

Answer No 1

(i) A rational number $\frac{p}{q}$ is an equivalence class of pairs (p, s) with $q \neq 0$, where

$$\frac{p}{q} = \frac{p'}{q'} \Leftrightarrow pq' = p'q$$

$$\text{Addition: } \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}$$

$$\text{Multiplication: } \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

(ii) For $\frac{p}{q}, \frac{r}{s} \in \varphi$:

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in \varphi, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} \in \varphi$$

Integers are closed under addition and multiplication, and $qs \neq 0$.

(iii) Associativity:

$$\text{Addition: } \left(\frac{p}{q} + \frac{r}{s} \right) + \frac{t}{u} = \frac{(ps+rq)u + qs}{qsu}$$

$$= \frac{psu + rqu + qs}{qsu}$$

$$= \frac{p}{q} + \left(\frac{r}{s} + \frac{t}{u} \right)$$

$$\text{Multiplication: } \left(\frac{p}{q} \cdot \frac{r}{s} \right) \cdot \frac{t}{u} = \frac{pr}{qs} \cdot \frac{t}{u} = \frac{prt}{qsu}$$

$$= \frac{p}{q} \cdot \left(\frac{r}{s} \cdot \frac{t}{u} \right)$$

(iv) Commutativity:

$$\text{Addition: } \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} = \frac{rq + ps}{sq}$$

$$= \frac{r}{s} + \frac{p}{q}$$

$$\text{Multiplication: } \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} = \frac{rp}{sq} = \frac{r}{s} \cdot \frac{p}{q}$$

(iv) Identities:

Additive identity: $0 = \frac{0}{1}$

$$\frac{p}{q} + 0 = \frac{p}{q} + \frac{0}{1} = \frac{p \cdot 1 + 0 \cdot q}{q \cdot 1} = \frac{p}{q}$$

Multiplicative identity: $1 = \frac{1}{1}$

$$\frac{p}{q} \cdot 1 = \frac{p}{q} \cdot \frac{1}{1} = \frac{p}{q}$$

(v) Inverse:

Additive inverse: For $\frac{p}{q} \in Q$,

$$-\frac{p}{q} = \frac{-p}{q}, \quad \frac{p}{q} + \frac{-p}{q} = 0$$

Multiplicative Inverse: For $\frac{p}{q} \notin Q, p \neq 0$

$$\left(\frac{p}{q}\right)^{-1} = \frac{1}{p}, \quad \frac{p}{q} \cdot \frac{1}{p} = 1$$