2 · Why

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2 · Why `eqW` + `recΔ` instead of the old "Boolean-encoding 5-operator" idea?

Criterion	recΔ + eqW (current plan)	"Boolean encoding" (OBool + and/or/not + cond + recursion?)
Termination proof	Easy: δ-height strictly drops; equality witness is 1-step.	Each Boolean op needs its *own* measure; conditional can *increase* size → SN proof messier.
Kernel size	Adds 2 constructors, 4 rules.	Needs at least 4 new constructors (OBool value, `oand`, `onot`, conditional) **plus** a recursion operator anyway.
Confluence risk	Minimal (orthogonal root rules).	Boolean conditionals overlap with merge/annihilation unless carefully ordered.
Gödel plateau	Built-in via `eqW`.	Must still invent an equality test that returns OBool → circles back to witness anyway.
Reviewer perception	"Primitive recursion + structural equality witness" is standard and respectable.	"Re-invented Boolean algebra inside non-Boolean calculus" looks like you smuggled *another* logic rather than eliminated it.
Axiom-free bragging rights	Kept: no truth table, just cancellation + equality witness.	Slightly diluted: OBool is basically a truth-table type; reviewers may call it a Boolean axiom in disguise.

Bottom line: **recΔ + eqW hits the sweet-spot**—smallest patch that removes Lean Bool/DecEq/Nat dependence *and* unlocks Σ_1 recursion.

1 · Zero-rewrite path: keep files, patch in place

Phase	File(s) touched	What you do	LOC delta
P1 (drop external Bool)	`Basic.lean` `Normalizer.lean`	❶ Introduce `eqW` constructor + one root-step rule. ❷ Replace every `if … then … else …` and `beq` in `normalize` / helpers with a *pattern match* on `eqW` result (`void` vs `integrate …`).	≈+40
P2 (add δ-recursor)	`Arithmetic.lean` (new) `Normalizer.lean`	❶ Add `recΔ` constructor + two rewrite rules. ❷ Re-encode `add` , `mul` , bounded search loops through `recΔ` instead of Nat recursion. ❸ Extend lex-measure with "δ-height" component (3 lines in measure definition, 1 lemma proof).	≈+80
P3 (purge DecidableEq/Nat in diagonal)	`Diagonal.lean`	❶ Replace `DecidableEq` code compare with `eqW` plateau test. ❷ Drop meta-level `Nat` counter for plateau; reuse `recΔ` with δ-budget.	–20
P4 (finish Bool-out)	grep-replace across repo	❶ Remove `open classical` . ❷ Delete `by_cases` , `decide` . ❸ Add tiny helper `oboolCond` built from merge & cancellation if really needed (rare).	–
P5 (update tests / CI)	`lakefile.toml` `test/*.lean`	❶ Add static audit: `grep -R "Bool|DecidableEq|Nat "` (spaces) fails CI. ❷ Add quick random-trace confluence harness using new rules.	+15

Everything else (proof-predicate skeleton, substitution predicate, confluence proof outline) stays structurally identical—only their *dependencies* swap from Lean mathlib Booleans to your own equality witness.

3 · Refactor tip-sheet (Lean specifics)

3522. **Constructor injection**
3523. **One-step rules added to `normalizeCore`**
3524. **Measure extension** (pseudo):
3525. **Replace every `DecidableEq`**
3526. **Bounded search with `recΔ`**

4 · Checklist to declare "Boolean/Numeral free" ✅

4957. `grep -R " Bool "` → empty.
4958. `grep -R "DecidableEq"` → empty.
4959. `grep -R " Nat "` → allowed **only** inside separate `TerminationProof.lean` (meta).
4960. `open classical` gone everywhere except the one termination proof file (where it is harmless meta).
4961. CI job runs `lake build`, `lake exe confluence-test`, `lake exe sn-test`.

4 Do you need to start over?

No. You can **refactor the existing Lean scripts**:

4169. **Introduce two new constructors** (`eqW`, `recΔ`) and 4 rewrite rules.
4170. **Replace every Lean `Bool` / `Nat` usage** with operator forms (`void`, `delta` -height, `eqW` plateaus).
4171. **Re-prove strong normalization** with a lex measure extended by δ-height; the old β/annihilation proofs stay intact.
4172. **Patch diagonal & provability** to use `eqW` witness and `recΔ` -bounded search.
4173. Run a CI script that fails on any `Bool`, `Nat`, `DecidableEq`, `open classical`, or `axiom`.

That is incremental, not a rewrite-from-scratch.

5 Boolean-encoding alternative?

The older idea—add a whole OBool algebra—works but **bulks the kernel**: you end up with four–six extra constructors (true, false, and, not, cond) *plus* some recursion anyway. Reviewers will say you merely re-invented Booleans rather than eliminated them. `eqW + recΔ` keeps the

kernel minimal and aligns with existing operator semantics, while still allowing an internal equality test essential for Gödel.

Macros (erased before final kernel checking) may exist for developer ergonomics, e.g. `numeral`, `add`, `mul`, variable encodings, quoting, etc. They must expand to pure `Trace` terms using the six constructors only.

3 · Refactor tip-sheet (Lean specifics)

Constructor injection

```
inductive Trace where
| void
| delta : Trace → Trace
| integrate : Trace → Trace
| merge : Trace → Trace → Trace
| eqW : Trace → Trace → Trace -- NEW
| recΔ : Trace → Trace → Trace → Trace -- NEW
| … -- keep var/lam/app/tag/pair as-is
```

Constructor injection

```
inductive Trace where
| void
| delta : Trace → Trace
| integrate : Trace → Trace
| merge : Trace → Trace → Trace
| eqW : Trace → Trace → Trace -- NEW
| recΔ : Trace → Trace → Trace → Trace -- NEW
| … -- keep var/lam/app/tag/pair as-is
```

One-step rules added to `normalizeCore`

```
| eqW a b =>
let na := normalizeCore a
let nb := normalizeCore b
match structuralCompare na nb with
| .same => Trace.void
| .diff => Trace.integrate (Trace.merge na nb)
```

```
| recΔ b s Trace.void => normalizeCore b
| recΔ b s (Trace.delta n) =>
let r := normalizeCore (Trace.recΔ b s n)
normalizeCore (Trace.app (normalizeCore s) r)
```

Measure extension (pseudo):

```
measure t :=
(betaSites t, annSites t, deltaHeight t, size t)
```

Replace every `DecidableEq`

```
-- before
if h : a = b then void else integrate (merge a b)
-- after
Trace.eqW a b
```

Bounded search with recΔ

```
def enumUpTo (B : Trace) (P : Trace → Trace) : Trace :=
Trace.recΔ Trace.void
(Trace.lam (Trace.app (Trace.merge
(Trace.app P (Trace.var 0)) -- merge cancels iff P(n)=void
(Trace.var 1))) ) -- accumulator
B
```

2. Candidate “5th Operator” Archetypes (Evaluation)

Label	Informal Role	Power Gain	Complexity	Risks to Minimality	Solves Which Claims
R (Primitive Recursor)	`R base step n` (unary δ-chain iterator)	High (PR closure)	Moderate	Adds arithmetic bias	C14, C23, strengthens EqNat completeness
Iter (Fold / Catamorphism)	`fold t f acc` general structural fold	Very High (arbitrary structural recursion)	High	May look like cheating ("meta recursion smuggled in")	Wide: substitution, proof enumeration
Mu (Well-Founded Fixpoint)	`mu F seed` reduces while measure drops	High (internal diagonal & derivability)	High	Harder termination proof	Diagonal, D1–D3
EqW (Equality Witness)	`eqW a b → void` iff nf(a)=nf(b)` else canonical non-void	Medium	Low	Focused, simple	Removes external Bool / DecEq reliance
Search (Bounded Existential)	`seek B P` tries P(0..B)	Medium	Low–Med	Narrow semantics	Σ_1 prov existential, provability bounding
Quote (Self Code)	`quote t` returns code trace	Medium	Low	Adds reflection primitive	Cleaner diagonal; still need recursion

Pairing Recursor (PR on pairs)	`R2 base step m n`	High	Moderate	More arity complexity	Encodes multi-arg PR without meta loops

Key insight: You *do not* need several; one *carefully shaped* operator that can *encode the rest* is better than scattering primitives.

3. Recommended Strategy

Pick **ONE of these two designs**:

Option A (Leanest, Safe Minimality): **Equality-Guided Primitive Recursor**

`recΔ`

Purpose: Supply *total primitive recursion on δ-chains***and** internal equality tests via its base case branch outcome—killing external Bool and enabling Σ_1 completeness for all PR predicates you actually need (Prov, SubF, substitution counters, size bounds).

Signature (core calculus extension)

```
css
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recΔ b s n
```

- `b` : base trace (value for 0)
- `s` : step operator expecting current accumulator
- `n` : δ-chain acting as unary numeral

Rewrite Rules (Left → Right)

3623. `recΔ b s void` → `b`

3624. `recΔ b s (delta n)` → `s (recΔ b s n)` (where `s x` is encoded as `app s x` or `merge (tag step) (pair s x)` depending on your encoding choice)

Side Conditions: None (structural). Termination obvious: δ-height decreases.

Measure Impact: Adds a "Δheight(n)" component—strictly drops each step.

Derivable Additions:

- Addition: `add m n := recΔ n (λr. delta r) m`
- Multiplication: `mul m n := recΔ void (λr. add r n) m`
- Equality witness: `eqNatW a b := recΔ (testBase a b) (λr. testStep r a b) (maxDepth a b)` (below I'll show simpler EqW alternative)

Pros:

- Still *arithmetically flavored* but tiny and orthodox (reviewers accept primitive recursion).
- Easy termination & confluence integration (orthogonal critical pairs).

Cons:

- Only handles unary loops—multi-argument primitive recursion requires pairing encodings (acceptable).

Option B (More Dramatic, "Breakthrough Branding"): **Reflective Fixpoint / Search Operator** `μΠ`

Signature:

```
r
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muΠ F seed M
```

- `F` : transformer (expects a code and a candidate)
- `seed` : initial trace
- `M` : measure budget (δ-chain) guaranteeing finitude

Rewrite Scheme:

4826. `muΠ F seed void` → `seed`

4827. `muΠ F seed (delta m)` → `stabilize(seed, F seed, muΠ F (F seed) m)`

Where `stabilize(x, Fx, rest)` reduces to `x` **if** an *internal equality witness* (see EqW below) shows `nf(x)=nf(Fx)`, otherwise `rest`.

Pros:

- Directly internalizes *bounded* fixpoint iteration, diagonal plateau, Löb prerequisites.
- Lets you *drop* several ad hoc enumeration gadgets.

Cons:

- Heavier termination argument (must show each non-stable iteration decrements `M`).
- Risk of reviewer saying you imported a reflective engine disguised as a primitive.

If you pick Option B you *still need* an equality witness primitive or to embed it inside `stabilize`. That effectively makes EqW implicit anyway.

4. Equality Witness Primitive (If NOT Using Recursor First)

Even with Option A you likely still want **EqW** because it surgically removes external Bool / DecEq. It is ultra-cheap:

Operator: `eqW a b`

Rewrite Rules (on fully normalized or as part of normalization):

5857. If pattern-structural match succeeds (constructor by constructor) → `void`.

5858. First mismatch position → `integrate (merge a b)` *(or any canonical non-void marker with both original subtraces embedded for refutational info).*

Implementation Tactic:

- Structural evaluator returns either `void` or `delta marker` but you can *stay purely rewrite* by expanding `eqW` into a one-step decision after recursively normalizing children.
- Confluence unaffected (deterministic, no overlapping left sides—only one root rule once children normalized).

Payoff:

- All prior `beq` usages become `eqW`.
- Diagonal plateau detection internal: test `eqW ψ (F ψ)`.

5. Which Combination Maximizes "Breakthrough" with Minimal Rework?

Combo	New Primitives	Covered Gaps	Risk	Marketing Strength
RecΔ + EqW	2	PR closure, Σ_1 repr., equality, diagonal plateau (still need simple iteration wrapper)	Low	"Minimal arithmetic + intrinsic equality"
μΠ (with baked EqW)	1 (but richer)	Equality, diagonal, search, partial recursion	Medium	"Single reflective fixpoint primitive"
EqW only	1	Equality / Bool removal	Low	Too small (doesn't solve Σ_1 completeness)

Recommended:RecΔ + EqW if you want *credibility first*; **μΠ** only if you're prepared to write a very clean termination + safety section so it doesn't look like a magical oracle.

6. Formal Specs (You Can Paste Into Draft)

6.1 Grammar Extension

```
go
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t ::= ... | recΔ t t t | eqW t t
```

6.2 Measures

Extend measure triple to quadruple: `μ(t) = (βSites(t), annSites(t), δHeight(t), size(t))`

- `recΔ b s n` contributes `δHeight(n)` to third component.
- Rule application strictly decreases third component (or earlier components if inside β or annihilation contexts).

6.3 Critical Pair Audit (New)

Overlap	Reason Safe
`recΔ` inside β redex body	β reduces first; measure: βSites drop.
`recΔ` and annihilation (integrate/delta)	Independent root symbols.
`eqW` vs structural normalizer	`eqW` only fires after children in NF; no overlap.

6.4 Key Lemmas (Lean Skeleton Names)

Lemma	Statement Intent
`recΔ_decrease`	`μ(recΔ b s (delta n)) > μ(s (recΔ b s n))`
`recΔ_total`	Termination & NF existence for all δ-chains.
`eqW_sound`	`eqW a b = void → nf a = nf b`.
`eqW_complete`	`nf a = nf b → eqW a b = void`.
`eqW_irrefl_nonvoid`	`eqW a b ≠ void → nf a ≠ nf b`.
`add_repr`	`add m n ≡ δ^{eval m + eval n} void`.
`mul_repr`	Analogous for multiplication.
`prov_sigma1_full`	Uses `recΔ` to bound enumeration internal.
`diag_fixed_internal`	Uses `eqW` as plateau test instead of external Bool.

7. Impact on Existing Claims

Claim (Cx)	Status After RecΔ + EqW (If Proved)
C16–C18 (freedom claims)	Move from NOT-COHERENT → PROVABLE ONCE Bool purged.
C14 Primitive Recursive Representability	Becomes NOW after addition + proofs.
C23 Σ_1 Completeness (restricted PR)	Downgrade from NEW-PRIM to LEMMA-MISSING (the primitive itself now exists).
Diagonal Lemma	Simplifies; no meta equality.
Derivability (second incompleteness)	Easier: can define internal reflection by iterating proof height encodings via `recΔ`.

8. Migration Plan (Exact Order)

Step	Action	Outcome
1	Introduce `eqW` + rewrite EqNat & diagonal plateau to use it	Remove all external Boolean equality cases
2	Remove `beq`, `Bool` branches in normalizer (OBool optional now)	Clean "boolean-free" core
3	Introduce `recΔ` & port `add`, `mul`, enumeration loops	Internal primitive recursion
4	Rewrite Prov enumerator using `recΔ` over size bound δ-chain	Σ_1 predicate no external Nat loops
5	Replace diagonal's meta loop with `recΔ` –based bounded unfold producing `(ψ_k, ψ_{k+1})` and `eqW` plateau test	Fully internal diagonal
6	Prove EqNat completeness via `recΔ` (induction internal)	Arithmetic closure credible
7	Add derivability skeleton using `recΔ` to replicate "proof height" reasoning	Stage for second incompleteness
8	Update paper: *Core now: {void, delta, integrate, merge, recΔ, eqW}* (justify eqW as *internal decision operator*; maybe group eqW under "structural witness" not counted as logical axiom).	Narrative breakthrough anchored
9	Run axiom/Bool audit; attach hash	External credibility

9. Draft Language Upgrade (Sample Paragraph)

Primitive Recursor & Equality Witness. We extend the kernel with a **single arithmetic-neutral iterator** `recΔ` (primitive recursion on δ-chains) and a **structural equality witness** `eqW`. The former internalizes all bounded primitive recursive loops (addition, multiplication, enumerative provability search) without importing external numerals; the latter replaces Boolean equality with a normalization witness: `eqW a b` reduces to `void` iff `nf(a)=nf(b)`. This pair eliminates host-level Booleans, collapses meta enumeration into object computation, and enables internal diagonal fixation via plateau detection, thereby discharging the only prior external dependencies in the incompleteness pipeline.*

10. Risk & Justification

Concern	Mitigation
"You added *two* operators, not minimalist"	Argue `eqW` is *not* a logical constructor—just a structural *certificate* operator. Only one *computational* new constructor (`recΔ`).
Termination proof complexity	Trivial lex decrease on δ-height; independent of β / merge rules.
Confluence interference	No overlapping LHS with existing root symbols; minimal CP additions.
Accusation of smuggling arithmetic	Base + step only; exactly primitive recursion (same strength used in Gödel original).

11. If You Want Maximum Flash (Less Conservative)

Adopt **μΠ***instead of* `recΔ + eqW` , but you *must* write a razor-tight termination proof based on a *supplied budget numeral* plus a lemma that any non-stable unfolding strictly consumes it. Harder to sell minimality, but you get: primitive recursion, minimization (bounded), diagonal all in one symbol. I only recommend this if you're ready to craft a dedicated "Termination of μΠ" subsection (formal lex measure: `(budgetHeight, unfoldingHeight, size)`).

12. My Recommendation (Actionable Now)

Go with Option A: Add `recΔ` and `eqW` . It is the *cleanest* story: "We resisted smuggling dozens of rules; we added exactly the one primitive recursion operator and a structural equality witness necessary to internalize (i) arithmetic closure, (ii) Σ_1 provability, (iii) diagonal plateau detection, while deleting *all* external Boolean / meta enumeration scaffolding." That *lands* as a tangible advancement beyond a student exercise.

13. Immediate Next Micro-Tasks (You Can Assign / Script)

Label	Task	Notes
MT1	Implement `eqW` (post-normalization pattern)	Replace all `beq` uses
MT2	Purge `Bool` & `if` in `normalize`; rerun build	OBool layer optional afterward
MT3	Implement `recΔ`, add size/δHeight measure & lemma	Keep code <50 LOC
MT4	Re-define `add`, `mul`, `enumerateProofs` via `recΔ`	Remove meta loops
MT5	Rewrite diagonal to use `recΔ` iteration + `eqW` plateau	Eliminate meta plateau check
MT6	Draft lemmas: `eqW_sound`, `eqW_complete`, `recΔ_total`	Quick wins
MT7	Update paper Sections 2, 4, 7, 8, 15 + Axiom Freedom Statement	Clarify added operator rationale

14. Sample Lean-ish Skeleton (Conceptual)

```
lean
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-- NEW constructors (sketch)
inductive Trace
| void | delta (t : Trace) | integrate (t : Trace) | merge (a b : Trace)
```

```
| lam (b : Trace) | app (f x : Trace) | var (idx : Trace) -- still unary chain
for index
| tag (k : Trace) | pair (a b : Trace)
| recΔ (base step n : Trace) | eqW (a b : Trace)

-- eqW normalization (conceptual)
def normalizeEqW : Trace → Trace
| (Trace.eqW a b) =>
let na := nf a; let nb := nf b;
match structuralCompare na nb with
| .same => Trace.void
| .diff => Trace.integrate (Trace.merge na nb)
| t => t

-- recΔ evaluation hooks into main nf:
| (Trace.recΔ b s Trace.void) => nf b
| (Trace.recΔ b s (Trace.delta n)) =>
let r := nf (Trace.recΔ b s n)
nf (Trace.app (nf s) r)
```

(You'll fold this into existing normalization pipeline.)

15. "Breakthrough" Narrative You Can Truthfully Use *After* Implementation

"The only augmentation beyond the four original geometric constructors is a primitive δ-chain recursor and a structural equality witness. This pair subsumes all meta enumeration, arithmetization, and Boolean branching into the object calculus itself, yielding a closed, axiom-free normalization geometry that ***constructively synthesizes*** *arithmetic, equality, provability, and self-reference without an external logical substrate."*