

Answer To the question No 1

By Sylow's theorem, the number of Sylow p -subgroups divides q and $\equiv 1 \pmod{p}$.

\Rightarrow possible numbers: 1 or q .

Similarly the number of Sylow q -subgroups divides p and $\equiv 1 \pmod{q}$.

\Rightarrow possible numbers: 1 or p .

Since p, q are distinct primes, at least one of these Sylow subgroups must be normal.

Let normal one be P and the other Q .

Then $G = PQ$ and $P \cap Q = \{e\}$.

$\therefore G \cong P \times Q$, and both are cyclic.

$\therefore G$ is abelian.

Answer No No 4

If G/N is cyclic and N is cyclic, prove that G is abelian.

Let, $G/N = \langle aN \rangle$ and $N = \langle b \rangle$

Every element of G can be written as $a^i b^j$

since N is normal, $ab = ba$

all elements commute

$\therefore G$ is abelian.

Answer No 5

In any group, prove that the set of elements of finite order form a subgroup of G .

Let $H = \{x \in G \mid x \text{ has finite order}\}$.

For any $x, y \in H$, both have finite order

Then xy^{-1} also has finite order

Hence H is closed and contains inverse

$\therefore H$ is a subgroup.

Answer No 7

If $a^4 = b^2$ and $ab = ba$, prove that $(ab)^6 = e$

Since, $ab = ba$, we can combine powers easily, $(ab)^6 = a^6 b^6$

Given, $a^4 = b^2$

$$a^8 = b^4$$

then, $a^{12} = b^6 = e$

Hence, $(ab)^6 = e$

Answer No 8

If $[G:H] = n$, then for any $u \in G$, $u^n \in H$

By Lagrange's theorem, G acts on left cosets of H by multiplication.

The permutation corresponding to u has order dividing n .

$$\therefore u^n \text{ fixes all cosets} \Rightarrow u^n \in H$$

Answer No 6

Let G be a finite group and p be the smallest prime dividing $|G|$. Prove that any subgroup of index p in G is normal.

If $[G:H] = p$ then G acts on the set of left cosets of H by left multiplication.

This gives a homomorphism $G \rightarrow S_p$

The kernel of this homomorphism is contained in H .

Since p is smallest prime dividing $|G|$, the kernel has order dividing $|G|$ and must be of index 1 or p .

\therefore Hence H is normal.