T_1, T_2, ..., T_N$. For the receiver swap, at each time T_{i-1}, the floating rate $R(T_{i-1}, T_i)$ is determined and paid out at time T_i against receiving the fixed rate K. If for example the floating rate is the 6M CIBOR, then the swap will have semi-annual payments. If the floating rate is the overnight rate then the swap is referred to as an OIS (Overnight Index Swap). In Denmark the OIS rate is also called the CITA (Copenhagen Interbank Tomorrow/Next Average) rate. Swaps with CIBOR as reference rate trade with maturities up to 30 years while CITA swaps typically trade with maturities up to one year.

If we assume the reference rate to be default-free, then we can value the floating rate leg by the following model independent argument: We consider first a floating rate bond that pays $R(T_{i-1}, T_i)\delta_i$ at each T_i, with $\delta_i = T_i - T_{i-1}$, and repays the notional, H, at time T_N. Immediately after the payment at time T_{N-1}, the bond will effectively be a zero coupon bond with payoff $(1 + R(T_{N-1}, T_N)\delta_i)H$ at time T_N. Since the reference rate is default-free, and by no-arbitrage, it must hold that investing an amount H at time T_{N-1}

[1] L^2 denotes the space of random variables with finite second moment.

in $P^{T_N}_{T_{N-1}}$ must yield the same return as the bond. Hence, the bond must have a value of H at time T_{N-1}. As the bond has a value of H at time T_{N-1}, a similar argument will hold for T_{N-2}. Performing this argumentation back to time T_0, the date of the first fixing, the floating rate note will have a value of H. The value of the bond at any $t < T_0$ will therefore be given by $P_t^{T_0}H$. Since the floating rate bond pays back the notional at maturity and the floating rate leg in the swap does not, the value of the floating rate leg in the swap must be $H(P_t^{T_0} - P_t^{T_N})$. The fixed leg pays the deterministic amount $K\delta_i$ at each payment date, so the value of these must be given by

$$\sum_{i=1}^{N} P_t^{T_i} H K \delta_i.$$

It is customary to initiate a swap at a value of zero. This is ensured by finding the value of K that makes the value of the floating rate leg equal to the value of the fixed rate leg. We will denote this K as the par swap rate and it will be given by

$$S(t, T_0, T_N) = \frac{P_t^{T_0} - P_t^{T_N}}{\sum_{i=1}^{N} P_t^{T_i} \delta_i}. \tag{2.22}$$

If $T_0 = t$, then we will denote the mapping $T \mapsto S(t, t, T)$ the swap curve. Two other important curves are the zero coupon curve and the forward rate curve that we will define below.

2.4 Zero Rates, Forward Rates and Curve Fitting

If swap rates are available in the market for every maturity, then we can in principle solve for all the zero coupon bonds. Say that the prices of these zero coupon bonds are available, then we will define the zero rate y_t^T as the continuously compounded yield satisfying

$$\bar{P}_t^T = \mathrm{e}^{-(T-t)y_t^T}, \tag{2.23}$$

or equivalently

$$y_t^T = -\frac{1}{T-t} \ln \bar{P}_t^T, \tag{2.24}$$

where $\bar{P}_t^T$ denotes the observed price in the market. Note that y_t^T contains the exact same amount of information as the zero coupon bonds themselves. However, yields can be more intuitive than prices since they take into account the time to maturity.

Another important interest rate is the forward rate. If for example funding is needed between two future points in time T and $T + \Delta$ where $\Delta > 0$, then we can lock in a future interest rate by buying one zero coupon bond maturing at time T against selling $\frac{\bar{P}_t^T}{\bar{P}_t^{T+\Delta}}$

zero coupon bonds maturing at time $T+\Delta$. The return earned per unit of time over the period $[T;T+\Delta]$ will be given by

$$F(t,T,T+\Delta)=\frac{1}{\Delta}\left(\frac{\bar{P}_t^T}{\bar{P}_t^{T+\Delta}}-1\right), \tag{2.25}$$

which we will denote the time t forward rate for the future period $[T;T+\Delta]$. If we let $\Delta\to 0^+$ in (2.25) then we get the instantaneous forward rate $f(t,T)$:

$$\begin{aligned} f(t,T) &= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\left(\frac{\bar{P}_t^T}{\bar{P}_t^{T+\Delta}}-1\right) \\ &= \lim_{\Delta\to 0^+}\frac{\bar{P}_t^T-\bar{P}_t^{T+\Delta}}{\Delta}\lim_{\Delta\to 0^+}\frac{1}{\bar{P}_t^{T+\Delta}} \\ &= -\frac{\partial\bar{P}_t^T}{\partial T}\frac{1}{\bar{P}_t^T} \\ &= -\frac{\partial\ln\bar{P}_t^T}{\partial T}. \end{aligned} \tag{2.26}$$

In all of the above we have assumed that zero coupon bond prices are available for all maturities. In reality though, only a very few zero coupon bonds are available. For this reason we will rather be using the liquid swap market to imply zero coupon bonds and thereof the zero rates and forward rates. It is customary to impose some kind of interpolation scheme for the yield curve in order to connect the market observed yields. Hagan & West (2006) [21] discuss a wide range of these interpolation schemes and in particular the so called Cubic Spline. Having observed a set of data point (τ_i,y_i) for $i\in\{1,2,...,n\}$ from some mapping $\tau\mapsto y(\tau)$ the Cubic Spline interpolates y by use of the following polynomial structure:

$$y(\tau)=a_i+b_i(\tau-\tau_i)+c_i(\tau-\tau_i)^2+d_i(\tau-\tau_i)^3 \quad \tau_i\le\tau\le\tau_{i+1}. \tag{2.27}$$

To ensure the Cubic Spline to pass through the observed points (τ_i,y_i) we need $a_i=y_i$. The Cubic Spline also ensures continuity of the function itself and its derivative by imposing the relations $y_{i+1}=y_i+b_i(\tau_{i+1}-\tau_i)+c_i(\tau_{i+1}-\tau_i)^2+d_i(\tau_{i+1}-\tau_i)^3$ and $b_{i+1}=b_i+c_i(\tau_{i+1}-\tau_i)+d_i(\tau_{i+1}\tau_i)^2$. For each i, we now have three equations and four unknowns. Different possibilities are suggested to complete this system of equations and one such is the Hermite Spline which is also discussed by Hagan & West. The Hermite Spline defines b_i as being the slope at τ_i for the quadratic passing through the points (τ_{i-1},y_{i-1}), (τ_i,y_i) and (τ_{i+1},y_{i+1}). The Hermite Spline therefore completes the system of equations and we can now use the scheme for interpolating zero rates. Since we do not observe the zero rates but instead the swap rates, we will have to find a way of computing zero rates from swap rates. If we rewrite the swap rate (2.22) for $T_0=t$ in terms of the longest zero coupon bond, then we get

$$P_t^{T_N} = \frac{1 - S(t,t,T_N)\sum_{i=1}^{N-1} P_t^{T_i}\delta_i}{1 + S(t,t,T_N)\delta_N}.$$

By the definition of the T_N zero rate in equation (2.24), we find the corresponding zero rate as

$$y_t^{T_N} = -\frac{1}{\tau_N}\ln\left(\frac{1 - S(t,t,T_N)\sum_{i=1}^{N-1} P_t^{T_i}\delta_i}{1 + S(t,t,T_N)\delta_N}\right), \tag{2.28}$$

where $\tau_N = T_N - t$. In order to find $y_t^{T_N}$ from the market swap quotes, Hagan & West suggested an algorithm where we choose an initial guess for $y_t^{T_N}$ for each of the maturities for which we have swap quotes available. Then we will interpolate the zero yields by Hermite Spline for all maturities entering equation (2.28) through the zero coupon bonds, $P_t^{T_i} = \mathrm{e}^{-\tau_i y_t(\tau_i)}$. From this set of zero rates, we will calculate a new set of zero rates through equation (2.28). Repeating this iterative procedure until the sum of absolute deviation of theoretical swap rates from market swap rates are less than 1 basis point results in a fast convergence.

When the Hermite Spline is applied to the zero rates, then we assure the zero curve to be differentiable and the forward curve to be continuous. As nothing assures the forward curve to be differentiable, we may end up with a non-smooth curve. If we combine equations (2.23) and (2.26), then we can find the forward curve as

$$f(t,T) = \frac{\partial}{\partial\tau}\left(\tau y(\tau)\right). \tag{2.29}$$

Hagan & West suggested to apply the Hermite Spline directly to the quantity $\tau y(\tau)$ which results in a more smooth forward curve. In figure 2, a set of market quotes as well as the bootstrapped zero and forward curves are presented for the method suggested by Hagan & West. Note that the zero curve should not pass through the market quotes; only the swap rates implied by the zero curve should. The forward curve will be above the zero curve whenever the slope of the zero curve is positive as can be seen from the figure. Whenever we refer to the zero curve or forward curve in the rest of this thesis we will mean the interpolated approximations described in the above.

2.5 Caps & Floors

Two important interest rate derivatives that will be of importance later are the so called caps and floors contracts. Since these products are highly sensitive towards changes in volatility, they are often used for calibration purposes. The cap is used to protect against increasing interest rates while the floor is used to protect against falling interest rates.

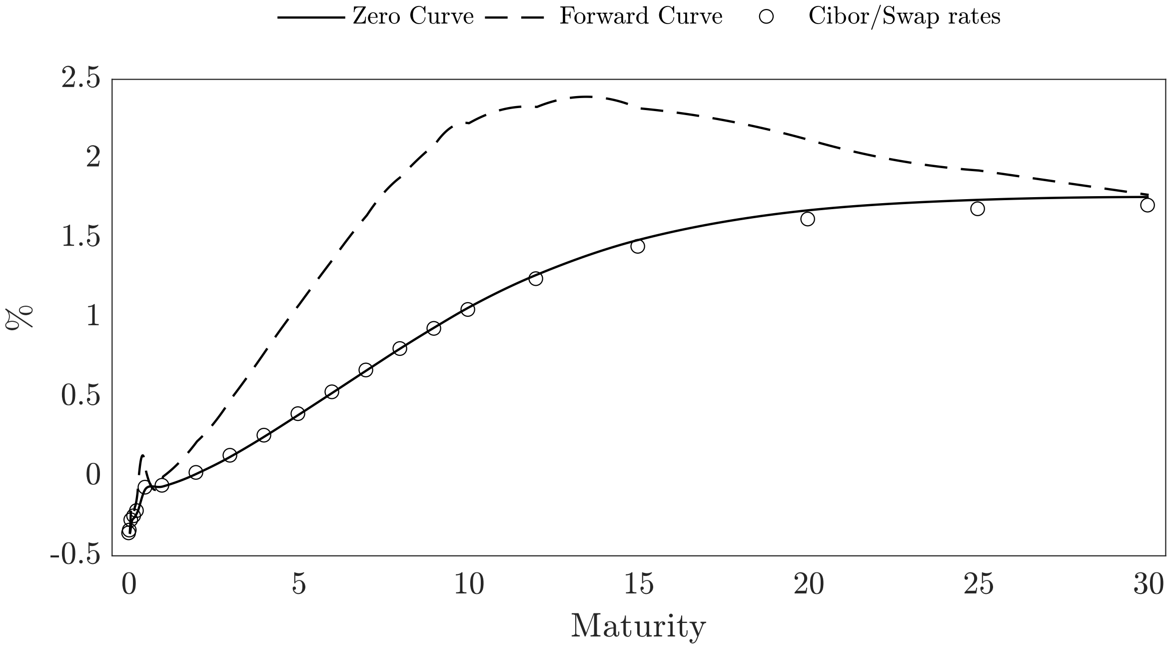

Figure 2: Danish zero and forward curves as of *2017-06-01*. The curves are bootstrapped from CIBOR and swap market quotes indicated by the scatter points. CIBOR rates are used from the one week tenor and out to the six month tenor. Swap rates are used from the one year tenor out to the 30 year tenor.

A typical cap contract could be specified to pay $\delta_i(R(T_{i-1},T_i)-K)^+$ at time T_i for $i=1,2,...,N$, where $R(T_{i-1},T_i)$ is some reference rate and K is the fixed cap rate. As can be seen from this payment structure, the cap is effectively a payer swap in which payments only take place if $R(T_{i-1},T_i)>K$. We will say that the cap contract is At-The-Money (ATM) if the cap rate equals the par swap rate, i.e. $K=S(t,T_0,T_N)$. If we consider a single payment $\delta_i(R(T_{i-1},T_i)-K)^+$ from a cap, also known as a caplet, then this amount will be known at time T_{i-1} and hence its value must be $\Pi(r_{T_{i-1}})=P^{T_i}_{T_{i-1}}\delta_i(R(T_{i-1},T_i)-K)^+$ at time T_{i-1}. We write the value of the payment as $\Pi(r_{T_{i-1}})$ to denote that it only depends on the realised value of the short rate at time T_{i-1}. Noting that the derivations of the term structure PDE of theorem (2.1) and the Feynman-Kac theorem (2.2) remain intact when changing the terminal condition to a function $\Pi(r_T)$, we can write the value of the caplet as follows

$$\mathbf{Caplet}_t^{T_{i-1}}=\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{T_{i-1}}r_s\mathrm{d}s}P^{T_i}_{T_{i-1}}\delta_i(R(T_{i-1},T_i)-K)^+\right]. \tag{2.30}$$

Writing $R(T_{i-1},T_i)$ in terms of $P^{T_i}_{T_{i-1}}$ we can rewrite (2.30) as follows:

$$\mathbf{Caplet}_t^{T_{i-1}}=\frac{1}{K^*}\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{T_{i-1}}r_s\mathrm{d}s}K^*\mathbb{1}\{K^*>P^{T_i}_{T_{i-1}}\}-\mathrm{e}^{-\int_t^{T_{i-1}}r_s\mathrm{d}s}P^{T_i}_{T_{i-1}}\mathbb{1}\{K^*>P^{T_i}_{T_{i-1}}\}\right], \tag{2.31}$$

where $K^*=\frac{1}{1+\delta K}$. The two terms of equation (2.31) can be viewed as two separate securities with payoff functions given by $K^*\mathbb{1}\{K^*>P^{T_i}_{T_{i-1}}\}$ and $\mathbb{1}\{K^*>P^{T_i}_{T_{i-1}}\}P^{T_i}_{T_{i-1}}$

respectively. In order to value each of these two it will make sense to make a change of measure. Consider first the price function, F_t^T, of an interest rate dependent derivative expiring at time T and let P_t^T denote the time t price of a zero coupon bond maturing at time T. By Ito's lemma we can find the process followed by the quantity F_t^T/P_t^T to be

$$\mathrm{d}\frac{F_t^T}{P_t^T} = \left[(\beta^T(t,r_t))^2 - \beta^T(t,r_t)\beta^F(t,r_t)\right]\frac{F_t^T}{P_t^T}\mathrm{d}t + (\beta^F(t,r_t) - \beta^T(t,r_t))\frac{F_t^T}{P_t^T}\mathrm{d}W_t^{\mathbb{Q}}, \quad (2.32)$$

where we have used that both F_t^T and P_t^T have drift r_t under $\mathbb{Q}$ and $\beta^F(t,r_t)$ indicates the volatility of F_t^T. Assuming $\beta^T(t,r_t)$ to be an adapted process then the Girsanov theorem tells us that we may define a new probability measure $\mathbb{Q}^T$ through the following Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \mathrm{e}^{\int_0^T \beta^T(t,r_t)\mathrm{d}W_t^{\mathbb{Q}} - \frac{1}{2}(\beta^T(t,r_t))^2\mathrm{d}t}. \quad (2.33)$$

Under this new measure we will have that

$$W_t^{\mathbb{Q}^T} = W_t^{\mathbb{Q}} - \int_0^t \beta^T(s,r_s)\mathrm{d}s \quad (2.34)$$

defines a Brownian motion under $\mathbb{Q}^T$. The Girsanov theorem may be found in e.g. Björk (2009) [31]. Inserting (2.34) in (2.32) we obtain

$$\mathrm{d}\frac{F_t^T}{P_t^T} = (\beta^F(t,r_t) - \beta^T(t,r_t))\frac{F_t^T}{P_t^T}\mathrm{d}W_t^{\mathbb{Q}^T}. \quad (2.35)$$

From equation (2.35) we now see that the quantity F_t^T/P_t^T has no drift under $\mathbb{Q}^T$. Hence, F_t^T/P_t^T must be a martingale under the $\mathbb{Q}^T$ measure and we may establish the following relation

$$F_t^T = P_t^T\mathbb{E}_t^{\mathbb{Q}^T}\left[\frac{F_T^T}{P_T^T}\right] = P_t^T\mathbb{E}_t^{\mathbb{Q}^T}\left[F_T^T\right], \quad (2.36)$$

where P_t^T is often called the numeraire asset. Using this technique for the first term in the expectation of (2.31) with $P_t^{T_{i-1}}$ as numeraire, we find

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{T_{i-1}} r_s\mathrm{d}s}K^*\mathbb{1}\{K^* > P_{T_{i-1}}^{T_i}\}\right] &= P_t^{T_{i-1}}K^*\mathbb{E}_t^{\mathbb{Q}^{T_{i-1}}}\left[\mathbb{1}\{K^* > P_{T_{i-1}}^{T_i}\}\right] \\ &= P_t^{T_{i-1}}K^*\mathbb{Q}_t^{T_{i-1}}\left(K^* > P_{T_{i-1}}^{T_i}\right). \quad (2.37)\end{aligned}$$

For the second term it will be convenient to use $P_t^{T_i}$ as a numeraire asset. Doing so and the second term of (2.31) may be written as follows

$$\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{T_{i-1}} r_s \mathrm{d}s}\mathbb{1}\{K^* > P_{T_{i-1}}^{T_i}\}P_{T_{i-1}}^{T_i}\right] = P_t^{T_i}\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{\mathbb{1}\{K^* > P_{T_{i-1}}^{T_i}\}P_{T_{i-1}}^{T_i}}{P_{T_{i-1}}^{T_i}}\right]$$
$$= P_t^{T_i}\mathbb{Q}_t^{T_i}\left(K^* > P_{T_{i-1}}^{T_i}\right). \tag{2.38}$$

Bringing the two terms back together, we find the price of the caplet to be

$$\mathbf{Caplet}_t^{T_{i-1}} = \frac{1}{K^*}\left[P_t^{T_{i-1}}K^*\mathbb{Q}_t^{T_{i-1}}\left(K^* > P_{T_{i-1}}^{T_i}\right) - P_t^{T_i}\mathbb{Q}_t^{T_i}\left(K^* > P_{T_{i-1}}^{T_i}\right)\right]. \tag{2.39}$$

Equation (2.39) has a very nice representation since we can determine the price of the caplet by calculating two probabilities under two seperate measures. The Girsanov theorem provides a linkage between these measures through equation (2.34), and by using this relation, we may find the relevant short rate dynamics to be used when calculating the probabilities as

$$\mathrm{d}r = (\alpha(t,r_t) - q_t\beta(t,r_t) + \beta(t,r_t)\beta^T(t,r_t))\mathrm{d}t + \beta(t,r_t)\mathrm{d}W_t^{\mathbb{Q}^T}, \quad T \in \{T_{i-1}, T_i\}. \tag{2.40}$$

When we have calculated the probabilities to be used, then all there is left to do is to sum up all the caplets to get the price of the cap. The price of the cap with maturity T_N becomes

$$\mathbf{Cap}_t^{T_N} = \frac{1}{K^*}\sum_{i=1}^{N} P_t^{T_{i-1}}K^*\mathbb{Q}_t^{T_{i-1}}\left(K^* > P_{T_{i-1}}^{T_i}\right) - P_t^{T}\mathbb{Q}_t^{T_i}\left(K^* > P_{T_{i-1}}^{T_i}\right). \tag{2.41}$$

The probabilities in (2.41) will be model dependent, so we must return to the computation of these when we have specified a short rate model. The corresponding floor contract simply pays $\delta(K - R(T_{i-1},T_i))^+$ instead of $\delta(R(T_{i-1},T_i) - K)^+$. Going long one cap and short one floor with same strike must therefore pay $R(T_{i-1},T_i) - K$, which is exactly the payment of a payer swap. Hence, we can price the interest rate floor by parity.

The interest rate curves and derivatives we have now defined will constitute our market, which we will assume to be available for the rest of the thesis. With our market in place, we can now turn to the callable mortgage bond.

3 Callable Mortgage Bonds

In this section we will look into the properties of the callable mortgage bond and build a model suited for the Danish market. Our baseline model will be the one suggested by Stanton (1995) [24], who applied the model to the US market. Since 1995, interest rates have found their way into the negative territory, meaning that we will have to make some necessary adjustments to the model later on. For the fixed rate callable mortgage bond, the borrower will commit himself to deliver some agreed stream of payments. This payment stream could for example be defined through an annuity type loan, possibly including periods of deferred amortisation. The borrower will also hold the option to terminate the payments at any given time against paying the remaining principal. Hence, effectively the borrower is short one non-callable bond and long one call option on the bond with strike equal to the remaining principal. In contrast to usual option theory, borrowers are assumed to be faced with some barriers preventing them from prepaying as it becomes optimal. These barriers will be described in the following.

3.1 Callability

Since the borrower is effectively short one non-callable bond and long one call option, we may write the value of the mortgage liabilities to the borrower as follows

$$M_t^\ell = B_t^T + V_t^\ell, \tag{3.1}$$

where B_t^T is the value of the non-callable bond and V_t^ℓ the value of the call option as seen from the borrower. When calling the option the borrower will be faced with a fee of size X per unit of remaining principal, F_t, where X may vary across borrowers. Think of X as the monetary costs associated with prepaying but also the non-monetary or implicit costs. If the borrower has to take time off from work in order to go to the bank, then this could be an example of an implicit cost held by the borrower. The strike value of the option therefore becomes $(1 + X)F_t$. The value of the mortgage bond as seen from the investors point of view will be given by

$$M_t^a = B_t^T + V_t^a. \tag{3.2}$$

M_t^ℓ and M_t^a differ as the holder of the mortgage bond does not receive the costs, X, associated with prepayment. If the mortgage liabilities are greater than the remaining principal plus prepayment costs, i.e. $M_t^\ell > (1 + X)F_t$, then it will be optimal to prepay the mortgage. Given that it is optimal to prepay the mortgage loan, we will assume that there is a given probability that the borrower will perform a prepayment over a given period of time. That borrowers will only prepay with a certain probability when it becomes optimal, may simply be ascribed to the fact that they cannot be expected to monitor the financial market continuously. Hence, we will assume that borrowers check for optimal prepayment at discrete points in time.

3.2 Prepayment

We will assume that the time at which a borrower checks for optimal prepayment is a stochastic event. Let $\tau \in \mathbb{R}_+$ denote the next time the borrower checks for optimal prepayment, and let $F(t) = \mathbb{P}(\tau < t)$ be the associated density. Then the so called hazard rate λ_t, defined by

$$\lambda_t = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}(t \leq \tau < t + h | \tau \geq t), \tag{3.3}$$

will fully describe the probability of checking for optimal prepayment. To see this we rewrite (3.3) as follows

$$\lambda_t = \lim_{h \to 0^+} \frac{1}{h} \frac{\mathbb{P}(\tau < t + h) - \mathbb{P}(\tau < t)}{\mathbb{P}(\tau > t)} = \lim_{h \to 0^+} \frac{F(t+h) - F(t)}{(1 - F(t))h} = \frac{\frac{\mathrm{d}F(t)}{\mathrm{d}t}}{1 - F(t)}, \tag{3.4}$$

which can be solved for F as $F(t) = 1 - \mathrm{e}^{-\int_0^t \lambda_s \mathrm{d}s}$. We see that an increasing λ_t increases the probability of checking for optimal prepayment. It could be justified that λ_t should be a function of one or more market variables, but Stanton simply assumes that the intensity parameter can be in two possible states. Specifically, λ_t is defined as follows:

$$\lambda_t = \begin{cases} \lambda_1 & \text{if } M_t^\ell < (1 + X_i) F_t \\ \lambda_1 + \lambda_2 & \text{if } M_t^\ell \geq (1 + X_i) F_t \end{cases}, \tag{3.5}$$

where $\lambda_1, \lambda_2 \geq 0$ are constants. That is, if it is not optimal to prepay, i.e. $M_t^\ell < (1+X_i)F_t$, then a prepayment will happen according to some baseline prepayment rate λ_1. If it is optimal to prepay, i.e. $M_t^\ell \geq (1 + X_i)F_t$, then λ_t will be increased to $\lambda_1 + \lambda_2$. The reason for this split is due to the way prepayments work in the US. In the US, borrowers cannot buy back their mortgage bond in the market if the price goes below par. Nevertheless, there might be exogenous reasons making it necessary for borrowers to prepay their loan before maturity. Since these suboptimal prepayments will expectedly happen less frequently compared to the optimal prepayments, it makes sense to assume different states of the intensity parameter λ_t. It is important to note, that the positive probability of investors receiving back their money at par, when the bond trades below, is a feature of the US market. In Denmark, borrowers may buy back the bonds linked to their loans, which provide them with an extra optionality to be discussed later.

As previously noted, X will be allowed to vary across borrowers. In this way we get a more nuanced view on the prepayment costs, or equivalently the required gain associated with prepayment. Stanton assumes the distribution of prepayment costs to follow a Beta distribution. The Beta distribution in defined over the interval $[0, 1]$, and allows for many different shapes of the prepayment cost distribution. This is a convenient feature since we have no good reason to impose any particular a priori shape on the distribution. Some of the many shapes that can be generate by the Beta distribution have been illustrated in figure 3 below by adjusting the distribution's only two parameters, α and β.

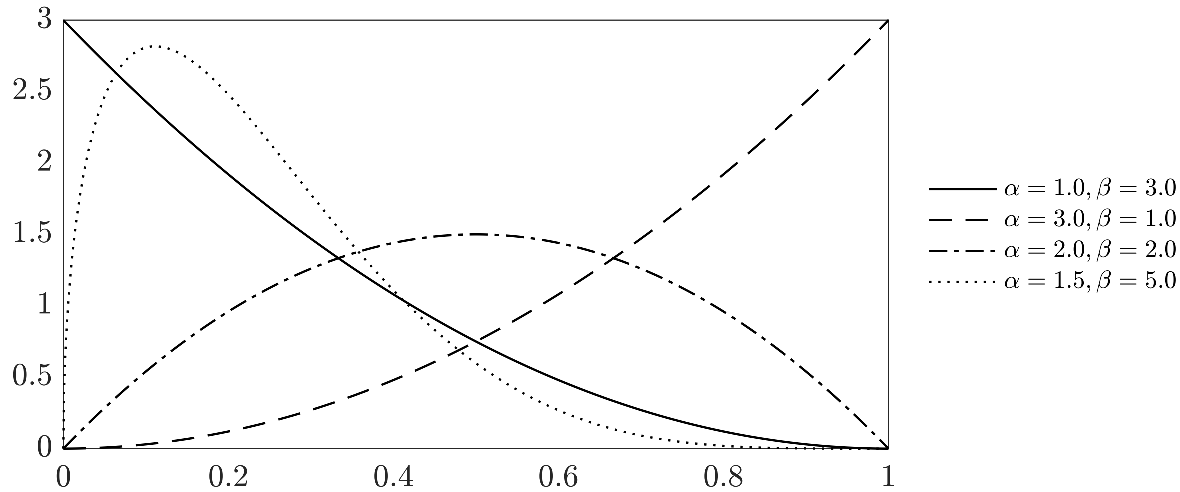

Figure 3: The probability density function of the Beta distribution for different parameter choices.

3.3 The Mortgage PDE

We will now derive a PDE for the mortgage bond and the mortgage liabilities. As we have introduced a new source of risk through prepayments, we cannot simply let the value of a callable mortgage bond depend only on time and the short rate, as was the case for the zero coupon bond. In fact, we will have to let the price depend on both time, the short rate and some new prepayment variable, y_t. We will define the prepayment variable y_t to be a Poisson process with intensity parameter λ_t. We will let $y_t = 0$ indicate that no prepayment has occurred and $y_t \geq 1$ indicate that prepayment has occurred. The mortgager's obligations terminate at $\tau = \inf\{t \in \mathbb{R} \mid y_t \geq 1\}$. At the time of prepayment, the mortgage liabilities will jump from $M^\ell_{\tau-}$ to $M^\ell_{\tau+} = (1+X_i)F_\tau$ while the mortgage bond will jump from $M^a_{\tau-}$ to $M^a_{\tau+} = F_\tau$. In order to properly handle functions of discontinuous processes, we will have to shortly introduce a variation of Ito's lemma for jump processes. We will say that Y_t is a jump process if it takes the form

$$Y_t = Y_0 + \int_0^t \mu(s,Y_s)\mathrm{d}s + \int_0^t \sigma(s,Y_s)\mathrm{d}W_s + J_t, \tag{3.6}$$

where $\mu(t,Y_t)$ and $\sigma(t,Y_t)$ are adapted processes, W_t is a standard Brownian motion and J_t is an adapted pure jump process. That is, at the jump times $\{\tau_1,\tau_2,...\}$, J_t will jump an amount $\mathrm{d}J_t = J_t - J_{t-}$, where both jump times and jumps themselves may be stochastic. For such a jump process there exists a natural extension of Ito's formula and we refer to theorems 11.5.1 and 11.5.4 in Shreve (2004) [30] for one- and two-dimensional versions of the formula as well as proofs. In the present case, we have two dimensions as both the short rate, r_t, and the prepayment variable, y_t, follow stochastic processes. Hence, we are in a special case of a two-dimensional jump process which leads to the following corollary to theorem 11.5.4 in Shreve (2004).

Corollary 3.1. *The price function of a security M_t depending on time, the short rate and the prepayment variable, i.e. $M_t = M(t, r_t, y_t)$, will satisfy*

$$\begin{aligned} M_t = M_0 &+ \int_0^t \frac{\partial M}{\partial t}(s, r_s,y_s)\mathrm{d}s + \int_0^t \frac{\partial M}{\partial r}(s, r_s,y_s)\mathrm{d}r_s + \int_0^t \frac{1}{2}\frac{\partial^2 M}{\partial r^2}(s, r_s,y_s)\,(\mathrm{d}r_s)^2 \\ &+ [M(\tau, r_\tau,y_\tau) - M(\tau-, r_{\tau-},y_{\tau-})]\mathbb{1}(\tau \leq t). \end{aligned} \tag{3.7}$$

We see that the only difference in (3.7) from the usual Ito's lemma is the square bracket, ensuring that when y_t jumps then so does M_t. If we write M_t on differential form and insert for r_t we get

$$\mathrm{d}M_t = \alpha_t^M M_t\mathrm{d}t + \beta_t^M M_t\mathrm{d}W_t^{\mathbb{P}} + (M_t - M_{t-})\mathrm{d}y_t, \tag{3.8}$$

where

$$\alpha_t^M = \frac{\frac{\partial M}{\partial t} + \alpha(t, r_t)\frac{\partial M}{\partial r} + \frac{1}{2}\beta^2(t, r_t)\frac{\partial^2 M}{\partial r^2}}{M_t} \quad \text{and} \quad \beta_t^M = \frac{\beta(t, r_t)\frac{\partial M}{\partial r}}{M_t}. \tag{3.9}$$

If the security is the callable mortgage bond then the value of the bond immediately after a jump will be the remaining principal, i.e. $M_\tau^a = F_\tau$. Likewise, for the mortgage liabilities we will have $M_\tau^\ell = (1 + X_i)F_\tau$. Since we can now distinguish M_t from M_{t-} we may replace $(M_t - M_{t-})\mathrm{d}y_t$ with $(F_t - M_t^a)\mathrm{d}y_t$ for the bond and $((1 + X_i)F_t - M_t^\ell)\mathrm{d}y_t$ for the liability. Since the callable mortgage bond pays out dividends at a rate C_t, the total gain from holding the security over an infinitesimal time-period must be $\mathrm{d}M_t^a + C_t\mathrm{d}t$. We would now like to hedge the interest rate risk of the callable bond by constructing a portfolio, V_t, consisting of one callable mortgage bond and $h_t^T = -\frac{\partial P_t^T}{\partial r}/\frac{\partial M_t^a}{\partial r}$ zero coupon bonds maturing at time T. Since the portfolio will pay dividends, we will assume that these are placed in the short rate. The amount placed in the short rate will be denoted h_t^A. The self-financing condition becomes

$$V_t = M_t^a + h_t^T P_t^T + h_t^A, \tag{3.10}$$

$$\mathrm{d}V_t = \mathrm{d}M_t^a + C_t\mathrm{d}t + h_t^T\mathrm{d}P_t^T + h_t^A r_t\mathrm{d}t. \tag{3.11}$$

Inserting the dynamics of the callable mortgage bond from (3.8) and the zero coupon bond from (2.6) we find the dynamics to be

$$\mathrm{d}V_t = \left(\alpha_t^M M_t^a + C_t + h_t^T \alpha_t^T P_t^T + h_t^A r_t\right) \mathrm{d}t + (F_t - M_t^a)\mathrm{d}y_t. \tag{3.12}$$

Equation (3.12) looks almost risk free in the sense that we have eliminated all terms including Brownian increments. However, we are still left with the prepayment risk from y_t, which we are not able to hedge. The problem is that we have introduced an idiosyncratic source of risk, as prepayment risk will vary across mortgage pools. Equation (3.12) is a dead end as seen from the perspective of arbitrage free pricing, as it defines a jump process with no other securities in the market depending on this exact source of risk. The same problem was encountered by Merton (1976) [25] when pricing options where the underlying stock price is discontinuous and Ingersoll (1977) [14] when pricing corporate convertible bonds. A pragmatic way of proceeding, used by both Merton and Ingersoll, is to replace $\mathrm{d}y_t$ by its expected value, namely $\mathbb{E}(\mathrm{d}y_t) = \lambda_t \mathrm{d}t$. Stanton, on the other hand, implicitly assumes that prepayments will only occur at payment dates. This means that we can safely put $\mathrm{d}y_t = 0$ between payment dates. Strictly speaking, y_t is also no longer a Poisson process in this case. With this in mind, we will set $\mathrm{d}y_t = 0$, meaning that between payment dates equation (3.12) becomes

$$\mathrm{d}V_t = \left(\alpha_t^M M_t + C_t \mathrm{d}t + h_t^T \alpha_t^T P_t^T + h_t^A r_t\right) \mathrm{d}t. \tag{3.13}$$

Since equation (3.13) has been left without any stochastic sources it should earn the risk free rate, i.e. $\mathrm{d}V_t = r_t V_t \mathrm{d}t$. Using this relation we get

$$\left(\alpha_t^M M_t + C_t + h_t^T \alpha_t^T P_t^T + h_t^A r_t\right) \mathrm{d}t = r_t(M_t^a + h_t^T P_t^T + h_t^A)\mathrm{d}t. \tag{3.14}$$

From our derivations of the term structure PDE in section 2.2 we know, that under the alternative measure $\mathbb{Q}$, the zero coupon bond has drift r_t. Hence, by inserting α_t^M, switching to the $\mathbb{Q}$ measure and using that $\alpha_t^T = r_t$ under $\mathbb{Q}$, equation (3.14) reduces to

$$\frac{\partial M^a}{\partial t} + (\alpha(t, r_t) - q_t \beta(t, r_t))\frac{\partial M^a}{\partial r} + \frac{1}{2}\beta^2(t, r_t)\frac{\partial^2 M^a}{\partial r^2} + C_t = r_t M_t^a. \tag{3.15}$$

What is left now is to specify the dynamics of the short rate. Stanton chooses to use the CIR model by Cox, Ingersoll & Ross (1985) [12], where the short rate is assumed to have the $\mathbb{P}$-dynamics,

$$\mathrm{d}r_t = \kappa(\mu - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t^{\mathbb{P}}. \tag{3.16}$$

This specification fits into our general short rate dynamics from equation (2.1) with $\alpha(t, r_t) = \kappa(\mu - r_t)$ and $\beta(t, r_t) = \sigma\sqrt{r_t}$. Recall that the $\mathbb{Q}$-dynamics take the form

$$\mathrm{d}r_t = (\alpha(t, r_t) - q_t\beta(t, r_t))\mathrm{d}t + \beta(t, r_t)\mathrm{d}W_t^{\mathbb{Q}}. \tag{3.17}$$

Inserting for $\alpha(t, r_t)$ and $\beta(t, r_t)$, the $\mathbb{Q}$-dynamics of the CIR model becomes

$$\mathrm{d}r_t = (\kappa(\mu - r_t) - q_t\sigma\sqrt{r_t})\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t^{\mathbb{Q}}. \tag{3.18}$$

As discussed earlier, it is now possible to either specify a structure for q_t or imply q_t from market prices. Cox, Ingersoll & Ross chooses to specify q_t as being linear in $\sqrt{r_t}$.[2] In this case we get

$$\mathrm{d}r_t = (\kappa\mu - (\kappa + q)r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t^{\mathbb{Q}} \tag{3.19}$$

for some constant q. With this specification, the mortgage PDE (3.15) becomes

$$\frac{\partial M^a}{\partial t} + (\kappa\mu - (\kappa + q)r_t)\frac{\partial M^a}{\partial r} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 M^a}{\partial r^2} + C_t = r_t M_t^a. \tag{3.20}$$

Similarly, the PDE obeyed by the mortgage liabilities will be given by

$$\frac{\partial M^\ell}{\partial t} + (\kappa\mu - (\kappa + q)r_t)\frac{\partial M^\ell}{\partial r} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 M^\ell}{\partial r^2} + C_t = r_t M_t^\ell. \tag{3.21}$$

Before solving equations (3.20) and (3.21), we will have to define the payments from C_t. Typically, the size as well as the time of the mortgage bond payments are scheduled, and this payment schedule is distributed by the mortgage institution to the investor. However, since the mortgage bonds from a pool are not issued all at once, the payment schedule per unit of notional will change over time as issuance and prepayments occur. We will disregard this fact and simply consider a stylised world with annuity type payment streams. Given a yearly coupon rate R, an initial principal F_0 and a time to maturity T, the annuity will pay a constant payment $\bar{Y}$ at each payment date t_i where $i \in \{1, 2, ..., N\}$ and $t_N = T$. Using that $F_{t_N} = 0$ we find that

$$\bar{Y} = \frac{\widetilde{R}}{1 - \left(1 + \widetilde{R}\right)^{-N}} F_0, \tag{3.22}$$

where $\widetilde{R} = \frac{R}{n}$ is the periodic interest rate when there are n payments per year. The

[2] More specifically, a general equilibrium model is specified where individuals have logarithmic utility functions, which implies the linear relationship in $\sqrt{r_t}$.

principal will now amortise according to the following equation

$$F_{t_i} = \left(1+\widetilde{R}\right)^i F_0 - \bar{Y}\frac{\left(1+\widetilde{R}\right)^i - 1}{\widetilde{R}}. \tag{3.23}$$

Both results follow by standard calculations. Since the payments happen discretely in time, a smart choice is to set C_t equal to a sum of so called delta functions. Inspired by Wilmot et. al. (1993) [23] we can define the Dirac delta function, $\delta(x)$, as the limit of $f(x)$ when $\varepsilon \to 0^+$ where

$$f(x) = \begin{cases} \frac{1}{2\varepsilon} & \mid x \mid \leq \varepsilon \\ 0 & \mid x \mid > \varepsilon \end{cases}. \tag{3.24}$$

The special thing about $f(x)$ is that $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ for any $\varepsilon > 0$. Define now

$$C_t = \sum_{i=1}^{N} \bar{Y}\delta(t-t_i). \tag{3.25}$$

Then the accumulated payments, D_t, will be given by

$$D_t = \int_0^t C_s \mathrm{d}s = \int_0^t \sum_{i=1}^{N} \bar{Y}\delta(s-t_i)\mathrm{d}s = \sum_{t_i \leq t} \bar{Y}. \tag{3.26}$$

We know that from holding the mortgage bond, we will be receiving the gain $\mathrm{d}M_t^a + C_t\mathrm{d}t$ over each time-period $[t; t+\mathrm{d}t]$. C_t will be zero everywhere except at payment dates, where $\bar{Y}$ is paid out. As payment dates and prepayment dates coincide, we get notationally challenged and for this reason we will explain the jump in asset values in words. Over the payment date t_i the bond value $M_{t_i-}^a$ will change to $M_{t_i}^a$ if prepayment does not occur, but if prepayment occurs, then $M_{t_i-}^a$ will drop to F_{t_i}. In both cases the payment $\bar{Y}$ is also delivered. Since this implies a stochastic boundary condition, Stanton makes the pragmatic assumption that $M_{t_i-}^a$ will be the expected value of the bond at time t_i plus the payment. We therefore have,

$$M_{t_i-}^a = (1-\mathbb{P}(\tau = t_i | \tau \geq t_i))M_{t_i}^a + \mathbb{P}(\tau = t_i | \tau \geq t_i)F_{t_i} + \bar{Y}. \tag{3.27}$$

Stanton assumes that the probability of prepayment at time t_i is given by $\mathbb{P}(\tau = t_i | \tau \geq t_i) = 1 - \mathrm{e}^{-\int_{t_i}^{t_i+1} \lambda_t \mathrm{d}t} \approx 1 - \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}}$ as would be the probability if prepayment happened continuously. We therefore end up with the boundary condition

$$M_{t_i-}^a = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}} M_{t_i}^a + (1-\mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})F_{t_i} + \bar{Y} \quad \forall i. \tag{3.28}$$

Hence, we may solve the mortgage PDEs over each interval $[t_{i-1}, t_i)$ with boundary conditions given by (3.28). We summarise our results in the following theorem.

Theorem 3.1. *A callable mortgage bond delivering discrete payments of size $\bar{Y}$ will satisfy the following PDEs over the half-open interval $[t_{i-1}, t_i)$*

$$\frac{\partial M^a}{\partial t} + (\kappa\mu - (\kappa+q)r_t)\frac{\partial M^a}{\partial r} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 M^a}{\partial r^2} = r_t M_t^a, \tag{3.29}$$
$$\frac{\partial M^\ell}{\partial t} + (\kappa\mu - (\kappa+q)r_t)\frac{\partial M^\ell}{\partial r} + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 M^\ell}{\partial r^2} = r_t M_t^\ell, \tag{3.30}$$

with boundary conditions given by

$$M_{t_i-}^a = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}} M_{t_i}^a + (1 - \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})F_{t_i} + \bar{Y}, \tag{3.31}$$
$$M_{t_i-}^\ell = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}} M_{t_i}^\ell + (1 - \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})(1+X)F_{t_i} + \bar{Y}, \tag{3.32}$$

where

$$\lambda_{t_i} = \begin{cases} \lambda_1 & \text{if } M_{t_i}^\ell < (1+X)F_{t_i} \\ \lambda_1 + \lambda_2 & \text{if } M_{t_i}^\ell \geq (1+X)F_{t_i} \end{cases}, \tag{3.33}$$

for $i \in \{0, 1, ..., N\}$ and $M_{t_N}^a = M_{t_N}^\ell = 0$.

It is not obvious how we should solve for the mortgage bond in the theorem above analytically. Therefore, we will now look at how to approximate the solution.

3.4 Pricing by Finite Differences

The solution to the PDEs (3.64) and (3.65) can be approximated by use of so called finite difference techniques. The idea is simply to approximate the partial derivatives with difference quotients. For a function $f = f(t,x)$ of time and space, we will discretise time as $\{t_0, t_1, ..., t_j, ..., t_J\} = \{0, \Delta t, ..., j\Delta t, ..., T\}$ where $\Delta t = \frac{T}{J}$, and space as $\{x_0, x_1, ..., x_i, ..., x_N\} = \{x_{\min}, x_{\min} + \Delta x, ..., x_{\min} + i\Delta x, ..., x_{\max}\}$ where $\Delta x = \frac{1}{N}(x_{\max} - x_{\min})$. With this discretisation we will denote by $f_{i,j}$ the function f evaluated in (t_j, x_i). The most common difference operators are now given in the following definition.

Definition 3.1. *The forward, backward, central and second-order central differences of $f(t,x)$ w.r.t. x are given by*

$$D_x^+ f_{i,j} = \frac{f_{i+1,j} - f_{i,j}}{\Delta x}, \tag{3.34}$$
$$D_x^- f_{i,j} = \frac{f_{i,j} - f_{i-1,j}}{\Delta x}, \tag{3.35}$$
$$D_x f_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}, \tag{3.36}$$

$$D_{xx} f_{i,j} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2}. \tag{3.37}$$

Different schemes exists applying different combinations of the above difference operators. The most common ones are the *explicit*, the *implicit* and the *Crank-Nicolson* schemes. The explicit scheme makes use of the backward difference for the time derivative as well as central and second order central differences for the first and second derivatives in space. The implicit scheme does the same in the space direction but makes use of a forward difference in the time direction. Finally, the Crank-Nicolson scheme averages the two methods. As shown in Seydel (2009) [27], the error from approximating by use of the Crank-Nicolson scheme is of order $\mathcal{O}(\Delta t^2)+\mathcal{O}(\Delta x^2)$ compared to the implicit and explicit methods, which are of first order in the time direction. The Crank-Nicolson method has therefore gained popularity, as it tends to be more stable than the other two methods. Defining $m^a_{i,j}$ as the approximation to M^a_t in (3.64), the Crank-Nicolson method becomes

$$\begin{aligned}\frac{1}{2}\Big[D_t^- m^a_{i,j+1} + \hat{\mu}_i D_r m^a_{i,j+1} + \frac{1}{2}\hat{\sigma}_i D_{rr} m^a_{i,j+1} + D_t^+ m^a_{i,j} + \hat{\mu}_i D_r m^a_{i,j} + \frac{1}{2}\hat{\sigma}_i D_{rr} m^a_{i,j}\Big] \\ = \frac{1}{2}(r_{i,j+1} m^a_{i,j+1} + r_{i,j} m^a_{i,j}),\end{aligned} \tag{3.38}$$

where $\hat{\mu}_i = \kappa\mu - (\kappa + q) r_{i,j}$ and $\hat{\sigma}_i = \sigma^2 r_{i,j}$. By inserting equations (3.34)-(3.37) and rearranging we can find the relation

$$A_i m^a_{i-1,j} + B_i m^a_{i,j} + C_i m^a_{i+1,j} = -A_i m^a_{i-1,j+1} + D_i m^a_{i,j+1} - C_i m^a_{i+1,j+1}, \tag{3.39}$$

where

$$A_i = \frac{1}{4\Delta r}\hat{\mu}_i - \frac{1}{4(\Delta r)^2}\hat{\sigma}_i, \tag{3.40}$$

$$B_i = \frac{1}{\Delta t} + \frac{1}{2(\Delta r)^2}\hat{\sigma}_i + \frac{1}{2} r_{i,j}, \tag{3.41}$$

$$C_i = -\frac{1}{4\Delta r}\hat{\mu}_i - \frac{1}{4(\Delta r)^2}\hat{\sigma}_i, \tag{3.42}$$

$$D_i = \frac{1}{\Delta t} - \frac{1}{2(\Delta r)^2}\hat{\sigma}_i - \frac{1}{2} r_{i,j+1}, \tag{3.43}$$

Define now the vector $\mathbf{m}^a_j$ with elements $[\mathbf{m}^a_j]_i = m^a_{i,j}$ for $i = 0, 1, ..., N$. Having computed $\mathbf{m}^a_{j+1}$, the equations (3.39)-(3.43) will define $N-1$ equations with $N+1$ unknowns when setting $i = 1, 2, ..., N-1$. We encounter problems when $i \in \{0,N\}$, since we get outside the grid by using the central differences. If we can come up with some boundary condition for $i \in \{0, N\}$, or equivalently $r \in \{r_{\min}, r_{\max}\}$, then it will be possible to complete the linear system of equations. A clever way to come up with boundary conditions is to consider the behaviour of either the function itself or its derivatives at high and low values of the short rate, as we might have additional information about the behaviour of the function at these values. If for example we assume the mortgage bond to have zero convexity for very high and low values of the short rate, then we may set the second derivative equal to zero in the PDEs (3.64) and (3.65). This will make the D_{rr} terms vanish in equation (3.38). The zero convexity can be justified since the embedded optionality will be either deep in or out of the money at these values. If also we replace $D_r m_{0,j}$ with $D_r^+ m_{0,j}$ and replace $D_r m_{N,j}$ with $D_r^- m_{N,j}$, then we avoid using values outside the grid. All in all we can establish the following two equations:

$$B_0 m_{0,j}^a + C_0 m_{1,j}^a = D_0 m_{0,j+1}^a - C_0 m_{1,j+1}^a, \tag{3.44}$$
$$A_N m_{N-1,j}^a + B_N m_{N,j}^a = -A_N m_{N-1,j+1}^a + D_N m_{N,j+1}^a, \tag{3.45}$$

where

$$B_0 = \frac{1}{\Delta t} + \frac{\hat{\mu}_0}{2\Delta r} + \frac{1}{2} r_{0,j}, \quad C_0 = -\frac{\hat{\mu}_0}{2\Delta r}, \quad D_0 = \frac{1}{\Delta t} - \frac{\hat{\mu}_0}{2\Delta r} - \frac{1}{2} r_{0,j},$$
$$A_N = \frac{\hat{\mu}_N}{2\Delta r}, \quad B_N = \frac{1}{\Delta t} - \frac{\hat{\mu}_N}{2\Delta r} + \frac{1}{2} r_{N,j} \quad \text{and} \quad D_N = \frac{1}{\Delta t} + \frac{\hat{\mu}_N}{2\Delta r} - \frac{1}{2} r_{N,j+1}.$$

Since we now have $N+1$ equations with $N+1$ unknowns, we may write our system of equations on matrix form as follows

$$\mathbf{A}\mathbf{m}_j^a = \mathbf{C}\mathbf{m}_{j+1}^a, \tag{3.46}$$

where

$$\mathbf{A} = \begin{bmatrix} B_0 & C_0 & 0 & 0 & 0 & \cdots & 0 \\ A_1 & B_1 & C_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & B_2 & C_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & A_N & B_N \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} D_0 & -C_0 & 0 & 0 & 0 & \cdots & 0 \\ -A_1 & D_1 & -C_1 & 0 & 0 & \cdots & 0 \\ 0 & -A_2 & D_2 & -C_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -A_{N-1} & D_{N-1} & -C_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & -A_N & D_N \end{bmatrix}.$$

Performing a similar discretisation for M_t^ℓ and we are able to solve for the bond price and mortgage liabilities iteratively. Algorithm 3.1 below gives a non-technical illustration of an implementation. The final thing that we have to do before using our finite difference solver is to choose the upper and lower values for r_t. Since we are in the CIR model, and negative interest rates cannot occur, a natural lower bound will be $r_{\min} = 0$. In order to come up with an upper bound, we will use a value so high that it is likely not to occur. One such value could be three standard deviations away from the mean for a horizon equal to the time to maturity of the bond considered. With the upper and lower values in place, we have completed our pricing PDE solver.

Algorithm 3.1: Finite Difference Implementation

$\mathbf{m}^a = 0$, $\mathbf{m}^\ell = 0$

$\mathbf{G} = \mathbf{A}^{-1}\mathbf{C}$

for $j = J$ ***to*** $j = 0$ **do**

  if *Payment date* **then**

    $B = \mathbb{1}(\mathbf{m}^\ell \geq (1+X)F_{t_j})$

    $\lambda = \lambda_1 + \lambda_2 B$

    $\mathbf{m}^a = \mathrm{e}^{-\frac{1}{n}\lambda}\mathbf{m}^a + (1 - \mathrm{e}^{-\frac{1}{n}\lambda})F_{t_j} + \bar{Y}$

    $\mathbf{m}^\ell = \mathrm{e}^{-\frac{1}{n}\lambda}\mathbf{m}^\ell + (1 - \mathrm{e}^{-\frac{1}{n}\lambda})F_{t_j}(1+X) + \bar{Y}$

  end

  $\mathbf{m}^a = \mathbf{G}\mathbf{m}^a$

  $\mathbf{m}^\ell = \mathbf{G}\mathbf{m}^\ell$

end

The pricing algorithm has been implemented in Matlab and is available through appendix A.6. Figure 4 below shows the price of a 30 year 12.5% annuity mortgage bond for varying prepayment intensities λ_2 and varying initial short rates. As is evident from the graph, higher prepayment intensities are associated with lower values of the mortgage bond. Especially at low interest rate levels the bond exhibits negative convexity for sufficiently high prepayment intensities. Not only is the model capable of generating negative convexity, but also will the duration of the bond be negative whenever the slope of the price function is positive. This phenomenon is a consequence of the costs associated with prepayment, since the prepayment costs constitute a barrier to the mortgager who will defer a potential prepayment to a time of lower interest rates.

Before estimating and applying the model to the Danish market for callable mortgage bonds, we will consider some reasonable and necessary adjustments to the model. The CIR model used by Stanton has been developed at a time where very low and even negative interest rates had never occurred. Since the CIR model is only defined for positive interest rates, we will have to choose a short rate model capable of generating negative interest rates. Secondly, Stanton assumes that if a borrower, for some exogenous reason, choses to prepay when the mortgage bond trades below par, then the borrower will still have to prepay at par. In Denmark, all mortgage bonds are born with the option to deliver back the mortgage bond to the mortgage institution in case a borrower want to terminate his payment obligations. In the next two sections we will investigate the effect of these two adjustments.

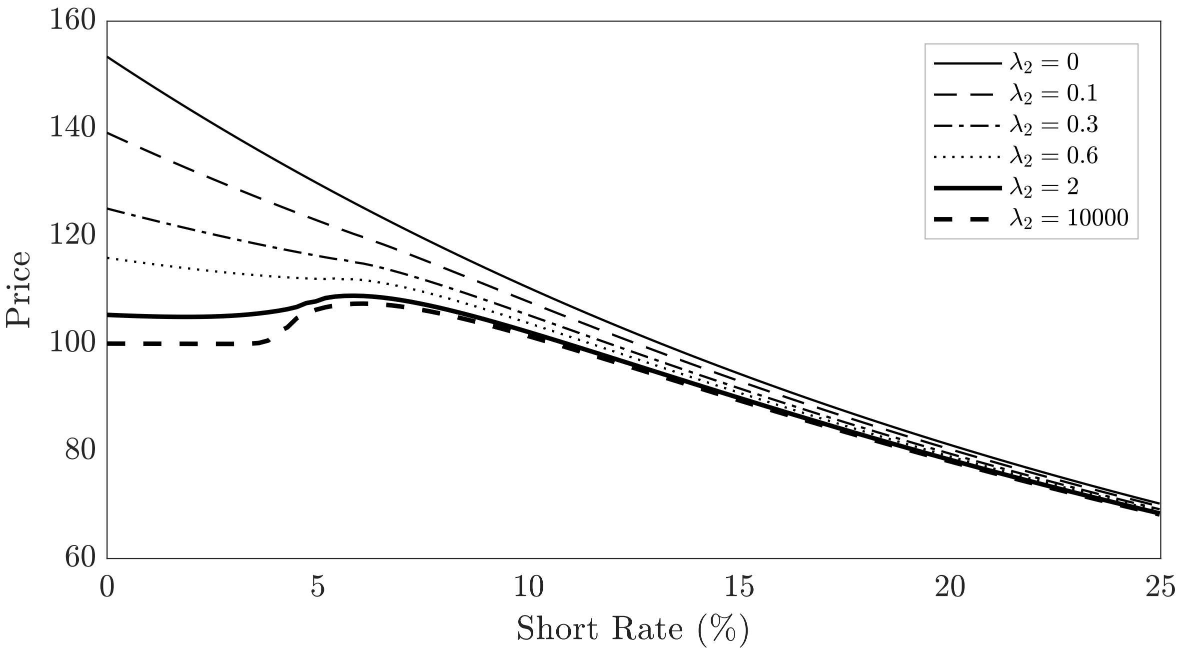

Figure 4: The price of a callable mortgage bond for varying prepayment intensities and initial short rate. The prices are calculated for a 30 year 12.5% mortgage with face value equal to 100. The parameters are those used by Stanton: $\kappa = 0.29368$, $\mu = 0.07935$, $\sigma = 0.11425$, $q = -0.12165$ and $X = 0.24$. λ_1 is set equal to zero in order to isolate the effect of varying λ_2.

3.5 The Extended Vasicek Model

In order to incorporate the presence of negative interest rates, we will introduce the extended Vasicek model by Hull & White (1990) [13]. In this model the short rate dynamics are assumed to take the form:

$$\mathrm{d}r_t = \kappa[\theta(t) - r_t]\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{P}}. \tag{3.47}$$

The model allows for a deterministic time dependent mean level, $\theta(t)$. We will see that with this specification, it will be possible for the model to fit the initial term structure of interest rates if $\theta(t)$ is chosen properly. In order to fit the model to market observables, our first job is to find the prices for zero coupon bonds in the model. From theorem (2.4) we must have that the price of a zero coupon bond will satisfy the following PDE

$$\frac{\partial P_t^T}{\partial t} + \kappa[\hat{\theta}(t) - r_t]\frac{\partial P_t^T}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_t^T}{\partial r^2} = r_t P_t^T, \quad (t, r_t) \in [0;T) \times \mathbb{R}, \tag{3.48}$$

where $\hat{\theta}(t) = \theta(t) - \frac{\sigma}{\kappa} q_t$. If we guess the general solution $P_t^T = \mathrm{e}^{-A(t,T) - B(t,T) r_t}$ and insert into equation (3.48), then we get

$$[-A'(t,T) - B'(t,T)r_t]P_t^T - \kappa[\hat{\theta}(t) - r_t]B(t,T)P_t^T + \frac{1}{2}\sigma^2 B^2(t,T)P_t^T = r_t P_t^T. \tag{3.49}$$

Dividing through by P_t^T and rearranging we obtain

$$-A'(t,T) - \kappa\hat{\theta}(t)B(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) - [1 + B'(t,T) - \kappa B(t,T)]r_t = 0. \tag{3.50}$$

If equation (3.50) has to hold for all r_t, and considering T as a parameter, then we must have that the following two ordinary differential equations must hold:

$$\frac{\mathrm{d}A(t,T)}{\mathrm{d}t} + \kappa\hat{\theta}(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) = 0, \tag{3.51}$$

$$1 + \frac{\mathrm{d}B(t,T)}{\mathrm{d}t} - \kappa B(t,T) = 0, \tag{3.52}$$

with terminal conditions $A(T,T) = B(T,T) = 0$. The solution to these ordinary differential equations can easily be shown to be

$$B(t,T) = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa(T-t)}), \tag{3.53}$$

$$A(t,T) = \kappa\int_t^T \hat{\theta}(s)B(s,T)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(t,T) + \frac{\sigma^2}{2\kappa^2}\left(B(t,T) - (T-t)\right), \tag{3.54}$$

see appendix A.1. If we at time zero observe zero coupon bond prices $\bar{P}_0^T$ for each maturity T, then the idea of Hull & White was to choose the function $\hat{\theta}(t)$ such that the model prices are consistent with the market prices. The function of $\hat{\theta}(t)$ ensuring this property can be shown to be given by

$$\hat{\theta}(t) = \bar{f}(0,t) + \frac{1}{\kappa}\frac{\mathrm{d}\bar{f}(0,t)}{\mathrm{d}t} + \frac{\sigma^2}{2\kappa^2}(1 - \mathrm{e}^{-2\kappa t}), \tag{3.55}$$

which implies that

$$A(t,T) = -\ln\left(\frac{\bar{P}_0^T}{\bar{P}_0^t}\right) - B(t,T)\bar{f}(0,t) + \frac{\sigma^2}{4\kappa}B^2(t,T)(1 - \mathrm{e}^{-2\kappa t}), \tag{3.56}$$

where we assume that $\bar{f}(0,t)$, the observed forward curve for the short rate, is differentiable. The derivations of equations (3.55) and (3.56) are extensive and for that reason also left in appendix A.2. To verify that we indeed fit the initial term structure, one can simply insert $A(0,T)$ into the formula for P_0^T using that $f(0,0) = r_0$.

At this point it is appropriate to have a discussion of what we have achieved by choosing $\hat{\theta}(t)$ according to (3.55). If we had let $\theta(t)$ be a constant, then we would be back in the original Vasicek model. Estimating the model parameters by maximum likelihood and agreeing on a specification of the market price of risk, we would be able to price zero coupon bonds and swaps by their closed form solutions. The problem with this approach is that the model most likely will not replicate the prices in the market. This is the reason why the literature and market practitioners have turned to the procedure of estimating model parameters by calibration. By this procedure we minimise the deviations of model prices from market prices by adjusting the model parameters. As is evident from this procedure, the model will lose its ability to describe the behaviour of the short rate, since the model parameters are not determined by any sound statistical estimation procedure. However, this might not necessarily be a problem. Instead of specifying the function $q_t = q(t, r_t)$ which influences the drift part of the short rate process, we can take the market as given and let the market specify the drift rate used for pricing. Since we have assumed that a full term structure of interest rates is available in the market, we may simply regard q_t, or equivalently $\hat{\theta}(t) = \theta(t) - \frac{\sigma}{\kappa}q_t$, as being chosen by the market. For this reason we will not have a problem with our choice of $\hat{\theta}(t)$, but we have to be aware that $\hat{\theta}(t)$ is deterministic and therefore our choice of $\hat{\theta}(t)$ today should also apply tomorrow. As discussed by Hull & White themselves this assumption is likely to fail and the market practice is simply to respecify $\hat{\theta}(t)$ each time a new curve is available. It is hard to justify this practice from a theoretical viewpoint, and the main argument for proceeding with this practice anyway is the following: If we are able to price the most simple fixed income securities in the market accurately, then we will hopefully also be pricing the more complex products, like the callable bond, more accurately.

With our choice of $\hat{\theta}(t)$ in place, we still need the parameters κ and σ. As discussed by Hull & White these could in principle also be functions of time, but in this thesis we will keep these as constants[3]. In reality we will often observe that interest rates exhibit stochastic volatility, which is of course in conflict with the constant volatility of our model. A pragmatic approach is therefore to imply the volatility and mean reversion from volatility dependent securities like the cap. In order to calibrate our model to the market for caps, we will need to be able to price these contracts. In the following we will show how to do so in the extended Vasicek model. Section 2.5 showed that the cap contract could be valued by computing the probabilities in the following valuation formula.

$$\mathbf{Cap}_t^{T_N} = \frac{1}{K^*}\sum_{i=1}^{N} P_t^{T_{i-1}} K^* \mathbb{Q}_t^{T_{i-1}}\left(K^* > P_{T_{i-1}}^{T_i}\right) - P_t^{T_i}\mathbb{Q}_t^{T_i}\left(K^* > P_{T_{i-1}}^{T_i}\right). \tag{3.57}$$

Since we have an explicit model now, we can find the dynamics under the $\mathbb{Q}^T$ measure. Using that $\beta^T(t, r_t) = -\sigma B(t, T)$ along with equation (2.40) and the relevant dynamics of the short rate becomes

$$\mathrm{d}r_t = [\kappa(\theta(t) - r_t) - \sigma^2 B(t, T)]\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}^T}.$$

To evaluate the probability of the event $\{K^* > P_T^S\}$ we will use the result from appendix A.3 saying that r_T is Gaussian under both $\mathbb{Q}^T$ and $\mathbb{Q}^S$. Since $P_T^S = \mathrm{e}^{-A(T,S)-B(T,S)r_T}$

[3] It is not the aim of this thesis to perfectly price exotic derivatives, so in order to keep the focus on the mortgage bonds it seems reasonable to keep these parameters constant

we will have that P_T^S is log-normal since $-A(T,S) - B(T,S)r_T$ is simply a linear combination of a Gaussian variable. Let $r_T|r_t \sim \mathcal{N}(\mu(t,T), \nu^2(t,T))$, then we must have that $\ln P_T^S|r_t \sim \mathcal{N}(\tilde{\mu}(t,T,S), \tilde{\nu}^2(t,T,S))$ where $\tilde{\mu}(t,T,S) = -A(T,S) - B(T,S)\mu(t,T)$ and $\tilde{\nu}^2(t,T,S) = B^2(T,S)\nu^2(t,T)$. For a log-normal random variable it holds that $\mathbb{E}_t^{\mathbb{Q}^T}[P_T^S] = \mathrm{e}^{\tilde{\mu}(t,T,S)+\frac{1}{2}\tilde{\nu}^2(t,T,S)}$ or equivalently $\tilde{\mu}(t,T,S) = \ln \mathbb{E}_t^{\mathbb{Q}^T}[P_T^S] - \frac{1}{2}\tilde{\nu}^2(t,T,S)$. Using this, we may compute the probability of $\{K^* > P_T^S\}$ under $\mathbb{Q}^T$ as follows:

$$
\begin{aligned}
\mathbb{Q}^T(K^* > P_T^S) &= \mathbb{Q}^T\left(\frac{\ln K^* - \tilde{\mu}(t,T,S)}{\tilde{\nu}(t,T,S)} > \frac{\ln P_T^S - \tilde{\mu}(t,T,S)}{\tilde{\nu}(t,T,S)}\right) \\
&= \Phi\left(\frac{\ln K^* - \tilde{\mu}(t,T,S)}{\tilde{\nu}(t,T,S)}\right) \\
&= \Phi\left(\frac{\ln\left(\frac{K^*}{\mathbb{E}_t^{\mathbb{Q}^T}[P_T^S]}\right)}{\tilde{\nu}(t,T,S)} + \frac{1}{2}\tilde{\nu}(t,T,S)\right),
\end{aligned} \tag{3.58}
$$

where $\Phi(\cdot)$ denotes the standard normal CDF. We now recall that $\frac{P_T^S}{P_T^T}$ is a martingale under $\mathbb{Q}^T$. Using this property and inserting for $\tilde{\nu}(t,T,S)$, the above probability becomes

$$
\mathbb{Q}^T(K^* > P_T^S) = \Phi\left(\frac{\ln\left(\frac{K^* P_t^T}{P_t^S}\right)}{B(T,S)\nu(t,T)} + \frac{1}{2}B(T,S)\nu(t,T)\right).
$$

The variance of the short rate is derived in appendix A.3 and is given by

$$
\nu^2(t,T) = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa(T-t)}\right).
$$

By similar argumentation, we can find the probability of $\{K^* > P_T^S\}$ under $\mathbb{Q}^S$. The only thing that will change is the expectation in (3.58), which should now be evaluated under the $\mathbb{Q}^S$-measure. Appendix A.4 evaluates this expectations and we have that

$$
\mathbb{E}^{\mathbb{Q}^S}\left[P_T^S\right] = \frac{P_t^S}{P_t^T}\mathrm{e}^{B^2(T,S)\nu^2(t,T)}. \tag{3.59}
$$

By replacing the expectation operator in equation (3.58) with (3.59) we find that

$$
\mathbb{Q}^S(K^* > P_T^S) = \Phi\left(\frac{\ln\left(\frac{K^* P_t^T}{P_t^S}\right)}{B(T,S)\nu(t,T)} - \frac{1}{2}B(T,S)\nu(t,T)\right). \tag{3.60}
$$

We will summarise the above results in the following theorem.

Theorem 3.2. *In the extended Vasicek model, the price of a cap contract is given by*

$$\mathbf{Cap}_t^{T_N} = \frac{1}{K^*} \sum_{i=1}^{N} P_t^{T_{i-1}} K^* \Phi(-d_{2,i}) - P_t^{T_i} \Phi(-d_{1,i}), \tag{3.61}$$

where

$$d_{1,i} = \frac{\ln\left(\frac{P_t^{T_i}}{K^* P_t^{T_{i-1}}}\right)}{B(T_{i-1},T_i)\nu(t,T_{i-1})} + \frac{1}{2} B(T_{i-1},T_i)\nu(t,T_{i-1}), \tag{3.62}$$

and

$$d_{2,i} = d_{1,i} - B(T_{i-1},T_i)\nu(t,T_{t-i}). \tag{3.63}$$

Theorem 3.2 gives us a closed form solution to the price of an interest rate cap. The price function can be seen as a sum of put options on zero coupon bonds with strike K^*, for which Jamshidian (1989) [8] was the first to derive a pricing formula equivalent to the above. It is now possible to calibrate the extended Vasicek model to market quotes. Since interest rate caps are not very liquid in the Danish market, we will use EURIBOR caps as a proxy. As the Danish krone is pegged to the Euro, and since EURIBOR and CIBOR rates tend to move together, we will expect the calibrated parameters to be a good approximation. Figure 5 shows an example of a calibration to Euro cap quotes on the 1$^{\text{st}}$ of June 2017. Given that we have seven quotes and two parameters to be calibrated, the calibration does rather well. The calibrated parameters are $\kappa = 4.916 \cdot 10^{-11}$ and $\sigma = 0.699\%$.

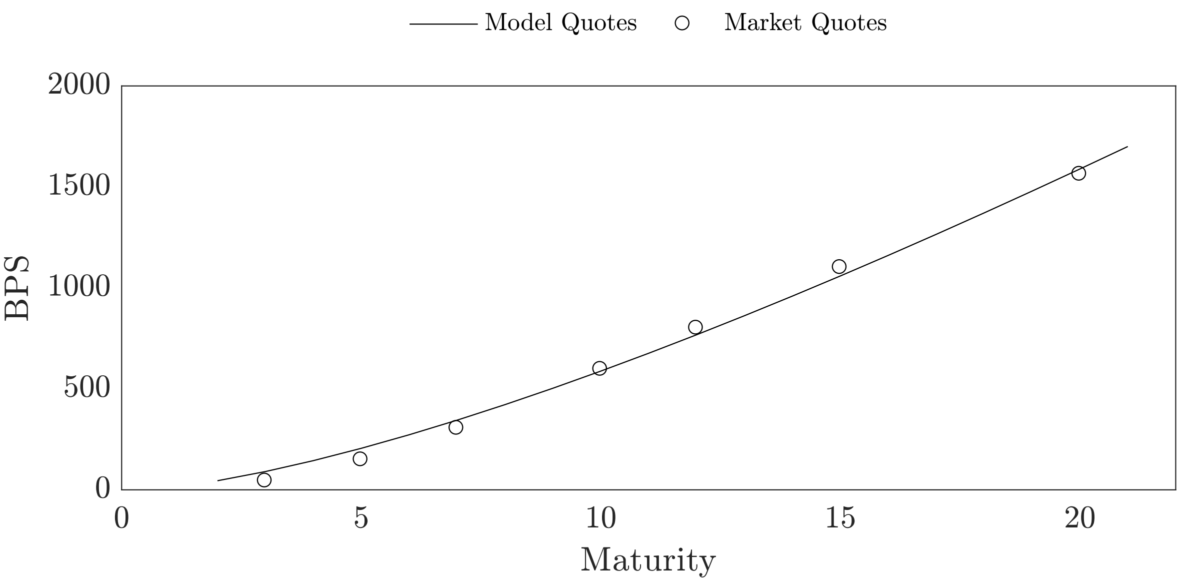

Figure 5: The extended Vasicek model calibrated to Euro cap quotes as of the 1$^{\text{st}}$ of June 2017. The calibrated parameters are $\kappa = 4.916 \cdot 10^{-11}$ and $\sigma = 0.699\%$. Market quotes are extracted from Bloomberg.

If we perform this calibration procedure over time, then we will end up with a time series of implied values for κ and σ. Figure 6 shows the evolution of these parameters. From the figure it is clear that the parameters vary over time but do not diverge or explode. The volatility is low as would be expected for very short interest rates, but the mean reversion hits the zero floor in times of low interest rate levels. The mean reversion could in practice be allowed to become negative, but the model becomes unrealistic for negative κ as the short rate will be a diverging process. Hence, it would be meaningless to allow for negative values of κ. An extremely low but positive κ is also not desirable since κ enters as a denominator in the equation for $\hat{\theta}(t)$ in (3.55) leading to an explosive behaviour of $\hat{\theta}(t)$. This calibration problem seems to be very common, see for example Brigo & Mercurio (2006) [5], and is most likely an indication that our model is too simple to describe the behaviour of the cap market. A pragmatic way of proceeding will be to estimate κ by maximum likelihood as if we were in the classic Vasicek model and then use this κ instead. Appendix A.5 derives the maximum likelihood estimator, and by using the one year swap rate as a proxy for the short rate dynamics we find a κ of 7.5%. Performing a recalibration using this fixed κ results in a slightly higher implied volatility, a way more reasonable mean reversion and a non-explosive $\hat{\theta}(t)$.

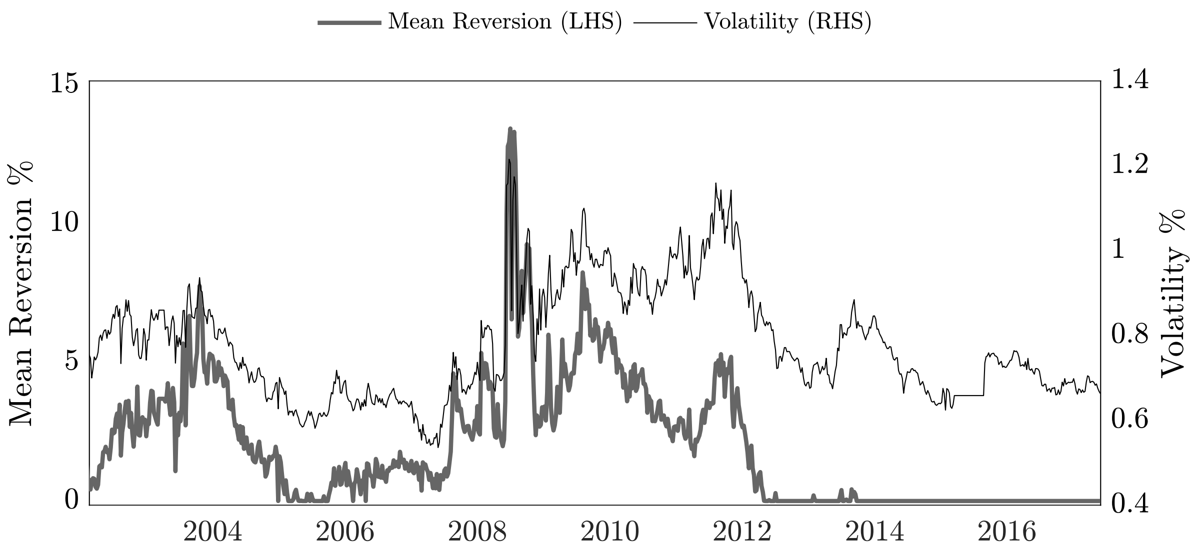

Figure 6: The extended Vasicek model calibrated to Euro cap quotes over time. Market quotes are extracted from Bloomberg.

3.6 The Delivery Option

With the model extended to take into account the current negative interest rate environment, we remain to incorporate the so called delivery option. We will make the assumption that borrowers are never forced to prepay the remaining notional F_t whenever $F_t > M_t^a$. In this scenario, we will allow the borrower to prepay M_t^a instead of F_t, corresponding to the borrower buying back the bond and delivering it to the mortgage institution. In either case, the borrower will still have to pay the proportional prepayment cost of X, such that the prepaid amount totals $(1+X)F_t$ if $F_t < M_t^a$ and $(1+X)M_t^a$ if $F_t > M_t^a$, or equivalently, $(1+X)\min(F_t, M_t^a)$. We will also collapse the prepayment intensities λ_1 and λ_2 into one prepayment intensity, λ. We do so, since there is no longer reason to distinguish the two scenarios, as the mortgage owners can now prepay whenever they find it optimal. Compared to the case with no delivery option, the bond must decrease in value when introducing the delivery option, the reason being that the investor will no longer receive $\min(F_t, M_t^a) \leq F_t$ in a prepayment event. With the new assumptions about the short rate dynamics and the delivery option, we may now modify theorem 3.1 as follows.

Theorem 3.3. *(The extended Stanton model) A callable mortgage bond will satisfy the following PDEs over the half-open interval* $[t_{i-1}, t_i)$

$$\frac{\partial M^a}{\partial t} + \kappa(\hat{\theta}(t) - r_t)\frac{\partial M^a}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 M^a}{\partial r^2} = r_t M_t^a, \tag{3.64}$$
$$\frac{\partial M^\ell}{\partial t} + \kappa(\hat{\theta}(t) - r_t)\frac{\partial M^\ell}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 M^\ell}{\partial r^2} = r_t M_t^\ell, \tag{3.65}$$

with boundary conditions given by

$$M_{t_i-}^a = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}} M_{t_i}^a + (1 - \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})\min(F_{t_i}, M_{t_i}^a) + \bar{Y}, \tag{3.66}$$
$$M_{t_i-}^\ell = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}} M_{t_i}^\ell + (1 - \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})(1+X)\min(F_{t_i}, M_{t_i}^a) + \bar{Y}, \tag{3.67}$$
$$\lambda_{t_i} = \lambda \mathbb{1}\{M_{t_i}^\ell \geq (1+X)\min(F_{t_i}, M_{t_i}^a)\}, \tag{3.68}$$

for $i \in \{1, 2, ..., N\}$ *and* $M_{t_N}^a = M_{t_N}^\ell = 0$.

Comparing theorem 3.3 to theorem 3.1 it might seem that not much has changed, but from a computational viewpoint, we have become challenged. Most importantly $\hat{\theta}(t)$ will have to be respecified each time a new swap curve is available and so will the implied volatility each time new cap quotes are available. From the finite difference implementation in section 3.4 we found that the matrices $\mathbf{A}$ and $\mathbf{B}$ in equation (3.46) were not changing over time. Hence, only a single matrix inversion of $\mathbf{A}$ was needed, and for that reason we could keep it outside the loop of algorithm 1. Since $\mathbf{A}$ and $\mathbf{B}$ will now change over time, they will have to be computed in every each loop.

Prices from the extended Stanton model have been illustrated in figure 7 for a 30 year 2.5% mortgage bond. From the figure it is clear how negative interest rates are no longer a problem. It can also be seen how the price of the bond decreases as the prepayment intensity increases since it becomes more likely that the borrower will prepay the bond when it becomes optimal. Finally, an increasing prepayment cost increases the price and adds negative convexity to the bond as borrowers will wait for interest rates to become lower before prepaying.

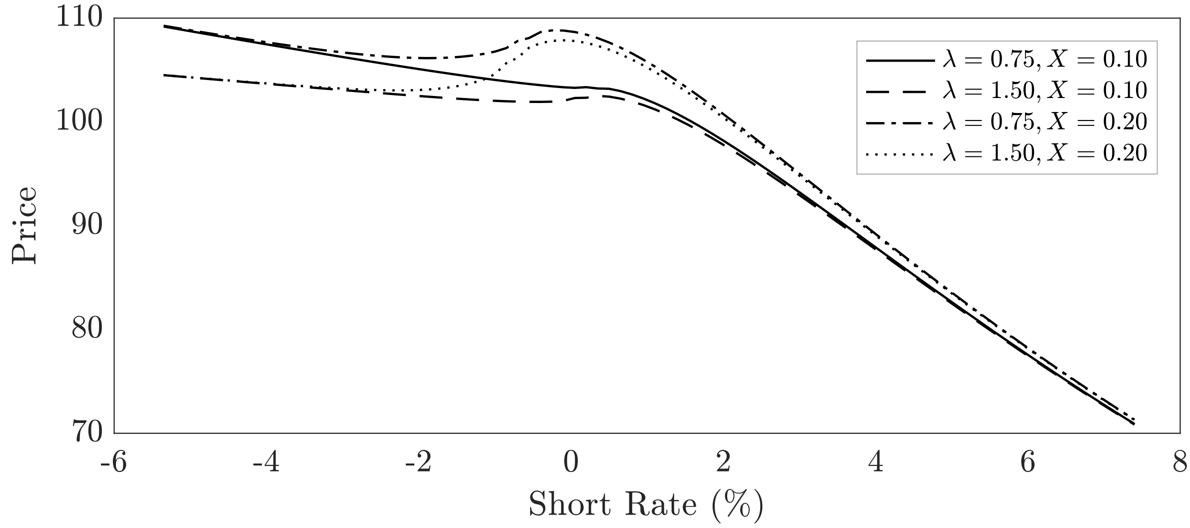

Figure 7: Callable mortgage bond in the extended Stanton model for varying prepayment intensities and prepayment costs. The prices are calculated for a 30 year 2.5% mortgage with face value equal to 100. The short rate model is calibrated to market CIBOR, swap and Cap quotes as of the 1st of June 2017.

Figure 8 below gives a similar illustration of an actual mortgage bond from Nykredit Realkredit. The figures are of course not directly comparable since each scatter point defines a new shorter bond while a new swap curve is available each day. However, when comparing figure 8 to figure 7 it is clear that the negative convexity of the model is also present in the market. That the market prices in 2016 and 2017 are above the ones in 2014 and 2015 for the same interest rate levels is due to a changing borrower distribution, which will be discussed in section 4.4.

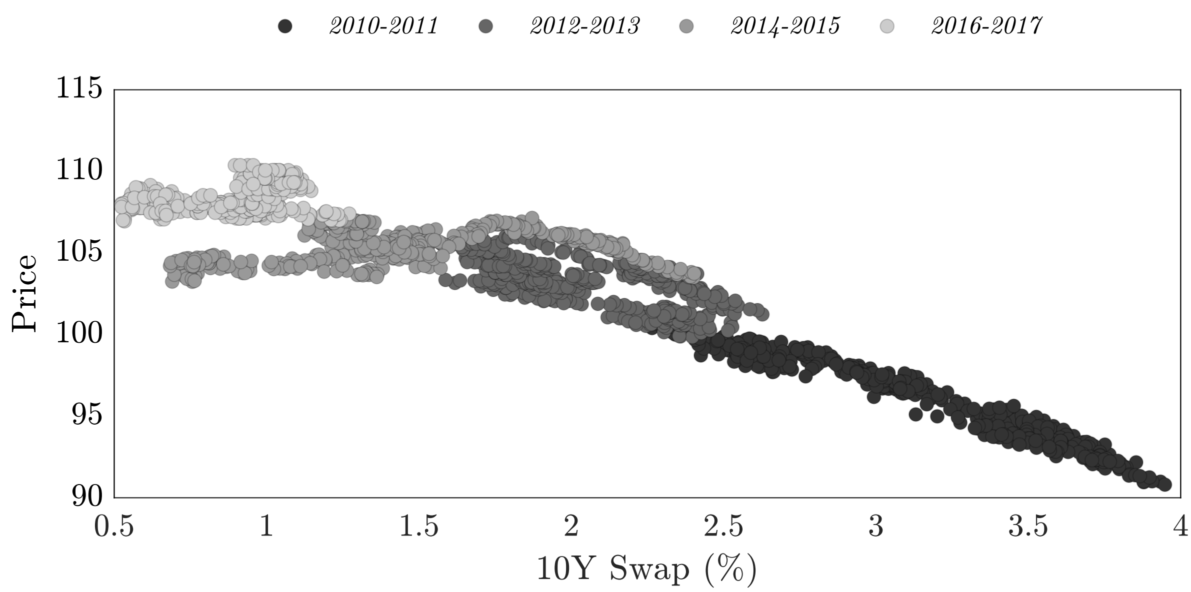

Figure 8: Market prices for a 30 year 4% coupon bond from Nykredit maturing in year 2041. (ISIN: DK0009775355).

4 Estimating Prepayments

4.1 The Danish Prepayment Data

Before we jump into the actual estimation of the model, we will have a look at the characteristics of the data used in the estimation procedure. In contrast to the mortgage market in the USA, the Danish market consists of rather few but very large mortgage pools. The large mortgage pools are a result of the demand for high liquidity in the underlying mortgage bonds. The Danish data are by law[4] published through the exchange and contains information on the borrower composition (CK92), preliminary prepayments (CK93), cash flows (CK94) and the final prepayments (CK95). The data is therefore very detailed and in practice, it is often seen that prepayment models are extended to take all kind of aspects of the data into account. In this thesis we will only consider the CK95 data, but one could for example extent the model to discriminate borrowers by loan size using the CK92 data. Figure 9 gives an example of the CK95 data for a 5% 30 year callable mortgage bond issued by Nykredit Realkredit. From the figure it is clear how the prepayments increase during times of falling interest rates as the mortgage liabilities increase during these periods. We also see a tendency that some borrowers are more quickly prepaying compared to others. This tendency is consistent with borrowers having different costs associated with prepaying, which will imply that interest rates can stay in a falling trend while prepayments keep occurring. In the next subsection we will look into how these prepayment rates may be estimated.

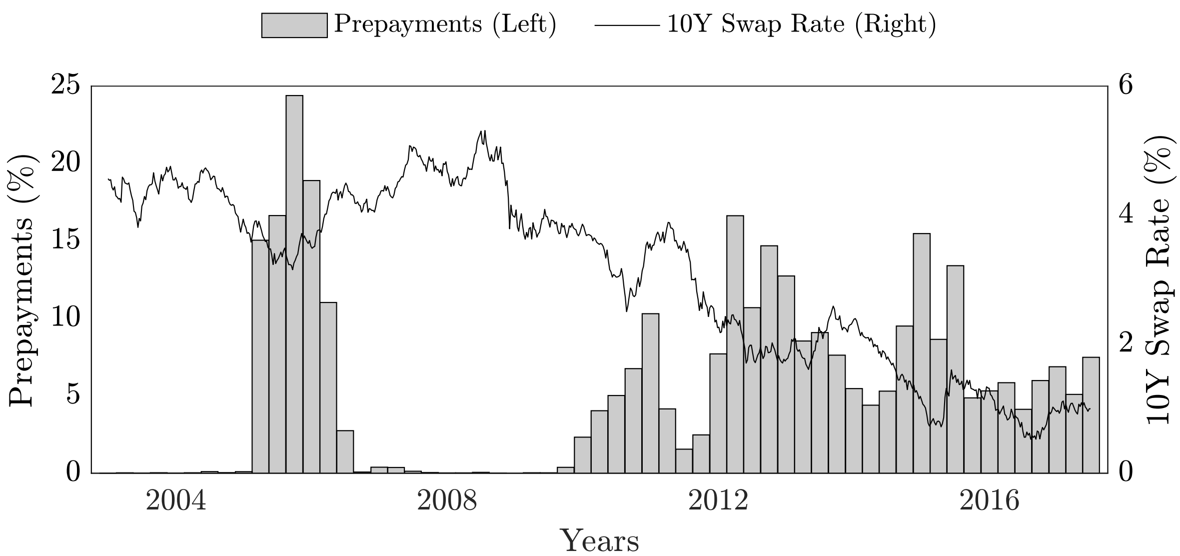

Figure 9: Actual prepayments (CK95) measured in percent of outstanding amount for a 5% 30 year mortgage bond maturing in 2035 from Nykredit Realkredit (ISIN: DK0009753469) (left) and the 10 year Danish swap rate (right). By the end of the issuing period the pool outstanding amount totalled DKK 65.5 bn.

4 See *Lov om værdipapirhandel §27a, stk.1.*

4.2 Generalised Method of Moments

In order to estimate the prepayment rates, we will be using the Generalised Method of Moments (GMM), which is an estimation technique proposed by Hansen (1982) [16]. The idea is to estimate the model directly by use of the moment conditions implied by the model. The method is often applicable when we have more moments than we have parameters to estimate or when Maximum Likelihood estimation does not suit the estimation problem. In our case we have only three parameters to estimate while we have plenty of data moments available through the prepayments. Therefore, it seems natural to try to come up with a set of moment restriction for the prepayments. We will start out by making the temporary assumption that we are able to monitor the prepayment behaviour of every single borrower in the pool. Each of these borrowers have an associated cost drawn independently from a beta distribution $\text{Beta}(\alpha, \beta)$. Specifically, we will let $X_{k,l}$ denote the cost held by borrower k in mortgage pool l for $k = 1, 2, ..., K_l$ and $l = 1, 2, ..., L$. Define the observation times $t_1, ..., t_i, ..., t_I$ and let $\delta_{i,k,l}$ be an indicator function taking the value one if borrower k in mortgage pool l prepays at time t_i. For each combination (r_{t_i}, t_i) we can calculate the value of the mortgage liabilities given some level of costs, X, and given some intensity λ. The level $X_i^* = X_i^*(\lambda)$ that ensures $M_{i,j}^{\ell}(X_i^*) = (1 + X_i^*)F_{t_i}$ will define the critical level of optimal prepayment. That is, if $X_{k,l} < X_i^*$, then it will be optimal for borrower k in mortgage pool l to prepay at time t_i. Define this event as $\Omega_{i,k,l}^X = \{X_{k,l} < X_i^*\}$. Define also the event $\Omega_{i,k,l}^{\lambda}$ as the event that borrower k in pool l checks whether prepayment is optimal at time t_i, which happens with probability $\mathbb{P}(\Omega_{i,k,l}^{\lambda}) = 1 - \mathrm{e}^{-\frac{1}{n}\lambda}$. Finally, the event that no prepayment has occurred up until time t_i can be written as $\Omega_{i,k,l}^{\delta} = \{\delta_{1,k,l} = \delta_{2,k,l} = \cdots = \delta_{i-1,k,l} = 0\}$. We may now decompose $\delta_{i,k,l}$ as follows

$$\delta_{i,k,l} = \mathbb{1}\{\Omega_{i,k,l}^{\lambda}\}\mathbb{1}\{\Omega_{i,k,l}^{X}\}\mathbb{1}\{\Omega_{i,k,l}^{\delta}\} \tag{4.1}$$

The above simply states that prepayment happens if the borrower checks whether prepayment is optimal while prepayment actually is optimal and prepayment has not yet occurred. Using that the event $\Omega_{i,k,l}^{\lambda}$ is independent of the events $\Omega_{i,k,l}^{X}$ and $\Omega_{i,k,l}^{\delta}$ the probability of the prepayment event (4.1) must be

$$\begin{aligned}\mathbb{P}(\delta_{i,k,l} = 1) &= \mathbb{P}(\Omega_{i,k,l}^{\lambda} \cap \Omega_{i,k,l}^{X} \cap \Omega_{i,k,l}^{\delta}) \\ &= \mathbb{P}(\Omega_{i,k,l}^{\lambda})\mathbb{P}(\Omega_{i,k,l}^{X} \cap \Omega_{i,k,l}^{\delta}) \end{aligned} \tag{4.2}$$

The event of not having prepaid in any previous period, $\Omega_{i,k,l}^{\delta}$, is of course highly correlated with the borrower's prepayment cost and hence the event $\Omega_{i,k,l}^{X}$, why we cannot easily separate these events. We will therefore have to compute the distribution of the borrower's prepayment cost over time. In the original paper by Stanton, this continuous distribution is approximated by a discrete distribution in order to avoid keeping track of a continuous distribution changing over time. It is however possible to derive and keep track of this distribution if we choose to sort our data in a smart way. Standing at time t_i we have observed the critical level of prepayment costs $X_1^*, X_2^*, ..., X_{i-1}^*$. If we now sort these in ascending order to obtain $\bar{X}_1 \leq \bar{X}_2 \leq \cdots \leq \bar{X}_{N_i-1}$ and define $\bar{X}_0 = 0$ and $\bar{X}_{N_i} = 1$, then we have discretised the domain of X in non-overlapping, and therefore mutually exclusive, intervals. By the law of total probability, we can now compute the following relation:

$$
\begin{aligned}
\mathbb{P}(\{X \leq x\} \cap \Omega_{i,k,l}^{\delta}) &= \sum_{h=1}^{N_i} \mathbb{P}(\{X \leq x\} \cap \Omega_{i,k,l}^{\delta} \cap \{\bar{X}_{h-1} \leq X \leq \bar{X}_h\}) \\
&= \sum_{h=1}^{N_i} \mathbb{P}(\{\bar{X}_{h-1} \leq X \leq \min(x,\bar{X}_h)\} \cap \Omega_{i,k,l}^{\delta}) \\
&= \sum_{h=1}^{N_i} \mathbb{P}(\bar{X}_{h-1} \leq X \leq \min(x,\bar{X}_h)) \mathbb{P}\left(\Omega_{i,k,l}^{\delta} \mid \bar{X}_{h-1} \leq X \leq \min(x,\bar{X}_h)\right) \\
&= \sum_{h=1}^{N_i} \left[\Psi(\min(x,\bar{X}_h); \alpha, \beta) - \Psi(\bar{X}_{h-1}; \alpha, \beta)\right]^+ \\
&\qquad \mathbb{P}\left(\Omega_{i,k,l}^{\delta} \mid \bar{X}_{h-1} \leq X \leq \min(x,\bar{X}_h)\right), \qquad (4.3)
\end{aligned}
$$

where we let $\Psi(x; \alpha, \beta)$ denote the cumulative distribution function for the Beta distribution with parameters α and β. Note that whether X is in $[\bar{X}_{h-1}; \min(x,\bar{X}_h)]$ or in $[\bar{X}_{h-1}; \bar{X}_h]$ will produce the same probability of not having prepaid before time t_i for $x > \bar{X}_{h-1}$. If a borrower has prepayment costs $X \in [\bar{X}_0; \bar{X}_1]$ then there has been $N_i - 1$ times where it has been optimal to prepay, each with probability $1 - \mathrm{e}^{-\frac{1}{n}\lambda}$. Hence, the probability of not having prepaid before time t_i must be $\mathrm{e}^{-\frac{N_i-1}{n}\lambda}$. If instead $X \in [\bar{X}_{h-1}; \bar{X}_h]$ then the probability of not having prepaid before time t_i is $\mathrm{e}^{-\frac{N_i-h}{n}\lambda}$. The probability (4.3) therefore becomes

$$
\mathbb{P}(\{X \leq x\} \cap \Omega_{i,k,l}^{\delta}) = \sum_{h=1}^{N_i} \left[\Psi(\min(x,\bar{X}_h),\alpha,\beta) - \Psi(X_{h-1},\alpha,\beta)\right]^+ \mathrm{e}^{-\frac{N_i-h}{n}\lambda}. \qquad (4.4)
$$

Inserting (4.4) in the prepayment probability (4.2) and using $x = X_i^*$ we can establish the probability of prepaying in any given period as

$$
P(\delta_{i,k,l} = 1) = (1 - \mathrm{e}^{-\frac{1}{n}\lambda}) \sum_{h=1}^{N_i} \left[\Psi(\min(X_i^*,\bar{X}_h),\alpha,\beta) - \Psi(X_{h-1},\alpha,\beta)\right]^+ \mathrm{e}^{-\frac{N_i-h}{n}\lambda}. \qquad (4.5)
$$

Before establishing a set of moment restrictions we note that borrowers have the same unconditional probability of prepaying. Hence, we may drop the k subscript from the probabilities and define $p_{i,l} = \mathbb{P}(\delta_{i,k,l} = 1)$. If we define $\eta_{i,l}$ as the number of borrowers prepaying at time t_i, then the proportion of the remaining borrowers prepaying at any time t_i must be

$$w_{i,l} = \frac{\eta_{i,l}}{K_l - \sum_{h=1}^{i-1} \eta_{h,l}} = \frac{\bar{p}_{i,l}}{1 - \sum_{h=1}^{i-1} \bar{p}_{h,l}},$$

where $\bar{p}_{i,l} = \frac{\eta_{i,l}}{K_l}$ is the proportion of the initial pool prepaying at time t_i and therefore also the sample counterpart to the prepayment probability in equation (4.5). We can therefore relax the assumption that we observe each and every borrower and simply compute moment restriction in terms of the prepayments. The theoretical first order moment for the prepayment rate $w_{i,l}$ will be given by

$$\mathbb{E}[w_{i,l}] = \frac{p_{i,l}}{1 - \sum_{h=1}^{i-1} p_{h,l}},$$

and hence we may define the function $f_{i,l} = f(w_{i,l}; \alpha, \beta, \lambda) = w_{i,l} - \mathbb{E}[w_{i,l}]$, which must have zero first order moment. Stanton chooses to take the average of $f_{i,l}$ over pools and use this average as the relevant moment in his GMM estimation. This is done since the data from the USA consists of a large number of small pools leading to an expectedly more stable estimate for the theoretical moment. In the Danish data the exact opposite is true. The Danish data consists of rather few but often very large mortgage pools. Hence, the prepayment rates announced from mortgage institutions are expected to represent rather accurate estimates of the theoretical moments.[5] Instead of taking the average over pools we will instead make use of all available prepayment rates, both over time and pools, to establish a vector of moment restrictions as follows:

$$\mathbf{F}(\mathbf{\Theta}) = \begin{bmatrix} f_{1,1} & f_{2,1} & \cdots & f_{i,l} & \cdots & f_{I-1,L} & f_{I,L} \end{bmatrix}',$$

where $\mathbf{\Theta} = [\alpha \quad \beta \quad \lambda]'$. The idea of GMM is now to estimate the parameters by minimising a quadratic form given some weighting matrix $\mathbf{W}$ as follows:

$$\hat{\mathbf{\Theta}} = \arg\min_{\mathbf{\Theta}} \mathbf{F}(\mathbf{\Theta})'\mathbf{W}\mathbf{F}(\mathbf{\Theta}). \tag{4.6}$$

The weighting matrix is used to allocate more weight to important moments. One could experiment with allocating more weight to moments where the prepayment rates stem from large pools since these prepayment rates expectedly will contain more information and less variation. Alternatively, one can specify an initial matrix $\mathbf{W}_0$ and estimate $\hat{\mathbf{\Theta}}_0$ given this matrix in a first step. From the estimate $\hat{\mathbf{\Theta}}_0$ we can then estimate a new weight matrix $\mathbf{W}_1$ in a second step. The optimal choice of the weight matrix turns out to be the inverse of the asymptotic covariance matrix of $\mathbf{F}(\mathbf{\Theta})$, see e.g. Hamilton (1994) [11], which

5 This is supported by looking at prepayments across pools for bonds with equal characteristics. These bonds will show highly similar prepayment behaviour when the pools are large.

is exactly the same as allocating the most weight to the moments with the least variation. An estimate for the asymptotic covariance matrix could be the sample counterpart, but we cannot compute this sample covariance matrix as we do not observe the individual borrowers prepayment decisions. To see this, assume that we observe a single pool in which we also observe each borrower. Then we could compute the prediction error for each borrower as a vector $\mathbf{F}_k(\mathbf{\Theta}) = [f_{1,k}, f_{2,k}, \cdots, f_{I,k}]$ with $f_{i,k} = p_i - \delta_{i,k}$. The moment condition would then be

$$\mathbf{F}(\mathbf{\Theta}) = \frac{1}{K}\sum_{k=1}^{K} \mathbf{F}_k(\mathbf{\Theta}),$$

and the sample covariance matrix would be

$$\hat{\Sigma} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{F}_k(\mathbf{\Theta})\mathbf{F}_k(\mathbf{\Theta})',$$

Assuming now that a law of large numbers applies, such that $\hat{\mathbf{\Theta}} \xrightarrow{p} \mathbf{\Theta}$, and assuming that the central limit theorem applies, then it can be shown, see Hamilton (1994) [11], that

$$\sqrt{K}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{D} \mathcal{N}(0,V),$$

where V is the asymptotic variance. V can be estimated by

$$\hat{V} = (\hat{D}\hat{\Sigma}^{-1}\hat{D}')^{-1},$$

where $\hat{D}$ is the Jacobian of $\mathbf{F}(\mathbf{\Theta})$ evaluated in $\hat{\mathbf{\Theta}}$. Since each element of $\mathbf{F}(\mathbf{\Theta})$ is relatively simple, the Jacobian can be determined by finding each of the gradients $\nabla p_i(\mathbf{\Theta})$. We therefore have that

$$\hat{D} = \begin{bmatrix} \nabla p_1(\mathbf{\Theta})|_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}} \\ \vdots \\ \nabla p_I(\mathbf{\Theta})|_{\mathbf{\Theta}=\hat{\mathbf{\Theta}}} \end{bmatrix},$$

which can be approximated by finite differences. It is clear how we can calculate the Jacobian without being able to monitor each borrower, but the sample covariance matrix, $\hat{\Sigma}$, depends on the individual contributions as well as the size of the pool, K. It is for this reason, that we cannot compute the standard errors for our parameter estimates when turning to real data.

In order to investigate the performance of the GMM estimation procedure, we will perform a stylised simulation experiment. We will consider a single pool with $K =$ 10,000 borrowers and draw the K prepayment cost levels from the Beta(α, β) distribution with $\alpha = 2$ and $\beta = 5$. The intensity parameter for how often borrowers check for optimal prepayment will be set to $\lambda = 0.5$. We will enforce fluctuations in the critical level of optimal prepayment, X_i^*, by setting $X_i^* = \min(\max(Y_{t_i}, 0), 1)$ where $Y_0 = 0$, $\mathrm{d}Y_t = 0.0001\mathrm{d}W_t$ and W_t is a standard Brownian motion. The level of X_i^* would in reality, of course, be governed by the fluctuations in interest rates, so choosing X_i^* to have this representation merely facilitates an easy illustration of the linkage between the level of prepayment and the level of X_i^*. Time is measured in years and we look at a 10 year period. Estimating the model by GMM using equation (4.6) results in the parameter estimates of table 1. The table has been equipped with 95% confidence bands produced as described above since we know K in the experiment. The results show that we get rather close to the true parameters of the data generating process, while the true parameters are within the 95% confidence bands. Figure 10 shows the generated prepayments, the estimated prepayments and the critical level of prepayment costs. When the critical level of prepayment costs increase it becomes more and more favourable for borrowers to prepay and therefore prepayments increase. The figure also shows the fitted number of prepayments which track the actual prepayments perfectly. Hence, if a pool is large enough, then we should not worry too much about the accuracy of the parameter estimates, but rather question ourselves whether the model is correctly specified.

α	β	λ
1.9587	4.9770	0.4499
[1.9873; 2.0127]	[4.8955; 5.1045]	[0.4852; 0.5148]

Table 1: GMM estimates for the simulated data. 95% confidence bands in parentheses.

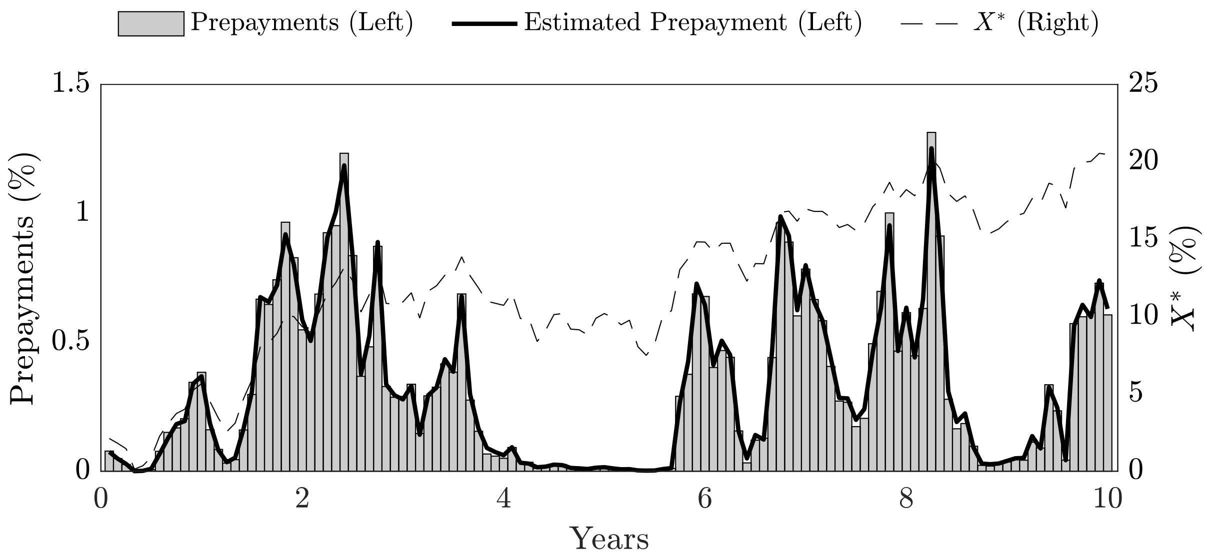

Figure 10: Simulated monthly prepayments measured in percent of outstanding amount, where prepayment costs are drawn from the Beta(2,5)-distribution. The prepayment intensity is set to $\lambda = 0.5$ and 10,000 borrowers are initially in the pool.

4.3 Implementation and Programmatic Architecture

We have now seen that the estimation procedure performs remarkably well under the simulated data. However, when taking the model to the market data, the estimation becomes increasingly more complex, and to see this it is worth thinking of the amount of calculations that the optimiser will be going through when iteratively evaluating the quadratic form (4.6). Figure 11 illustrate the complexity of the estimation problem. The optimiser will send an initial guess to the quadratic form that will compute the theoretical prepayment rates. In order to compute the theoretical prepayment rates, we have to find X_i^* for all relevant time points t_i by inverting the pricing function using a bisection algorithm. Each time we make a call to the pricing function from the bisection algorithm, we will be solving a new PDE by Crank-Nicolson. Also, in order to use the finite difference machinery, we have to calibrate the short rate model to the current term structure of interest rates as well as quoted cap prices. The bisection algorithm will converge quickly but still increase the computational burden as it requires us to evaluate the pricing function multiple times. Calculating X^* for all time points and all pools therefore results in some heavy calculations. When all the values of X^* are known, then we may finally calculate the model prepayments given the parameters α and β of the Beta-distribution.

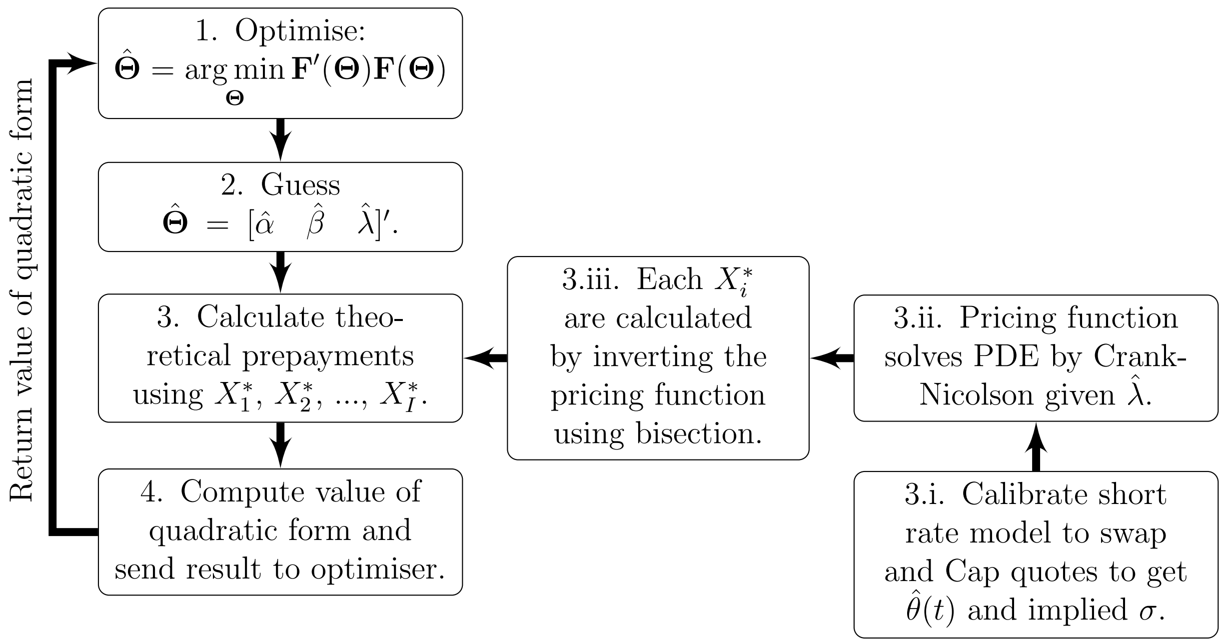

Figure 11: Flow chart illustrating the difficulties of having the PDE pricing mechanism inside the estimation procedure. Each time the optimiser changes $\hat{\lambda}$, there will be calculated new values for each X_i^*, which is a heavy process.

As should be evident from the above, changing any parameter value will result in a tremendous workload since the optimiser has to recalculate all of the above each time it changes a parameter. In order to bring down the computational burden to a manageable level, it has been of fundamental importance to make use of object oriented programming. In order to easily handle the market data a class called `market` has been built. `market` contains all relevant market data; curves, prices, prepayment rates and implied volatilities for every day. Secondly, a class called `cStantonHW` for pricing in the extended Stanton model has been built. `cStantonHW` is available through appendix A.7, but two important features are worth noting here. First, `cStantonHW` has been made a handle class. This means that when curves have been calibrated and finite difference matrices have been calculated, then we can easily reference these each time we perform calculations for a new bond without recalculating everything. Second, `cStantonHW` has been equipped with a public method called `cStantonHW.prepare`. This method takes care of all possible one time costs associated with pricing; calibrating, setting model parameters and calculating and inverting finite difference matrices. Evaluating the pricing function in `cStantonHW` takes approximately 1.5 seconds for the first evaluation while the second evaluation takes only 0.05 seconds, meaning that the initial costs associated with pricing takes up 1.45 seconds or 96.7% of the computational time. Since no of the parameters λ, α or β enters the finite difference matrices nor has anything to do with the calibrations, we can easily change these parameters and extract a new price in 0.05 seconds. However, this is still not sufficient. Though we can get prices in 0.05 seconds we still have to invert the liability function to get X^* across time and pools. The bisection technique has been set to take a maximum of 15 evaluation, which produces a highly accurate value for X^*. For a ten year data set with quarterly prepayment rates, this results in a maximum of 600 evaluations or equivalently 30 seconds per mortgage pool. This is not an appropriate evaluation time to have inside an optimiser. Instead we will bring the computation of X^* outside the optimiser. Since X^* is a function of λ, which the optimiser iterates over, we will calculate X^* for a discrete, but reasonable, set of values for λ. Consequently, we obtain a data matrix of values for X^* with both a time direction and a λ-direction. When the optimiser then changes the value of λ, it should simply interpolate using this data matrix. The improved estimation procedure is illustrated in figure 12.

Implementing the estimation procedure in line with the above considerations and estimating the model based on a single 5% 30 year mortgage results, as expected, in a slow one time cost of setting up the optimisation problem, but a fast optimisation itself. The solver converges in 0.14 seconds and the estimated parameters are $\hat{\alpha} = 0.3376$, $\hat{\beta} = 3.6704$ and $\hat{\lambda} = 0.5656$. In figure 13 the actual prepayments as well as model prepayments are presented. From the figure it is clear, that the model predicts increasing prepayments in times of low interest rates and vice versa. The general interest rate level, represented by the 10 year swap rate, and the predicted prepayment rates are highly linked, whereas the actual prepayment rates may seem just slightly lagged. This may be due to the fact that the model evaluates the mortgage liabilities on the exact date of the prepayment while borrowers will in reality evaluate and decide on their prepayment decisions during

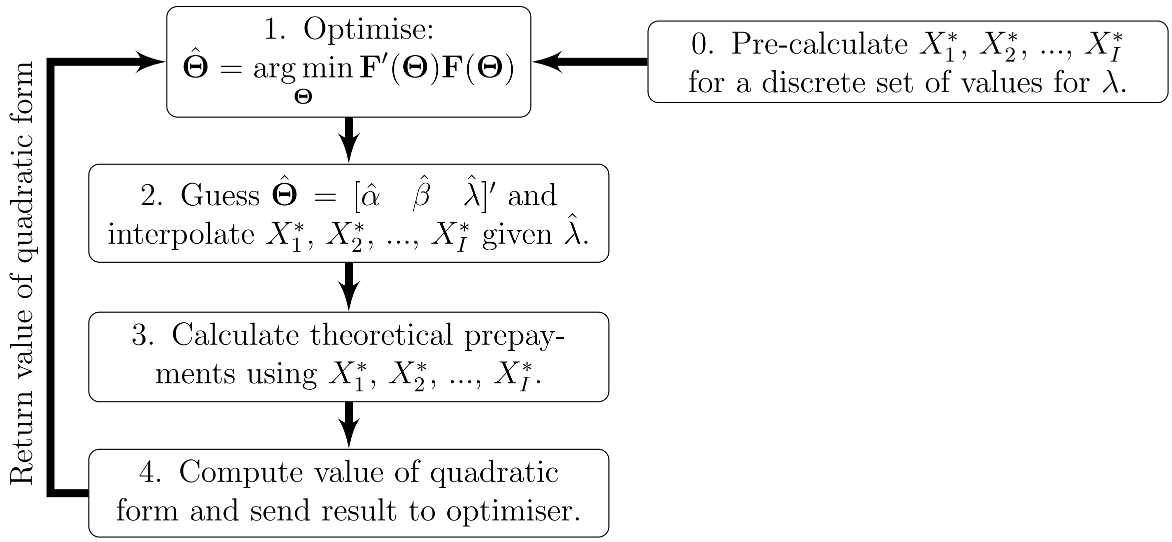

Figure 12: Flow chart illustrating the improved estimation procedure. Each time the optimiser changes $\hat{\lambda}$ there will be interpolated values for X_1^*, X_2^*, ..., X_I^* which is a quick process. Now the computational burden only depends on how fine a grid we use for λ and not the number of times the optimiser evaluates the objective function.

the months up to the prepayment date. Finally, it is worth noting that the model underestimates the prepayment rates during 2005 and 2006 while it overestimates during 2010 and 2011. The exact reason for the poor performance during these years is hard to determine. In both time periods the model predicts increasing prepayment rates in line with the falling interest rates. Hence, the model predictions are consistent with the theory, and to explain the deviations from the observed prepayment rates may require other explanatory factors independent of the interest rate level.

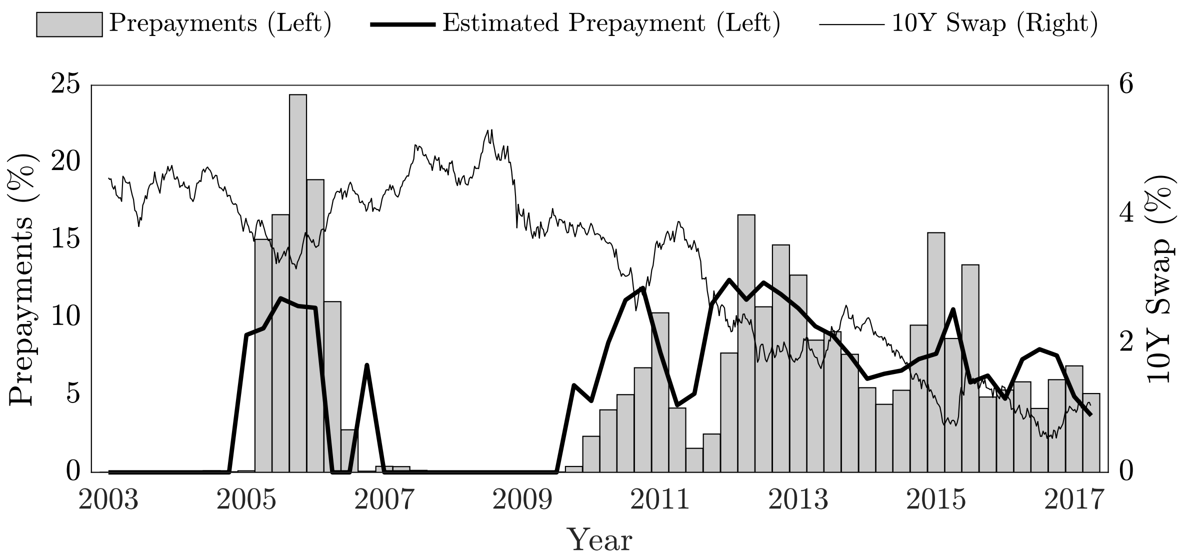

Figure 13: Actual and estimated prepayments measured in percent of outstanding amount for a 5% 30 year mortgage bond from Nykredit Realkredit (ISIN: DK0009753469) with maturity 2035. The estimated parameters are $\hat{\alpha} = 0.3376$, $\hat{\beta} = 3.6704$ and $\hat{\lambda} = 0.5656$.

We have now seen how the model performs for a single mortgage pool and the natural thing to do is to estimate the model for a larger sample. High-quality data is available over time for the Danish mortgage bonds, so we will estimate the model across 84 different bonds, which are presented in table 2. The bonds chosen are all 30 year amortising bonds maturing between year 2024 and 2044, and the prepayment data are available from year 1996 to 2017.

	Coupon					
	8%	7%	6%	5%	4%	Total
Nykredit Realkredit	4	5	7	6	4	26
Nordea Kredit	0	2	3	5	4	14
BRFkredit	3	2	7	5	4	21
Realkredit Danmark	2	4	7	6	4	23

Table 2: Number of bonds used in estimation sorted by mortgage institution and coupon.

Having calculated all the X_i^*'s across time and pools, the relatively large estimation procedure converges in only 2 minutes and 12 seconds resulting in the parameter estimates $\hat{\alpha} = 0.0612$, $\hat{\beta} = 2.5411$ and $\hat{\lambda} = 0.5773$. The parameters of the Beta-distribution indicate that the average prepayment cost is approximately 2.6% while a prepayment intensity of $\hat{\lambda} = 0.5773$ corresponds to a probability of 13.4% that a borrower will check for optimal prepayment within a quarter. Both of these numbers seem reasonable. The prepayment estimation errors for each quarter are presented in figure 14 below. From the figure we can see, admittedly, that there is rather large variation in the prediction errors. The average prepayment error is 0.29% and the skewness of these errors is 2.05, indicating that the model is right on average and that the largest errors occur when prepayments are very large.

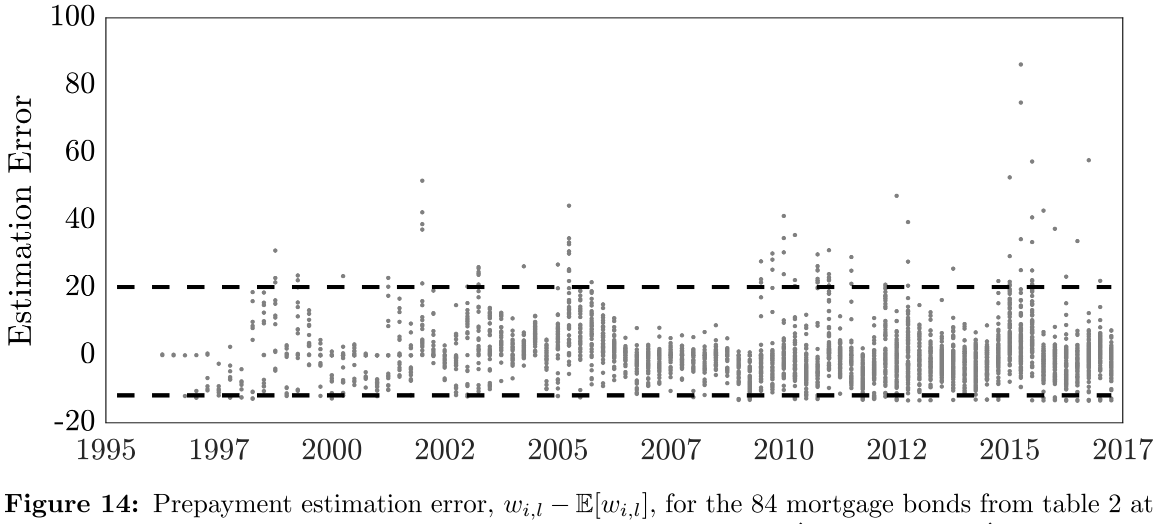

Figure 14: Prepayment estimation error, $w_{i,l} - \mathbb{E}[w_{i,l}]$, for the 84 mortgage bonds from table 2 at each quarter in the sample. The estimated parameters are $\hat{\alpha} = 0.0612$, $\hat{\beta} = 2.2511$ and $\hat{\lambda} = 0.5773$. 95% of the prepayment errors are within the dashed lines corresponding to the range [-11.9; 20.2]

Another way to investigate the models accuracy is to estimate the model parameters for half of the sample and use these parameters to predict the second half of the sample. Though our sample stretches from year 1996 to 2017, we will divide the sample as of the 1st of January 2011, since at this point in time we have approximately 50% of the available prepayment rates in each subsample. The estimated parameters are now $\hat{\alpha} = 0.1776$, $\hat{\beta} = 4.3242$ and $\hat{\lambda} = 0.6997$, which is not very far from the full sample estimation in terms of the implied borrower distribution. Figure 15 below illustrates the averaged observed prepayment rates for the whole sample as well as the averaged fitted prepayment rates. The prepayment rates for the second part of the sample has been estimated by the parameters of the first sample indicated by the dotted line. From the fitted values it is clear that the model captures the overall movements in prepayments, and the previously mentioned skewness in prepayment errors can bee seen from the underestimations during periods of very high prepayments. Finally, the out of sample estimates for the second half of the sample performs very good as the model still manages to capture the overall prepayment tendencies after 2011.

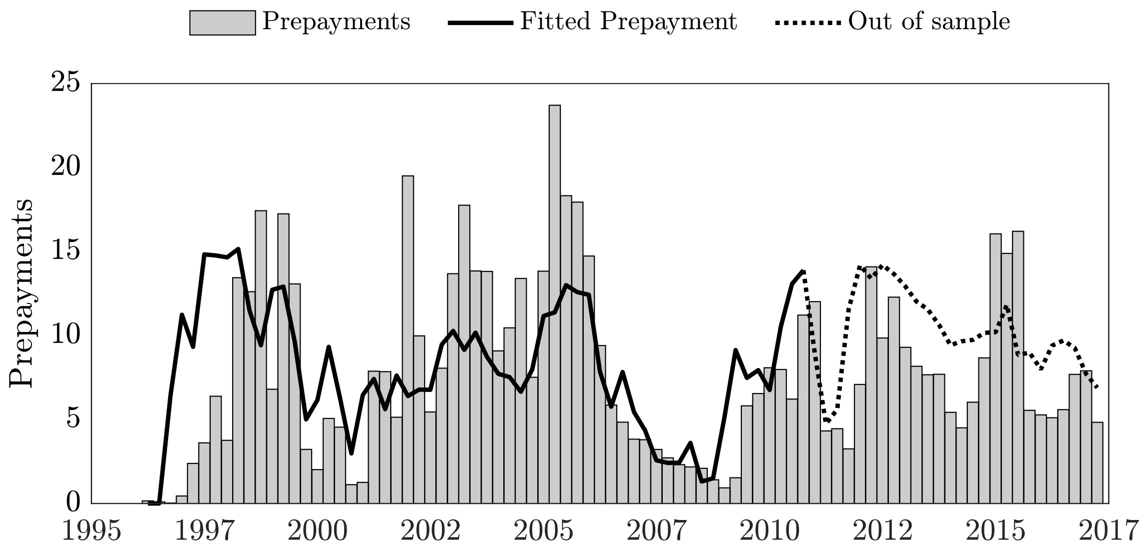

Figure 15: Actual and estimated prepayments measured in percent of outstanding amount averaged across the 84 mortgage bonds from table 2. The estimated parameters for the split sample are $\hat{\alpha} = 0.1776$, $\hat{\beta} = 4.3242$ and $\hat{\lambda} = 0.6997$. The out of sample estimated prepayments use these parameters.

4.4 Implied Borrower Distribution

Until this point every bond price we have calculated has been taking a fixed prepayment cost level X as an input, as if each bond were linked to one single borrower. Given that we have now estimated the parameters of the distribution of X it will be natural to calculate bond prices by taking into account the time-varying distribution of prepayment costs. In order to calculate the conditional distribution of the borrowers' prepayment costs, we have to compute the probability $\mathbb{P}(X \leq x|\Omega^{\delta}_{i,k,l}) = \frac{1}{\mathbb{P}(\Omega^{\delta}_{i,k,l})}\mathbb{P}(\{X \leq x\} \cap \Omega^{\delta}_{i,k,l})$. The second part of this probability is known from equation (4.4), but we still need to compute the probability entering the denominator of the fraction. Note that we can rewrite the event $\Omega^{\delta}_{i,k,l}$ as follows

$$\begin{aligned}\Omega^{\delta}_{i,k,l} &= \{\delta_{i-1,k,l} = 0\} \cap \{\delta_{i-2,k,l} = 0\} \cap \cdots \cap \{\delta_{1,k,l} = 0\} \\ &= \{\{\delta_{i-1,k,l} = 1\} \cup \{\delta_{i-2,k,l} = 1\} \cup \cdots \cup \{\delta_{1,k,l} = 1\}\}^{C},\end{aligned}$$

where the superscript C denotes the complement. Then we get

$$\mathbb{P}(\Omega^{\delta}_{i,k,l}) = 1 - \sum_{m=1}^{i-1} p_{m,l},$$

and combining this with equation (4.4), the borrower distribution must be

$$\mathbb{P}(X \leq x|\Omega^{\delta}_{i,k,l}) = \frac{1}{1 - \sum_{m=1}^{i-1} p_{m,l}} \sum_{h=1}^{N} \left[\Psi(\min(x,\bar{X}_h); \alpha, \beta) - \Psi(\bar{X}_{h-1}; \alpha, \beta)\right]^{+} \mathrm{e}^{-\frac{N-h}{n}\lambda}. \tag{4.7}$$

To find the density function, we note that only one term in the summation of equation (4.7) contains x. If $x \in [\bar{X}_{h-1},\bar{X}_h]$ then we can differentiate equation (4.7) to find the density, $f^X_t(x)$, of the borrower distribution at time t:

$$f^X_t(x) = \frac{1}{1 - \sum_{m=1}^{i-1} p_{m,l}} \psi(x; \alpha, \beta)\mathrm{e}^{-\frac{N-h}{n}\lambda}, \quad x \in [\bar{X}_{h-1}; \bar{X}_h], \quad t \in [t_i; t_{i+1}), \tag{4.8}$$

where $\psi(x; \alpha, \beta)$ denotes the density function of the Beta distribution. Note that the density is unchanged in-between prepayment dates. Figure 16 shows the initial as well as final distribution of borrowers' prepayment costs in the simulation experiment from section 4.2. Recall from figure 10 that X^* mainly stayed below 20% meaning that it has never been optimal to prepay for all the borrowers with prepayment costs greater than 20%. From figure 16 it is clear how the shape of the density is intact beyond 20% while very little probability mass is left for prepayment costs below 20%. Since more and more probability mass moves to larger values of X the value of the mortgage bond should increase as the borrowers left in the pool are expected to have relatively high levels of prepayment costs. That the model exhibits this feature is nice since it replicates what is typically known as the *burnout effect*. The burnout effect is the effect of increasing bond prices due to borrowers not prepaying when there, from a purely rational perspective, has been optimal prepayment possibilities several times. This is exactly the phenomenon we saw in figure 8.

When we turn to the actual data then we do not observe the individual borrowers, but we can still plot the analytical densities over time for the estimated parameters. Figure

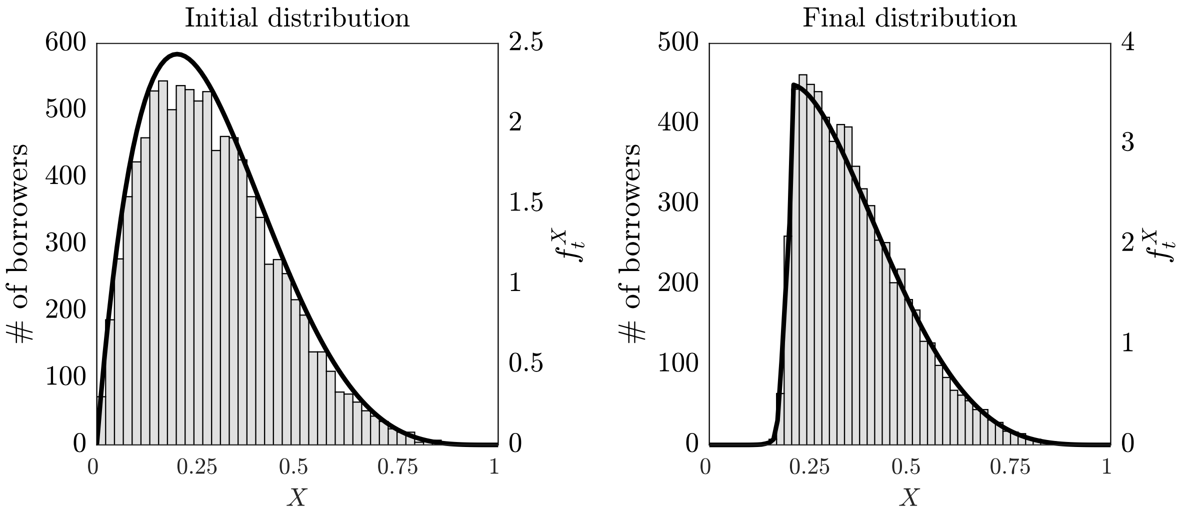

Figure 16: Initial and final distributions of the transaction cost level from the simulation experiment in section 4.2. The bars indicate the number of borrowers (Left) and the solid black line indicates the analytical density (Right) evaluated in the estimated parameters.

17 illustrates the time-varying density of the borrower distribution for the same mortgage bond as in figure 13. The density as of the 1st of January 2004 shows the initial distribution as no prepayments has yet occurred. A total of 90% of the borrowers are estimated to have prepayment costs less than 26%, which does not seem that unreasonable. When reaching the 1st of January 2016, 90% of the borrowers are estimated to have prepayment costs less than 65%. The pool is therefore estimated to consist of borrowers that are rather slow at prepaying in the sense that they require a relatively large gain in order to perform a prepayment.

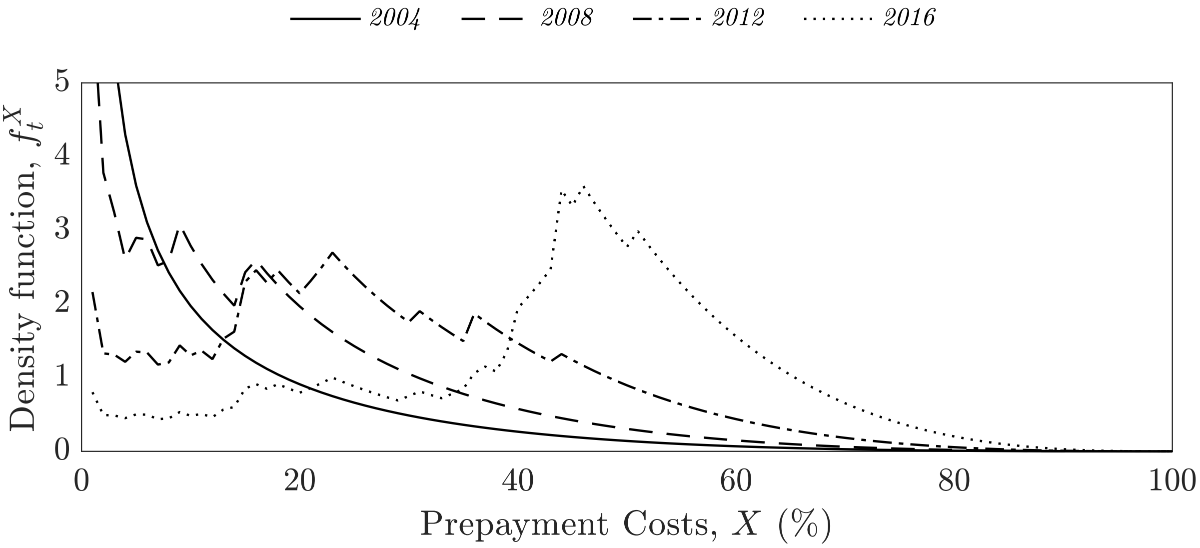

Figure 17: Estimated borrower distributions for a 5% 30 year mortgage bond from Nykredit Realkredit (ISIN: DK0009753469) with maturity 2035. The densities are computed as of the 1st of January for different years illustrating how the location of the probability mass changes over time as prepayments occur.

5 Key Figures

Through sections 2, 3 and 4, we have now established a model for pricing of callable mortgage bonds. From the viewpoint of an investor, the price is of course far from the only interesting figure to monitor. In order to assess whether a particular mortgage bond is an interesting investment opportunity, it will be necessary to identify the bond's risk and return profile. Within fixed income portfolio management the most common way to evaluate the risk and return profile is to calculate the duration and convexity of the bond and compare these to either the expected return or the swap spread. In the following we will go through the relevant key figures in the context of callable mortgage bonds.

5.1 Model Prices

In order to put a price on a bond linked to a whole pool rather than just a single mortgage loan, we will perceive the mortgage pool as a portfolio of bonds, where each bond is linked to one single borrower with fixed prepayment cost X. The value of such a portfolio will simply be the weighted sum of the individual bonds, with the weights being the amount outstanding for each type of bond. As we do not know the amount outstanding for each of these theoretical bonds, we will instead use the implied borrower distribution known from equation (4.7). The value of the bond therefore becomes

$$M_t = \mathbb{E}_t^{\mathbb{P}}[M_t^a(X)] = \int_0^1 M_t^a(x) f_t^X(x) \mathrm{d}x. \tag{5.1}$$

Since equation (5.1) contains an integral over the function M_t^a, which will be evaluated by finite differences, we will have to approximate the integral by a finite sum. If we partition the interval $[0;1]$ into N ranges each of size $\frac{1}{N}$, then a reasonable approximation will be

$$M_t \approx \sum_{i=1}^{N} M_t^a \left(\frac{2i-1}{2N} \right) \mathbb{P} \left(\frac{i-1}{N} \leq X \leq \frac{i}{N} \right), \tag{5.2}$$

where the probability is easily calculated in the borrower distribution from equation (4.7). The weighted price from equation (5.2) has been calculated over time for a 30 year 6% coupon bond from BRFkredit maturing the 1st of October 2026. The resulting model prices as well as the observed market prices are presented in figure 18. The actual and estimated prepayments are also presented in the figure. From the figure it is clear how the model tracks the actual bond price very close until 2008, where the model price starts to deviate from the market price. The first thing to note is that the model should not fit the observed prices perfectly unless everybody in the market are using the exact same prepayment model with the exact same parameters to price mortgage bonds. However, we should expect the model to perform very well in the beginning of the life of the bond, since at this point in time the bond trades below par and no noteworthy prepayments have occurred. This means that the prepayment option is out of the money and the difference between a callable and a non-callable bond will be modest. Since all banks will use models calibrated to the same market swap quotes, the non-callable prices should be close. That the prices start to deviate in the years after 2008 may be due to the highly model dependent burnout effect. As of the 1st of January 2008 there was only 8% of the initial pool left. These borrowers will be in the right tail of the borrower distribution, which undoubtedly must be varying over models and therefore produce different prices.

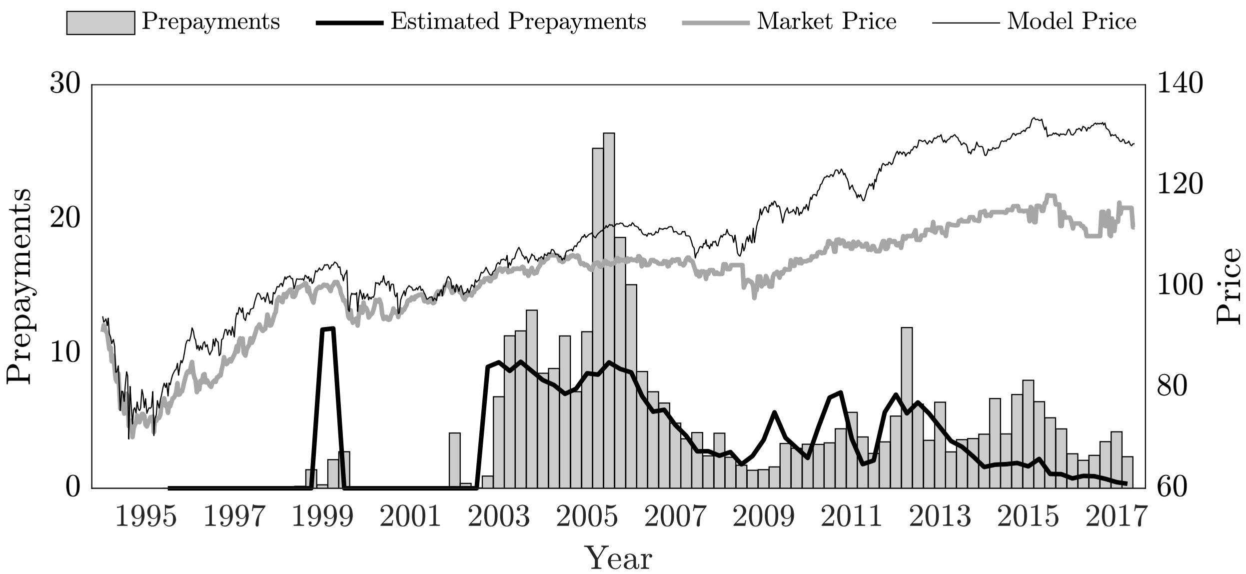

Figure 18: Model prices and market prices for a 6% 30 year mortgage bond from BRFkredit maturing the 1st of October 2026 (ISIN: DK0009334575). The parameter used are those of the full sample estimation and prepayments are measured in percent of outstanding amount.

In figure 19 below, the model prices have been shown for varying prepayment costs alongside the borrower distribution. The prices and the borrower distribution are calculated as of the 1st of June 2017, at which point the model and market prices deviate a lot. From the figure it is clear that our model price is highly sensitive towards the chosen distribution, since the right tail will be scaled up due to the previous periods' extensive prepayment activity. A lot of weight is therefore applied to the very high prices causing the model to overestimate the price in comparison with the market. Effectively, the model is converging towards pricing the bond as if it was a non-callable bond. This is of course not very satisfying and it is indeed a drawback of the models simple structure. The difference between model prices from market prices are often summarised by the so called *option adjusted spread* (OAS) to be defined below.

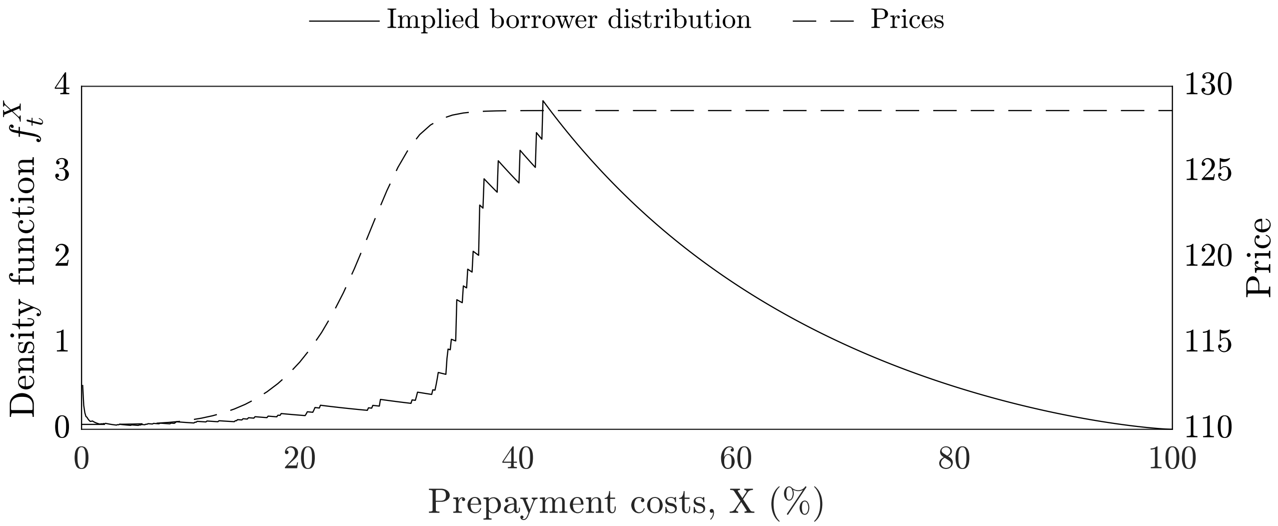

Figure 19: Bond prices for varying prepayment costs as of the 1st of June 2017 for a 6% 30 year mortgage bond from BRFkredit maturing the 1st of October 2026 (ISIN: DK0009334575). The implied borrower distribution illustrates which bond prices are allocated the most weight.

5.2 Option Adjusted Spread

The option adjusted spread is typically defined as the spread above the locally risk free rate ensuring that model prices, M_t^{Mdl}, and market prices, M_t^{Mkt}, equate. Formally this means, that one would have to solve for the spread, s, in the equation

$$M_t^{\text{Mdl}}(r_t + s) = M_t^{\text{Mkt}}(r_t). \tag{5.3}$$

However, nothing ensures that the model price in equation (5.3) is a monotonically decreasing function in the short rate. Hence, there will in general not be a unique solution for the OAS. When pricing is performed through Monte Carlo or binomial trees, the OAS is often assumed not to influence the future cash flows but only the discounting, in which case it is possible to define the OAS uniquely, see e.g Kupiec & Kah (1999) [22]. When pricing by finite differences, we are not performing any discounting; we are simply solve a PDE. However, as pointed out in Levin & Davidson (2005) [1] and He et. al (2013) [32], we can assume that the bond holder is compensated for the prepayment risk by adjusting the bond return to $r_t + s$ under $\mathbb{Q}$, that is

$$\mathrm{d}M_t^a = (r_t + s)M_t^a \mathrm{d}t + \beta_t^M M_t^a \mathrm{d}W_t^{\mathbb{Q}}.$$

Using these dynamics instead and the mortgage PDE will be

$$\frac{\partial M^a}{\partial t} + \kappa(\hat{\theta}(t) - r_t)\frac{\partial M^a}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 M^a}{\partial r^2} = (r_t + s)M_t^a. \tag{5.4}$$

That the spread now enters the PDE (5.4) is inconvenient, since we have to find the spread by inverting the pricing function numerically. For this reason we will have to compute new finite difference matrices each time s is changed. Luckily, we know from the Feynman-Kac theorem, that the solution to the above PDE for $t \in [t_{i-1}, t_i)$ must be

$$M_t^a = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{t_i}(r_u+s)\mathrm{d}u} M_{t_i}^a\right] = \mathrm{e}^{-(t_i-t)s}\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^{t_i} r_u \mathrm{d}u} M_{t_i}^a\right].$$

Since we can factor out the term including the spread, we can reuse our PDE solver as if there was no spread, if just we remember to change the boundaries to take the spread into account. That is, when we have solved the PDE by finite differences over the interval $[t_i, t_{i+1})$, then we will use the modified boundary condition as follows:

$$M_{t_i-}^a = \mathrm{e}^{-\frac{1}{n}\lambda_{t_i}}\mathrm{e}^{-\frac{1}{n}s}M_{t_i}^a + (1-\mathrm{e}^{-\frac{1}{n}\lambda_{t_i}})\min(F_{t_i}, \mathrm{e}^{-\frac{1}{n}s}M_{t_i}^a) + \bar{Y}.$$

Since the pricing function is now a monotonically decreasing function in s, the OAS has become a uniquely defined quantity.

At the end of the day, the OAS is just a transformation of the pricing error from the price domain into the short rate domain. The reason for performing such a transformation is, that the OAS is being interpreted by practitioners as an excess return over the short rate[6]. To understand if this is a reasonable interpretation or not, we should readdress the discussion of incompleteness in the mortgage bond market. If we decompose the callable bond into a non-callable bond and a call option, then the non-callable bond can be priced indisputably by static arbitrage arguments while the call option cannot since the prepayment risk is unhedgeable. The price formation of the call option will therefore be a consequence of the market's prepayment expectations rather than a result of arbitrage pricing. If we believe that our model is a perfect description of borrowers prepayment behaviour, then we may interpret the OAS as some kind of expected excess return. However, if the OAS is a consequence of our model being misspecified, then it is nothing but a pricing error. Whether the former or the latter is the most reasonable explanation is a question open for discussion, but the academic literature tends to conclude that the OAS is a result of a misspecified model, see e.g. Kupiec & Kah (1999) [22] or Dahl et. al (2009) [10] for more detailed discussions.

5.3 Duration and Convexity

The classic duration measures like Macaulay or Fisher-Weil duration are no longer appropriate measures when applied to callable bonds. Both measures assumes that the future cash flows are deterministic, but for the callable bond there is great uncertainty associated

6 Transforming the price deviation into the short rate domain corresponds to finding the yield to maturity for a non-callable bond as yields are often easier to compare than prices.

with the future cash flows. Instead we will use our pricing machinery to calculate the relevant sensitivities. Recall that by Ito's lemma, the evolution of the callable mortgage bond must be

$$\mathrm{d}M_t = \frac{\partial M_t}{\partial t}\mathrm{d}t + \frac{\partial M_t}{\partial r}\mathrm{d}r_t + \frac{1}{2}\frac{\partial^2 M_t}{\partial r^2}(\mathrm{d}r_t)^2. \tag{5.5}$$

So if we want to manage the risk of the bond, then it is natural calculate the derivatives in equation (5.5), since they fully describe the changes in M_t from changes in time and the short rate. Using the first derivative in the short rate, we will define the option adjusted duration (OAD) as

$$D(t, r_t) = -\frac{1}{100}\frac{\partial M_t}{\partial r}. \tag{5.6}$$

The OAD interprets as the loss occurring from a 1%-point increase in the short rate. This of course only holds locally, and for large changes in the short rate, the duration becomes misleading when the second order derivative is different from zero. For this reason it is custom to also manage the second order risk known as the option adjusted convexity (OAC). We will define the OAC as

$$K(t, r_t) = \frac{1}{100^2}\frac{\partial^2 M_t}{\partial r^2}. \tag{5.7}$$

The OAC can be rewritten in terms of the OAD, in which case we get

$$K(t, r_t) = -\frac{1}{100}\frac{\partial D(t, r_t)}{\partial r}. \tag{5.8}$$

We can now see that the OAC interprets as the fall in duration from a 1%-point increase in the short rate. As described in section 3.4, we can approximate the derivatives entering the duration and convexity by central and second order central differences:

$$D(t, r_t) \approx -\frac{1}{100}\frac{M(t, r_t + \Delta r) - M(t, r_t - \Delta r)}{2\Delta r} \tag{5.9}$$

$$K(t, r_t) \approx \frac{1}{100^2}\frac{M(t, r_t + \Delta r) - 2M(t, r_t) + M(t, r_t - \Delta r)}{(\Delta r)^2}. \tag{5.10}$$

When computing these risk measures, we are faced with a familiar problem in derivatives pricing. Derivatives are most often associated with non smooth payoff functions, which have a tendency to introduce oscillations in prices and risk sensitivities, when pricing is performed by finite differences. In the case of the callable mortgage bond, the non-smoothness is introduced each time we reach a prepayment date. At the prepayment dates we will calculate λ_t by equation (3.68), which we will restate below.

$$\lambda_{t_i} = \lambda \mathbb{1}\{M^{\ell}_{t_i} \geq (1+X)\min(F_{t_i}, M^{a}_{t_i})\}. \tag{5.11}$$

Since we will use λ_t to compute the expected value of the bond prices immediately after each prepayment date, the prices will not only become non-smooth, they will also be discontinuous. As has been evident from previous price graphs, the prices themselves have not been showing abnormal behaviour, but if we look at the sensitivities in figure 20, then these are suffer from oscillating behaviour.

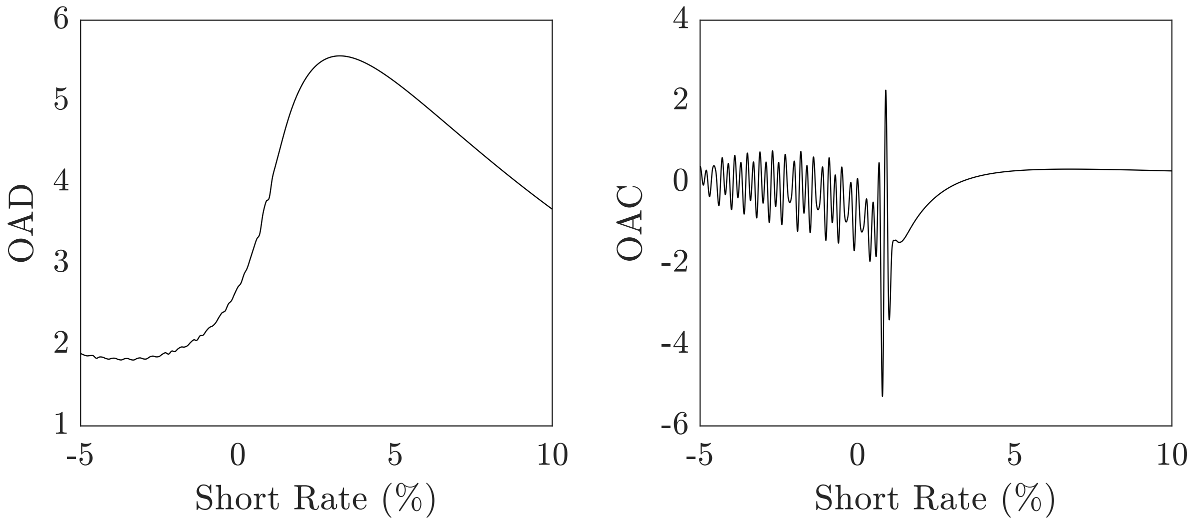

Figure 20: Option adjusted duration and convexity for a 30 year 2.5% mortgage bond calibrated to the market as of the 1st of June 2017. The number of time steps in the finite difference grid has been set to 12 per year corresponding to step sizes of one month each.

The oscillating behaviour in figure 20 has been widely studied and a commonly used solution to the problem is the so called *Rannacher time-stepping*. Rannacher (1984) [26] studied the oscillations and found that if the very first time-step in the Crank-Nicolson solver is replaced by two half time-steps of implicit schemes then it would be possible to recover the second order accuracy of the Crank-Nicolson scheme. The Rannacher time-stepping has been implemented in line with Giles (2010) [18] who analyses the problem in the Black-Scholes setting and Wadea et. al. (2007) [3] who made use of the method in pricing barrier options, where discontinuities occur repeatedly. In our case the discontinuities also occur repeatedly in each quarter, so we will instead take three steps in the time direction per quarter, and let the first step be two half time-steps with the implicit scheme. In figure 21 the risk measures have been illustrated again by use of the repeated Rannacher time-stepping. From the figure it is clear that the risk measures have become more stable than was the case without Rannacher time-stepping. Though heavily reduced, the OAC still exhibits oscillating behaviour.

As the source of the oscillations reoccur every quarter, and we use time-steps with the size of one month, one could try to increase the number of steps in the time direction

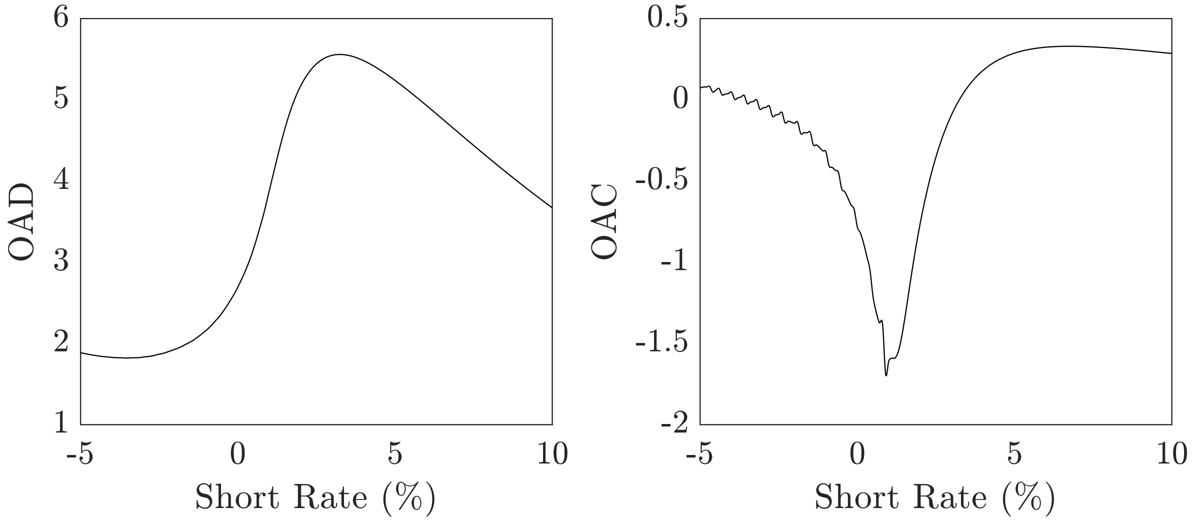

Figure 21: Option adjusted duration and convexity for a 30 year 2.5% mortgage bond calibrated to the market as of the 1st of June 2017. The number of time steps in the finite difference grid has been set to 12 per year corresponding to step sizes of one month each. The Rannacher time-stepping has been applied after each prepayment date.

in order to achieve higher accuracy. This improves the situation somewhat, but still leave us with oscillating effects as the discontinuities occur too frequently. An alternative approach is therefore to smooth the boundary conditions. The idea of smoothing the boundary conditions was used by Heston & Zhou (2000) [29] who replaced the boundary values with locally averaged values. Another example of use is Rein (2002) [17] who approximates the discontinuous Heaviside function by a smooth counterpart. The latter example is closely related to our present case, and inspired by Rein we will therefore approximate the indicator function in (3.68) by the following smoothed alternative.

$$G_\epsilon(x) = \begin{cases} 0 & \text{if} \quad x < -\epsilon \\ \frac{1}{2} + \frac{1}{2\epsilon}x + \frac{1}{2\pi}\sin\left(\frac{\pi}{\epsilon}x\right) & \text{if} \quad |x| \leq \epsilon \\ 1 & \text{if} \quad x > \epsilon \end{cases}, \tag{5.12}$$

where ϵ is a constant chosen sufficiently small to replicate the behaviour of the indicator function and sufficiently large to stay smooth when implemented in the code[7]. Replacing the indicator function in equation (3.68) with the smoothed function $G(x)$ we get that

$$\lambda_{t_i} \approx \lambda G_\epsilon\left(M^\ell_{t_i} - (1+X)\min(F_{t_i}, M^a_{t_i})\right). \tag{5.13}$$

Applying (5.13) in the finite difference solver results in perfectly smoothed risk figures as

[7] $G(x)$ has been implemented to take vector inputs and apply equation (5.12) to each element of the input vector. ϵ has been chosen as the difference between the smallest and the largest value of the input vector divided by five. Also $G(x)$ has only been applied locally to 50 neighbouring values at points where the indicator should change sign. This has been done to not influence regions where the liabilities are close to the prepayment amount.

can be seen from figure 22 below. Like a non-callable bond, the duration is low for high interest rates since the future cash flows are worth less in a high interest rate environment. Conversely, the OAD is low for very low interest rates due to the increased prepayment activity at low interest rates. The OAC is mainly negative except for very low and very high short rate levels. In the region of negative OAC, an increasing short rate will reduce the level of prepayments and cash flows are thereby pushed into the future, increasing the duration of the bond.

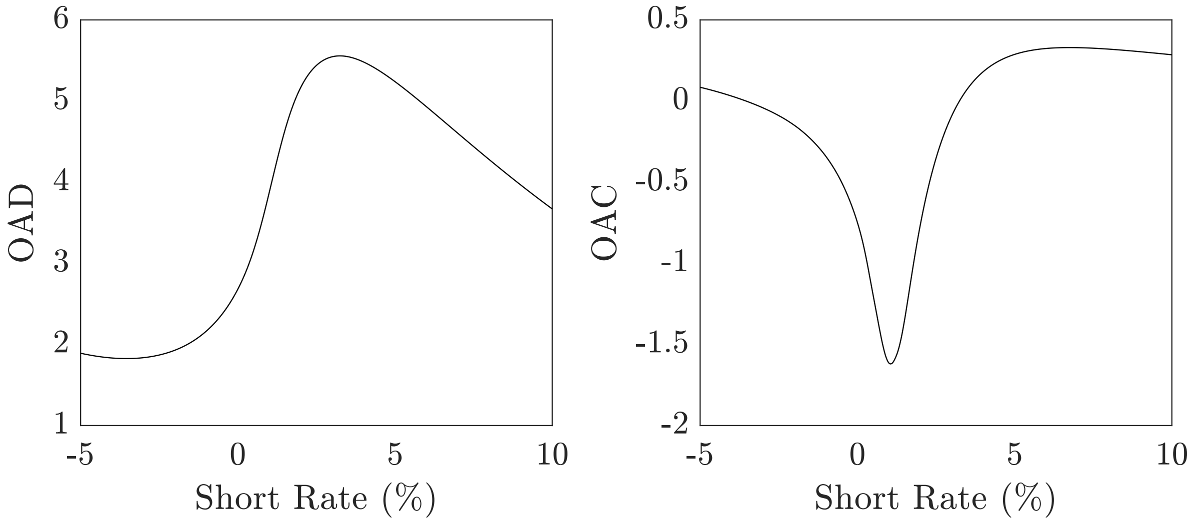

Figure 22: Option adjusted duration and convexity for a 30 year 2.5% mortgage bond calibrated to the market as of the 1st of June 2017. The number of time steps in the finite difference grid has been set to 12 per year corresponding to step sizes of one month each. Rannacher time-stepping has been applied in combination with the smoothing technique.

Back test

In order to test if the risk figures produced by the model are sensible, we will implement a backtest procedure. If we discretise equation (5.5) then the evolution of the callable mortgage bond can be approximated by the following equation

$$\Delta M_{t_i} = \vartheta(t_{i-1}, r_{t_{i-1}})\Delta t - D(t_{i-1}, r_{t_{i-1}})100\Delta r_{t_i} + \frac{1}{2}K(t_{i-1}, r_{t_{i-1}})(100\Delta r_{t_i})^2 + \Delta\varepsilon_i \quad (5.14)$$

where $\varepsilon_i = \sum_{j=1}^{i} \Delta\varepsilon_i$ is the accumulated errors from the approximation and $\vartheta(t, r_t)$ denotes the time-value from holding the bond. The time-value is found by finite differences as

$$\vartheta(t, r) = \frac{M(t + \Delta t, r_t) - M(t, r_t)}{\Delta t}. \quad (5.15)$$

If the risk figures produced by our model are a good description of the mortgage bonds sensitivities towards changes in time and the short rate, then we would expect the accumulated errors of equation (5.14) to stay reasonably low. If we use the six month CIBOR rate as our proxy for the short rate in equation (5.14) and approximate the evolution of the 6% 30 year mortgage bond from section 5.1, then we get the results in figure 23. The figure reveals the rather devastating result, that the hedge series, which is the time series obtained from the duration and convexity approximation, does a rather poor job at describing the price evolution. There seems to be two possible sources of this bad performance; either our model is producing bad risk figures over time or the correctness of our fundamental assumption, that the short rate is the driving factor for all interest rate dependent securities, is dubious. In the below we will question the latter.

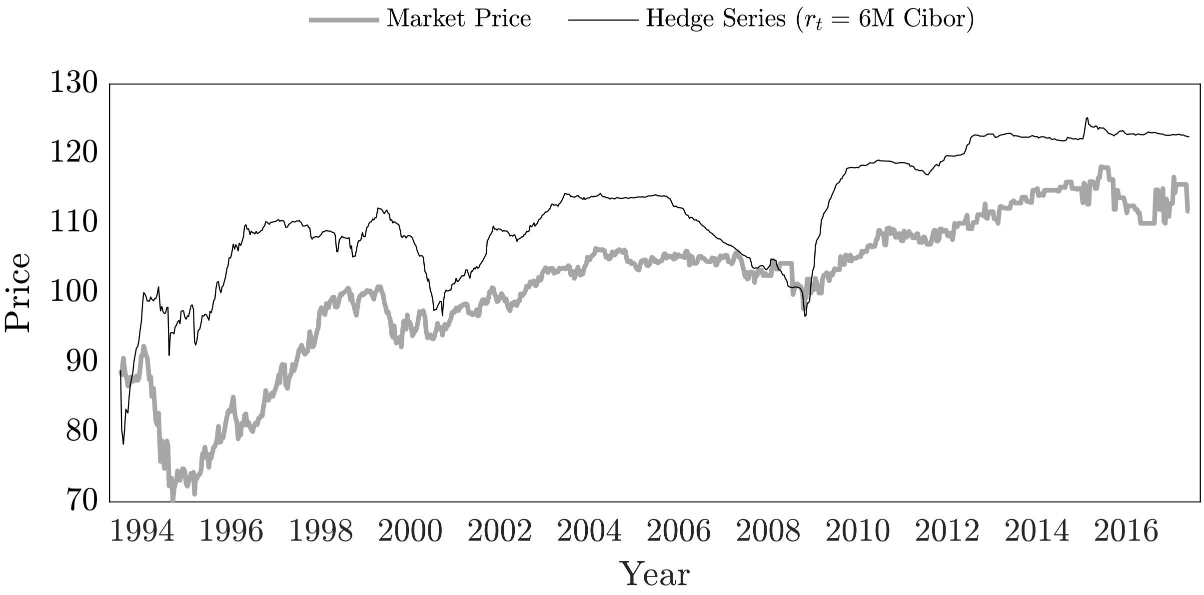

Figure 23: Market price and accumulated price evolution from the duration and convexity approximation in equation (5.14) for a 6% 20 year mortgage bond from BRFkredit maturing the 1st of October 2026 (ISIN: DK0009334575). The short rate is approximated by the six month CIBOR rate.

In our one-factor framework it is assumed that one single factor determines the shape of the yield curve implying perfect correlation across all yields on the curve. This is of course a strict assumption, and the reason for the assumption is, that keeping the model simple results in an elegant pricing framework. One typical way to examine the number of factors influencing the yield curve is by means of so called *Principal Component Analysis* (PCA). Let $\mathbf{y}_t$ be a vector of yields for different maturities at time t. Then we can define $\mathbf{\Sigma} = \text{Cov}(\Delta\mathbf{y}_t)$ as the covariance matrix of the changes in yields. PCA then simply uses that $\mathbf{\Sigma}$ must be positive semi-definite, in which case we can perform a spectral decomposition

$$\mathbf{\Sigma} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1},$$

where $\mathbf{V}$ is a matrix with eigenvectors as columns and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues along the diagonal. Multiplying any eigenvector onto $\mathbf{y}_t$ will produce a *factor* that will be uncorrelated with all other factors. Let $x_i = \mathbf{y}_t' v_i$ and $x_j = \mathbf{y}_t' v_j$ be the i'th and j'th factors where v_i and v_j are the i'th and j'th eigenvectors, then these factors will have zero covariance as the eigenvectors are orthogonal. Let $y_{i,t}$ be the i^{th} element of $\mathbf{y}_t$, then the total variance of the system must be

$$\sum_{i=1}^{n} \mathbb{V}(\Delta y_{i,t}) = \text{tr}(\mathbf{\Sigma}) = \text{tr}(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}) = \text{tr}(\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}) = \sum_{i=1}^{n} \lambda_i,$$

where λ_i is the i'th eigenvalue. The idea of PCA is now to determine which factors are primarily driving the variation in the system. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be ordered in descending order. Then the proportion of the total variation described by the j'th factor must be given by the quantity $\frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$. If the first $k \leq N$ factors describe the majority of the variation, then this might be a sign that only k factors are needed to explain the system.

As we have already computed the zero coupon yield curve for each day back to 1994, we will perform the PCA analysis on the monthly changes for different maturities. The results are presented in table 3. From the table it can be seen how only the first few factors account for the majority of the total variation in the yield curve. However, the first factor account for as much as 81.6% of the variation, which is in favour of our choice of a one-factor model.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Eigenvalue	0.3952	0.0483	0.0269	0.009	0.0037	0.0007	0.0003	0.0002
Acc. Explained	81.6%	91.6%	97.1%	99.0%	99.8%	99.9%	100.0%	100.0%

Table 3: PCA applied to monthly changes in the Danish zero coupon yield curve bootstrapped from CIBOR and swap quotes from 1994 to 2017. Eight maturities were used (1Y, 2Y, 3Y, 5Y, 10Y, 15Y, 20Y & 30Y). The eigenvalues as well as their accumulated explanatory power have been shown.

We have now seen that the assumption of the yield curve movements being described by a single factor is not too harsh an assumption. However, if we can identify a point on the yield curve that is more correlated with the bond, then we might be able to choose a more proper factor. Table 4 below illustrates the correlation matrix for changes in different points on the yield curve as well as the previously used bond from BRFkredit. From the table we see that the short end of the curve has the least correlation with the bond while the five and 10 year yields are more negatively correlated. There might therefore be chances that a yield with a longer maturity will perform better when chosen as explanatory factor. Say we instead choose to use the 10 year yield, then to find the dynamics of y_t^T we recall from equation (2.24) that

$$y_t^T = -\frac{1}{T-t} \ln \bar{P}_t^T.$$

Inserting the affine form $\bar{P}_t^T = \mathrm{e}^{-A(t,T)-B(t,T)r_t}$ for the zero coupon bond we get

$$y_t^T = \frac{1}{T-t}\left[A(t,T) + B(t,T)r_t\right],$$

where $A(t,T)$ and $B(t,T)$ are given by equations (3.53)-(3.54). An application of Ito's lemma then yields

$$\mathrm{d}y_t^T = \frac{\partial y_t^T}{\partial t}\mathrm{d}t + \frac{\partial y_t^T}{\partial r}\mathrm{d}r_t + \frac{1}{2}\frac{\partial^2 y_t^T}{\partial r^2}(\mathrm{d}r_t)^2.$$

Noting that the second derivative in r_t is zero, inserting for the first derivative in r_t and manipulating the time-derivative, yields the relation

$$\mathrm{d}r_t = \frac{T-t}{B(t,T)}\left[\mathrm{d}y_t^T + \frac{\partial y_t^T}{\partial(T-t)}\mathrm{d}t\right]. \tag{5.16}$$

In equation (5.16) the $B(t,T)$-function is known from equation (3.53), $\mathrm{d}y_t^T$ can be approximated from the discrete changes in the 10 year yield, and finally the derivative in time to maturity can be found as the slope of the yield curve.

	1Y	2Y	3Y	5Y	10Y	15Y	20Y	30Y	Bond
1Y	1.000	0.874	0.734	0.628	0.506	0.677	0.709	0.447	-0.384
2Y	0.874	1.000	0.959	0.885	0.742	0.787	0.775	0.688	-0.499
3Y	0.734	0.959	1.000	0.963	0.839	0.792	0.747	0.789	-0.547
5Y	0.628	0.885	0.963	1.000	0.934	0.842	0.765	0.835	-0.572
10Y	0.506	0.742	0.839	0.934	1.000	0.895	0.795	0.873	-0.555
15Y	0.677	0.787	0.792	0.842	0.895	1.000	0.975	0.849	-0.485
20Y	0.709	0.775	0.747	0.765	0.795	0.975	1.000	0.826	-0.426
30Y	0.447	0.688	0.789	0.835	0.873	0.849	0.826	1.000	-0.487
Bond	-0.384	-0.499	-0.547	-0.572	-0.555	-0.485	-0.426	-0.487	1.000

Table 4: Correlation matrix for the monthly changes in the Danish zero coupon yield curve as well as a 6% 30 year mortgage bond from BRFkredit (ISIN: DK0009334575) for the period 1994 to 2017.

Discretising equation (5.16) as described above, we get

$$\Delta r_{t_i} \approx \frac{10}{B(t_i, t_i+10)}\left[\Delta y_{t_i}^{t_i+10} + \frac{y_{t_i}^{t_i+15} - y_{t_i}^{t_i+5}}{10}\Delta t\right], \tag{5.17}$$

where we have used the five and 15 year yields to approximate the slope at the 10 year yield. Equation (5.17) may then be calculated over time, and the obtained short rate changes, Δr_{t_i}, may subsequently be used in equation (5.14). It is clear, that whether we describe the dynamics of the mortgage bond price in terms of the short rate, or any other point on the yield curve, is equivalent within our theoretical framework. Nonetheless, this seems not to be the case in practise. Figure 24 below illustrates the dynamics for the approximation in equation (5.14) corrected to take the dynamics of y_t^{t+10} as an input. The hedge series, which is generated by the exact same duration and convexity measures as previously used, now replicates the market price dynamics much better. In theory we should not obtain this result but in practise we do. The reason for this finding may be due to our extensive use of the forward curve in the pricing process. Recall that to match the initial term structure the function $\hat{\theta}(t)$ from the extended Vasicek model had to be specified according to equation (3.55), which is restated below:

$$\hat{\theta}(t) = \bar{f}(0,t) + \frac{1}{\kappa}\frac{\mathrm{d}\bar{f}(0,t)}{\mathrm{d}t} + \frac{\sigma^2}{2\kappa^2}(1 - \mathrm{e}^{-2\kappa t}).$$

Since we make use of $\hat{\theta}(t)$ for all $t \in [0;T]$, where T is the maturity of the bond, the price will be a function of the whole forward curve. By manipulating relation (2.26) we can easily show that

$$y_t^T = \frac{1}{T-t}\int_t^T f(t,s)\mathrm{d}s.$$

That is, the yield for any point of the curve will be an average of all the instantaneous forward rates up to and including the maturity of the yield itself. This means that when we are using the 10 year yield as a driving factor then we are implicitly also including all the forward rates out to 10 year, which we have argued to enter the pricing function through $\hat{\theta}(t)$. Therefore, it does not seem too strange, that the 10 year yield is a better choice.

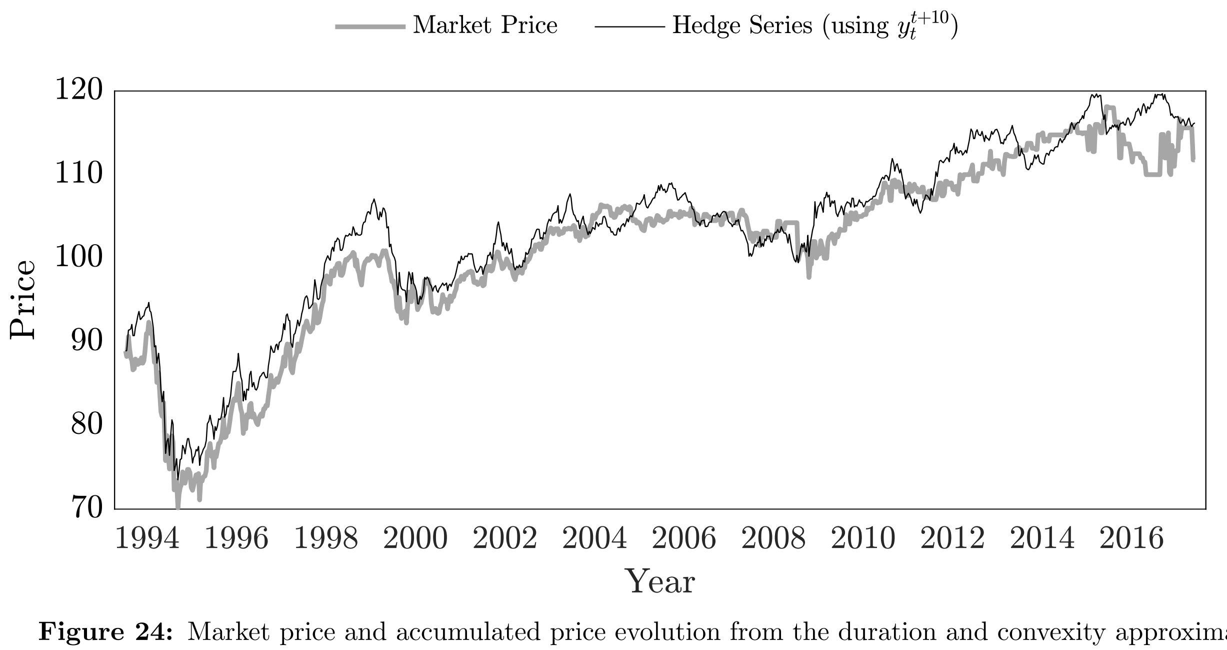

Figure 24: Market price and accumulated price evolution from the duration and convexity approximation in equation (5.14) for a 6% 30 year mortgage bond from BRFkredit maturing the 1st of October 2026 (ISIN: DK0009334575). The changes in the short rate has been corrected to take the dynamics of the 10 year yield as an input.

5.4 Expected Returns

Having seen that the risk measures produced by the model are indeed good at describing the risk profile, we will now turn to the expected return of a callable mortgage bonds. Assume that a payment date has just occurred and we want to know the proceeds from a one year investment in the bond. Over the next year there will be four payment dates at which the bond will pay both coupons, amortisation and prepayments. The materialised prepayment rates will depend on the evolution of the short rate, so we will have to both forecast the short rate and therefrom the prepayment rates. At each payment date we will, assumedly, reinvest the total payment from the bond. This reinvestment will take place to the price occurring immediately after the payment date, so we will have to calculate new prices at each of these dates. Instead of forecasting the short rate for the next four quarters, banks typically calculate the expected return given some shock to the short rate. In this way it becomes easier to grasp the risk and return profile of the bond. In table 5 below the one year accumulated expected return as well as the one year accumulated prepayments have been calculated for different bond segments. The numbers have been calculated given different shocks to the short rate, and the spreads applied to match the initial market prices have been held constant. A few selected bonds from table 5 have also been illustrated in figure 25 below.

	Short rate shock (Bp.)										
	-300	-200	-100	-50	-25	0	25	50	100	200	300
1.5'47	12.31%	10.94%	7.89%	5.14%	3.68%	2.19%	0.68%	−0.83%	−3.86%	−9.78%	−15.39%
	42.86%	42.09%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2'47	8.18%	7.19%	5.85%	4.91%	3.94%	2.35%	1.07%	−0.26%	−3.02%	−8.64%	−14.12%
	43.25%	42.75%	41.88%	41.15%	22.37%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2.5'47	5.19%	4.40%	3.46%	2.87%	2.53%	2.14%	1.70%	1.17%	−1.58%	−6.65%	−11.91%
	43.51%	43.17%	42.61%	42.19%	41.92%	41.60%	41.22%	38.36%	0.00%	0.00%	0.00%
3'41	5.84%	5.54%	4.63%	3.77%	3.22%	2.60%	1.91%	1.15%	−1.18%	−6.02%	−10.95%
	40.09%	36.80%	31.79%	29.28%	28.18%	27.03%	25.94%	25.11%	0.00%	0.00%	0.00%
3'44	3.25%	2.78%	2.19%	1.76%	1.50%	1.21%	0.90%	0.55%	−0.25%	−3.61%	−8.20%
	43.06%	42.19%	40.83%	40.16%	39.85%	39.52%	39.19%	38.89%	38.23%	0.00%	0.00%
3'47	3.64%	2.93%	2.18%	1.76%	1.53%	1.28%	1.01%	0.71%	−0.00%	−3.83%	−8.63%
	43.68%	43.45%	43.07%	42.80%	42.63%	42.43%	42.20%	41.93%	41.15%	0.00%	0.00%
3.5'44	2.78%	2.33%	1.83%	1.48%	1.28%	1.05%	0.81%	0.55%	−0.04%	−1.92%	−5.66%
	43.45%	42.85%	41.85%	41.39%	41.17%	40.94%	40.73%	40.53%	40.12%	25.78%	0.00%
3.5'47	3.57%	2.86%	2.15%	1.77%	1.56%	1.35%	1.12%	0.87%	0.30%	−2.10%	−6.76%
	43.79%	43.64%	43.38%	43.19%	43.08%	42.95%	42.79%	42.62%	42.18%	13.21%	0.00%
4'35	4.68%	5.12%	4.77%	4.00%	3.45%	2.82%	2.09%	1.30%	−0.38%	−4.02%	−8.45%
	36.26%	30.54%	22.49%	18.94%	17.42%	16.23%	15.45%	14.93%	14.20%	12.61%	0.00%
4'38	4.14%	4.70%	4.55%	3.88%	3.41%	2.83%	2.16%	1.42%	−0.19%	−3.80%	−8.56%
	38.60%	33.54%	26.08%	22.78%	21.32%	20.03%	19.14%	18.60%	17.80%	15.21%	0.00%
4'41	2.45%	2.71%	2.76%	2.45%	2.22%	1.92%	1.56%	1.14%	0.19%	−2.08%	−6.23%
	42.43%	40.24%	36.68%	35.07%	34.32%	33.59%	33.05%	32.71%	32.22%	31.40%	0.00%

Table 5: One year accumulated returns conditional on different shocks to the short rate as of the 1st of January 2017. The returns have been calculated across mortgage pools with outstanding amounts exceeding DKK 500m. The accumulated prepayment rates have been underset the accumulated returns. The numbers have been averaged across mortgage institutions within segments. A segment has been defined as all the bonds with the same coupon and maturity.

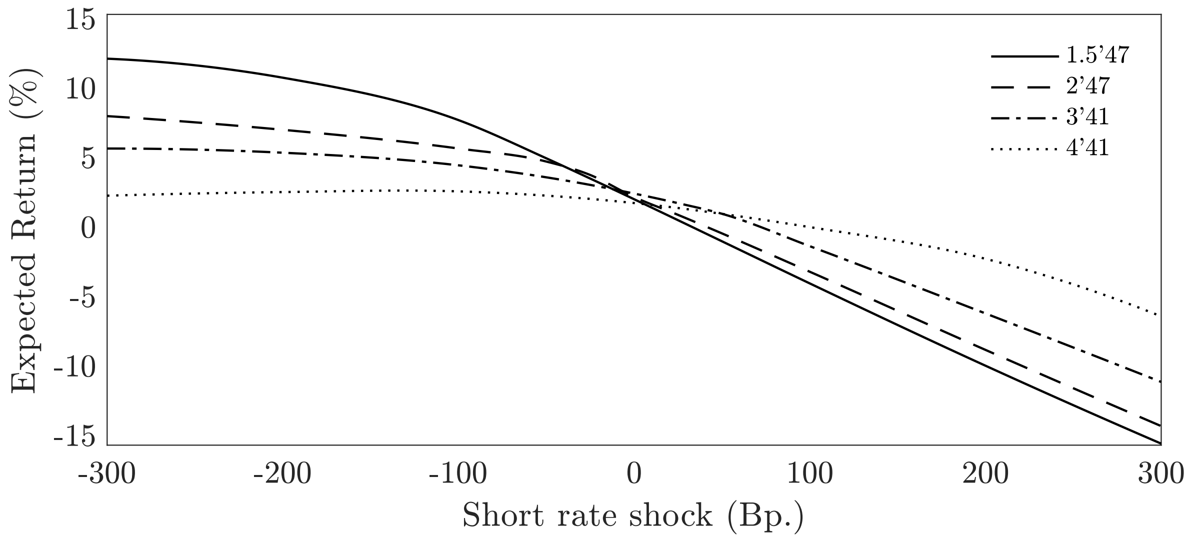

Figure 25: One year accumulated returns conditional on different shocks to the short rate as of the 1st of January 2017 for different bond segments.

From the table it can be seen how the very low coupons are the most risky ones, since they exhibit the most variation in returns from changes in the short rate. We also see that the bonds maturing in year 2047 have a tendency to rapidly go from zero prepayments to approximately 40% prepayments. This is due to the bonds being very young and therefore the borrower distribution is still very close to the initial borrower distribution. Recall that the estimated parameters implied that the average prepayment costs were only 2.6%, meaning that the borrowers generally require a low gain in order to prepay. For this reason, small changes in the short rate level can make it favourable for almost the entire pool to prepay. In case it is optimal to prepay there will be a probability of $\mathbb{P}(\Omega^\lambda) = 1 - e^{-\lambda} \approx 0.44$ that prepayments will occur over a year, which is why we see that the prepayments are suddenly jumping to 40%. Finally, we see from figure 25 that all bonds are bending off as rates are falling. This is of course due to the increased prepayments, but it is noteworthy that the expected return of the 4% coupon bond is falling for significantly large drops in the short rate.

Now that we have calculated the expected returns it is of course worth testing whether there is a relationship between expected returns and the OAS. Figure 26 below shows this relation for different bond segments. From the figure it seems that there might indeed be a relation between the OAS and the expected return. It is not a clear-cut relation but a positive trend seems to be present. The slope coefficient of the dashed OLS line is 0.0119 and the correlation coefficient between the OAS and the expected returns is 0.88. The comparison is of course biased since the OAS enters the pricing function and therefore also the expected return. However, the OAS only enters in a way that ensures that the model price today is equal to the market price and does not influence the model prepayments. So as long as we are confident that our model is a good description of borrowers' prepayment behaviour, we could perceive the OAS as an excess return.

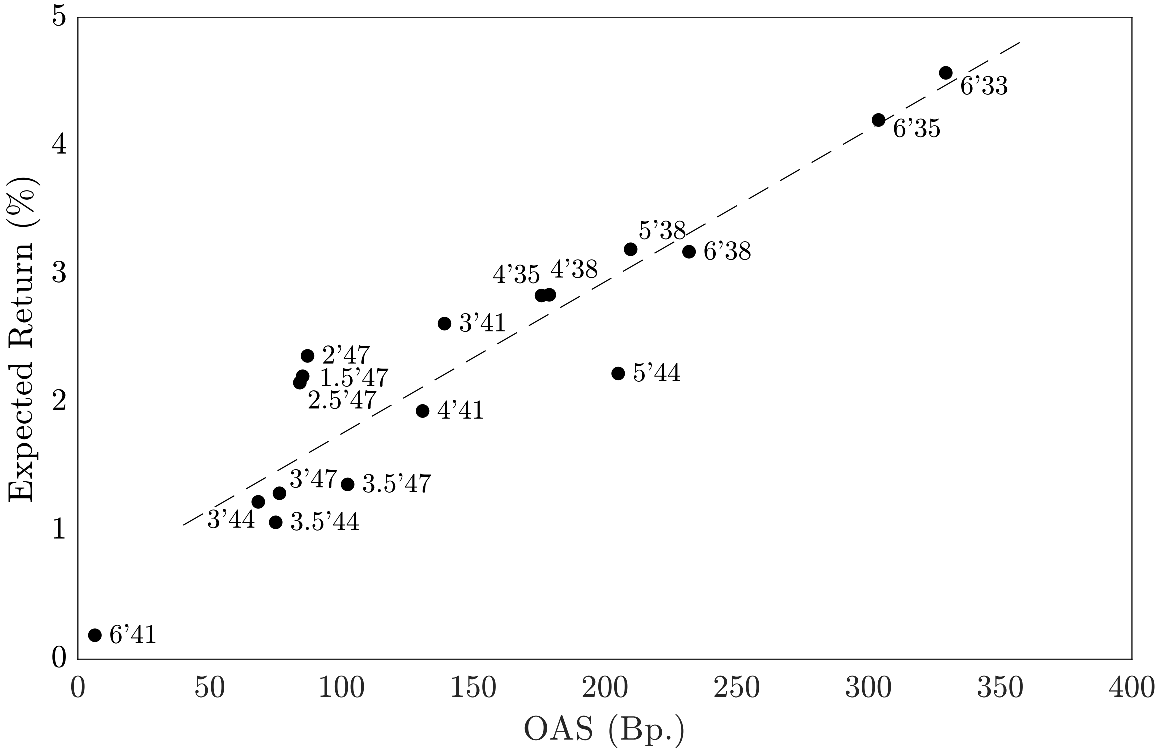

Figure 26: Expected returns from an unchanged short rate compared to the model OAS. The dashed line is the result of an OLS estimation. The estimated relation is $y = 0.5709 + 0.0119x$.

6 Structural Changes

Having seen that the model can produce sensible key figures, we should now try to challenge the model. In this section we will therefore look into some of the structural changes that have occurred over the past years. The aim is to identify relevant factors that either have been or could be disrupting the analysis carried out in this thesis. The most obvious place to begin searching will be the financial crisis, in which mortgage backed securities played a large role.

6.1 The Housing Market

During the years from 2007 to 2009 the housing market was subject to a large decline of around 25% for single family houses and 30% for apartments. Figure 27 illustrates these declines in terms of a price index for the Danish housing market. It is clear how a household taking out a mortgage loan prior to these large declines was unquestionably subject to solvency problems. The solvency of a household is typically measured in terms of the Loan to Value ratio (LTV). As the name suggests, it is the loan value divided by the value of the dwelling. Denoting the value of the borrowers dwelling by H_t then we have that

$$\mathrm{LTV}_t = \frac{F_t}{H_t}\frac{\min(M_t, 100)}{100},$$

since the loan value should take the delivery option into account.

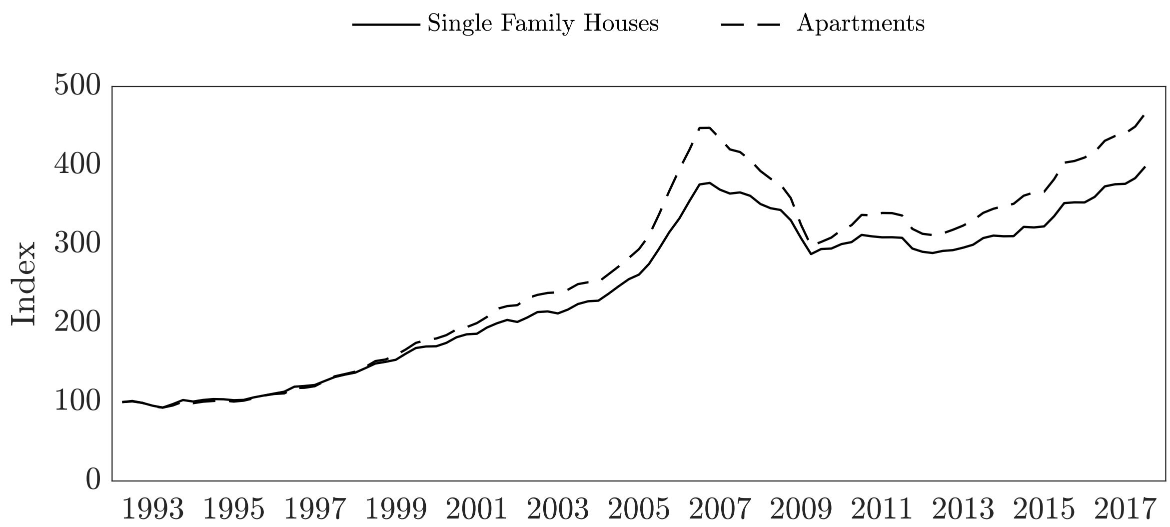

Figure 27: Price indices for the Danish housing market. Source: Finans Danmark.

If the LTV ratio moves above 100%, then the borrower will de facto be insolvent. In this scenario nothing will happen as long as the borrower continues to meet his mortgage obligations. However, if the borrower defaults, then the mortgage institution may be exposed to a loss. If credit risk was indeed a relevant factor to the model, then we should expect the market price to deviate from the model price during times of increasing LTV ratios. Over the period from 2006 to 2008 there was issued fixed rate callable mortgage bonds worth approximately DKK 160 billion with maturity in 2038. An example of one of these is a 4% bond from Realkredit Danmark. The maturity of this bond is the 1st of January 2038 meaning that the bond closed for issuance the 1st of January 2008. Assuming that a borrower took out a loan in the middle of the issuance period, the 1st of July 2006, with an initial LTV of 80%, we will calculate the evolution of the LTV. The value of the borrowers dwelling is assumed to follow the price index of a single family houses from figure 27, while the loan will amortise as an annuity. Figure 28 shows the evolution of the LTV of this theoretical borrower over the financial crisis as well as the market and model prices. As can be seen from the figure, the LTV ratio moves above 100% in 2009 indicating the possibility of technical insolvency among borrowers in the pool. We know from previous comparisons of model and market prices that the model tends to overestimate the burn out effect, but it seems that the overestimation during the rapid increase in LTV in late 2008 heavily coincides with the large increase in deviation of model prices from market prices. This suggests that the market might be pricing some credit risk that the model does not account for.

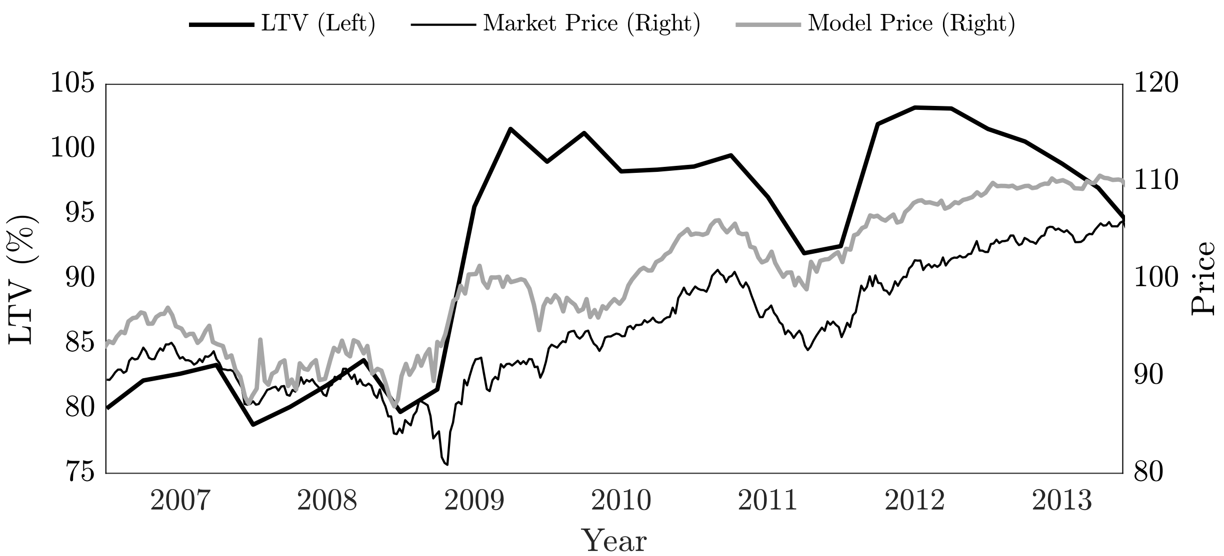

Figure 28: LTV ratio for a theoretical borrower in the the pool of a 4% 30 year bond from Realkredit Danmark maturing in 2038. The borrower starts out having a LTV ratio of 80%

6.2 Mortgage Fees

In order to stabilise the mortgage system during the years after the financial crisis, a lot of pressure has been put on mortgage banks to increase their solvency. Meanwhile, the interest rate levels have been dramatically decreasing, implying a transition of borrowers from the traditional fixed rate mortgages to the adjustable rate mortgages. These two factors in combination have made it necessary for the mortgage institutions to increase the fee rates paid in addition to the coupon payments. Figure 29 illustrates the evolution in fee rates since the financial crisis. From the figure it is clear how the level of fees has almost doubled over the past 10 years. Due to the transition of borrowers to the more risky adjustable rate mortgages, the mortgage institutions have started to discriminate on the fees applied to the different loans. The figure clearly illustrates how the two type of loans have diverged in terms of fee rates in recent years.

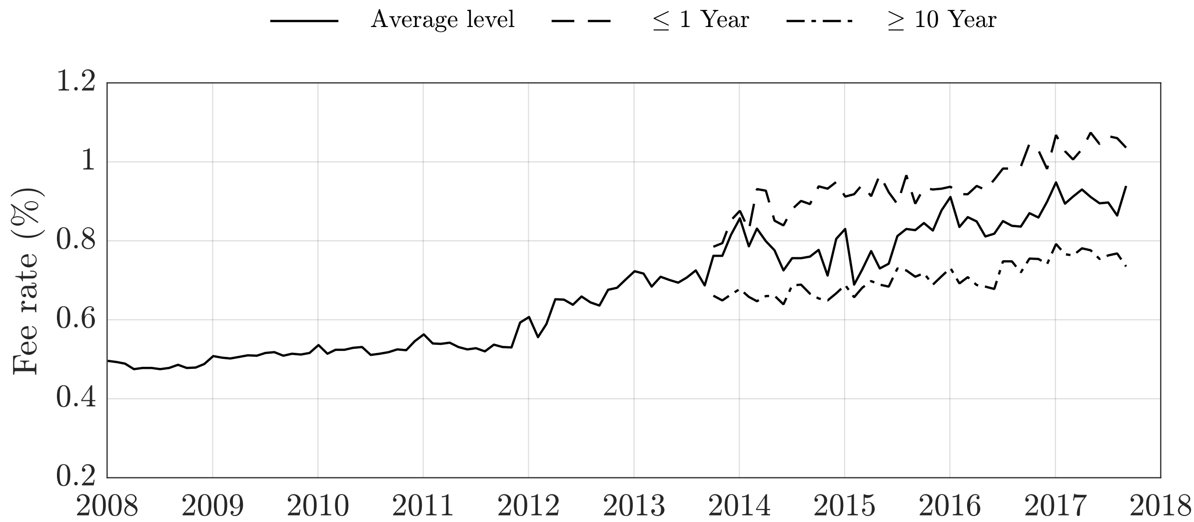

Figure 29: Fee rates on Danish mortgage loans by coupon fixing frequency. Before 2008 the general fee level has been stable for several years at 0.5%, while in 2013 the mortgage institutions started to largely discriminate fee rates by fixing frequency. The data is extracted from the Danish central bank, Nationalbanken. Data sorted by fixing frequency is only available from the 1st of October 2013, why these timeseries start at this point in time.

As we have not previously incorporated the mortgage fees in the model, we will have to do so in order to measure the effect from this discrimination in fees. We will make the assumption that a borrower with a fixed rate mortgage will pay a fee rate ϕ_1, while a borrower with a three month adjustable rate mortgage will pay a fee rate ϕ_2. Now recall that if a borrower with a callable mortgage wants to prepay, then he will pay $G_t = (1+X)\min(F_t, M_t^a)$. Up until now the condition for optimal prepayment has been that $G_t < M_t^\ell$. This condition, of course, does not take into account that the borrower will have to refinance the total amount G_t. If we make the simplifying assumption that the borrower may only refinance by use of the adjustable rate mortgage, then the liabilities held by the borrower immediately after refinancing must be $\widetilde{M}_t^\ell = G_t + \frac{1}{4}\phi_2 G_t P_t^{t+\frac{1}{4}}$. To see this recall that an adjustable rate bond will always trade at par when reaching a payment date, so in the absence of fees the liabilities must be $\widetilde{M}_t^\ell = G_t$. When a fee rate applies then the bond will still trade at par but the borrower will also have to discount the fee paid at the next fixing date. If we assume that over the next quarter the fees accrue according to standard money market conventions, then the fee payment must be $\frac{1}{4}\phi_2 G_t$.

Discounting this payment and adding the value of the bond we get $\widetilde{M}_t^\ell = G_t + \frac{1}{4}\phi_2 G_t P_t^{t+\frac{1}{4}}$. Hence, we should modify the prepayment condition such that a prepayment happens if $M_t^\ell > \widetilde{M}_t^\ell$.

With the above changes implemented in the model, we will now consider how the model behaves by suddenly discriminating on fee levels. Inspired by figure 29 we will let each type of loan have fees of size $\phi_1 = \phi_2 = 0.8\%$ before discrimination and fees of size $\phi_1 = 0.65\%$ and $\phi_2 = 0.95\%$ after discrimination. We will then calculate the price impact from the change for a 30 year 2.5% callable mortgage bond. Figure 30 illustrates the price impact from changing the fee structure. The price change might be modest but the result is clear. As the borrower's only refinancing alternative has become more expensive, there will now be less optimal opportunities to prepay why the prepayment activity is lowered. Since the investor does not receive the fees paid there will be no changes to the payment streams he is entitled to receive. However, as the prepayment activity has decreased, the value of the bond is increasing.

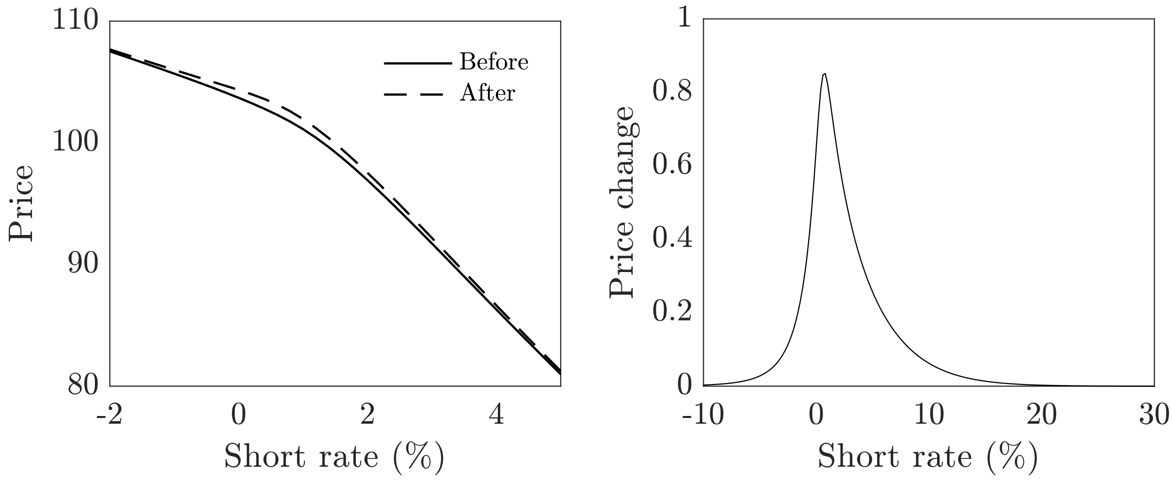

Figure 30: Prices before and after the increased discrimination in fee rates between fixed and adjustable rate mortgages. The fixed rate mortgage is a 30 year 2.5% annuity.

That the borrower can only refinance into the three month adjustable rate mortgage is of course a highly simplified assumption. In theory we could take into account a whole basket of refinancing alternatives in order to measure the gain from refinancing into any alternative available in the mortgage market. This would be a meaningful model refinement since it allows for a more realistic prepayment decision making.

7 Conclusion

This thesis presents a structural model for pricing of callable mortgage bonds. The model is extended to suit the Danish market for callable mortgage bonds by allowing for the delivery option embedded in Danish mortgage bonds. To make the model consistent with the current interest rate environment, the short rate is chosen to follow the dynamics of the extended Vasicek model by Hull & White. The model is calibrated to market swap and CIBOR quotes to fit the term structure of interest rates. As the extended Vasicek model assumes constant volatility, the model is also calibrated to Euro cap quotes in order to extract implied volatilities. Using the model prepayment rates and the market prepayment rates for a large sample of mortgage bonds, the model is estimated by GMM. The GMM procedure results in reasonable parameter estimates and the model prepayments track the actual prepayments closely. The burn out effect from the model tends to be too pronounced when the remaining mortgage pool becomes small. This results in the model overestimating the bond prices compared to the ones observed in the market. In terms of risk figures, empirical evidence suggests that the durations and convexities generated by the model accurately describe the evolution in market prices. The accumulated error from approximating the price evolution by use of model risk figures is stable over time, why the same can be concluded for the risk figures themselves. Finally, it is found that structural changes in mortgage fees as well as the housing market may have an effect on the price formation of mortgage bonds. These factors are not accounted for in the model but are suggested as possible extensions. The overall conclusion is that the final model does a good job in describing both prices and prepayments given that it only relies on three prepayment parameters and a calibrated short rate model.

A Appendix

A.1 Vasicek ODEs

We have to find the solutions to the ODEs of equations (3.51) and (3.52). If we guess a solutions $B(t,T) = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa(T-t)})$ and insert into equation (3.52) we see that the ODE is satisfied

$$1 + \frac{\mathrm{d}B(t,T)}{\mathrm{d}t} - \kappa B(t,T) = 1 - \mathrm{e}^{-\kappa(T-t)} - \kappa\frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa(T-t)}\right) = 0$$

The boundary condition $B(T,T) = \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa(T-T)}\right) = 0$ is also satisfied. Inserting $B(t,T)$ into the ODE (3.51) and integrating from t to T we get

$$A(T,T) - A(t,T) + \kappa\int_t^T \hat{\theta}(s)B(s,T)\mathrm{d}s - \frac{1}{2}\sigma^2\int_t^T B^2(s,T)\mathrm{d}s = 0$$

Using that $A(T,T) = 0$ and rearranging we get

$$A(t,T) = \kappa\int_t^T \hat{\theta}(s)B(s,T)\mathrm{d}s - \frac{1}{2}\sigma^2\int_t^T B^2(s,T)\mathrm{d}s \tag{A.1}$$

We now evaluate the integral over $B^2(t,T)$:

$$\begin{aligned}
\int_t^T B^2(s,T)\mathrm{d}s &= \int_t^T \frac{1}{\kappa^2}\left(1 + \mathrm{e}^{-2\kappa(T-s)} - 2\mathrm{e}^{-\kappa(T-s)}\right)\mathrm{d}s \\
&= \frac{1}{\kappa^2}\left((T-t) + \frac{1}{2\kappa} - \frac{1}{2\kappa}\mathrm{e}^{-2\kappa(T-t)} - \frac{2}{\kappa} + \frac{2}{\kappa}\mathrm{e}^{-\kappa(T-t)}\right) \\
&= \frac{1}{\kappa^2}\left((T-t) - \frac{1}{2\kappa}\left(1 + \mathrm{e}^{-2\kappa(T-t)} - 2\mathrm{e}^{-\kappa(T-t)}\right) - B(t,T)\right) \\
&= \frac{1}{\kappa^2}\left((T-t) - \frac{\kappa}{2}B^2(t,T) - B(t,T)\right).
\end{aligned} \tag{A.2}$$

Inserting (A.2) in (A.1) and we find the expression for $A(t,T)$:

$$A(t,T) = \kappa\int_t^T \hat{\theta}(s)B(s,T)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(t,T) + \frac{\sigma^2}{2\kappa^2}\left(B(t,T) - (T-t)\right). \tag{A.3}$$

A.2 Fitting the Initial Termstructure

We want to confirm that the function for $\hat{\theta}(t)$ given in equation (3.55) indeed fits the initial term structure. We start out by differentiating $A(0,t)$ with the purpose of isolating $\hat{\theta}(t)$:

$$\frac{\mathrm{d}A(0,t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa\int_0^t \hat{\theta}(s)B(s,t)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(0,t) + \frac{\sigma^2}{2\kappa^2}(B(0,t) - t)\right)$$

$$
\begin{aligned}
&= \kappa \int_0^t \hat{\theta}(s)\mathrm{e}^{-\kappa(t-s)}\mathrm{d}s + \frac{\sigma^2}{2\kappa}B(0,t)\frac{\mathrm{d}B(0,t)}{\mathrm{d}t} + \frac{\sigma^2}{2\kappa^2}\frac{\mathrm{d}B(0,t)}{\mathrm{d}t} - \frac{\sigma^2}{2\kappa^2} \\
&= \kappa \int_0^t \hat{\theta}(s)\mathrm{e}^{-\kappa(t-s)}\mathrm{d}s - \frac{\sigma^2}{2\kappa^2}(\mathrm{e}^{-\kappa t} - \mathrm{e}^{-2\kappa t}) - \frac{\sigma^2}{2\kappa^2}\mathrm{e}^{-\kappa t} - \frac{\sigma^2}{2\kappa^2} \\
&= \kappa \int_0^t \hat{\theta}(s)\mathrm{e}^{-\kappa(t-s)}\mathrm{d}s - \frac{\sigma^2}{2}B^2(0,t)
\end{aligned}
$$

We now find the second derivative to be

$$
\begin{aligned}
\frac{\mathrm{d}^2A(0,t)}{\mathrm{d}t^2} &= \kappa\hat{\theta}(t) - \kappa^2 \int_0^t \hat{\theta}(s)\mathrm{e}^{-\kappa(t-s)}\mathrm{d}s - \sigma^2 B(0,t)\mathrm{e}^{-\kappa t} \\
&= \kappa\hat{\theta}(t) - \kappa\frac{\mathrm{d}A(0,t)}{\mathrm{d}t} - \kappa\frac{\sigma^2}{2}B^2(0,t) - \sigma^2 B(0,t)\mathrm{e}^{-\kappa t} \\
&= \kappa\hat{\theta}(t) - \kappa\frac{\mathrm{d}A(0,t)}{\mathrm{d}t} - \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa t}).
\end{aligned} \tag{A.4}
$$

In the above we have applied Leibnitz's rule[8] stating that for a function $f = f(x,t)$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\mathrm{d}t = f(x,v(x))\frac{\mathrm{d}v(x)}{\mathrm{d}x} - f(x,u(x))\frac{\mathrm{d}u(x)}{\mathrm{d}x} + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\mathrm{d}t.
$$

We may now isolate $\hat{\theta}(t)$ by rearranging (A.4):

$$
\hat{\theta}(t) = \frac{\mathrm{d}A(0,t)}{\mathrm{d}t} + \frac{1}{\kappa}\frac{\mathrm{d}^2A(0,t)}{\mathrm{d}t^2} + \frac{\sigma^2}{2\kappa^2}(1-\mathrm{e}^{-2\kappa t}). \tag{A.5}
$$

Since we want to choose $\hat{\theta}(t)$ such that market prices equals model prices, we must have

$$
\bar{P}_0^t = \mathrm{e}^{-A(0,t)-B(0,t)r_0} \quad \Leftrightarrow \quad A(0,t) = -\ln\bar{P}_0^t - B(0,t)r_0
$$

Differentiating $A(0,t)$ once and twice we obtain

$$
\begin{aligned}
\frac{\mathrm{d}A(0,t)}{\mathrm{d}t} &= -\frac{\frac{\mathrm{d}\bar{P}_0^t}{\mathrm{d}t}}{\bar{P}_0^t} - r_0\mathrm{e}^{-\kappa t} = \bar{f}(0,t) - r_0\mathrm{e}^{-\kappa t}, \\
\frac{\mathrm{d}^2A(0,t)}{\mathrm{d}t^2} &= \frac{\mathrm{d}\bar{f}(0,t)}{\mathrm{d}t} + \kappa r_0\mathrm{e}^{-\kappa t},
\end{aligned}
$$

where we have to assume that the observed forward curve is differentiable. Inserting these derivatives in (A.5) we get

$$
\hat{\theta}(t) = \bar{f}(0,t) + \frac{1}{\kappa}\frac{\mathrm{d}\bar{f}(0,t)}{\mathrm{d}t} + \frac{\sigma^2}{2\kappa^2}(1-\mathrm{e}^{-2\kappa t}), \tag{A.6}
$$

8 See for example Sydsætter (2002) [15] formula 6.1.1.

which is exactly equation (3.55). Inserting this in the expression for $A(t,T)$ we get

$$\begin{aligned}
A(t,T) &= \int_t^T \bar{f}(0,s)(1-\mathrm{e}^{-\kappa(T-s)})\mathrm{d}s + \frac{1}{\kappa}\int_t^T \frac{\mathrm{d}\bar{f}(0,s)}{\mathrm{d}t}(1-\mathrm{e}^{-\kappa(T-s)})\mathrm{d}s \\
&+ \int_t^T \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa s})B(s,T)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(t,T) + \frac{\sigma^2}{2\kappa^2}\left(B(t,T)-(T-t)\right) \\
&= -\ln\left(\frac{\bar{P}_0^T}{\bar{P}_0^t}\right) - \int_t^T \bar{f}(0,s)\mathrm{e}^{-\kappa(T-s)}\mathrm{d}s + \frac{1}{\kappa}(\bar{f}(0,T)-\bar{f}(0,t)) \\
&- \frac{1}{\kappa}\int_t^T \frac{\mathrm{d}\bar{f}(0,s)}{\mathrm{d}t}\mathrm{e}^{-\kappa(T-s)}\mathrm{d}s + \int_t^T \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa s})B(s,T)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(t,T) \\
&+ \frac{\sigma^2}{2\kappa^2}\left(B(t,T)-(T-t)\right)
\end{aligned}$$

Using partial integration we have that

$$\int_t^T \frac{\mathrm{d}\bar{f}(0,s)}{\mathrm{d}t}\mathrm{e}^{-\kappa(T-s)}\mathrm{d}s = \bar{f}(0,T) - \bar{f}(0,t)\mathrm{e}^{-\kappa(T-t)} - \kappa\int_t^T \bar{f}(0,s)\mathrm{e}^{-\kappa(T-s)}\mathrm{d}s.$$

Inserting this in the expression for $A(t,T)$ yields

$$\begin{aligned}
A(t,T) &= -\ln\left(\frac{\bar{P}_0^T}{\bar{P}_0^t}\right) - B(t,T)\bar{f}(0,t) + \int_t^T \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa s})B(s,T)\mathrm{d}s + \frac{\sigma^2}{4\kappa}B^2(t,T) \\
&+ \frac{\sigma^2}{2\kappa^2}\left(B(t,T)-(T-t)\right)
\end{aligned}$$

We will now evaluate the last integral:

$$\begin{aligned}
\int_t^T \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa s})B(s,T)\mathrm{d}s &= \frac{\sigma^2}{2\kappa^2}\int_t^T (1-\mathrm{e}^{-2\kappa s})(1-\mathrm{e}^{-\kappa(T-s)})\mathrm{d}s \\
&= \frac{\sigma^2}{2\kappa^2}\int_t^T 1-\mathrm{e}^{-\kappa(T-s)} - \mathrm{e}^{-2\kappa s} + \mathrm{e}^{-\kappa(T+s)}\mathrm{d}s \\
&= \frac{\sigma^2}{2\kappa^2}\Big[(T-t) - \frac{1}{\kappa} + \frac{1}{\kappa}\mathrm{e}^{-\kappa(T-t)} + \frac{1}{2\kappa}\mathrm{e}^{-2\kappa T} - \frac{1}{2\kappa}\mathrm{e}^{-2\kappa t} \\
&\quad - \frac{1}{\kappa}\mathrm{e}^{-2\kappa T} + \frac{1}{\kappa}\mathrm{e}^{-\kappa(T+t)}\Big] \\
&= \frac{\sigma^2}{2\kappa^2}\Big[(T-t) - B(t,T) - \frac{1}{2\kappa}\mathrm{e}^{-2\kappa T} - \frac{1}{2\kappa}\mathrm{e}^{-2\kappa t} + \frac{1}{\kappa}\mathrm{e}^{-\kappa(T+t)}\Big] \\
&= \frac{\sigma^2}{2\kappa^2}\Big[(T-t) - B(t,T) - \frac{\kappa}{2}\mathrm{e}^{-2\kappa t}B^2(t,T)\Big]
\end{aligned}$$

Now we can insert into $A(t,T)$ and the result follows:

$$A(t,T) = -\ln\left(\frac{\bar{P}_0^T}{\bar{P}_0^t}\right) - B(t,T)\bar{f}(0,t) + \frac{\sigma^2}{4\kappa}B^2(t,T)(1-\mathrm{e}^{-2\kappa t}).$$

A.3 The Variance of the Short Rate

We will determine the distribution of the short rate in the extended Vasicek model under the $\mathbb{Q}^T$ measure. To do so we apply Ito's lemma to the quantity $\mathrm{e}^{at}r_t$:

$$\begin{aligned}
\mathrm{d}(\mathrm{e}^{at}r_t) &= a\mathrm{e}^{at}r_t\mathrm{d}t + \mathrm{e}^{at}\mathrm{d}r_t \\
&= a\mathrm{e}^{at}r_t\mathrm{d}t + \mathrm{e}^{at}[\kappa(\hat{\theta}(t) - r_t) - \sigma^2 B(t,T)]\mathrm{d}t + \mathrm{e}^{at}\sigma\mathrm{d}W_t^{\mathbb{Q}^T}
\end{aligned}$$

Choosing $a = \kappa$ and integrating from t to T and we obtain

$$\mathrm{e}^{\kappa T}r_T - \mathrm{e}^{\kappa t}r_t = \int_t^T \kappa\mathrm{e}^{\kappa s}\hat{\theta}(s) - \mathrm{e}^{\kappa s}\sigma^2 B(s,T)\mathrm{d}s + \sigma\int_t^T \mathrm{e}^{as}\mathrm{d}W_s^{\mathbb{Q}^T}$$

Moving $\mathrm{e}^{\kappa t}$ to the right hand side and dividing by $\mathrm{e}^{\kappa T}$ we obtain

$$r_T = \mathrm{e}^{-\kappa(T-t)}r_t + \mathrm{e}^{-\kappa T}\int_t^T \kappa\mathrm{e}^{\kappa s}\hat{\theta}(s) - \mathrm{e}^{\kappa s}\sigma^2 B(s,T)\mathrm{d}s + \sigma\mathrm{e}^{-\kappa T}\int_t^T \mathrm{e}^{\kappa s}\mathrm{d}W_s^{\mathbb{Q}^T}$$

Since $\mathrm{e}^{\kappa t}$ is just a deterministic function, the stochastic integral will be normally distributed. The variance of r_T will therefore be given by

$$\begin{aligned}
\mathbb{V}_t^{\mathbb{Q}^T}[r_T] &= \mathbb{V}_t^{\mathbb{Q}^T}\left[\sigma\mathrm{e}^{-\kappa T}\int_t^T \mathrm{e}^{\kappa s}\mathrm{d}W_s^{\mathbb{Q}^T}\right] \\
&= \sigma^2\mathrm{e}^{-2\kappa T}\int_t^T \mathrm{e}^{2\kappa s}\mathrm{d}s \\
&= \frac{\sigma^2}{2\kappa}\mathrm{e}^{-2\kappa T}(\mathrm{e}^{2\kappa T} - \mathrm{e}^{2\kappa t}) \\
&= \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa(T-t)}).
\end{aligned}$$

A.4 The Expected Value Under $\mathbb{Q}^S$.

We would like to evaluate the expectation $\mathbb{E}^{\mathbb{Q}^S}[P_T^S]$. We know that any price process deflated by P_t^S is a martingale under $\mathbb{Q}^S$ and in particular by equation (2.35) we have that

$$\mathrm{d}\frac{P_t^T}{P_t^S} = (\beta^T(t,r_t) - \beta^S(t,r_t))\frac{P_t^T}{P_t^S}\mathrm{d}W_t^{\mathbb{Q}^S} \tag{A.7}$$

Since we are in the extended Vasicek model we have $\beta^T(t,r_t) = -\sigma B(t,T)$ and $\beta^S(t,r_t) = -\sigma B(t,S)$. Inserting these we get

$$\mathrm{d}\frac{P_t^T}{P_t^S} = \sigma(B(t,S) - B(t,T))\frac{P_t^T}{P_t^S}\mathrm{d}W_t^{\mathbb{Q}^S}. \tag{A.8}$$

Since $P_T^T = 1$ we must have $\mathbb{E}^{\mathbb{Q}^S}[P_T^S] = \mathbb{E}^{\mathbb{Q}^S}\left[\frac{P_T^S}{P_T^T}\right]$. To evaluate this expectation we will need the distribution of $\frac{P_t^S}{P_t^T}$. By Ito's lemma and equation (A.8) we have that

$$\mathrm{d}\frac{P_t^S}{P_t^T} = \sigma^2(B(t,S) - B(t,T))^2\frac{P_t^S}{P_t^T}\mathrm{d}t - \sigma(B(t,S) - B(t,T))\frac{P_t^S}{P_t^T}\mathrm{d}W_t^{\mathbb{Q}^S}.$$

Taking the logarithm of $\frac{P_t^S}{P_t^T}$ and using again Ito's lemma we get

$$\mathrm{d}\ln\frac{P_t^S}{P_t^T} = \frac{1}{2}\sigma^2(B(t,S) - B(t,T))^2\mathrm{d}t - \sigma(B(t,S) - B(t,T))\mathrm{d}W_t^{\mathbb{Q}^S}.$$

Integrating from t to T and rearranging yields

$$\frac{P_T^S}{P_T^T} = \frac{P_t^S}{P_t^T}\mathrm{e}^{\int_t^T \frac{1}{2}\sigma^2(B(u,S)-B(u,T))^2\mathrm{d}u - \sigma\int_t^T(B(u,S)-B(u,T))\mathrm{d}W_u^{\mathbb{Q}^S}}.$$

The inside of the exponential function will be Gaussian, so we will determine the mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. The mean can be found as

$$\begin{aligned}
\hat{\mu} &= \int_t^T \frac{1}{2}\sigma^2(B(u,S) - B(u,T))^2\mathrm{d}u \\
&= \frac{\sigma^2}{2\kappa^2}\int_t^T(\mathrm{e}^{-\kappa(T-u)} - \mathrm{e}^{-\kappa(S-u)})^2\mathrm{d}u \\
&= \frac{\sigma^2}{2\kappa^2}(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa S})^2\int_t^T \mathrm{e}^{2\kappa u}\mathrm{d}u \\
&= \frac{\sigma^2}{4\kappa^3}(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa S})^2(\mathrm{e}^{2\kappa T} - \mathrm{e}^{2\kappa t}) \\
&= \frac{\sigma^2}{4\kappa^3}\mathrm{e}^{-2\kappa T}\left(1 - \mathrm{e}^{-\kappa(S-T)}\right)^2\left(\mathrm{e}^{2\kappa T} - \mathrm{e}^{2\kappa t}\right) \\
&= \frac{\sigma^2}{4\kappa^3}(1 - \mathrm{e}^{-\kappa(S-T)})^2(1 - \mathrm{e}^{-2\kappa(T-t)}) \\
&= \frac{1}{2}B^2(T,S)\nu^2(t,T).
\end{aligned}$$

The variance can be found as

$$\begin{aligned}
\hat{\sigma}^2 &= \mathbb{V}_t^{\mathbb{Q}^S}\left(-\sigma\int_t^T B(u,S) - B(u,T)\mathrm{d}W_u^{\mathbb{Q}^S}\right) \\
&= \sigma^2\int_t^T(B(u,S) - B(u,T))^2\mathrm{d}u \\
&= 2\hat{\mu}.
\end{aligned}$$

Let $X \sim \mathcal{N}(\hat{\mu},\hat{\sigma}^2)$, then we can now evaluate the desired expectation:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}^S}\left[\frac{P_T^S}{P_T^T}\right] &= \frac{P_t^S}{P_t^T}\mathbb{E}\left[\mathrm{e}^X\right] \\
&= \frac{P_t^S}{P_t^T}\mathrm{e}^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2} \\
&= \frac{P_t^S}{P_t^T}\mathrm{e}^{B^2(T,S)\nu^2(t,T)}.
\end{aligned}$$

A.5 Vasicek MLE

By similar calculations as in A.3 and using that $\theta(t) = \theta$ is a constant, we can find

$$r_T = \mathrm{e}^{-\kappa(T-t)}r_t + \theta[1-\mathrm{e}^{-\kappa(T-t)}] + \sigma\mathrm{e}^{-\kappa T}\int_t^T \mathrm{e}^{\kappa s}\mathrm{d}W_s^{\mathbb{P}}.$$

The conditional mean is given by

$$\mathbb{E}_t[r_T] = \mathrm{e}^{-\kappa(T-t)}r_t + \theta[1-\mathrm{e}^{-\kappa(T-t)}],$$

and appendix A.3 derived the variance as

$$\mathbb{V}_t\left[r_T\right] = \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa(T-t)}).$$

Since the stochastic integral is Gaussian, we will have that the increments are also Gaussian, and the conditional density will therefore be

$$f_{r_{t_i}}(x|r_{t_{i-1}},\kappa,\theta,\sigma) = \frac{1}{\sqrt{2\pi\nu_i}}\mathrm{e}^{-\frac{(x-\mu_i)^2}{2\nu_i}},$$

where

$$\mu_i = \mathrm{e}^{-\kappa\Delta t_i}r_{t_{i-1}} + \theta[1-\mathrm{e}^{-\kappa\Delta t_i}], \quad \nu_i = \frac{\sigma^2}{2\kappa}(1-\mathrm{e}^{-2\kappa\Delta t_i}) \quad \text{and} \quad \Delta t_i = t_i - t_{i-1}$$

The log likelihood is therefore given by

$$\log\mathcal{L}\left(\kappa,\theta,\sigma \mid \{r_{t_i}\}_{i=0}^N\right) = -\frac{N}{2}\ln(2\pi) - \sum_{i=1}^N \ln v_i + \frac{(r_{t_i}-\mu_i)^2}{2\nu_i^2}. \tag{A.9}$$

Performing an estimation in Matlab for the period 1996 to 2017 with the 1Y swap rate as a proxy for the dynamics of the short rate and we get the results in table 6.

κ	θ	σ
7.4977%	0.0827%	0.8074%

Table 6: Vasicek parameter estimates.

A.6 cStanton

The original Stanton model has been implemented in Matlab as a class called `cStanton`. Below is only the main function of the model, namely the pricing function. As can be seen, the pricing function itself only takes up 85 lines of code. The code has been commented to make it easier to read.

```
        function [M_a,M_l,space,time]=pricingFD(obj)
            % Calculate annuity and payment dates
            annuitySchedule=obj.schedule;
            Y=annuitySchedule(:,4);
            principal=annuitySchedule(:,5);

            % Create grid
            time=(obj.t0:obj.dt:obj.T)';
            space=(obj.r_min:obj.dr:obj.r_max)';

            % Terminal conditions
            M_a=zeros(obj.N+1,1);
            M_l=zeros(obj.N+1,1);

            % Mu and Sigma
            Mu=obj.kappa*obj.mu-(obj.kappa+obj.q)*space;
            Sig=obj.sig^2*space;

            % A B C D
            A=1/(4*obj.dr)*Mu-1/(4*obj.dr^2)*Sig;A=A(:);
            B=1/obj.dt+1/2*Sig*1/(obj.dr^2)+1/2*space;B=B(:);
            C=-1/(4*obj.dr)*Mu-1/(4*obj.dr^2)*Sig;C=C(:);
            D=1/obj.dt-1/(2*obj.dr^2)*Sig-1/2*space;D=D(:);

            % Correcting boundaries (zero convexity)
            B(1)=1/obj.dt+Mu(1)/(2*obj.dr)+1/2*space(1);
            C(1)=-Mu(1)/(2*obj.dr);
            D(1)=1/obj.dt-Mu(1)/(2*obj.dr)-1/2*space(1);
            A(end)=Mu(end)/(2*obj.dr);
            B(end)=1/obj.dt-Mu(end)/(2*obj.dr)+1/2*space(end);
            D(end)=1/obj.dt+Mu(end)/(2*obj.dr)-1/2*space(end);

            % Left Hand Side Matrix
```

```
        LHS1=[[zeros(1,obj.N);diag(A(2:end))],zeros(obj.N+1,1)];
        LHS2=diag(B);
        LHS3=[zeros(obj.N+1,1),[diag(C(1:end-1));zeros(1,obj.N)]];
        LHS=LHS1+LHS2+LHS3;

        % Right Hand Side Matrix
        RHS1=[[zeros(1,obj.N);diag(-A(2:end))],zeros(obj.N+1,1)];
        RHS2=diag(D);
        RHS3=[zeros(obj.N+1,1),[diag(-C(1:end-1));zeros(1,obj.N)]];
        RHS=RHS1+RHS2+RHS3;

        % Invert LHS
        invLHS=inv(LHS);

        for j=obj.J+1:-1:1

            % Check if date is paydate
            if mod(j-1,obj.n2)==0

                % Get index for payment/principal
                index=(j-1)/obj.n2+1;

                % If paydate then calculate new principal
                if j==1
                    F_t=obj.F0;
                else
                    F_t=principal(index);
                end

                % Calculate Lambda
                b=M_l>(1+obj.X)*F_t;
                LAMBDA=obj.lambda1+obj.lambda2*b;
                LAMBDA2=1-exp(-1/(obj.n)*LAMBDA);

                % Take "expected value"
                M_l=M_l.*(1-LAMBDA2)+LAMBDA2.*F_t*(1+obj.X);
                M_a=M_a.*(1-LAMBDA2)+LAMBDA2.*F_t;

                % Add payment
                M_a=M_a+Y(index);
                M_l=M_l+Y(index);
            end

            % Calculate new values of liabilities
            f=RHS*M_l;
            M_l=(invLHS)*f;

```

```
                % Calculate new values of bond
                f=RHS*M_a;
                M_a=(invLHS)*f;
            end
        end
```

A.7 cStantonHW

Since the implementation of the extended Stanton model is the core of the thesis, the whole class `cStantonHW` has been included. Again the code is commented in order to make it readable.

```
classdef cStantonHW < matlab.mixin.Copyable
    % Pricing in the Stanton (1995) model

    properties (Access = public)
        % Preallocate non-dependent model parameters
        X=[];
        F0=[];
        n=[];
        T=[];
        lambda=[];
        Alpha=[];
        Beta=[];
        delivery=true;
        feeRateShort=0.00;
        feeRateLong=0.00;
    end

    properties (Access = private)
        % FD matrices
        matrixContainer={};

        % Private Model parameters
        thetaPrivate=[];
        XstarPrivate=[];
        RPrivate=[];
        OASPrivate=[];

        % Private market
        mkt=[];

        % Preallocate finite difference settings
        % Set discretisation parameters
        n2=1; % Time steps between payment dates
```

```
        t0=0;
        r_min=-0.1;
        r_max=0.3;
        N=200;
        RannacherTimeStepping=true;
        smoothPayoff=true;
        RannacherSteps=2;

        % Prices, OAS, OAD & OAC
        prices=[];
        OAD=[];
        OAC=[];
        MOAD=[];
        MOAC=[];
        Carry=[];

        % CDF
        privateCDF=[];
        privateCDFgrid=(0:0.05:1)';

    end

    properties (Access = private, Dependent)
        % Annuity settings
        R_tilde=[];
        schedule=[];

        % FD time and space
        time=[];
        space=[];

        % FD settings
        J=[];
        dt=[];
        dr=[];

        % Distributions
        CDF=[];
    end

    properties (Access = public, Dependent)
        % Preallocate dependent model parameters
        kappa=[];
        sigma=[];
        theta=[];
        Xstar=[];
        R=[];
```

```

        % Dependent market
        market=[];

        % Time to maturity
        maturity=[];

        % FD settings
        Rannacher=true;
        smoothing=true;

        % OAS
        OAS=[];
    end

    methods (Access = public)
        function obj=cStantonHW(varargin)
            % Set user provided properties
            for i=1:2:nargin
                switch lower(varargin{i})
                    %%% Model parameters
                    case {'x','prepaymentcosts'}
                        % Set transaction costs
                        obj.X=varargin{i+1};
                    case {'f0','principal','notional'}
                        % Set principal
                        obj.F0=varargin{i+1};
                    case {'n','terms'}
                        % Set terms
                        obj.n=varargin{i+1};
                    case {'r','coupon'}
                        % Set coupon rate
                        obj.R=varargin{i+1};
                    case {'t','maturity'}
                        % Set maturity
                        obj.T=varargin{i+1};
                    case 'lambda'
                        % Set prepayment intensity
                        obj.lambda=varargin{i+1};
                    case {'a','alpha'}
                        % Set A
                        obj.Alpha=varargin{i+1};
                    case {'b','beta'}
                        % Set B
                        obj.Beta=varargin{i+1};
                    case 'market'
                        obj.mkt=varargin{i+1};
```

```

                    % Finite difference parameters
                case 'timesteps'
                    obj.n2=varargin{i+1};
                case 'spacesteps'
                    obj.N=varargin{i+1};
                case 'rannacher'
                    obj.Rannacher=varargin{i+1};
                case 'smoothing'
                    obj.smoothing=varargin{i+1};
                case 'feerateshort'
                    obj.feeRateShort=varargin{i+1};
                case 'feeratelong'
                    obj.feeRateLong=varargin{i+1};
            end
        end
    end

    function [expectedReturn,accumulatedPrepayments,scenarios]=...
            expReturn(obj,shortRate,scenarios)
        % Get price today
        P0=obj.getPrice(shortRate);

        % Save maturity and Xstar
        saveT=obj.maturity;
        saveXstar=obj.Xstar;

        % Define scenarios
        if nargin<3
            scenarios=(-100:10:100)';
        end

        % Preallocate return vector
        expectedReturn=nan(size(scenarios));
        accumulatedPrepayments=nan(size(scenarios));

        % Loop over scenatios
        for j=1:size(scenarios,1)
            % Calculate X* for the four next draws and calc prices along
            % the way
            XstarForecast=nan(4,1);
            P=nan(4,1);
            for i=1:4
                obj.maturity=saveT-0.25*i;
                XstarForecast(i)=obj.getXstar(shortRate+scenarios(j)...
                    /10000);
                obj.Xstar=[saveXstar;XstarForecast(1:i)];
```

```
            P(i)=obj.getPrice(shortRate+scenarios(j)/10000);
        end

        % Prepayments
        temp=modelPrepayments(obj.Alpha,...
            obj.Beta,...
            obj.lambda,...
            [saveXstar;XstarForecast],...
            4);
        prepayments=temp(end-3:end);

        % Calculate capital evolution
        notional=100;
        for i=1:4
            % Remaining payment dates
            payDates=(saveT-i*0.25)*obj.n;

            % Calc payment
            payment=obj.R_tilde/(1-(1+obj.R_tilde)^(-payDates))*...
                notional;

            % Calc interest
            interest=notional*obj.R_tilde;

            % Calc amortisation
            amortisation=payment-interest;

            % Total draw
            draw=amortisation*(1-prepayments(i))+...
                notional*prepayments(i);

            % Total payment
            totalPayment=interest+draw;

            % Subtract prepayments and amortisationfrom notional
            notional=notional-draw;

            % Reinvest payment
            notional=notional+totalPayment/P(i)*100;

        end

        % Calculate return
        expectedReturn(j)=P(4)*notional/(P0*100)-1;

        % Calc accumulate prepayments
        accumulatedPrepayments(j)=1-prod(1-prepayments);
```

```

            % Reset maturity and Xstar
            obj.maturity=saveT;
            obj.Xstar=saveXstar;
        end
    end

    function val=getPrice(obj,shortRate)
        % Check if prices are available
        if isempty(obj.prices)
            weightedPrices(obj);
        end

        % Hermite interpolate
        val=nan(size(shortRate));
        for i=1:size(val,1)
            val(i)=hermiteInterpolation(obj.space,obj.prices,...
                shortRate(i));
        end
    end

    function [oad,oac,moad,moac,carry]=keyfigures(obj,shortRate)
        % Check if prices are available
        if isempty(obj.prices)
            weightedPrices(obj);
        end

        % Hermite interpolate keyfigures
        oad=nan(size(shortRate));
        for i=1:size(oad,1)
            oad(i)=hermiteInterpolation(obj.space,obj.OAD,...
                shortRate(i));
        end
        if nargout>1
            oac=nan(size(shortRate));
            for i=1:size(oac,1)
                oac(i)=hermiteInterpolation(obj.space,obj.OAC,...
                    shortRate(i));
            end
        end
        if nargout>2
            moad=nan(size(shortRate));
            for i=1:size(moad,1)
                moad(i)=hermiteInterpolation(obj.space,obj.MOAD,...
                    shortRate(i));
            end
        end
```

```
        if nargout>3
            moac=nan(size(shortRate));
            for i=1:size(moac,1)
                moac(i)=hermiteInterpolation(obj.space,obj.MOAC,...
                    shortRate(i));
            end
        end
        if nargout>4
            carry=nan(size(shortRate));
            for i=1:size(carry,1)
                carry(i)=hermiteInterpolation(obj.space,obj.Carry,...
                    shortRate(i));
            end
        end
    end

    function val=getOAS(obj,shortRate,marketPrice)
        % Calc for OAS=0
        obj.OAS=0;
        weightedPrices(obj);
        modelPrice=hermiteInterpolation(obj.space,obj.prices,shortRate);


        % Set initial bounds
        lb=0;ub=0;
        if modelPrice>marketPrice
            while modelPrice>marketPrice
                % Price at lower bound
                Plb=modelPrice;

                % If upper bound is too low then increase it
                ub=ub+0.05;

                % Calc for OAS=ub
                obj.OAS=ub;
                weightedPrices(obj);
                modelPrice=hermiteInterpolation(obj.space,obj.prices,...
                    shortRate);

                % Price at upper bound
                Pub=modelPrice;
            end
            lb=ub-0.05;
        else
            while modelPrice<marketPrice
                % Price at upper bound
                Pub=modelPrice;
```

```
                % If upper bound is too low then increase it
                lb=lb-0.01;

                % Calc for OAS=ub
                obj.OAS=lb;
                weightedPrices(obj);
                modelPrice=hermiteInterpolation(obj.space,obj.prices,...
                    shortRate);

                % Price at lb
                Plb=modelPrice;
            end
            ub=lb+0.01;
        end

        deviation=1;
        while abs(deviation)>0.01
            %obj.OAS=0.5*(lb+ub);
            obj.OAS=lb+(ub-lb)/(Pub-Plb)*(marketPrice-Plb);
            weightedPrices(obj);
            modelPrice=hermiteInterpolation(obj.space,obj.prices,...
                shortRate);
            deviation=modelPrice-marketPrice;
            if deviation>0
                lb=obj.OAS;
                Plb=modelPrice;
            else
                ub=obj.OAS;
                Pub=modelPrice;
            end
        end
        val=obj.OAS;
    end

    function val=getXstar(obj,shortRate,lb,ub)
        % if bounds are provided then use these
        if nargin<3
            lb=0;
            ub=1;
        end
        l=101;
        k=0;
        mb=0.5*(lb+ub);
        obj.X=mb;
        while abs(l-100*(1+obj.X))>0.01&&k<14
            mb=0.5*(lb+ub);
```

```
                obj.X=mb;
                [~,Ml,r]=obj.pricingFD;
                l=hermiteInterpolation(r,Ml,shortRate);
                if l>100*(1+obj.X)
                    lb=mb;
                else
                    ub=mb;
                end
                k=k+1;

                % Terminate while loop if we hit 0 og 1
                if round(mb,3)==0
                    mb=0;break;
                elseif round(mb,3)==1
                    mb=1;break;
                end
            end

            % Save result
            val=mb;
        end


        function [M_a,M_l,space,time]=pricingFD(obj)
            % Calculate annuity and payment dates
            annuitySchedule=obj.schedule;
            Y=annuitySchedule(:,4);
            fee=annuitySchedule(:,5)*obj.feeRateLong/obj.n;
            principal=annuitySchedule(:,5);

            % Terminal conditions
            M_a=zeros(obj.N+1,1);
            M_l=zeros(obj.N+1,1);

            % Get OAS
            if isempty(obj.OAS)
                spread=0;
            else
                spread=obj.OAS;
            end
            % Prepare for pricing
            obj.prepare;

            for j=obj.J+1:-1:1

                % Check if date is paydate
                if mod(j-1,obj.n2)==0
```

```

            % Get index for payment/principal
            index=(j-1)/obj.n2+1;

            % OAS discounting
            if j<(obj.J+1)
                M_a=exp(-1/obj.n*spread)*M_a;
            end

            % If paydate then calculate new principal
            if j==1
                F_t=obj.F0;
            else
                F_t=principal(index);
            end

            % Delivery: min(F,M).  No-Delivery: F
            if obj.delivery
                G=min(F_t,M_a);
            else
                G=F_t;
            end

            % Adjust for fees on 3M ARM
            P=exp(-obj.space*obj.n);
            G_l=(1+0.25*obj.feeRateShort*P).*(1+obj.X).*G;

            % Calculate Lambda (by smoothing)
            b=G_l<M_l;b=b+0;
            if obj.smoothing
                input=M_l-G_l;
                jumps=[0;diff(b)];
                IX=(1:obj.N+1)'.*abs(jumps);IX(abs(jumps)==0)=[];
                for i=1:size(IX,1)
                    minIX=max(IX(i)-25,1);
                    maxIX=min(IX(i)+24,size(b,1));
                    if minIX==1
                        b(minIX:maxIX)=...
                            obj.smoothIndicator(...
                            input(1:maxIX-minIX+1));
                    elseif maxIX==size(IX,1)
                        b(minIX:maxIX)=...
                            obj.smoothIndicator(...
                            input(size(b,1)-(maxIX-minIX)+1:end));
                    else
                        b(minIX:maxIX)=...
                            obj.smoothIndicator(input(minIX:maxIX));
```

```
                            end
                        end
                    end

                    LAMBDA=b.*(1-exp(-1/(obj.n)*obj.lambda));

                    % Take "expected value"
                    if j>1
                        M_l=M_l.*(1-LAMBDA)+LAMBDA.*G_l;
                        M_a=M_a.*(1-LAMBDA)+LAMBDA.*G;
                    end

                    % Add payment
                    M_a=M_a+Y(index);
                    M_l=M_l+Y(index)+fee(index);
                end

                % FD unless we are at t=0
                if j>1
                    % Calculate new values of liabilities
                    M_l=obj.matrixContainer{j,1}*M_l;

                    % Calculate new values of bond
                    M_a=obj.matrixContainer{j,1}*M_a;
                end
            end

            % Set output
            space=obj.space;
            time=obj.time;

        end

        % Prepare for pricing.
        function prepare(obj)

            % Get forward curve and its derivative
            if isempty(obj.theta)
                forward=obj.mkt.forwardCurve(obj.time);
                derivative=obj.mkt.forwardCurveDiff(obj.time);

                % The derivative may explode due to strange behaviour of
                % the short end of the curve. Hence, we set the derivative
                % in [0;1) equal to the derivative at time t=1.
                B=obj.time<1;
                derivative(B)=derivative(sum(B));

```

```
                % Get theta
                obj.theta=forward+1/obj.kappa*derivative...
                    +obj.sigma^2/(2*obj.kappa)...
                    *(1-exp(-2*obj.kappa*obj.time));
            end

            % If FD matrices are empty then recalculate them
            if isempty(obj.matrixContainer)
                obj.updateFDmatrices;
            end
        end

    end

    % Private methods
    methods(Access = private)
        function weightedPrices(obj)

            % Get borrower distribution
            x=(obj.privateCDFgrid(2:end)+obj.privateCDFgrid(1:end-1))./2;
            probabilities=obj.CDF(2:end)-obj.CDF(1:end-1);

            % Get prices
            M=nan(obj.N+1,size(x,1));
            for i=1:size(x,1)
                obj.X=x(i);
                [M(:,i)]=pricingFD(obj);
            end
            % Weight prices
            obj.prices=M*probabilities;

            % Calc MOAD
            temp=-(obj.prices(3:end)-obj.prices(1:end-2))./(2*obj.dr*100);
            obj.OAD=[temp(1);temp;temp(end)];
            obj.MOAD=obj.OAD./obj.prices*100;

            % Calc MOAC
            temp=(obj.prices(3:end)-2*obj.prices(2:end-1)...
                +obj.prices(1:end-2))./(obj.dr^2*100^2);
            obj.OAC=[temp(1);temp;temp(end)];
            obj.MOAC=obj.OAC./obj.prices*100;

            % Get prices at t+1
            M=nan(obj.N+1,size(x,1));
            saveT=obj.T;obj.T=saveT-0.25;
            for i=1:size(x,1)
                obj.X=x(i);
```

```
                [M(:,i)]=pricingFD(obj);
            end
            obj.T=saveT;

            % Calc carry
            obj.Carry=(M*probabilities-obj.prices)./0.25;

        end

        function updateFDmatrices(obj)
            % Preallocation
            obj.matrixContainer=cell(obj.J+1,1);
            r=obj.space;

            for j=obj.J+1:-1:1
                if ((j-1)/obj.n2)==round((j-1)/obj.n2)&&obj.Rannacher

                    % Perform Rannacher startup
                    dt05=obj.dt/obj.RannacherSteps;

                    % Mu and sigma
                    Mu=obj.kappa*(obj.theta(j)-obj.space);
                    SIG=obj.sigma^2;

                    % A B C D vectors
                    A=-1/(2*obj.dr)*Mu+1/2*SIG*1/(obj.dr^2);A=A(:);
                    B=-1/dt05-1/obj.dr^2*SIG-r;B=B(:);
                    C=1/(2*obj.dr)*Mu+1/(2*obj.dr^2)*SIG;C=C(:);
                    D=-1/dt05*ones(obj.N+1,1);D=D(:);

                    % Correcting boundaries (zero convexity)
                    B(1)=-1/dt05-Mu(1)/obj.dr-r(1);
                    C(1)=Mu(1)/obj.dr;
                    D(1)=-1/dt05;
                    A(end)=-Mu(end)/obj.dr;
                    B(end)=-1/dt05+Mu(end)/obj.dr-r(end);
                    D(end)=-1/dt05;

                    % Left Hand Side Matrix
                    LHS1=[[zeros(1,obj.N);diag(A(2:end))],zeros(obj.N+1,1)];
                    LHS2=diag(B);
                    LHS3=[zeros(obj.N+1,1),[diag(C(1:end-1));...
                        zeros(1,obj.N)]];
                    LHS=LHS1+LHS2+LHS3;

                    % Right Hand Side Matrix
                    RHS=diag(D);
```

```
                % Invert LHS and multiply to RHS
                obj.matrixContainer{j,1}=(LHS\RHS)^obj.RannacherSteps;
            else
                % Mu and sigma
                Mu=obj.kappa*(obj.theta(j)-obj.space);
                SIG=obj.sigma^2;

                % A B C D vectors
                A=1/(4*obj.dr)*Mu-1/(4*obj.dr^2)*SIG;A=A(:);
                B=1/obj.dt+1/2*SIG*1/(obj.dr^2)+1/2*r;B=B(:);
                C=-1/(4*obj.dr)*Mu-1/(4*obj.dr^2)*SIG;C=C(:);
                D=1/obj.dt-1/(2*obj.dr^2)*SIG-1/2*r;D=D(:);

                % Correcting boundaries (zero convexity)
                B(1)=1/obj.dt+Mu(1)/(2*obj.dr)+1/2*r(1);
                C(1)=-Mu(1)/(2*obj.dr);
                D(1)=1/obj.dt-Mu(1)/(2*obj.dr)-1/2*r(1);
                A(end)=Mu(end)/(2*obj.dr);
                B(end)=1/obj.dt-Mu(end)/(2*obj.dr)+1/2*r(end);
                D(end)=1/obj.dt+Mu(end)/(2*obj.dr)-1/2*r(end);

                % Left Hand Side Matrix
                LHS1=[[zeros(1,obj.N);diag(A(2:end))],zeros(obj.N+1,1)];
                LHS2=diag(B);
                LHS3=[zeros(obj.N+1,1),[diag(C(1:end-1));...
                    zeros(1,obj.N)]];
                LHS=LHS1+LHS2+LHS3;

                % Right Hand Side Matrix
                RHS1=[[zeros(1,obj.N);diag(-A(2:end))],...
                    zeros(obj.N+1,1)];
                RHS2=diag(D);
                RHS3=[zeros(obj.N+1,1),[diag(-C(1:end-1));...
                    zeros(1,obj.N)]];

                % Invert LHS and multiply to RHS
                obj.matrixContainer{j,1}=LHS\(RHS1+RHS2+RHS3);
            end
        end
    end

    function y=smoothIndicator(~,x)
        one=ones(size(x));
        if all(x==0)
            y=one;
        else
```

```
                eps=(max(x)-min(x))/5;
                y=(0.5+1/(2*eps)*x+1/(2*pi)*...
                    sin(pi/eps*x)).*(x<=eps&-eps<=x)+one.*(x>eps);
            end
        end

        function clearFDmatrices(obj)
            obj.matrixContainer=[];
        end

        function clearPrices(obj)
            obj.prices=[];
            obj.MOAD=[];
            obj.MOAC=[];
        end
    end

    % Methods for private dependent properties.
    methods
        function val=get.R_tilde(obj)
            val=obj.R/obj.n;
        end

        function val=get.schedule(obj)
            val=annuity(obj.R,obj.F0,obj.T,obj.n);
        end

        function val=get.time(obj)
            val=(obj.t0:obj.dt:obj.T)';
        end

        function val=get.space(obj)
            val=(obj.r_min:obj.dr:obj.r_max)';
        end

        function val=get.J(obj)
            val=obj.T*obj.n*obj.n2;
        end

        function val=get.dt(obj)
            val=obj.T/obj.J;
        end

        function val=get.dr(obj)
            val=(obj.r_max-obj.r_min)/obj.N;
        end

```

```
        function set.CDF(obj,inputVal)
            obj.privateCDF=inputVal;
        end

        function val=get.CDF(obj)
            if ~isempty(obj.privateCDF)
                val=obj.privateCDF;
            else
                val=nan(size(obj.privateCDFgrid));
                for i=1:size(obj.privateCDFgrid,1)
                    [~,val(i)]=...
                        borrowerDistribution(obj.privateCDFgrid(i),...
                        obj.Xstar,obj.lambda,obj.Alpha,obj.Beta,obj.n);
                end
                obj.privateCDF=val;
            end
        end
    end

    % Get/Set for public properties
    methods
        function val=get.kappa(obj)
            val=obj.mkt.kappa;
        end
        function val=get.sigma(obj)
            val=obj.mkt.sigma;
        end
        function set.theta(obj,inputVal)
            obj.thetaPrivate=inputVal;
            obj.clearFDmatrices;
            obj.clearPrices;
        end
        function val=get.theta(obj)
            val=obj.thetaPrivate;
        end
        function set.OAS(obj,inputVal)
            obj.OASPrivate=inputVal;
            obj.clearPrices;
        end
        function val=get.OAS(obj)
            val=obj.OASPrivate;
        end
        function set.Xstar(obj,inputVal)
            obj.XstarPrivate=inputVal;
            obj.clearPrices;
            obj.privateCDF=[];
        end
```

```
        function val=get.Xstar(obj)
            val=obj.XstarPrivate;
        end
        function set.R(obj,inputVal)
            obj.OASPrivate=[];
            obj.RPrivate=inputVal;
            obj.clearPrices;
        end
        function val=get.R(obj)
            val=obj.RPrivate;
        end

        function set.market(obj,inputVal)
            obj.mkt=inputVal;
            obj.XstarPrivate=[];
            obj.thetaPrivate=[];
            obj.OASPrivate=[];
            obj.clearFDmatrices;
            obj.clearPrices;
        end

        function set.maturity(obj,inputVal)
            % Clear stuff if we select a longer maturity
            if inputVal>obj.T
                obj.thetaPrivate=[];
                obj.clearFDmatrices;
            end
            obj.clearPrices;
            obj.T=inputVal;
        end
        function val=get.maturity(obj)
            val=obj.T;
        end

        function set.Rannacher(obj,inputVal)
            obj.RannacherTimeStepping=inputVal;
            obj.clearFDmatrices;
            obj.clearPrices;
        end
        function val=get.Rannacher(obj)
            val=obj.RannacherTimeStepping;
        end

        function set.smoothing(obj,inputVal)
            obj.smoothPayoff=inputVal;
            obj.clearPrices;
        end
```

```
        function val=get.smoothing(obj)
            val=obj.smoothPayoff;
        end
    end
end
```

References

[1] Alexander Levin and Andrew Davidson *Prepayment Risk- and Option-Adjusted Valuation of MBS*, 2005.

[2] Alexander J. McNeil, Rudiger Frey and Paul Embrechts *Quantitative Risk Management*, 2015.

[3] B.A. Wadea, A.Q.M. Khaliq, M. Yousuf, J. Vigo-Aguiar & R. Deininger *On smoothing of the CrankNicolson scheme and higher order schemes for pricing barrier options*, 2007.

[4] Bjarne Astrup Jensen *Rentesregning 6. udgave*, 2013.

[5] Damiano Brigo & Fabio Mercurio *Interest Rate Models - Theory and Practice*, 2006.

[6] ECBC *2017 ECBC European Covered Bond Fact Book*, 2017

[7] Europa-parlamentets og Rådets Forordning (EU) nr. 575/2013 af 26. juni 2013 *om tilsynsmssige krav til kreditinstitutter og investeringsselskaber og om ændring af forordning (EU) nr. 648/2012*, 2013.

[8] Farshid Jamshidian *An exact bond option formula*, 1989.

[9] Fischer Black, Myron Scholes and Robert C. Merton, *The Pricing of Options and Corporate Liabilities*, 1973.

[10] Henrik Dahl, Claus Hvidegaard & Henrik Jørgensen *Hvordan fordamper OAS? Volatilitetsbias i realkreditmodeller*, 2009.

[11] James D. Hamilton *Time series analysis*, 1994.

[12] John Cox, Jonathan Ingersoll & Stephen Ross *A Theory of the Term Structure of Interest Rates*, 1985.

[13] John Hull & Alan White *Pricing Interest-Rate-Derivative Securities*, 1990.

[14] Jonathan Ingersoll *An Examination of Corporate Call Policies on Convertible Securities*, 1977.

[15] Knut Sydsætter, Atle Seierstad & Arne Strøm *Matematisk Analyse*, 2002.

[16] Lars Peter Hansen *Large Sample Properties of Generalized Method of Moments Estimators*, 1982.

[17] Martin Rein *Drop-surface interactions*, 2002.

[18] Michael B. Giles *Crank-Nicolson Scheme*, 2010.

[19] Nationalbanken, article *Stabil Udvikling i Udlandets Ejerandele af Danske Obligationer*, 28$^{\text{th}}$ of August 2017.

[20] Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski and Diana E. Woodward, *Managing Smile Risk*, 2002.

[21] Patrick Hagan & Graeme West *Interpolation Methods for Curve Construction*, 2006.

[22] Paul Kupiec & Adama Kah *On the origin and interpretation of OAS*, 1999.

[23] Paul Wilmot, Jeff Dewynne & Sam Howison *Option Pricing - Mathematical Models and Consumption*, 1993.

[24] Richard Stanton *Rational Prepayment and the Valuation of Mortgage-Backed Securities*, 1995.

[25] Robert Merton *Option Pricing When Underlying Stock Returns are Discontinuous*, 1976.

[26] Rolf Rannacher *Finite element solution of diffusion problems with irregular data*, 1984.

[27] Rudiger U. Seydel *Tools for Computational Finance*, 2009.

[28] Steven L. Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, 1993.

[29] Steven L. Heston & Guofu Zhou *On the rate of convergence of discrete-time contingent claims*, 2000.

[30] Steven Shreve *Stochastic Calculus for Finance II*, 2004.

[31] Thomas Björk *Arbitrage Theory in Continuous Time*, 2009.

[32] Xuefei He, Simon Acomb and Ser-Huang Poon *Pricing and Risk Measures of Mortgage Backed Securities with PDE Method*, 2013.