Answer To the question No 1

By Sylow's theorem, the number of Sylow p- subgroups divides q and $\equiv 1 \mod p$.

$\Rightarrow$ possible numbers: 1 or q.

Similarly the number of Sylow q-subgroups p and $\equiv 1 \mod q$.

$\Rightarrow$ possible numbers: 1 or p

Since p, q are distinct primes, at least one of these Sylow subgroups must be normal

let normal one be P and the other Q.

Then $G = PQ$ and $P \cap Q = \{e\}$

$\therefore G \cong P \times Q$, and both are cyclick

$\therefore G$ is abelian.

Ancwer No No 4

If G/N is eyelic and N is cyclic, prove that G is abelian.

let, $G/N = \langle aN \rangle$ and $N = \langle b \rangle$

Frery element of G can be written as $a^i b^j$

since N is normal, $ab = ba$

all elements compute

$\therefore$ G is abelian.

Answer No 5

In any group, prove that the set of element of finite order form a subgroup of G.

Let $H = \{x \in G \mid x$ has finite order$\}$.

For any $x, y \in H$, both has finite order

Then xy^{-1} also has finite order

Hence H is closed and contains inverse

$\therefore$ H is a subgroup.

Answer No 7

If $a^4 = b^2$ and $ab = ba$, prove that $(ab)^6 = e$

Since, $ab = ba$, we can combine powers easily, $(ab)^6 = a^6 b^6$

Given, $a^4 = b^2$

$a^8 = b^4$

then, $a^{12} = b^6 = e$

Hence, $(ab)^6 = e$

Answer No 8

If $[G:H] = n$, then for any $x \in G$, $x^n \in H$

By lagrange's theorem, G acts on left cosets of H by multiplication.

The permutation corresponding to x has order deviding n.

$\therefore x^n$ fixes all cosets $\Rightarrow x^n \in H$

Answer No 6

Let G be a finite group and p be the smallest prime dividing $|G|$. Prove that any subgroup of index p in G is normal.

If $[G:H] = p$ then G acts on the set of left cosets of H by lef multiplication.

This gives a homomorphism $G \longrightarrow S_p$

The kernel of this homomorphism is contained in H.

Since p is smallest prime dividing $|G|$. the kernel has order dividing $|G|$ and must be of index 1 or p.

$\therefore$ Hence H is normal.