برنامه‌نویسی پویا

بزرگترین زیردنباله مشترک (Longest Common Subsequence)

بلند ترین زیر دنباله مشترک

Longest common subsequence (LCS)

- یک رشته DNA شامل یک دنباله از مولکول‌هایی است که به آنها Base می‌گویند.
برخی از این Baseها عبارتند از

Guanine, Cytosine, Thymine, Adenine

- هرکدام از این Baseها با حرف اول خود نشان داده می‌شوند.
  - ✓ به عنوان مثال $\{A,G,C,T\}$ می‌تواند یک رشته DNA باشد.

بلند ترین زیر دنباله مشترک

Longest common subsequence (LCS)

- شباهت دو رشته DNA را می‌توان به یکی از سه روش زیر پیدا کرد:
  - ✓ حداقل تعداد تغییراتی روی Baseها که یک رشته را به رشته دیگر تبدیل کند.
  - ✓ طول زیررشته مشترک دو رشته
  - ✓ طول زیردنباله مشترک دو رشته
    - برای دو رشته $S1$ و $S2$، رشته جدیدی مانند $S3$ پیدا کنید طوریکه، Baseها ظاهر شوند ، اما ترتیب حضور این base ها در S_1 و S_2 باید همان ترتیب حضور در S_3 باشد.

S1 = ACCGGTCGAGTGCGCGGAAGCCGGCCGAA

S2 = GTCGTTCGGAATGCCGTTGCTCTGTAAA

S3 = GTCGTCGGAAGCCGGCCGAA

زیردنباله مشترک

- یک زیردنباله حاوی حروف مشترک دو دنباله است که متوالی نیستند ولی ترتیب حضور در دنباله‌های اولیه را حفظ کرده‌اند.

$X: ABCBDAB$

$Y: BDCABA$

CS1: BCBA

CS2: BCA

CS3: CA

CS1: BDAB

چگونه مسئله را حل کنیم

$X:\ ABCBDAB$
$Y: BDCABA$

$X:\ ABCBDABC$
$Y: BDCABAC$

رابطه بازگشتی

Theorem 15.1 (Optimal substructure of an LCS)
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 , \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j , \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > 0 \text{ and } x_i \neq y_j . \end{cases}$$

بزرگترین زیردنباله مشترک (LCS)

```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let b[1..m, 1..n] and c[0..m, 0..n] be new tables
for i = 1 to m
    c[i, 0] = 0
for j = 0 to n
    c[0, j] = 0
for i = 1 to m
    for j = 1 to n
        if x_i == y_j
            c[i, j] = c[i − 1, j − 1] + 1
            b[i, j] = "↖"
        elseif c[i − 1, j] ≥ c[i, j − 1]
            c[i, j] = c[i − 1, j]
            b[i, j] = "↑"
        else c[i, j] = c[i, j − 1]
            b[i, j] = "←"
return c and b
```

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

```
PRINT-LCS(b, X, i, j)
    if i == 0 or j == 0
        return
    if b[i, j] == "↖"
        PRINT-LCS(b, X, i − 1, j − 1)
        print x_i
    elseif b[i, j] == "↑"
        PRINT-LCS(b, X, i − 1, j)
    else PRINT-LCS(b, X, i, j − 1)
```

i \ j		0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

مثال

- حال راه‌حل مذکور را روی یک مثال ببینیم:
  - ✓ X = ABCB
  - ✓ Y = BDCAB
- به راحتی می‌توان دریافت که:

$$LCS(X, Y) = BCB$$

LCS Example (0)

X = ABCB;   m = |X| = 4
Y = BDCAB; n = |Y| = 5
Allocate array c[6,5]

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi						
1	**A**						
2	**B**						
3	**C**						
4	**B**						

LCS Example (1)

for i = 1 to m $\qquad$ c[i,0] = 0

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi						
1	**A**	**0**					
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

LCS Example (2)

for j = 0 to n   c[0,j] = 0

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**					
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

LCS Example (3)

case i=1 and   j=1

A != B

but, c[0,1]>=c[1,0]

so c[1,1] = c[0,1], and b[1,1] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑				
2	B	0					
3	C	0					
4	B	0					

LCS Example (4)

case i=1 and   j=2

A != D

but, c[0,2]>=c[1,1]

so c[1,2] = c[0,2], and b[1,2] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	A	**0**	**0** ↑	**0** ↑			
2	B	**0**					
3	C	**0**					
4	B	**0**					

LCS Example (5)

case i=1 and j=3 4
A != C
but, c[0,3]>=c[1,2]
so c[1,3] = c[0,3], and b[1,3] = ↑

			j	0	1	2	3	4	5
	i			Yj	B	D	C	A	B
0	0		Xi	0	0	0	0	0	0
0	1		A	0	0 ↑	0 ↑	0 ↑		
	2		B	0					
	3		C						
			B						

LCS Example (6)

case i=1 and   j=4

A = A

so c[1,4] = c[0,2]+1, and b[1,4] = ↖

i		0	1	2	3	4	5
	j	0	1	2	3	4	5
		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0** ↑	**0** ↑	**0** ↑	**1** ↖	
2	**B**	**0**					
3	**C**	**0**					
4	**B**	**0**					

LCS Example (7)

case i=1 and j=5

A != B

this time c[0,5]<c[1,4]

so c[1,5] = c[1, 4], and b[1,5] = ←

j		0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0					
3	C	0					
4	B	0					

LCS Example (8)

case i=2 and j=1
B = B
so c[2, 1] = c[1, 0]+1, and b[2, 1] = ↖

i	j	0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖				
3	C	0					
4	B	0					

LCS Example (9)

case i=2 and j=2

B != D

and c[1, 2] < c[2, 1]

so c[2, 2] = c[2, 1] and b[2, 2] = ←

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←			
3	C	0					
4	B	0					

LCS Example (10)

case i=2 and   j=3

B != D

and c[1, 3] < c[2, 2]

so c[2, 3] = c[2, 2] and b[2, 3] = ←

		j	0	1	2	3	4	5
i			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B		0	1 ↖	1 ←	1 ←		
3	C		0					
4	B		0					

LCS Example (11)

case i=2 and j=4

B != A

and c[1, 4] = c[2, 3]

so c[2, 4] = c[1, 4] and b[2, 2] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	
3	C	0					
4	B	0					

LCS Example (12)

case i=2 and j=5

B = B

so c[2, 5] = c[1, 4]+1 and b[2, 5] = ↖

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0					
4	B	0					

LCS Example (13)

case i=3 and j=1
C != B
and c[2, 1] > c[3,0]
so c[3, 1] = c[2, 1] and b[3, 1] = ↑

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B		0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C		0	1 ↑				
4	B		0					

LCS Example (14)

case i=3 and j= 2

C != D

and c[2, 2] = c[3, 1]

so c[3, 2] = c[2, 2] and b[3, 2] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑			
4	B	0					

LCS Example (15)

case i=3 and j= 3

C = C

so c[3, 3] = c[2, 2]+1 and b[3, 3] = ↖

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖		
4	B	0					

LCS Example (16)

case i=3 and j= 4

C != A

c[2, 4] < c[3, 3]

so c[3, 4] = c[3, 3] and b[3, 3] = ←

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖	2 ←	
4	B	0					

LCS Example (17)

case i=3 and j= 5

C != B

c[2, 5] = c[3, 4]

so c[3, 5] = c[2, 5] and b[3, 5] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖	2 ←	2 ↑
4	B	0					

LCS Example (18)

case i=4 and   j=1

B = B

so c[4, 1] = c[3, 0]+1 and b[4, 1] = ↖

i	j	0	1	2	3	4	5
		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0** ↑	**0** ↑	**0** ↑	**1** ↖	**1** ←
2	**B**	**0**	**1** ↖	**1** ←	**1** ←	**1** ↑	**2** ↖
3	**C**	**0**	**1** ↑	**1** ↑	**2** ↖	**2** ←	**2** ↑
4	**B**	**0**	**1** ↖				

LCS Example (19)

case i=4 and j=2

B != D

c[3, 2] = c[4, 1]

so c[4, 2] = c[3, 2] and b[4, 2] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖	2 ←	2 ↑
4	B	0	1 ↖	1 ↑			

LCS Example (20)

case i=4 and j= 3

B != C

c[3, 3] > c[4, 2]

so c[4, 3] = c[3, 3] and b[4, 3] = ↑

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖	2 ←	2 ↑
4	B	0	1 ↖	1 ↑	2 ↑		

LCS Example (21)

case i=4 and   j=4

B != A

c[3, 4] = c[4, 3]

so c[4, 4] = c[3, 4] and b[3, 5] = ↑

i		j: 0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0 ↑	0 ↑	0 ↑	1 ↖	1 ←
2	B	0	1 ↖	1 ←	1 ←	1 ↑	2 ↖
3	C	0	1 ↑	1 ↑	2 ↖	2 ←	2 ↑
4	B	0	1 ↖	1 ↑	2 ↑	2 ↑	

LCS Example (22)

case i=4 and   j=5

B= B

so c[4, 5] = c[3, 4]+1 and b[4, 5] = ↖

i		j	0	1	2	3	4	5
			Yj	**B**	**D**	**C**	**A**	**B**
0	Xi		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0** ↑	**0** ↑	**0** ↑	**1** ↖	**1** ←
2	**B**		**0**	**1** ↖	**1** ←	**1** ←	**1** ↑	**2** ↖
3	**C**		**0**	**1** ↑	**1** ↑	**2** ↖	**2** ←	**2** ↑
4	**B**		**0**	**1** ↖	**1** ↑	**2** ↑	**2** ↑	**3** ↖

Finding LCS

i	j	0	1	2	3	4	5
		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**B**	**0**	**1**	**1**	**2**	**2**	**3**

Finding LCS (2)

LCS (reversed order): **B C B**

LCS (straight order): **B C B**

(this string turned out to be a palindrome)

	j	0	1	2	3	4	5
i		Yj	**B**	**D**	**C**	**A**	**B**
0	Xi	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**B**	**0**	**1**	**1**	**2**	**2**	**3**