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DeepLearning.AI

Math for Machine Learning

Linear algebra - Week 4

W4 Lesson 1

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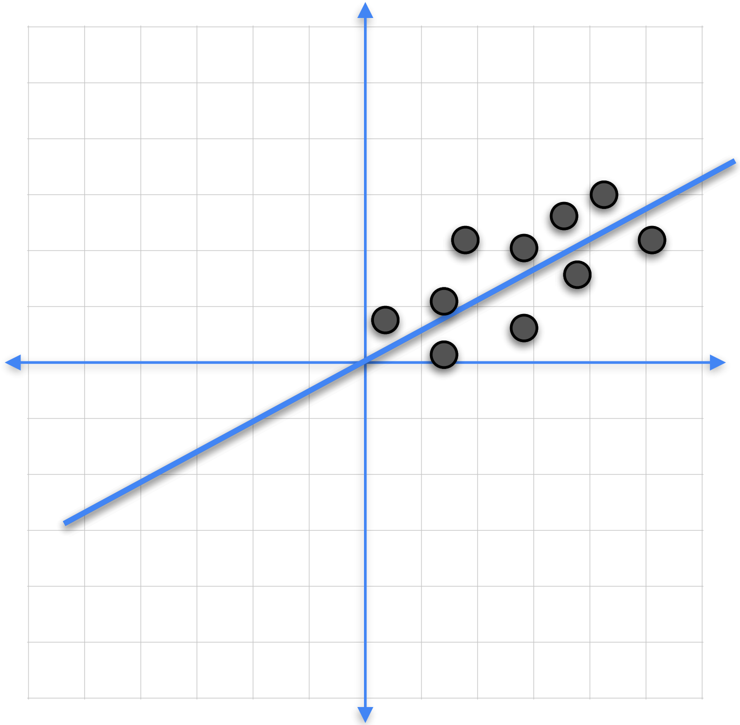
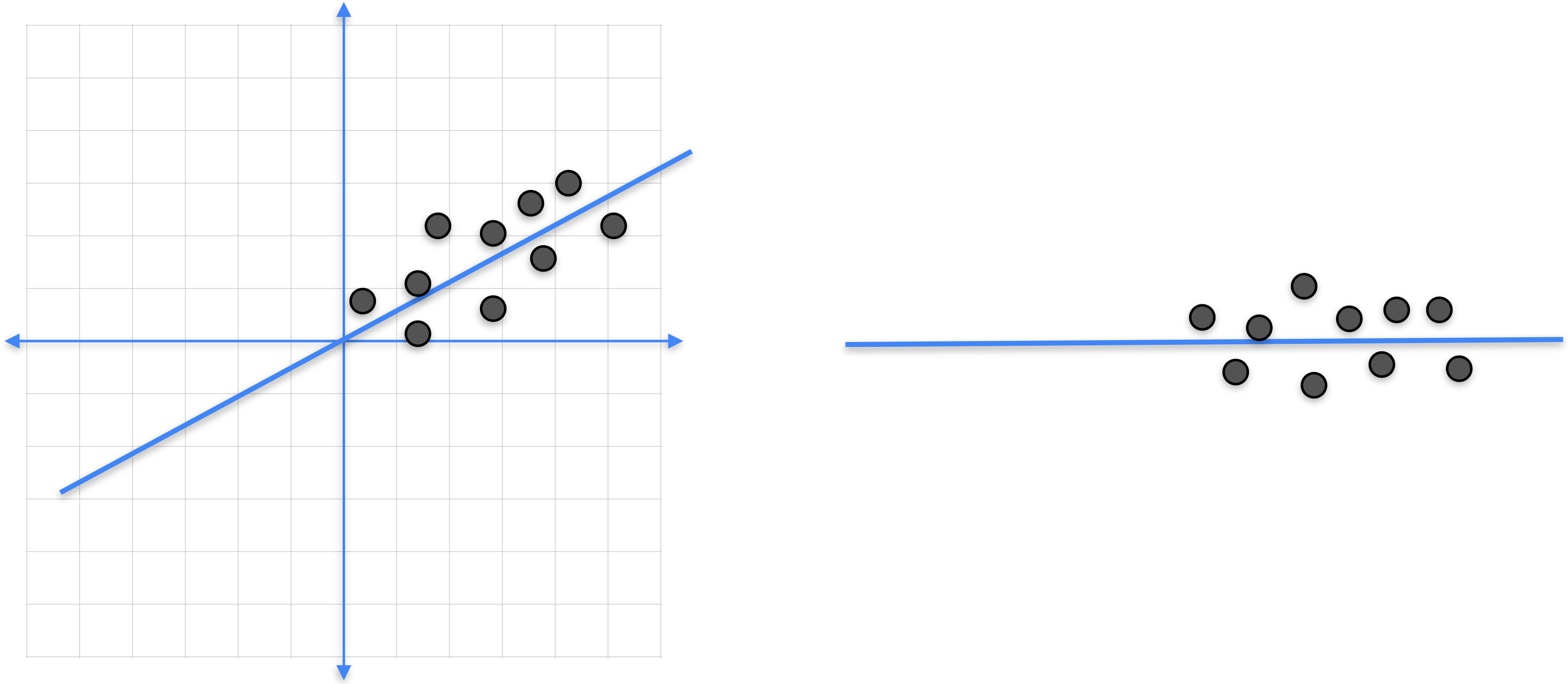
Determinants and Eigenvectors

Machine learning motivation

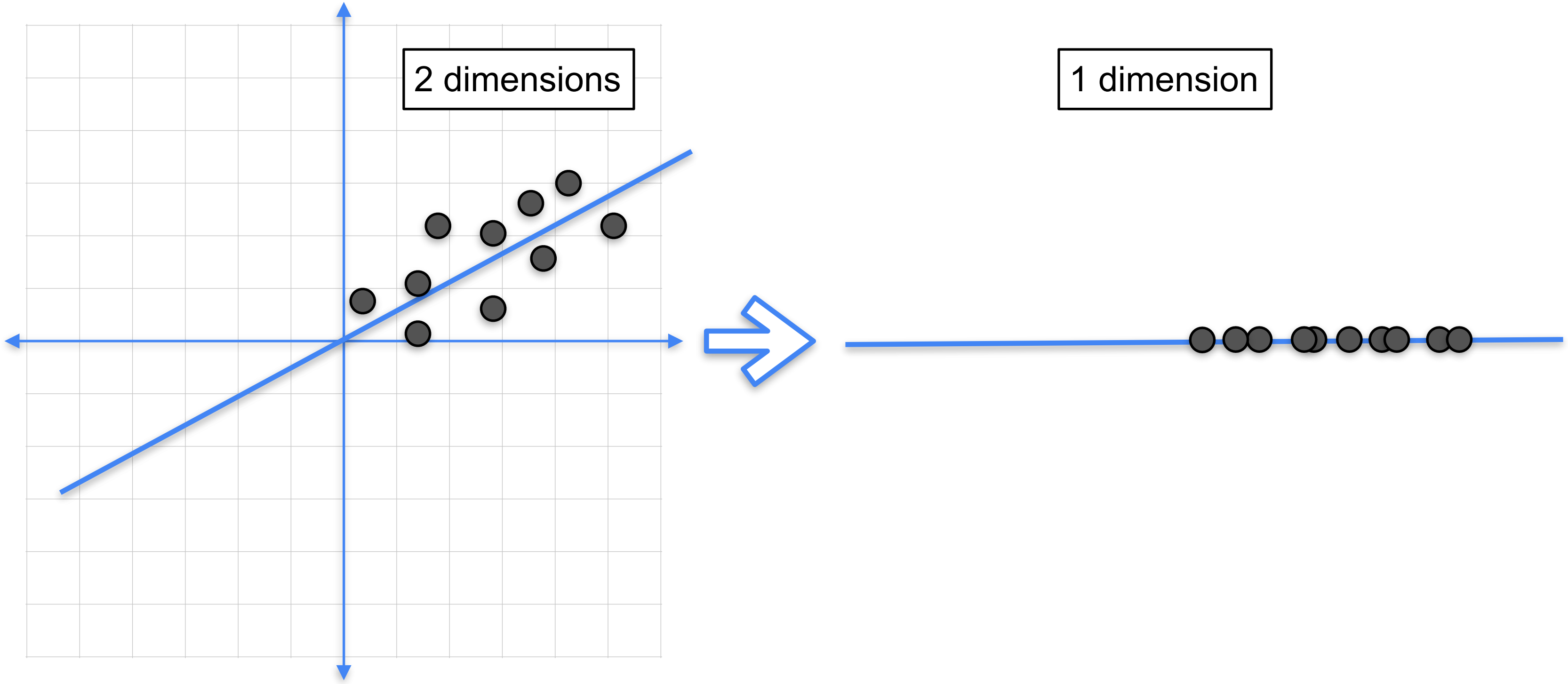
Principal Component Analysis



Principal Component Analysis

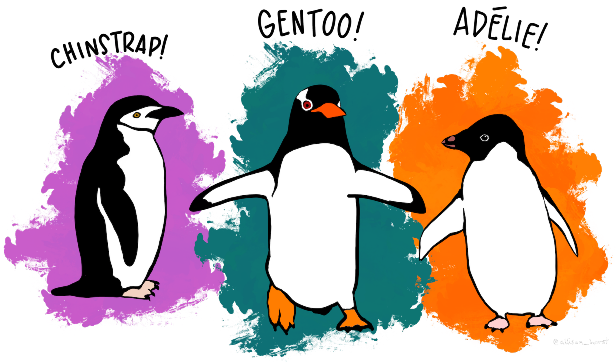


Principal Component Analysis



Principal Component Analysis

- Reduce dimensions (columns) of dataset
- Preserve as much information as possible



species	culmen_length_mm	culmen_depth_mm	flipper_length_mm	body_mass_g
Adelie	40.6	17.2	187.0	3475.0
Adelie	38.9	17.8	181.0	3625.0
Adelie	35.7	16.9	185.0	3150.0
Gentoo	50.0	15.3	220.0	5550.0
Adelie	34.5	18.1	187.0	2900.0

→

PC1	PC2	species
1.353843	-0.422253	Adelie
1.760446	-0.350965	Adelie
2.005766	-1.113797	Adelie
-2.585758	0.061768	Gentoo
2.438111	-0.786227	Adelie

What to expect?

Linear transformation

1

2

3

Characterize your transformation

What to expect?

Linear transformation

1 Singular / Non-singular

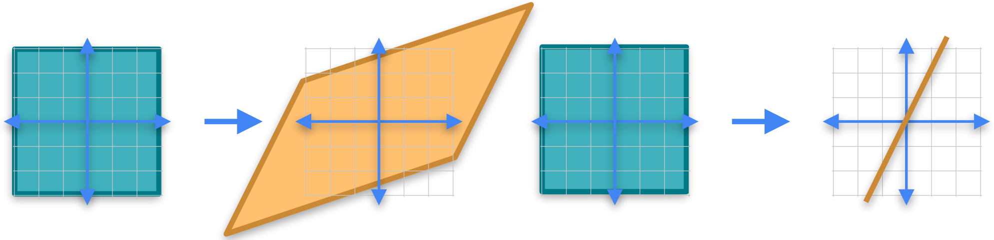
2

3

Characterize your transformation

3	1
1	2

1	1
2	2



What to expect?

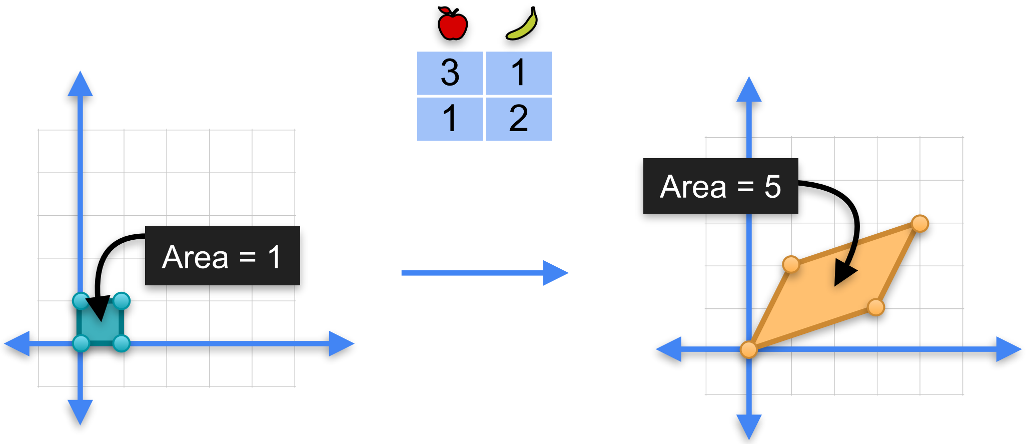
Linear transformation

1

2 Determinant as area

3

Characterize your transformation



What to expect?

Linear transformation

1

2

3 Properties of determinant

Characterize your transformation

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

Det = 5 Det = 3 Det = 15 = 5 · 3

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

Det = 5 $\text{Det} = 0.2 = \frac{1}{5}$

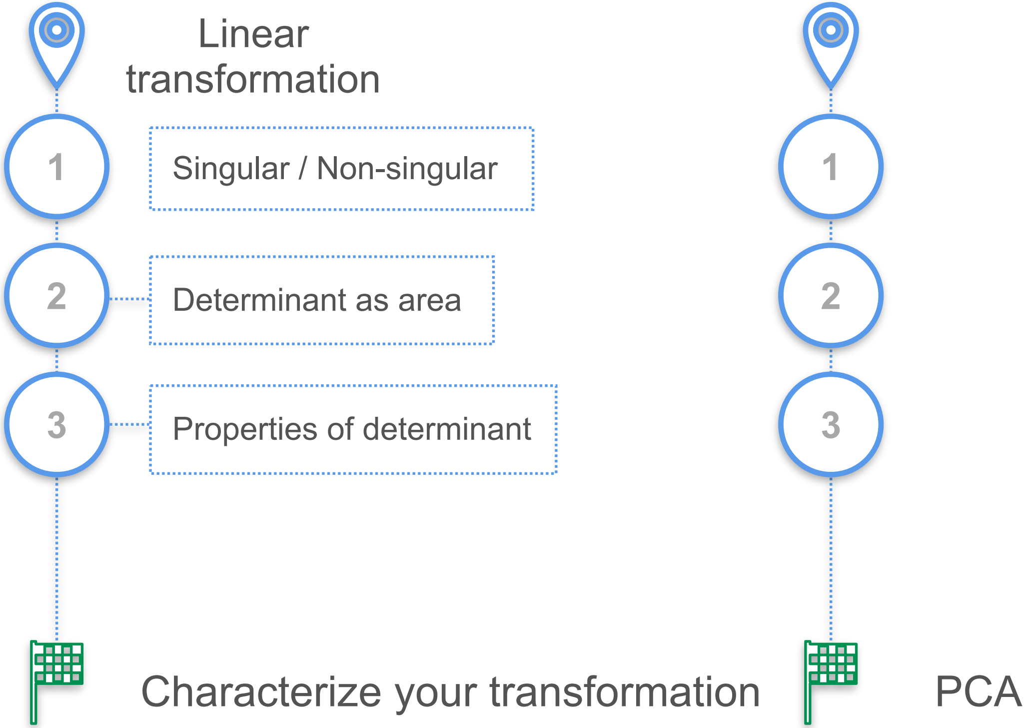
What to expect?

Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

Characterize your transformation

What to expect?



What to expect?

Linear transformation

1 Singular / Non-singular

2 Determinant as area

3 Properties of determinant

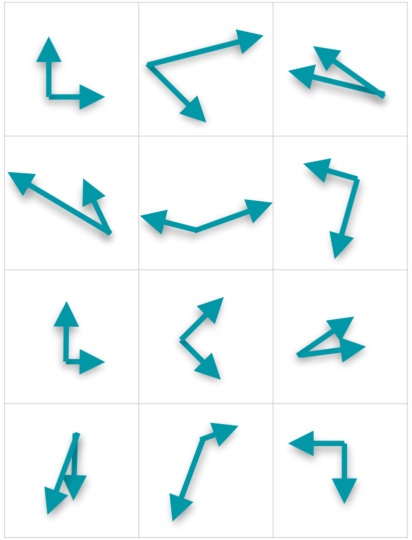
Characterize your transformation

1 Basis

2

3

PCA



What to expect?

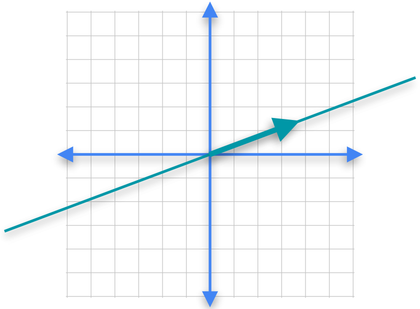
Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

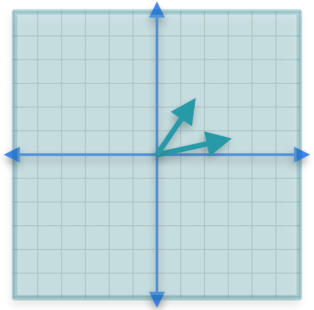
Characterize your transformation

1
2. Span
3

PCA



1 element
Dimensions: 1



2 elements
Dimensions: 2

What to expect?

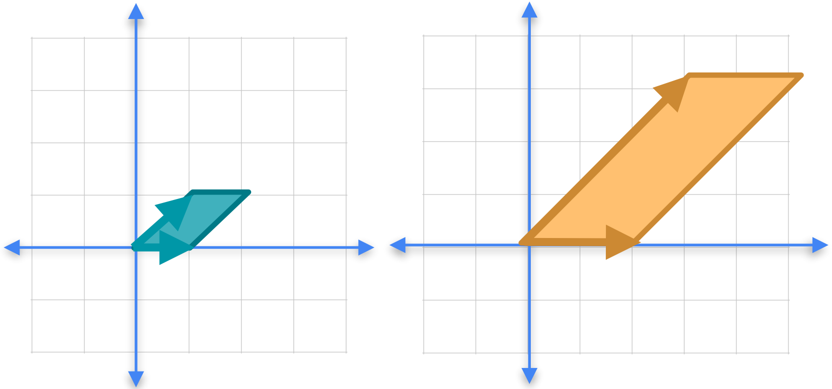
Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

Characterize your transformation

1.
2.
3. Eigenvalues and Eigenvectors

PCA



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$(1,0) \rightarrow (2,0)$

$$Av_1 = \lambda_1 v_1$$

What to expect?

Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

Characterize your transformation

1. Basis
2. Span
3. Eigenvalues and Eigenvectors

PCA

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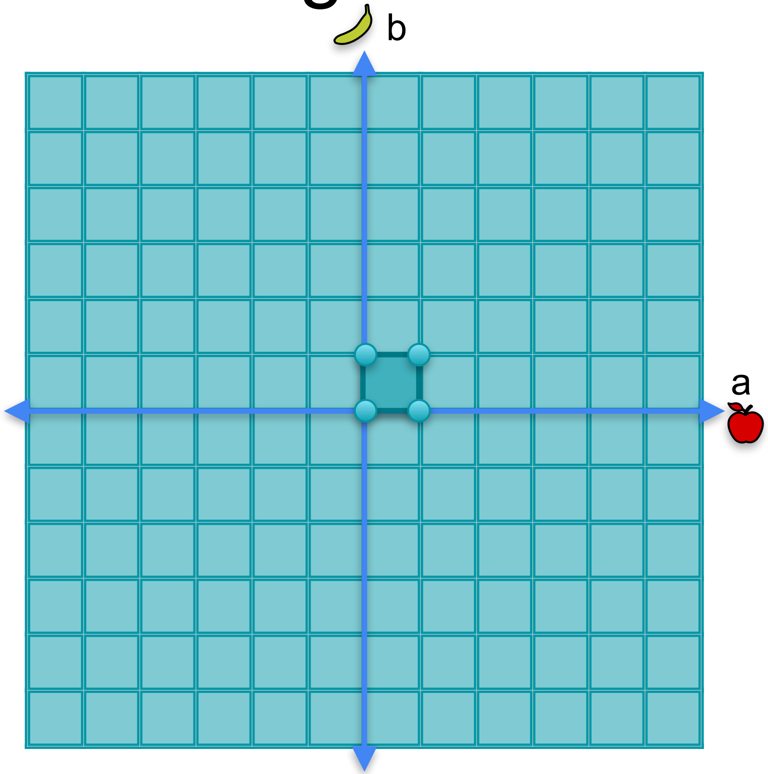
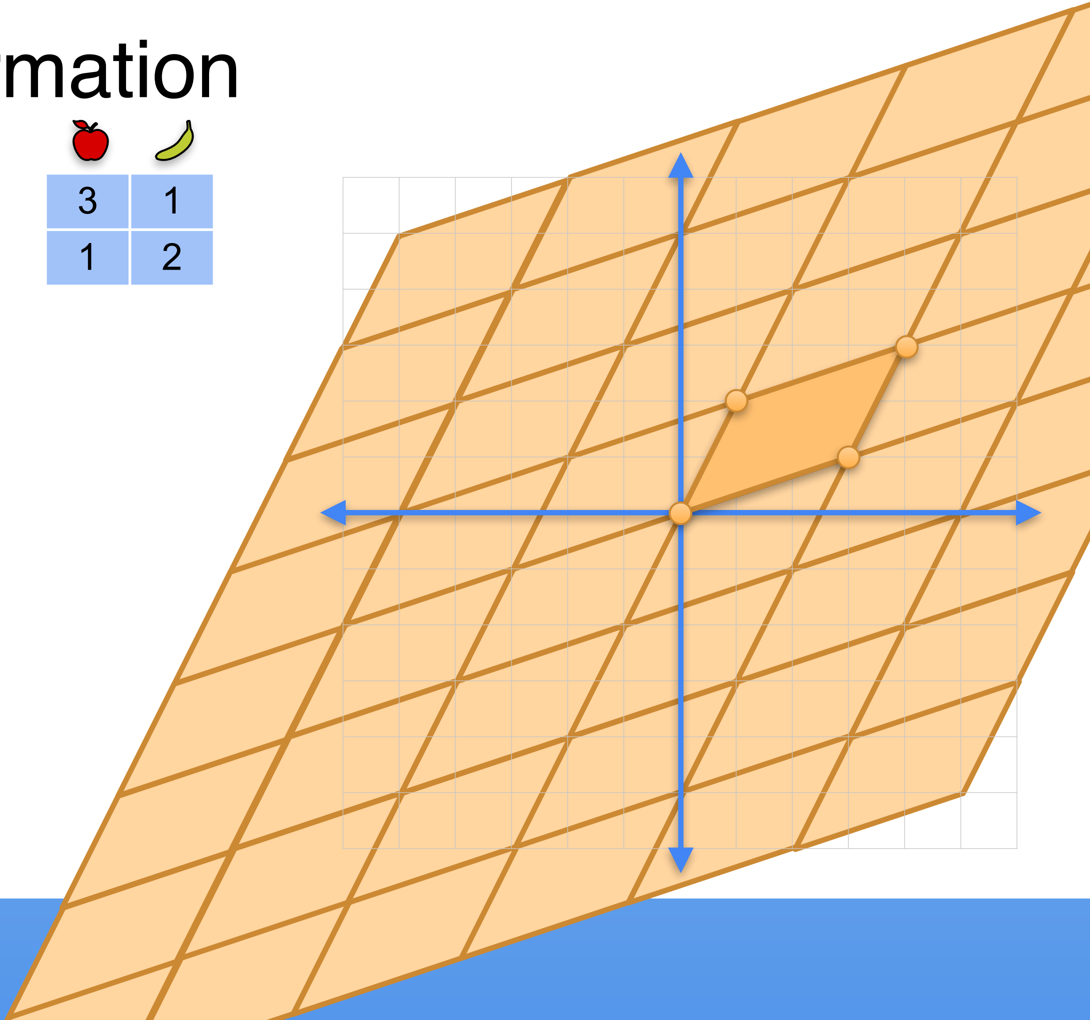
Determinants and Eigenvectors

Singularity and rank of linear transformations

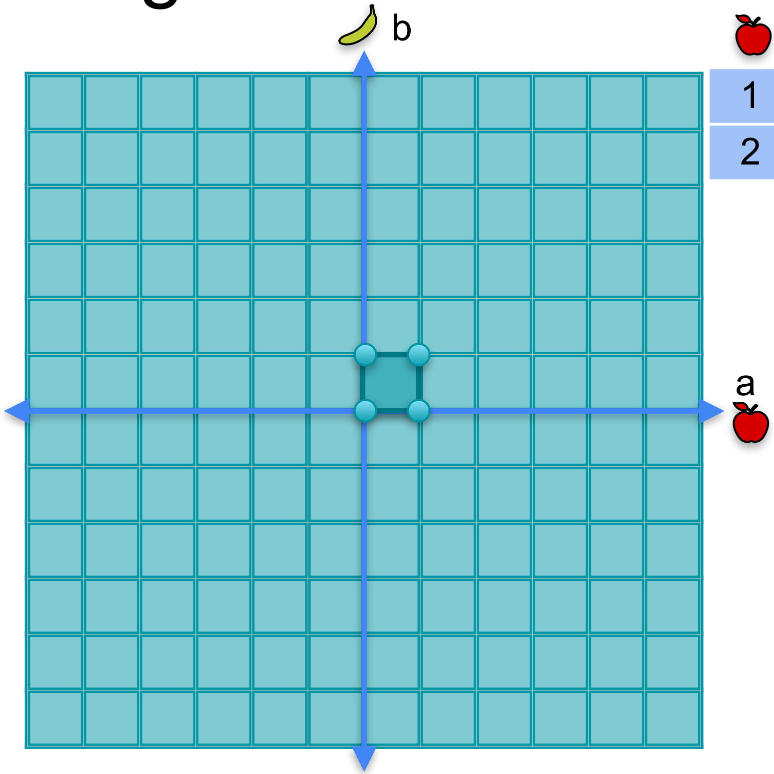
Non-singular transformation



🍎	🍌
3	1
1	2

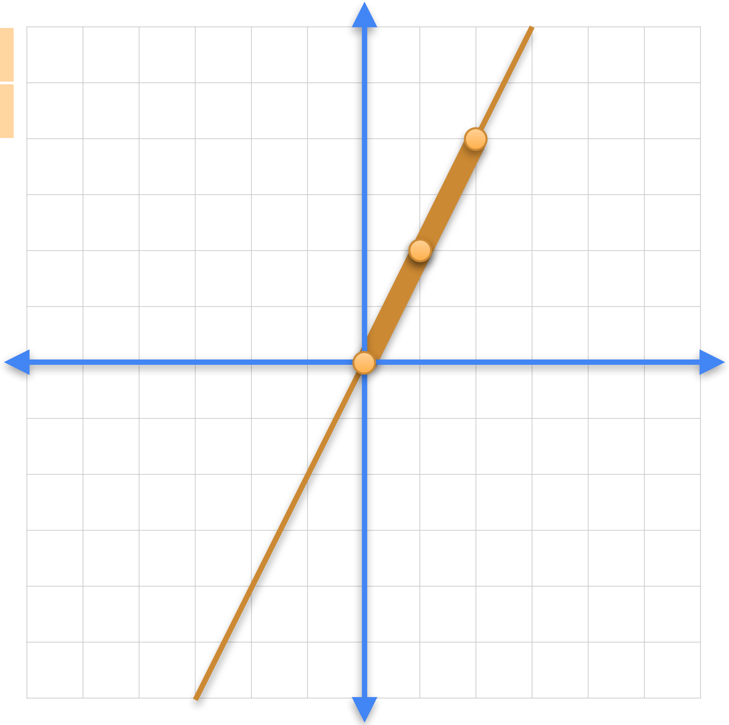


Singular transformation

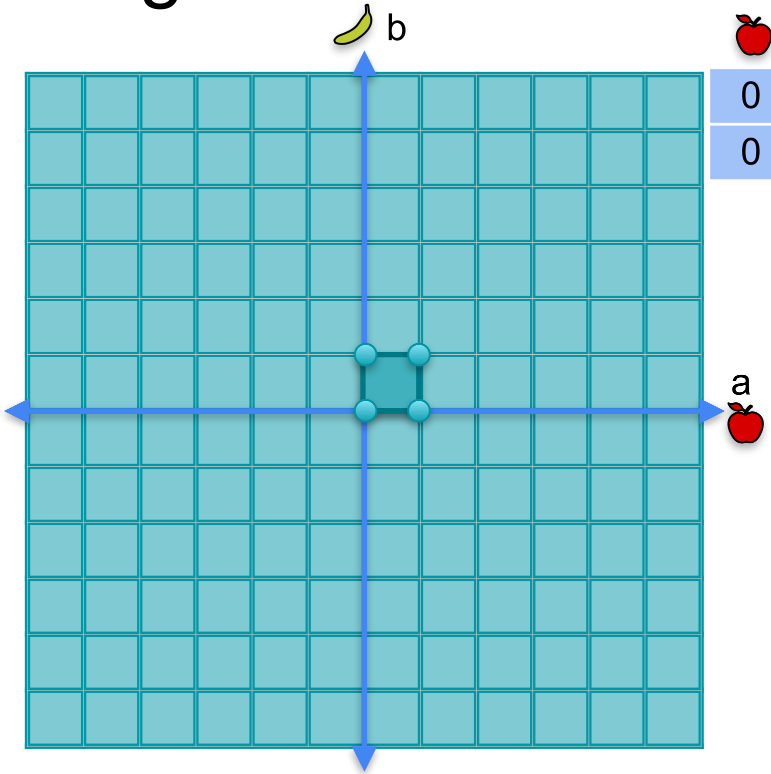


$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

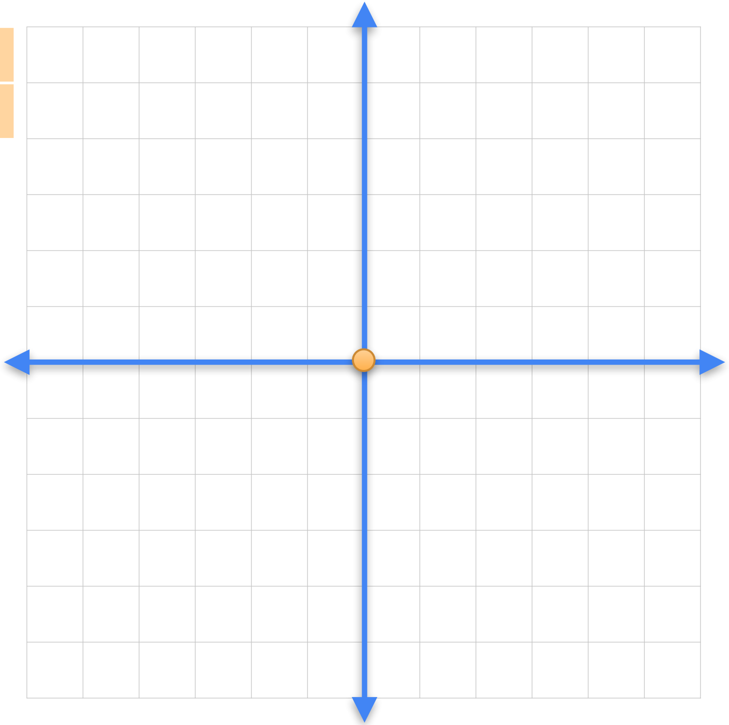
(0,0) → (0,0)
(1,0) → (1,2)
(0,1) → (1,2)
(1,1) → (2,4)



Singular transformation



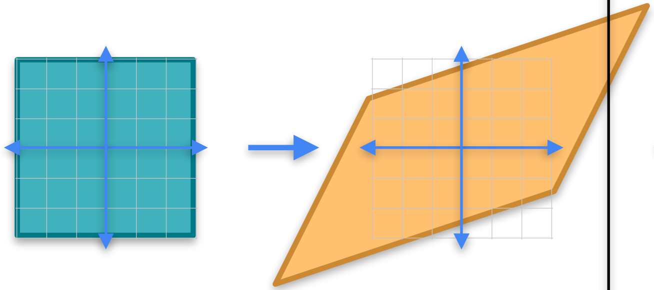
$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$



Singular and non-singular transformations

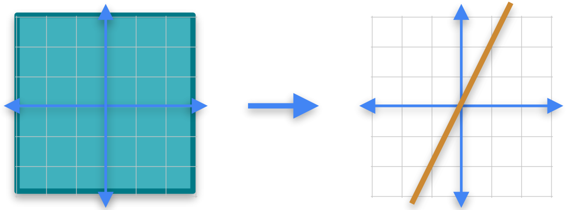
Non-singular

3	1
1	2



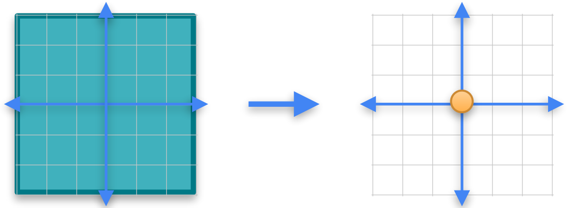
Singular

1	1
2	2



Singular

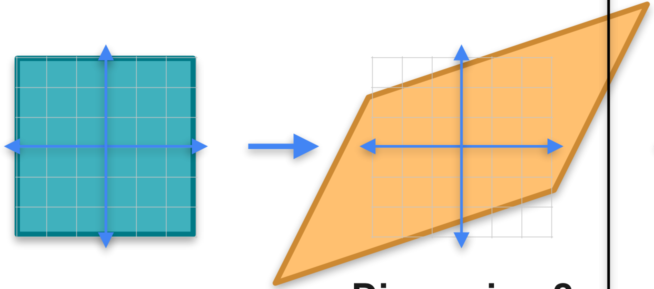
0	0
0	0



Rank of linear transformations

Rank 2

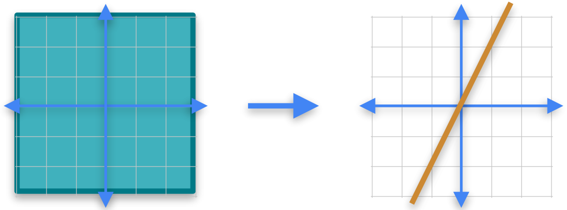
3	1
1	2



Dimension 2

Rank 1

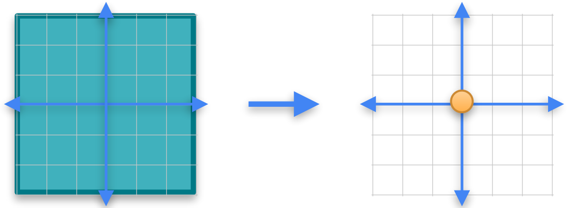
1	1
2	2



Dimension 1

Rank 0

0	0
0	0



Dimension 0

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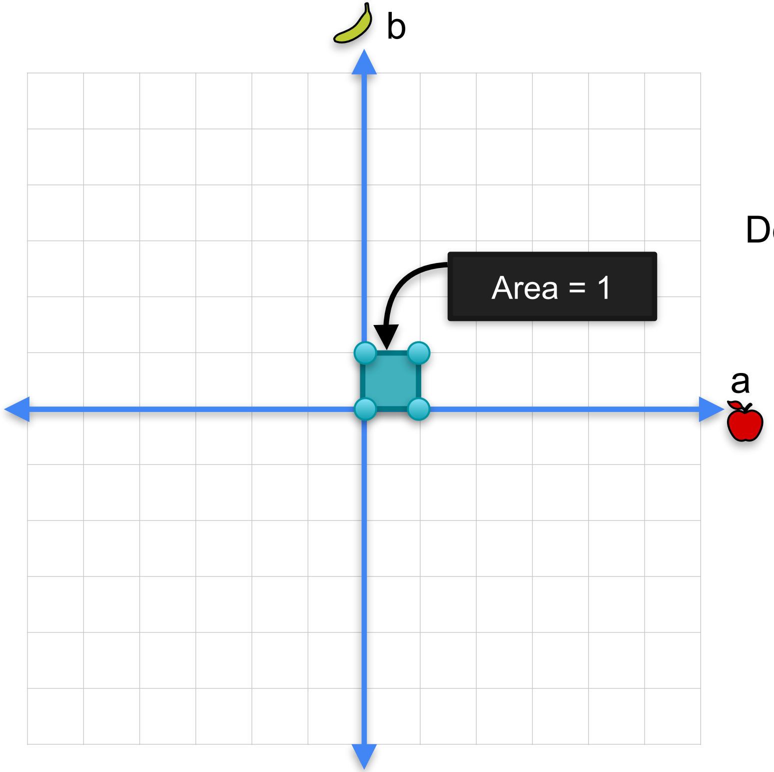
Determinants and Eigenvectors

Determinant as an area

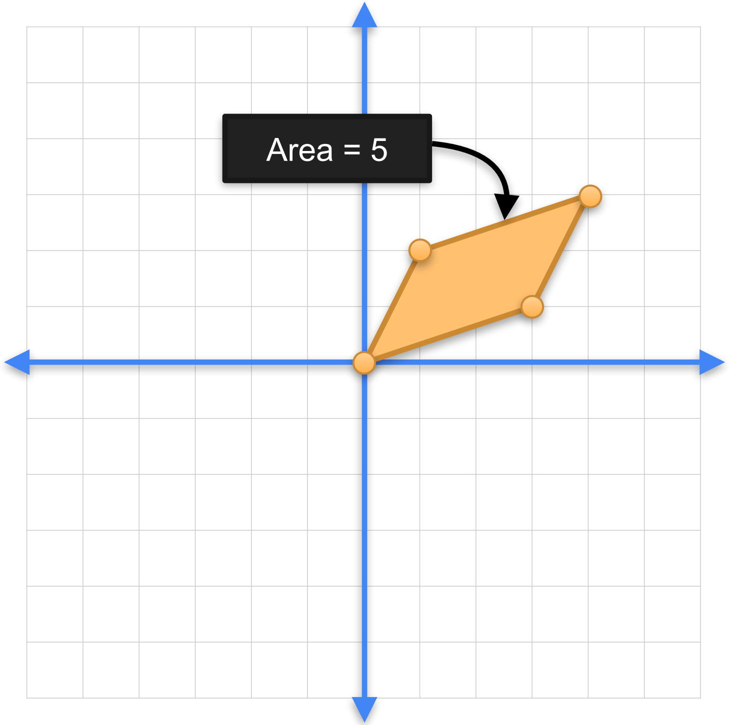
Determinant as an area



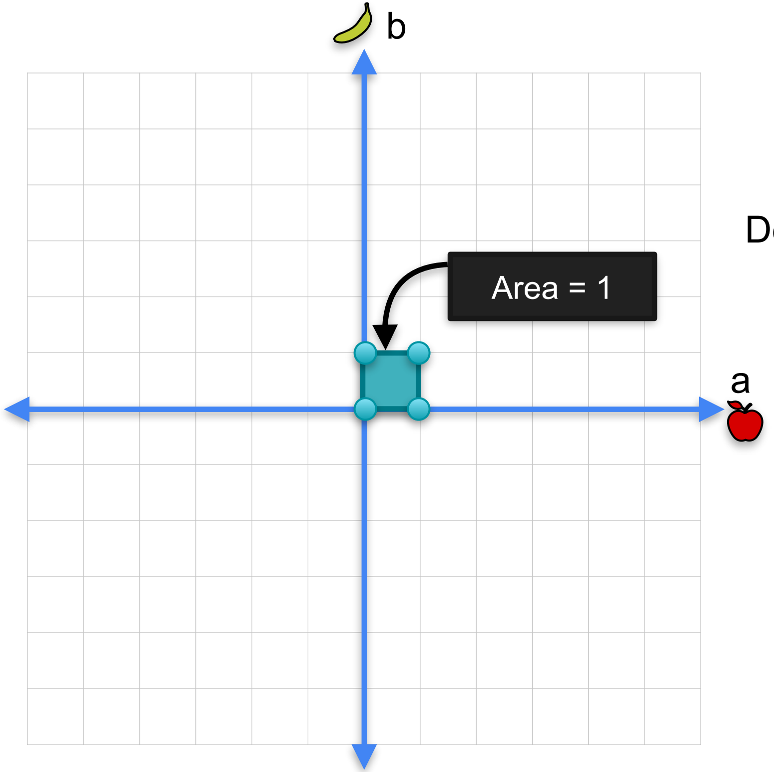
3	1
1	2

Det = $3 \cdot 2 - 1 \cdot 1$

Det = 5



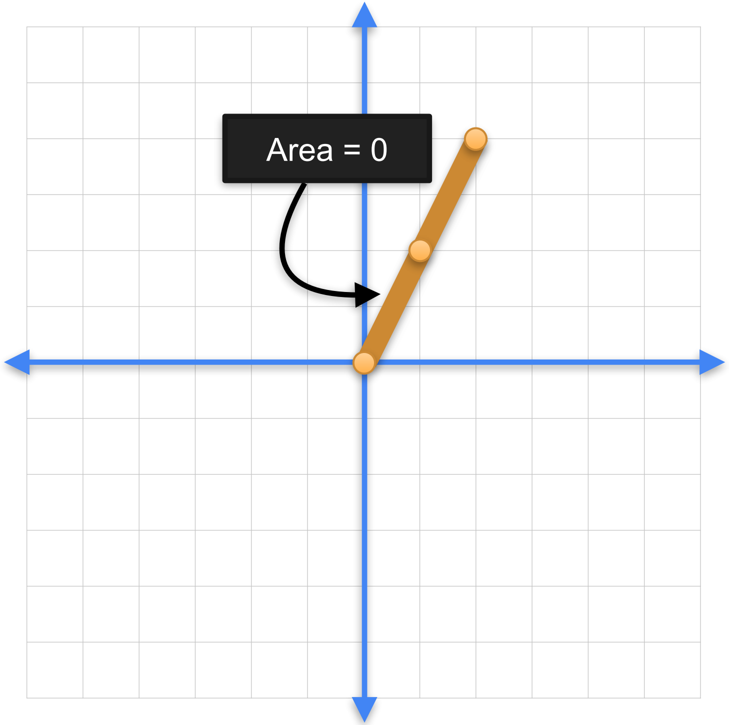
Determinant as an area



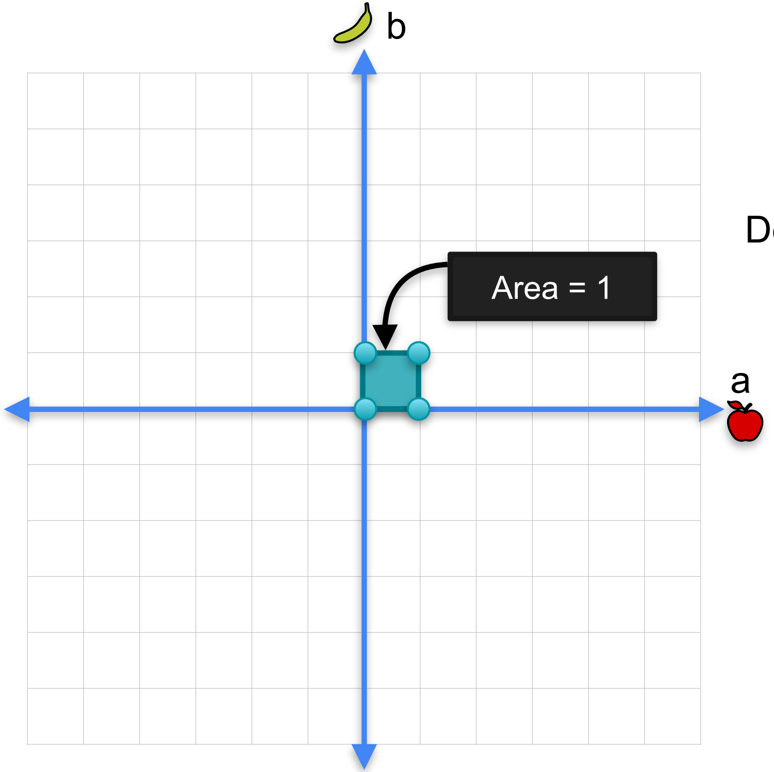
🍎	🍌
1	1
2	2

Det = $1 \cdot 2 - 1 \cdot 2$

Det = 0



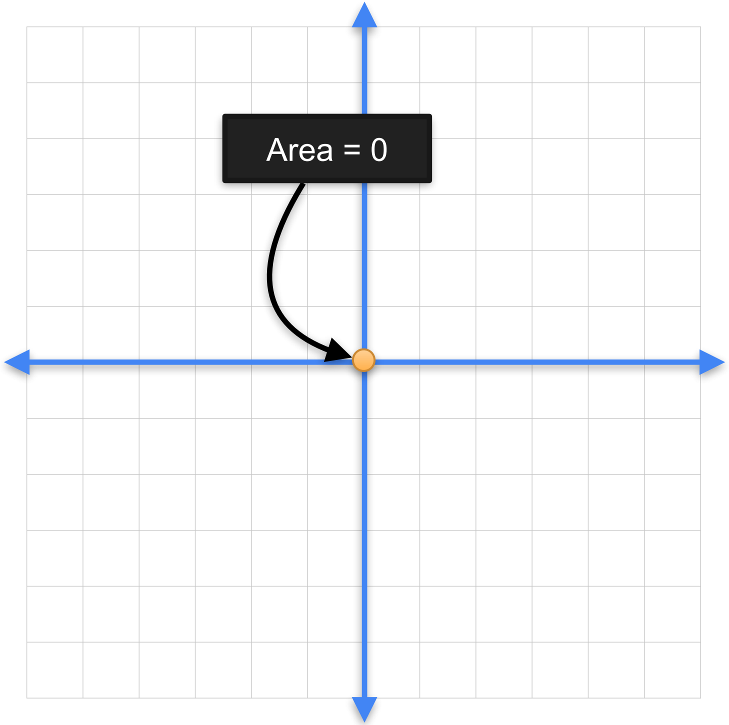
Determinant as an area



$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

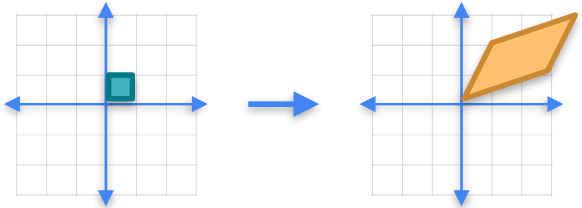


Determinant as an area

Non-singular

3	1
1	2

Determinant = 5

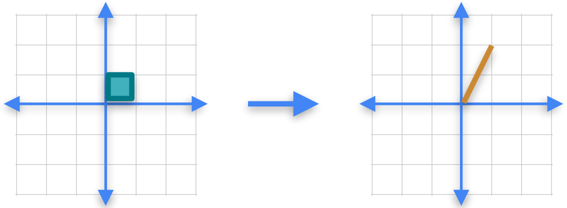


Area = 5

Singular

1	1
2	2

Determinant = 0

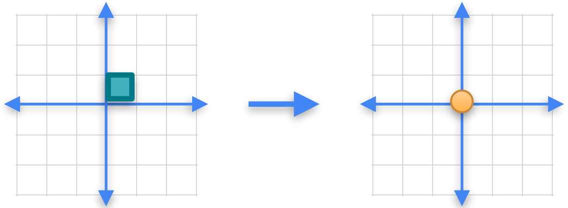


Area = 0

Singular

0	0
0	0

Determinant = 0



Area = 0

Negative determinants?

3	1
1	2

Det = $3 \cdot 2 - 1 \cdot 1$

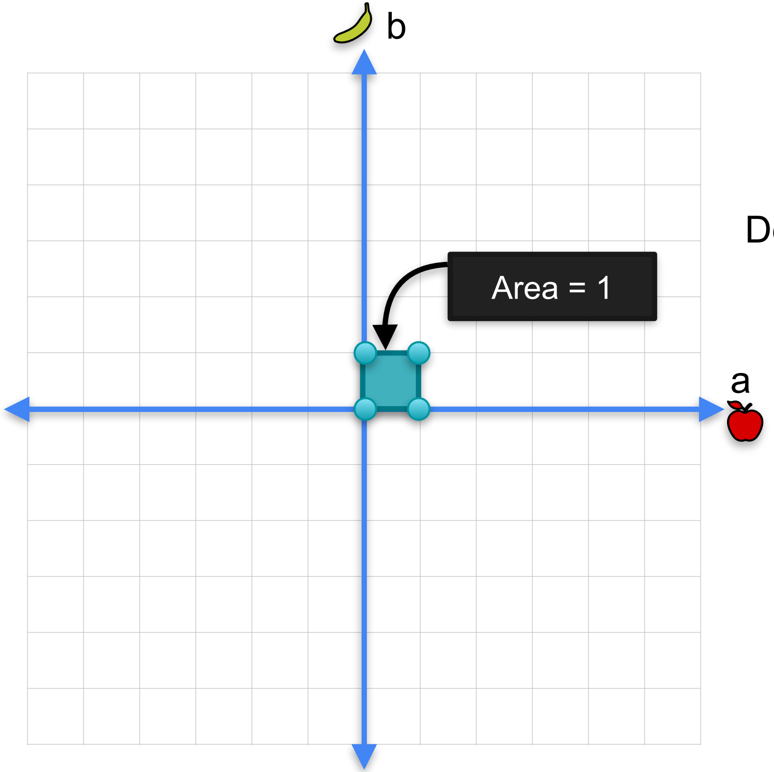
Det = 5

1	3
2	1

Det = $1 \cdot 1 - 3 \cdot 2$

Det = -5

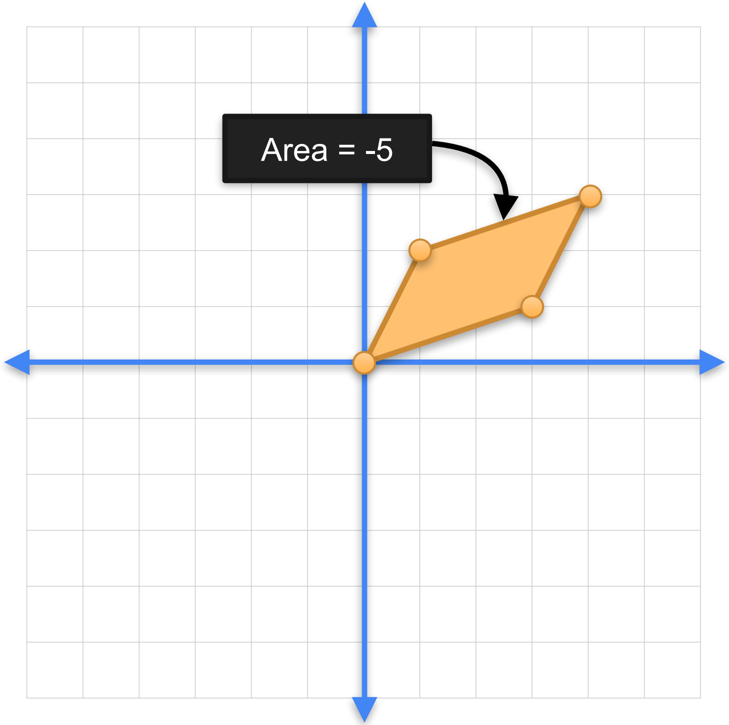
Determinant as an area



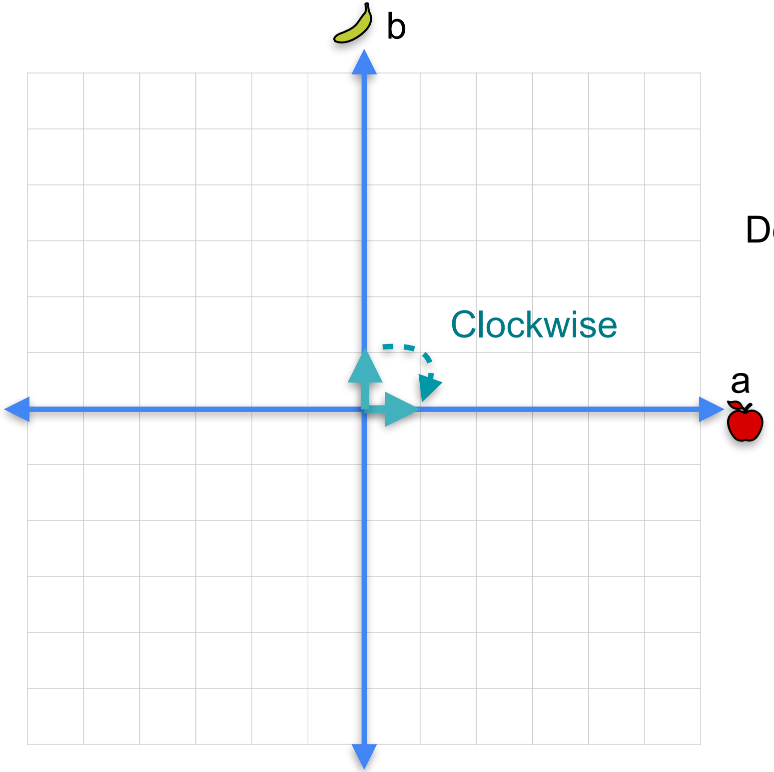
🍎	🍌
1	3
2	1

Det = $1 \cdot 1 - 3 \cdot 2$

Det = -5



Determinant as an area

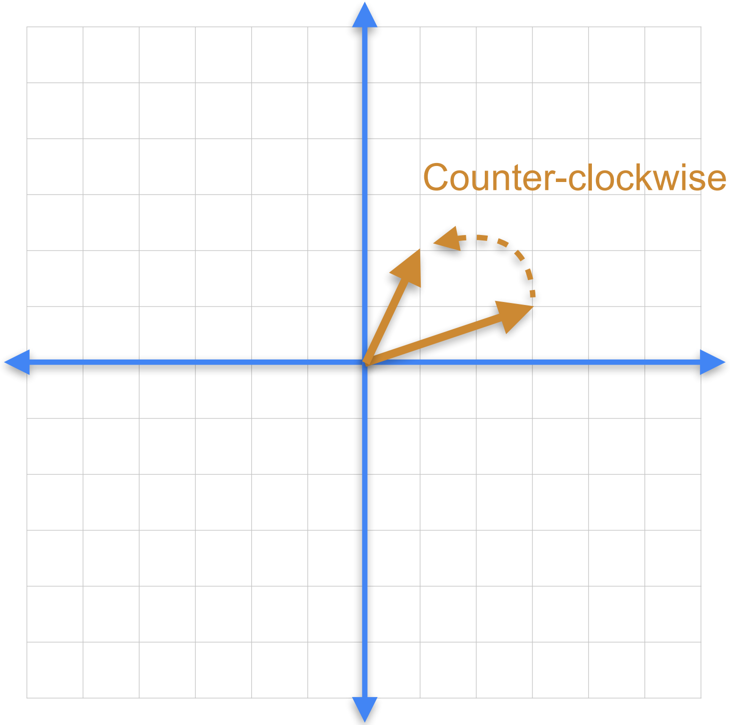


1	3
2	1

Det = $1 \cdot 1 - 3 \cdot 2$

Det = -5

Negative



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Determinants and Eigenvectors

Determinant of a product

Determinant of a product

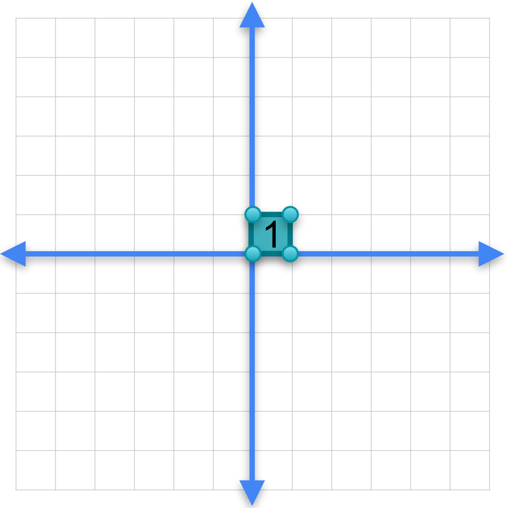
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 7 & 6 \end{bmatrix}$$

det = 5	det = 8	det = 40
$3 \cdot 2 - 1 \cdot 1$	$5 \cdot 2 - 2 \cdot 1$	$16 \cdot 6 - 8 \cdot 7$

Determinant of a product

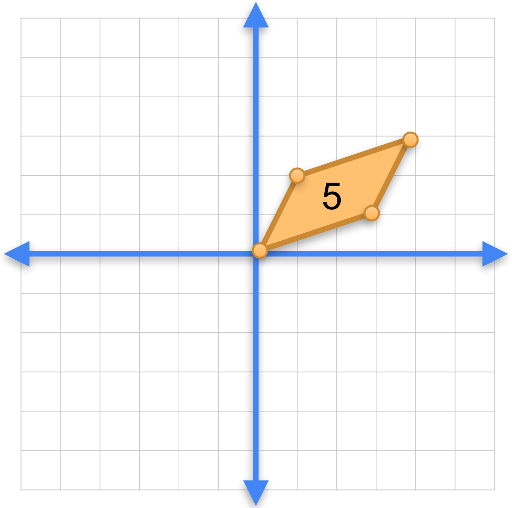
$$\det(AB) = \det(A)\det(B)$$

Determinant of a product



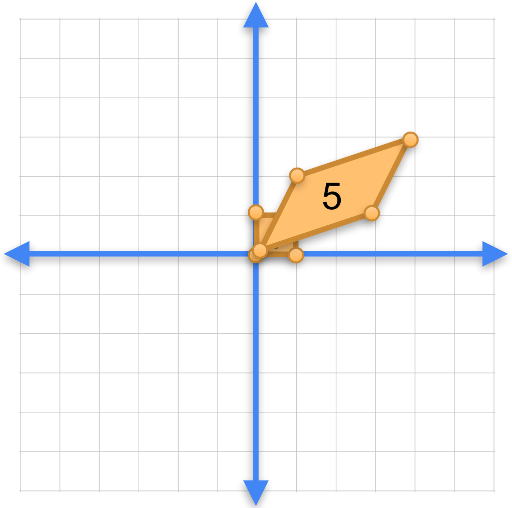
3	1
1	2

Det = 5



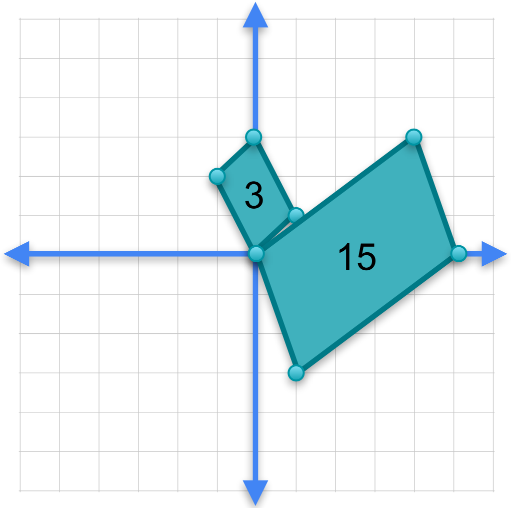
Area blows up by 5

Determinant of a product



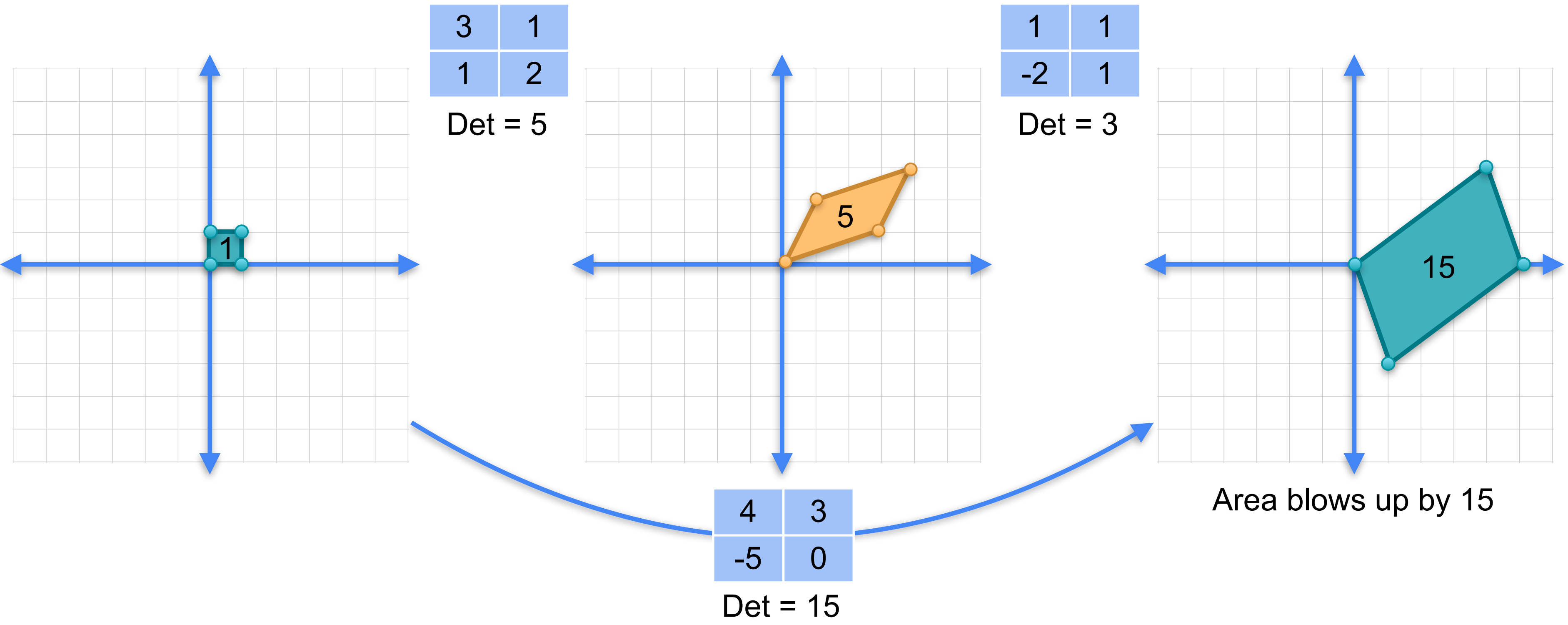
1	1
-2	1

Det = 3



Area blows up by 3

Determinant of a product



Quiz

- The product of a singular and a non-singular matrix (in any order) is:
    - Singular
    - Non-singular
    - Could be either one

Solution

- If A is non-singular and B is singular, then det(AB) = det(A) x det(B) = 0, since det(B) = 0. Therefore det(AB) = 0, so AB is **singular**.

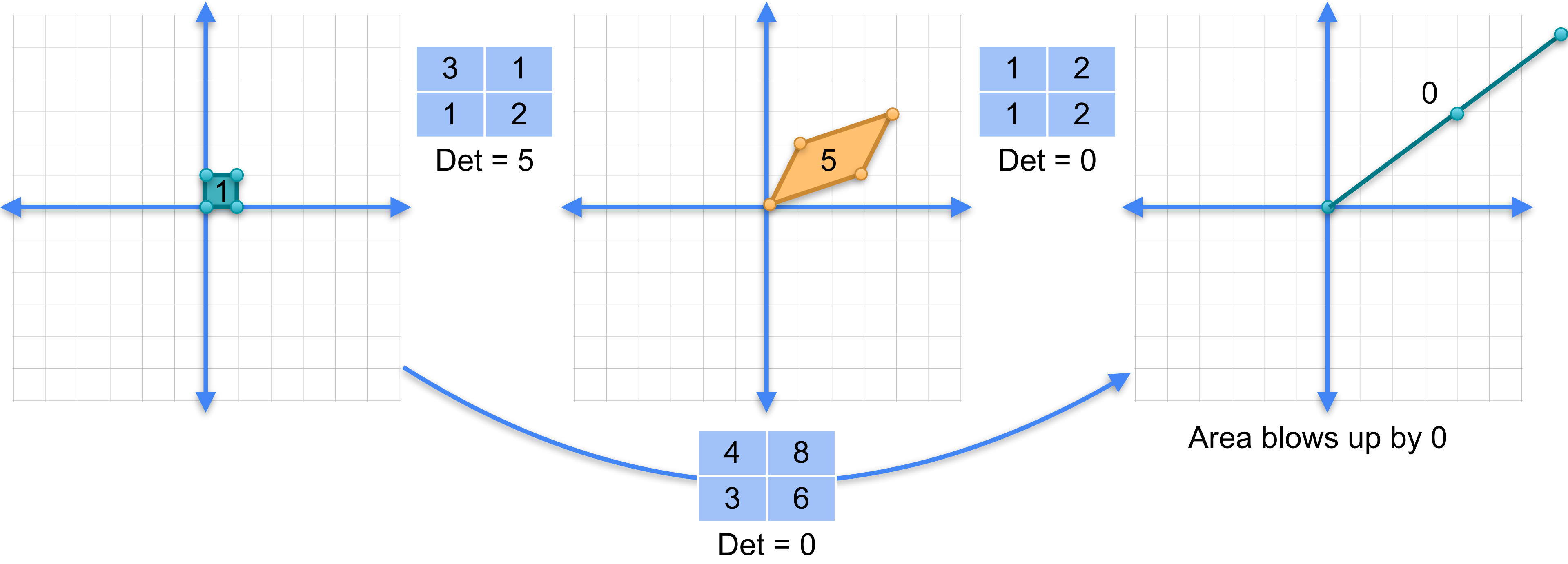
When one factor is zero

$$5 \cdot 0 = 0$$

When one factor is singular…

Non-singular			Singular			Singular	
3	1		1	2	=	4	8
1	2		1	2		3	6
Det = 5			Det = 0			Det = 0	

If one factor is singular…



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Determinants and Eigenvectors

Determinant of inverse

Quiz

- Find the determinants of the following matrices

0.4	-0.2
-0.2	0.6

0.25	-0.25
-0.125	0.625

Solution

$$\text{Det}\begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det}\begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5 det = 0.2

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8 det = 0.125

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

det = 0 det = ???

$$0^{-1} = ???$$

Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Why?

$$\det(AB) = \det(A)\det(B)$$

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(I) = \det(A)\det(A^{-1})$$

$$1 \qquad \frac{1}{\det(A)}$$

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Determinant of the identity matrix

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

W4 Lesson 2

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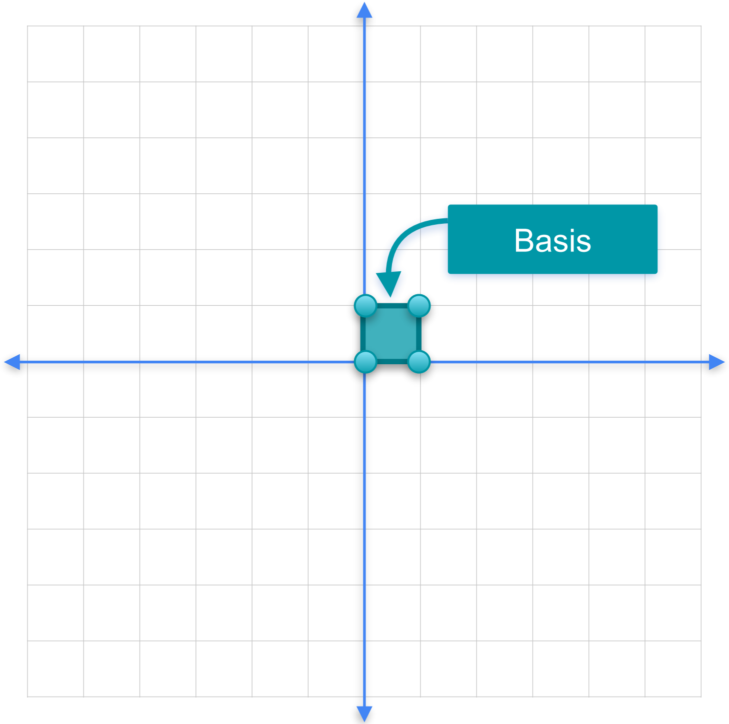
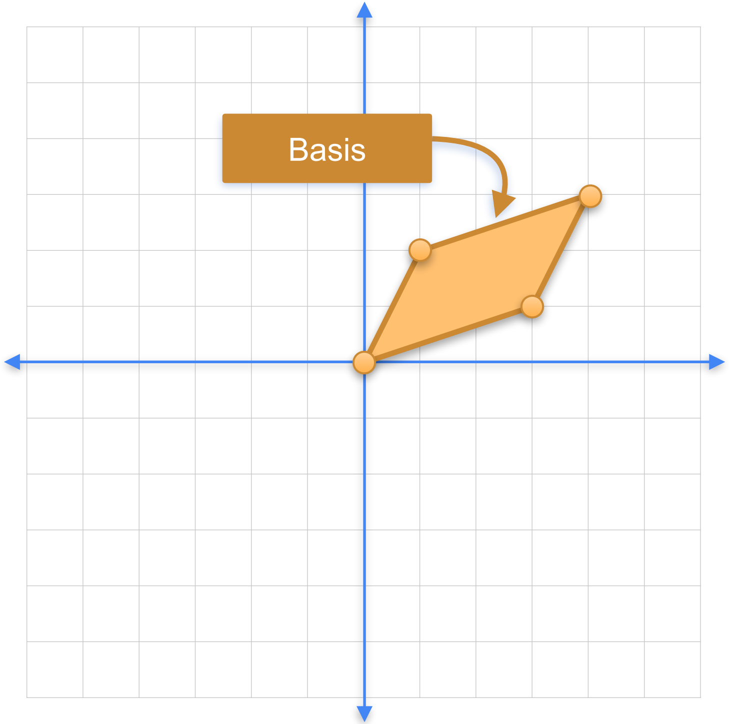
Determinants and Eigenvectors

Bases

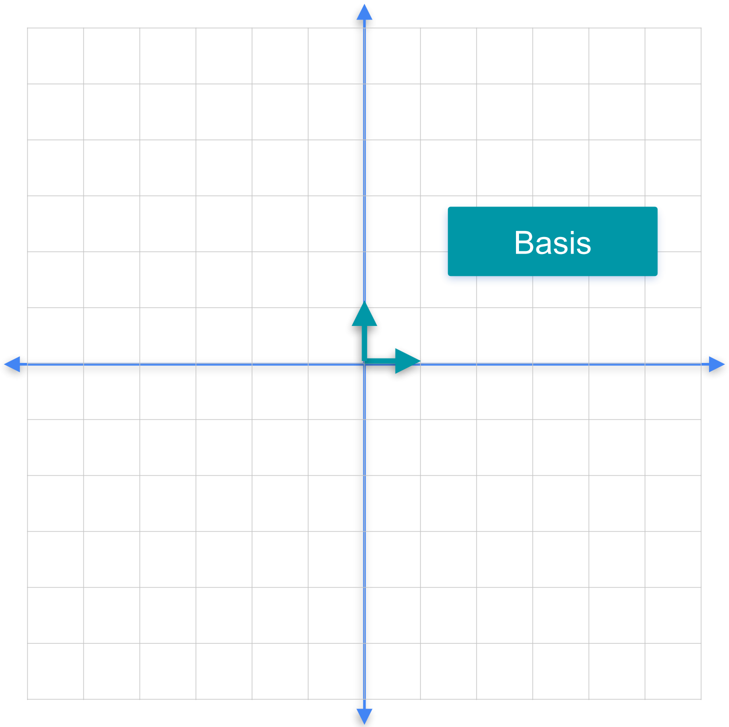
Bases



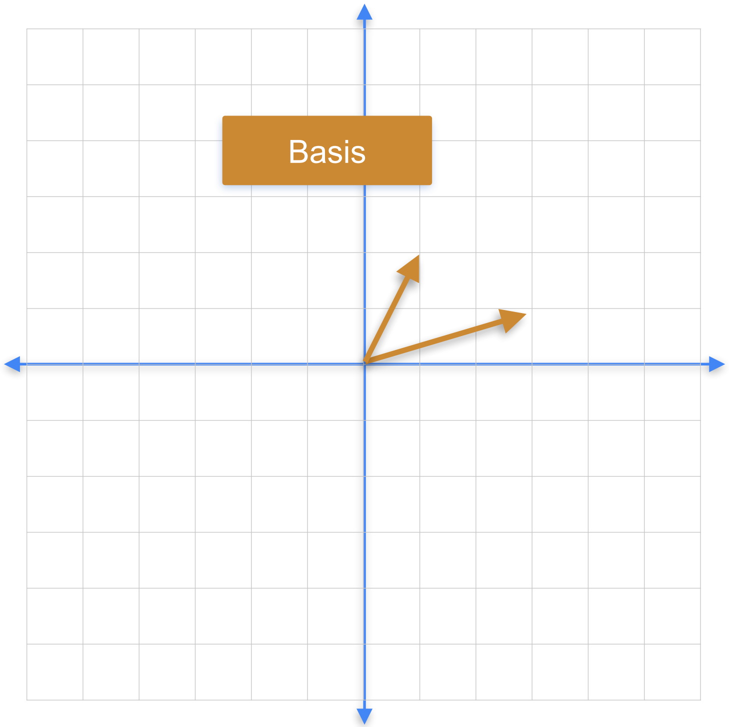
3	1
1	2



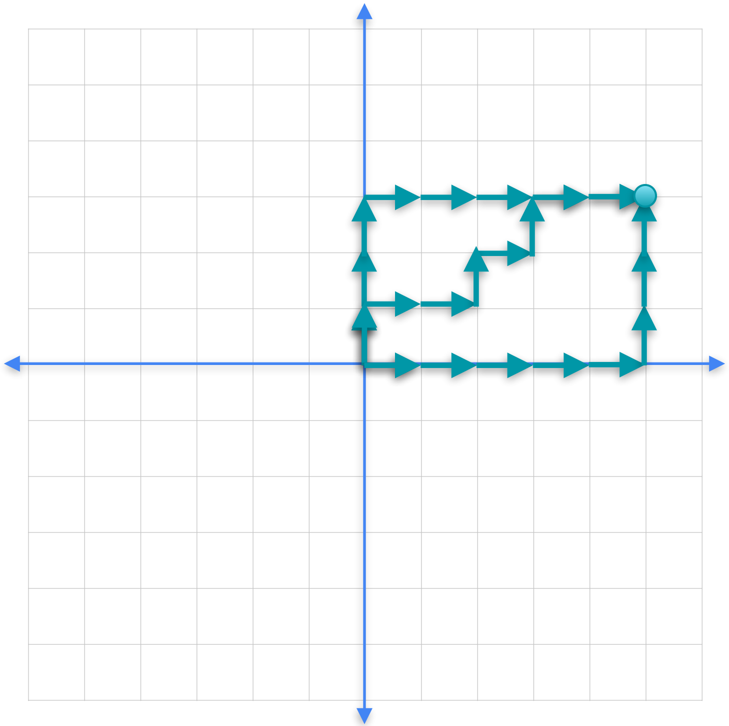
Bases



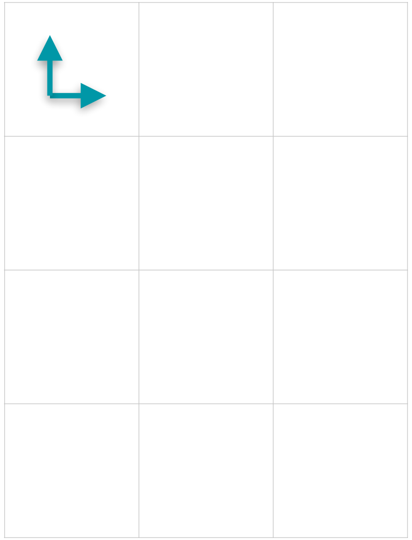
3	1
1	2



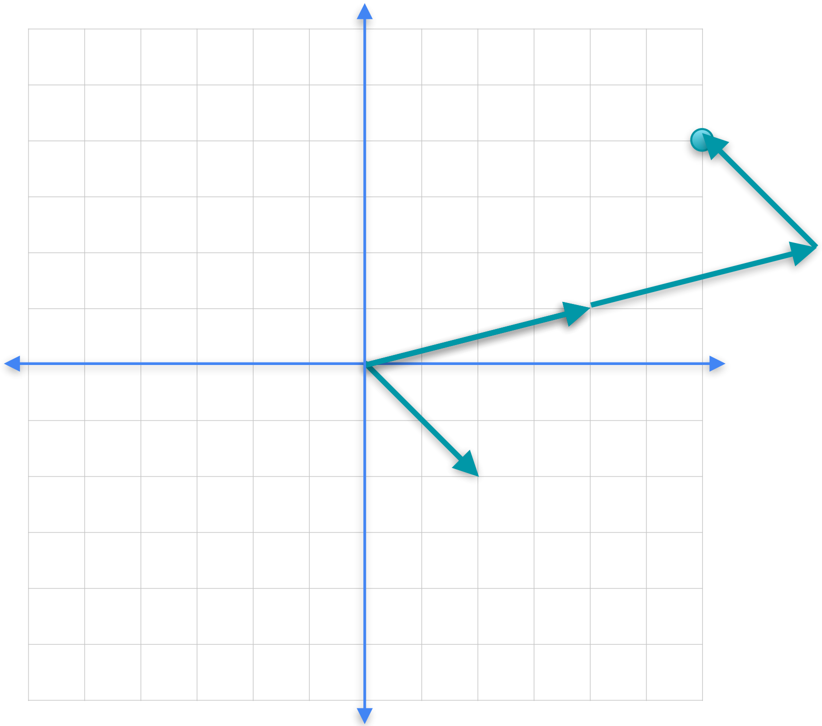
Bases

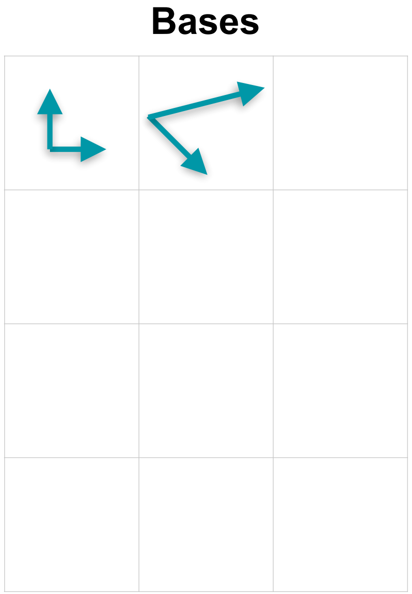


Bases

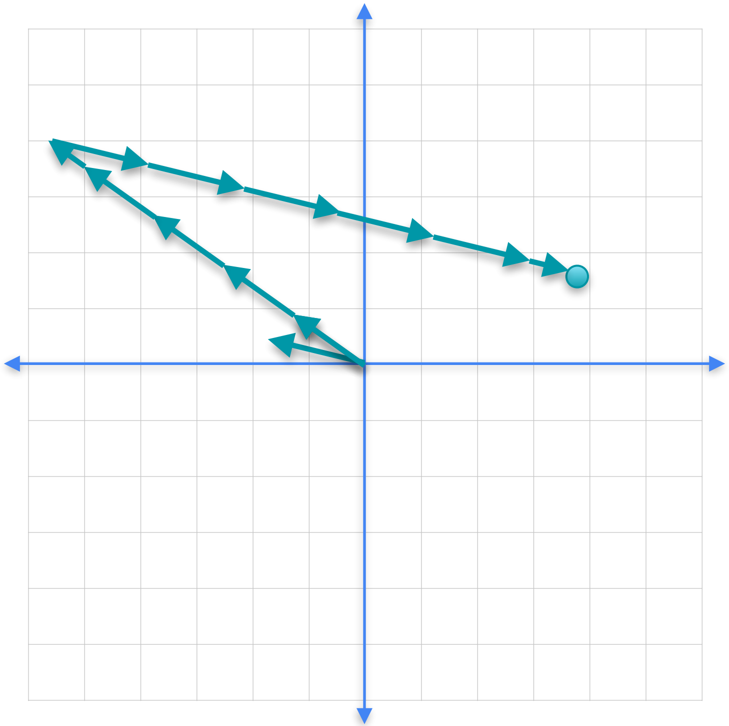


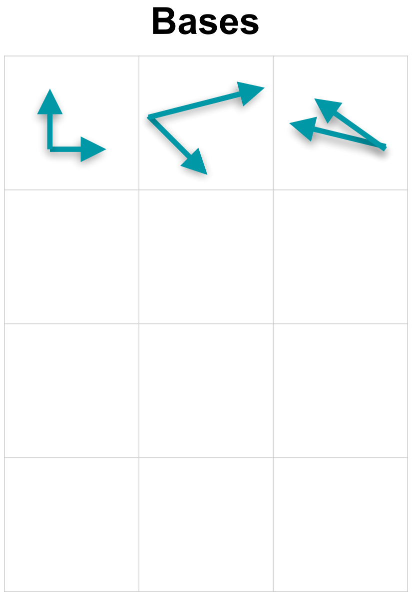
Bases



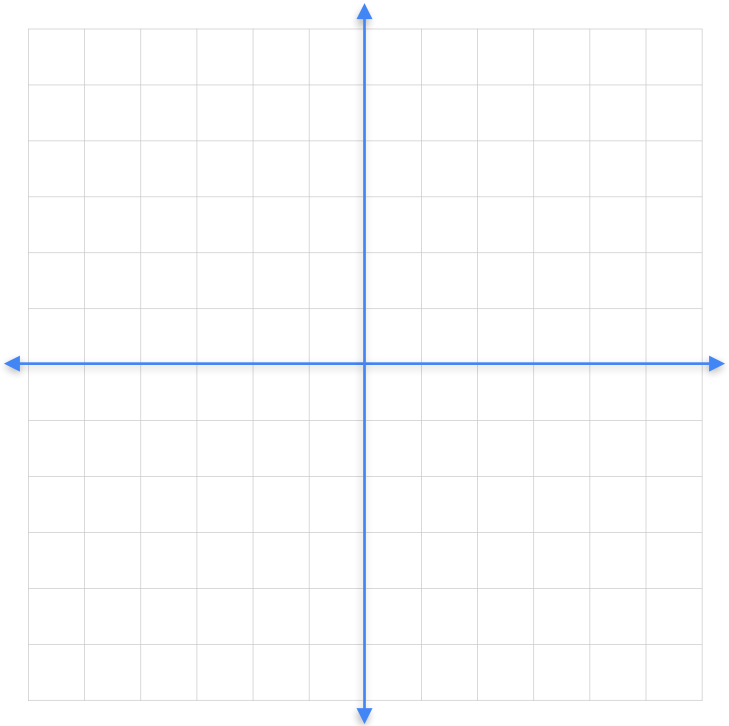


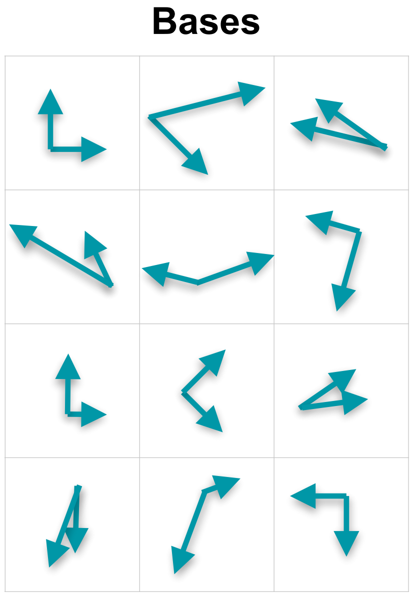
Bases



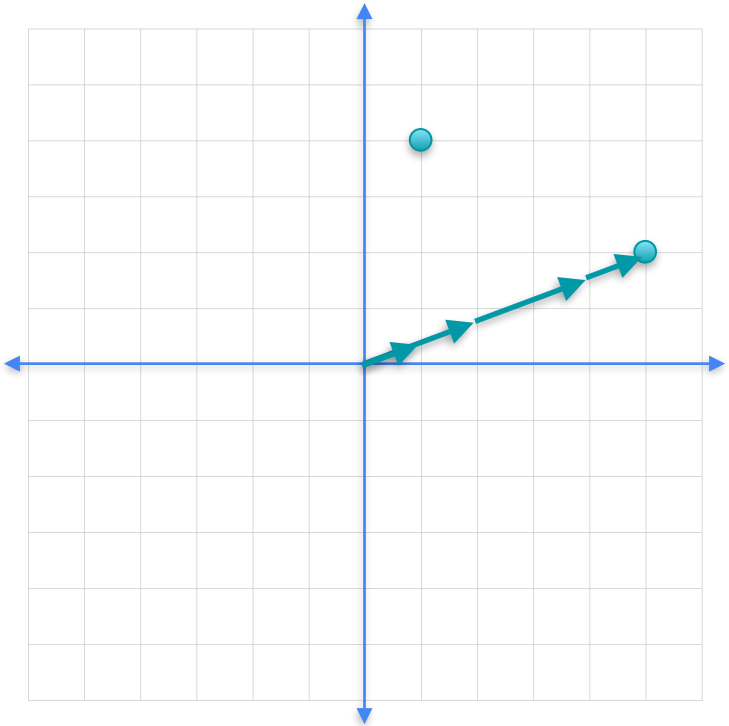


Bases

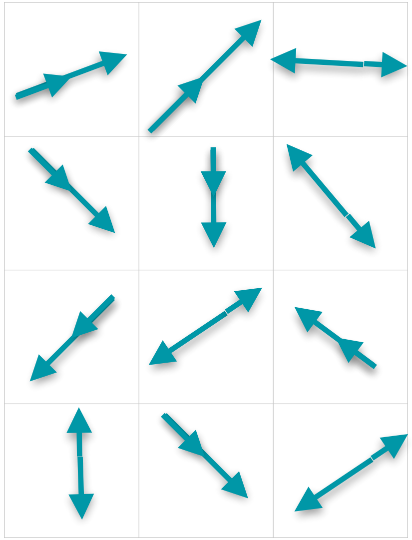




What is not a basis?



Not bases



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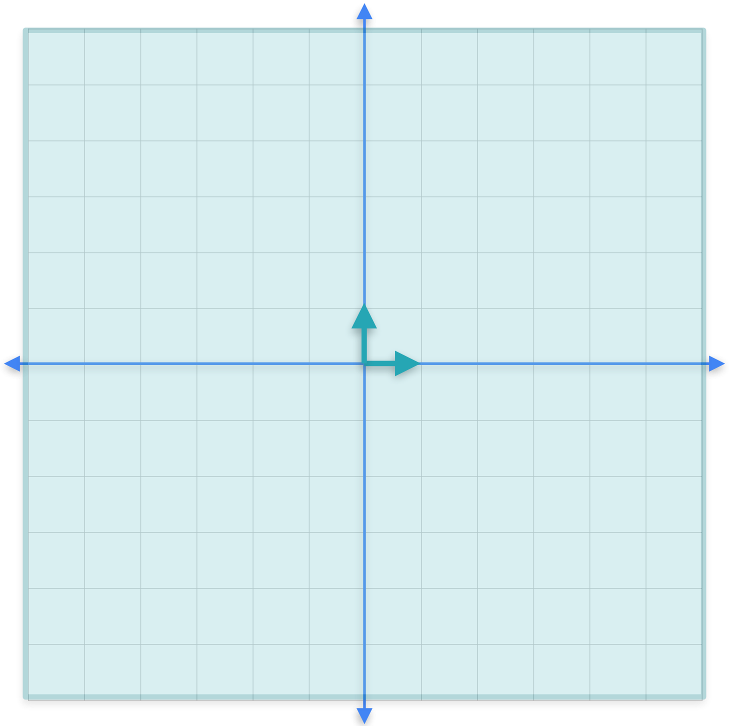
Determinants and Eigenvectors

Span

Span



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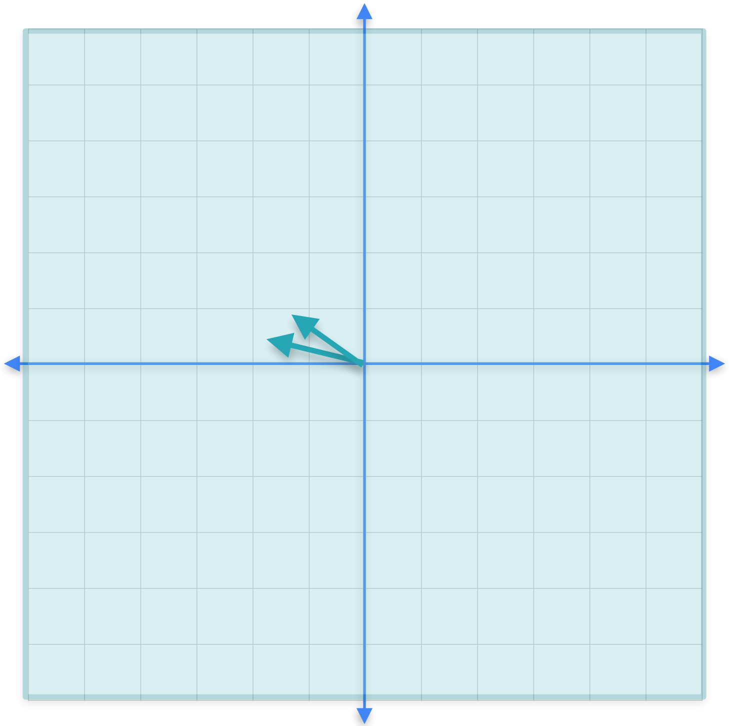
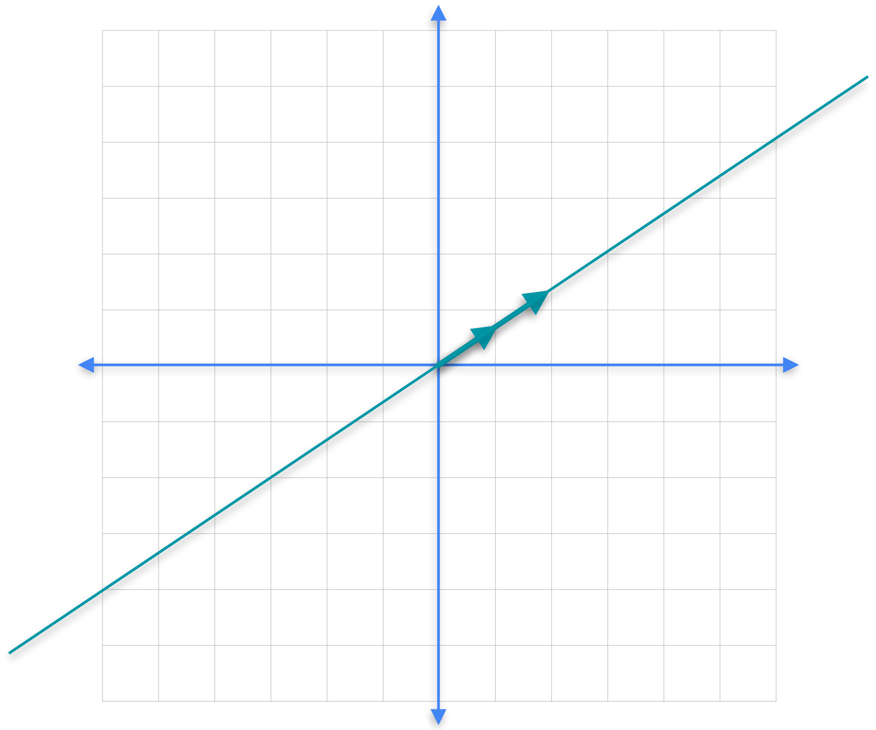


Span



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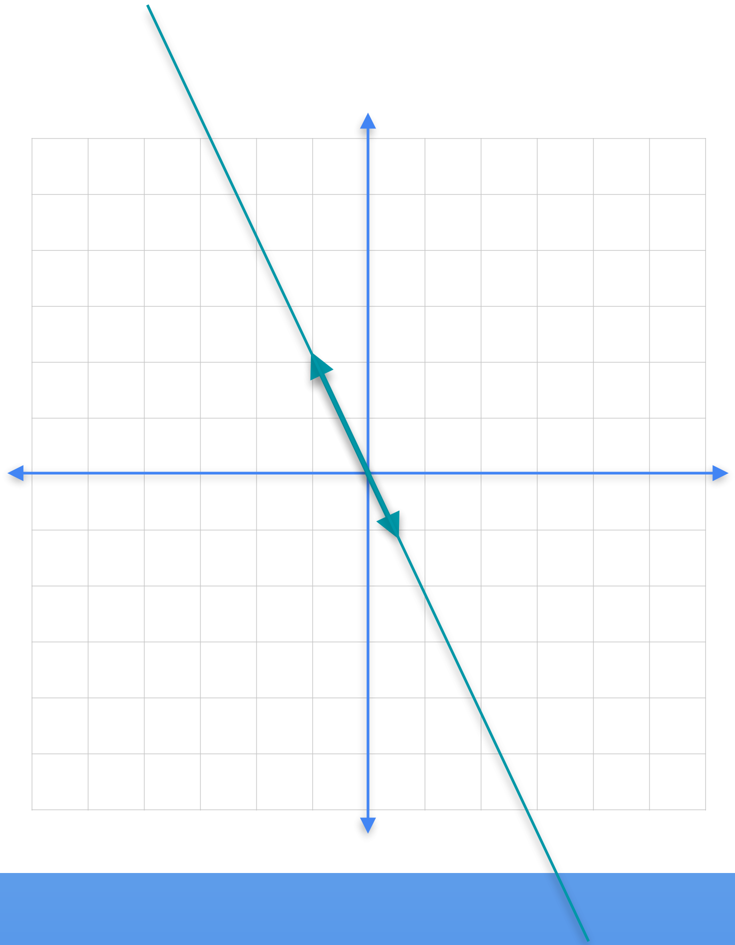
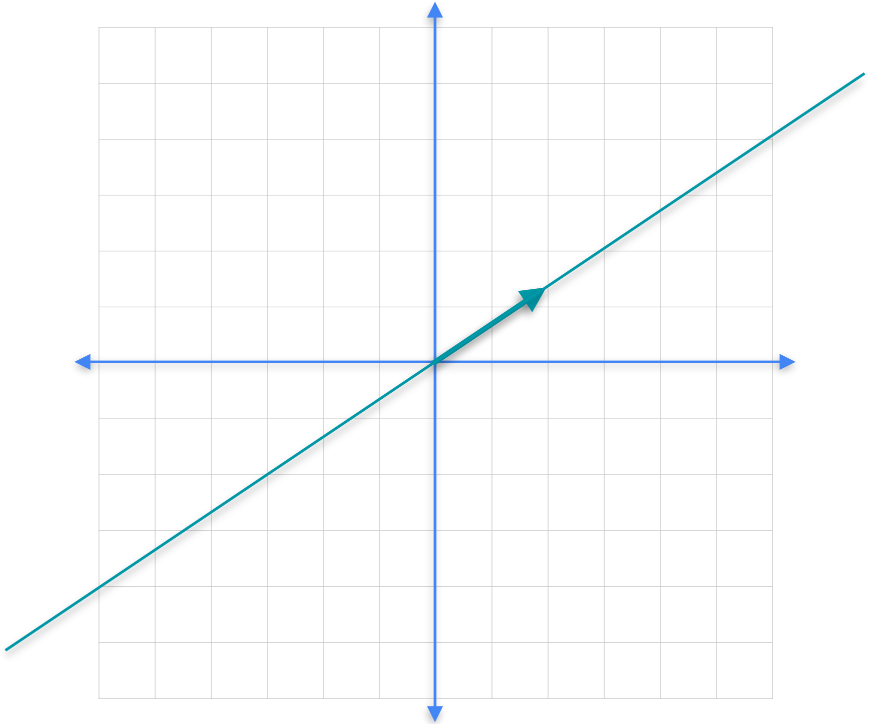
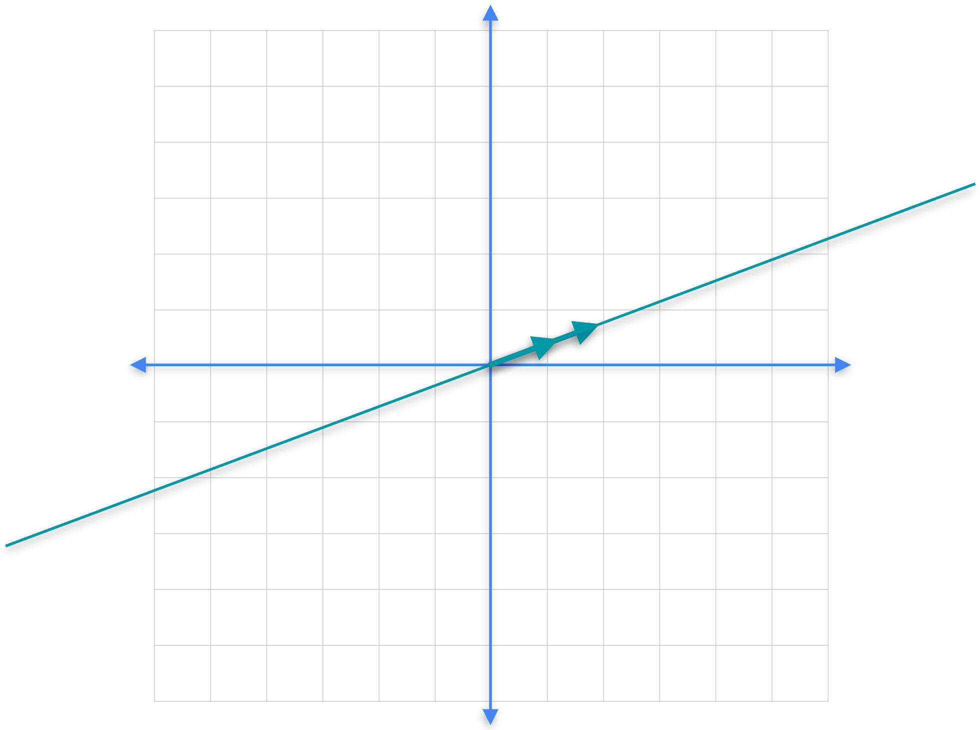
Span



Span

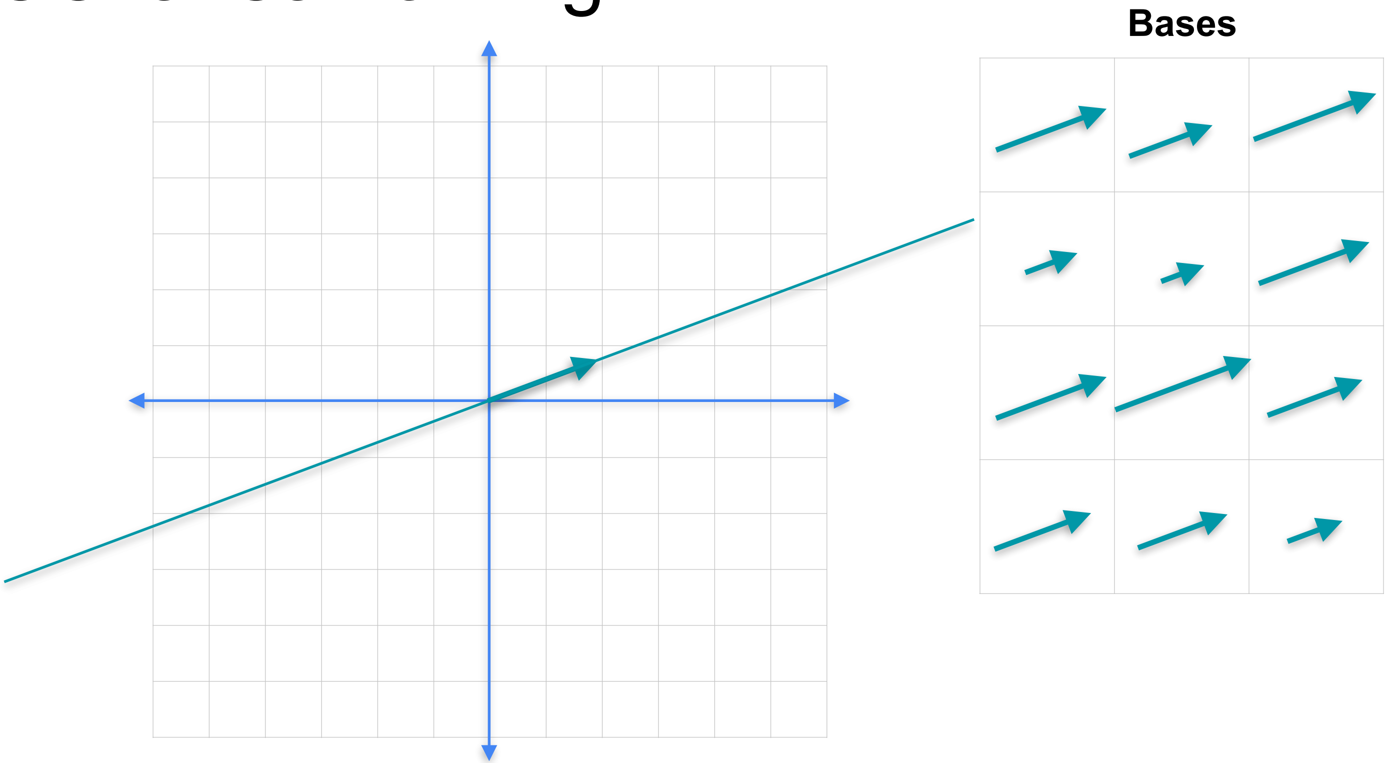


Is this a basis?

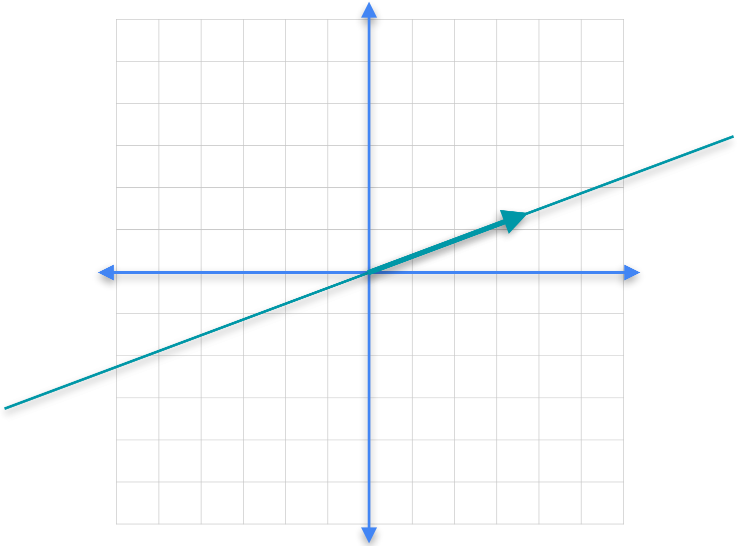


No

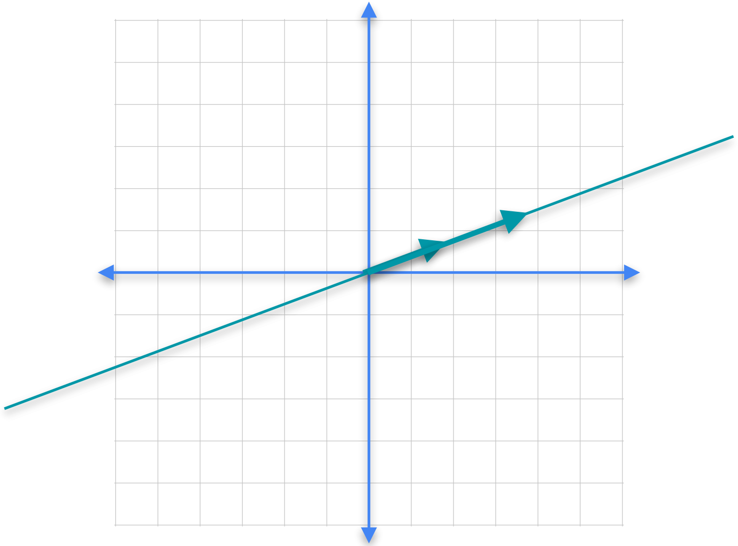
Is this a basis for something?



A basis is a minimal spanning set

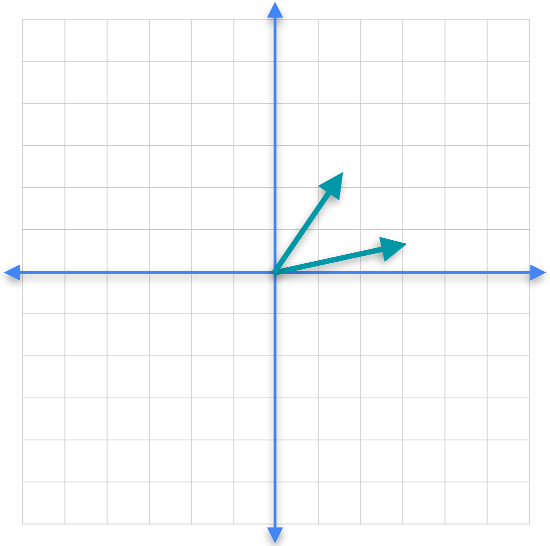


Basis

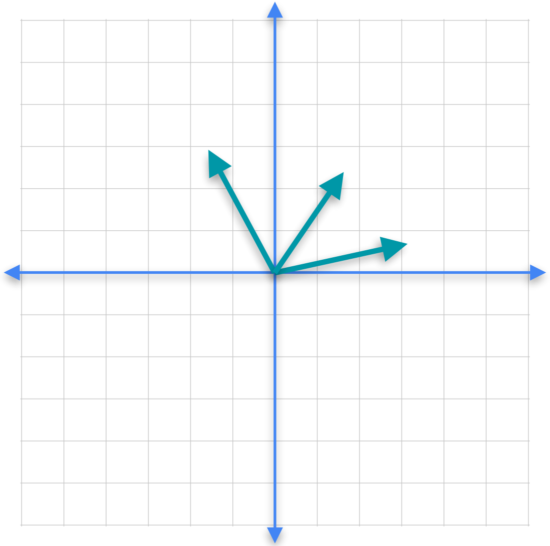


Not a basis

A basis is a minimal spanning set

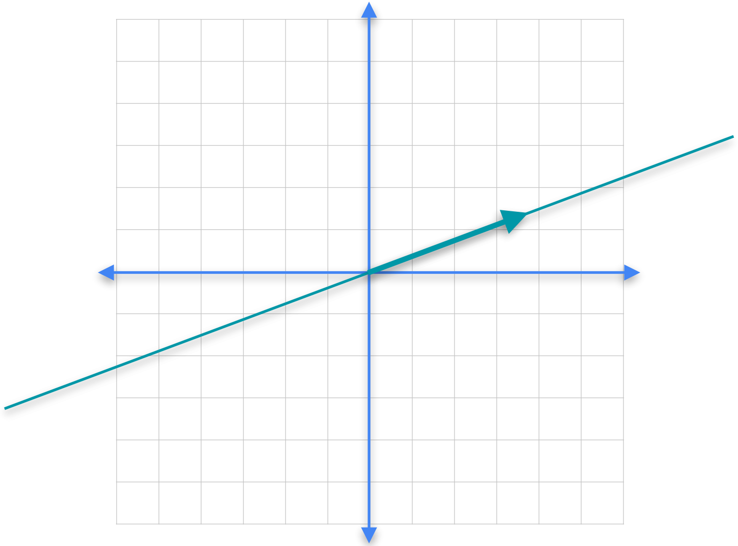


Basis

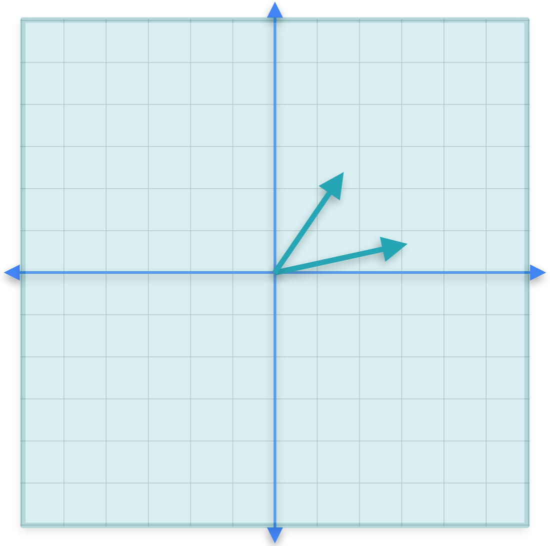


Not a basis

Number of elements in the basis is the dimension

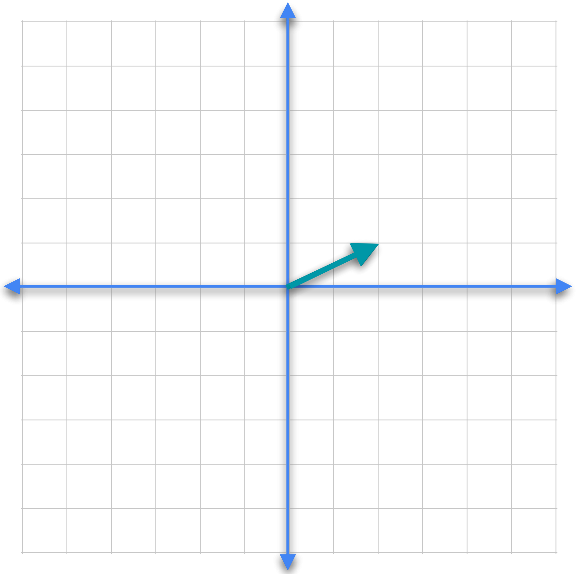


Dimensions: 1
1 element in the basis



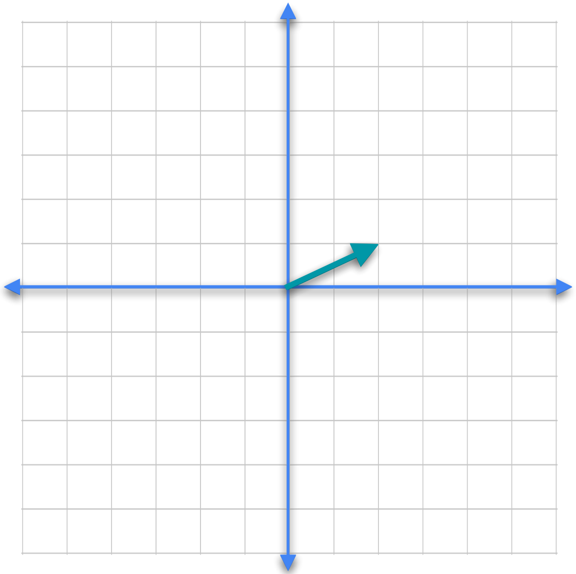
Dimensions: 2
2 elements in the basis

Linearly independent and linearly dependent vectors

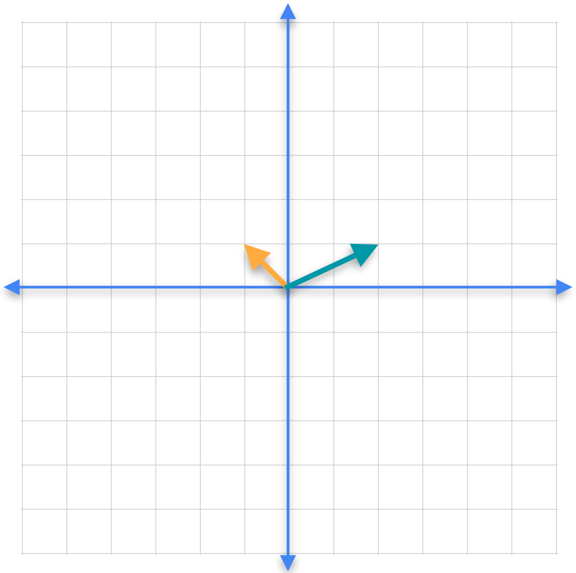


Linearly independent

Linearly independent and linearly dependent vectors

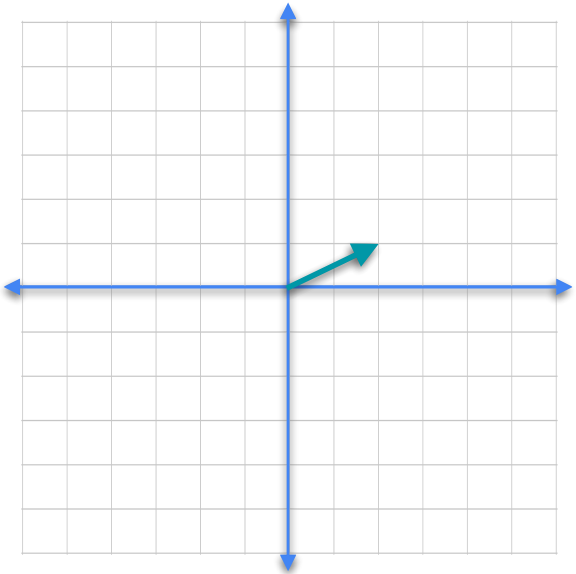


Linearly independent

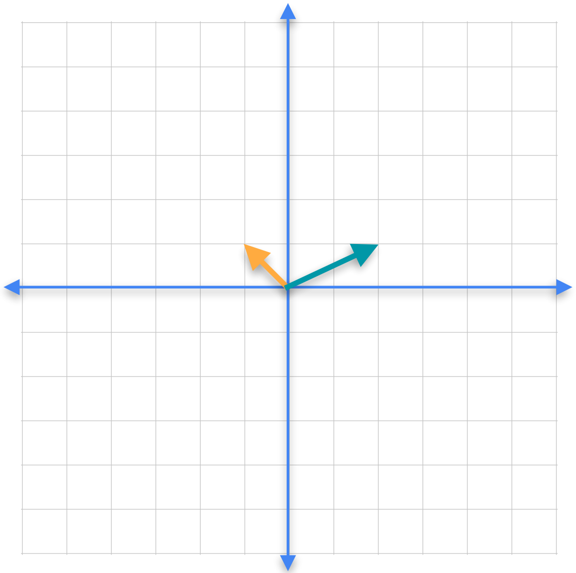


Linearly independent

Linearly independent and linearly dependent vectors

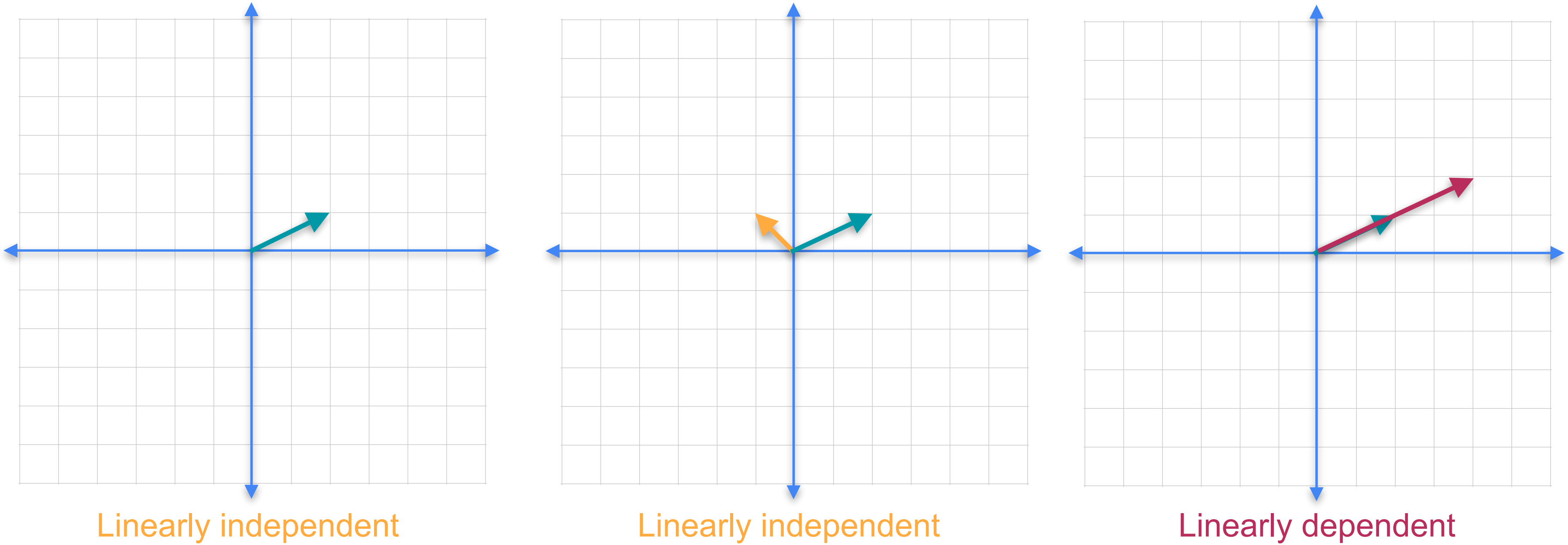


Linearly independent

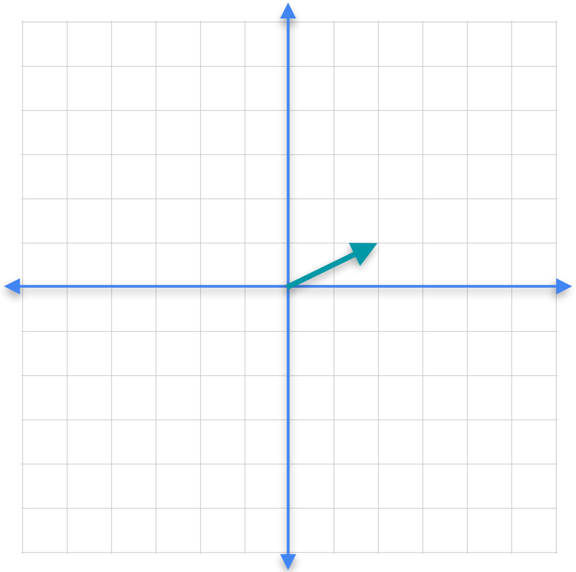


Linearly independent

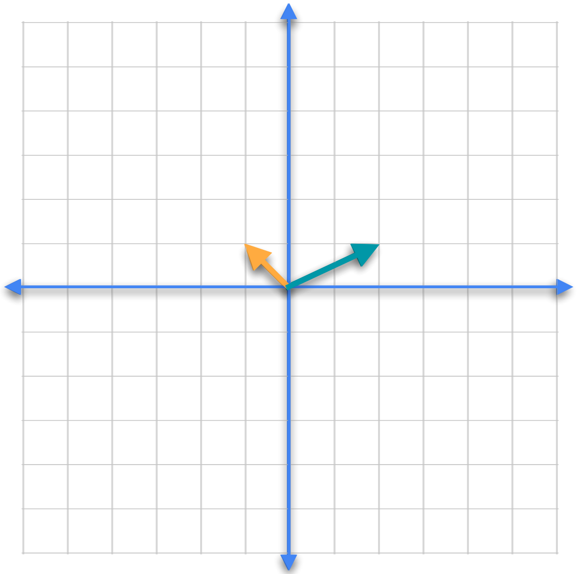
Linearly independent and linearly dependent vectors



Linearly independent and linearly dependent vectors

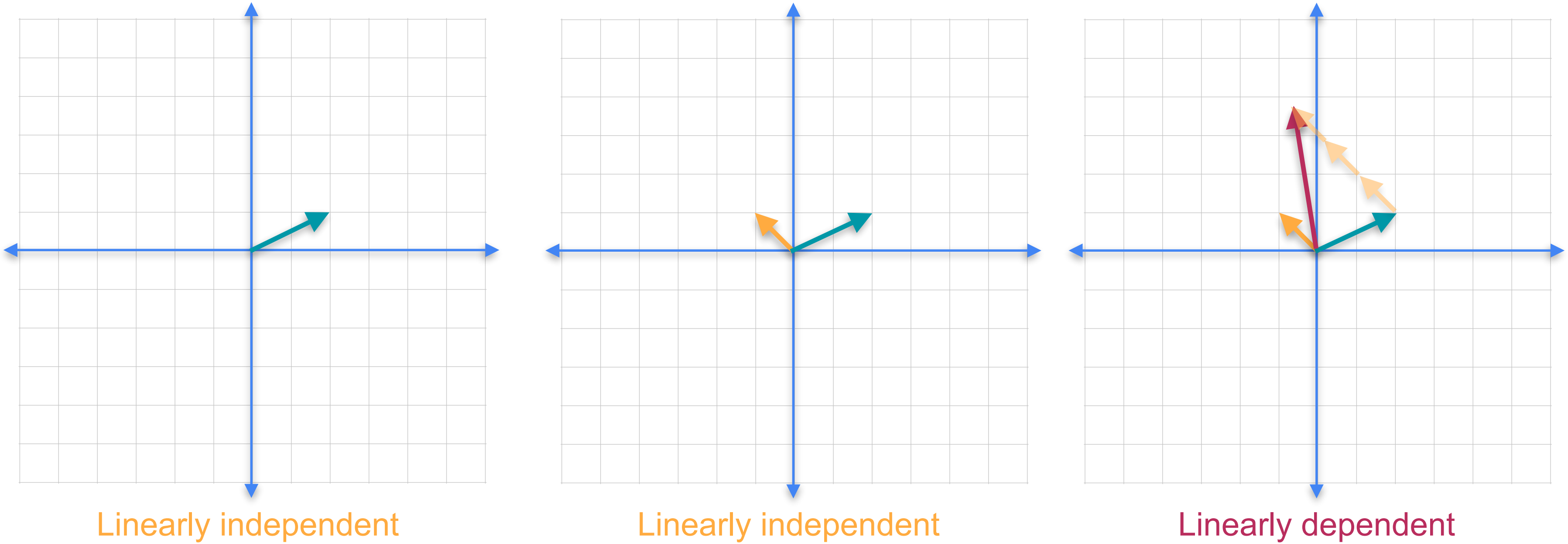


Linearly independent

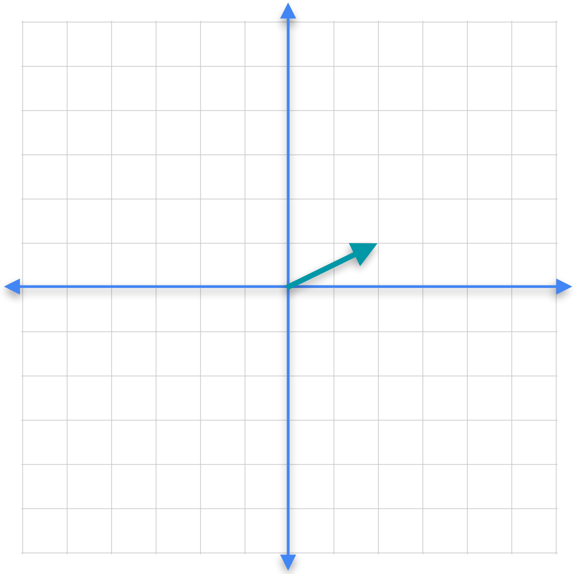


Linearly independent

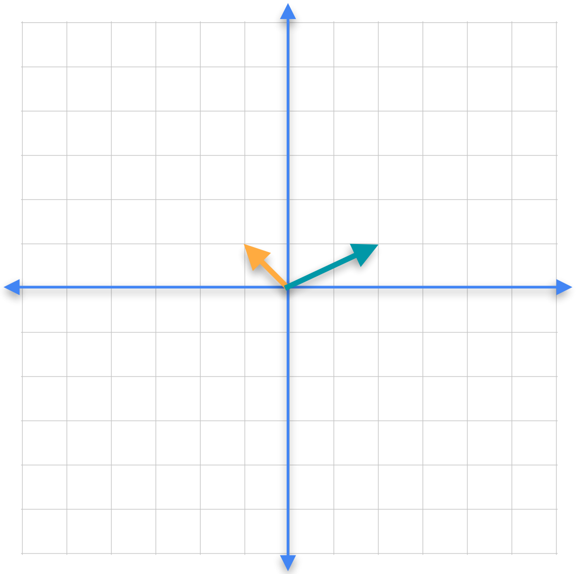
Linearly independent and linearly dependent vectors



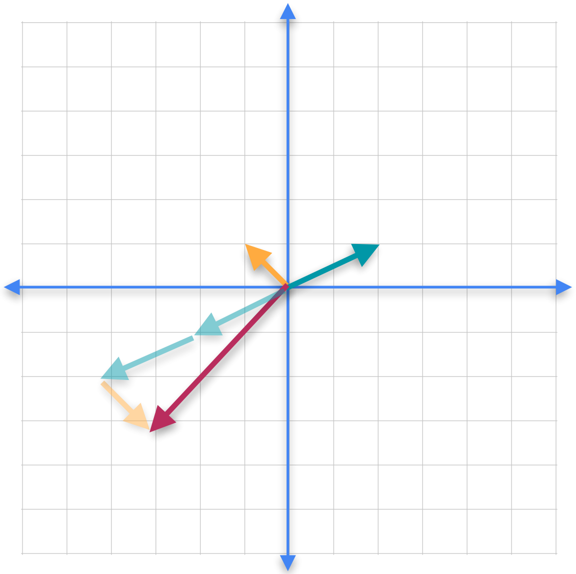
Linearly independent and linearly dependent vectors



Linearly independent

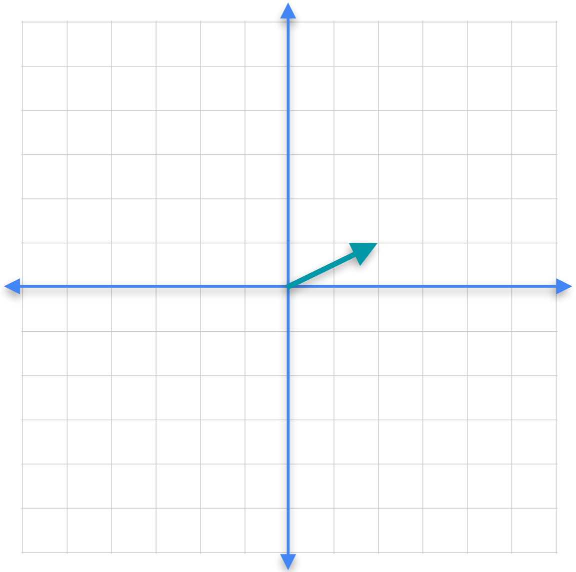


Linearly independent

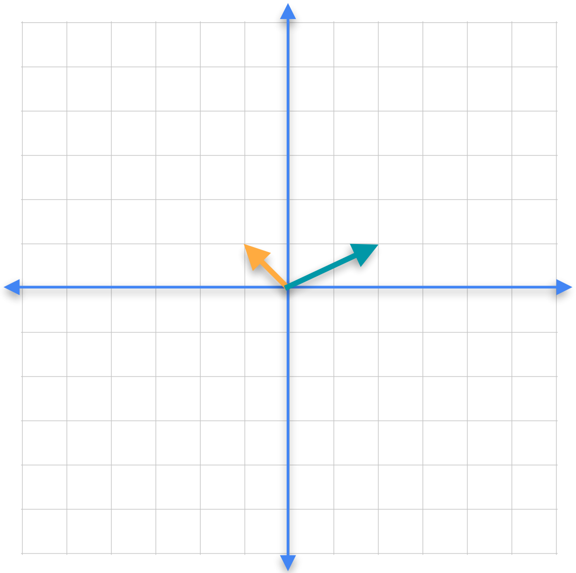


Linearly dependent

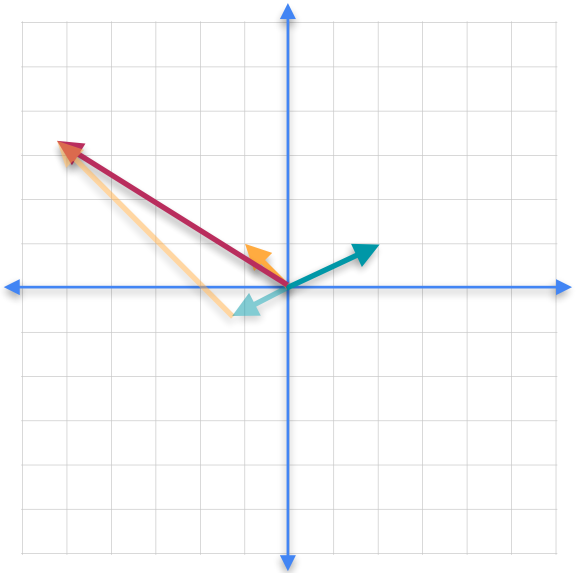
Linearly independent and linearly dependent vectors



Linearly independent

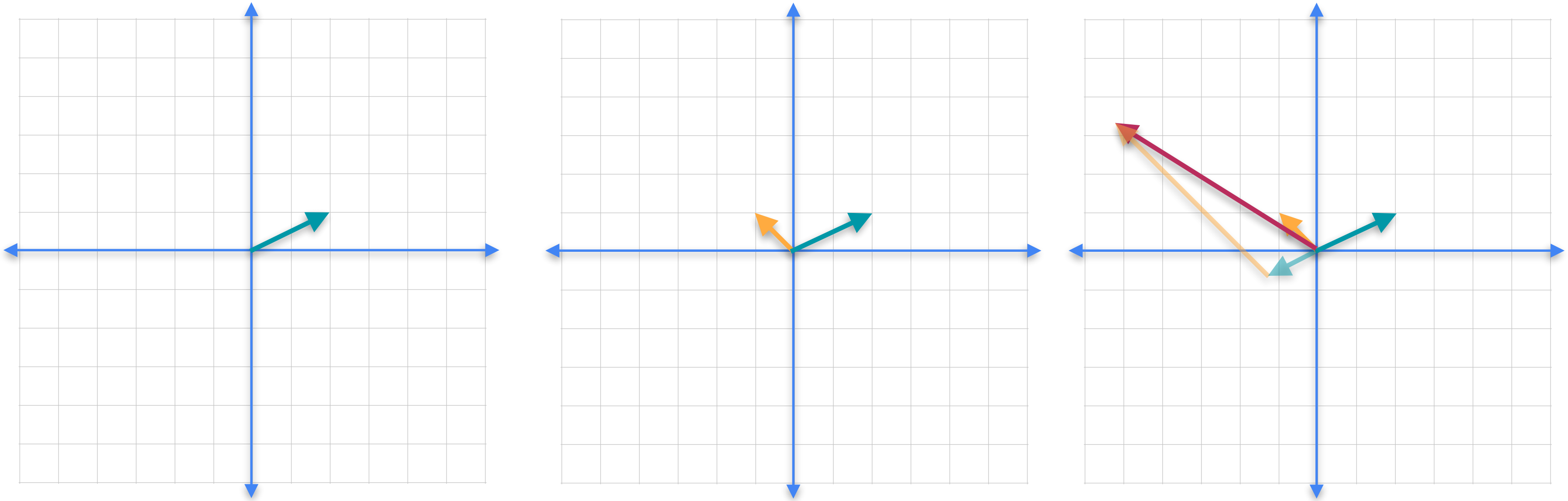


Linearly independent



Linearly dependent

Linearly independent and linearly dependent vectors



Linearly independent | Linearly independent | Linearly dependent

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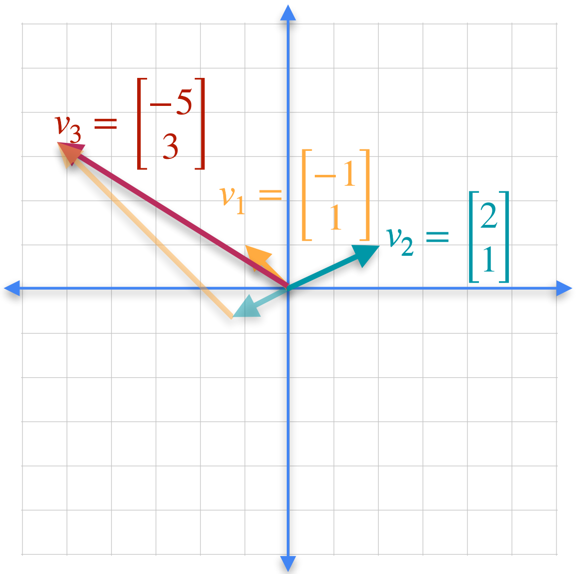
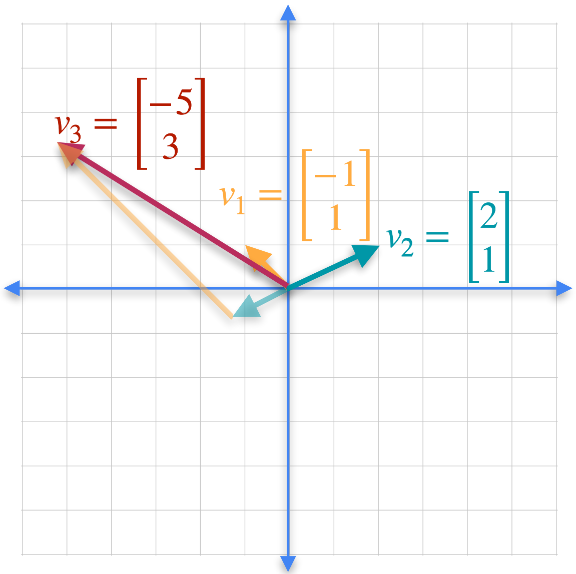
Let's see how to check for linear dependence



Linearly dependent

$\alpha \quad + \beta \quad =$

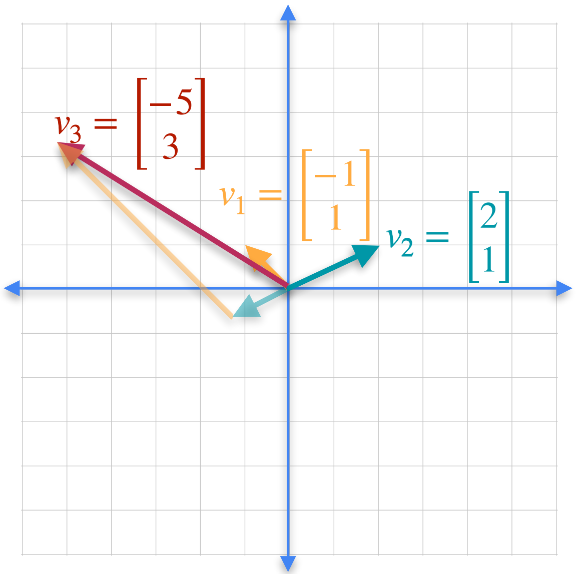
Let's see how to check for linear dependence



Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

Let's see how to check for linear dependence



Linearly dependent

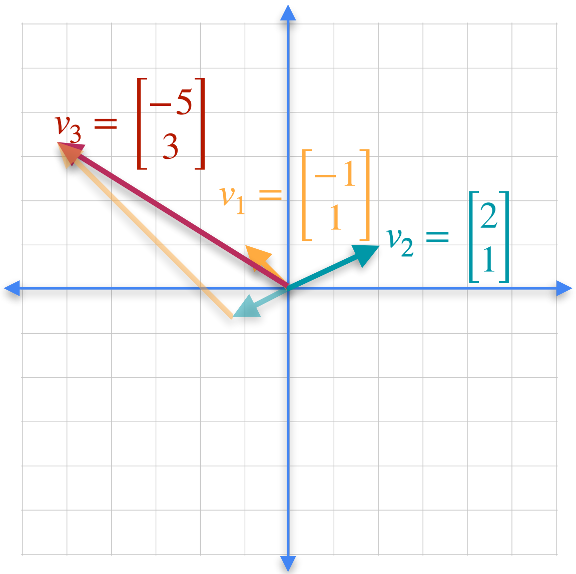
$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1 $-\alpha + 2\beta = -5$

2 $\alpha + \beta = 3$

Let's see how to check for linear dependence



Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

v_3 is a linear combination of v_1 and v_2

$\boxed{1}$ $-\alpha + 2\beta = -5$

$\boxed{2}$ $\alpha + \beta = 3$

$\boxed{1} + \boxed{2}$

$$3\beta = -2 \longrightarrow \beta = -\frac{2}{3}$$

$\boxed{2}$

$$\alpha - \frac{2}{3} = 3 \longrightarrow \alpha = \frac{11}{3}$$

Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \qquad\qquad -1 \qquad\qquad =$$

Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} -1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Not a basis!

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

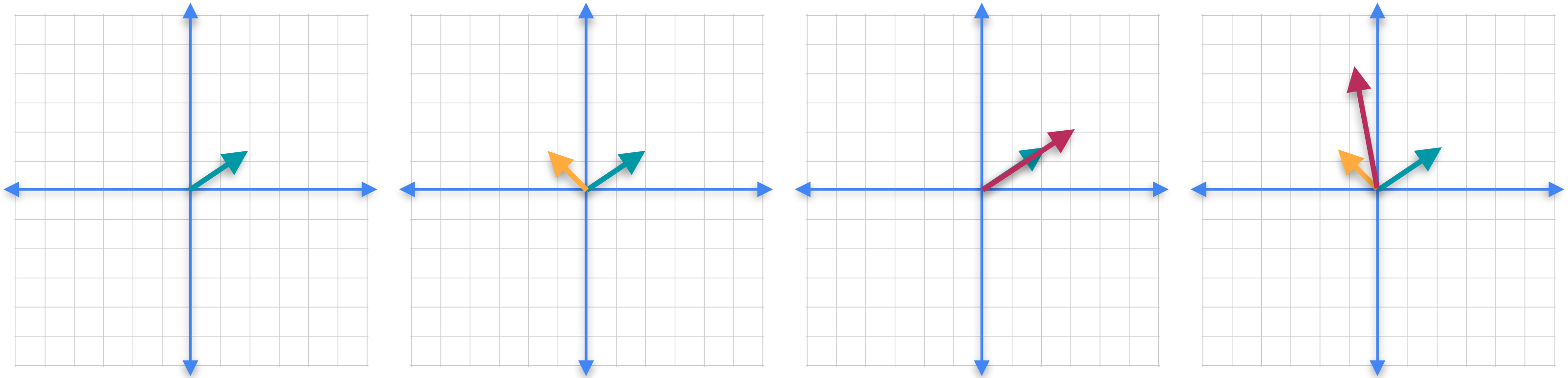
$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Basis: a formal definition

A basis is a set of vectors that:

- Spans a vector space
- Is linearly independent

Not all sets of N vectors are a basis for N-dimensional space



Spans a line
Linearly independent
Is a basis

Spans the plane
Linearly independent
Is a basis

Spans a line
Linearly dependent
Not a basis

Spans the plane
Linearly dependent
Not a basis

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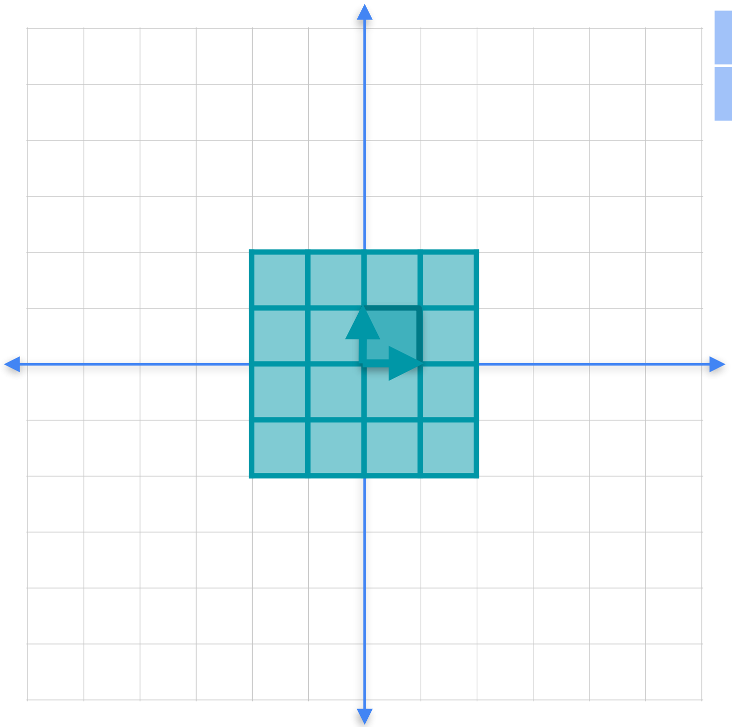
Determinants and Eigenvectors

Eigenbasis

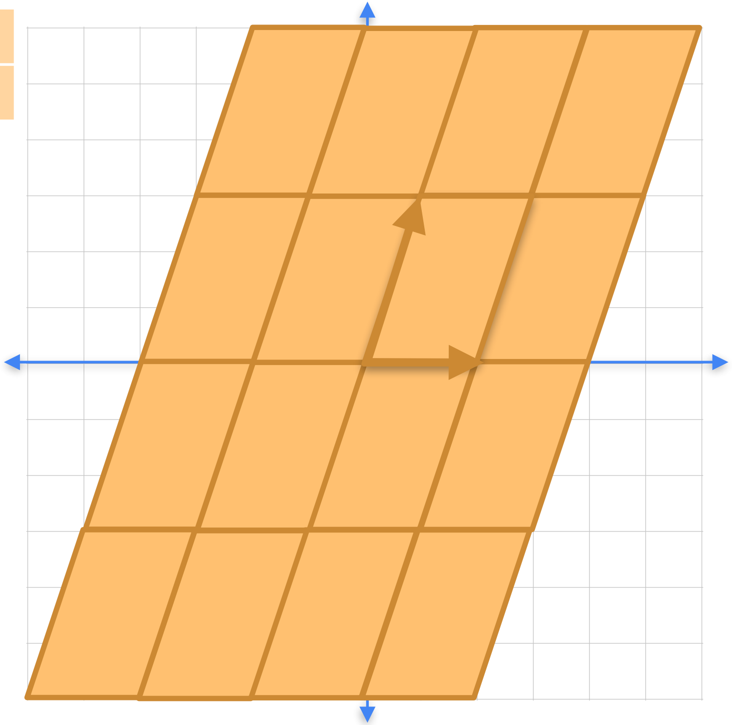
Basis



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

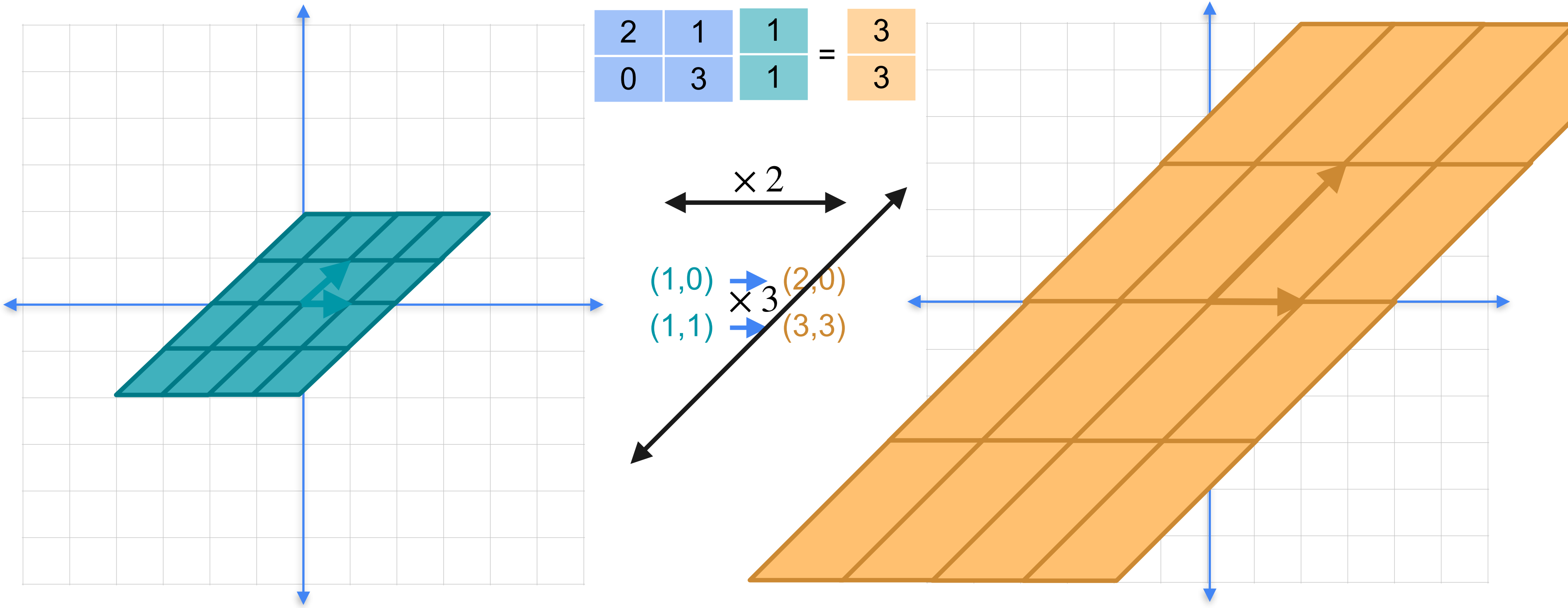
$(1,0) \rightarrow (2,0)$

$(0,1) \rightarrow (1,3)$

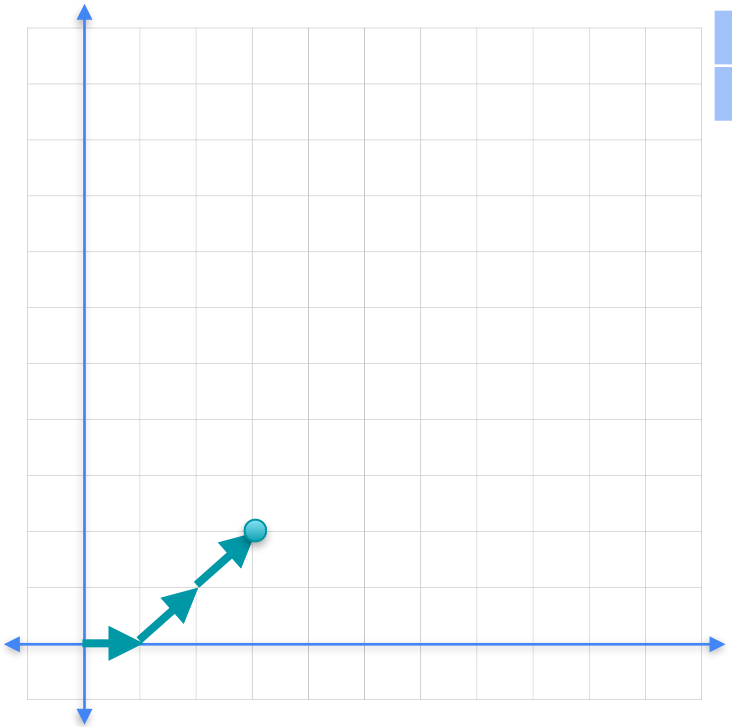


Eigenbasis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

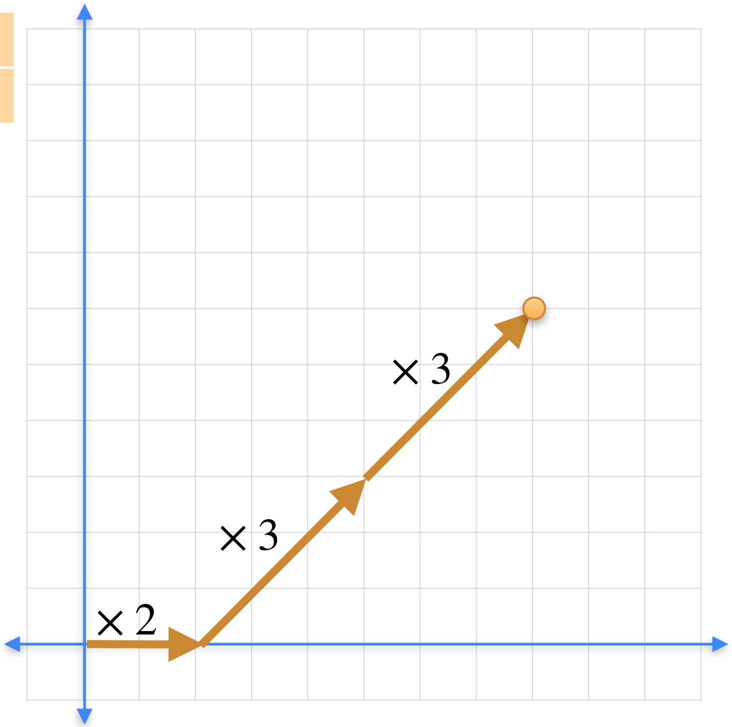


Eigenbasis



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

(3,2) → (8,6)



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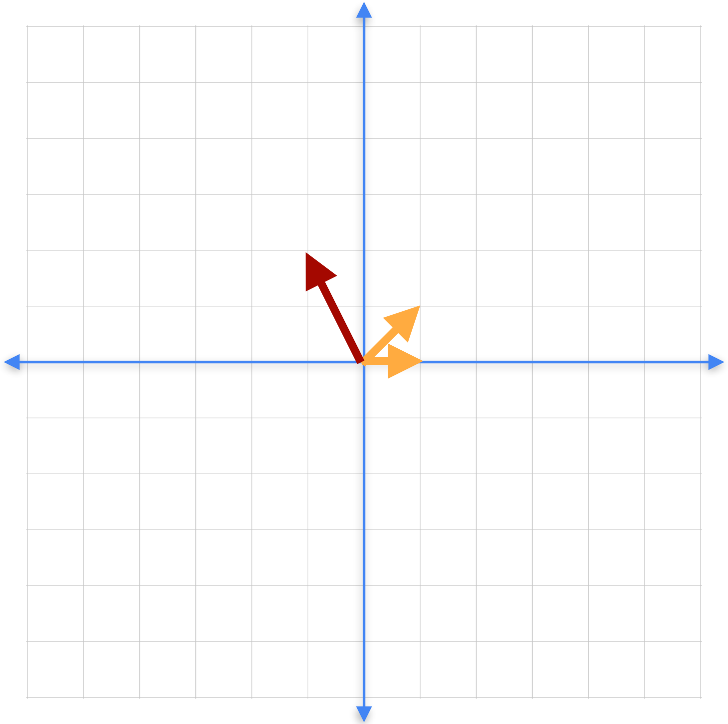
Determinants and Eigenvectors

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

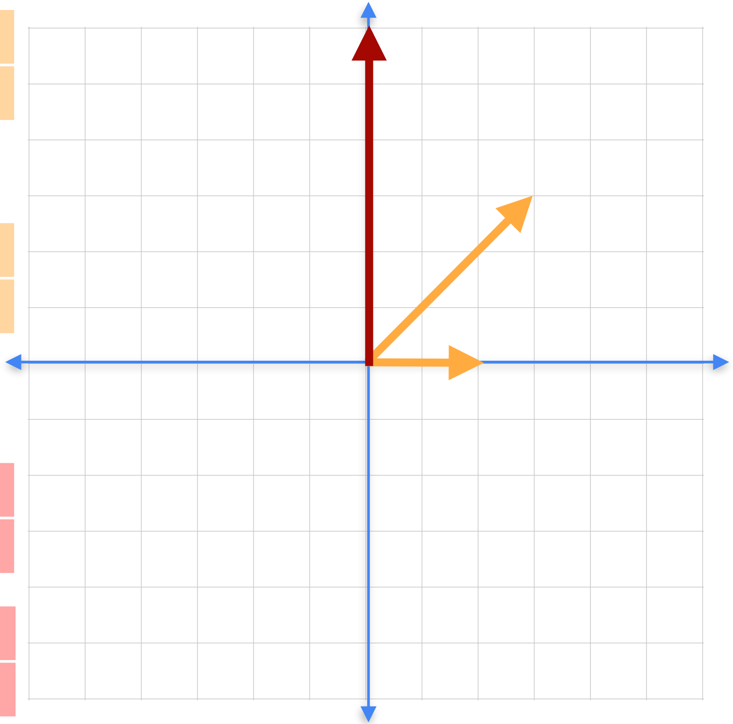


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Eigenvalues and eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

8 multiplications 2 multiplications

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Matrix Multiplication
More work

Scalar Multiplication
Less work

$$Av_1 = \boxed{\lambda_1 v_1}$$ First pair

$$Av_2 = \boxed{\lambda_2 v_2}$$ Second pair

Eigenvalues Eigenvectors

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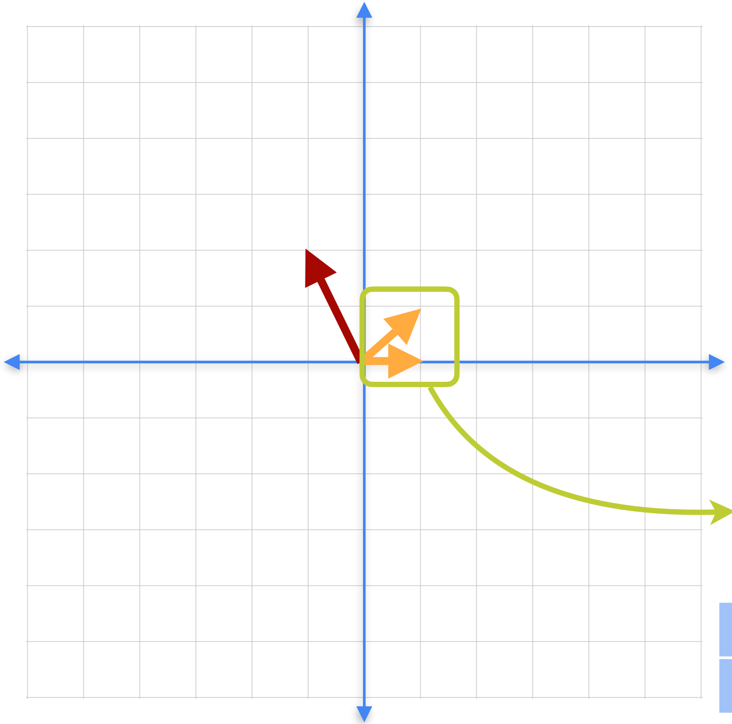
Eigenvalues and eigenvectors



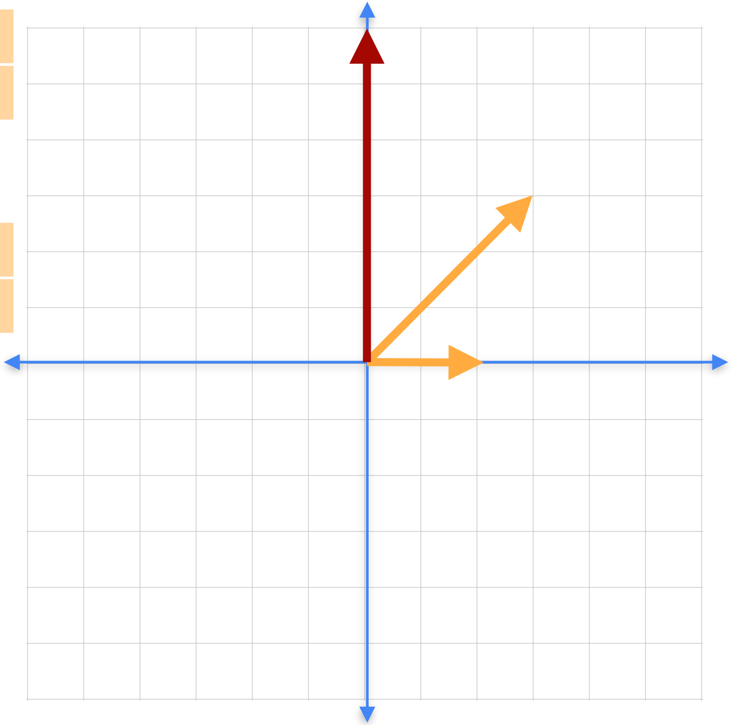
$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}$$

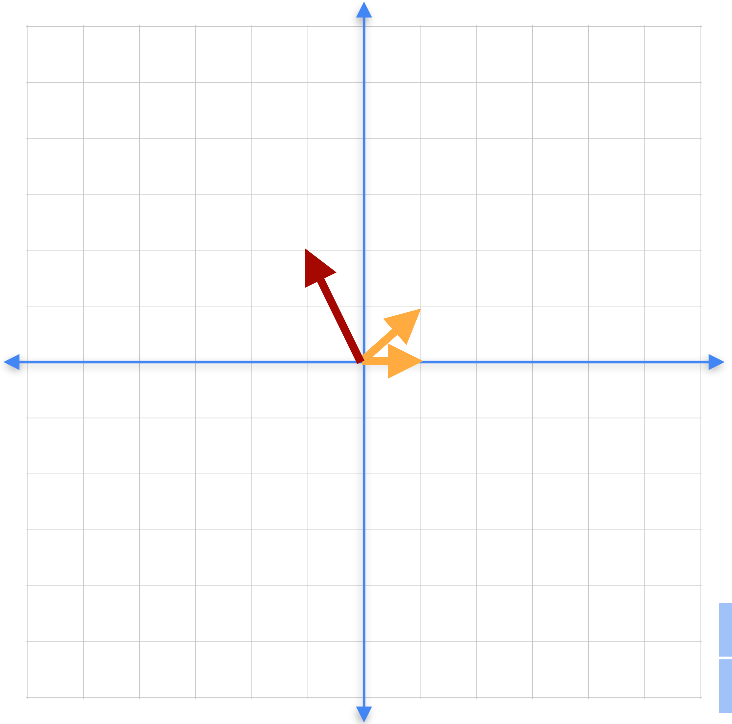
$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$

form a basis

$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$$



Eigenvalues and eigenvectors

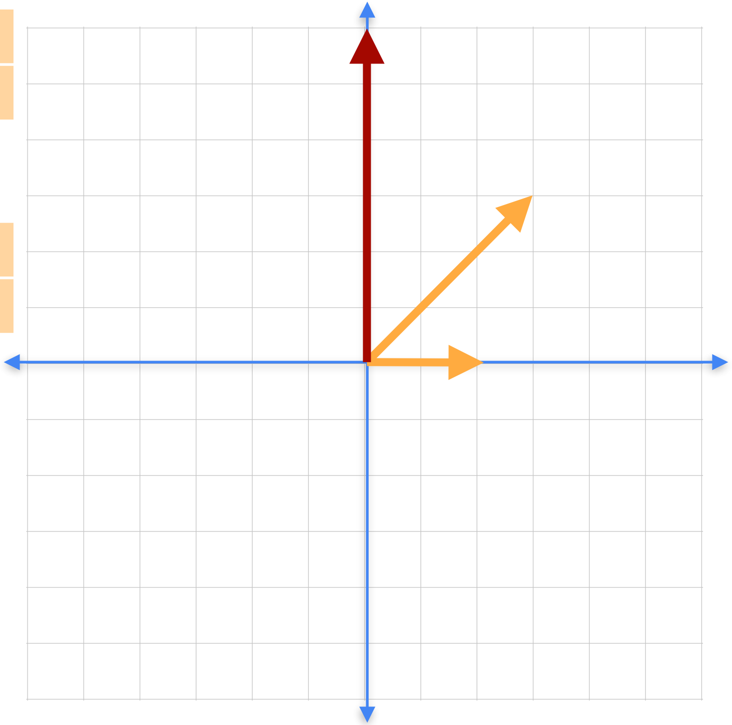


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

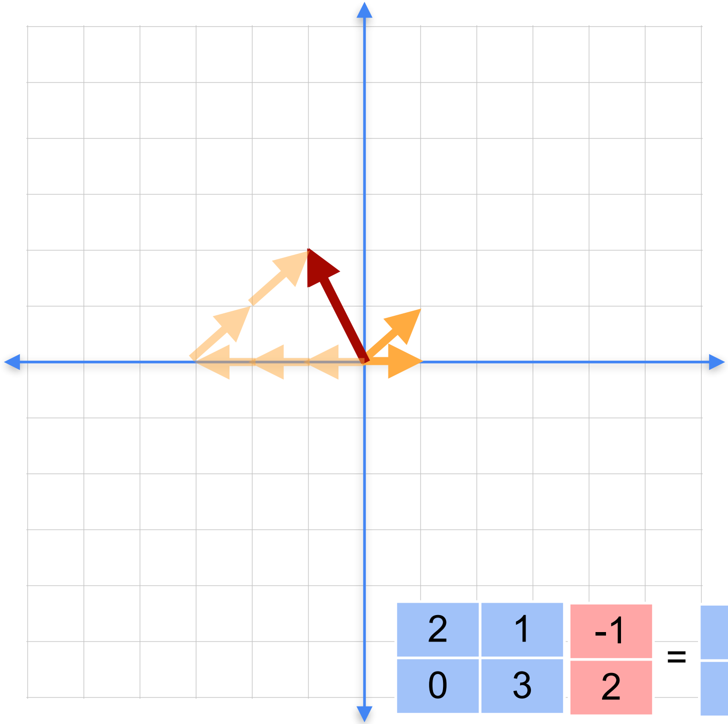
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



Eigenvalues and eigenvectors

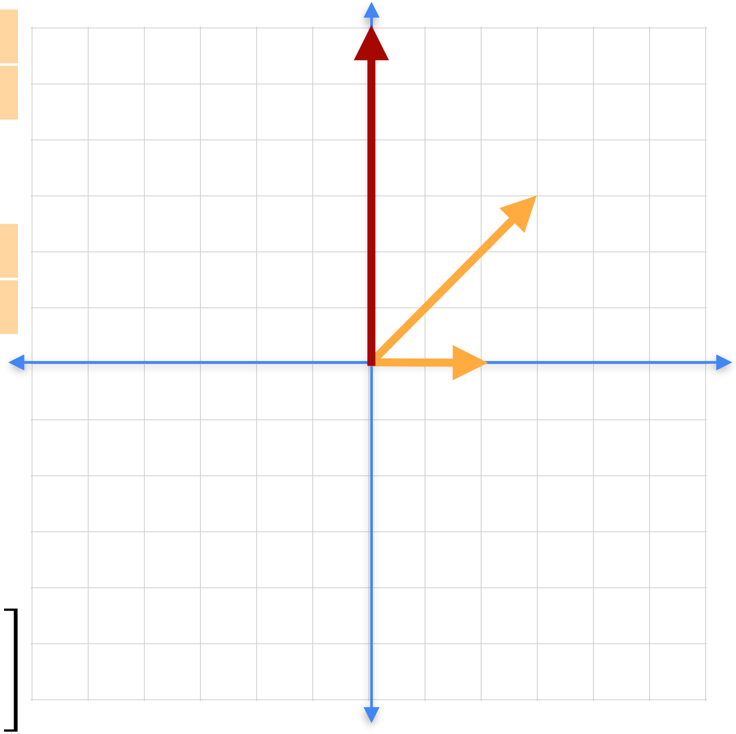


$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}$$

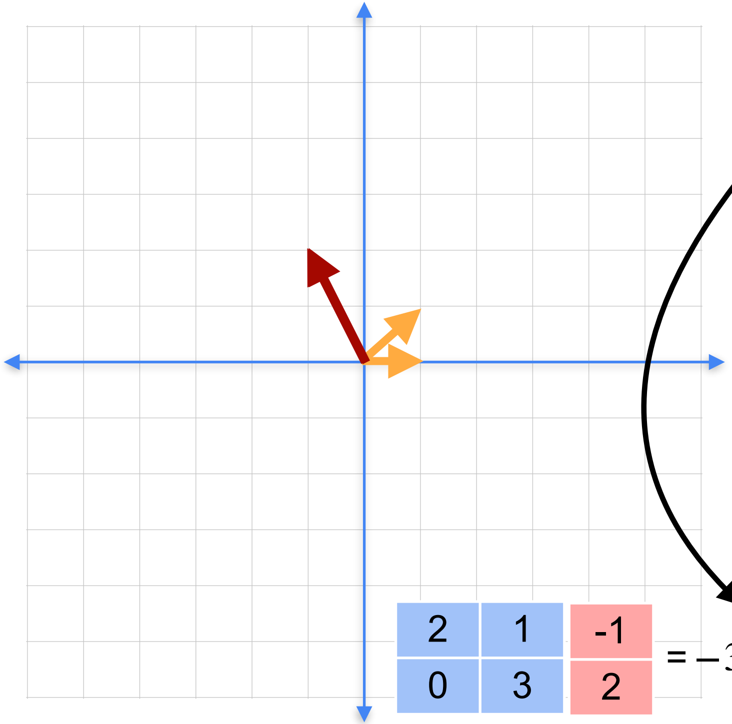
$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}$$

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ form a basis

$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2&1\\0&3\end{bmatrix}\left[-3 \qquad +2 \qquad\right]$$



Eigenvalues and eigenvectors



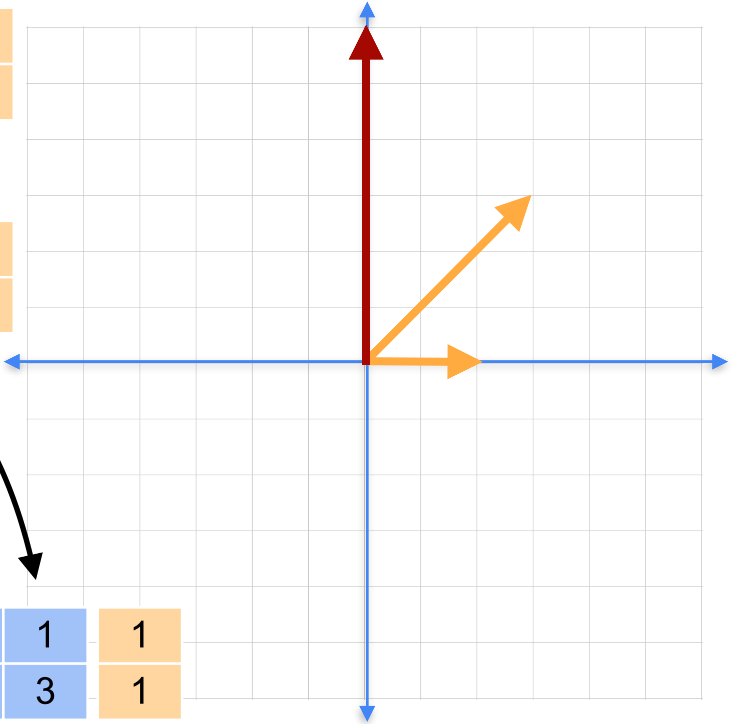
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

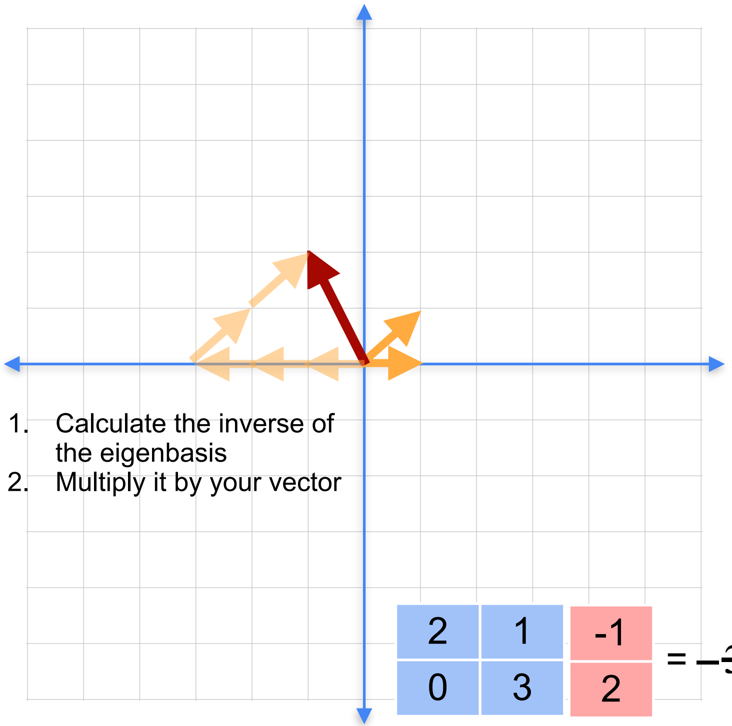
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

form a basis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Eigenvalues and eigenvectors



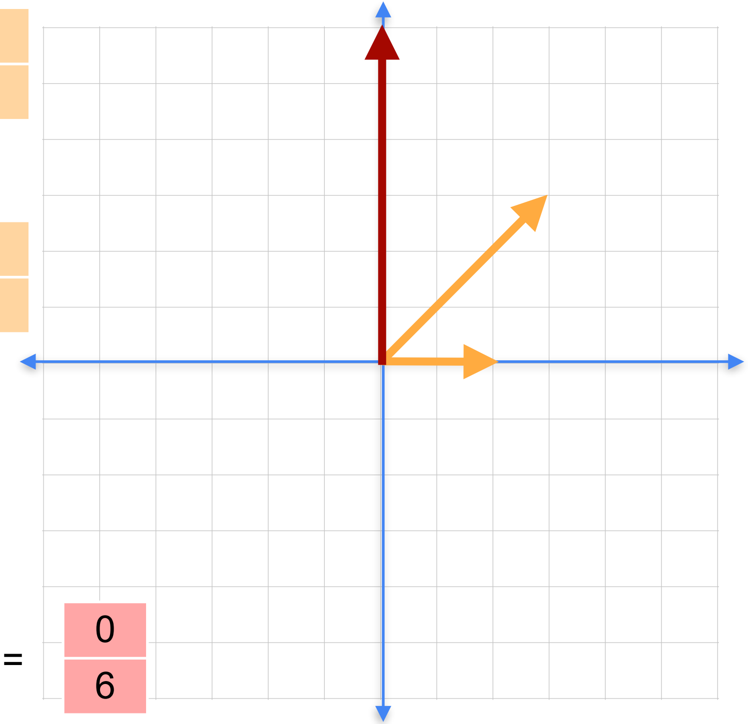
1. Calculate the inverse of the eigenbasis
2. Multiply it by your vector

$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}$$

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$

form a basis



$$\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = -\cancel{3}\,6\begin{bmatrix}1\\0\end{bmatrix} + \cancel{2}\,6\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$$

Eigenvalues and eigenvectors

- $Av = \lambda v$ for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation

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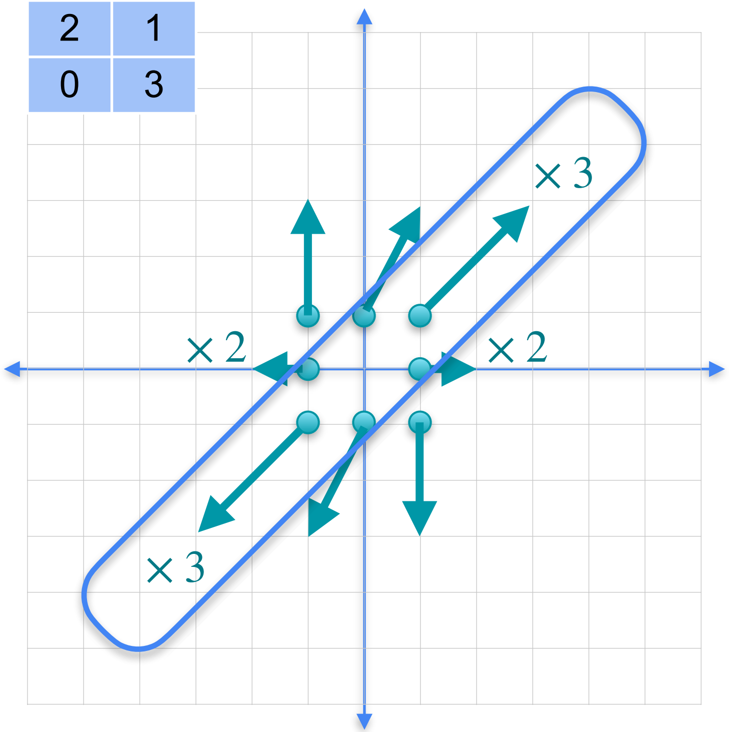
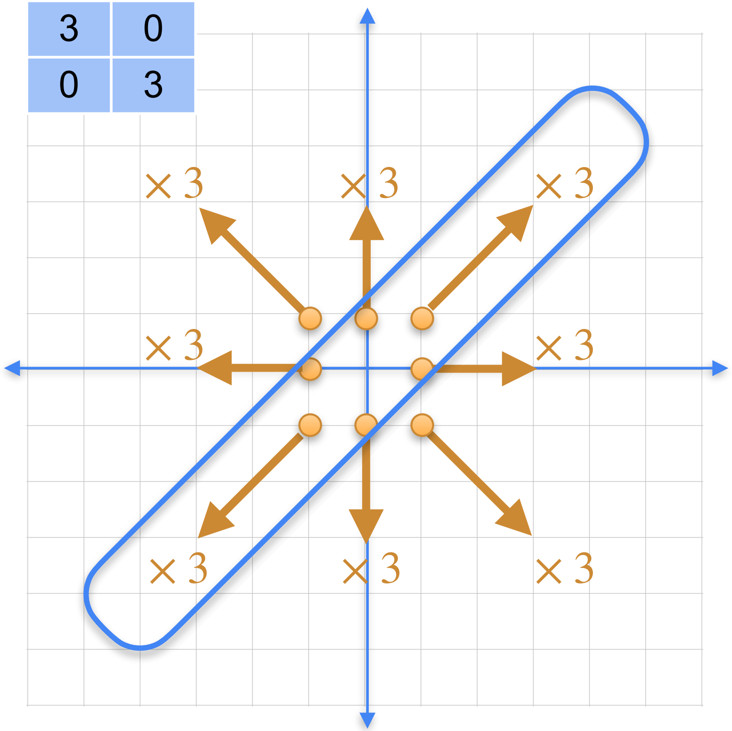
Determinants and Eigenvectors

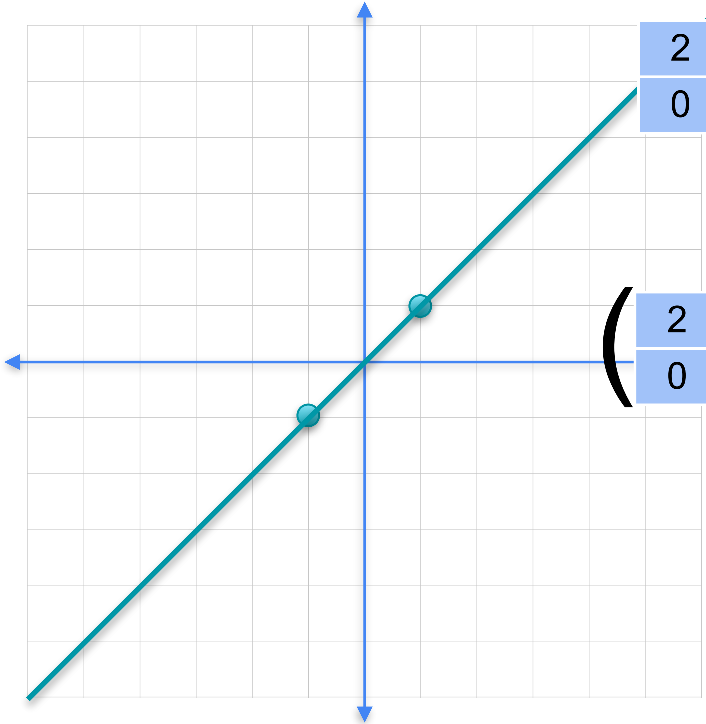
Calculating eigenvalues and eigenvectors

Finding eigenvalues





Finding eigenvalues



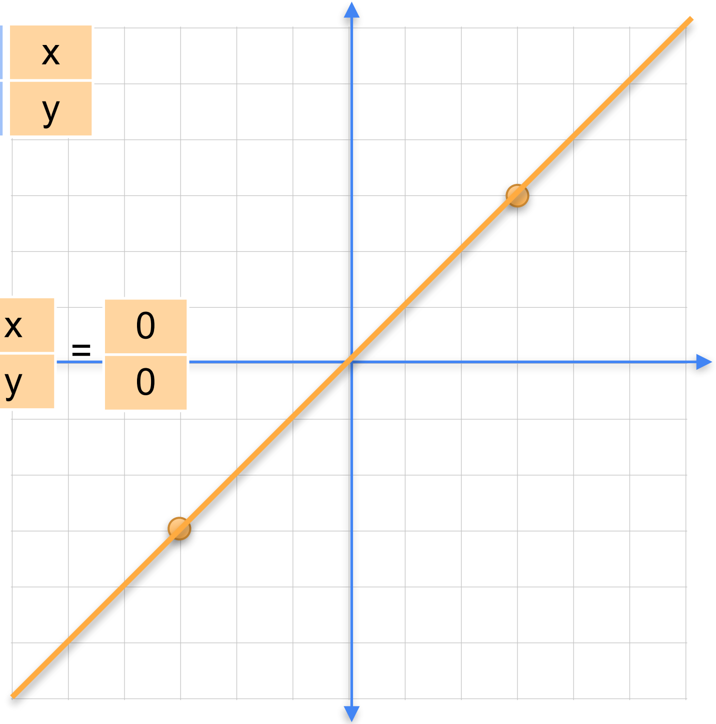
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many points

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

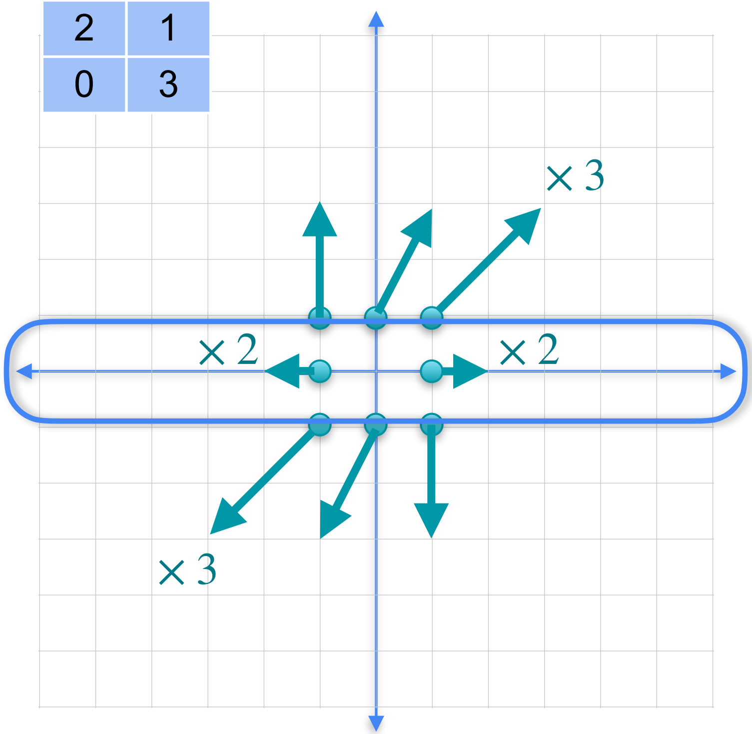
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

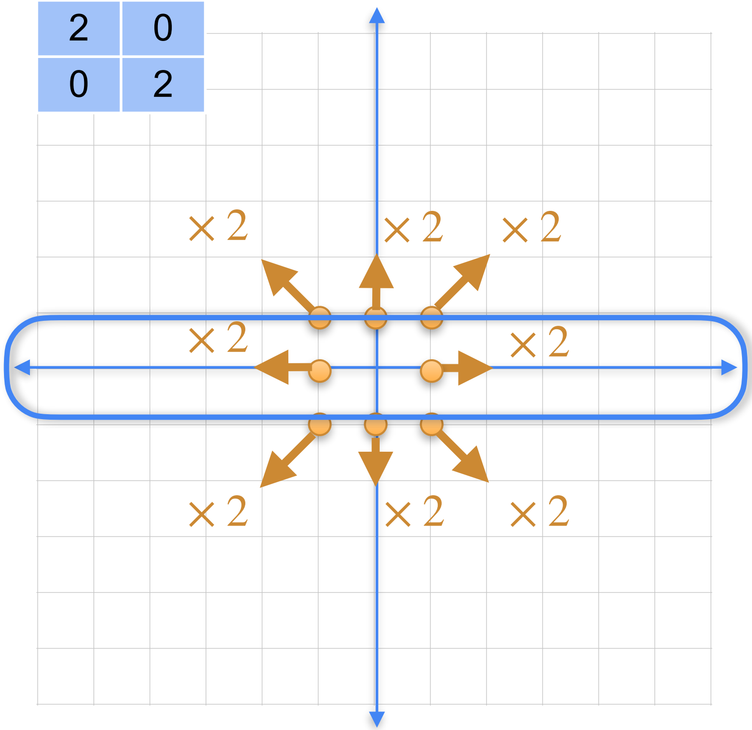


Finding eigenvalues

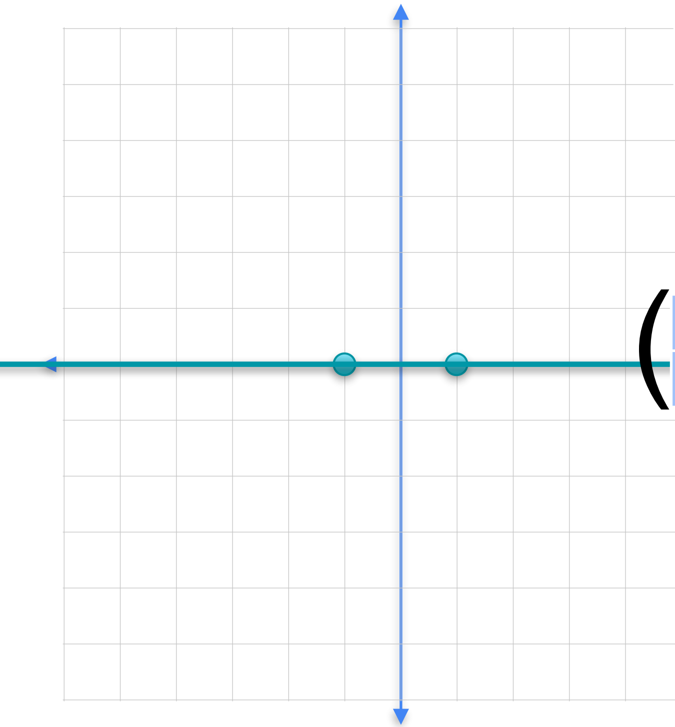
2	1
0	3



2	0
0	2



Finding eigenvalues



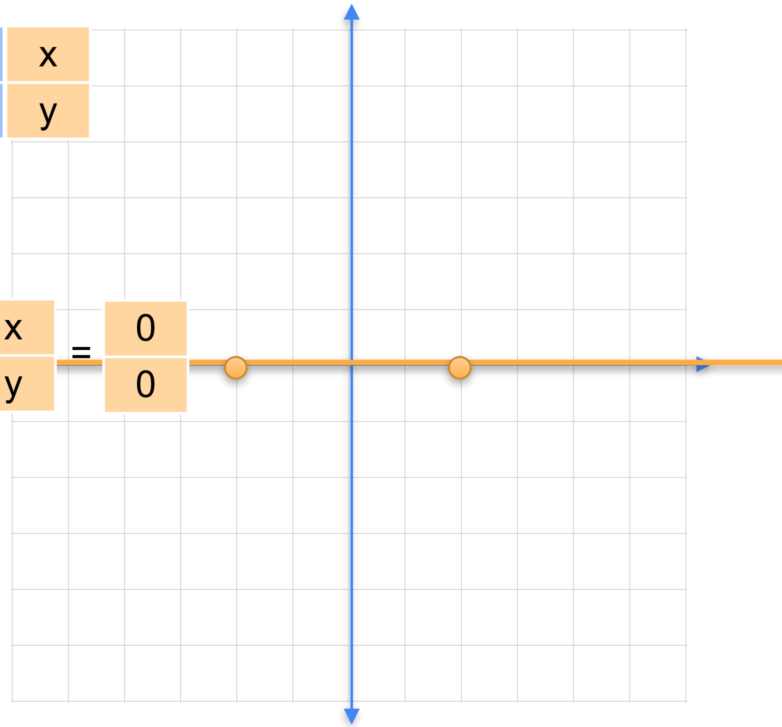
$$\begin{bmatrix}2&1\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

For infinitely many points

$$\left(\begin{bmatrix}2&1\\0&3\end{bmatrix}-\begin{bmatrix}2&0\\0&2\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\det\begin{bmatrix}0&1\\0&1\end{bmatrix}=0$$



Finding eigenvalues

If λ is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$
$$\lambda = 3$$

Finding eigenvectors

Eigenvalues: $\lambda = 2$, $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} 2x + y &= 2x \\ 0x + 3y &= 2y \end{aligned} \qquad \begin{aligned} x &= 1 \\ y &= 0 \end{aligned} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} 2x + y &= 3x \\ 0x + 3y &= 3y \end{aligned} \qquad \begin{aligned} x &= 1 \\ y &= 1 \end{aligned} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3

- The characteristic polynomial is $\det \begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$

- Which factors as $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are $\begin{aligned} \lambda &= 11 \\ \lambda &= 1 \end{aligned}$

Finding eigenvalues

$A =$

2	1	-1
1	0	-3
-1	-3	0

$\lambda I =$

λ	0	0
0	λ	0
0	0	λ

Characteristic polynomial: $\det(A - \lambda I) = 0$

det

2 - λ	1	-1
1	-λ	-3
-1	-3	-λ

$= 0$

$$(2-\lambda)\lambda^2 \quad +3 \quad +3 \quad -9(2-\lambda) \quad +\lambda \quad +\lambda \quad = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda+3)(\lambda-1)(\lambda-4) = 0$$

Eigenvalues: $-3, 1, 4$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues: $-3, 1, \boxed{4}$

$$Av = \lambda v$$

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues: $-3, 1, \boxed{4}$

$$Av = \lambda v$$

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4x_1 \\ x_1 - 3x_3 &= 4x_2 \\ -x_1 - 3x_2 &= 4x_3 \end{aligned}$$

$$\begin{array}{lrcl} R_1 & -2x_1 + x_2 - x_3 & = & 0 \\ R_2 & x_1 - 4x_2 - 3x_3 & = & 0 \\ R_3 & -x_1 - 3x_2 - 4x_3 & = & 0 \end{array}$$

$$\begin{gathered} R_2 + R_3 \\ -7x_2 - 7x_3 = 0 \\ x_2 = -x_3 \end{gathered} \qquad \begin{gathered} 3R_1 + R_3 \\ -7x_1 - 7x_3 = 0 \\ x_1 = -x_3 \end{gathered}$$

$$\begin{aligned} x_1 &= k \\ x_2 &= k \\ x_3 &= -k \end{aligned}$$

infinite solutions of this form

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

this works!

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 2 \\ x_3 &= -2 \end{aligned}$$

so does this!

Eigenvector: $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 1$ $\lambda_3 = -3$

Eigenvectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Note on dimensions

Eigenvalues → Determinant → Square Matrix

9	4
4	3

✓

9	4	5
4	3	-2

✗

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Determinants and Eigenvectors

On the number of eigenvectors

Number of eigenvectors

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

3 by 3 matrix $\xrightarrow{?}$ 3 distinct eigenvalues $\xrightarrow{?}$ 3 distinct eigenvectors

Eigenvalues	$\lambda_1 = 4$	$\lambda_2 = 1$	$\lambda_3 = -3$
Eigenvectors	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial = $\det(A - \lambda I)$ = det $\begin{bmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0.5 \\ 0 & 0 & 2 - \lambda \end{bmatrix}$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0 = 0$$

Eigenvalues: $4, \boxed{2,\ 2}$ Repeated eigenvalue

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

$x_1 = 0$

$x_2 =$ any number

$x_3 = 0$

Eigenvector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Point in different directions

Different eigenvectors

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

	$\lambda_1 = 4$	$\lambda_2 = 2$	$\lambda_3 = 2$
Eigenvalues	$\lambda_1 = 4$	$\lambda_2 = 2$	$\lambda_3 = 2$
Eigenvectors	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$A =$

2	0	0
-1	4	-0.5
4	0	2

Characteristic polynomial = $\det(A - \lambda I)$ = det

2-λ	0	0
1	4-λ	0.5
-4	0	2-λ

$$(2-\lambda)^2(4-\lambda) + 0 + 0 - 0 - 0 - 0$$

Eigenvalues: $4, \boxed{2,\ 2}$ Repeated eigenvalue

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$x_1 = 0$ $\quad x_3 = 0$ $\quad x_2 =$ any number

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

$x_1 = 0$ $\quad$ $x_3 = 4x_2$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 4 \end{aligned} \qquad \begin{bmatrix} 0 \\ 0.5 \\ 2 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 0.5 \\ x_3 &= 2 \end{aligned}$$

On the same line

Same eigenvector

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$x_1 = 0$ $\quad\quad$ $x_3 = 4x_2$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 2$ $\lambda_3 = 2$

Eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

Can't create an eigenbasis from this matrix

Summary

a	b
c	d

a	b	c
d	e	f
g	h	i

Eigenvalues

λ_1, λ_2

$\lambda_1, \lambda_2, \lambda_3$

If $\lambda_1 \neq \lambda_2$ → 2 eigenvectors (2 different directions)

If $\lambda_1 = \lambda_2$ → 1 eigenvector (1 direction)
→ 2 eigenvectors (2 different directions)

If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ → 3 eigenvectors (3 different directions)

If $\lambda_1 = \lambda_2 \neq \lambda_3$ → 2 eigenvectors (2 different directions)
→ 3 eigenvectors (3 different directions)

If $\lambda_1 = \lambda_2 = \lambda_3$ → 1 eigenvector (1 direction)
→ 2 eigenvectors (2 different directions)
→ 3 eigenvectors (3 different directions)

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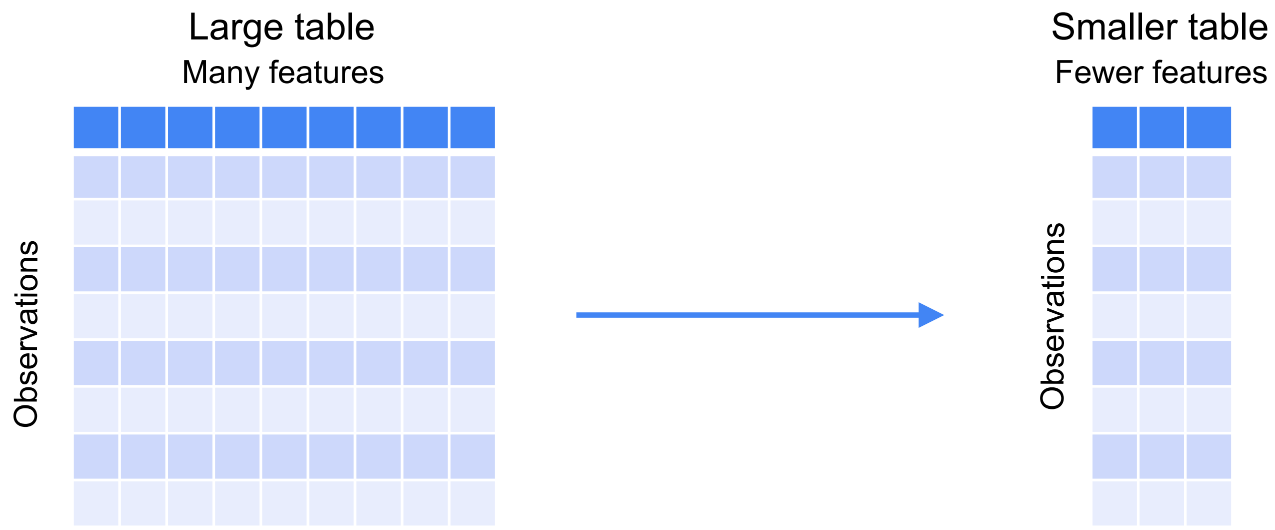
Determinants and Eigenvectors

Dimensionality reduction annd projection

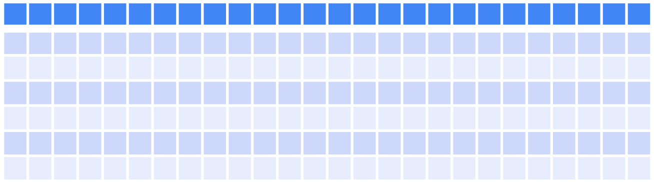
Dimensionality Reduction

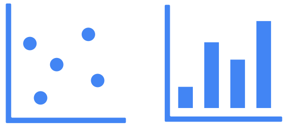
- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



Dimensionality Reduction

- Leads to smaller datasets
- Easier to visualize





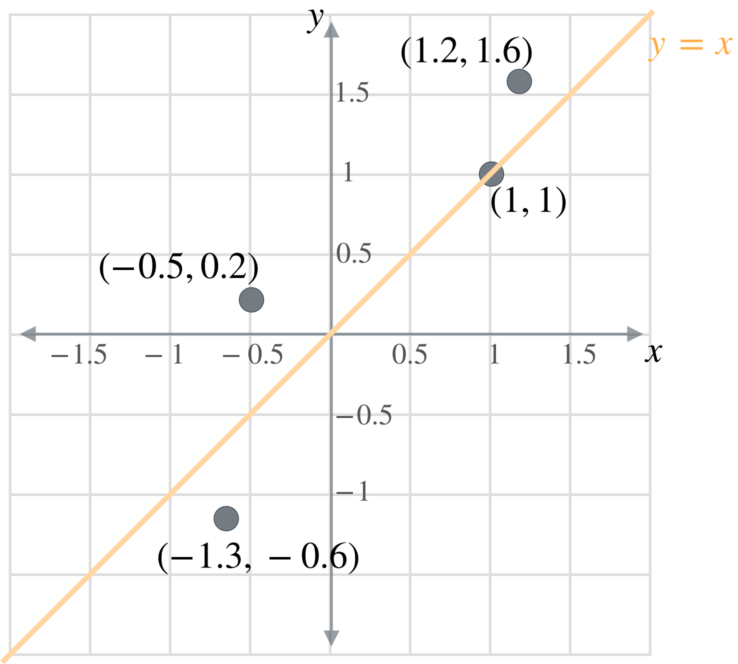
Customer Age	Account Age	Days Since Login	Total Purchases	Total $ Spent
23	1 month	10 days	1	$100
71	45 months	2 days	5	$150
54	30 months	15 days	2	$70
36	22 months	12 days	4	$210

Easy approach - just delete columns

Loses valuable information

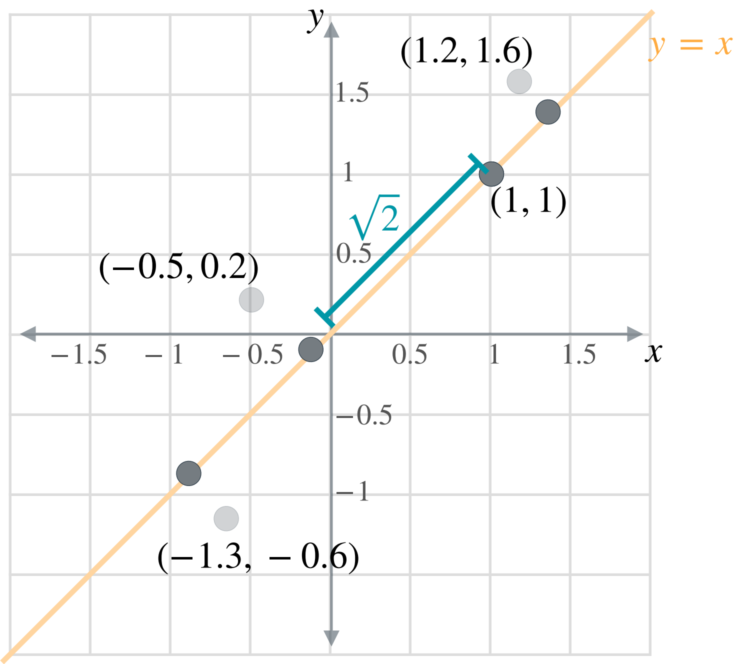
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



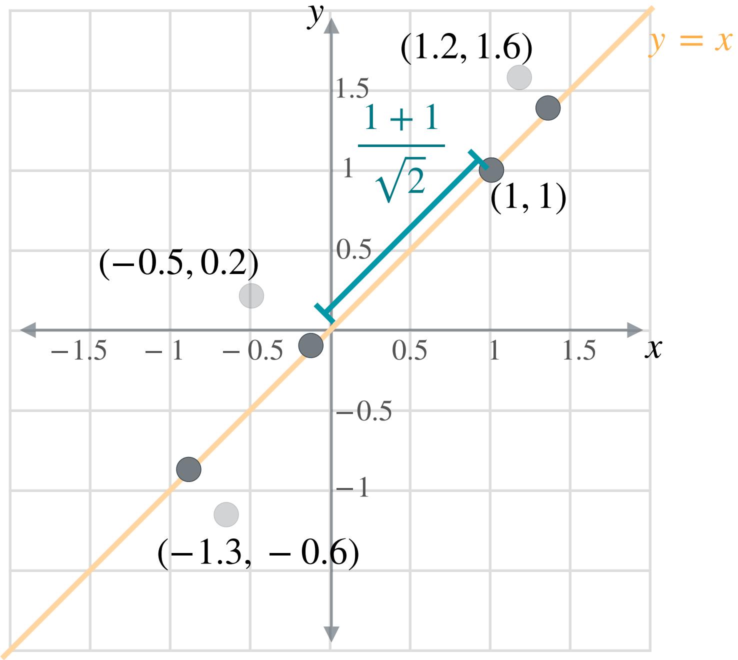
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



Projections

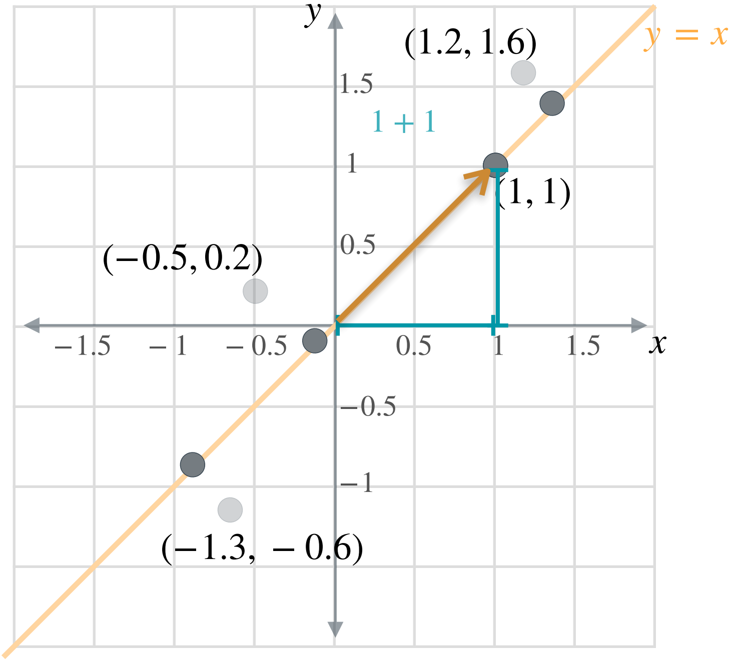
x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

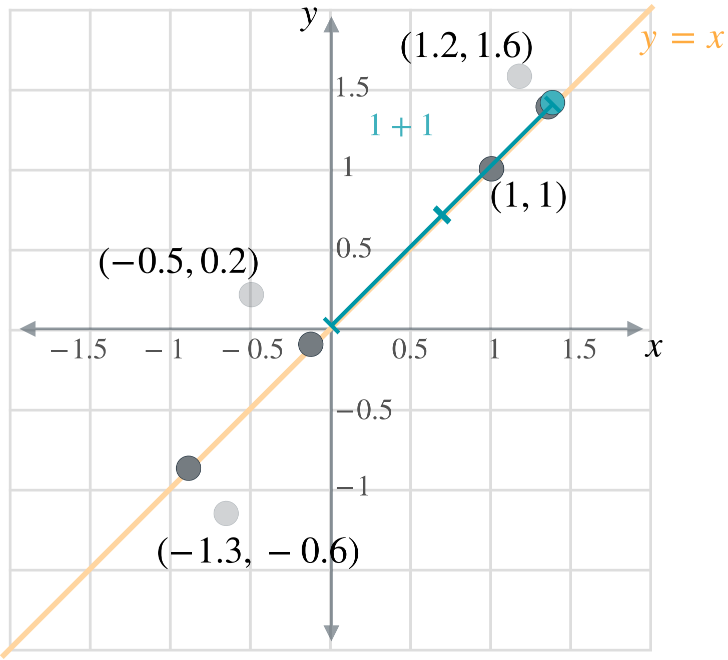
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1 + 1)$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

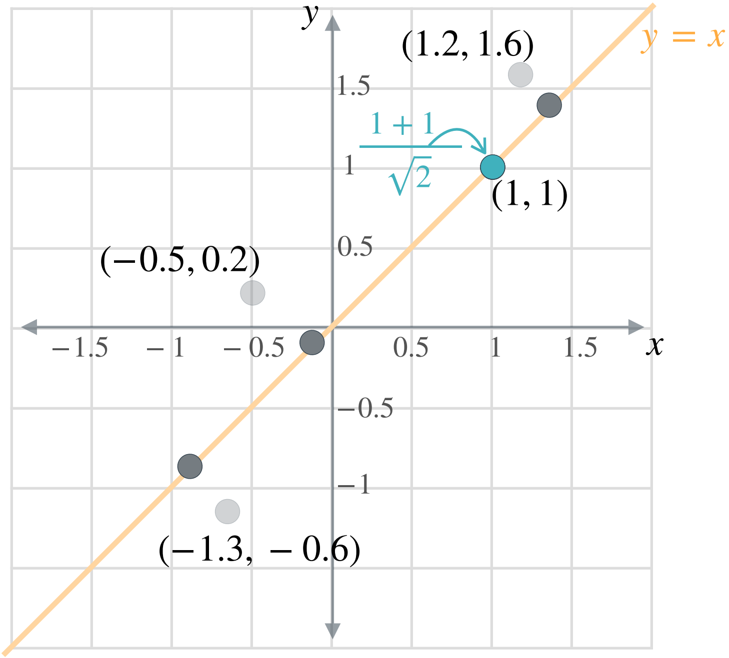
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{(1+1)}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = (1+1)/\sqrt{2}$$



Projections

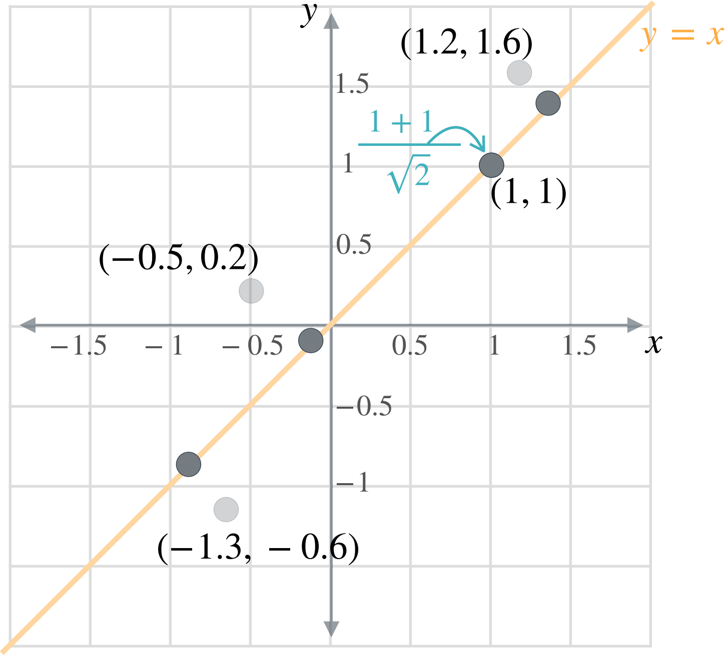
x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

Norm of 1

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$(1+1)/\sqrt{2}$$

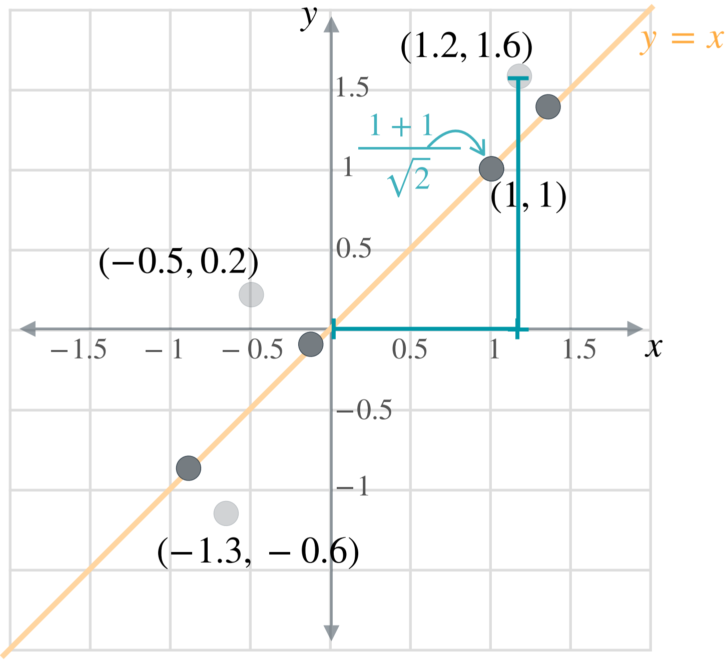
$$\frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

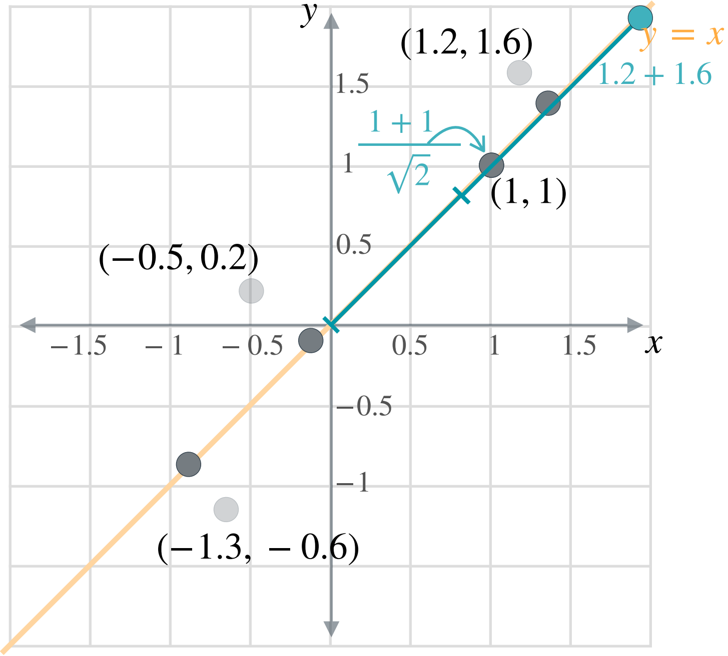
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

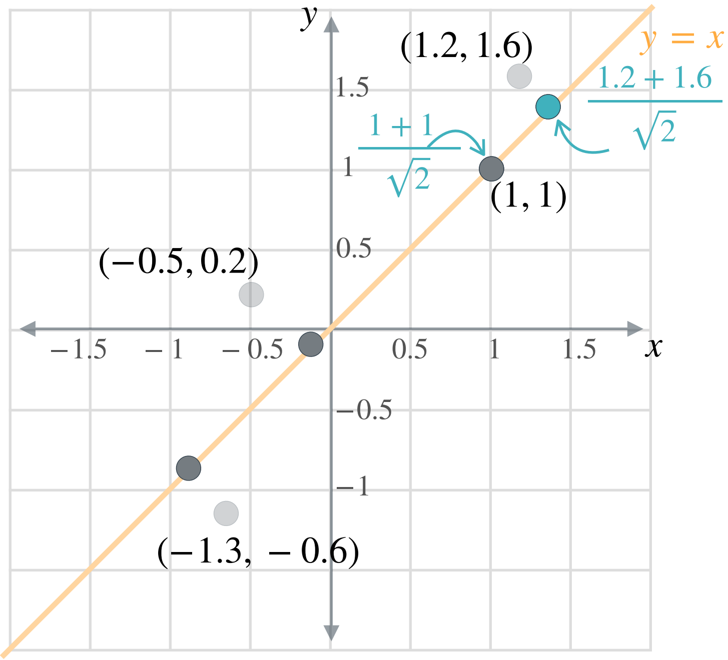
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6) \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

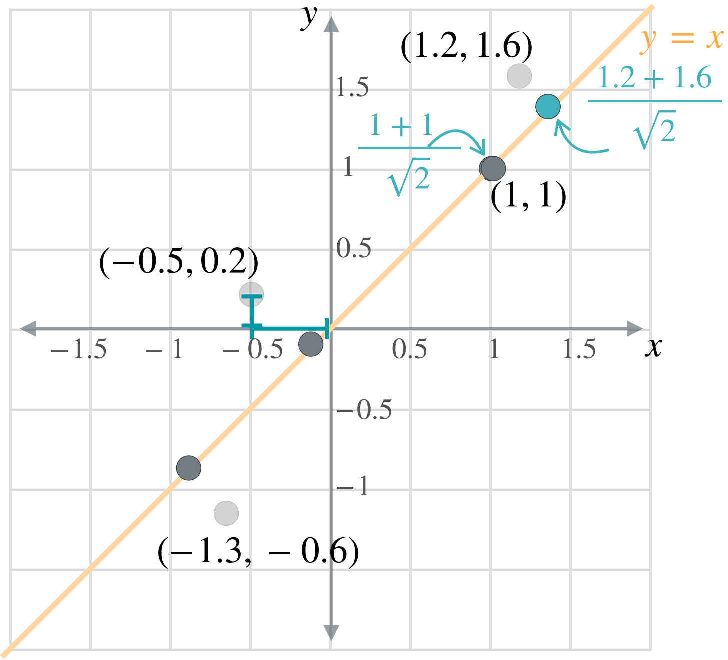
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

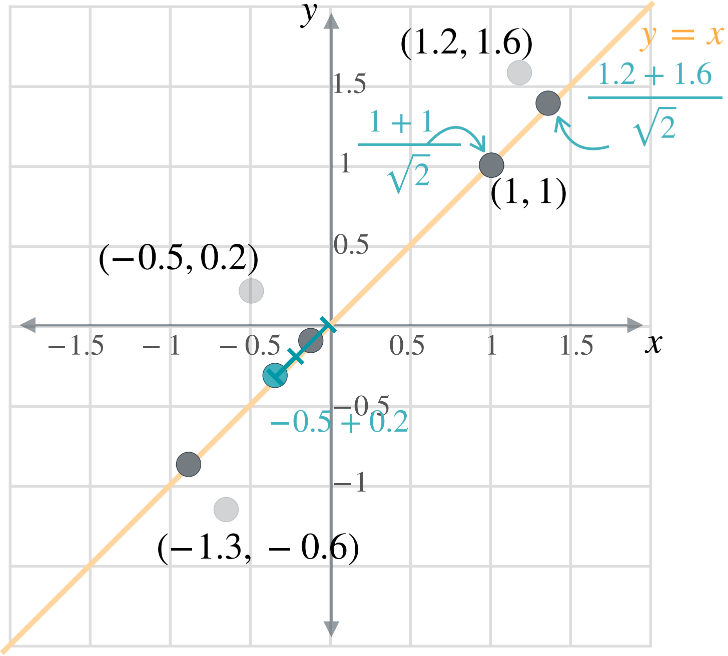
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ \ \end{bmatrix}$$



Projections

	x	y
	1.0	1.0
	1.2	1.6
→	-0.5	0.2
	-1.3	-0.6

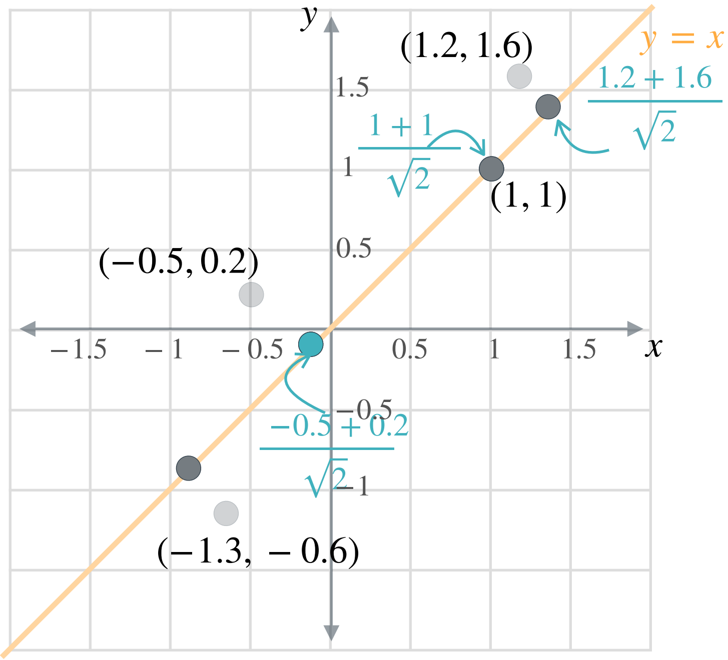
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2) \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

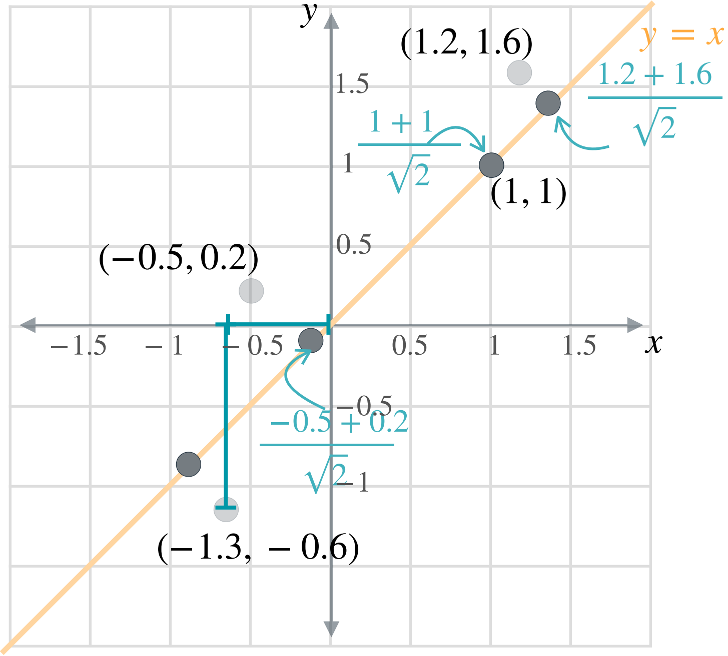
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

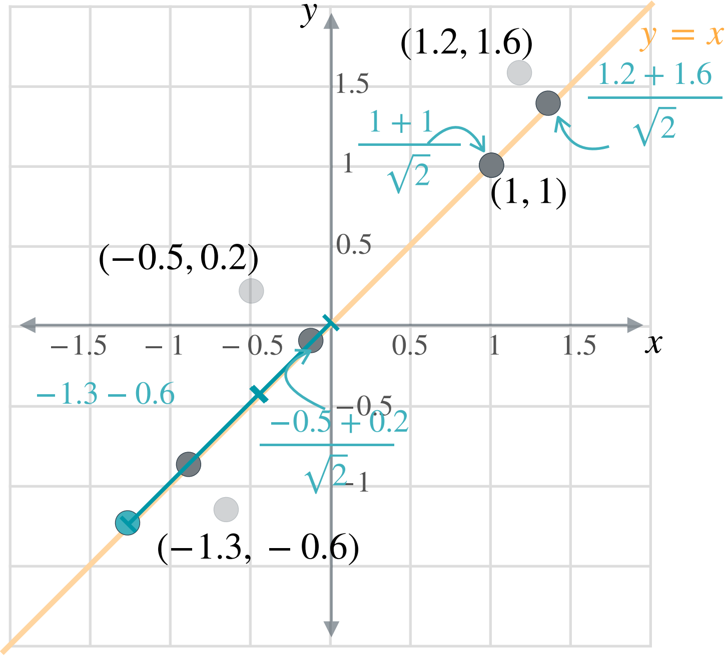
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ (-1.3,-0.6) \end{bmatrix}$$



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

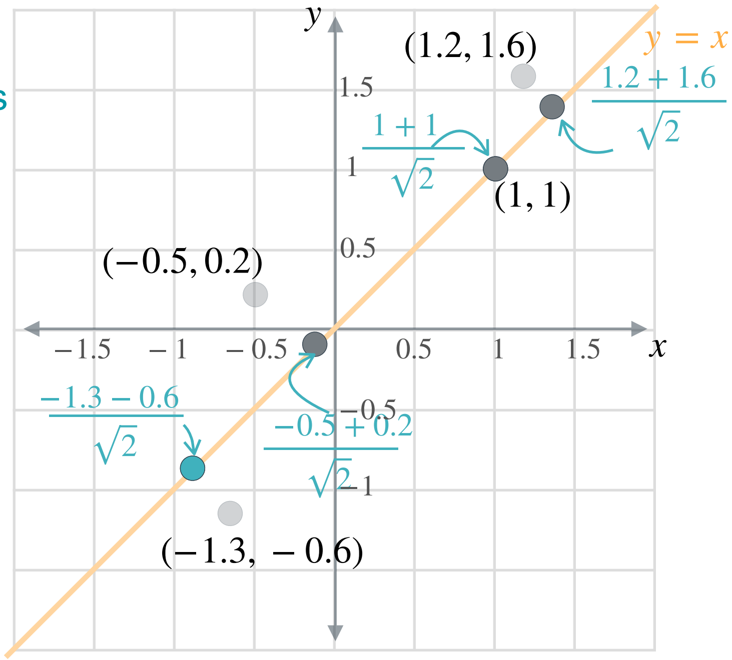
Final coordinates

$(1+1)/\sqrt{2}$

$(1.2+1.6)/\sqrt{2}$

$(-0.5+0.2)/\sqrt{2}$

$(-1.3,-0.6)/\sqrt{2}$



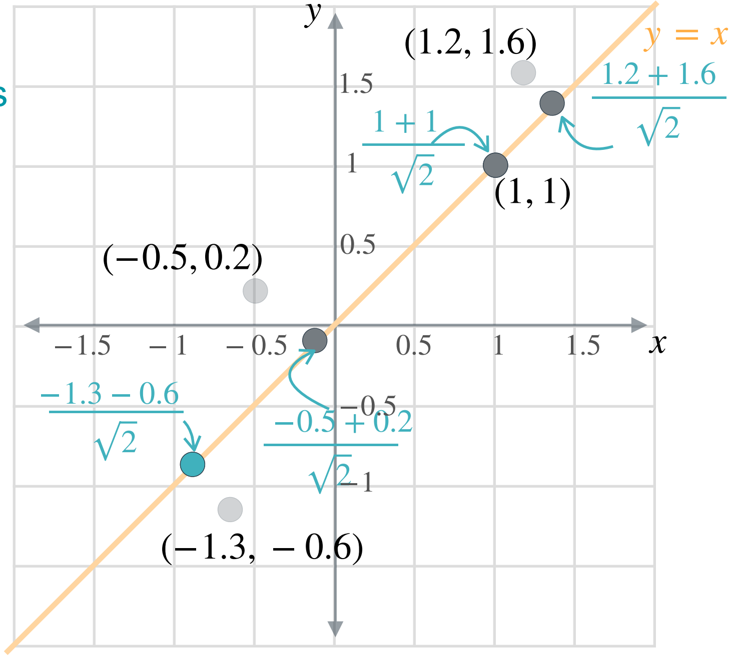
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
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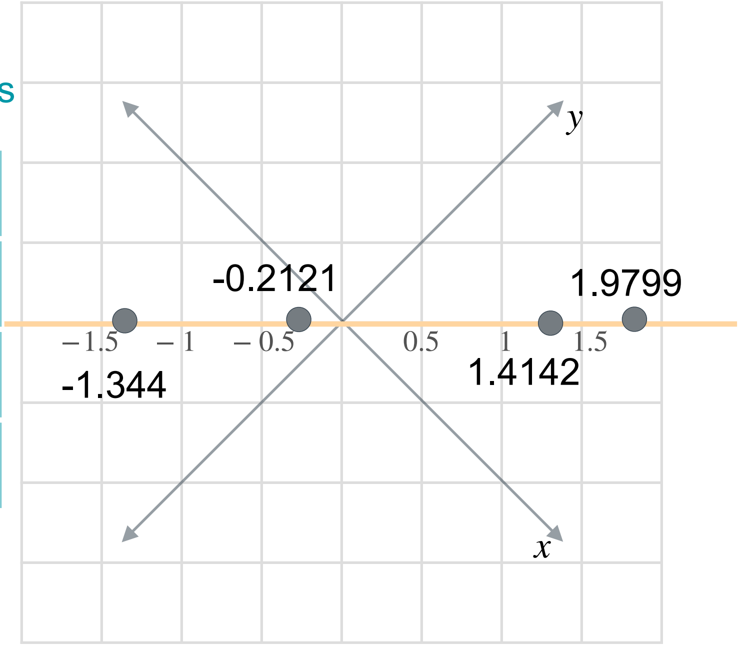

Projections

To project a matrix A onto a vector v

$$A_P = A \frac{v}{\|v\|_2}$$

$$\underset{r \times 1}{} \qquad \underset{r \times c}{} \qquad \underset{c \times 1}{}$$

$r \times 1$ $\qquad$ $r \times c$ $\qquad$ $c \times 1$

Projections

To project a matrix A onto vectors v_1 and v_2

$$\underset{r \times 2}{A_P} = \underset{r \times c}{A} \underset{c \times 2}{\overbrace{\left[\frac{v_1}{\|v_1\|_2} \; \frac{v_2}{\|v_2\|_2} \right]}^{V}}$$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = A \overbrace{\begin{bmatrix} \dfrac{v_1}{\|v_1\|_2} & \dfrac{v_2}{\|v_2\|_2} \end{bmatrix}}^{V}$$

$r \times 2 \qquad \boxed{r} \times c \qquad c \times \boxed{2}$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = AV$$

$$r \times 2 \qquad r \times c \quad c \times 2$$

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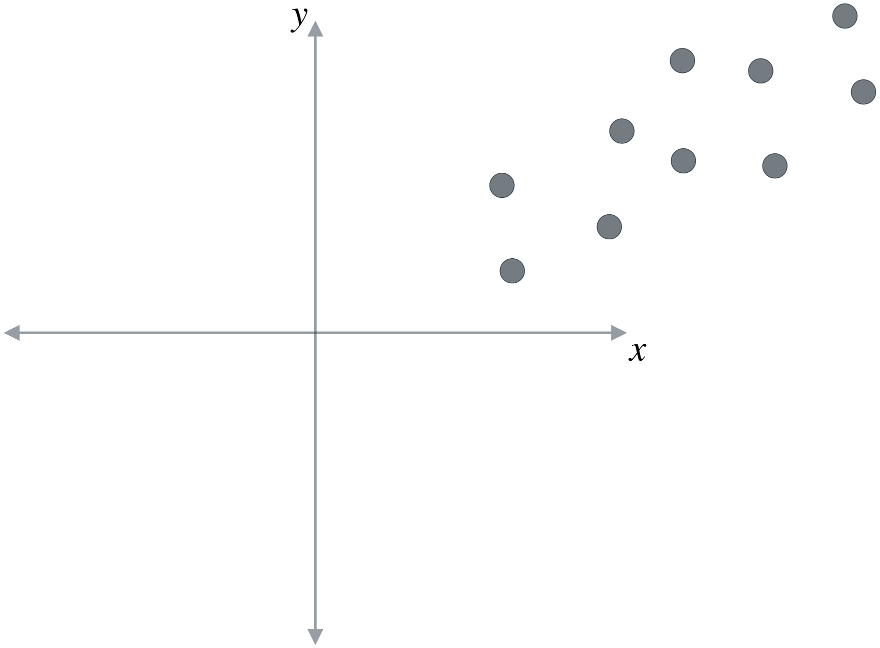
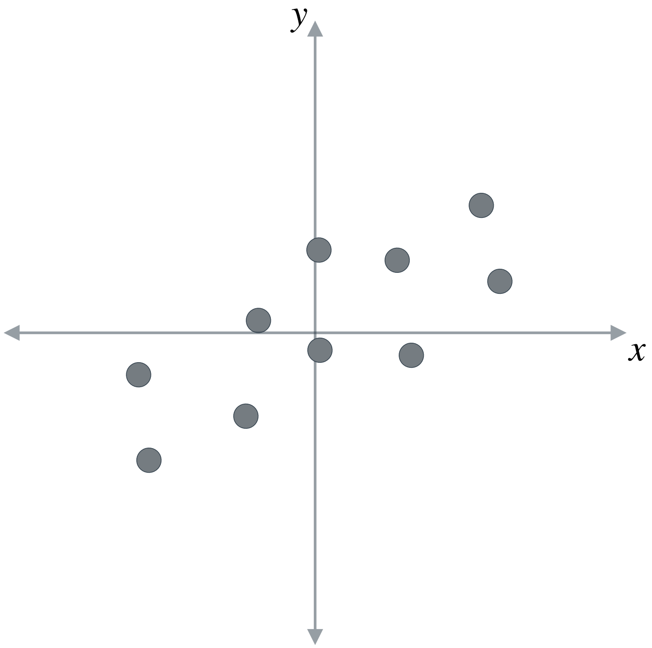
Determinants and Eigenvectors

Motivating PCA

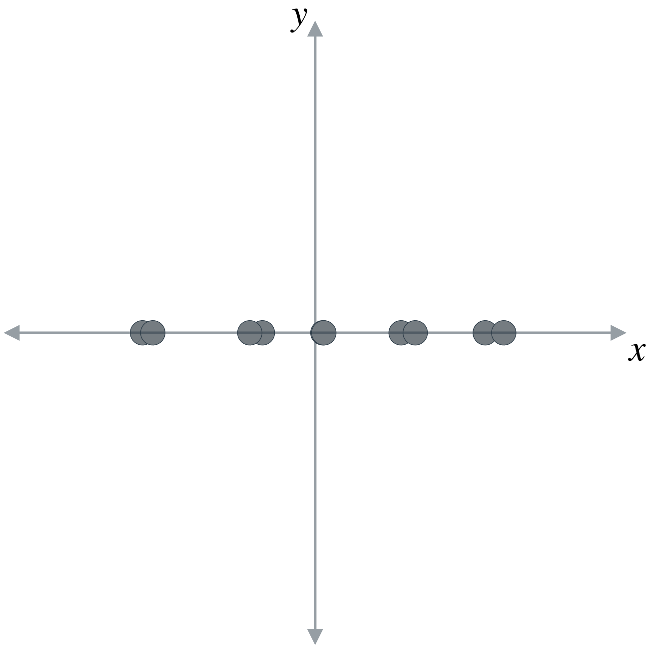
Dimensionality Reduction



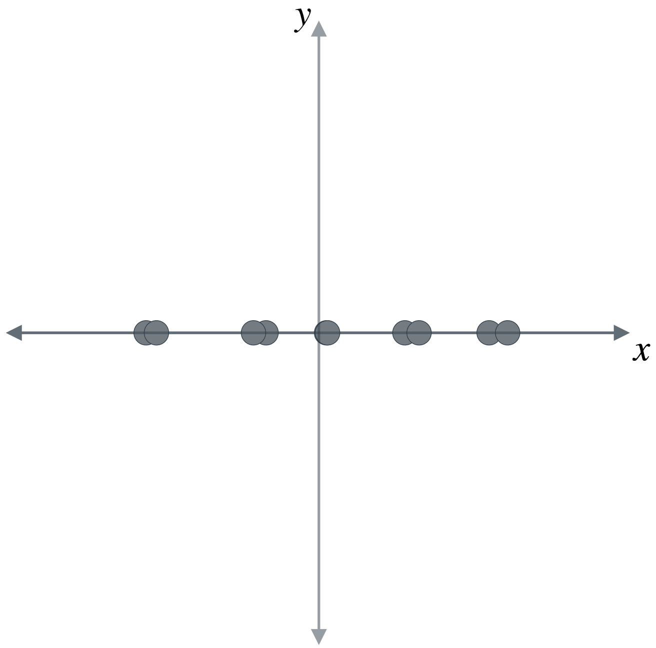
Principal Component Analysis (PCA)



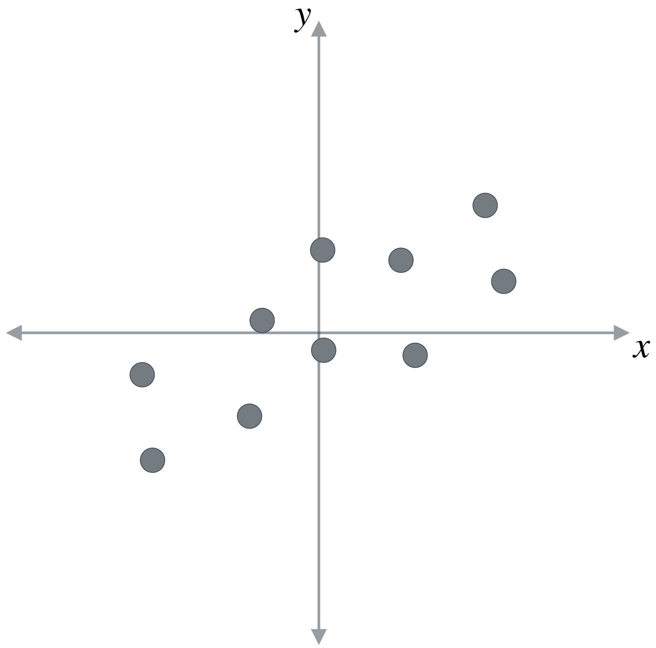
Principal Component Analysis (PCA)

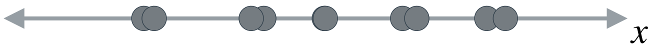


Principal Component Analysis (PCA)

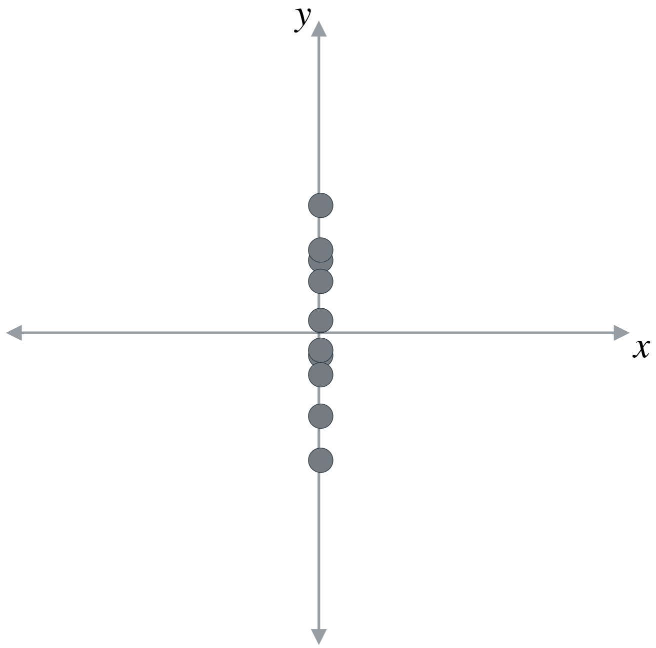


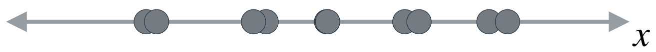
Principal Component Analysis (PCA)



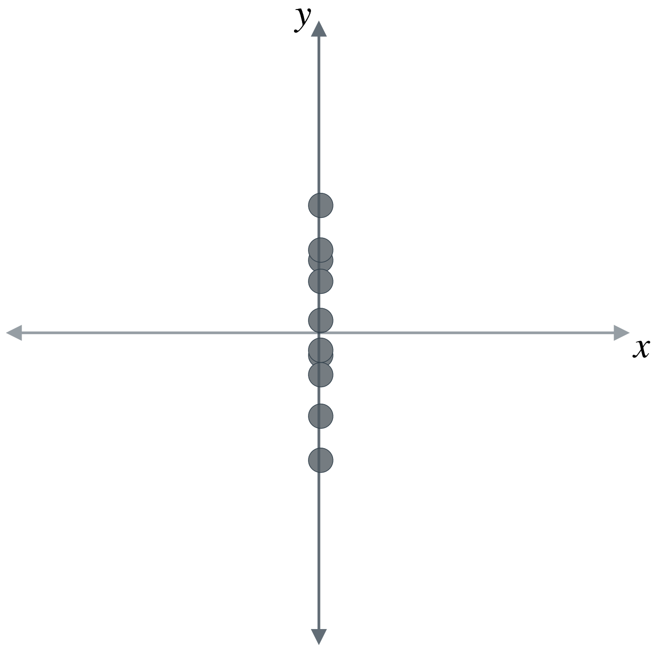


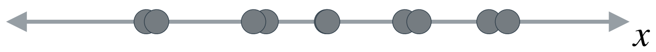
Principal Component Analysis (PCA)



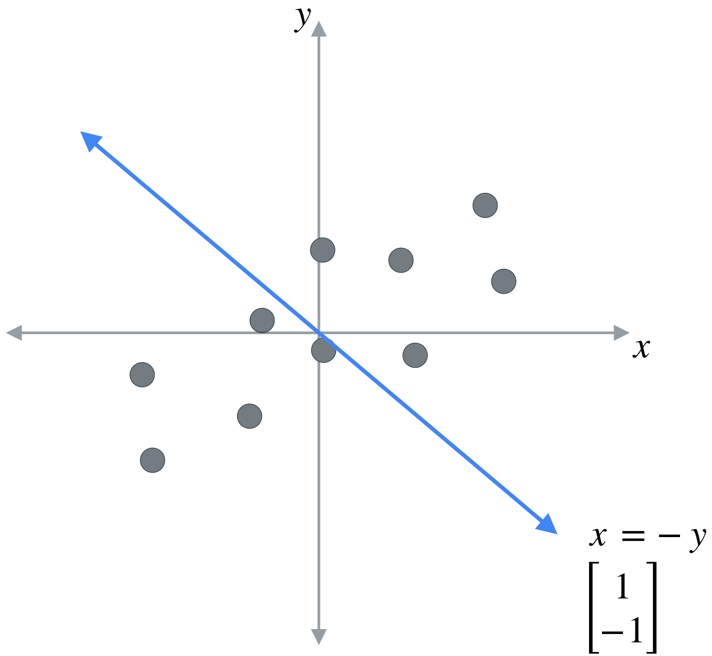


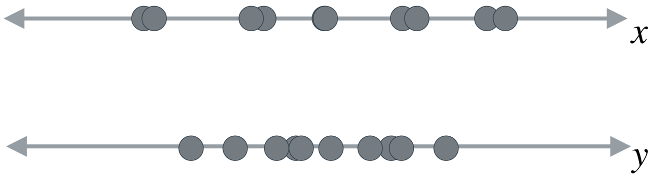
Principal Component Analysis (PCA)



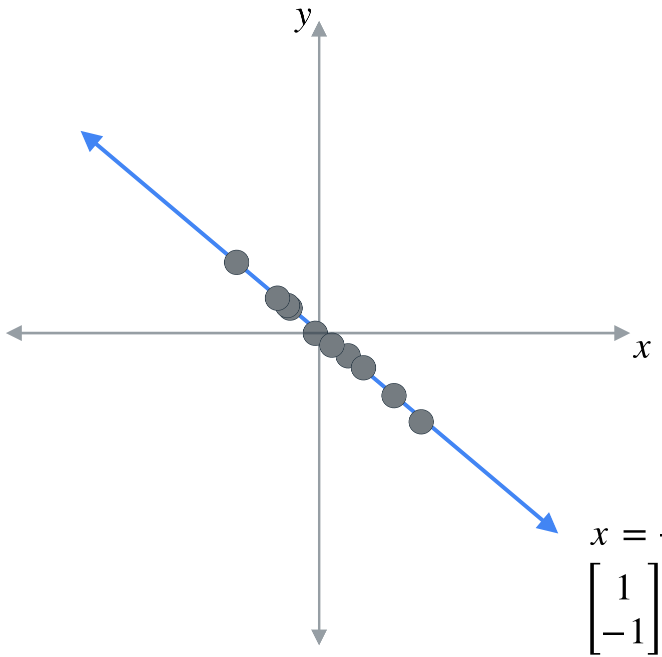


Principal Component Analysis (PCA)



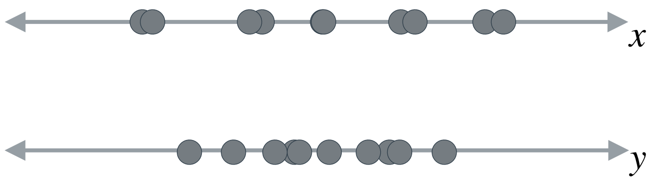


Principal Component Analysis (PCA)

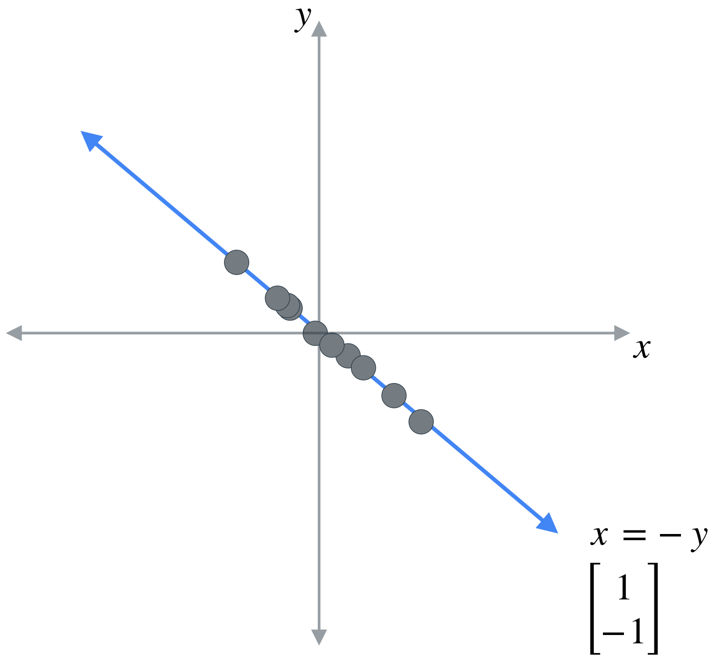


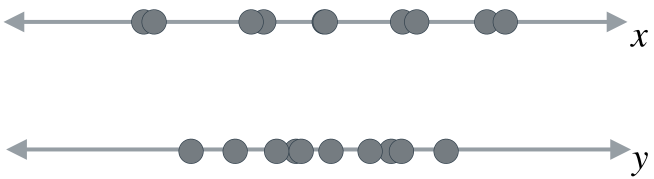
$$x = -y$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

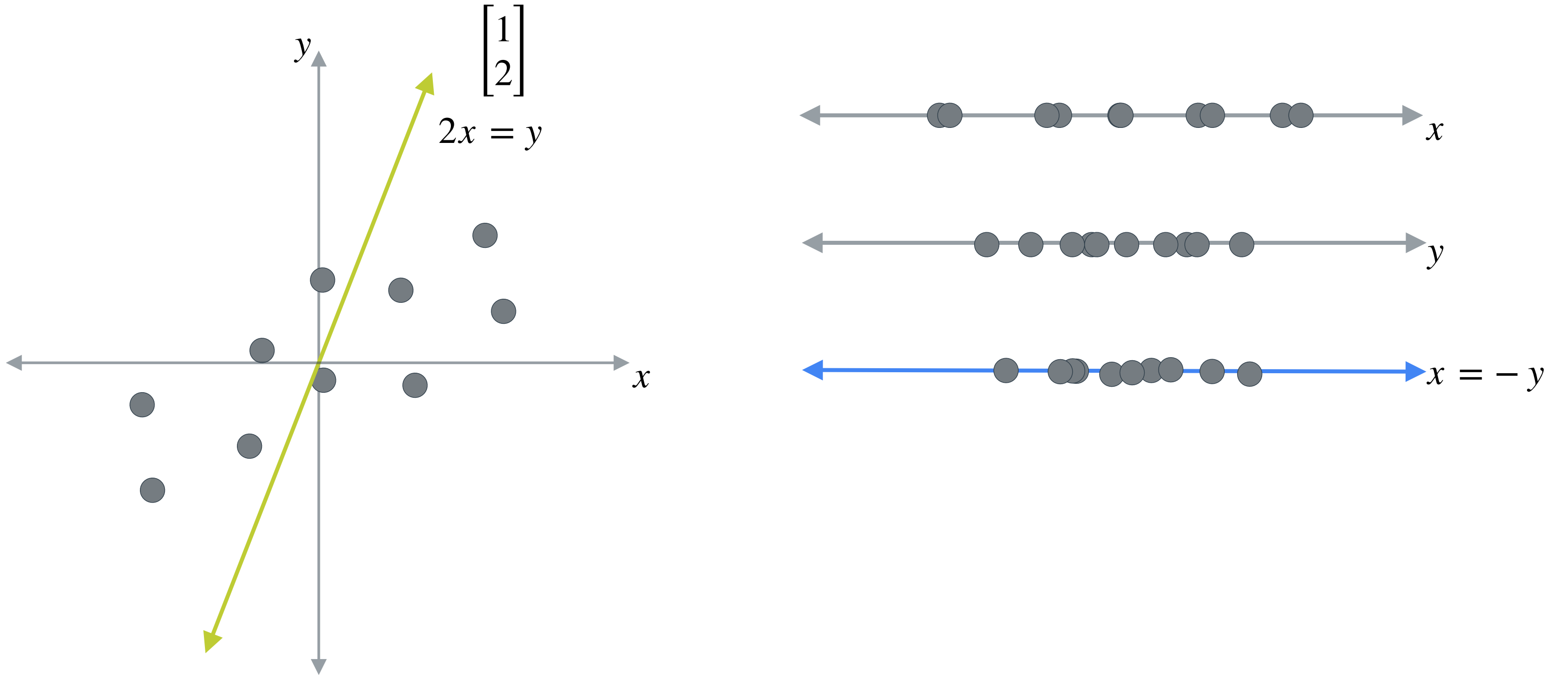


Principal Component Analysis (PCA)





Principal Component Analysis (PCA)



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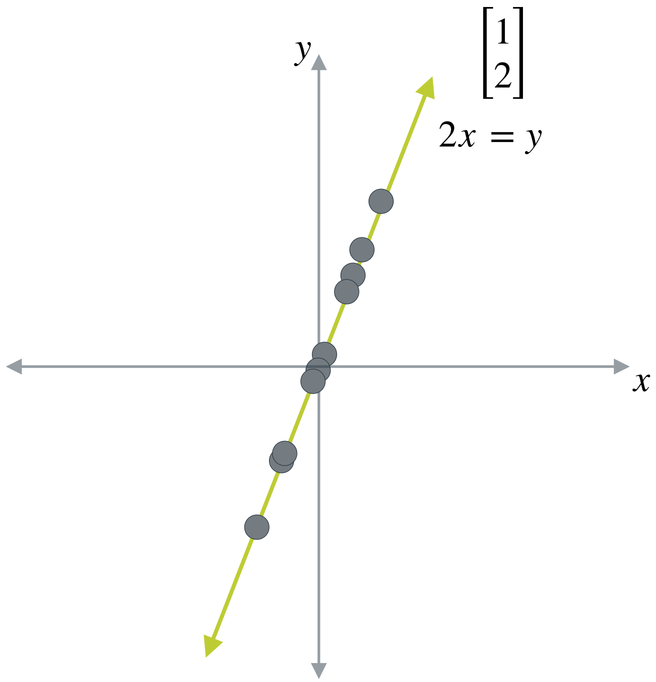

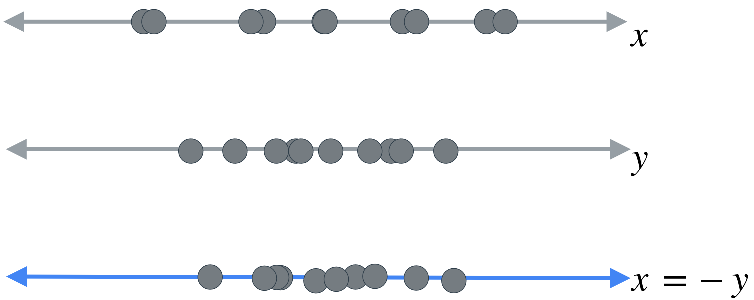

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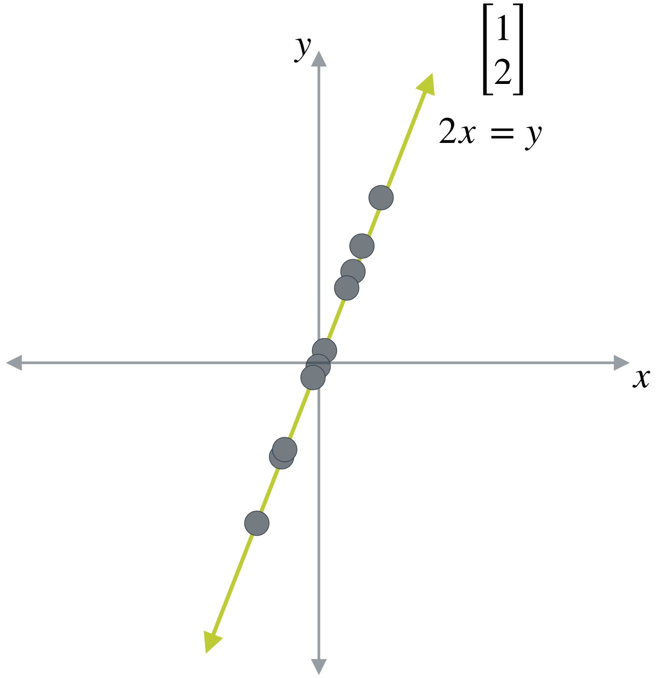

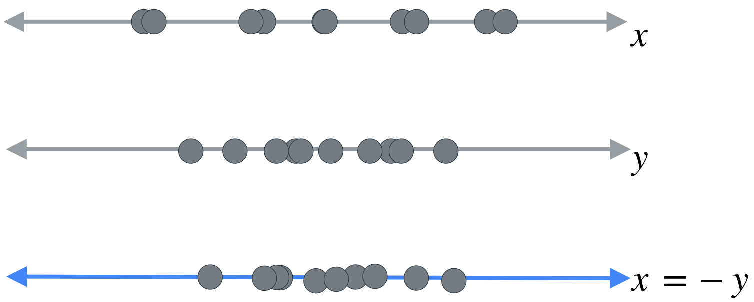

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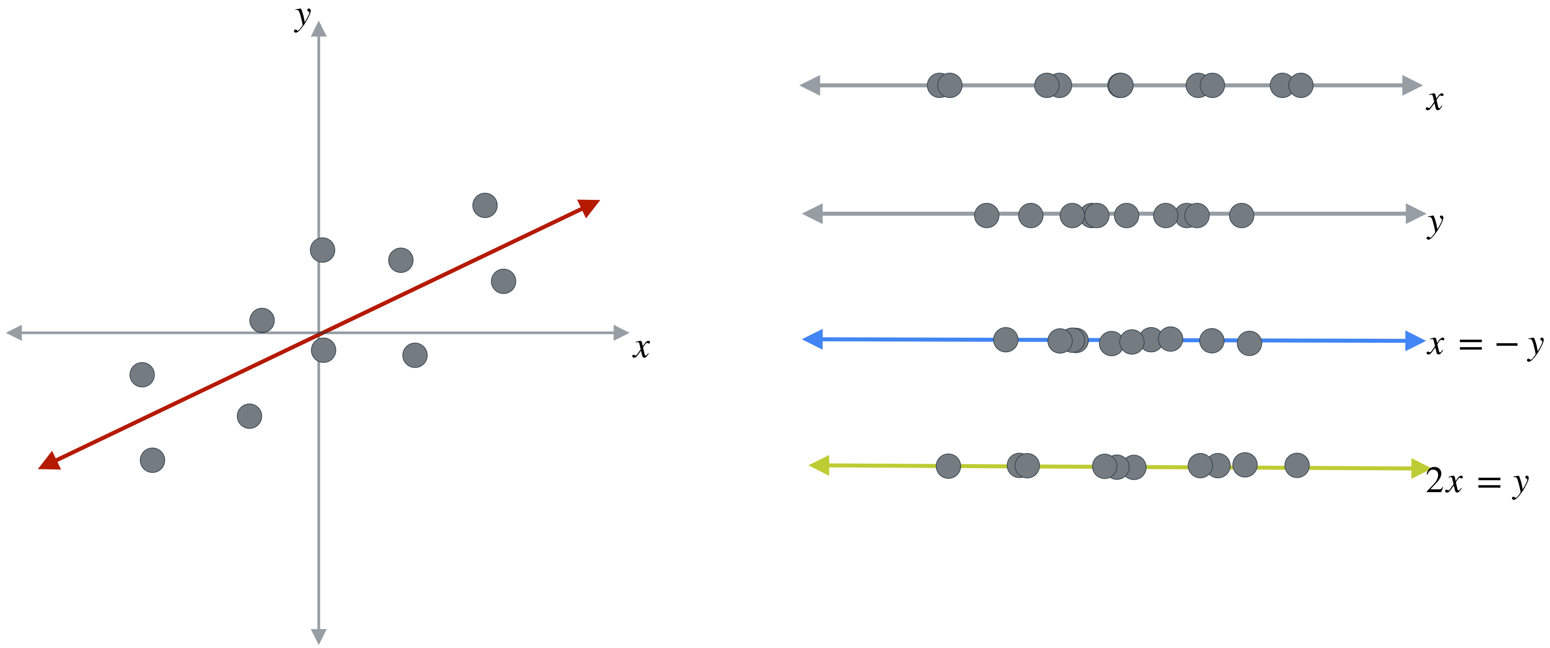

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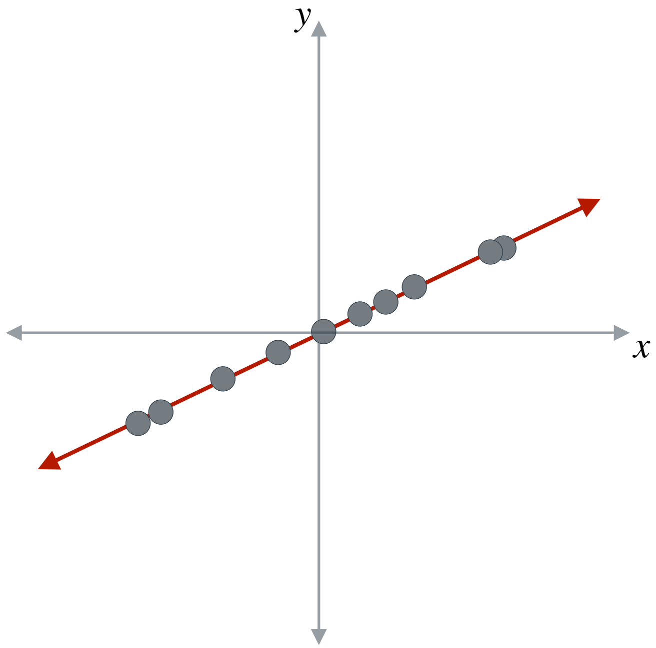

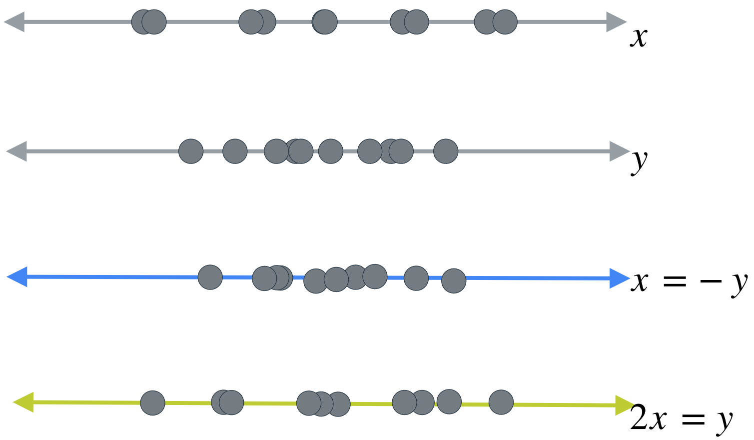

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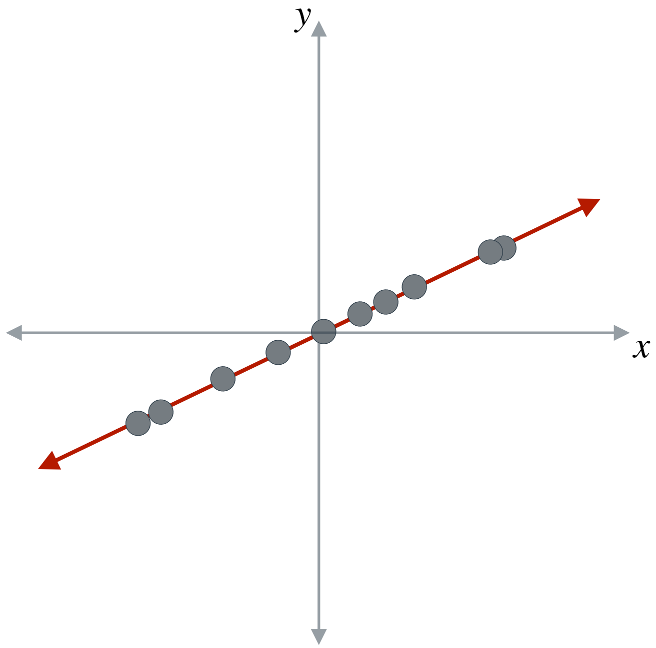

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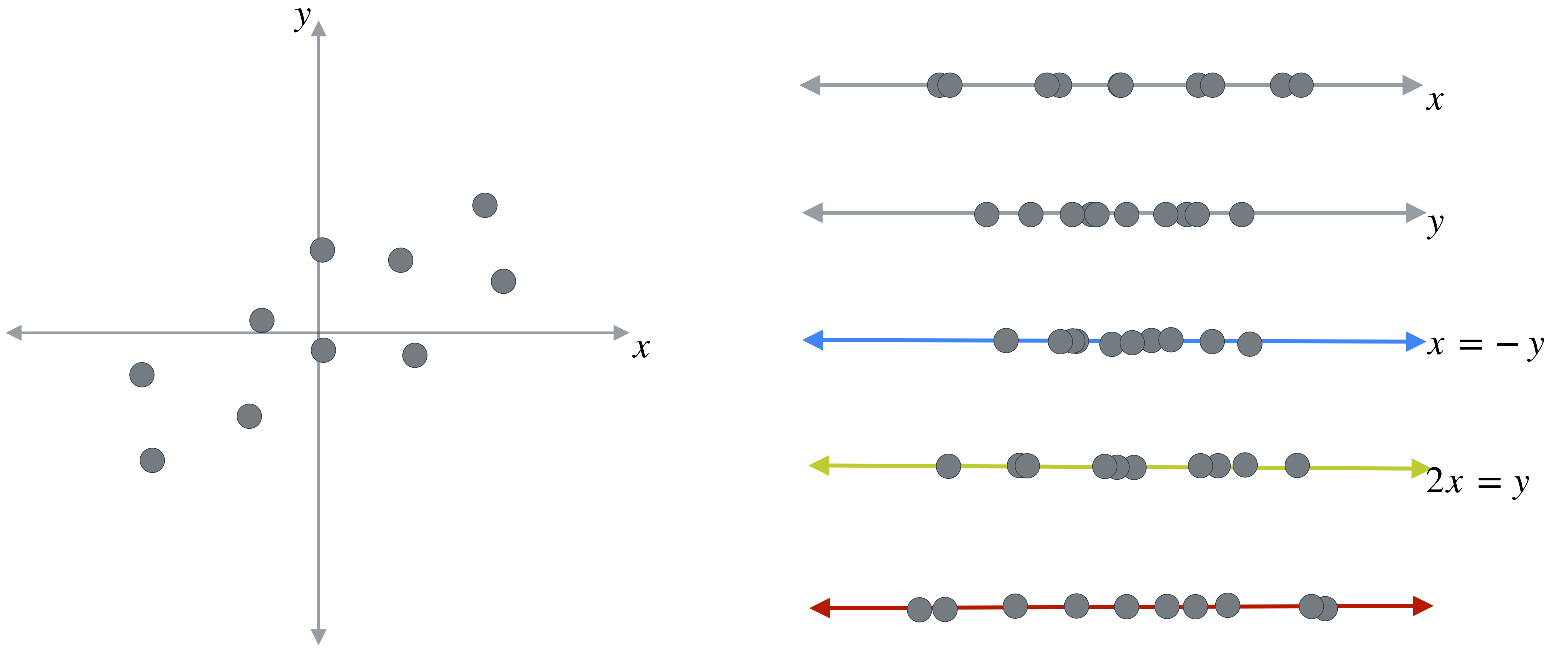

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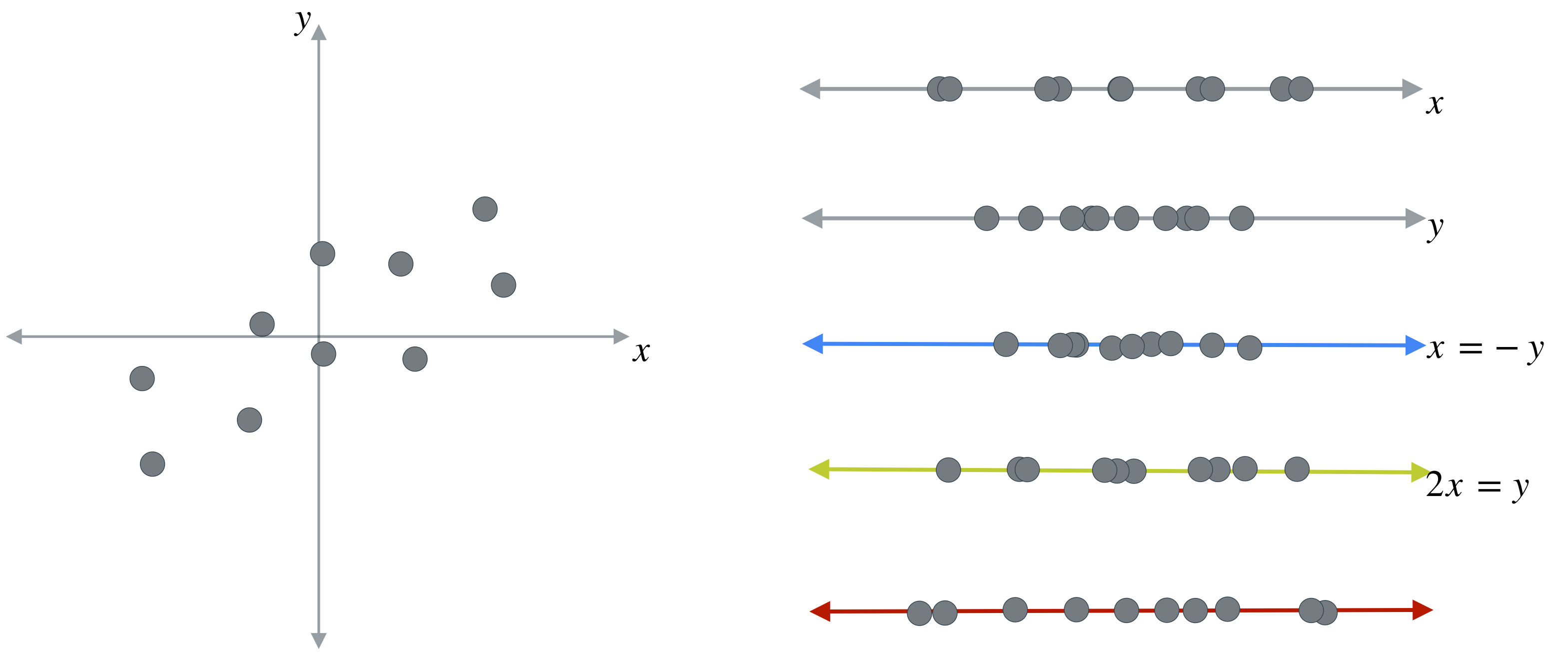
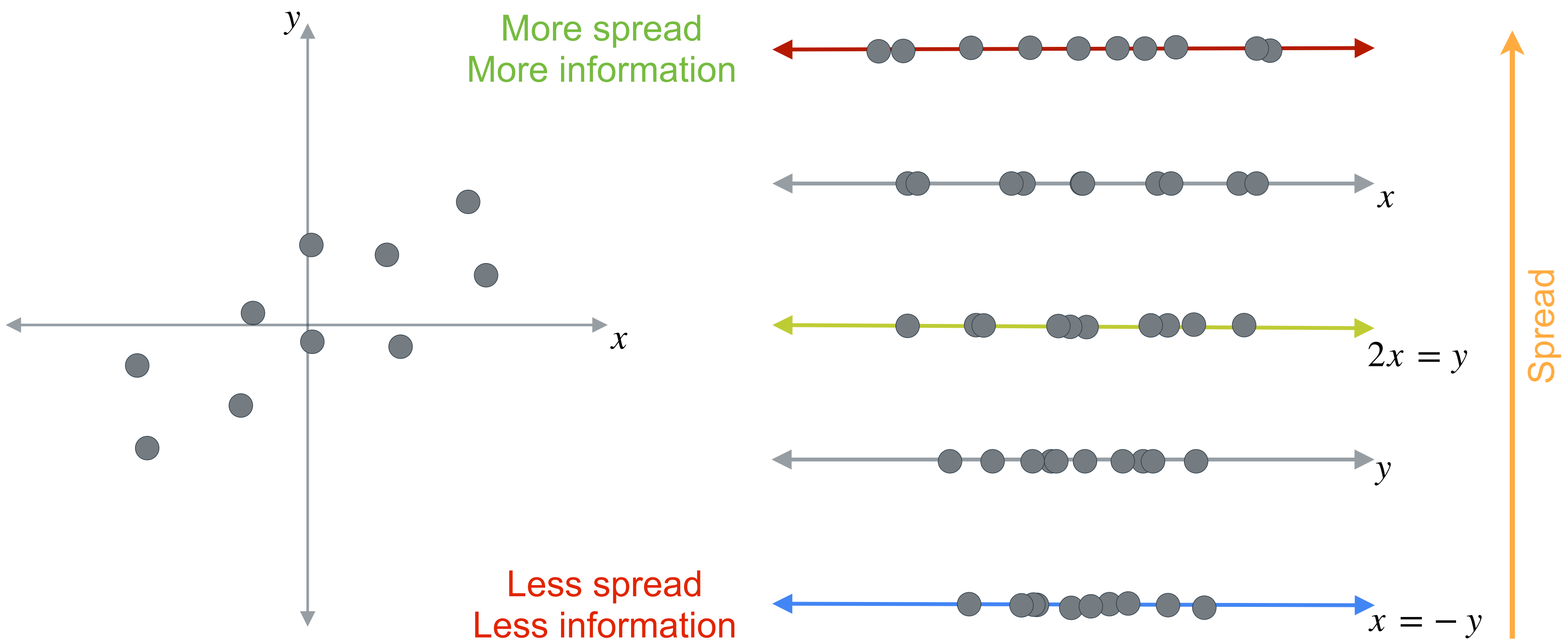
Principal Component Analysis (PCA)

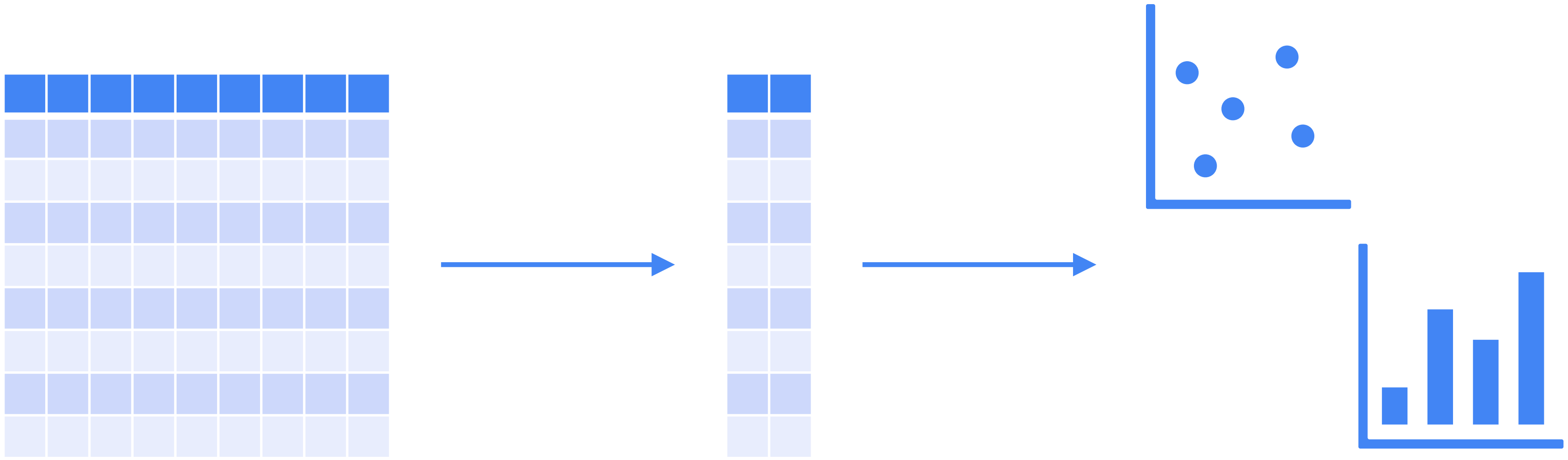


Principal Component Analysis (PCA)



Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization



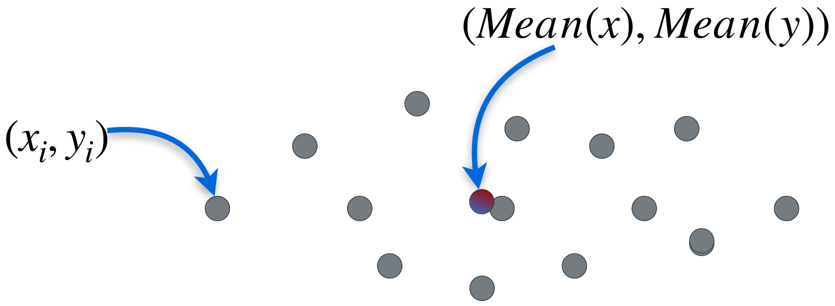
DeepLearning.AI

Determinants and Eigenvectors

Variance and covariance

Mean

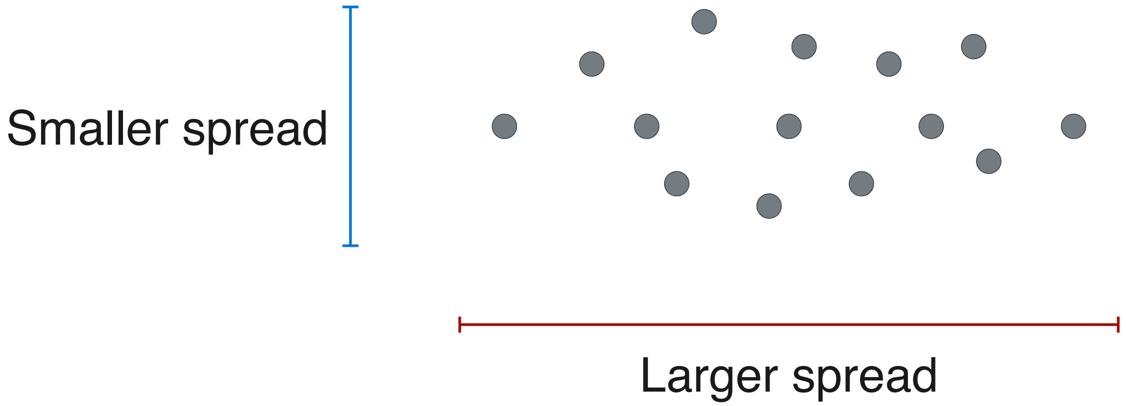
"The average of the data"



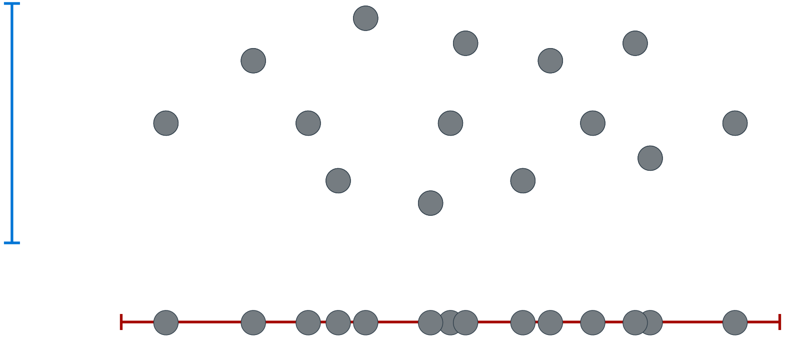
$$Mean(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$Mean(y) = \frac{1}{n}\sum_{i=1}^{n} y_i$$

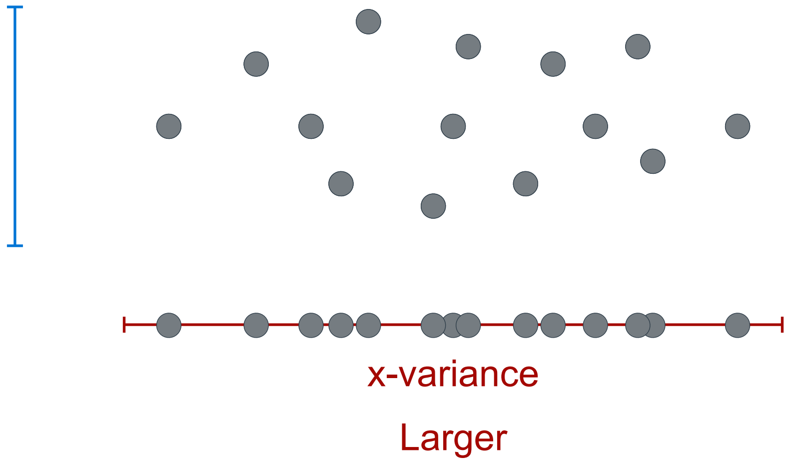
Variance



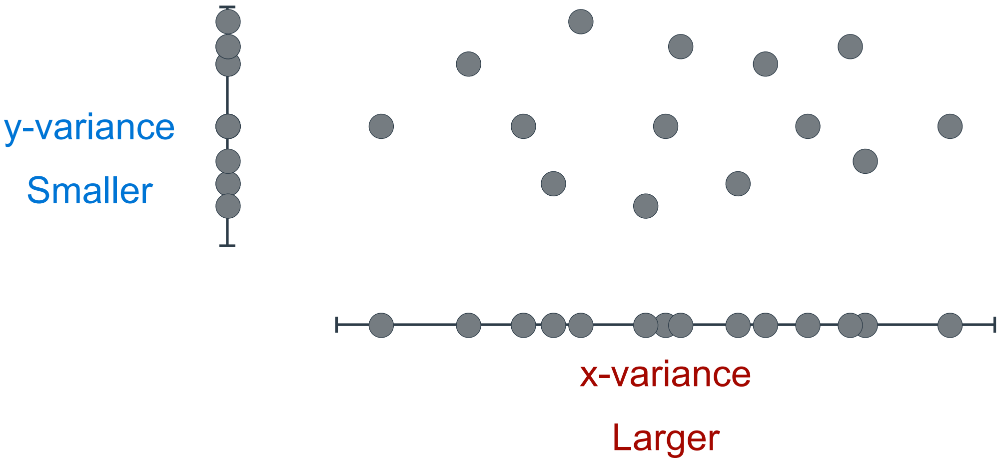
Variance



Variance



Variance



Variance

$$Variance(x) = \boxed{\frac{1}{n-1}}\boxed{\sum_{i=1}^{n}}(x_i - Mean(x))^2 \quad = 16$$

	x_i	x_i - $Mean(x)$	$(x_i - Mean(x))^2$
1	10	1	1
2	4	-5	25
3	11	2	4
4	14	5	25
5	6	-3	9

$\longrightarrow$ 64

$$Mean(x) = 9$$

Variance

$$Variance(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - Mean(x))^2$$

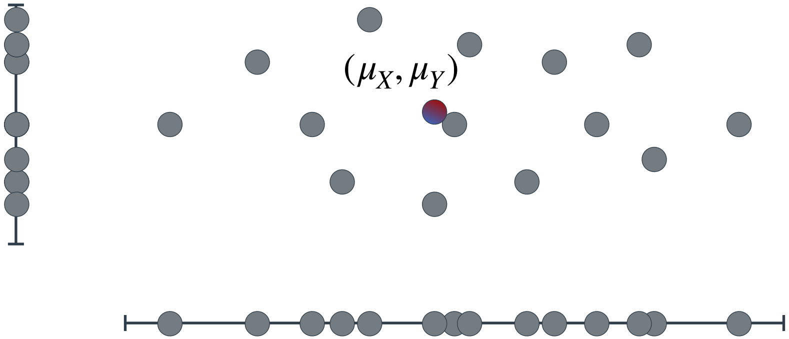
$$Var(x) \qquad\qquad \mu_x$$

"The average squared distance from the mean"

Variance

$$Var(y) = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \mu_Y)^2$$

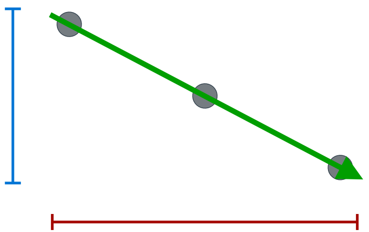
y-variance
Smaller



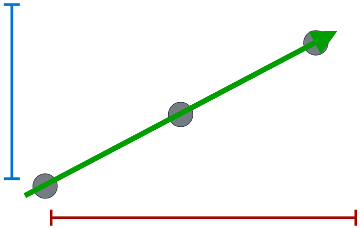
x-variance
Larger

$$Var(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

Problem



Solution: Covariance



Negative covariance

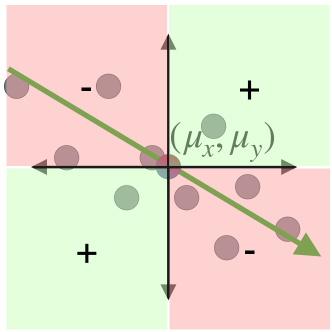
Positive covariance

Covariance

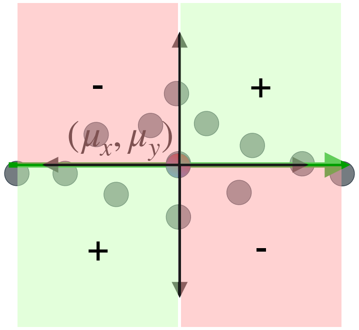
"Take the average"

$$Cov(x,y) = \boxed{\frac{1}{n-1}\sum_{i=1}^{n}}\boxed{(x_i-\mu_x)(y_i-\mu_y)}$$

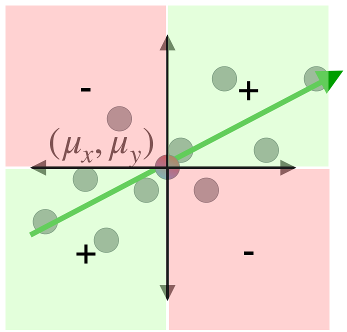
$$Var(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)^2(x_i-\mu_x)$$



negative
covariance



covariance zero
(or very small)

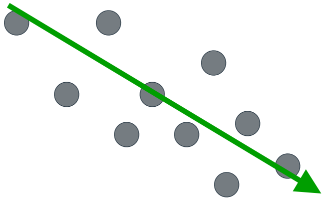


positive
covariance

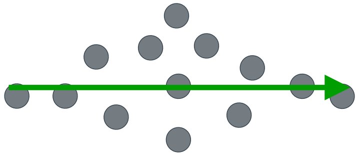
Covariance

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)$$

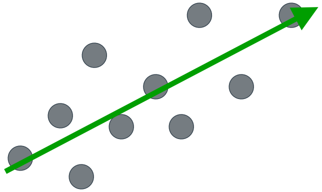
"The direction of the relationship between two variables"



negative covariance



covariance zero (or very small)



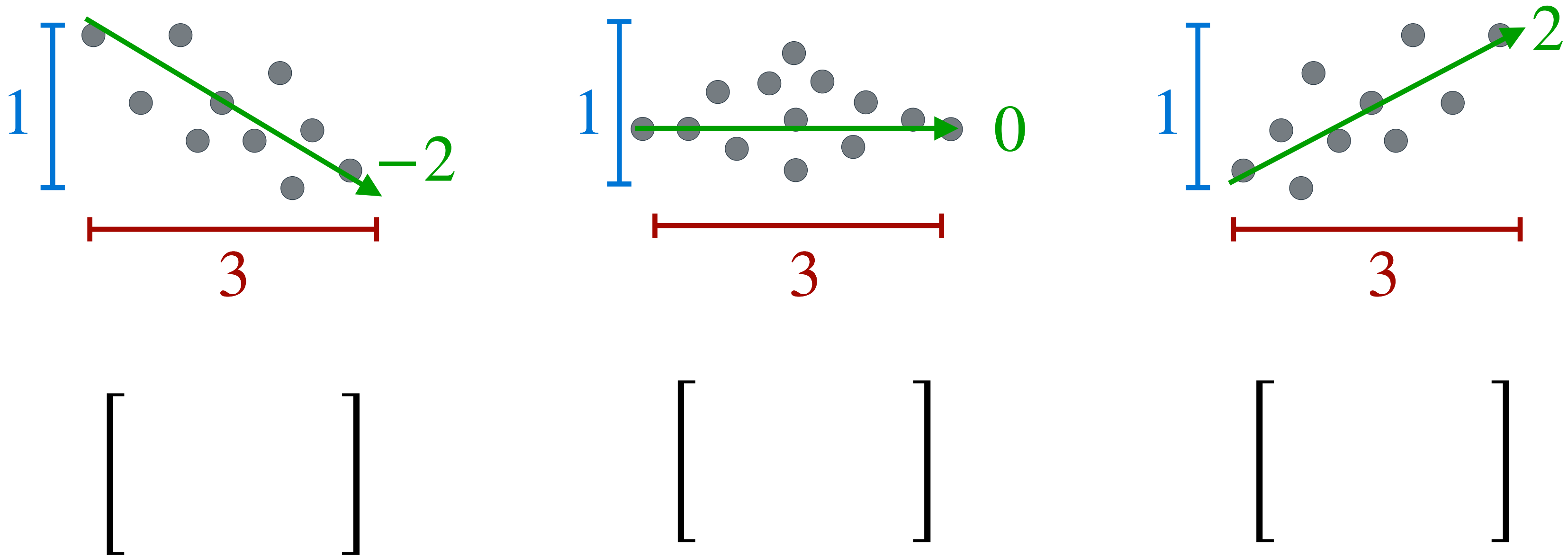
positive covariance

DeepLearning.AI

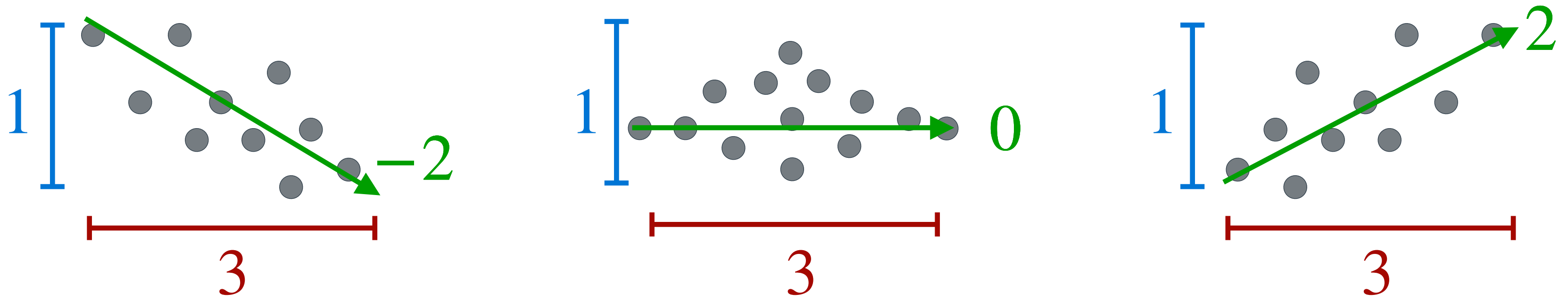
Determinants and Eigenvectors

The covariance matrix

Covariance matrix

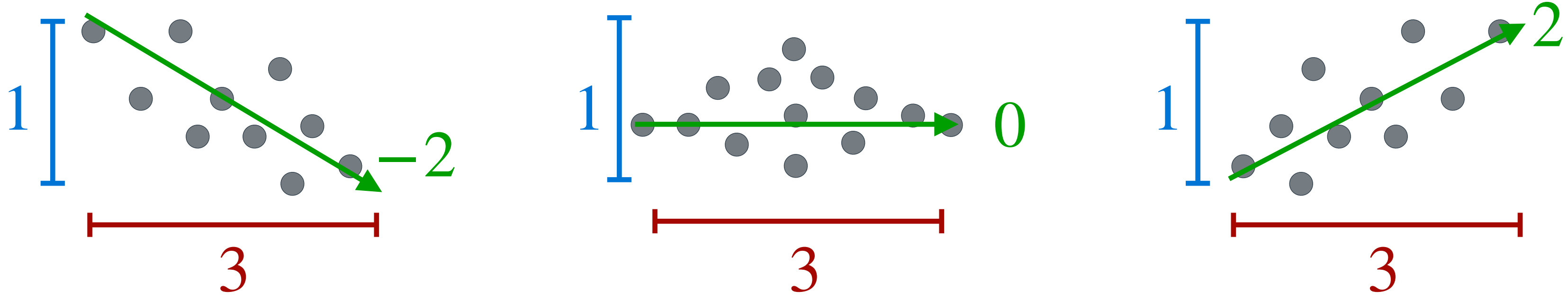


Covariance matrix



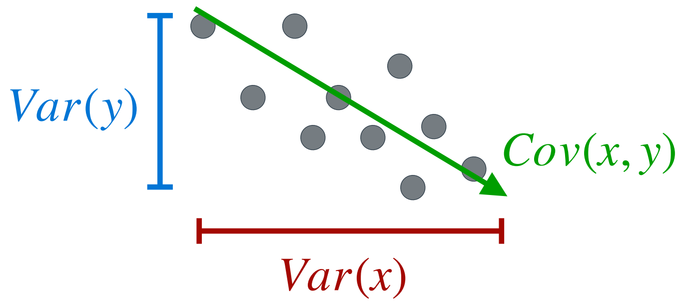
$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

Covariance matrix



$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

Covariance matrix



$$C = \begin{array}{c c} & \begin{array}{cc} x & \quad y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} Var(x) & Cov(x, y) \\ Cov(y, x) & Var(y) \end{bmatrix} \end{array}$$

$$Cov(x, x) = Var(x)$$

Covariance matrix

$$\begin{bmatrix} \boxed{x_1 \quad y_1} \\ x_2 \quad y_2 \\ \vdots \quad \vdots \\ x_n \quad y_n \end{bmatrix}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(\quad - \quad)^T(\quad - \quad)$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) \quad = \frac{1}{n-1}\left(\qquad - \qquad \right)^T\left(\qquad - \qquad \right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) \quad = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\underbrace{\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix}}_{\boxed{2} \times n} \underbrace{\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}}_{n \times \boxed{2}}$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) \quad = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$(x_1-\mu_x)(x_1-\mu_x) + (x_2-\mu_x)(x_2-\mu_x) + \ldots + (x_n-\mu_x)(x_n-\mu_x)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$\sum_{i=1}^{n}(x_i-\mu_x)^2 = Var(x)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \end{bmatrix}$$

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)^2 = Var(x)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \end{bmatrix}$$

$$(x_1-\mu_x)(y_1-\mu_y) + (x_2-\mu_x)(y_2-\mu_y) + \dots + (x_n-\mu_x)(y_n-\mu_y)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \end{bmatrix}$$

$$\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y) = Cov(x,y)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ & \end{bmatrix}$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y) = Cov(x,y)$$

Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & \end{bmatrix}$$

Covariance matrix

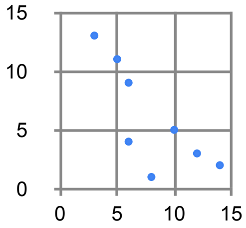
$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & Var(y) \end{bmatrix}$$

Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \qquad C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$



Similar X and Y variances
Negative covariance

A

x	y
10	5
12	3
6	9
6	4
5	11
14	2
8	1
3	13

$\mu_x = 8 \quad \mu_y = 6$

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$(A - \mu)^T$

$\frac{1}{7}$

$x - \mu_x$	2	4	-2	-2	-3	6	0	-5
$y - \mu_y$	-1	-3	3	-2	5	-4	-5	8

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$$C = \begin{bmatrix} 14 & -11.86 \\ -11.86 & 19.71 \end{bmatrix}$$

Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \qquad C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate $A - \mu$
4. $\frac{1}{n-1}\left(A-\mu\right)^T\left(A-\mu\right)$ gives the covariance matrix C

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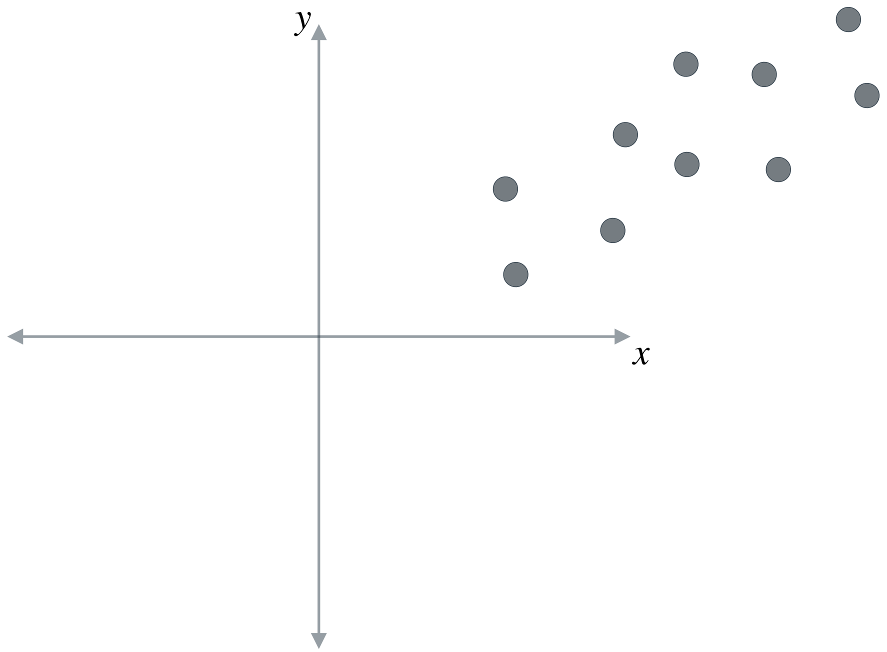
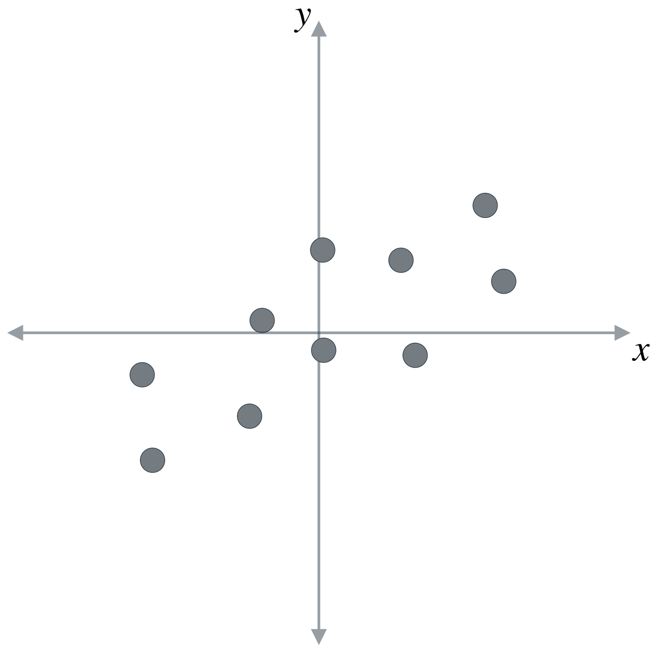
Determinants and Eigenvectors

PCA - Overview

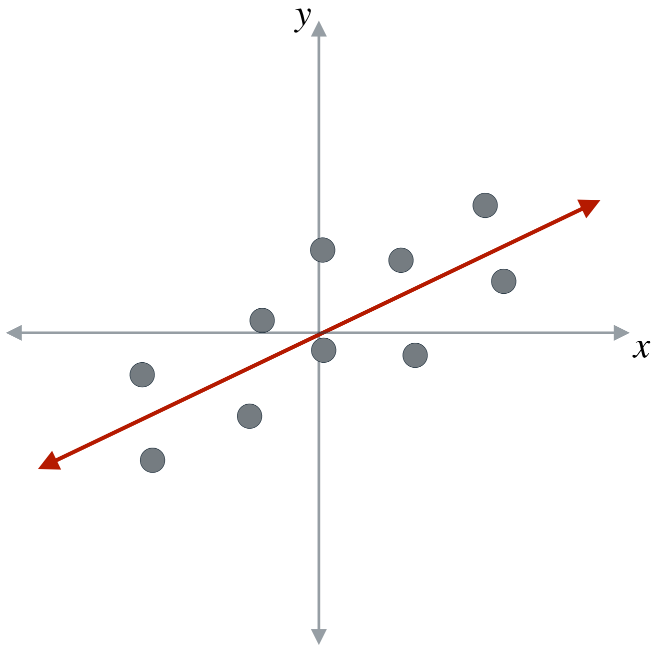
Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



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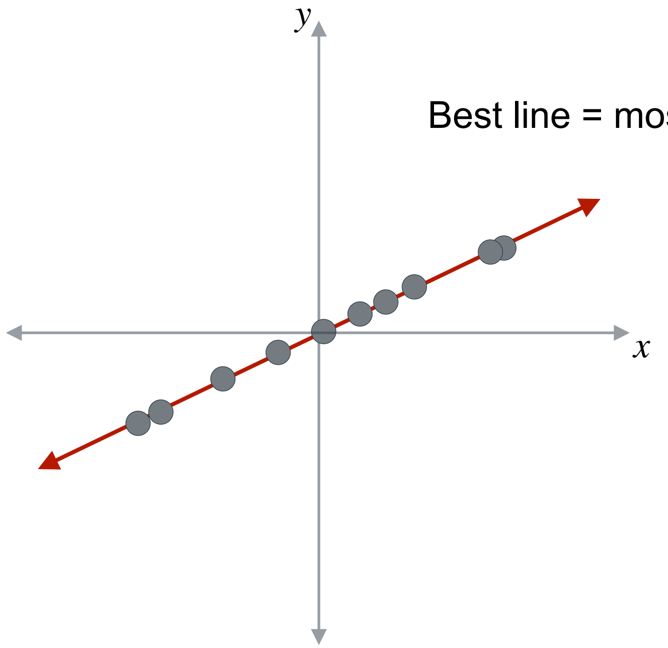
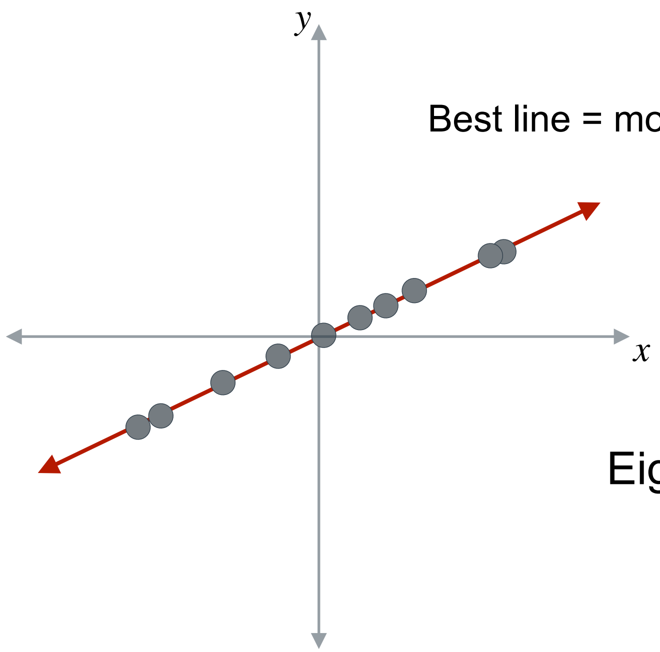
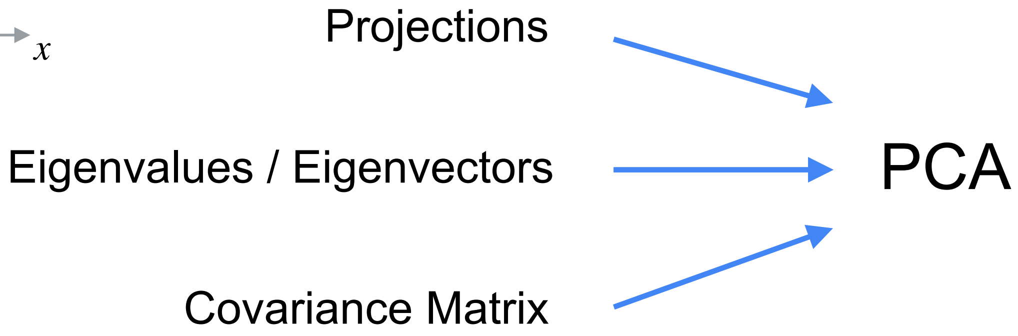


Best line = most variance

Principal Component Analysis (PCA)





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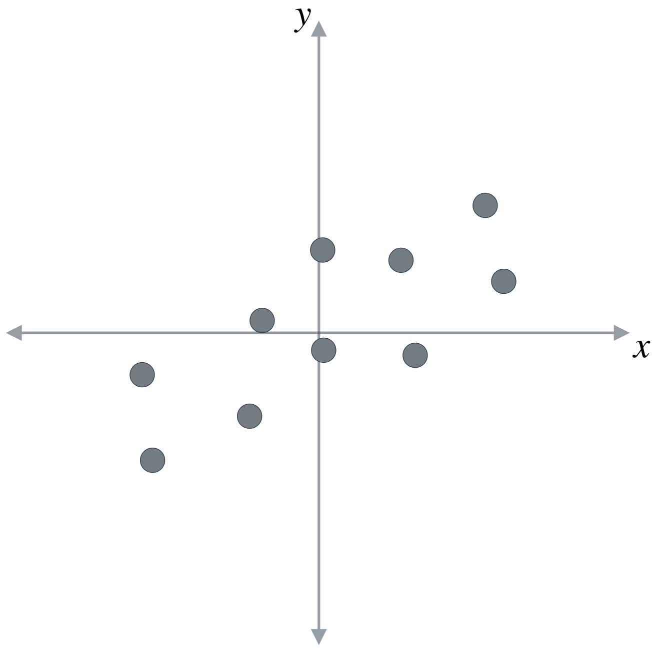

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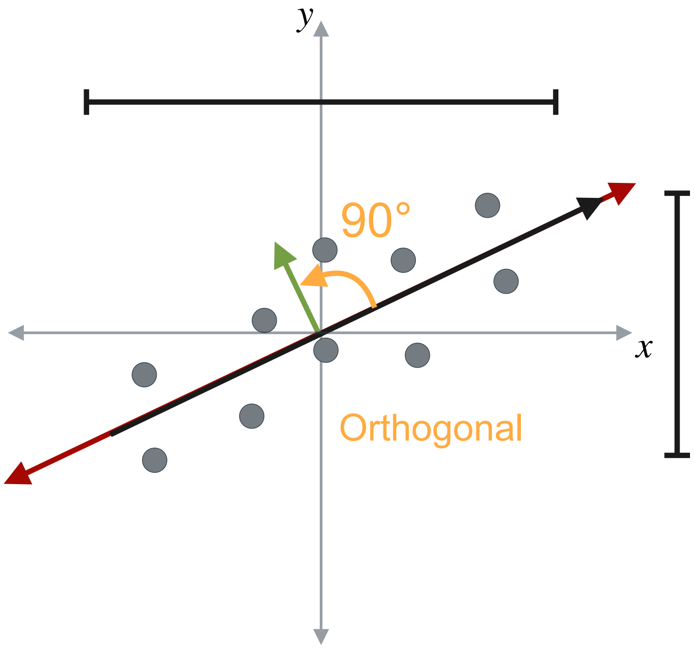
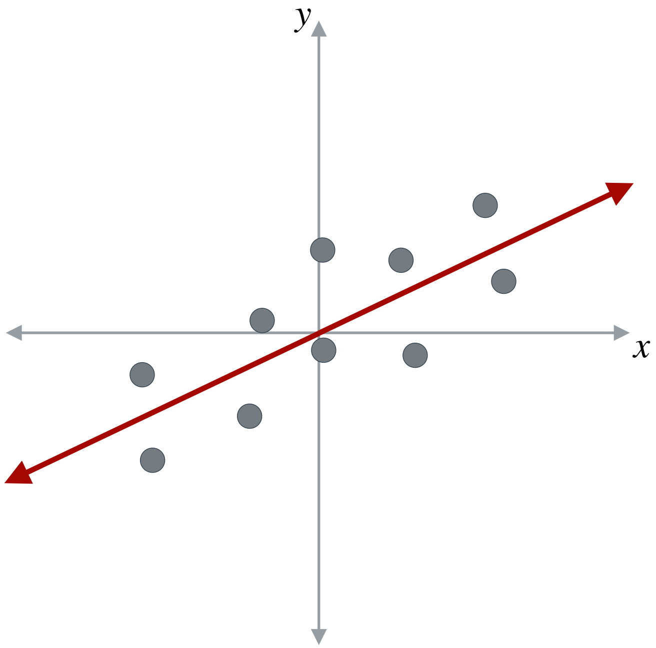


$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\cancel{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$ Eigenvectors (direction)

11 $\cancel{1}$ Eigenvalues (magnitude)

Principal Component Analysis (PCA)



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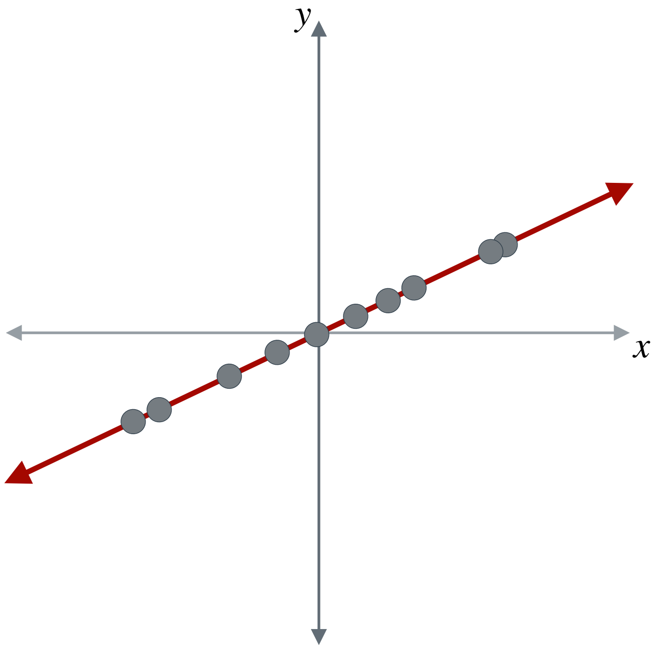
Eigenvectors
(direction)

11

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



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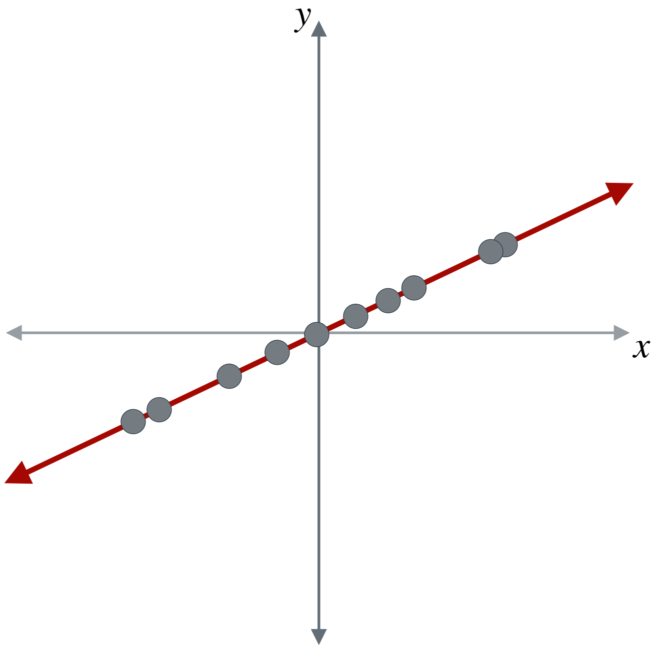
Eigenvectors
(direction)

11

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



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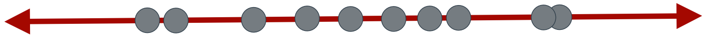
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors (direction)

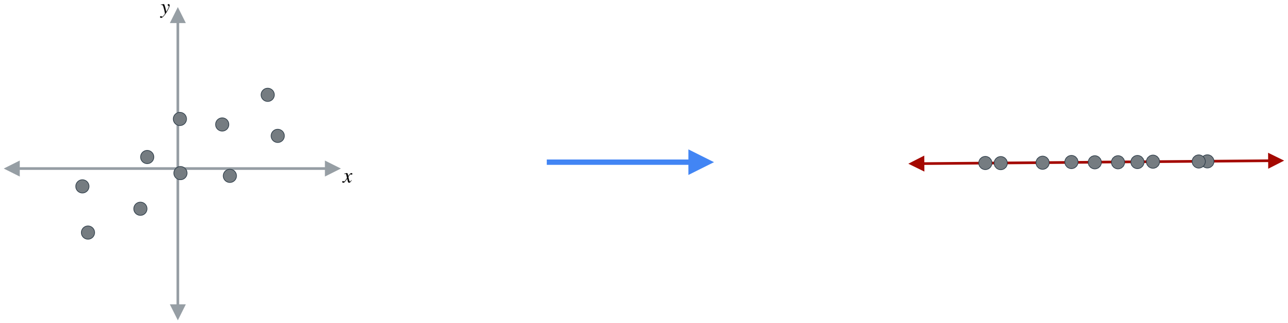
$$11$$

Eigenvalues (magnitude)

Principal Component Analysis (PCA)



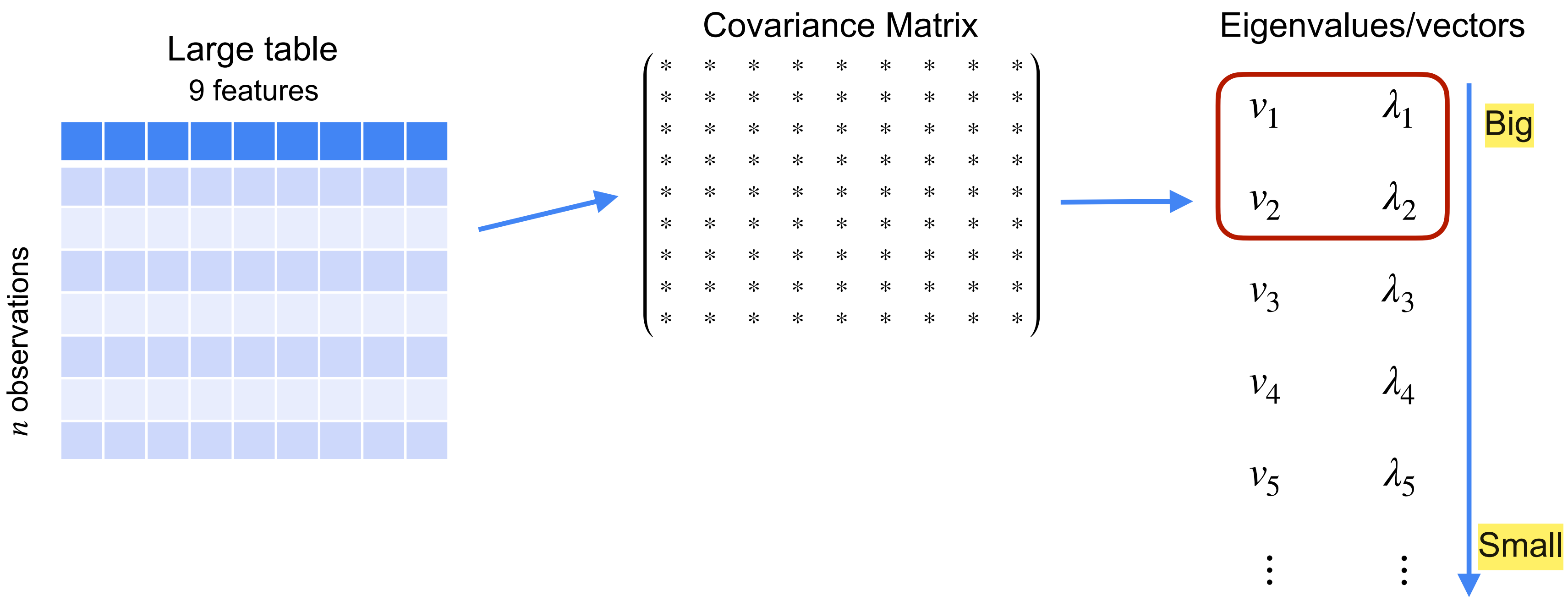
Principal Component Analysis (PCA)



x	y

z

PCA: Principal Component Analysis



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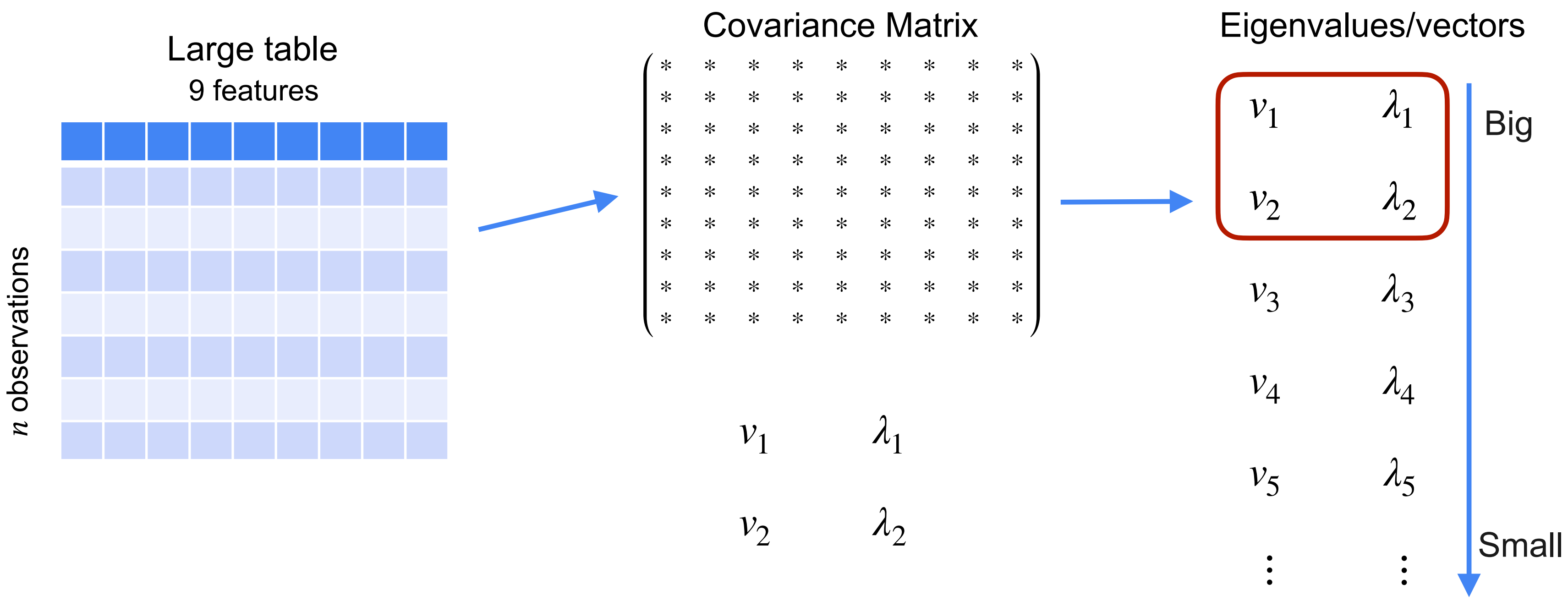
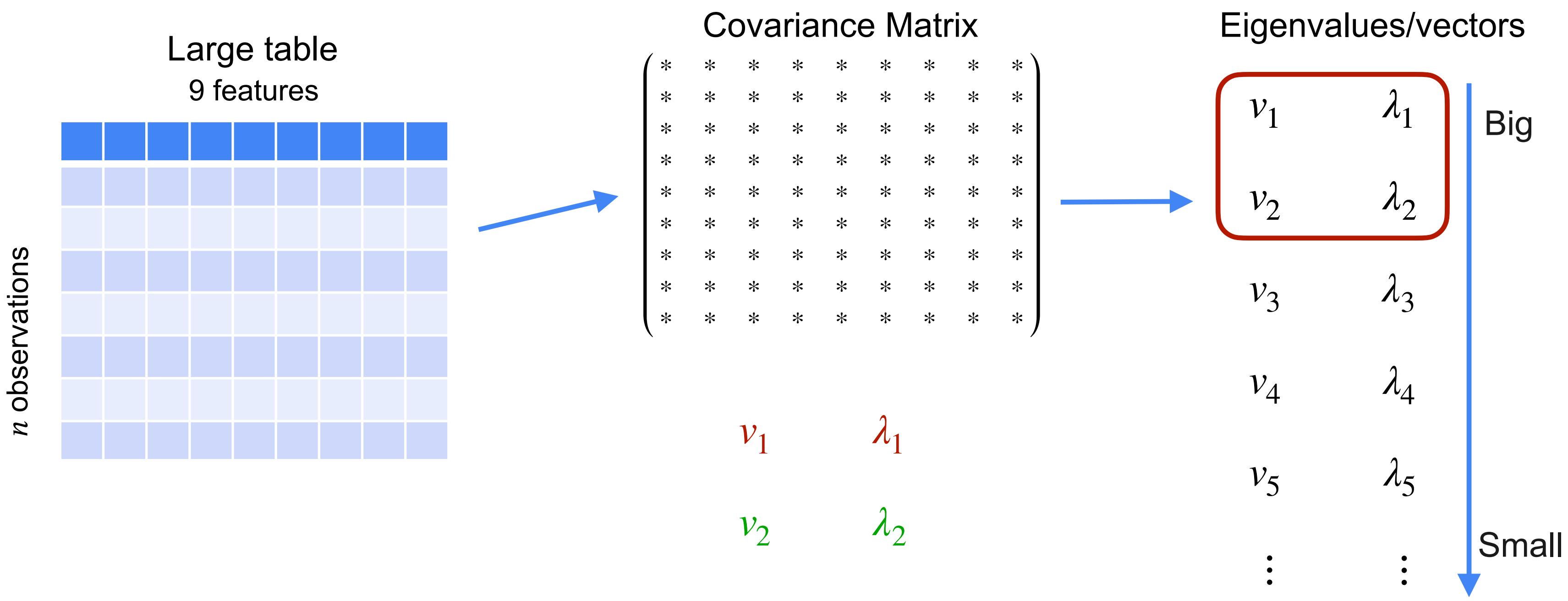
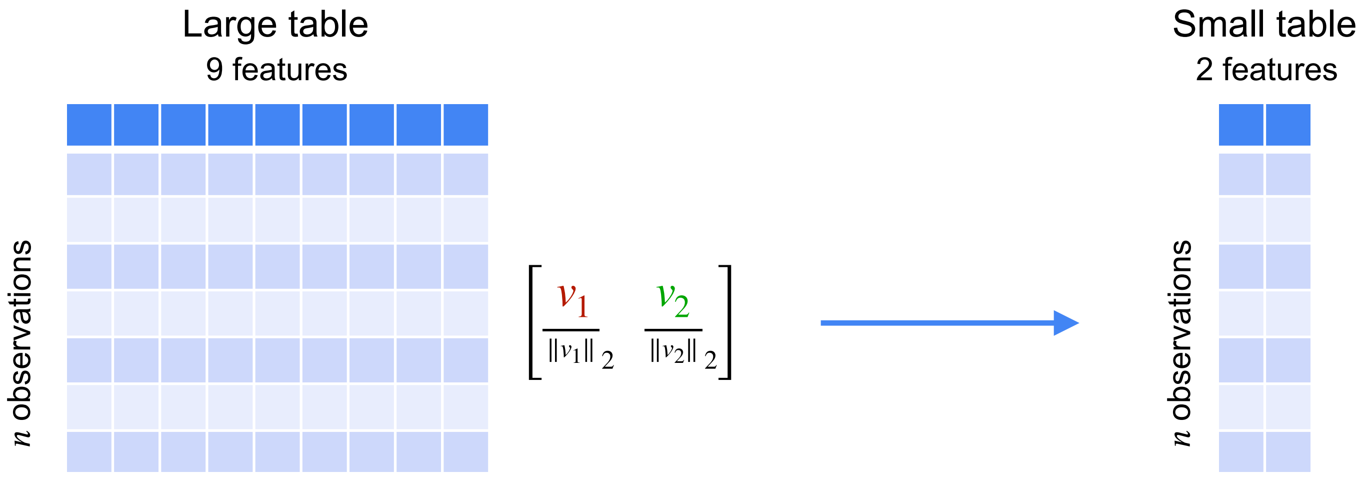


PCA: Principal Component Analysis



PCA: Principal Component Analysis



$$\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}$$

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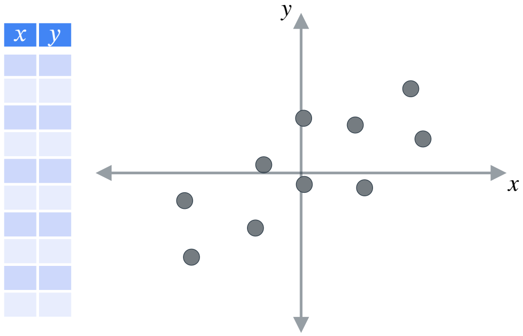
Determinants and Eigenvectors

PCA - Why it works

PCA: Why It Works

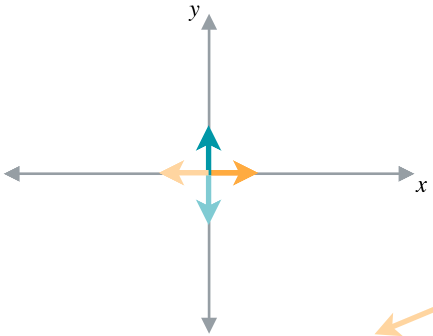


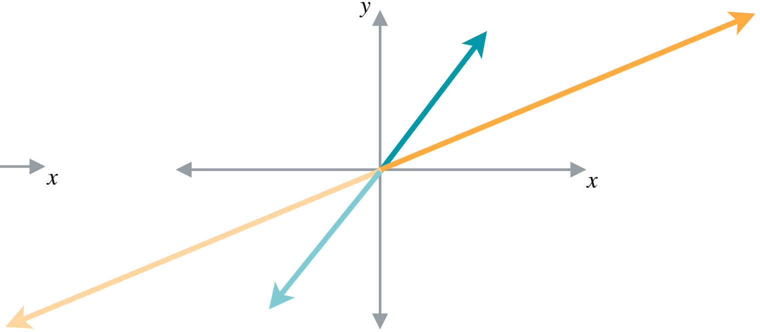
Covariance matrix

$$C$$

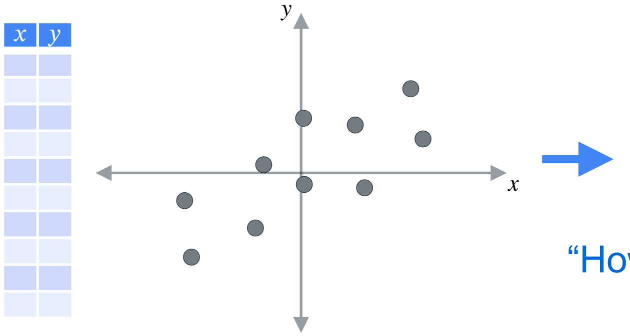
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"





PCA: Why It Works



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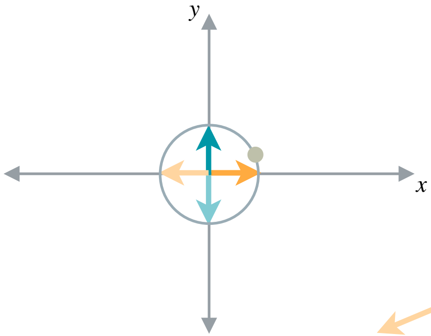

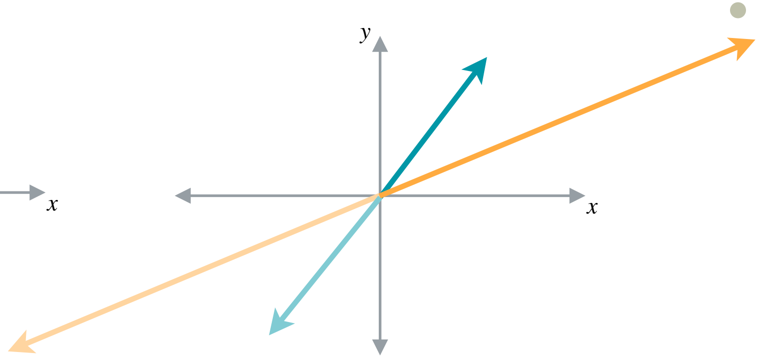

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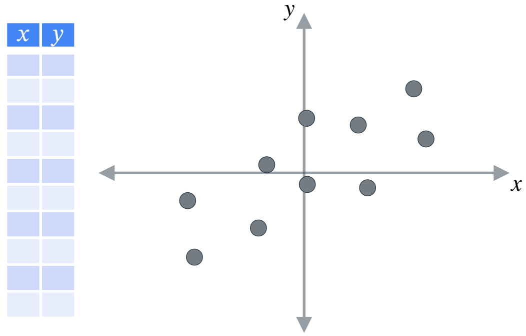

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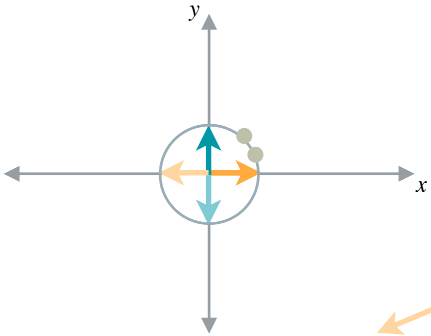

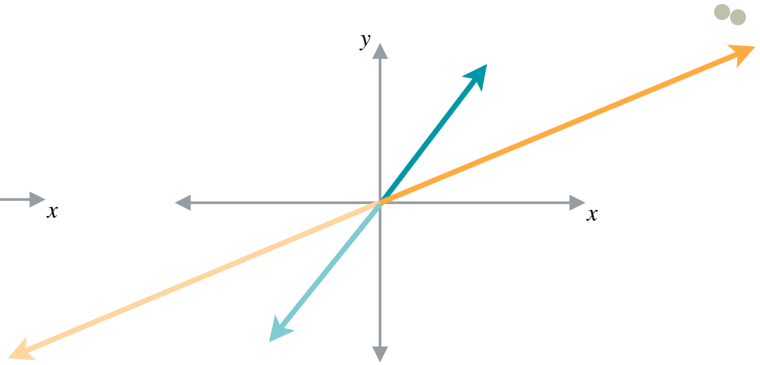

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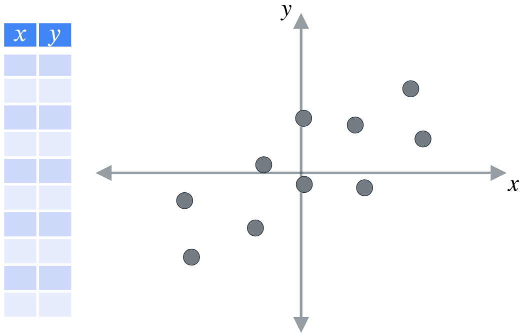

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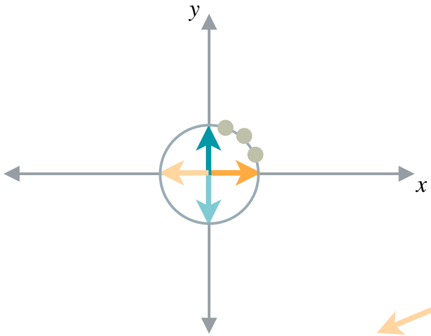

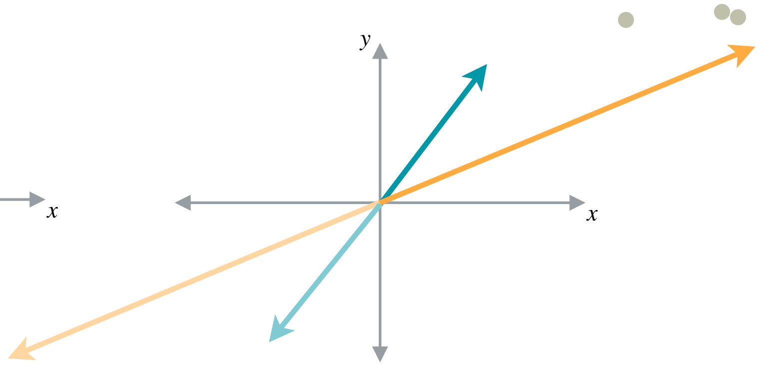

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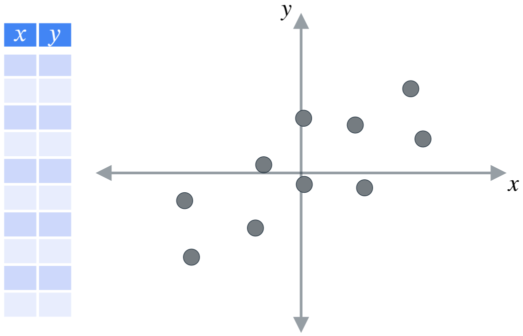

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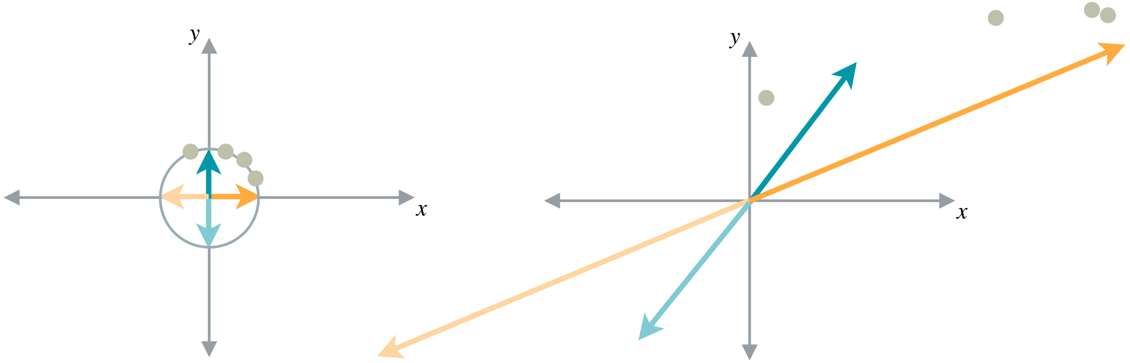

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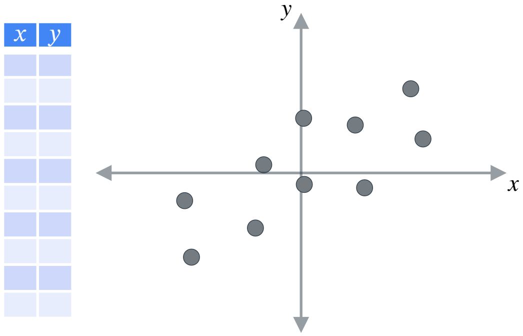

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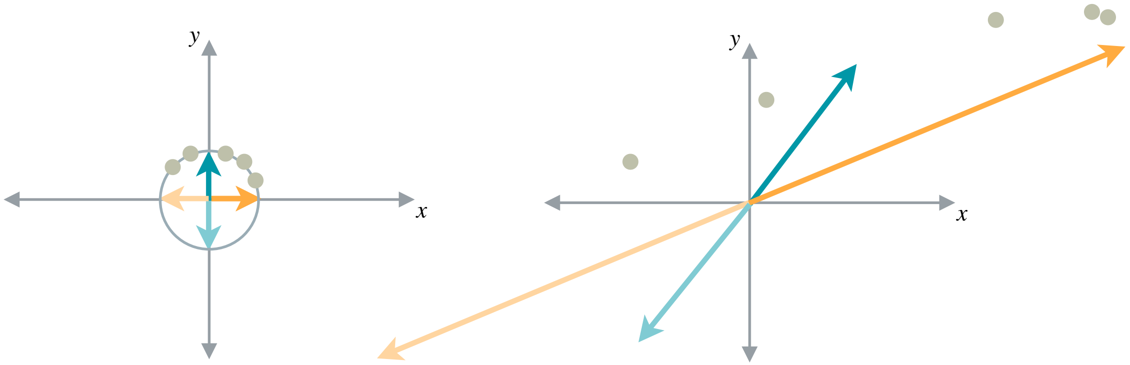

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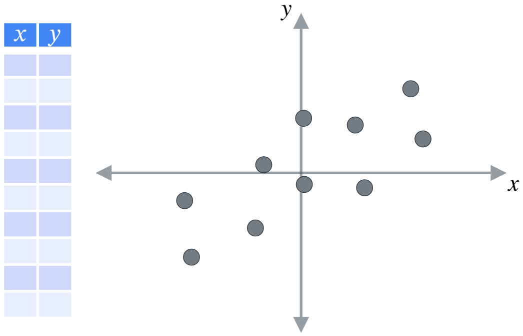

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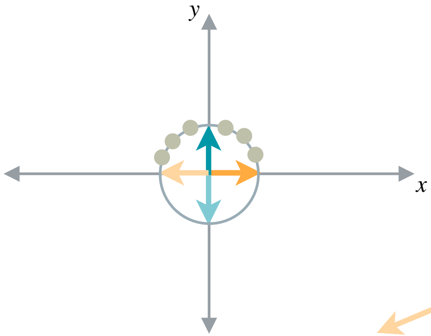

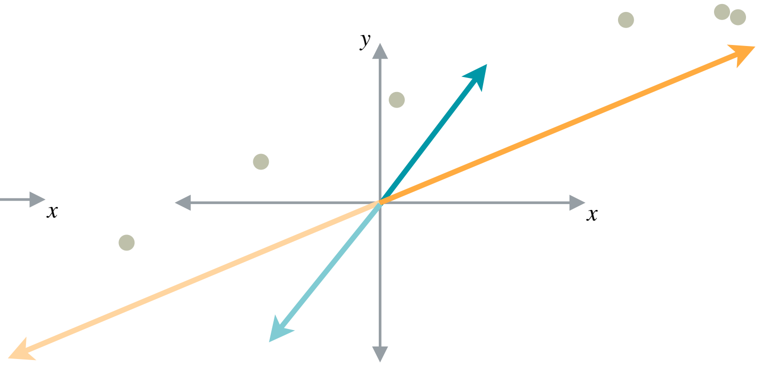

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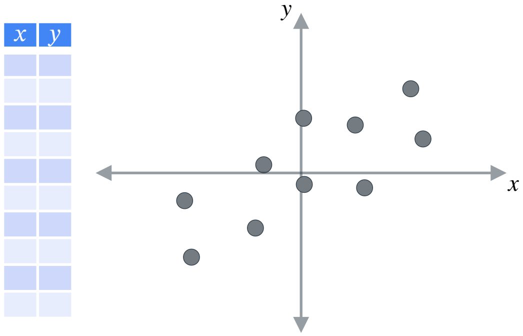

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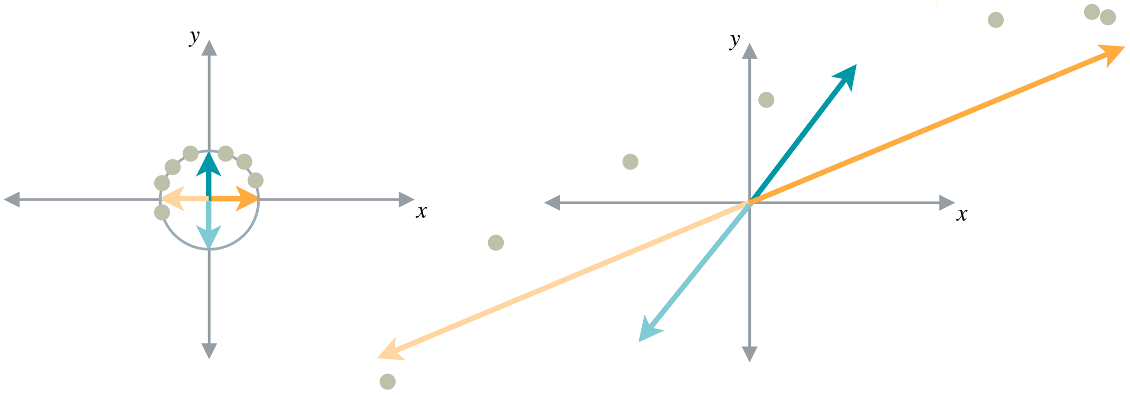

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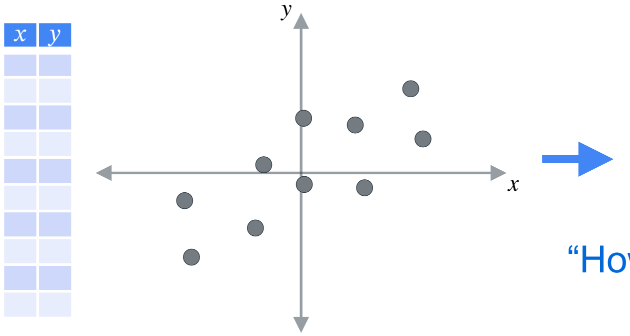

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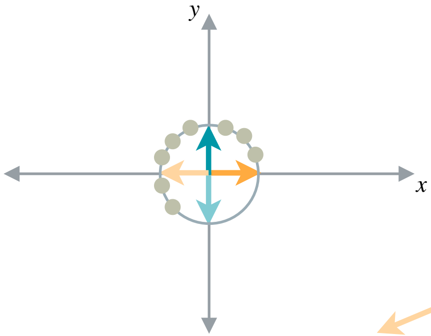

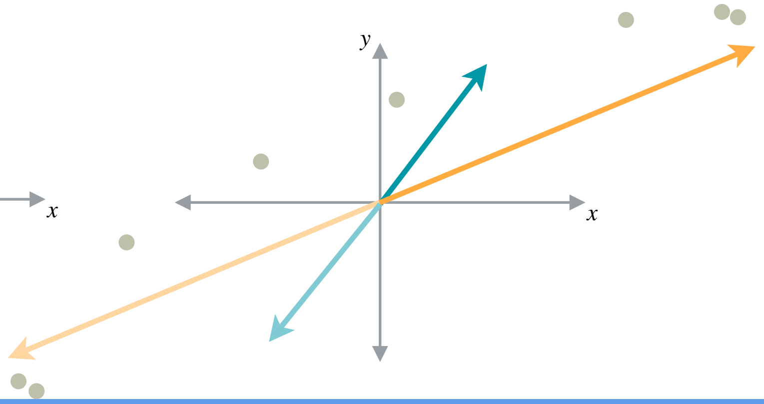

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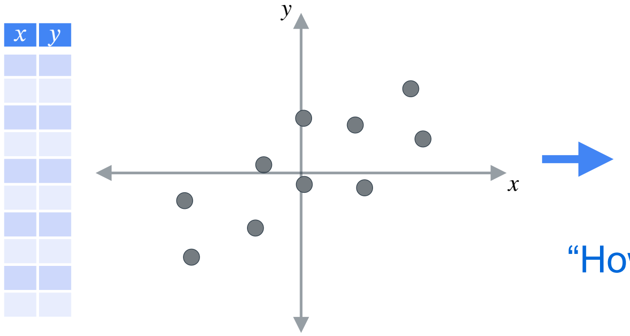

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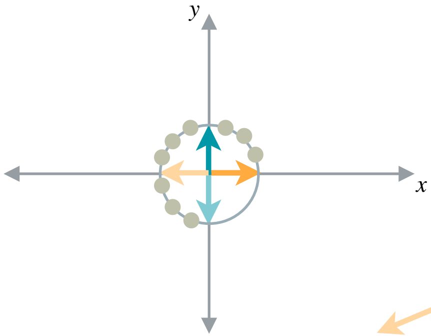

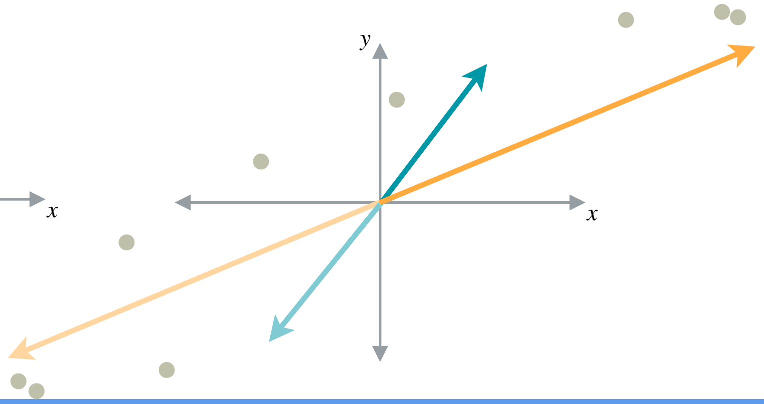

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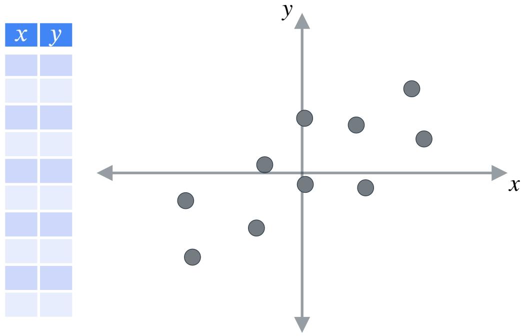

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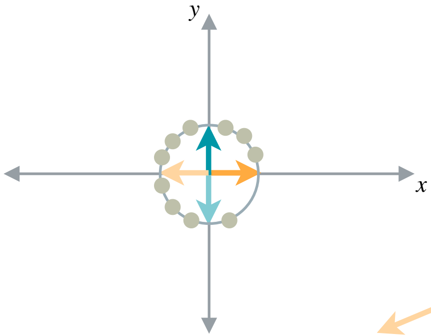

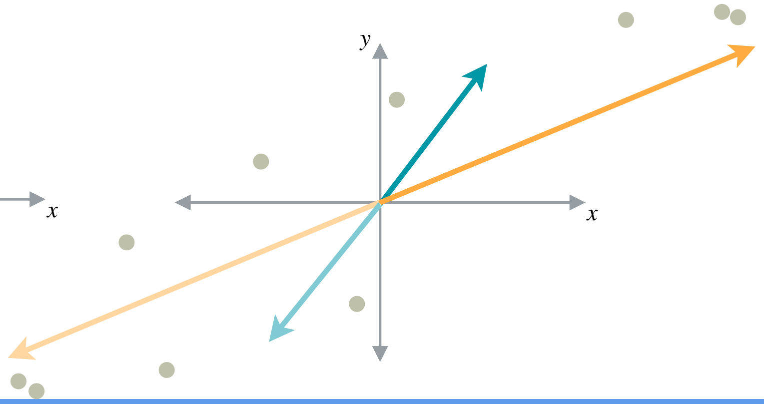

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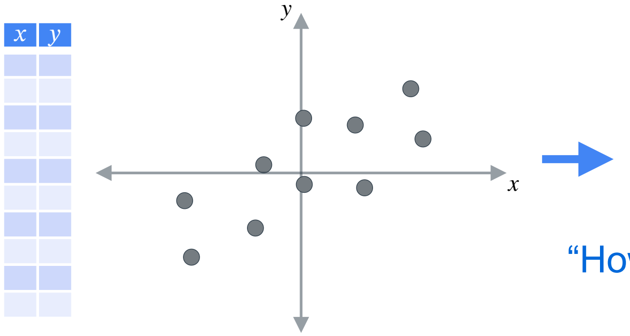

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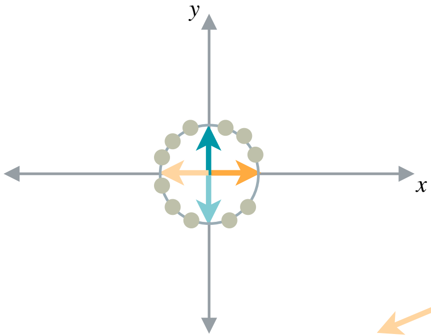

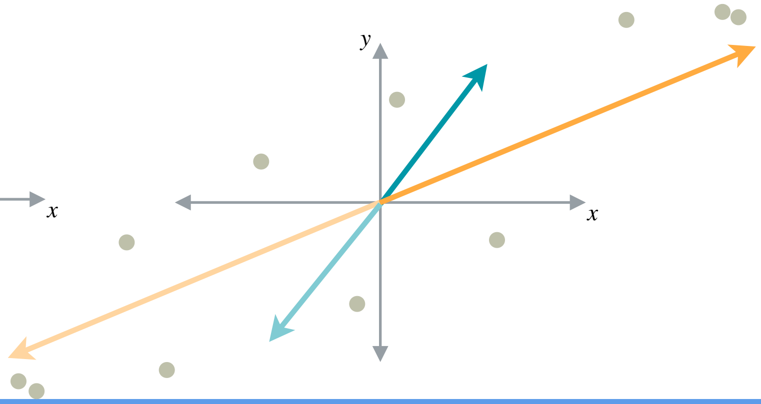

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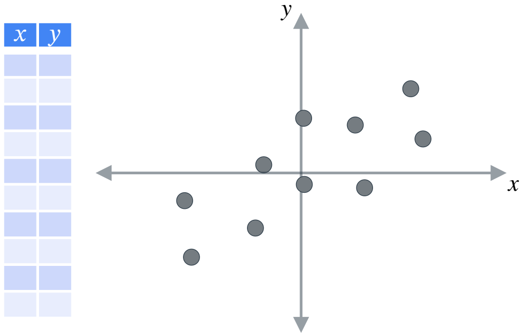

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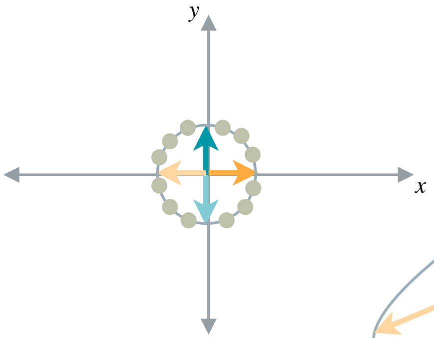

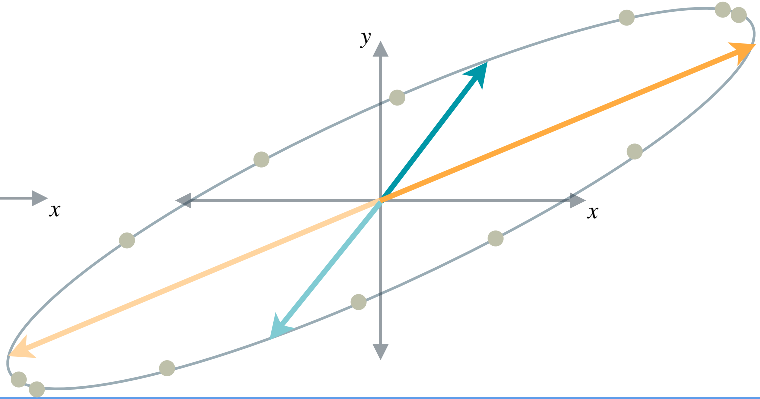

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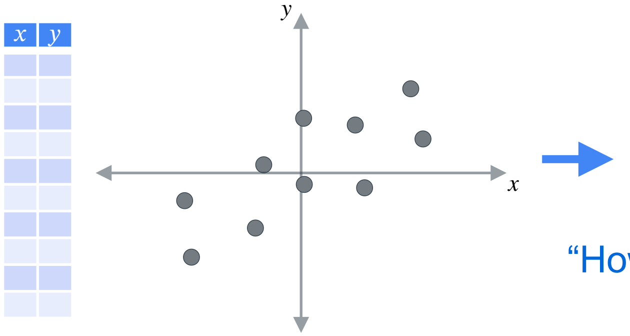

Covariance matrix

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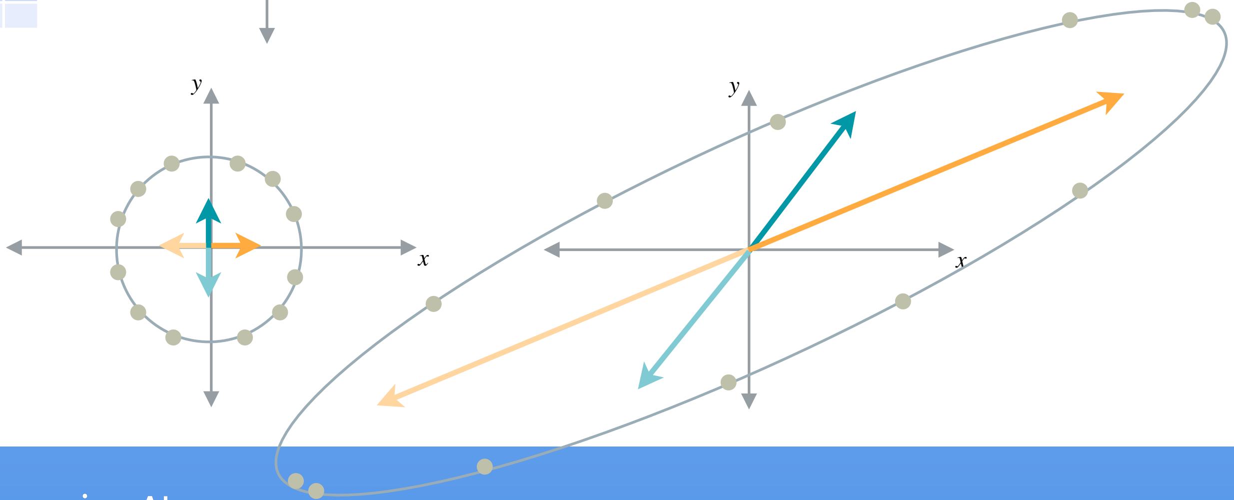

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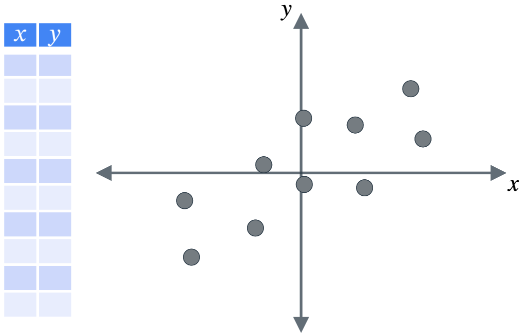

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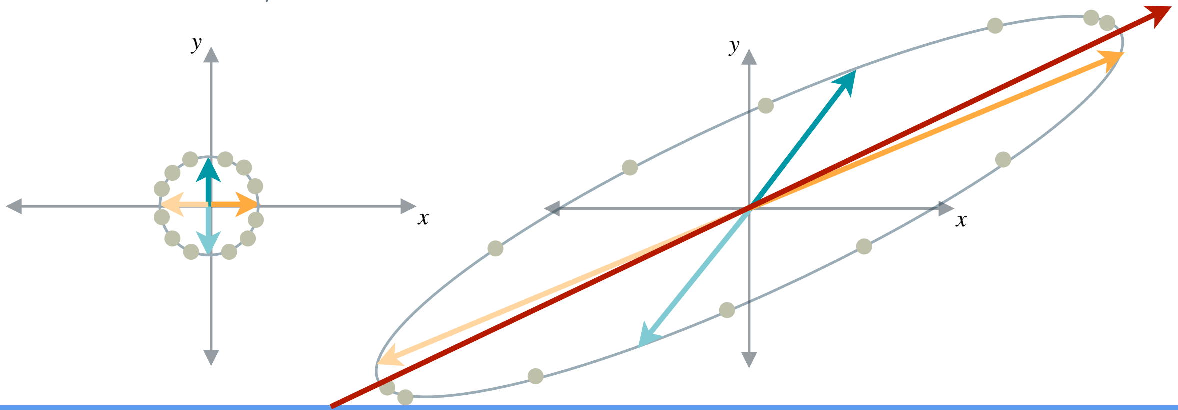

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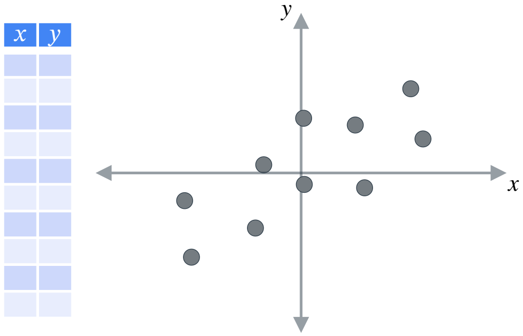


Covariance matrix

C

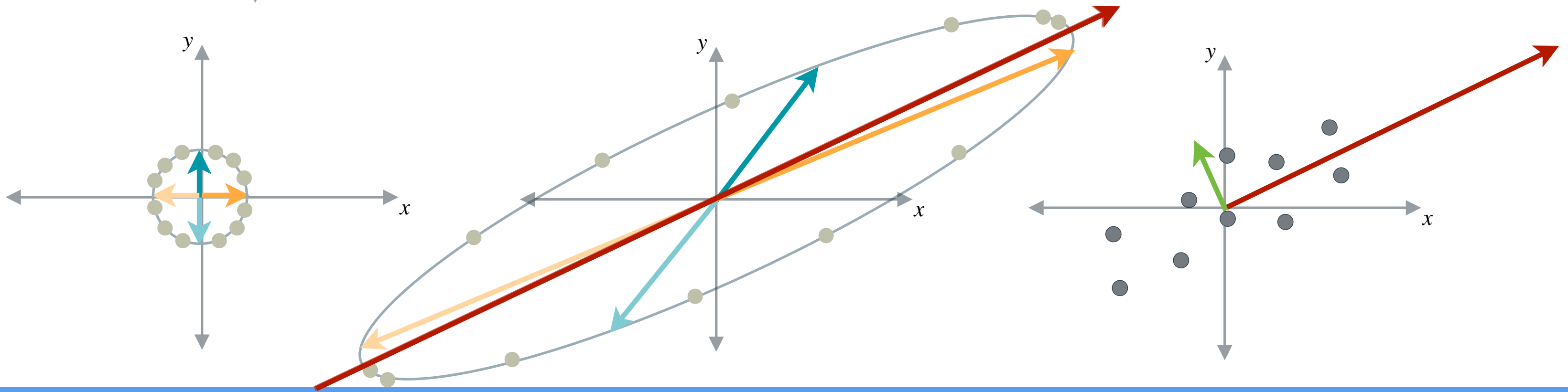
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"

Eigenvalues and Eigenvectors

$$\begin{matrix} 11 & 1 \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{matrix}$$

"Everything is just stretches"



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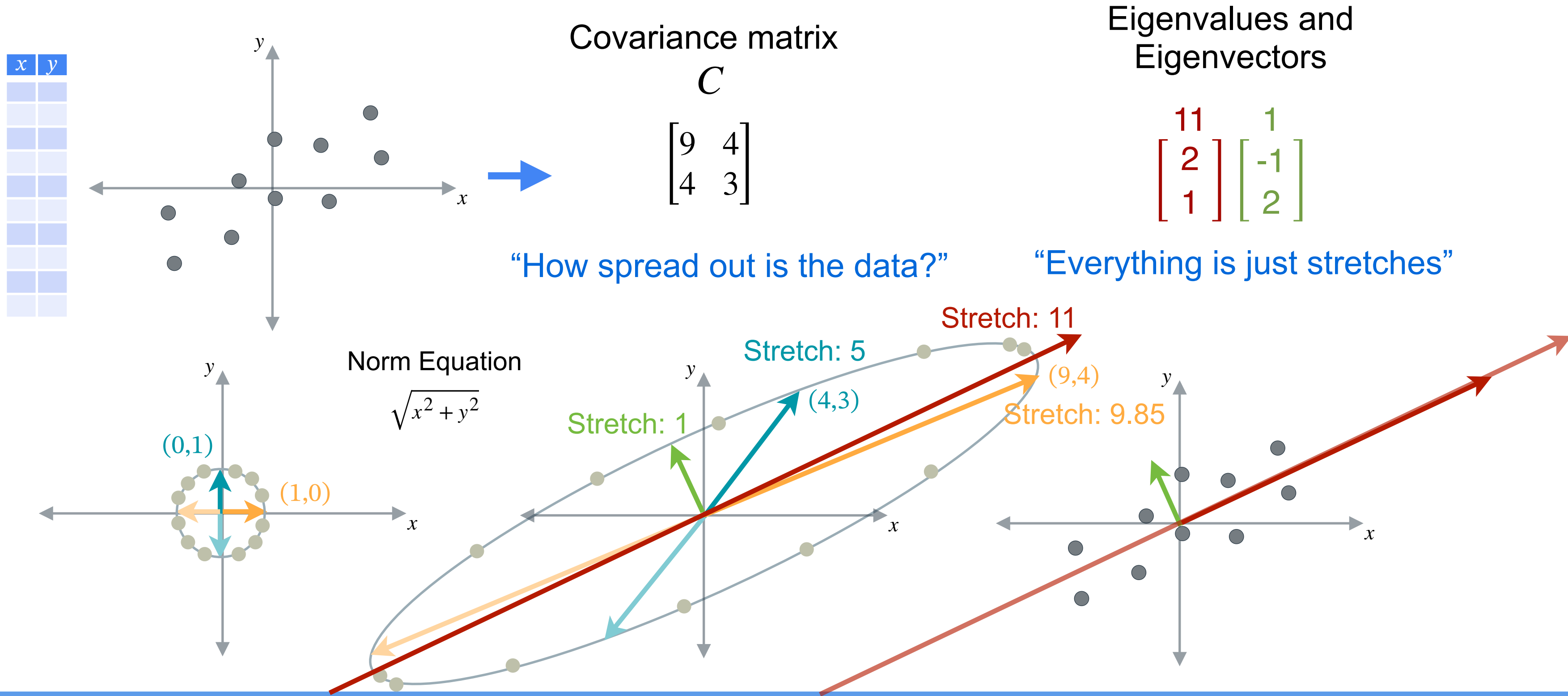

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Determinants and Eigenvectors

PCA - Mathematical formulation

PCA Mathematical formulation

You have n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

1 Create matrix

$$\begin{array}{cc} & \text{5 variables} \\ X = & \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{15} \\ x_{21} & x_{22} & \cdots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n5} \end{bmatrix} \end{array} n \text{ Observations}$$

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \cdots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \cdots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \cdots & x_{n5} - \mu_5 \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

3 Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X-\mu)^T(X-\mu) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) & Cov(X_1, X_4) & Cov(X_1, X_5) \\ Cov(X_1, X_2) & Var(X_2) & Cov(X_2, X_3) & Cov(X_2, X_4) & Cov(X_2, X_5) \\ Cov(X_1, X_3) & Cov(X_2, X_3) & Var(X_3) & Cov(X_3, X_4) & Cov(X_3, X_5) \\ Cov(X_1, X_4) & Cov(X_2, X_4) & Cov(X_3, X_4) & Var(X_4) & Cov(X_4, X_5) \\ Cov(X_1, X_5) & Cov(X_2, X_5) & Cov(X_3, X_5) & Cov(X_4, X_5) & Var(X_5) \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

4 Calculate Eigenvectors and Eigenvalues

Big	λ_1	v_1
	λ_2	v_2
	λ_3	v_3
	λ_4	v_4
Small	λ_5	v_5

5 Create Projection Matrix

$$V = \begin{bmatrix} \frac{}{\|v_1\|_2} & \frac{}{\|v_2\|_2} \end{bmatrix}$$

6 Project Centered Data

$$X_{PCA} = (X - \mu)V$$

Determinants and Eigenvectors

Conclusion