ordinal-toolkit.md — FINAL (2025-07-29)

ordinal-toolkit.md — OperatorKernel O6

Version 2025-07-29 — authoritative, no placeholders; aligns with AGENT.md (same date)

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0 Scope

This toolkit consolidates **all ordinal facts, imports, name-prefix rules, and μ-measure patterns** required by the OperatorKernelO6 meta proofs (SN, confluence, arithmetic). It is the single source of truth for ordinal API usage and module locations. If a symbol is not listed here (or in AGENT.md §8), treat it as **out-of-scope** and raise a **CONSTRAINT BLOCKER**.

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1 Import & Library Audit (authoritative)

> Use exactly these modules; the right-hand column clarifies *what is found where*. Generic ordered-monoid lemmas must **not** be used for ordinal multiplication unless explicitly noted.

Area	Correct import	Contains / Notes
WF/Acc	`Init.WF`	`WellFounded`, `Acc`, `InvImage.wf`, `Subrelation.wf`
Prod lex orders	`Mathlib.Data.Prod.Lex`	`Prod.Lex` for lexicographic measures
Ordinal basics	`Mathlib.SetTheory.Ordinal.Basic`	`omega0_pos`, `one_lt_omega0`, `lt_omega0`, `nat_lt_omega0`
Ordinal arithmetic	`Mathlib.SetTheory.Ordinal.Arithmetic`	`Ordinal.add_*`, `Ordinal.mul_*`, `Ordinal.mul_lt_mul_of_pos_left`, `Ordinal.mul_le_mul_iff_left`, primed `mul_le_mul_left'`/`mul_le_mul_right'`, `le_mul_right`
Ordinal exponentiation	`Mathlib.SetTheory.Ordinal.Exponential`	`opow`, `opow_add`, `Ordinal.opow_le_opow_right`, `isNormal_opow`
Successor helpers	`Mathlib.Algebra.Order.SuccPred`	`Order.lt_add_one_iff`, `Order.add_one_le_of_lt`
ℕ-casts (order bridges)	`Mathlib.Data.Nat.Cast.Order.Basic`	`Nat.cast_le`, `Nat.cast_lt`
Tactics	`Mathlib.Tactic.Linarith`, `Mathlib.Tactic.Ring`	`linarith`, `ring` (both whitelisted)
Generic monoid inequality	`Mathlib.Algebra.Order.Monoid.Defs`	**Generic** `mul_le_mul_left` — do **not** use it for ordinal products.

Qualification rule (must appear verbatim at call-sites):

- **Exponent (≤-mono):** call `Ordinal.opow_le_opow_right` (never the bare name).
- **Exponent (<-mono at base ω):** use the **local** theorem `opow_lt_opow_right` defined in §2.4 (since upstream removed `Ordinal.opow_lt_opow_right`).
- **Products:** prefer `Ordinal.mul_lt_mul_of_pos_left` and `Ordinal.mul_le_mul_iff_left` (or `mul_le_mul_left'`/`mul_le_mul_right'`) — these are the **ordinal** APIs.
- **Successor bridge:** call `Order.lt_add_one_iff` / `Order.add_one_le_of_lt` with the `Order.` prefix.

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2 Toolkit Lemma Catalogue (names, signatures, modules)

> All entries compile under Mathlib 4 (≥ v4.8) + this project's local bridges. Nothing here is hypothetical.

2.1 Basics & Positivity

- `omega0_pos : 0 < omega0` — *module:* `SetTheory.Ordinal.Basic`
- `one_lt_omega0 : 1 < omega0` — *module:* `SetTheory.Ordinal.Basic`
- `lt_omega0 : o < omega0 ↔ ∃ n : ℕ, o = n` — *module:* `SetTheory.Ordinal.Basic`
- `nat_lt_omega0 : ∀ n : ℕ, (n : Ordinal) < omega0` — *module:* `SetTheory.Ordinal.Basic`

2.2 Addition & Successor

- `add_lt_add_left  : a < b → c + a < c + b` — *module:* `SetTheory.Ordinal.Arithmetic`
- `add_lt_add_right : a < b → a + c < b + c` — *module:* `SetTheory.Ordinal.Arithmetic`
- `add_le_add_left  : a ≤ b → c + a ≤ c + b` — *module:* `SetTheory.Ordinal.Arithmetic`
- `add_le_add_right : a ≤ b → a + c ≤ b + c` — *module:* `SetTheory.Ordinal.Arithmetic`
- `Order.lt_add_one_iff : x < y + 1 ↔ x ≤ y` — *module:* `Algebra.Order.SuccPred`
- `Order.add_one_le_of_lt : x < y → x + 1 ≤ y` — *module:* `Algebra.Order.SuccPred`

Absorption on infinite right addends

- `Ordinal.one_add_of_omega_le : omega0 ≤ p → (1 : Ordinal) + p = p`
- `Ordinal.nat_add_of_omega_le : omega0 ≤ p → (n : Ordinal) + p = p`

2.3 Multiplication (Ordinal-specific)

- `Ordinal.mul_lt_mul_of_pos_left : a < b → 0 < c → c * a < c * b`
- `Ordinal.mul_le_mul_iff_left   : c * a ≤ c * b ↔ a ≤ b`
- Primed monotone helpers: `mul_le_mul_left'`, `mul_le_mul_right'` (convenient rewriting forms).
- `le_mul_right : 0 < b → a ≤ b * a`.

> **Note:** `mul_le_mul_left` without a trailing apostrophe comes from `Algebra.Order.Monoid.Defs` and is **generic** (ordered monoids). Do **not** use it to reason about ordinal multiplication.

2.4 Exponentiation (ω-powers & normality)

- `opow_add : a ^ (b + c) = a ^ b * a ^ c` — split exponents.
- `opow_pos : 0 < a → 0 < a ^ b` — positivity of powers.
- `Ordinal.opow_le_opow_right : 0 < a → b ≤ c → a ^ b ≤ a ^ c` — **use fully-qualified**.

Local strict-mono for ω-powers (replacement for deprecated upstream lemma):

```lean
/-- Strict-mono of ω-powers in the exponent (base `omega0`). --/
@[simp] theorem opow_lt_opow_right {b c : Ordinal} (h : b < c) :
  omega0 ^ b < omega0 ^ c := by
  simpa using
    ((Ordinal.isNormal_opow (a := omega0) one_lt_omega0).strictMono h)
```

Why this is correct: `isNormal_opow` states that, for `a > 1`, the map `b ↦ a ^ b` is normal (continuous, strictly increasing). With `a := omega0` and `one_lt_omega0`, `strictMono` yields exactly `<` from `<` in the exponent, which is what we need in μ-decrease proofs.

2.5 Cast bridges (ℕ ↔ Ordinal)

```lean
@[simp] theorem natCast_le {m n : ℕ} : ((m : Ordinal) ≤ (n : Ordinal)) ↔ m ≤ n := Nat.cast_le
@[simp] theorem natCast_lt {m n : ℕ} : ((m : Ordinal) < (n : Ordinal)) ↔ m < n := Nat.cast_lt
```

2.6 Finite vs. infinite split helper

```lean
theorem eq_nat_or_omega0_le (p : Ordinal) : (∃ n : ℕ, p = n) ∨ omega0 ≤ p := by
  classical
  cases lt_or_ge p omega0 with
  | inl h  => rcases (lt_omega0).1 h with ⟨n, rfl⟩; exact Or.inl ⟨n, rfl⟩
  | inr h  => exact Or.inr h
```

Absorption shorthands

```lean
theorem one_left_add_absorb {p : Ordinal} (h : omega0 ≤ p) : (1 : Ordinal) + p = p :=
  by simpa using (Ordinal.one_add_of_omega_le (p := p) h)

theorem nat_left_add_absorb {n : ℕ} {p : Ordinal} (h : omega0 ≤ p) : (n : Ordinal) + p = p :=
  by simpa using (Ordinal.nat_add_of_omega_le (p := p) (n := n) h)
```

2.7 Two-sided product monotonicity (derived helper)

```lean
/-- Two–sided monotonicity of `(*)` for ordinals, built from one–sided lemmas. -/
 theorem ord_mul_le_mul {a b c d : Ordinal} (h₁ : a ≤ c) (h₂ : b ≤ d) :
    a * b ≤ c * d := by
  have h₁' : a * b ≤ c * b := by
    simpa using (mul_le_mul_right' h₁ b)
  have h₂' : c * b ≤ c * d := by
    simpa using (mul_le_mul_left'  h₂ c)
  exact le_trans h₁' h₂'
```

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3 μ-Measure Playbook (used across all rule proofs)

Goal form: for each kernel rule `Step t u`, show `mu u < mu t`. Typical shape reduces to chains like

```
ω^κ * (x + 1) ≤ ω^(x + κ')
```

Standard ladder (repeatable):

1. **Assert base positivity:** `have ωpos : 0 < omega0 := omega0_pos`.
2. **Lift inequalities through exponents:** use `Ordinal.opow_le_opow_right ωpos h` for `≤`, and the local `opow_lt_opow_right` for `<`.
3. **Split exponents/products:** `rw [opow_add]` to turn exponent sums into products so product monotonicity applies cleanly.
4. **Move (≤) across products:** use `Ordinal.mul_le_mul_iff_left`, `mul_le_mul_left'`, `mul_le_mul_right'`; for `<` use `Ordinal.mul_lt_mul_of_pos_left` with a positive left factor.
5. **Absorb finite addends:** once `omega0 ≤ p`, rewrite `(n:Ordinal) + p = p` (or `1 + p = p`).
6. **Bridge successor:** convert `x < y + 1` ↔ `x ≤ y` via `Order.lt_add_one_iff`; introduce `x + 1 ≤ y` via `Order.add_one_le_of_lt` when chaining.
7. **Clean arithmetic noise:** `simp` for associativity/neutral elements; `ring` or `linarith` only for integer-arithmetic side-conditions (both tactics are whitelisted).

Critical correction for `recΔ b s n` (μ-rules):

Do **not** try to relate `mu s` and `mu (delta n)`. They are **independent parameters**; the inequality `mu s ≤ mu (delta n)` is **false in general**. A simple counterexample (compiles in this codebase):

```lean
def s : Trace := delta (delta void)      -- μ s begins with a higher ω-tower
def n : Trace := void                     -- μ (delta n) is strictly smaller
-- here: mu s > mu (delta n)
```

Structure μ-decrease proofs without assuming any structural relation between `s` and `n` beyond what the rule’s right-hand side entails.

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4 Order.succ vs `+ 1` (bridge & hygiene)

Lean will often rewrite `p + 1` to `Order.succ p` in goals. Work with the `Order` lemmas:

- `Order.lt_add_one_iff : x < y + 1 ↔ x ≤ y`
- `Order.add_one_le_of_lt : x < y → x + 1 ≤ y`

Keep the `Order.` prefix to avoid name resolution issues. Avoid inventing `succ_eq_add_one`—rely on these bridges instead.

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5 Do-Not-Use / Deprecated in this project

- **Generic** `mul_le_mul_left` (from `Algebra.Order.Monoid.Defs`) on ordinal goals. Use

`Ordinal.mul_*` APIs instead.
- Old paths `Mathlib.Data.Ordinal.*` — replaced by `Mathlib.SetTheory.Ordinal.*`.
- `Ordinal.opow_lt_opow_right` (upstream removed). Use the **local** `opow_lt_opow_right` defined in §2.4.
- `le_of_not_lt` (deprecated) — use `le_of_not_gt`.

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6 Minimal import prelude (copy-paste)

```lean
import Init.WF
import Mathlib.Data.Prod.Lex
import Mathlib.SetTheory.Ordinal.Basic
import Mathlib.SetTheory.Ordinal.Arithmetic
import Mathlib.SetTheory.Ordinal.Exponential
import Mathlib.Algebra.Order.SuccPred
import Mathlib.Data.Nat.Cast.Order.Basic
import Mathlib.Tactic.Linarith
import Mathlib.Tactic.Ring
open Ordinal
```

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7 Ready-made snippets

Nat-sized measure (optional helper):

```lean
@[simp] def size : Trace → Nat
| void => 1
| delta t => size t + 1
| integrate t => size t + 1
| merge a b => size a + size b + 1
| recΔ b s n => size b + size s + size n + 1
| eqW a b => size a + size b + 1

theorem step_size_decrease {t u : Trace} (h : Step t u) : size u < size t := by
  cases h <;> simp [size]; linarith
```

WF via ordinal μ:

```lean
def StepRev : Trace → Trace → Prop := fun a b => Step b a

theorem strong_normalization_forward
  (dec : ∀ {a b}, Step a b → mu b < mu a) : WellFounded (StepRev Step) := by
  have wfμ : WellFounded (fun x y : Trace => mu x < mu y) := InvImage.wf (f := mu) Ordinal.lt_wf
  have sub : Subrelation (StepRev Step) (fun x y => mu x < mu y) := by intro x y h; exact dec h
  exact Subrelation.wf sub wfμ
```

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8 Cross-file consistency notes

- This toolkit and **AGENT.md (2025-07-29)** are **synchronized**: imports, prefixes, do-not-use list, and the μ-rule correction are identical. If you edit one, mirror the change here.
- Cite lemma modules explicitly in comments or nearby text in code reviews to prevent regressions (e.g., “`Ordinal.mul_lt_mul_of_pos_left` – from `SetTheory.Ordinal.Arithmetic`”).

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9 Checklist (before sending a PR)

- [ ] Imports ⊆ §6, no stray module paths.
- [ ] All exponent/product/`+1` lemmas called with **qualified** names as in §1.
- [ ] μ-proofs avoid any relation between `μ s` and `μ (δ n)` in `recΔ b s n`.
- [ ] Tactics limited to `simp`, `linarith`, `ring`.
- [ ] No generic `mul_le_mul_left` on ordinal goals; use `Ordinal.mul_*` API.
- [ ] SN proof provides μ-decrease on all 8 rules; WF via `InvImage.wf`.
- [ ] Normalize-join confluence skeleton compiles (`normalize`, `to_norm`, `norm_nf`, `nfp`).

End of file.