Test Flight Problem Set

Question 1

Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

Let m and n take the values $1, 2, 3, 4, \cdots$ and consider the following equations

$$\begin{aligned} 3(1) + 5(1) &= 8, \\ 3(1) + 5(2) &= 13, \\ 3(2) + 5(1) &= 11, \\ 3(2) + 5(2) &= 16, \\ 3(3) + 5(1) &= 13. \end{aligned}$$

Clearly, $\neg(\exists m \in \mathcal{N})\neg(\exists n \in \mathcal{N})[3m + 5n = 12]$.

Therefore, the statement $3m + 5n = 12$ is false.

Question 2

Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Let n be the smallest integer and consider the numbers

$$n, n + 1, n + 2, n + 3, n + 4.$$

Summing these integers we get

$$\begin{aligned} n + n + 1 + n + 2 + n + 3 + n + 4 &= 5n + 10 \\ &= 5(n + 2). \end{aligned}$$

But $(n + 2) \in \mathbb{Z} \quad \forall\, n \in \mathbb{Z}$ and $5(n + 2)$ is divisible by 5. Hence, for any $n \in \mathbb{Z}$, the sum of any consecutive integers is divisible by 5. Thus, the statement is true.

Question 3

Say whether the following is true or false and support your answer by a proof: For any integer n,the number $n^2 + n + 1$ is odd.

By definition, a number m is odd if it can be expressed in the form $m = 2k + 1$ with $k \in \mathbb{Z}$.

Let

$$\begin{aligned} m &= n^2 + n + 1 \\ &= n(n+1) + 1. \end{aligned}$$

For any n, t=a number $n(n+1)$ is even since it is a product of an odd and even number. Hence, $m = n^2 + n + 1$ is odd.

Question 4

We prove that every odd natural number is of one of the forms $4n + 1 or 4n + 3$, where n is an integer.

n has the form, $n = 2k + 1$, $\forall k \in \mathbb{Z}$. We consider the cases when k is odd or even.

Case 1: For k odd, $\exists q \in \mathbb{Z}$ so that

$$\begin{aligned} n &= 2(2q + 1) + 1 \\ &= 4q + 3. \end{aligned}$$

Case 2: For k even, $\exists q \in \mathbb{Z}$, such that $k = 2q$. Thus

$$\begin{aligned} n &= 2(2q) + 1 \\ &= 4q + 1. \end{aligned}$$

From this argument, we conclude that any odd number has the form $4n + 1$ or $4n + 3$.

Question 5

We prove that for any integer n, at least one of the integers $n, n + 2, n + 4$ is divisible by 3.

Proof. Clearly, if an integer n is divisible by 3, it leaves a remainder of $r \in \{0, 1, 2\}$.

We consider three cases.

Case 1: For $r = 0$. That is

$$n \equiv (0 \mod 3).$$

Hence n is divisible by 3.

Cases 2: for $r = 1$.

$$n \equiv (1 \mod 3)$$

Add 2 both side to obtain

$$n + 2 \equiv (3 \mod 3) \equiv (0 \mod 3).$$

Hence, $n + 2$ is divisible by 3.

Case 3: for $r = 2$. we have that

$$n \equiv (2 \mod 3).$$

Adding 4 both sides, we get

$$n + 4 \equiv (6 \mod 3) \equiv (0 \mod 3)$$

Hence, $n + 4$ is divisible by 3.

From the three cases above, we conclude that for any $n \in \mathbb{Z}$, at least one of $n, n+2, n+4$ is divisible by 3. □

Question 6

A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes',pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof. In question 5, we showed that for any $n \in \mathbb{Z}$, $n, n+2, n+4$ is divisible by 3. Now, any prime triple can be expressed in the form

$$n, n+2, n+4 \text{ with } n > 1.$$

Thus, at least one of them is divisible by 3. To the contrary, suppose 3, 5, 7 are not the only triple, then

$$n, n+2, n+4$$

are prime numbers with $n > 3$. Since n is prime, it is not a multiple of 3.

Further, $n+2$ and $n+4$ will be multiples of 3 and so, not prime. This leads to a contradiction, hence $2, 3, 5$ is the only triple possible. □

Question 7

We prove that for any natural number n

$$2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2 \tag{1}$$

Proof. We prove by induction. Let $n = 1$, then

$$2^1 = 2^2 - 2 = 2.$$

Hence, the statement is true for $n = 1$. Assume that the statement holds for any $n \geq 1$, then it should also be true for $n + 1$. Thus we deduce that

$$2 + 2^2 + 2^3 + \cdots + 2^{n+1} = 2(2^{n+1} - 1)$$

To this end, add 2^{n+1} to both side of equation (1), and simplify the LHS, we have

$$
\begin{aligned}
2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} &= 2 + 2^2 + 2^3 + \cdots + 2^n + 2 \times 2^n \\
&= 2^{n+1} - 2 + 2 \times 2^{n+1} \\
&= 2 \times 2^{n+1} - 2 \\
&= 2(2^{n+1} - 1).
\end{aligned}
$$

Hence, the statement holds for $n + 1$. Therefore, by induction, statement holds. $\square$

Question 8

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ ends to the limit ML.

Proof. By definition, a sequence $\{a_n\}_{n=1}^{\infty}$ is said to converges to the limit L, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - L| < \varepsilon.$$

Choosing $\varepsilon = \frac{\varepsilon}{M}$ with $M > 0$. Since ML is the limit of $\{Ma_n\}_{n=1}^{\infty}$, we have that

$$
\begin{aligned}
|Ma_n - ML| &= |M(a_n - L)| \\
&= M|a_n - L| \\
&< M\frac{\varepsilon}{M} \\
&= \varepsilon.
\end{aligned}
$$

Hence, ML is the desired limit of the sequence, $\{Ma_n\}_{n=1}^{\infty}$. $\square$

Question 9

Given an infinite collection $An, n = 1, 2, \cdots$ of intervals of the real line, their intersection is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, \cdots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Example 1. Let $A_1 = (0, 1),\ A_2 = \left(0, \frac{1}{2}\right),\ A_3 = \left(0, \frac{1}{3}\right),\ \cdots, A_n = \left(0, \frac{1}{k}\right)$. Clearly,

$$A_k \subset, \cdots, A_3 \subset A_2 \subset A_1,$$

and

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$

Proof. We prove by contradiction. Suppose on the contrary that

$$\alpha \in \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right).$$

Clearly,

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

By ε-definition of limits, choose $\varepsilon > 0 (\alpha = \varepsilon)$. Then $\therefore$ N such that $n > N$, and

$$\left|\frac{1}{n-0}\right| = \left|\frac{1}{n}\right| < \alpha.$$

Thus, $\forall\ n \in \mathbb{Z},\ 0 < \frac{1}{n} < \alpha$. Thus, $\alpha \notin \left(0, \frac{1}{n}\right)$ which is contradictory to our assumption that

$$\alpha \in \left(0, \frac{1}{n}\right)$$

$$\therefore \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$

□

Question 10

Give an example of a family of intervals $A_n, n = 1, 2, 3, \cdots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Example 2. Let $A_1 = [0, 1),\ A_2 - \left[0, \frac{1}{2}\right), A_3 = \left[0, \frac{1}{3}\right), \cdots, A_n = \left[0, \frac{1}{n}\right).$

Clearly

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

Proof. The proof follows directly from the argument in question 9. This does not lead to a contradiction since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \in \bigcap_{n=1}^{\infty} A_n.$$

□