

Protograph Based LDPC Codes with Minimum Distance Linearly Growing with Block Size

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Abstract— We propose several LDPC code constructions that simultaneously achieve good threshold and error floor performance. Minimum distance is shown to grow linearly with block size (similar to regular codes of variable degree at least 3) by considering ensemble average weight enumerators. Our constructions are based on projected graph, or protograph, structures that support high-speed decoder implementations. As with irregular ensembles, our constructions are sensitive to the proportion of degree-2 variable nodes. A code with too few such nodes tends to have an iterative decoding threshold that is far from the capacity threshold. A code with too many such nodes tends to not exhibit a minimum distance that grows linearly in block length. In this paper we also show that precoding can be used to lower the threshold of regular LDPC codes. The decoding thresholds of the proposed codes, which have linearly increasing minimum distance in block size, outperform that of regular LDPC codes. Furthermore, a family of low to high rate codes, with thresholds that adhere closely to their respective channel capacity thresholds, is presented. Simulation results for a few example codes show that the proposed codes have low error floors as well as good threshold SNR performance.

I. INTRODUCTION

Low-density parity-check (LDPC) codes were proposed by Gallager [1] in 1962. After introduction of turbo codes by Berrou et al [2] in 1993, researchers revisited LDPC codes, and extended the work of Gallager using the code graphs introduced by Tanner [3] in 1981. After 1993 there have been many contributions to the design and analysis of LDPC codes; see for example [12], [14], [15], [5], [16], [17], [20] and references there. Recently a flurry of work has been conducted on the design of LDPC codes with imposed sub-structures, starting with the introduction of multi-edge type codes in [11] and [13].

Repeat-Accumulate (RA) [6], Irregular Repeat-Accumulate (IRA) [7] and recently Accumulate-Repeat-Accumulate (ARA) [19] codes were proposed as simple subclasses of LDPC codes with fast encoder structures. For high-speed decoding, it is advantageous for an LDPC code to be constructed from a protograph [8] or a projected graph [10]. A protograph is a Tanner graph with a relatively small number of nodes. A “copy-and-permute” operation [8] can be applied to the protograph to obtain larger derived graphs of various sizes. This operation consists of first making N copies of the protograph,

and then permuting the endpoints of each edge among the N variable and N check nodes connected to the set of N edges copied from the same edge in the protograph. The derived graph is the graph of a code N times as large as the code corresponding to the protograph, with the same rate and the same distribution of variable and check node degrees. LDPC codes with protograph structure are subclass of multi edge type LDPC codes. As an example for protograph based LDPC codes we consider the rate-1/3 Repeat-Accumulate (RA) code depicted in Fig. 1(a). For this code the minimum E_b/N_0 threshold with iterative decoding is 0.502 dB. This code has a protograph representation shown in Fig. 1(b), as long as the interleaver π is chosen to be decomposable into permutations along each edge of the protograph. The iterative decoding threshold is unchanged despite this constraint imposed by the protograph. The protograph consists of 4 variable nodes and 3 check nodes, connected by 9 edges. Three variable nodes are connected to the channel and are shown as dark filled circles. One variable node is not connected to the channel (i.e., it is punctured) and is depicted by a blank circle. The three check nodes are depicted by circles with a plus sign inside.

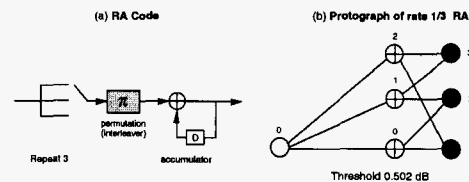


Fig. 1. (a) A rate-1/3 RA code with repetition 3, and (b) its corresponding protograph.

RA, IRA, and ARA codes, with suitable definitions of their interleavers, all have simple protograph representations and thus are amenable to both high-speed encoding and decoding. In [21] further extensions of RA, IRA, and ARA codes, all constructed from simple loop-free encoding modules were provided. These extensions provide greater flexibility to construct codes with lower decoding thresholds. However for certain applications low error floor performance is required.

II. PRECODED REGULAR LDPC CODES

Classic regular LDPC codes, in addition to simplicity, have low error floor performance. However, their iterative decoding thresholds are high. For example the (3,6) regular LDPC codes have an iterative decoding threshold of 1.11 dB while their ensemble asymptotic minimum distances grows like $0.023 n$ as n goes to infinity. For comparison the asymptotic minimum distance of random codes grows as $0.11 n$. We express the normalized logarithmic asymptotic weight distribution of a code as $r(\delta) = \frac{\ln(A_d)}{n}$ where d is Hamming distance, $\delta = \frac{d}{n}$, and A_d is the ensemble weight distribution. The first zero crossing of this function (i.e. $r(\delta_{min}) = 0$ for $\delta_{min} > 0$) if exists then it indicates the non-zero normalized minimum distance of the code and therefore $d_{min} = \delta_{min} \times n$. Different methods to compute the asymptotic weight enumerators for LDPC codes with protograph structure presented in [23], [24], and [25]. The asymptotic weight distribution of (3,6) LDPC and rate 1/2 random codes are shown in Fig. 2.

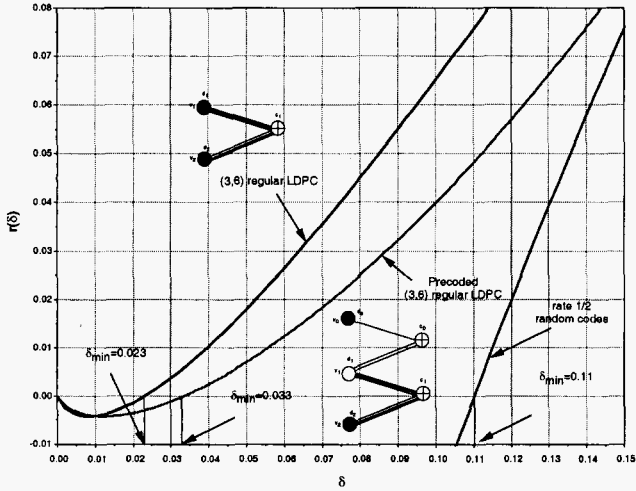


Fig. 2. Asymptotic weight distributions and zero crossings for (3,6) regular and rate 1/2 LDPC family

Precoding places a degree 1 variable node between a constraint node and a higher degree variable (forming an accumulator) which is then optionally erased. Precoding often lowers the iterative decoding threshold of a given protograph without altering its rate [19]. An example of a family of precoded regular LDPC codes is shown in Fig. 3. As shown in the table, precoding gain decreases as code rate increases. However, we note that in general iterative decoding thresholds for very high code rate regular LDPC codes are already satisfactory.

III. ACCUMULATE REPEAT JAGGED ACCUMULATE (ARJA) CODES

As shown in [19], Accumulate Repeat Accumulate (ARA) Codes have reasonable thresholds. However their asymptotic ensemble minimum distance does not grow with n . Consider the rate-1/2 systematic punctured RA code with repetition 3, and puncturing period 3, shown in Fig. 4. In [19] it

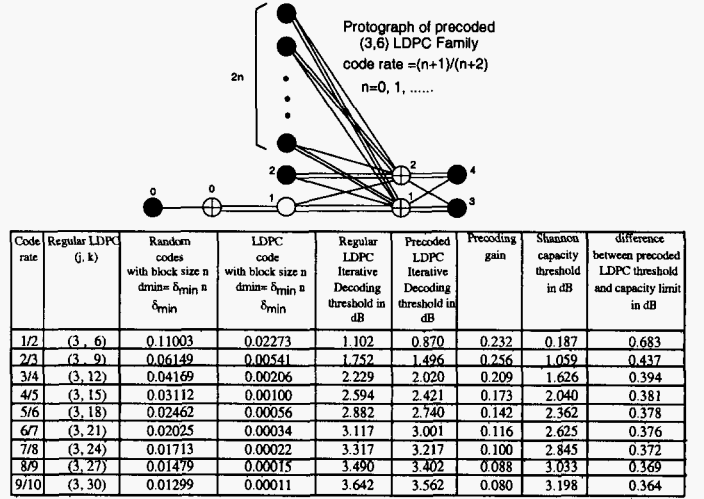


Fig. 3. Precoded LDPC family with associated iterative decoding thresholds and zero crossings.

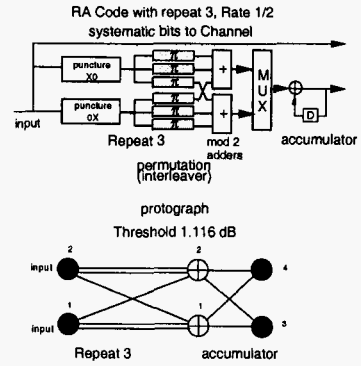


Fig. 4. A systematic rate 1/2 RA code and the corresponding protograph.

was shown that the threshold can be further improved by precoding the repetition code with an accumulator. The design of the precoder in [19] was guided by an analysis of the extrinsic SNR behavior of repetition codes and punctured accumulator codes using density evolution. The use of a rate-1 accumulator as a precoder dramatically improves the extrinsic SNR behavior of a repetition 3 outer code in the high extrinsic SNR region, and hence improves the threshold. An RA code with an accumulator precoder is called an Accumulate-Repeat-Accumulate (ARA) code [19].

An example of a simple rate-1/2 ARA code and its corresponding threshold is shown in Fig. 5. The ARA encoder in Fig. 5 uses a punctured accumulator as the precoder. A parallel decoder architecture based on decoding one copy of the protograph (one page) per clock cycle per half-iteration is shown in Fig. 6.

In an ARA code protograph the number of degree 2 variable nodes is equal to the number of inner checks (checks that are connected to these degree 2 variable nodes). If we decrease the number of degree 2 variable nodes with respect to inner checks, then the ensemble asymptotic minimum distance of code may grow with n . For example if we replace 50% of

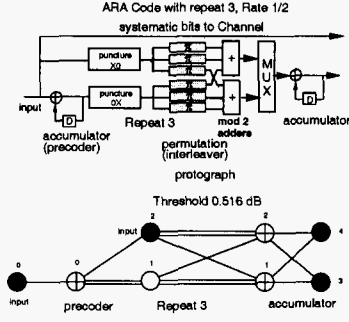


Fig. 5. A rate-1/2 ARA code and its protograph.

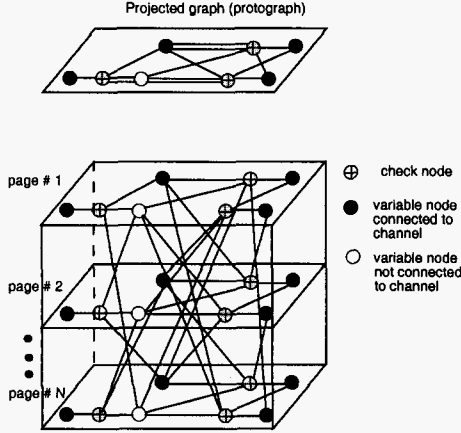


Fig. 6. Parallel LDPC decoder architecture for the rate-1/2 ARA protograph code in Fig. 5.

degree 2 variable nodes with degree 3 variable nodes, then the minimum distance grows with n . We call such constructed codes ARJA codes.

An example of a simple rate-1/2 ARJA code, its protograph, and the corresponding threshold are shown in Fig. 7. Following the computational method of [23] we have found $\delta_{min} = 0.015$ for this protograph.

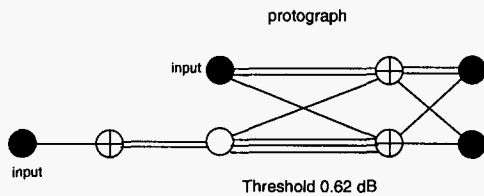
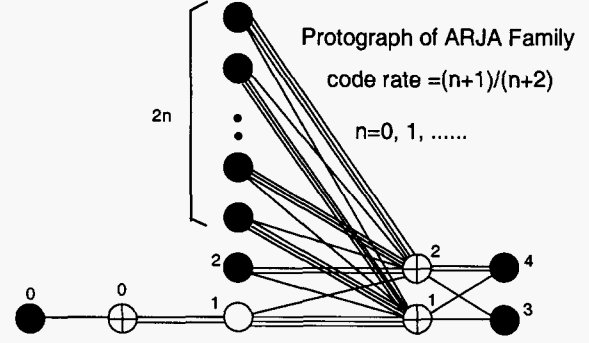


Fig. 7. A rate-1/2 ARJA code and its protograph. The asymptotic minimum distance over block size of this code is $\delta_{min} = 0.015$

IV. RATE-COMPATIBLE CODE FAMILIES

The codes defined thus far in the paper also allow the construction of code families derived from rate-compatible protographs. For example, an ARJA code family (based on the rate 1/2 code of Fig. 7) is shown in Fig. 8 for rates 1/2 and higher. This ARJA code family uses an accumulator without puncturing as the precoder. The higher code rates are constructed by using repetition codes for a portion of the



Code Rate	Protograph Threshold	Capacity	Difference
1/2	0.628	0.187	0.441
2/3	1.450	1.059	0.39
3/4	2.005	1.626	0.37
4/5	2.413	2.040	0.37
5/6	2.733	2.362	0.37
6/7	2.993	2.625	0.36
7/8	3.209	2.845	0.36

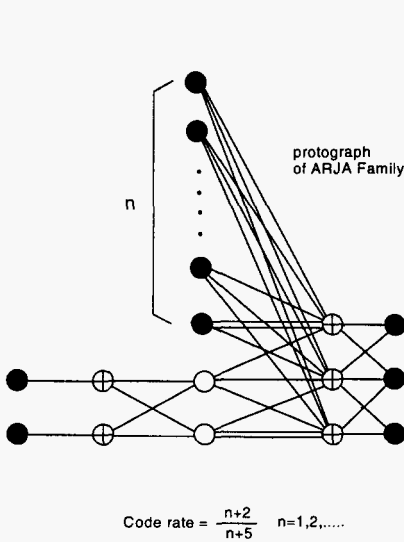
Fig. 8. Protograph of ARJA family with rates 1/2 and higher.

input bits and then adding permuted versions of the repetition to the Jagged accumulator. The thresholds achieved by the family compared to the corresponding capacity limits are also shown in Fig. 8. If we increase the degree 2 variable nodes to 2/3 of the inner checks we can further reduce the iterative decoding threshold at the expense of lowering the linearity coefficient that represent the growth of minimum distance with respect to n . An example of such family is shown in Fig. 9. Note that this family does not satisfy the relation $\lambda'(0)\rho'(1) < 1$, where $\lambda(x)$, $\rho(x)$ are the degree distributions for variable and constraint nodes. However, we note that for a protograph this condition is only a sufficient, but not a necessary, condition for minimum distance growing with n . Specifically, the ensemble asymptotic minimum distance over block size for this protograph is a small, but positive number $\delta_{min} = 0.004$.

We can also construct higher code rates from a rate 1/2 base protograph by puncturing. An example of such an approach is shown in Fig. 10.

V. LOW-RATE ARJA-TYPE LDPC CODES

For a given number of nodes and checks in a protograph one can search over all possible connections between variable and check nodes to obtain a protograph with the lowest threshold. Rather than searching we propose the following constructions that extend the ARJA families to low rates. We use the ARJA protograph shown in Fig. 7 to construct lower rate codes. The constructions in Figures 11 and 12 can be regarded as hybrid concatenated codes [22] where the outer code is a repetition



Code Rate	Proto graph Threshold	Capacity Threshold	Difference
1/2	0.490	0.187	0.303
4/7	0.800	0.530	0.270
5/8	1.078	0.815	0.263
2/3	1.328	1.059	0.269
7/10	1.548	1.272	0.276
8/11	1.744	1.459	0.285
3/4	1.920	1.626	0.294
10/13	2.077	1.777	0.300
11/14	2.221	1.914	0.307
4/5	2.353	2.040	0.313
13/16	2.474	2.156	0.318
14/17	2.585	2.263	0.322
5/6	2.689	2.362	0.327
16/19	2.785	2.455	0.330
17/20	2.874	2.543	0.331
6/7	2.957	2.625	0.332
19/22	3.036	2.702	0.334
20/23	3.109	2.775	0.334
7/8	3.181	2.845	0.336
22/25	3.249	2.910	0.339
23/26	3.313	2.973	0.340
8/9	3.372	3.033	0.339

Fig. 9. Protograph of ARJA-5-checks (2/3 of inner checks are degree 2) family with rates 1/2 and higher. The asymptotic minimum distance over block size for rate 1/2 code is $\delta_{min} = 0.004$

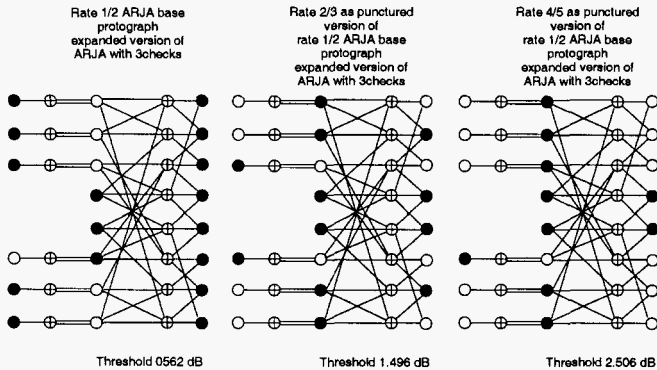


Fig. 10. Protograph of ARJA family with rates 1/2, 2/3 and 4/5 by puncturing.

code, the inner code is a jagged accumulator with possible puncturing, and the parallel code is a low-density generator matrix (LDGM) code. The simplest version of an LDGM code is implemented via differentiator or a single-parity-check code with 3 inputs and one parity bit. In our construction we used the ARJA family due to its low threshold and error floor performance.

This construction produces a rate-1/3 ARJA protograph having 7 variables and 5 checks with one variable punctured Fig. 11. The two checks on the right are still connected to two variables forming a jagged accumulator. The single check and single degree-1 variable on the left representing the precoder is also untouched. Thus the rate 1/2 ARJA base protograph is unchanged to preserve the code family structure. We used LDGM codes in parallel concatenation (similar to hybrid concatenation) to construct lower rate protographs. Figures 11 and 12 show our constructed protographs in the ARJA family, and the corresponding thresholds and Shannon capacities for rates 1/3 and 1/4. Other low rate codes can be obtained simi-

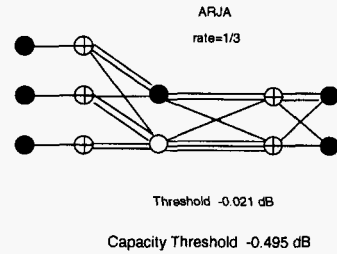


Fig. 11. Rate-1/3 ARJA protograph.

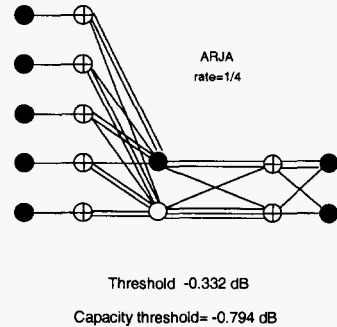


Fig. 12. Rate-1/4 ARJA protograph.

larly using the proposed construction method. Furthermore, if we remove the constraint that minimum distance should grow linearly with block size then codes with still lower thresholds can be obtained [22].

VI. SIMULATION RESULTS

The BER/FER vs. SNR advantage afforded by precoding is demonstrated using a regular (3,6) construction with input block size $k = 4096$ in Fig. 13. A measure of the plot at FER = 10^{-5} indicates a precoding gain essentially indistinguishable from the 0.23 dB predicted by density evolution. Performance curves for $k = 1024$ length codes with rate 1/4, 1/3, 1/2, 2/3, and 4/5 from the ARJA of Fig. 8 family are plotted in Fig. 14. Given the relatively short block length, these code exhibit exceptional threshold performance and error floors have yet to be observed. Fig. 15 shows the performance simulation results for $k = 4096$, rate 1/2 and higher of ARJA code family. All simulations were performed on a field-programmable gate array (FPGA) implementation of an LDPC decoder developed at JPL.

VII. CONCLUSION

In this paper we have introduced a new ensemble of structured codes that exhibit good threshold performance as well as a minimum distance, that for an average instance from the ensemble, increases with blocklength. This work has in addition demonstrated the value of precoding as a technique for reducing the iterative decoding threshold while maintaining rate.

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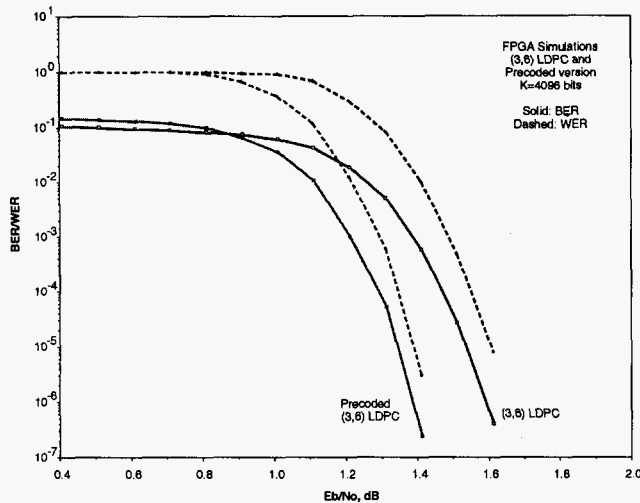


Fig. 13. Performance comparison of (3,6) LDPC and precoded (3,6) LDPC codes with input block size $k = 4096$.

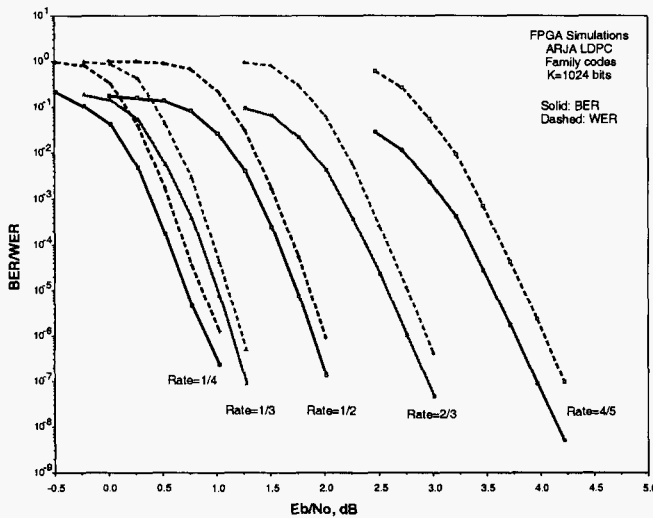


Fig. 14. Performance of ARJA code family of rates 1/4 and higher with input block size $k = 1024$.

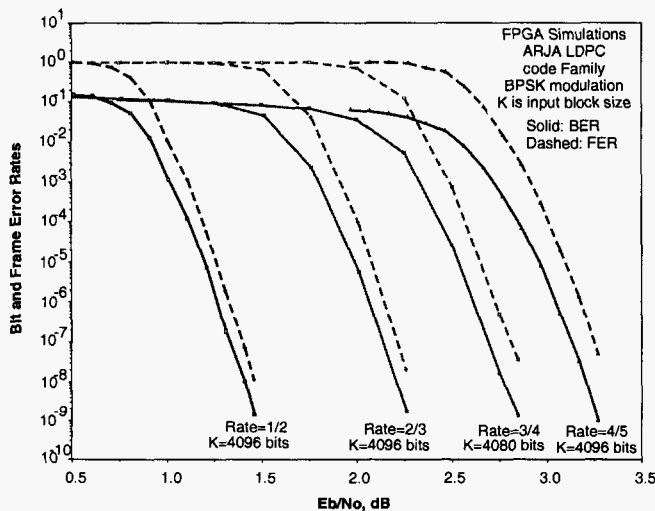


Fig. 15. Performance of ARJA code family with input block size $k = 4096$.

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