EXAMPLE 2.1 Find the convolution of two finite duration sequences:

$$h(n) = a^n u(n) \quad \text{for all } n$$
$$x(n) = b^n u(n) \quad \text{for all } n$$

(i) When $a \neq b$
(ii) When $a = b$

Solution: The impulse response $h(n)$ and the input $x(n)$ are zero for $n < 0$, i.e. both $h(n)$ and $x(n)$ are causal.

$$\therefore \qquad y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{(n-k)} = a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k$$

$$= a^n \left[\frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \left(\frac{b}{a}\right)}\right] \qquad [\text{when } a \neq b]$$

When $a = b$

$$y(n) = a^n[1 + 1 + 1 + \cdots + n + 1 \text{ terms}] = a^n(n + 1)$$

EXAMPLE 2.2 Find $y(n)$ if $x(n) = n + 3 \quad \text{for } 0 \le n \le 2$

$$h(n) = a^n u(n) \quad \text{for all } n$$

Solution: We have $$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Given $$x(n) = n + 3 \quad \text{for } 0 \le n \le 2$$
$$h(n) = a^n u(n) \quad \text{for all } n$$

$h(n) = 0$ for $n < 0$, so the system is causal. $x(n)$ is a causal finite duration sequence whose value is zero for $n > 2$. Therefore,

$$y(n) = \sum_{k=0}^{2} x(k)h(n-k)$$

$$= \sum_{k=0}^{2} (k + 3)a^{n-k}u(n-k)$$

$$= 3a^n u(n) + 4a^{n-1}u(n-1) + 5a^{n-2}u(n-2)$$

EXAMPLE 2.3 Determine the response of the system characterized by the impulse response $h(n) = (1/3)^n u(n)$ to the input signal $x(n) = 3^n u(n)$.

Solution: Given $x(n) = 3^n u(n)$ and $h(n) = \left(\frac{1}{3}\right)^n u(n)$

A causal signal is applied to a causal system

$$\therefore \qquad y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} 3^k \left(\frac{1}{3}\right)^{n-k}$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} 3^k \times 3^k$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} (3^2)^k$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{1 - (3^2)^{n+1}}{1-3^2}\right]$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{9^{n+1} - 1}{9 - 1}\right] = \left(\frac{1}{3}\right)^n \left[\frac{9^{n+1} - 1}{8}\right]$$

EXAMPLE 2.4 Determine the response of the system characterized by the impulse response $h(n) = 2^n u(n)$ for an input signal $x(n) = 3^n u(n)$.

Solution: Given $x(n) = 3^n u(n)$ and $h(n) = 2^n u(n)$

Since both $x(n)$ and $h(n)$ are causal, we have

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} 3^k \cdot 2^{n-k} = 2^n \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k = 2^n \left[\frac{1 - \left(\frac{3}{2}\right)^{n+1}}{1 - \frac{3}{2}}\right]$$

$$= 2^n \left[\frac{\left(\frac{3}{2}\right)^{n+1} - 1}{\frac{1}{2}}\right] = 2^{n+1}\left[\left(\frac{3}{2}\right)^{n+1} - 1\right]$$

EXAMPLE 2.5 Find the convolution of

$$x(n) = \cos n\pi\, u(n),\ \ h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Solution: Given $x(n) = \cos n\pi\, u(n) = (-1)^n u(n),\ h(n) = \left(\frac{1}{2}\right)^n u(n)$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} (-1)^k \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (-1)^k (2)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (-2)^k = \left(\frac{1}{2}\right)^n \frac{1 - (-2)^{n+1}}{1 - (-2)}$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1 + 2(-2)^n}{3}\right] = \frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}(-1)^n \quad \text{for } n > 0$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{3}(-1)^n u(n)$$

EXAMPLE 2.6 Find the convolution of

$$x(n) = u(n), \quad h(n) = u(n-3)$$

Solution: Given $x(n) = u(n), \quad h(n) = u(n-3)$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} u(k)\, u(n-3-k)$$

$$= \sum_{k=0}^{n-3} (1)(1) = n - 2$$

EXAMPLE 2.7 Consider a system with unit sample response

$$h(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases}, \text{ or equivalently } h(n) = a^n u(n).$$

Find the response to an input $x(n) = u(n) - u(n - N)$.

Solution: We know that the output $y(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

From this we can observe that to obtain the nth value of the output sequence we must form the product $x(k)\, h(n - k)$ and sum the values of the resulting sequence. The two component sequences are shown in Figure 2.2 as a function of k, with $h(n - k)$ shown for several values of n.

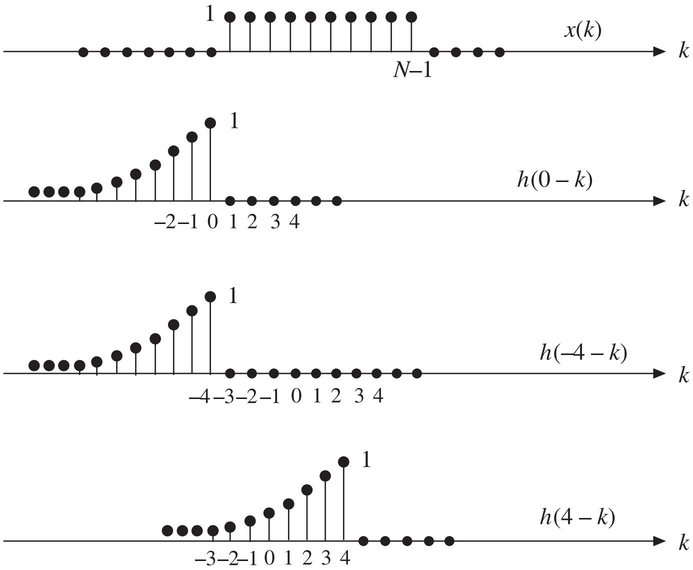

Figure 2.2 Component sequences in evaluating the convolution sum with $h(n - k)$ shown for several values of n.

As we see in Figure 2.2, for $n < 0$, $h(n - k)$ and $x(k)$ have no nonzero samples that overlap, and consequently $y(n) = 0$, $n < 0$. For n greater than or equal to zero but less than N, $h(n - k)$ and $x(k)$ have nonzero samples that overlap from $k = 0$ to $k = n$; thus for $0 \le n < N$,

$$y(n) = \sum_{k=0}^{n} a^{n-k} = a^n \frac{1 - a^{-(n+1)}}{1 - a^{-1}} = \frac{1 - a^{n+1}}{1 - a}, \quad 0 \le n < N$$

For $N - 1 \le n$, the nonzero samples that overlap extend from $k = 0$ to $k = N - 1$ and thus

$$y(n) = \sum_{k=0}^{N-1} a^{n-k} = a^n \frac{1 - a^{-N}}{1 - a^{-1}} = a^{n-(N-1)} \left[\frac{1 - a^N}{1 - a}\right], \quad N \le n$$

The response $y(n)$ is sketched in Figure 2.3.

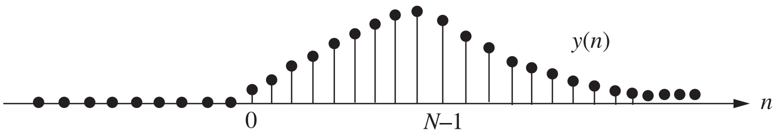

Figure 2.3 Response of the system with unit sample response $h(n) = a^n u(n)$ to the input $u(n) - u(n - N)$.

EXAMPLE 2.8 Find the convolution of

$$x(n) = \left(\frac{1}{2}\right)^n u(n), \quad h(n) = u(n) - u(n - 10)$$

Solution: Given $x(n) = \left(\frac{1}{2}\right)^n u(n), \quad h(n) = u(n) - u(n - 10)$

For $n < 10$, i.e., for $n \le 9$,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\,(1) \quad \text{for } 0 \le n \le 9$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \qquad 0 \le n \le 9$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^n \left[\frac{1 - \left(\frac{1}{2}\right)^{-(n+1)}}{1 - \left(\frac{1}{2}\right)^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^n [2^{n+1} - 1] \qquad 0 \le n \le 9$$

For $n \ge 10$, i.e., for $n > 9$,

$$y(n) = \sum_{k=n-9}^{n} \left(\frac{1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^{n-9} + \left(\frac{1}{2}\right)^{n-9+1} + \cdots + \left(\frac{1}{2}\right)^{n-9+9}$$

$$= \left(\frac{1}{2}\right)^{n-9} \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^9\right]$$

$$= \left(\frac{1}{2}\right)^{n-9}\left[\frac{1-\left(\frac{1}{2}\right)^{9+1}}{1-\frac{1}{2}}\right] \quad \text{for } n > 9$$

$$= 2\left(\frac{1}{2}\right)^{n-9} - 2\left(\frac{1}{2}\right)^{n+1}$$

$$= \left(\frac{1}{2}\right)^{n}[2^{10} - 1]$$

EXAMPLE 2.9 Find the convolution of

$$x(n) = \left(\frac{1}{3}\right)^{-n} u(-n-1) \quad \text{and} \quad h(n) = u(n-1)$$

Solution: Let $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

$x(k) = \left(\frac{1}{3}\right)^{-k} u(-k-1)$ and $h(k) = u(k-1)$ are plotted as shown in Figure 2.4. From Figure 2.4 [(a) and (b)] we can find that $x(k) = 0$ for $k > -1$ and $h(n-k) = 0$ for $k > n-1$.

For $n - 1 < -1$, i.e., for $n < 0$, the interval of summation is from $k = -\infty$ to $n - 1$.

$$\therefore \qquad y(n) = \sum_{k=-\infty}^{n-1}\left(\frac{1}{3}\right)^{-k} \quad \text{for } n-1 \le -1 \text{ or for } n \le 0$$

$$= \left(\frac{1}{3}\right)^{-(n-1)} + \left(\frac{1}{3}\right)^{-(n-2)} + \cdots$$

$$= \left(\frac{1}{3}\right)^{-(n-1)}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^{2} + \cdots\right]$$

$$= 3^{n-1}\left[\frac{1}{1-1/3}\right] = 0.5(3)^{n}$$

For $n - 1 > -1$, i.e. for $n > 0$, the interval of summation is from $k = -\infty$ to $k = -1$.

$$\therefore \qquad y(n) = \sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^{-k} = \sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k} = \frac{1/3}{1-1/3} = 0.5$$

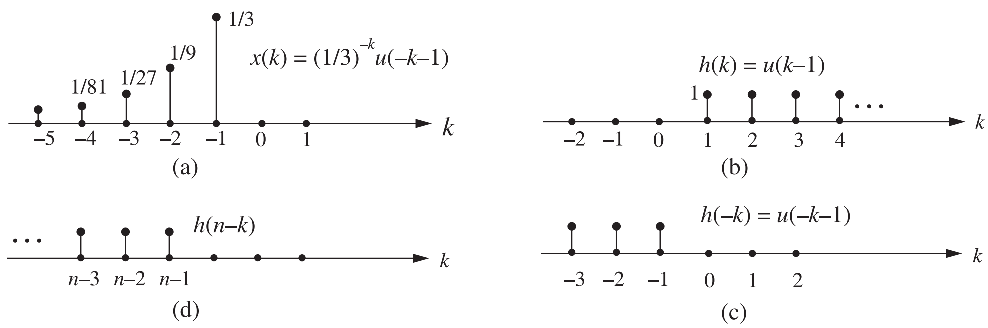

Figure 2.4 Component sequences in evaluating the convolution sum.

Figure 2.5 shows the plot of $y(n)$ for all values of n.

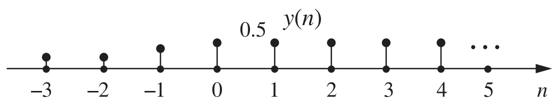

Figure 2.5 Plot of $y(n) = x(n) * h(n)$.

Unit step response

The unit step response can be obtained by exciting the input of the system by a unit step sequence, i.e. $x(n) = u(n)$. It can be easily expressed in terms of the impulse response using convolution sum. Let the impulse response of the discrete-time system be $h(n)$. Then the step response $s(n)$ can be obtained using the convolution sum:

$$s(n) = h(n) * u(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

Since $u(n - k) = 0$ for $k > n$ and $u(n - k) = 1$ for $k \le n$,

$$s(n) = \sum_{k=-\infty}^{n} h(k)$$

That is the step response is the running sum of the impulse response.

For a causal system,

$$s(n) = \sum_{k=0}^{n} h(k)$$

EXAMPLE 2.10 Evaluate the step response for the LTI system represented by the following impulse response:

(a) $h(n) = \delta(n) - \delta(n-2)$

(b) $h(n) = \left(\frac{1}{4}\right)^n u(n)$

(c) $h(n) = nu(n)$

(d) $h(n) = u(n)$

Solution:

(a) Given $h(n) = \delta(n) - \delta(n-2)$

The step response $s(n) = h(n) * u(n)$

$$= [\delta(n) - \delta(n-2)] * u(n)$$

$$= \delta(n) * u(n) - \delta(n-2) * u(n)$$

$$= u(n) - u(n-2)$$

(b) Given $h(n) = \left(\frac{1}{4}\right)^n u(n)$

$$s(n) = \left(\frac{1}{4}\right)^n u(n) * u(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 4^k$$

$$= \left(\frac{1}{4}\right)^n \left[\frac{1-4^{n+1}}{1-4}\right]$$

(c) Given $h(n) = nu(n)$

$$s(n) = h(n) * u(n)$$

$$= nu(n) * u(n)$$

$$= \sum_{k=0}^{n} k\, u(k) u(n-k)$$

$$= \sum_{k=0}^{n} k$$

(d) Given $h(n) = u(n)$

$$s(n) = u(n) * u(n) = \sum_{k=-\infty}^{\infty} u(k) u(n-k)$$

$$= \sum_{k=0}^{n} 1 = n+1$$

$\therefore \qquad s(n) = (n + 1)$

2.4 CONVOLUTION OF FINITE SEQUENCES

In practice, we often deal with sequences of finite length, and their convolution may be found by several methods. The convolution $y(n)$ of two finite-length sequences $x(n)$ and $h(n)$ is also of finite length and is subject to the following rules, which serve as useful consistency checks:

1. The starting index of $y(n)$ equals the sum of the starting indices of $x(n)$ and $h(n)$.
2. The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ and $h(n)$.
3. The length L_y of $y(n)$ is related to the lengths L_x and L_h of $x(n)$ and $h(n)$ by $L_y = L_x + L_h - 1$.

2.5 METHODS TO COMPUTE THE CONVOLUTION SUM OF TWO SEQUENCES *x*(*n*) AND *h*(*n*)

2.5.1 Method 1 Linear Convolution Using Graphical Method

Step 1: Choose the starting time n for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$ is a good choice.

Step 2: Express both the sequences $x(n)$ and $h(n)$ in terms of the index k.

Step 3: Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by n to the right if n is positive and to the left if n is negative to obtain $h(n - k)$.

Step 4: Multiply the two sequences $x(k)$ and $h(n - k)$ element by element and sum the products to get $y(n)$.

Step 5: Increment the index n, shift the sequence $h(n - k)$ to the right by one sample and perform Step 4.

Step 6: Repeat Step 5 until the sum of products is zero for all remaining values of n.

2.5.2 Method 2 Linear Convolution Using Tabular Array

Let $x_1(n)$ and $x_2(n)$ be the given N sample sequences. Let $x_3(n)$ be the N sample sequence obtained by linear convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$:

Step 1: Change the index from n to k, and write $x_1(k)$ and $x_2(k)$.

Step 2: Represent the sequences $x_1(k)$ and $x_2(k)$ as two rows of tabular array.

Step 3: Fold one of the sequences. Let us fold $x_2(k)$ to get $x_2(-k)$.

Step 4: Shift the sequence $x_2(-k)$, q times to get the sequence $x_2(q-k)$. If q is positive, then shift the sequence to the right and if q is negative, then shift the sequence to the left.

Step 5: The sample of $x_3(n)$ at $n = q$ is given by

$$x_3(q) = \sum_{k=0}^{N-1} x_1(k)\, x_2(q-k)$$

Determine the product sequence $x_1(k)x_2(q-k)$ for one period.

Step 6: The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e. $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

2.5.3 Method 3 Linear Convolution Using Tabular Method

Given $x(n) = \{x_1, x_2, x_3, x_4\}$, $h(n) = \{h_1, h_2, h_3, h_4\}$

The convolution of $x(n)$ and $h(n)$ can be computed as per the following procedure.

Step 1: Write down the sequences $x(n)$ and $h(n)$ as shown in Table 2.1.

Step 2: Multiply each and every sample in $h(n)$ with the samples of $x(n)$ and tabulate the values.

Step 3: Group the elements in the table by drawing diagonal lines as shown in table.

Step 4: Starting from the left sum all the elements in each strip and write down in the same order.

$$y(n) = x_1h_1,\ x_1h_2 + x_2h_1,\ x_1h_3 + x_2h_2 + x_3h_1,\ x_1h_4 + x_2h_3 + x_3h_2$$
$$+ x_4h_1,\ x_2h_4 + x_3h_3 + x_4h_2,\ x_3h_4 + x_4h_3,\ x_4h_4$$

Step 5: Mark the symbol $\uparrow$ at time origin ($n = 0$).

TABLE 2.1 Table for Computing $y(n)$

$h(n)$ \ $x(n)$	x_1	x_2	x_3	x_4
h_1	x_1h_1	x_2h_1	x_3h_1	x_4h_1
h_2	x_1h_2	x_2h_2	x_3h_2	x_4h_2
h_3	x_1h_3	x_2h_3	x_3h_3	x_4h_3
h_4	x_1h_4	x_2h_4	x_3h_4	x_4h_4

2.5.4 Method 4 Linear Convolution Using Matrices

If the number of elements in $x(n)$ are N_1 and in $h(n)$ are N_2, then to find the convolution of $x(n)$ and $h(n)$ form the following matrices:

1. Matrix H of order $(N_1 + N_2 - 1) \times N_1$ with the elements of $h(n)$
2. A column matrix X of order $(N_1 \times 1)$ with the elements of $x(n)$
3. Multiply the matrices H and X to get a column matrix Y of order $(N_1 + N_2 - 1)$ that has the elements of $y(n)$, the convolution of $x(n)$ and $h(n)$.

$$\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ h(N_2-1) & h(N_2-2) & \cdots & h(0) \\ 0 & h(N_2-1) & \cdots & h(1) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & h(N_2-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x(N_1-1) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y(N_1+N_2-1) \end{bmatrix}$$

$$H \qquad X = Y$$

2.5.5 Method 5 Linear Convolution Using the Sum-by Column Method

This method is based on the idea that the convolution $y(n)$ equals the sum of the (shifted) impulse responses due to each of the impulses that make up the input $x(n)$. To find the convolution, we set up a row of index values beginning with the starting index of the convolution and $h(n)$ and $x(n)$ below it. We regard $x(n)$ as a sequence of weighted shifted impulses. Each element (impulse) of $x(n)$ generates a shifted impulse response [Product with $h(n)$], starting at its index (to indicate the shift). Summing the response (by columns) gives the discrete convolution. Note that none of the sequences is flipped. It is better (if only to save paper) to let $x(n)$ be the shorter sequence. The starting index (and the marker location corresponding to $n = 0$) for the convolution $y(n)$ is found from the starting indices of $x(n)$ and $h(n)$.

2.5.6 Method 6 Linear Convolution Using the Flip, Shift, Multiply, and Sum Method

The convolution sum may also be interpreted as follows. We flip $x(n)$ to generate $x(-n)$ and shift the flipped signal $x(-n)$ to line up its last element with the first element of $h(n)$. We then successively shift $x(-n)$ (to the right) past $h(n)$, one index at a time, and find the convolution at each index as the sum of the pointwise products of the overlapping samples. One method of computing $y(n)$ is to list the values of the flipped function on a strip of paper and slide it along the stationary function, to better visualize the process. This technique has