GeoPDES

an Octave/Matlab Software for Research on IGA

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GRADUATE SCHOOL computational engineering

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Computational Electromagnetics Research Group at GSCE

Motivation and Introduction

Starting point: different codes, different problems, different developers.

Primary goal: uniform implementation of the codes, clear and easy to use, for didactic purposes.

In the presentation we will see two different packages.

- The NURBS toolbox, for geometry definition and manipulation.
- GeoPDEs, for isogeometric analysis, discretisation of PDEs.

Both packages are implemented in Octave (compatible with Matlab), open source and free.

Contributors

- Main developer:
  - **Rafael Vázquez** from IMATI-CNR (Pavia, IT) and EPFL (Lausanne, CH).
- List of contributors:
  - **Viacheslav Balobanov**, Bernoulli-Euler beam, error in H^2 norm.
  - **Andrea Bressan**, implementation of SubGrid methods and other contributions for Stokes problem.
  - **Elena Bulgarello**, contributions to the 3D div-conforming splines.
  - **Adriano Côrtes**, Nitsche's method for enforcing Dirichlet conditions with div-conforming splines.
  - **Jacopo Corno**, beta testing and geometry of the Tesla cavity.
  - **Luca Dedè**, advection-diffusion equation.
  - **Carlo de Falco**, contributions to the design of the code, implementation and maintenance of oct-files, implementation of many low level functions, and constant advice and support.
  - **Sara Frizziero**, contributions to the 3D div-conforming splines.
  - **Eduardo M. Garau**, algorithms and implementation of adaptive methods with hierarchical B-splines.
  - **Monica Montardini**, collocation methods.
  - **Marco Pingaro**, implementation of the bilaplacian and the Kirchhoff-Love plate.
  - **Alessandro Reali**, contributions to the original design of the code.
  - **Anna Tagliabue**, advection-diffusion equation.
  - **Lorenzo Tamellini**, collocation methods.

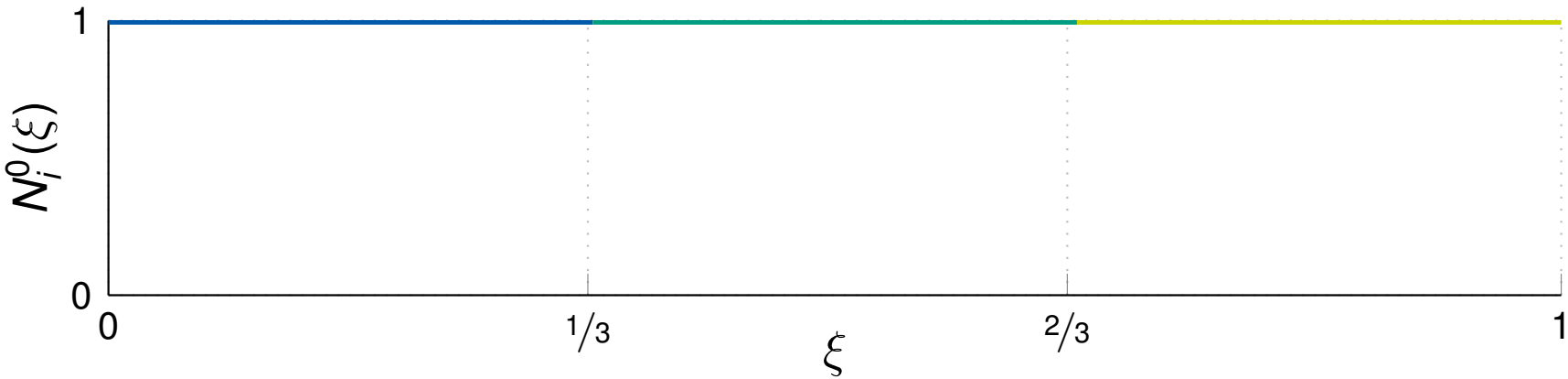
Outline

B-Spline Basis Functions

Given a *non-uniform knot vector* $\Xi = [\xi_1, ..., \xi_{n+p+1}]$, in the parametric domain $[0, 1]$, B-Spline basis functions are:

$$N_i^0(\xi) = \begin{cases} 1 \text{ if } \xi_i \leq \xi < \xi_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

$$N_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi).$$



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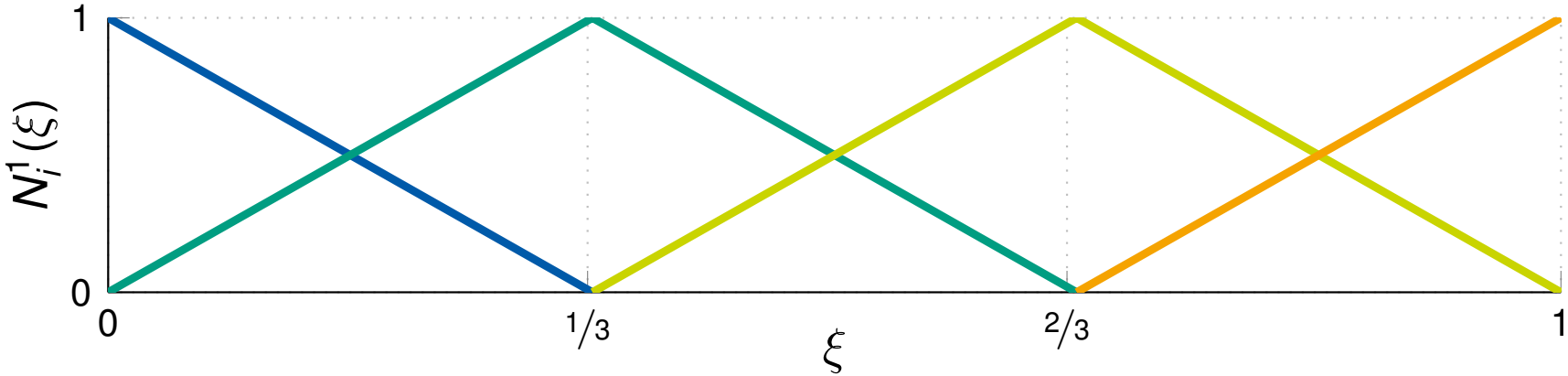

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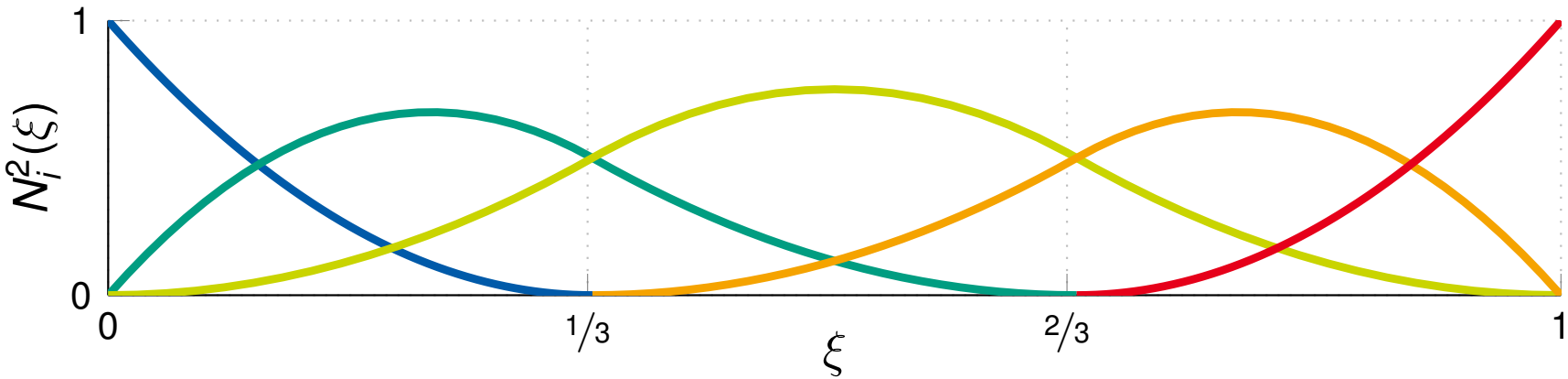
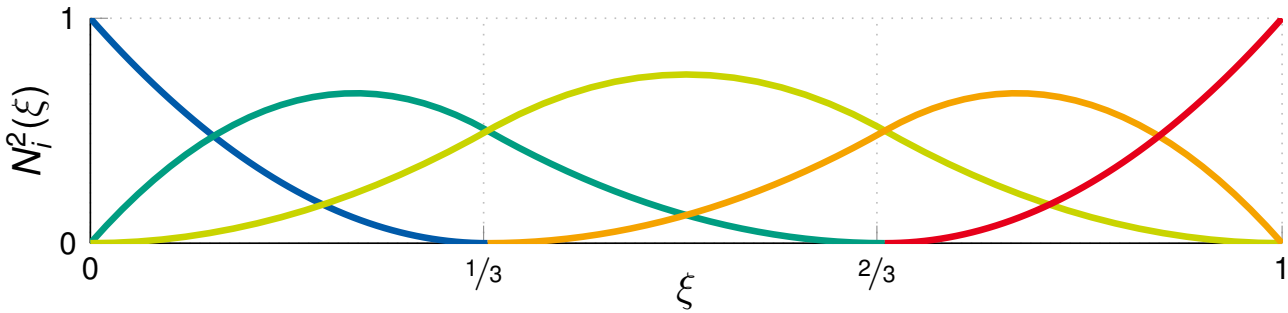


B-Splines: Main Properties



B-spline basis functions have the following properties:

- N_i^p are non-negative with partition-of-unity property
- Locally linearly independent on each knot span (ξ_i, ξ_{i+1})
- Support of N_i^p is (ξ_i, ξ_{i+p+1})
- Piecewise polynomials of degree p, and regularity at most $p - 1$

The regularity at ξ_i is controlled by the knot multiplicity m_i

- Number of continuous derivatives: $p - m_i$, with $m_i \leq p + 1$
- For $m_i = p$, B-Splines become interpolatory at the knot
- Multiple knots also reduce the support of B-Splines

B-Splines: Basis Functions Evaluation

To compute B-spline basis functions, the data we need are:

- Degree `p` (order $p+1$)
- Knot vector `knt`, including knot repetitions (length = m)
- Number of basis functions `ncp` ($n_{cp} = m - p - 1$)

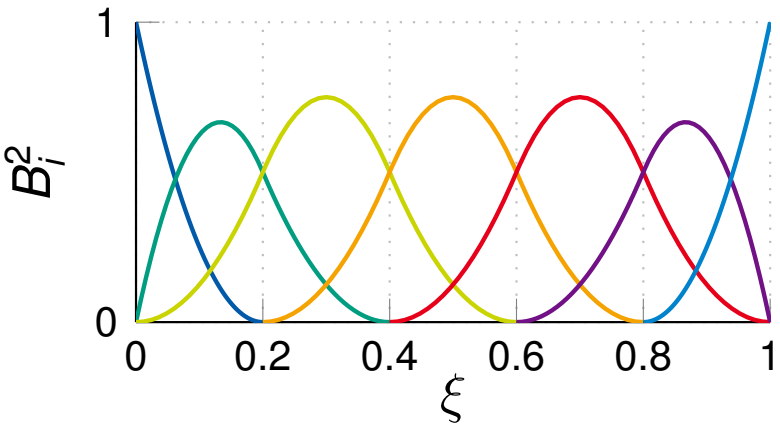
In the NURBS toolbox, they are evaluated using **`basisfun`**.
The derivatives can be computed using **`basisfunder`**.

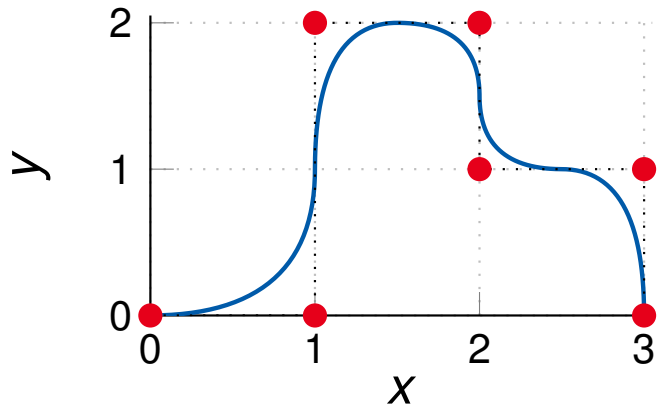
```
iv   = findspan (ncp-1, p, x, knt);
N    = basisfun (iv, x, p, knt);
Nder = basisfunder (iv, p, x, knt, nder);
```

For each point, only the $p+1$ non-vanishing functions are computed.
The global numbering is recovered using **`numbasisfun`**.

```
num = numbasisfun (iv, x, p, knt) + 1;  % Compute the numbering
```

B-Spline Curves: Definition





A B-Spline curve in $\mathbb{R}^d$ is defined as a linear combination of B-Splines:

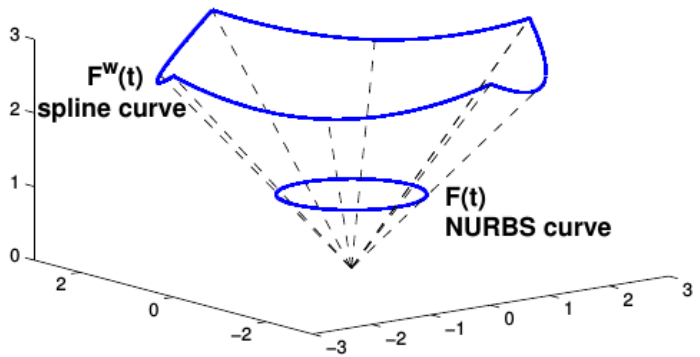
$$\mathbf{F}(\xi) = \sum_{i=0}^{n} \mathbf{C}_i N_i^p(\xi)$$

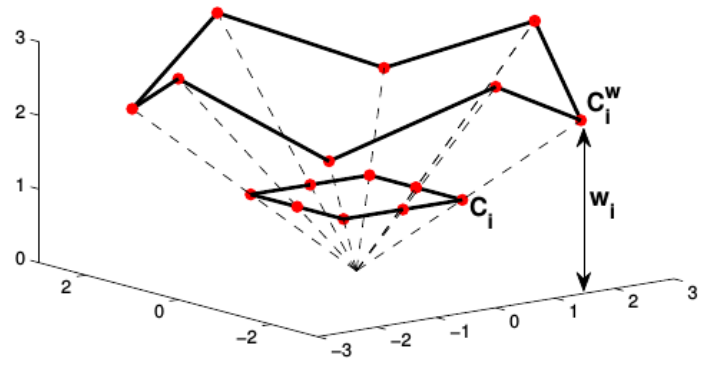
To define the parametrisation **F** we only need:

- The basis functions N_i^p, given by the knot vector
- The control points $\mathbf{C}_i \in \mathbb{R}^d$

NURBS Curves: Definition

A NURBS object in $\mathbb{R}^d$ is the projection of a B-Spline object in $\mathbb{R}^{d+1}$





In practice, a weight w_i is associated to each B-Spline function:

$$w_i = (\mathbf{C}_i^w)_{d+1}, \qquad R_i^p = \frac{N_i^p w_i}{\sum_{j=1}^{n} N_j^p w_j} \qquad \mathbf{F}(\xi) = \sum_{i=0}^{n} \mathbf{C}_i R_i^p(\xi)$$

- NURBS basis functions are piecewise rational polynomials
- They maintain most of the properties of B-splines
- Exact representation of conic sections

NURBS Curves: Definition in the NURBS Toolbox

To define B-Splines (or NURBS) curves, we use the command **`nrbmak`**.
It requires as input the knot vector and the control points.

```
crv = nrbmak (coefs, knt);
```

It creates a Matlab structure, which contains the following fields:

- **number**: the number of control points, `ncp`.
- **coefs**: control points coordinates (for NURBS also the weights).
- **order**: the degree plus one, $p + 1$.
- **knots**: the knot vector, `knt`.

The curve can be visualised with the commands **`nrbplot`**.

```
nrbplot (crv, 1000);
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Important: coefs contains the **"weighted" control points**, i.e. $\mathbf{C^w}$.

NURBS Curves: Evaluation

Other functions for visualisation based on the previous structure: **`nrbkntplot`** and **`nrbctrlplot`**.

```
nrbnktplot (crv);  % Plot the curve with knots on it
nrbctrlplot (crv); % Plot the curve with the control polygon
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The parametrisation **F** is evaluated with **`nrbeval`**:

```
x = 0:0.001:1;
F = nrbeval (nrb, x);
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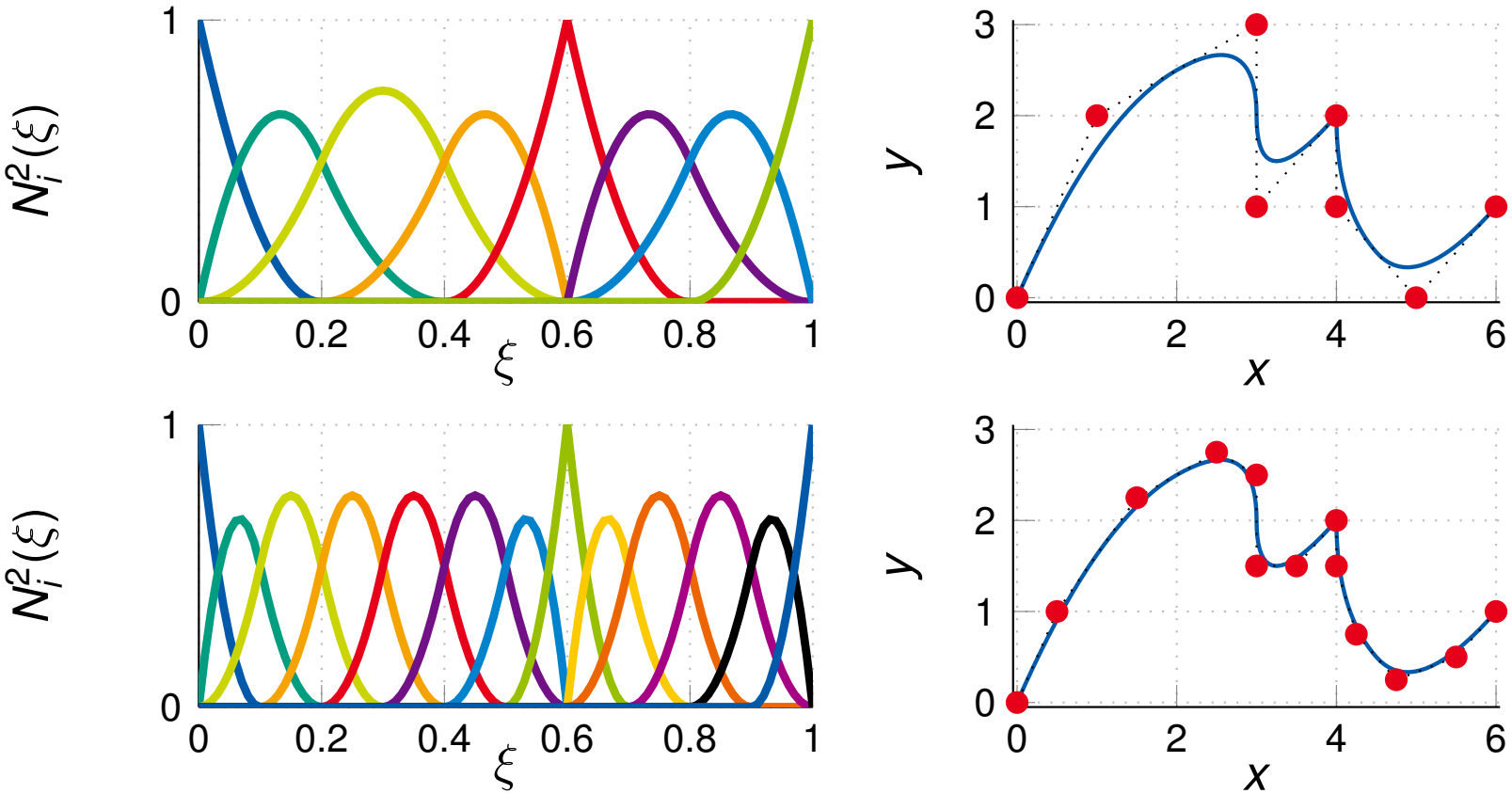
For IGA, we will also need the derivatives of the parametrisation, $\mathbf{F}'(\xi)$. They are computed with **`nrbderiv`** and **`nrbdeval`**:

```
dcrv = nrbderiv (crv);
[F, dF] = nrbdeval (crv, dcrv, x);
```

The second derivative can be computed in a very similar way.

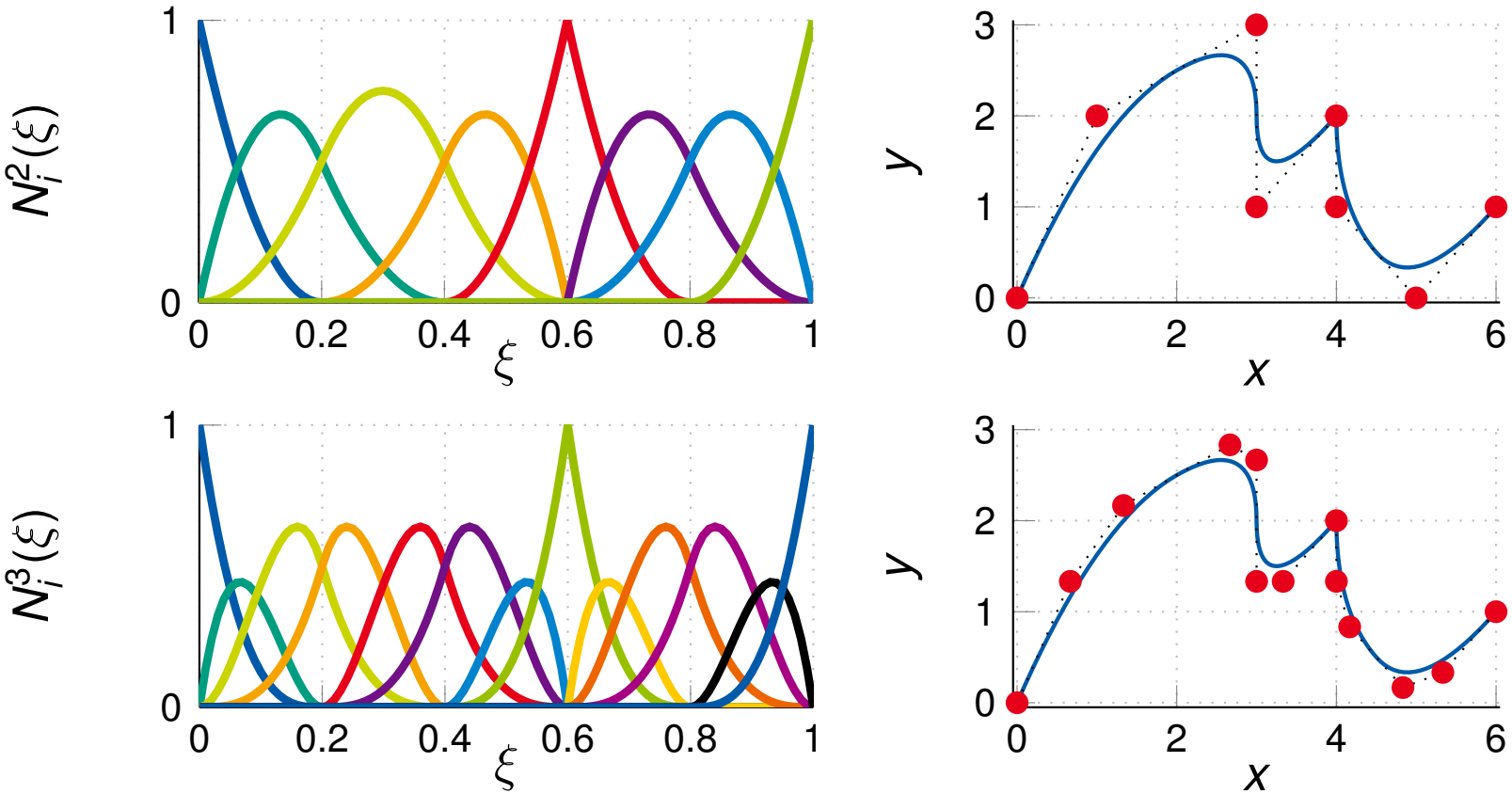
Knot Insertion (*h*-refinement)

Knot insertion: add more knots maintaining the parametrisation **F**.
In CAD it is used to add detail in some regions.
In IGA it is the analogous of *h*-refinement for FEM.



Degree Elevation (*p*-refinement)

Degree elevation: raise the degree maintaining the parametrisation **F**.
In IGA it is the analogous of *p*-refinement for FEM.



NURBS Toolbox: Knot Insertion and Degree Elevation

Knot insertion and degree elevation are implemented in the toolbox. The first input and the output are NURBS structures.

- For knot insertion, the function is **`nrbkntins`**. For instance, to add once the knot 1/10, and twice the knot 3/10:

```
new_crv = nrbkntins (crv, [1/10, 3/10, 3/10]);
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- For degree elevation, the function is **`nrbdegelev`**. For instance, to raise the degree by 2:

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- *k*-refinement can be obtained as a combination of both.
- If you want to perform a uniform refinement, the toolbox contains a function (**`kntrefine`**) to find the new knots to be inserted.

Surfaces and Volumes

The NURBS structure is also valid for surfaces and volumes.

It contains the knot vector in each direction, and a grid of control points.

Most of the functions for curves also work for surfaces and volumes: **`nrbmak`**, **`nrbctrlplot`**, **`nrbeval`**, **`nrbderiv`**, **`nrbkntins`**, **`nrbdegelev`**, ...

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It is possible to create the NURBS by hand, using **`nrbmak(coefs,knt)`**.

- The knot vectors are given in a cell-array, of dimension 2 or 3.
- The control points are in an array of size (4, n1, n2) or (4, n1, n2, n3).
- The fourth coordinate is always the weight, equal to one for B-splines.

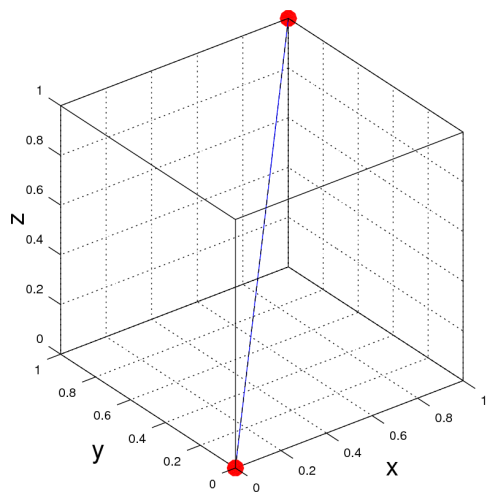
It is easier to create the geometry with functions of the toolbox.

Simple Geometric Entities

The starting point should be one of the simplest geometries.

- **`nrbline`**: creates a straight line in $\mathbb{R}^d$
- **`nrb4surf`**: creates a bilinear surface in $\mathbb{R}^d$ (including planar surfaces)
- **`nrbcirc`**: creates an arc of circunference in $\mathbb{R}^2$
- **`nrbcylind`**: creates the surface of a cylinder, in $\mathbb{R}^3$

```
1 crv = nrbline ([0 0 0], [1 1 1]);
  nrbctrlplot (crv);
```



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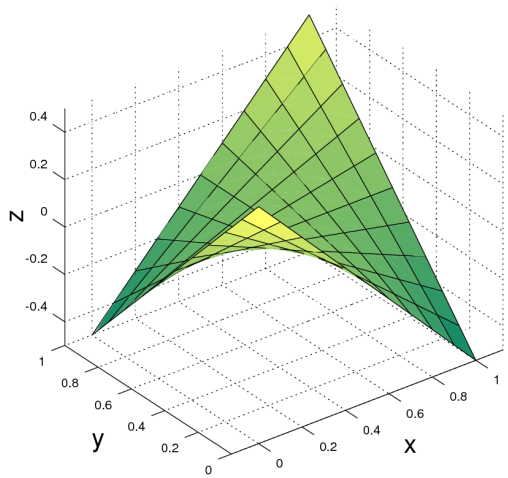
```
srf = nrb4surf ([0 0 0.5], [1 0 -0.5],
                [0 1 -0.5], [1 1 0.5]);
nrbplot (srf, [10, 10]);
```



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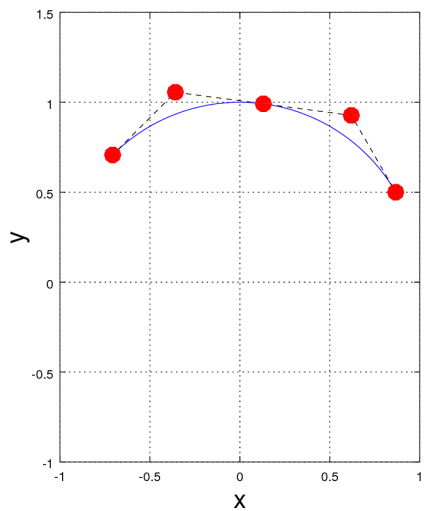
```
1 crv = nrbcirc (1, [0 0], pi/6, 3/4*pi);
  nrbctrlplot (crv);
```



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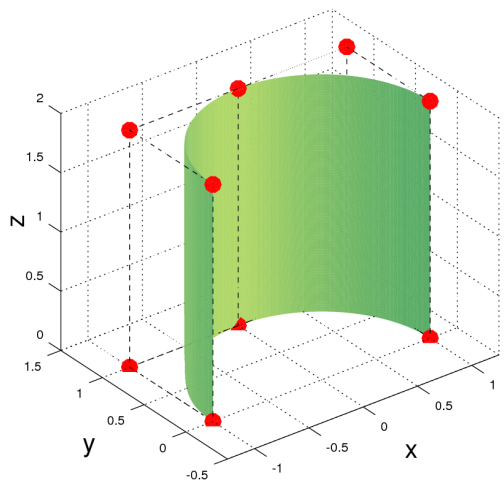
```
cyl = nrbcylind (2, 1, [0 0], 0, pi);
nrbctrlplot (cyl)
```

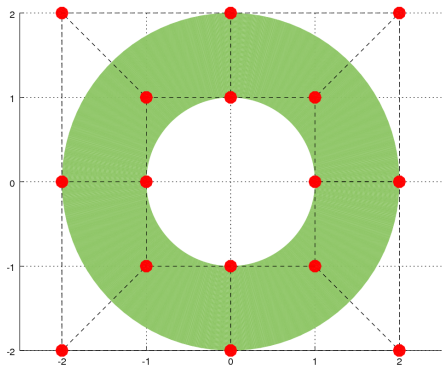


Revolution and Extrusion

The toolbox allows to create geometries by revolution and extrusion.

`nrbrevolve`: create a surface by revolution of a curve, or create a volume by revolution of a surface.

```
crv = nrbline ([1 0], [2 0]);
srf = nrbrevolve (crv, [0 0 0], [0 0 1], 2*pi);
```



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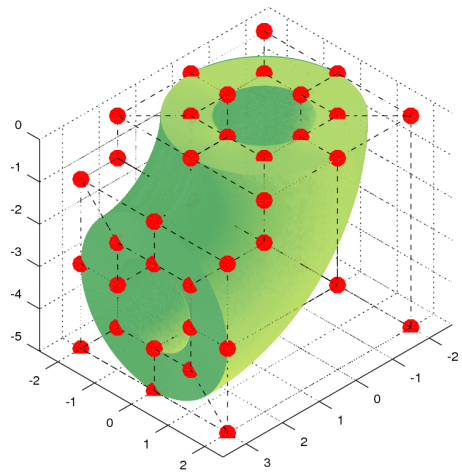
```

```
vol = nrbrevolve (srf, [3 0 0], [0 -1 0], pi/2);
```



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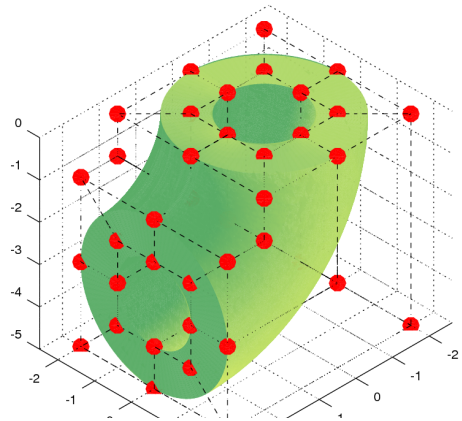
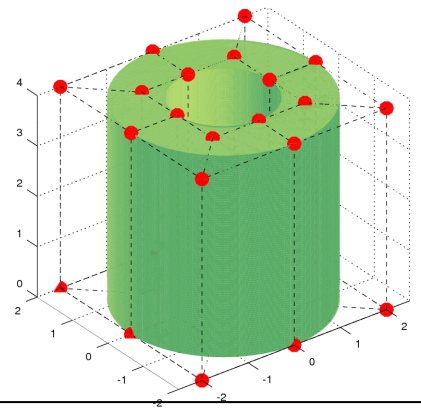


`nrbextrude`: create surface or volume by extrusion

```
vol = nrbextrude (srf, [0 0 4]);
```

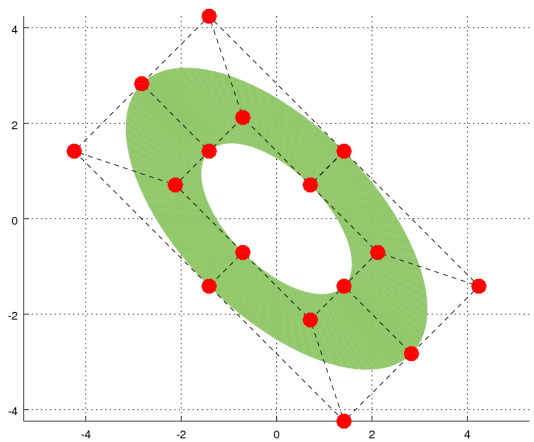


Transformations

Translation, rotation and scaling are applied with a transformation matrix.

`nrbtform`: modify a NURBS structure using a transformation matrix.

The matrix can be constructed using **`vectrans`**, **`vecrot`**, **`vecscale`**.



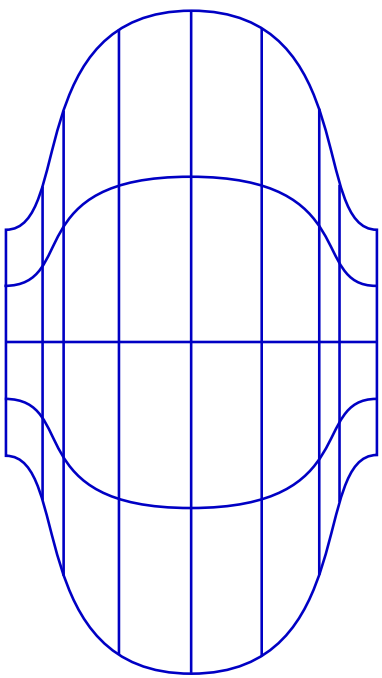

```
crv = nrbline ([1 0], [2 0]);
srf = nrbrevolve (crv, [0 0 0], [0 0 1], 2*pi);

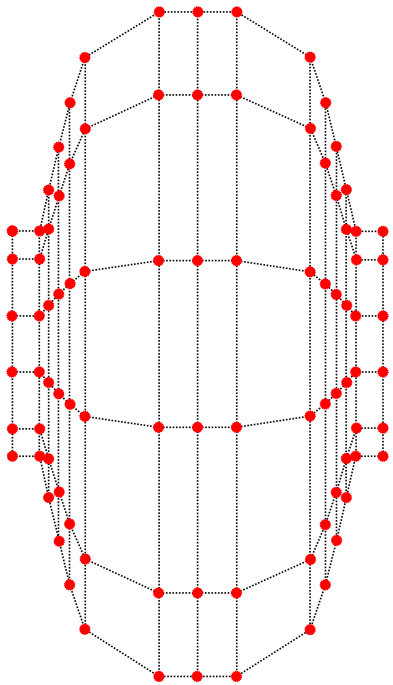
srf = nrbtform (srf, vecscale ([1 2 1]));
srf = nrbtform (srf, vecrot (pi/4, [0 0 1]))
```

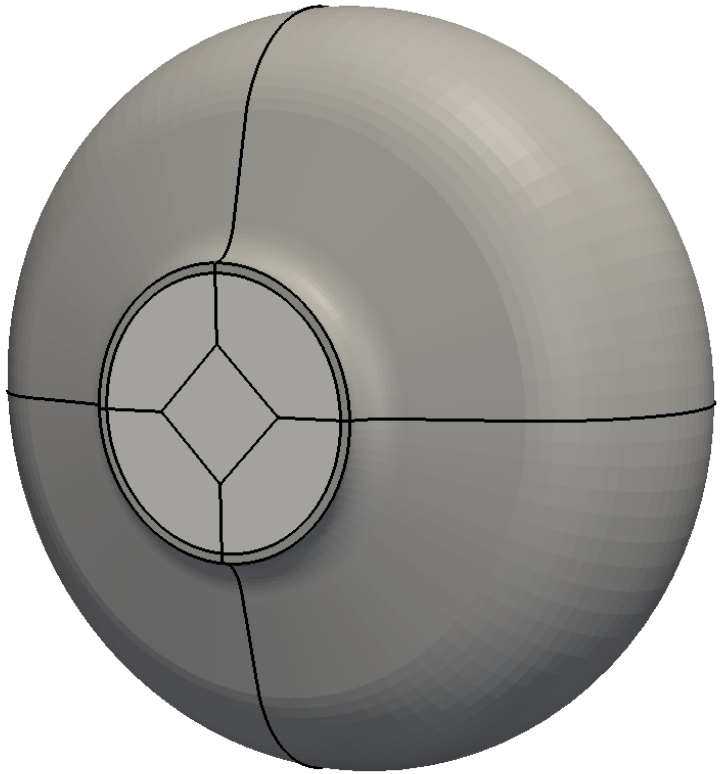
Other Useful Functions

- **`nrbruled`**: construct a ruled surface given two NURBS curves.
- **`nrbcoons`**: construct a Coons surface given four NURBS curves that define the boundary.
- **`nrbextract`**: collect the four boundary curves of a NURBS surface, or collect the six boundary surfaces of a NURBS volume. The output is an array of NURBS structures.
- **`nrbreverse`**: reverse the evaluation direction in a chosen dimension.
- **`nrbtransp`**: transpose a surface, changing the two directions.
- **`nrbpermute`**: generalisation of the previous function to volumes.
- **`nrbexport`**: save NURBS to a text file.
- **`nrb2iges`**: save NURBS to a text file compliant with the IGES format.

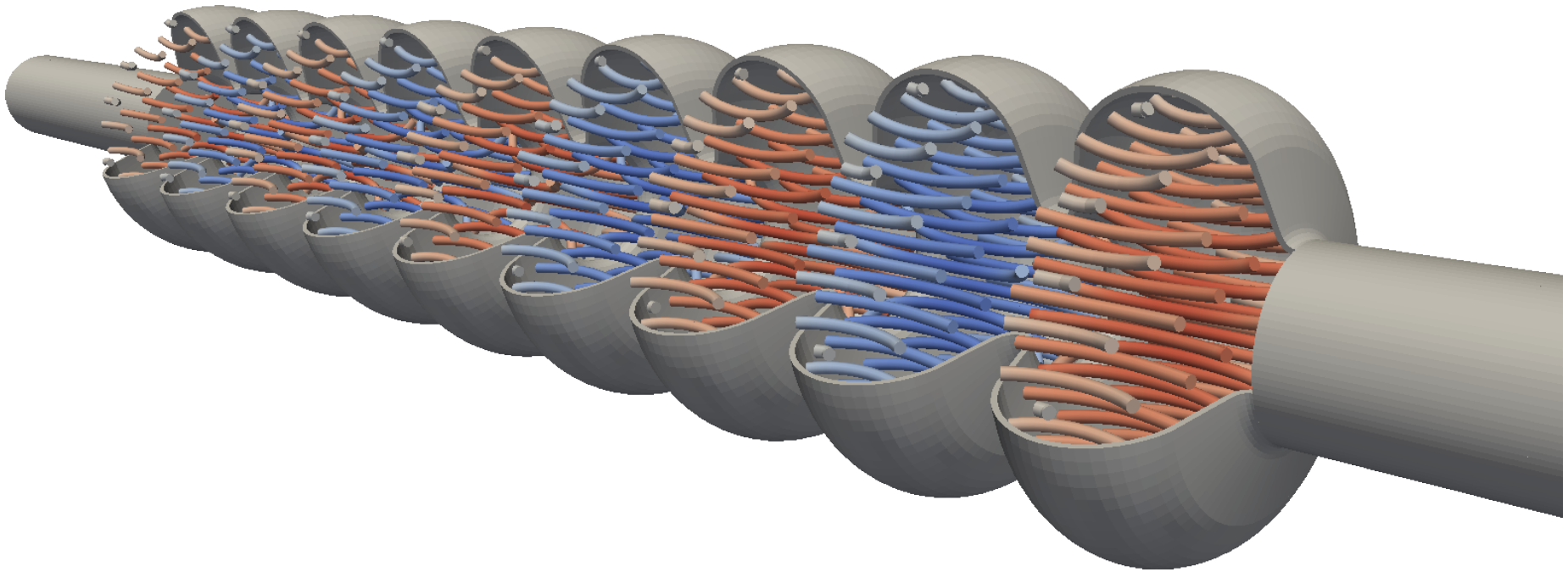
A Complicated Example



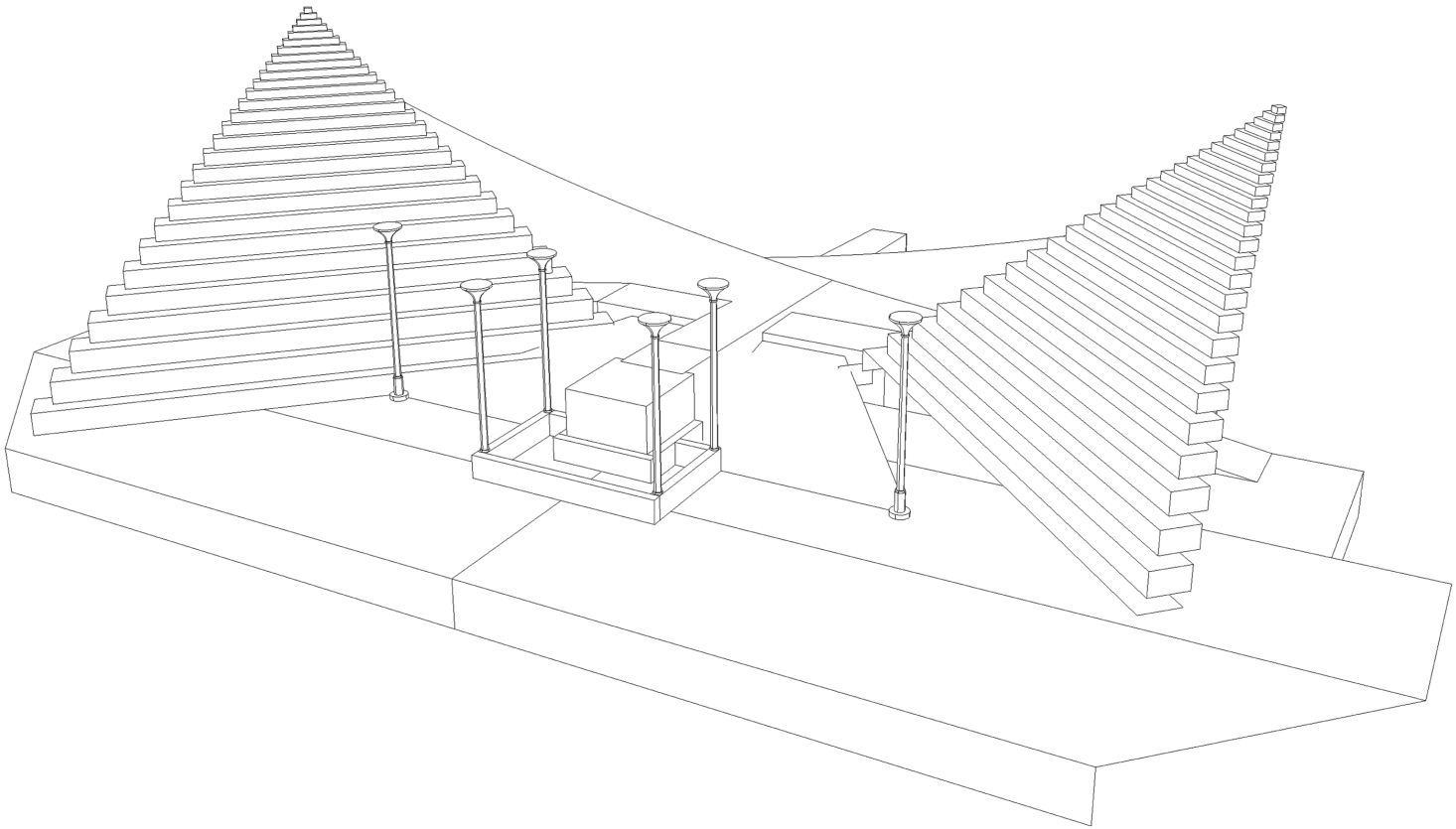




A Complicated Example



A Complicated Example



NURBS Toolbox: Conclusions

- It serves for basis function evaluation and geometry manipulation.
- It supports curves, surfaces and volumes.
- All the functions are well documented. Read the help if needed and check out The NURBS book.
- The NURBS structure can be used directly in GeoPDEs.
- Remember: it works with homogeneous (weighted) coordinates.

A list of all the functions in the toolbox is available online.

`https://octave.sourceforge.io/nurbs/`
`http://rafavzqz.github.io/geopdes/`

L. Piegl and W. Tiller. *The NURBS book*. Springer Science & Business Media, 2012.

Outline

A Simple Example: $\Delta u = f$ on a Quarter Ring

```
geo = geo_load ('ring_refined.mat');

knots = geo.nurbs.knots;
[qn, qw] = msh_set_quad_nodes (knots, msh_gauss_nodes (geo.nurbs.order));
msh = msh_cartesian (knots, qn, qw, geo);

space  = sp_nurbs (geo.nurbs, msh);

mat = op_gradu_gradv_tp (space, space, msh);
rhs = op_f_v_tp (space, msh,
    @(x, y) (8-9*sqrt(x.^2+y.^2)).*sin(2*atan(y./x))./(x.^2+y.^2));

drchlt_dofs = [];
for iside = 1:4
  drchlt_dofs = union (drchlt_dofs, space.boundary(iside).dofs);
end
int_dofs = setdiff (1:space.ndof, drchlt_dofs);

u = zeros (space.ndof, 1);
u(int_dofs) = mat(int_dofs, int_dofs) \ rhs(int_dofs);

sp_to_vtk (u, space, geo, [20 20], 'laplace_solution.vts', 'u')
err = sp_l2_error (space, msh, u,
    @(x,y)(x.^2+y.^2-3*sqrt(x.^2+y.^2)+2).*sin(2.*atan(y./x)))
```

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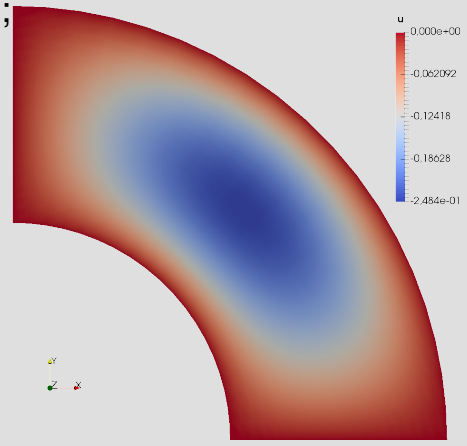

IGA in an Abstract Framework

- Problem to solve at the continuous level.

Abstract Framework:

$$a(u, v) = (f, v) \qquad \forall v \in V$$

Simple Example: Poisson Equation

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f \cdot v \qquad \forall v \in H_0^1(\Omega)$$

IGA in an Abstract Framework

- Problem to solve at the continuous level.
- Parametric domain and parametrisation of the physical domain.

Abstract Framework:

$$\mathbf{F} : \widehat{\Omega} \to \Omega, \ \mathbf{F} \text{ known and computable}$$

Simple Example: Poisson Equation

$$\widehat{\Omega} = [0, 1]^d \text{ and } \mathbf{F} \text{ is a NURBS}$$

IGA in an Abstract Framework

- Problem to solve at the continuous level.
- Parametric domain and parametrisation of the physical domain.
- Discrete problem and spaces in the parametric and physical domain.

Abstract Framework:

$V_h = \{v_h : \iota(v_h) = \widehat{v}_h \in \widehat{V}_h\}$, where ι is a pull-back depending on $\mathbf{F}$ and $\widehat{V}_h = \text{span}\{\widehat{v}_j\}_{j=1}^{N_h}$ is a finite-dimensional and computable space

Simple Example: Poisson Equation

$$V_h = \{v_h : v_h \circ \mathbf{F} = \widehat{v}_h \in \widehat{V}_h\}, \text{ with } \widehat{V}_h = \text{span}\{R_j\}_{j=1}^{N_h} \text{ a space of NURBS}$$

IGA in an Abstract Framework

- Problem to solve at the continuous level.
- Parametric domain and parametrisation of the physical domain.
- Discrete problem and spaces in the parametric and physical domain.
- Construct and solve a linear system to find the discrete solution.

Abstract Framework:

$$u_h = \sum_{i=1}^{N_h} \alpha_i v_i \qquad \sum_{i=1}^{N_h} \alpha_i a(v_i, v_j) = (f, v_j),\ j = 1, \dots, N_h$$

Simple Example: Poisson Equation

$$u_h = \sum_{i=1}^{N_h} \alpha_i R_i \qquad \sum_{i=1}^{N_h} \alpha_i \int_\Omega \nabla R_i \cdot \nabla R_j = \int_\Omega f R_j,\ j = 1, \dots, N_h$$

IGA in an Abstract Framework

To numerically compute the integrals, we define a partition $\widehat{\Omega} = \cup_{k=1}^{N_e} \widehat{K}_k$, and on each element $\widehat{K}_k$ a quadrature rule $\{(\widehat{\mathbf{x}}_{l,k}, w_{l,k})\}_{l=1}^{n_k}$

$$\int_{\Omega} \phi(\mathbf{x}) \approx \sum_{k=1}^{N_e} \sum_{l=1}^{n_k} w_{l,k} \phi(\mathbf{F}(\widehat{\mathbf{x}}_{l,k})) \left|\det(D\mathbf{F}(\widehat{\mathbf{x}}_{l,k}))\right|$$

IGA in an Abstract Framework

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$$\int_{\Omega} \phi(\mathbf{x}) \approx \sum_{k=1}^{N_e} \sum_{l=1}^{n_k} w_{l,k} \phi(\mathbf{F}(\widehat{\mathbf{x}}_{l,k})) |\det(D\mathbf{F}(\widehat{\mathbf{x}}_{l,k}))|$$

We compute the stiffness matrix as

$$A_{ij} \approx \sum_{k=1}^{N_e} \sum_{l=1}^{n_k} w_{l,k} \nabla v_j(\mathbf{F}(\widehat{\mathbf{x}}_{l,k})) \cdot \nabla v_i(\mathbf{F}(\widehat{\mathbf{x}}_{l,k})) |\det(D\mathbf{F}(\widehat{\mathbf{x}}_{l,k}))|$$

and recall that $\nabla v(\mathbf{F}(\widehat{\mathbf{x}}_{l,k})) = D\mathbf{F}^{-\top} \widehat{\nabla} \widehat{v}_i(\widehat{\mathbf{x}}_{l,k}))$.

IGA in an Abstract Framework

Summarising, the ingredients we need are:

- A partition of $\widehat{\Omega}$ and a quadrature rule (nodes and weights)
- The evaluation of **F** and the Jacobian $D\mathbf{F}$ at the quadrature points
- Value of the shape functions at the "mapped" quadrature points.
- A routine to put everything together and compute the matrices.

GeoPDEs: Main Structure

GeoPDEs has been implemented following the abstract framework.

The code is based on three main structures:

- Geometry: the parametrization **F** and its derivatives.
- Mesh: the partition of the domain and the quadrature rule.
- Space: the shape functions of the discrete space V_h.

Implementation is based on Matblab/Octave classes.

Only 1D information is stored to reduce memory consumption. Matrix assembly routines exploit tensor product structure to be efficient.

`geometry` Structure

`geo_load`: reads a NURBS geometry and returns a geometry structure.

Computation of the parametrisation **F** and its derivatives.

- **`map`**: function handle to compute **F** at given points in $\widehat{\Omega}$
- **`map_der`**: function handle to compute the Jacobian of **F**
- **`map_der2`**: (optional) function handle to compute the 2^{nd} derivatives of **F**

The NURBS Toolbox is used for the evaluation.

The computation of the geometry is separated from the shape functions.

- Necessary for non-isoparametric discretisations
- Geometry evaluations can be made in the coarsest given geometry

`msh_cartesian` Class

```
knots = geo.nurbs.knots;
[qn, qw] = msh_set_quad_nodes (knots, msh_gauss_nodes (geo.nurbs.order));
msh = msh_cartesian (knots, qn, qw, geo);
```

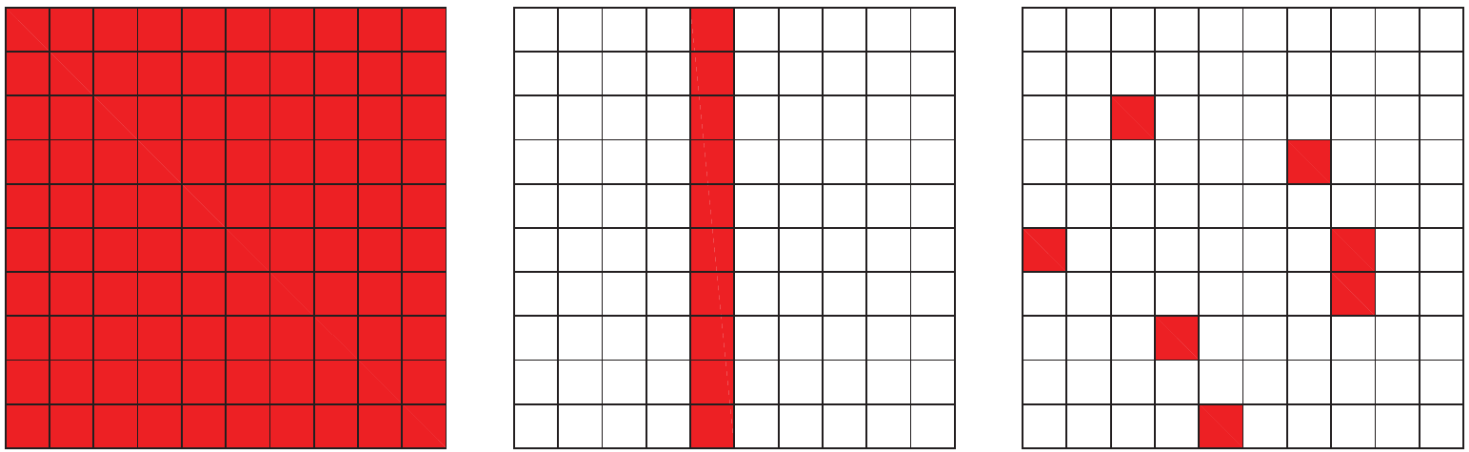
- Collect the knots of the geometry
- Set the quadrature nodes and weights in each parametric direction
- Call the **`msh_cartesian`** constructor

`msh_cartesian` Class Properties

Name	Type	Size	Description
`ndim`	scalar	1×1	Dimension of the parametric domain
`rdim`	scalar	1×1	Dimension of the physical space in which the domain is embedded
`nel`	scalar	1×1	Number of elements of the mesh
`nel_dir`	array	1×ndim	Number of elements in each parametric direction
`nqn`	scalar	1×1	Number of quadrature points per element
`nqn_dir`	array	1×ndim	Number of quadrature points per element, in each parametric direction
`qn`	cell array	1×ndim (nqn_dir×nel_dir)	Coordinates of the quadrature nodes in each parametric direction
`qw`	cell array	1×ndim (nqn_dir×nel_dir)	Quadrature weights in each parametric direction
`breaks`	cell array	1×ndim	Breakpoints that determine the elements, in each parametric direction
`boundary`	**`msh_cartesian`** array	1×(2*ndim)	A mesh object for each boundary side

`msh_cartesian` Methods

- **`msh_precompute`**: compute the quadrature rule, the parametrization **F** and its derivatives, for all quadrature points and all elements in the grid.
- **`msh_evaluate_col`**: compute the same quantities in one "column" of the mesh, i.e., fixing the element number in the first parametric direction.
- **`msh_evaluate_element_list`**: compute the same quantities in a given list of elements.



The output of these methods is a structure (see **`ex07_mesh.m`**).

`sp_scalar` Class

The `nurbs` field in the geometry structure already contains the necessary information to define the space in the parametric domain: the knot vector, the degree and the weights.

The **`sp_scalar`** class provides two constructors: **`sp_nurbs`** and **`sp_bspline`**.

```
1 space  = sp_nurbs (geo.nurbs, msh);
```

NURBS weights are internally handled.

Only univariate information are stored.

`sp_scalar` Class Properties

Name	Type	Size	Description
`space type`	string		Either spline or NURBS
`knots`	cell array	1×ndim	Knot vector in each parametric direction
`degree`	matrix	1×ndim	Degree in each parametric direction
`weights`	NDArray	ndof dir	Weight associated to each basis function, (only NURBS spaces)
`ndof`	scalar	1 × 1	Total number of degrees of freedom
`ndof_dir`	matrix	1×ndim	Number of degrees of freedom in each parametric direction
`nsh_max`	scalar	1 × 1	Maximum number of non-vanishing functions per element
`nsh_dir`	matrix	1×ndim	Maximum number of non-vanishing functions per element in each parametric direction
`sp_univ`	space struct	1×ndim	Univariate spline spaces in each parametric direction, with basis functions evaluated at points given by **`msh.qn`**
`transform`	string		Either *grad-preserving* or *integral-preserving*
`constructor`	function handle		Function handle to generate the same discrete space in a different Cartesian grid
`boundary`	**`sp_scalar`** array	1×(2*ndim)	A space object for each boundary side. Empty for *integral-preserving* transformation
Properties for boundary spaces			
`dofs`	array	1×ndof	Global numbering of the functions on each boundary. Empty for *integral-preserving* transformation.

`sp_scalar` Class Properties

- Due to the weights, the multivariate NURBS space is not the tensor product of univariate NURBS spaces. Thus, we store univariate spline spaces (**`sp_univ`**), plus the coefficients w_i (**`weights`**).

- It is possible to define also a mapping that preserves integrals

$$V_h = \left\{ v_h = \frac{1}{|D\mathbf{F}|} \left(\widehat{v}_h \circ \mathbf{F}^{-1} \right), \ \widehat{v}_h \in \widehat{V}_h \right\}$$

- Analogous methods to the mesh class.

Matrix and Vector Assembly

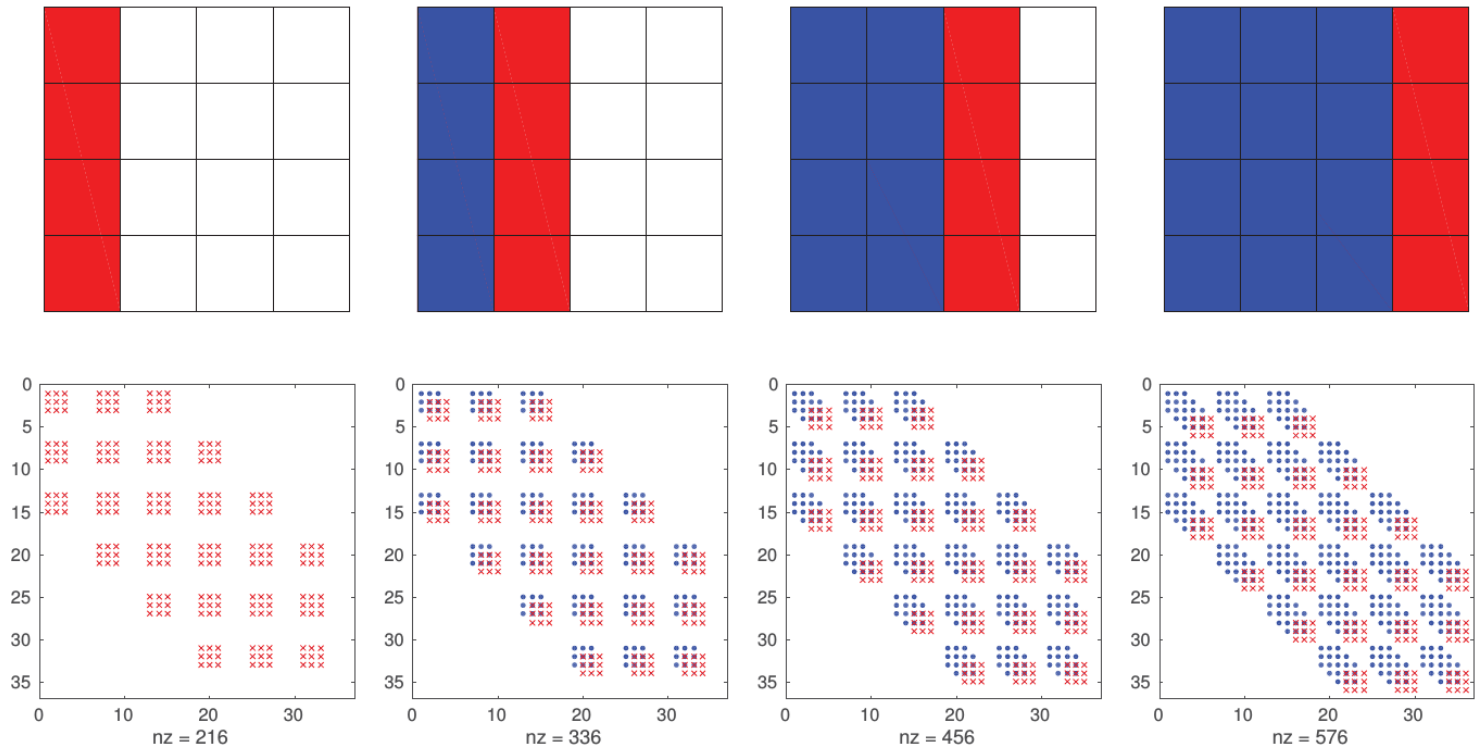
```
1 mat = op_gradu_gradv_tp (space, space, msh);
  rhs = op_f_v_tp (space, msh,
3     @(x, y) (8-9*sqrt(x.^2+y.^2)).*sin(2*atan(y./x))./(x.^2+y.^2));
```

Matrix and Vector Assembly

```
mat = op_gradu_gradv_tp (space, space, msh);
rhs = op_f_v_tp (space, msh,
    @(x, y) (8-9*sqrt(x.^2+y.^2)).*sin(2*atan(y./x))./(x.^2+y.^2));
```

```
function A = op_gradu_gradv_tp (space1, space2, msh, coeff)
A = sparse (space2.ndof, space1.ndof);
for iel = 1:msh.nel_dir(1)
  msh_col = msh_evaluate_col (msh, iel);
  sp1_col = sp_evaluate_col (space1, msh_col, 'value', false, 'gradient', true);
  sp2_col = sp_evaluate_col (space2, msh_col, 'value', false, 'gradient', true);
  for idim = 1:msh.rdim
    x{idim} = reshape (msh_col.geo_map(idim,:,:), msh_col.nqn, msh_col.nel);
  end
  A = A + op_gradu_gradv (sp1_col, sp2_col, msh_col, coeff (x{:}));
end
```

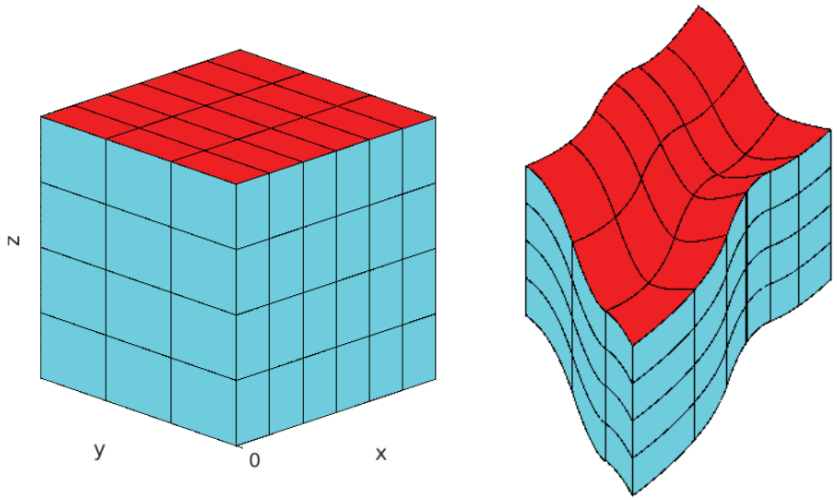
Matrix and Vector Assembly



- Low level assembly in `oct` files (C++) for efficiency
- To improve efficiency it would be possible to:
  - Assemble matrices and rhs at the same time
  - Exploit symmetry (if trial and test space are the same)

Boundary Objects

- $\mathbf{F}_\Gamma : \widehat{\Gamma} \to \Gamma$ restriction of $\mathbf{F}$ to one side
- Restriction of the space on the boundary, i.e. trace, by considering the relevant 1D knot vectors



- **`msh_cartesian.boundary`** and **`sp_scalar.boundary`** are arrays of objects of the same class
- **`sp_scalar.boundary.dofs`**: local-to-global numbering

N.B. Boundary objects contain only boundary info. Quantities requiring volumetric information (e.g. normal derivatives) cannot be computed using only boundary structures.

Imposing Boundary Conditions

Non-Homogeneous Neumann Boundary Condition

$$g_i = \int_{\Gamma_N} g_N R_i$$

We exploit the same operators on the boundary objects:

```
for side = nmnn_side
  dofs = space.boundary(side).dofs;
  rhs(dofs) = rhs(dofs) + op_f_v_tp (space.boundary(side), msh.boundary(side), g);
end
```

Imposing Boundary Conditions

Non-Homogeneous Dirichlet Boundary Condition

Lifting $\tilde{u}_h$ such that $\tilde{u}_h|_{\Gamma_D} = g_D$ (ends up on the rhs).

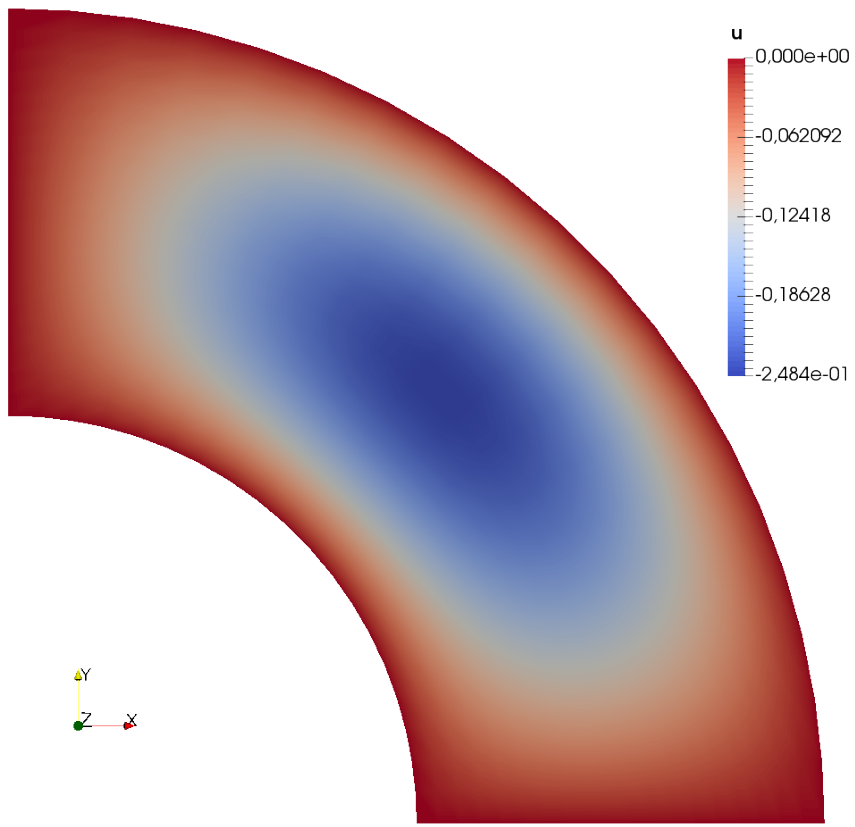
We use L^2 projection to impose the condition weakly, i.e. find $\tilde{u}_h \in V_{h,\Gamma_D}$ s.t.

$$\int_{\Gamma_N} \tilde{u}_h v_h = \int_{\Gamma_N} g_D v_h \qquad \forall v_h \in V_{h,\Gamma_D}$$

```
u = zeros (space.ndof, 1);
drchlt_dofs = [];
for side = drchlt_sides
  side_dofs = space.boundary(side).dofs;
  drchlt_dofs = union (drchlt_dofs, side_dofs);
  M(side_dofs, side_dofs) = M(side_dofs, side_dofs)
    + op_uv_tp (space.boundary(side), space.boundary(side), msh.boundary(side),
     f_one) ;
  rhs_drchlt(side_dofs) = rhs_drchlt(side_dofs)
    + op_fv_tp (space.boundary(side), msh.boundary(side), g);
end
u(drchlt_dofs) = M(drchlt_dofs,drchlt_dofs) \ rhs_drchlt(drchlt_dofs);
int dofs = setdiff (1:space.ndof, drchlt_dofs);
rhs(int_dofs) = rhs(int_dofs) - mat(int_dofs,drchlt_dofs) * u(drchlt_dofs);
```

Solve the Linear System

```
1 u(int_dofs) = mat(int_dofs,int_dofs) \ rhs(int_dofs);
```



Vector Field Spaces

- Define a NURBS space for each component in the reference domain:

$$\left\{ \begin{bmatrix} \widehat{N}_i \\ 0 \end{bmatrix} \right\}_{i=1}^{N} \bigcup \left\{ \begin{bmatrix} 0 \\ \widehat{N}_i \end{bmatrix} \right\}_{i=1}^{N}$$

- Map to the physical space with an appropriate transformation

$$\mathbf{v}_h = \widehat{v}_h \circ \mathbf{F}^{-1} \qquad \text{grad-preserving}$$

$$\mathbf{v}_h = \left(D\mathbf{F}^{+}\right)^{\top} \left(\widehat{\mathbf{v}}_h \circ \mathbf{F}^{-1}\right) \qquad \text{curl-preserving}$$

$$\mathbf{v}_h = \frac{1}{|D\mathbf{F}|} D\mathbf{F} \left(\widehat{\mathbf{v}}_h \circ \mathbf{F}^{-1}\right) \qquad \text{div-preserving}$$

`sp_vector` Class

```
space_scalar = sp_nurbs (geometry.nurbs, msh);
for idim = 1:msh.rdim
  scalar_spaces{idim} = space_scalar;
end
space = sp_vector (scalar_spaces, msh, 'grad-preserving');
```

- Properties and methods analogous to the scalar case
- Boundary spaces obtained by the natural trace operators induced by each transformation:
  - grad-preserving: $\gamma \mathbf{v} = \mathbf{v}$
  - curl-preserving: $\gamma_{\mathbf{t}} \mathbf{v} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$. Boundary is a curl-preserving **`sp_vector`** space with only tangential components
  - div-preserving: $\gamma_{\mathbf{n}} \mathbf{v} = \mathbf{v} \cdot \mathbf{n}$. Boundary its a **`sp_scalar`** class with integral preserving transformation

Maxwell Eigenvalue Problem

Find $\mathbf{E}_h \in V_{0,h}$ and $\lambda_h \in \mathbb{R}$ s.t.

$$\int_\Omega \nabla \times \mathbf{E}_h \cdot \nabla \times \mathbf{v}_h = \lambda_h \varepsilon_0 \mu_0 \int_\Omega \mathbf{E}_h \cdot \mathbf{v}_h \qquad \forall \mathbf{v}_h \in V_{0,h}$$

After discretisation we obtain a generalised eigenvalue problem:

$$\mathbf{Ke} = \lambda \mathbf{Me}$$

with **e** the electric field DoFs, **K** and **M** the curl-curl and mass matrices

Maxwell Eigenvalue Problem

```
mu0  = 1;
eps0 = 1;
degree     = [2 2 2];
regularity = degree - 1;
nsub       = [5 5 5];
nquad      = degree + 1;
% Create a Unit Cube and load the Geometry
cube = nrbextrude (nrb4surf ([0 0], [0 1], [1 0], [1 1]), [0 0 1]);
geo = geo_load (cube);
% Define Hcurl conforming knot vectors
[knots, zeta] = kntrefine (geo.nurbs.knots, nsub, degree, regularity);
[knots_hcurl, degree_hcurl] = knt_derham (knots, degree, 'Hcurl');
% Construct the Mesh
rule = msh_gauss_nodes (nquad);
[qn, qw] = msh_set_quad_nodes (zeta, rule);
msh = msh_cartesian (zeta, qn, qw, geo);
% Construct the Space
scalar_spaces = cell (msh.ndim, 1);
for idim = 1:msh.ndim
  scalar_spaces{idim} = sp_bspline (knots_hcurl{idim}, degree_hcurl{idim}, msh);
end
space = sp_vector (scalar_spaces, msh, 'curl-preserving');
```

Maxwell Eigenvalue Problem

```
% Assemble the matrices
K = op_curlu_curlv_tp (space, space, msh);
M = op_u_v_tp (space, space, msh, @(x,y) mu0*eps0*ones (size(x)));

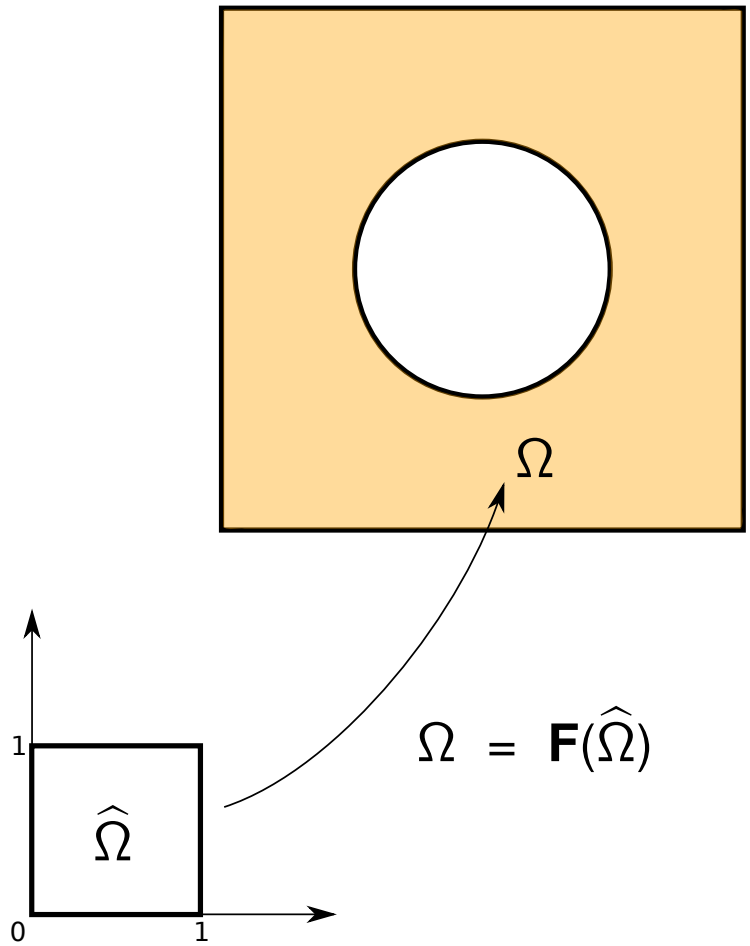
% Apply PEC Boundary Conditions
drchlt_dofs = [];
for iside = 1:numel (space.boundary)
  drchlt_dofs = union (drchlt_dofs, space.boundary(iside).dofs);
end
int_dofs = setdiff (1:space.ndof, drchlt_dofs);

% Solve the Eigenvalue Problem
eigf = zeros (space.ndof, numel(int_dofs));
[eigf(int_dofs, :), lambda] = eig (full (K(int_dofs, int_dofs)), full (M(int_dofs,
     int_dofs)));
lambda = diag (lambda);
nzeros = numel (find (lambda < 1e-10));

fprintf('First Computed Eigenvalues:\n')
disp (lambda(nzeros+1:nzeros+17))
```

Multipatch Setting

Not all geometries can be represented as a regular transformation of $[0, 1]^d$

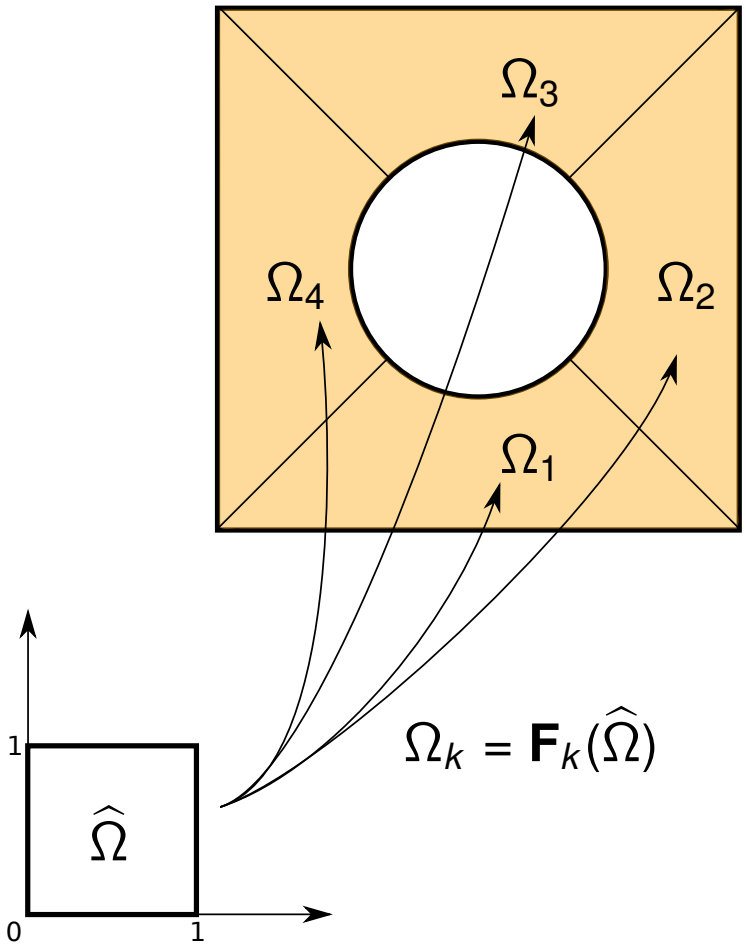


Multipatch Setting

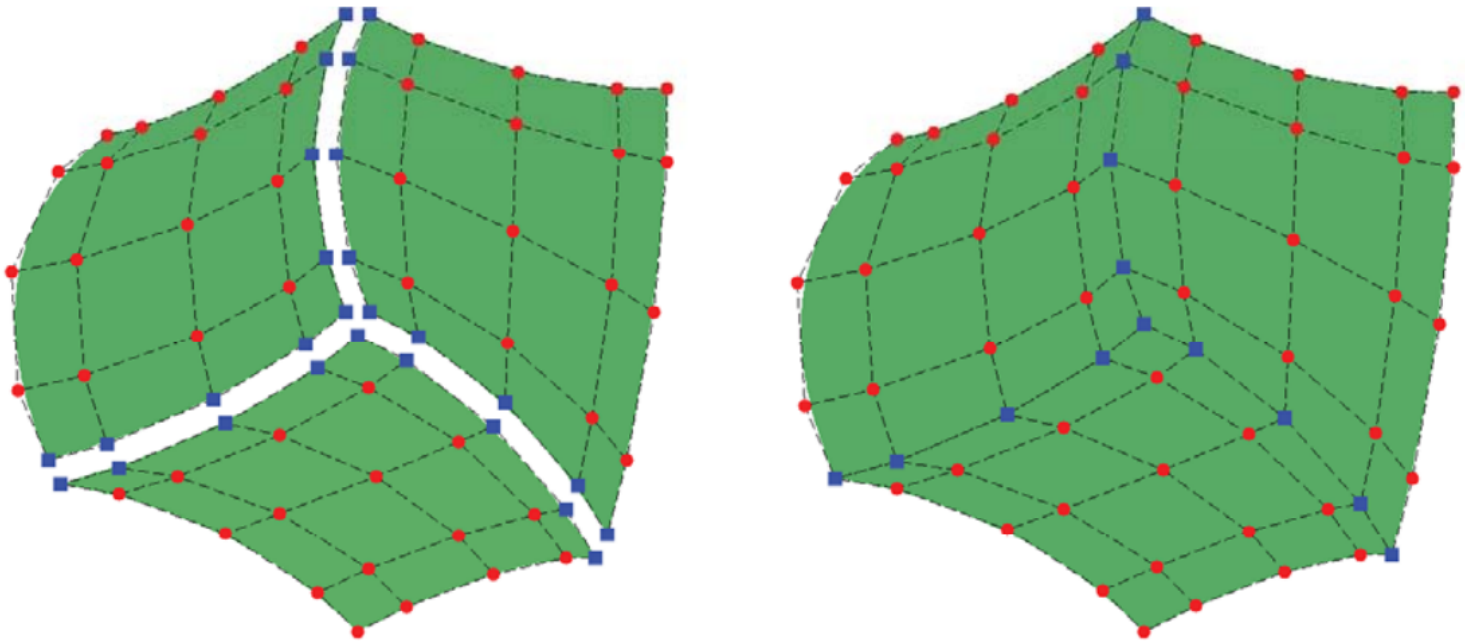
Not all geometries can be represented as a regular transformation of $[0,1]^d$

Multipatch:

- Subdivide the domain in k hexahedral patches
- Each $\Omega_k = \mathbf{F}_k(\widehat{\Omega})$
- Matching discretisation at the interfaces
- One-to-one matching of DoFs

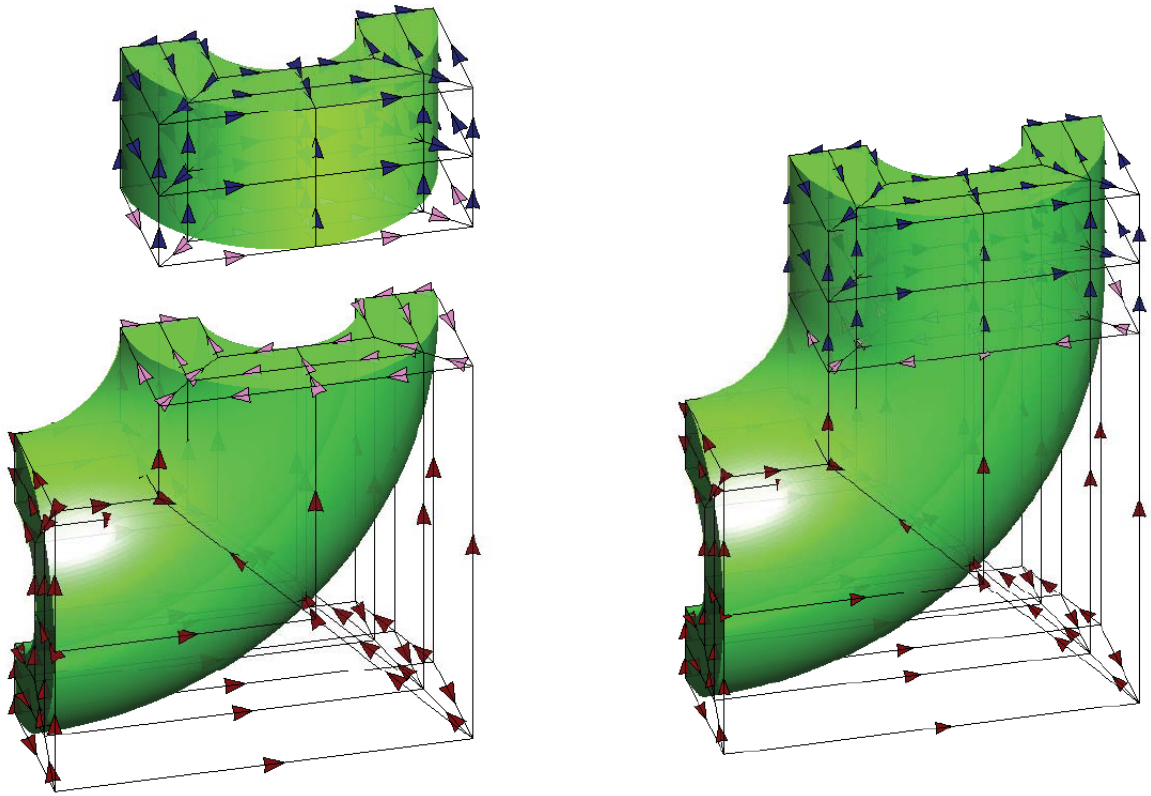


Multipatch Setting - Scalar Spaces



- Conforming meshes across patches
- One-to-one correspondance of boundary functions
- Enforce continuity $V_h = \{v \in C^0 : v|_{\Omega_i} \in V_h^{(i)},\ i = 0, \dots, N_{\text{ptc}}\}$ to obtain a discrete subspace of $H^1(\Omega)$
- Matching is enforced using a local-to-global indexing

Multipatch Setting - Vector Spaces



- Glue only tangential components for $H(\mathrm{curl}, \Omega)$
- Glue only normal components for $H(\mathrm{div}, \Omega)$
- Orientation needs to be taken into account

Multipatch in GeoPDEs

- **`nrbmultipatch`**: given an array of NURBS entities, computes the information about interfaces and boundaries

```
[interfaces, boundary] = nrbmultipatch (nurbs);
```

- **`msh_multipatch`**: analogous to the single patch case. It stores
  - number of patches
  - total number of elements
  - meshes associated to each patch (**`msh_cartesian`** objects)
- **`sp_multipatch`**: analogous to the single patch case. It stores
  - total number of basis functions
  - space objects for each patch
  - **`gnum`** cell array, relates the local numbering to the global one
  - **`dofs_ornt`** contains the orientation information

```
msh   = msh_multipatch (local_meshes, boundaries);
space = sp_multipatch (local_spaces, msh, interfaces, boundary_interfaces);
```

- Boundary of multipatch entities are multipatch entities

Available Operators

- Methods present in both **sp_scalar** and **sp_vector**:
  op_f_v_tp , **op_gradu_gradv_tp**, **op_u_v_tp**
- Methods only present in the class **sp_scalar**:
  op_gradgradu_gradgradv_tp, **op_laplaceu_laplacev_tp**, **op_mat_stab_SUPG_tp**, **op_rhs_stab_SUPG_tp**, **op_vel_dot_gradu_v_tp**
- Methods only present in the class **sp_vector**:
  op_curlu_curlv_tp , **op_curlu_v_tp** , **op_div_v_q_tp**, **op_su_ev_tp**, **op_v_gradp_tp**
- Methods present in the class **sp_multipatch**:
  op_f_v_mp, **op_gradu_gradv_mp**, **op_u_v_mp**, **op_curlu_curlv_mp**, **op_div_v_q_mp**, **op_su_ev_mp**, **op_v_gradp_mp**

Conclusions

GeoPDEs is an open source and free Matlab implementation of IGA.

- Particularly useful for teaching purposes and for new researchers
- Can serve as a rapid prototyping tool to test new ideas
- Different problems are already implemented
- Many examples and a technical report with detailed explanations

As of v.3.1.0 Hierarchical refinement is available!

Software download and information: http://rafavzqz.github.io/geopdes/

Mailing list: geopdes-users@lists.sourceforge.net

Useful References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J.A. Cottrell, T.J.R. Hughes, and Y. Bazilevs, *Isogeometric analysis: Toward integration of CAD and FEA*, Wiley, 2009.


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L. Piegl and W. Tiller, *The NURBS book*, 2 ed., Springer, 1997.


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