Asymptotic Analysis of Function Pairs

1 Introduction

For each pair of functions $(f(n), g(n))$, we determine whether:

- $f(n) \in O(g(n))$ (Big-O),
- $f(n) \in \Omega(g(n))$ (Big-Omega),
- $f(n) \in \Theta(g(n))$ (Big-Theta).

We use two methods for proof:

1. **Limit (ratio) test**:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0, & f(n) \in o(g(n)) \subseteq O(g(n)), \\ c > 0, & f(n) \in \Theta(g(n)), \\ \infty, & f(n) \in \omega(g(n)) \subseteq \Omega(g(n)). \end{cases}$$

2. **Formal Definition**: We show the existence of constants $c > 0$ and n_0 such that for all $n > n_0$,

$$f(n) \leq cg(n) \quad \text{(proving } O(g(n))\text{)}, \quad f(n) \geq cg(n) \quad \text{(proving } \Omega(g(n))\text{)}.$$

2 Analysis of Each Pair

2.1 Pair (a)

$$f(n) = 100n + \log n, \quad g(n) = n + (\log n)^2$$

2.1.1 Limit Test

For large n, the dominant term in both functions is n. Taking the limit:

$$\lim_{n\to\infty} \frac{100n + \log n}{n + (\log n)^2}$$

Factor out n from numerator and denominator:

$$\lim_{n\to\infty} \frac{n\left(100 + \frac{\log n}{n}\right)}{n\left(1 + \frac{(\log n)^2}{n}\right)}$$

Since $\frac{\log n}{n} \to 0$ and $\frac{(\log n)^2}{n} \to 0$, the ratio converges to:

$$\frac{100}{1} = 100.$$

Since the limit is a constant, we conclude:

$$f(n) \in \Theta(g(n)).$$

2.1.2 Formal Definition

We find constants c_1, c_2 such that:

$$c_1(n + (\log n)^2) \leq 100n + \log n \leq c_2(n + (\log n)^2).$$

Choosing $c_1 = 50$, $c_2 = 101$, and $n_0 = 16$, we satisfy the conditions, proving:

$$f(n) \in \Theta(g(n)).$$

2.2 Pair (b)

$$f(n) = \log n, \quad g(n) = \log(n^2) = 2\log n.$$

2.2.1 Limit Test

$$\lim_{n\to\infty} \frac{\log n}{2\log n} = \frac{1}{2}.$$

Since this is a positive constant, we conclude:

$$f(n) \in \Theta(g(n)).$$

2.2.2 Formal Definition

Since $g(n) = 2\log n$, we have:

$$\log n = \frac{1}{2}g(n).$$

Setting $c_1 = \frac{1}{2}, c_2 = 1, n_0 = 2$, we satisfy:

$$c_1 g(n) \le f(n) \le c_2 g(n).$$

Thus, $f(n) \in \Theta(g(n))$.

2.3 Pair (c)

$$f(n) = \frac{n^2}{\log n}, \quad g(n) = n(\log n)^2.$$

2.3.1 Limit Test

$$\lim_{n\to\infty} \frac{\frac{n^2}{\log n}}{n(\log n)^2} = \lim_{n\to\infty} \frac{n^2}{\log n} \cdot \frac{1}{n(\log n)^2} = \lim_{n\to\infty} \frac{n}{(\log n)^3}.$$

Since n eventually outgrows $(\log n)^3$, the limit is ∞, meaning:

$$f(n) \in \Omega(g(n)).$$

2.3.2 Formal Definition

We need:

$$\frac{n^2}{\log n} \geq cn(\log n)^2.$$

Rewriting:

$$\frac{n^2/\log n}{n(\log n)^2} = \frac{n}{(\log n)^3} \geq c.$$

Since this term grows indefinitely, for large enough n_0, we satisfy $f(n) \geq cg(n)$, proving:

$$f(n) \in \Omega(g(n)).$$

2.4 Pair (d)

$$f(n) = (\log n)^{\log n}, \quad g(n) = \frac{n}{\log n}.$$

A useful rewrite:

$$f(n) = n^{\log(\log n)}.$$

2.4.1 Limit Test

$$\frac{f(n)}{g(n)} = \frac{n^{\log(\log n)}}{n/\log n} = n^{\log(\log n)-1} \cdot \log n.$$

Since $n^{\log(\log n)-1}$ grows super-polynomially, we get:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

Thus, $f(n) \in \Omega(g(n))$.

2.4.2 Formal Definition

Rewriting:

$$n^{\log(\log n)-1} \log n \geq c.$$

Since this term goes to ∞, choosing $c = 1$ and large enough n_0 satisfies the conditions, proving:

$$f(n) \in \Omega(g(n)).$$

2.5 Pair (e)

$$f(n) = \sqrt{n}, \quad g(n) = (\log n)^5.$$

2.5.1 Limit Test

$$\lim_{n \to \infty} \frac{\sqrt{n}}{(\log n)^5}.$$

Since even a small power of n (such as $n^{1/2}$) eventually outgrows any power of $\log n$, this limit is ∞, implying:

$$f(n) \in \Omega(g(n)).$$

2.5.2 Formal Definition

To show $f(n) \in \Omega(g(n))$, we require:

$$\frac{\sqrt{n}}{(\log n)^5} \geq c.$$

Since this term grows indefinitely, for large enough n_0, we can satisfy $f(n) \geq cg(n)$. Therefore:

$$f(n) \in \Omega(g(n)).$$

2.6 Pair (f)

$$f(n) = n \cdot 2^n, \quad g(n) = 3^n.$$

2.6.1 Limit Test

$$\lim_{n\to\infty} \frac{n2^n}{3^n} = \lim_{n\to\infty} n \left(\frac{2}{3}\right)^n.$$

Since $\left(\frac{2}{3}\right)^n$ shrinks exponentially to 0, the product also tends to 0. Thus:

$$f(n) \in O(g(n)).$$

2.6.2 Formal Definition

We require:

$$n2^n \leq c3^n.$$

Rearranging:

$$n \left(\frac{2}{3}\right)^n \leq c.$$

Since $n\left(\frac{2}{3}\right)^n \to 0$, we can find n_0 large enough to satisfy $f(n) \leq cg(n)$, proving:

$$f(n) \in O(g(n)).$$

2.7 Pair (g)

$$f(n) = 2^{\sqrt{\log n}}, \quad g(n) = \sqrt{n}.$$

2.7.1 Limit Test

$$\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{\sqrt{n}}.$$

Rewriting the terms using exponentials:

$$\frac{\exp(\sqrt{\log n}\ln 2)}{\exp(\frac{1}{2}\log n)} = \exp\left(\ln 2 \cdot \sqrt{\log n} - \frac{1}{2}\log n\right).$$

For large n, the exponent:

$$\ln 2 \cdot \sqrt{\log n} - \frac{1}{2}\log n$$

becomes negative and unbounded. Thus:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

So $f(n) \in O(g(n))$.

2.7.2 Formal Definition

We need:

$$2^{\sqrt{\log n}} \leq c\sqrt{n}.$$

Rewriting:

$$\frac{2^{\sqrt{\log n}}}{\sqrt{n}} \leq c.$$

Since this ratio tends to 0, we can choose $c = 1$ and find n_0 such that the inequality holds, proving:

$$f(n) \in O(g(n)).$$

3 Summary of Classifications

Pair	$f(n)$	$g(n)$	Classification
(a)	$100n + \log n$	$n + (\log n)^2$	$\Theta(g(n))$
(b)	$\log n$	$2 \log n$	$\Theta(g(n))$
(c)	$\frac{n^2}{\log n}$	$n(\log n)^2$	$\Omega(g(n))$
(d)	$(\log n)^{\log n}$	$\frac{n}{\log n}$	$\Omega(g(n))$
(e)	$\sqrt{n}$	$(\log n)^5$	$\Omega(g(n))$
(f)	$n2^n$	3^n	$O(g(n))$
(g)	$2^{\sqrt{\log n}}$	$\sqrt{n}$	$O(g(n))$