# Watts–Strogatz Small-World Model: Simulation and Analysis

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**Abstract**

We implement and analyze the Watts–Strogatz model for small-world networks. Using numerical simulations, we measure degree distributions, clustering coefficients, and approximate average shortest path lengths across a range of rewiring probabilities $p$ and demonstrate the small-world transition: networks retain high clustering while their characteristic path length rapidly decreases at low $p$.

# 1 Introduction

## 1.1 Networks: The Foundation of Complex Systems

Networks are fundamental structures that describe relationships between entities in complex systems. From social connections between individuals to neural pathways in the brain, from the World Wide Web to protein interactions in cells, networks provide a powerful mathematical framework for understanding how components interact within larger systems. In network theory, we represent these systems as graphs consisting of nodes (vertices) that represent individual entities, connected by edges (links) that represent relationships or interactions between them. The study of networks has revealed that many real-world systems exhibit distinctive structural properties that distinguish them from simple random arrangements. These properties profoundly influence how information, diseases, behaviors, and other phenomena spread through the network, making network topology a critical factor in understanding system dynamics.

## 1.2 The Small-World Phenomenon

One of the most striking discoveries in network science is the small-world phenomenon – the observation that in many large networks, any two nodes can be connected through a surprisingly short chain of intermediate connections. This concept gained widespread attention through Stanley Milgram's famous "six degrees of separation" experiments in the 1960s, which suggested that any two people in the United States could be connected through approximately six social acquaintances. The small-world phenomenon reflects a fascinating property of many real networks: they simultaneously exhibit high local clustering (nodes tend to be connected to their neighbors' neighbors, forming tight-knit groups) and short average path lengths between any two nodes in the network. This combination seems paradoxical – how can a network maintain strong local structure while still allowing for efficient global connectivity?

### 1.3 The Watts-Strogatz Model: Bridging Order and Randomness

In 1998, Duncan Watts and Steven Strogatz introduced a groundbreaking model that elegantly explains how the small-world phenomenon emerges in real networks. The Watts-Strogatz model demonstrates how networks can transition from highly ordered, regular structures to completely random arrangements, passing through a small-world regime that captures the essential features observed in many natural and social systems. The model begins with a regular ring lattice where each node connects to its nearest neighbors, creating high clustering but long path lengths. Through a simple rewiring process – randomly reconnecting a fraction of edges with a certain probability – the model shows how just a small amount of randomness can dramatically reduce path lengths while preserving most of the local clustering structure. This creates networks that are "small worlds" – locally clustered like regular lattices but globally connected like random networks. The Watts-Strogatz model has profound implications across disciplines. It helps explain how rumors spread rapidly through social networks, how synchronization emerges in neural networks, how diseases propagate through populations, and how innovations diffuse through organizations. By providing a mathematical framework that bridges the gap between complete order and complete randomness, the model has become a cornerstone of modern network science and has inspired countless applications in physics, biology, sociology, and computer science. Understanding these concepts is essential for anyone seeking to comprehend the structure and dynamics of complex interconnected systems in our increasingly networked world.

## 2 Model and Methods

### 2.1 Ring lattice construction

The Watts–Strogatz model starts from a regular ring lattice of $n$ vertices. Each vertex is placed on a circle and connected to its $k$ nearest neighbors, $k/2$ on each side. This initial graph is regular (every vertex has degree $k$), strongly clustered, and has a relatively large characteristic path length due to the locality of connections.

### 2.2 Rewiring procedure

To introduce randomness while preserving the number of edges, each edge that connects vertex $i$ to its clockwise neighbor at distance $j \in \{1, \dots, k/2\}$ is rewired with probability $p$. Rewiring replaces that edge by an edge from $i$ to a uniformly random vertex $j'$ chosen so that $j' \neq i$ and $j'$ is not already a neighbor of $i$. This single-pass rewiring (performed once for every such link) interpolates between a regular lattice ($p = 0$) and an approximately random graph ($p = 1$). For small nonzero $p$ a few long-range shortcuts appear, dramatically decreasing typical distances while leaving local clustering largely intact: the hallmark of the small-world regime.

### 2.3 Metrics

We compute three complementary metrics.

**Degree distribution.** The degree distribution is the empirical distribution of node degrees $k_i$. At $p = 0$ the distribution is a delta at $k$; as $p$ increases the distribution broadens and, for $p \to 1$, tends toward the degree distribution of a random graph with mean degree $\langle k \rangle = k$.

**Local and average clustering coefficient.** The local clustering coefficient of vertex $i$ is defined by

$$C_i = \frac{2T_i}{k_i(k_i - 1)},$$

where $T_i$ is the number of triangles that contain $i$ and $k_i$ is the degree of $i$. The global average clustering coefficient is

$$C = \frac{1}{n} \sum_{i=1}^{n} C_i.$$

For the ring lattice one obtains a high clustering coefficient:

$$C(0) = \frac{3(k-2)}{4(k-1)} \quad (\text{for } n \gg k),$$

which depends on $k$ but not on $n$. Random rewiring destroys triangles and reduces $C$.

**Average shortest path length (BFS; sampling).** The graph distance $d(u, v)$ is the length of the shortest path between $u$ and $v$. The average shortest path length is

$$\langle L \rangle = \frac{1}{n(n-1)} \sum_{u \neq v} d(u, v).$$

We compute distances with Breadth-First Search (BFS). Because exact computation requires BFS from every node and scales as $O(n(n+m))$, we approximate $\langle L \rangle$ for large graphs by sampling: we run BFS from $S$ randomly chosen sources (typical $S = 100$) and average the finite distances returned. In practice the ring lattice scales as $\langle L \rangle_{p=0} \approx n/(2k)$ while an equivalent random graph has $\langle L \rangle \sim \ln n / \ln k$; the small-world regime is the window of small $p$ in which $\langle L \rangle$ falls from the lattice value toward the random-graph value while $C$ remains close to $C(0)$.

# 3 Simulations

## 3.1 Simulation settings

All simulations in this work were performed using the following baseline parameters unless otherwise stated:

- Number of nodes: $n = 1000$.
- Initial degree (ring lattice): $k = 10$ (even).
- Rewiring probability: sweep over $p \in [0, 1]$, using logarithmically spaced values for small $p$ and including $p = 0, 1$.
- Realizations per $p$: $R = 5$ (used to compute mean and standard deviation).
- BFS sampling per realization: $S = 100$ source nodes (or $S = \min(100, n)$) to approximate $\langle L \rangle$.
- Random seeds: fixed seeds for example figures; distinct seeds across realizations to estimate sample variance.

These choices balance computational cost with statistical stability. In particular, $n = 1000$, $k = 10$, $R = 5$, and $S = 100$ give clear small-world signatures while keeping run time appropriate for our project.

## 3.2 Measurement procedure

For each rewiring probability $p$ we generate $R$ independent realizations of the Watts–Strogatz graph. For every realization we compute:

1. The degree distribution $P(k)$ (histogram of node degrees).
2. Local clustering coefficients $C_i$ and the average clustering coefficient

$$C = \frac{1}{n} \sum_{i=1}^{n} C_i.$$

3. An approximation to the average shortest path length $\langle L \rangle$ by performing BFS from $S$ randomly sampled source nodes and averaging the finite distances. BFS from a single source costs $O(n+m)$; sampling provides a large speed-up relative to full all-pairs BFS at the cost of a modest statistical error.

### 3.3 Results: diagnostic plots

The main diagnostics are shown in Figures 1 and 2. Each point is the mean over $R$ realizations; error bars indicate one standard deviation.

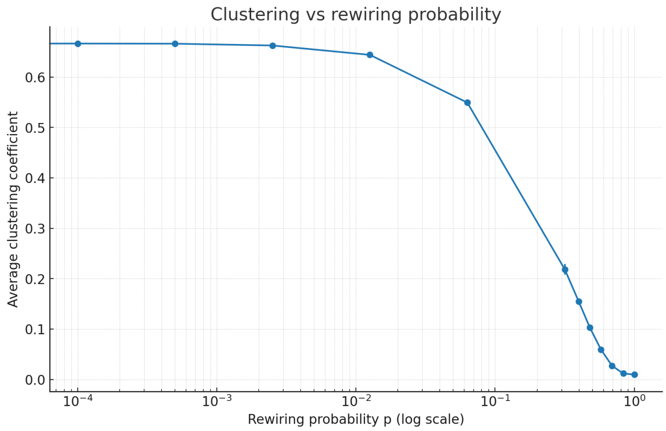



Figure 1: Average clustering coefficient $C$ vs. rewiring probability $p$.

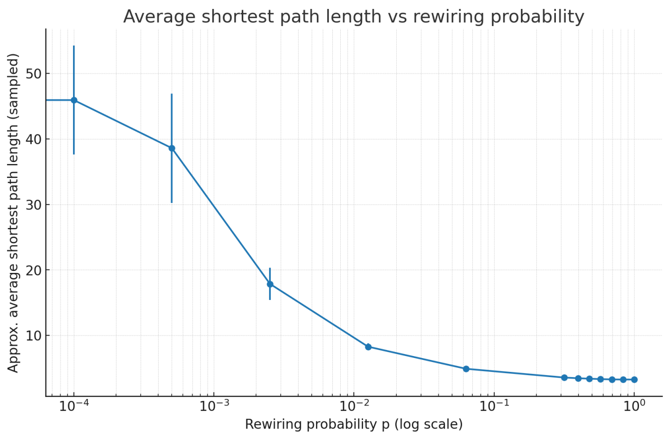



Figure 2: Approximate average shortest path length $\langle L \rangle$ vs. rewiring probability $p$.

### 3.4 Representative distributions and visualizations

To illustrate local structure and topology visually, we include representative degree histograms and circle-layout network visualizations in the Appendix. The PNG files generated by the notebook are:

- Degree histograms: `degree_hist_p_0_0.png`, `degree_hist_p_0_01.png`, `degree_hist_p_0_1.png`, `degree_hist_p_1_0.png`.
- Network visualizations (circle layout, n=200): `network_vis_p_0_0.png`, `network_vis_p_0_01.png`, `network_vis_p_1_0.png`.

See Appendix for the figures and captions.

### 3.5 Notes on reproducibility and parameter choices

All parameters are defined at the top of the Jupyter notebook. Example figure-producing cells set explicit random seeds to make the visuals reproducible; the parameter sweep uses different seeds per realization to sample variance. To refine statistical estimates, increase $R$ or $S$ at the cost of longer runtime.

## 4 Results

The numerical results from the sweep are shown in Table 1.

| $p$ | $C$ (mean) | $C$ (std) | $\langle L \rangle$ (mean) | $\langle L \rangle$ (std) |
|---:|---:|---:|---:|---:|
| 0 | 0.6667 | 0.0000 | 50.400 | 0.000 |
| 0.0001 | 0.6664 | 0.0004 | 45.918 | 8.303 |
| 0.000501 | 0.6661 | 0.0005 | 38.581 | 8.336 |
| 0.002512 | 0.6625 | 0.0011 | 17.866 | 2.435 |
| 0.012589 | 0.6440 | 0.0032 | 8.282 | 0.508 |
| 0.063096 | 0.5493 | 0.0032 | 4.908 | 0.055 |
| 0.316228 | 0.2186 | 0.0109 | 3.571 | 0.019 |
| 0.398107 | 0.1550 | 0.0056 | 3.467 | 0.016 |
| 0.47863 | 0.1032 | 0.0030 | 3.395 | 0.013 |
| 0.57544 | 0.0595 | 0.0022 | 3.334 | 0.004 |
| 0.691831 | 0.0272 | 0.0016 | 3.286 | 0.014 |
| 0.831764 | 0.0120 | 0.0017 | 3.267 | 0.005 |
| 1 | 0.0098 | 0.0009 | 3.268 | 0.014 |

Table 1: Sweep over rewiring probabilities: average clustering coefficient $C$ and approximate average shortest path length $\langle L \rangle$ (mean and standard deviation over realizations).

The numerical experiments confirm the canonical Watts–Strogatz small-world transition. Figure 1 shows the average clustering coefficient $C$ as a function of the rewiring probability $p$, and Figure 2 shows the corresponding approximate average shortest path length $\langle L \rangle$. At $p = 0$ (the regular ring lattice) networks exhibit high clustering and large path lengths. In contrast, at $p = 1$ (fully rewired) clustering is low while $\langle L \rangle$ is small, consistent with random-graph behavior. Importantly, for small nonzero $p$ (e.g. $p$ on the order of $10^{-3}$–$10^{-2}$) the networks retain a clustering coefficient close to the lattice value while $\langle L \rangle$ already drops by a large factor toward the random-graph value. This coexistence of high clustering and small characteristic path length, observed in the numerical sweep (see Table 1 for representative values at $p = 0$, $p \approx 10^{-3}$, $p \approx 10^{-2}$, and $p = 1$), is the hallmark of the small-world regime described by Watts and Strogatz.

# 5 Discussion

## 5.1 Implications and relation to real-world networks

The Watts–Strogatz construction isolates a simple, robust mechanism that explains how networks can be both locally clustered and globally short-path. Many empirical systems (social contacts, transportation grids, neuronal networks, and collaboration graphs) display precisely this combination: strong local cohesion (triangles and cliques) together with short average distances. In these systems, a relatively small number of long-range connections ("shortcuts") permit rapid global traversal while leaving most local connectivity patterns unchanged.

## 5.2 Limitations

Several limitations of our simulations and of the model should be noted:

- **Finite-size effects.** Our experiments use finite $n$ (for example $n = 1000$) so that quantitative values of $C$ and $\langle L \rangle$ depend on system size. To confirm universality of observations, results should be compared across different $n$.

- **Sampling for path length.** The reported average path lengths are approximations obtained by running BFS from $S$ sampled source nodes. Sampling significantly reduces computational cost but introduces statistical error; this is why we report means and standard deviations across realizations and sampling seeds.

- **Model simplicity.** The Watts–Strogatz model produces relatively homogeneous degree sequences and does not capture many features of real networks such as heavy-tailed degree distributions, directed or weighted edges, community structure, or temporal growth.

- **Implementation variants.** Different implementations (rewiring vs. adding shortcuts; Newman–Watts variant) yield quantitatively different outcomes. The particular rewiring procedure must be stated explicitly for reproducibility.

### 5.3 Possible extensions

Natural extensions to bridge the gap between the WS toy model and empirical networks include:

- *Weighted edges:* assign weights (e.g., travel times, strength of ties) and study weighted shortest paths.
- *Directed graphs:* many networks (citation, web, follower graphs) are directed; distances and clustering must then be adapted.
- *Community structure:* plant or detect modules and study how inter-community shortcuts control global distances and modularity.
- *Degree heterogeneity:* mix the WS rewiring with preferential-attachment mechanisms to introduce hubs and study their effect on distance and clustering.
- *Analytical approximations:* compare empirical curves to theoretical estimates: a lattice scales like $\langle L \rangle_{p=0} \sim n/(2k)$ while random graphs have $\langle L \rangle \sim \ln n / \ln k$; perturbative approximations and mean-field formulas can be used to interpolate in between.

### 5.4 A concrete and fun illustration: Six Degrees of Kevin Bacon

The well-known "Six Degrees of Kevin Bacon" parlor game frames actor separation as a small-world question: two actors are adjacent if they co-starred in a movie; the Bacon number of an actor is the shortest-path distance to Kevin Bacon. Empirically, most actors have a small Bacon number (often 1–4), which reflects the small-world nature of the Hollywood co-starring network.

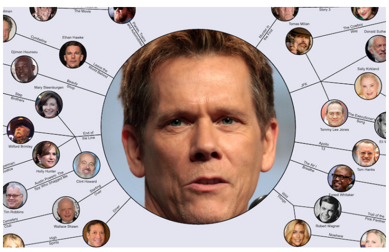


Figure 3: Six Degrees of Kevin Bacon

Interpreting this with Watts–Strogatz intuition:

- Clusters form among actors who frequently work together (high local clustering).
- A small number of actors who cross genres, studios, or geographic production areas create long-range shortcuts connecting distant clusters.

Consequently, global distances collapse while local clustering remains high, exactly the small-world phenomenon captured by the WS model.

### 5.5 Concluding remark

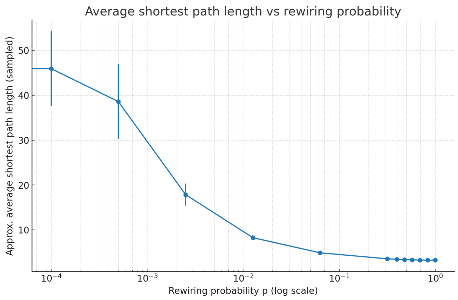
The Watts–Strogatz model is valuable because it isolates a single, intuitive mechanism (the introduction of a small number of long-range links into a clustered lattice) that explains how many large, complex systems can be locally dense yet globally navigable. The model is a natural baseline for more elaborate models and for empirical comparisons.

## 6 Conclusion

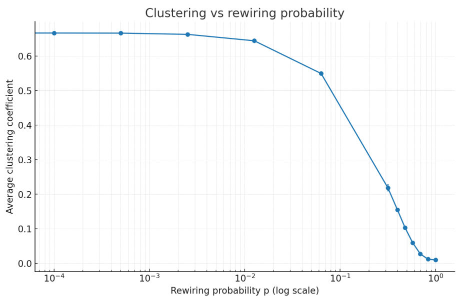
We implemented and analyzed the Watts–Strogatz small-world model using numerical simulations and demonstrated the characteristic small-world transition. By sweeping the rewiring probability $p$ and measuring the degree distribution, the local and average clustering coefficient $C$, and an approximate average shortest path length $\langle L \rangle$ (obtained via BFS with sampling), we observed the canonical behaviour: at $p = 0$ the network is highly clustered with long path lengths, at $p = 1$ it resembles a random graph with low clustering and short paths, and, crucially, for small nonzero $p$ the network simultaneously retains high clustering while $\langle L \rangle$ falls dramatically. This coexistence of local cohesion and global navigability is the small-world phenomenon. Our results are robust across realizations for the chosen parameters ($n = 1000, k = 10$) though subject to finite-size and sampling limitations; extensions such as weighted or directed links, planted community structure, or degree heterogeneity would deepen the comparison with empirical networks. The accompanying notebook and figures provide reproducible code and diagnostics so the experiments can be readily varied and extended.

# Appendices

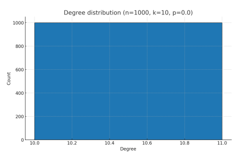
## Figures



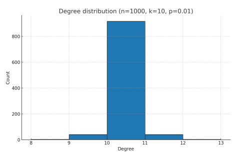

(a) Average path length vs $p$



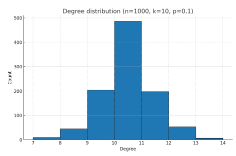

(b) Clustering coefficient vs $p$



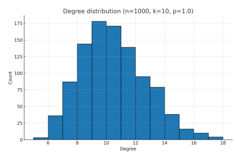
(c) $p = 0.0$



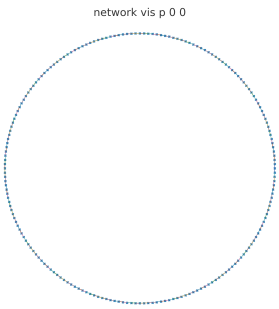
(d) $p = 0.01$



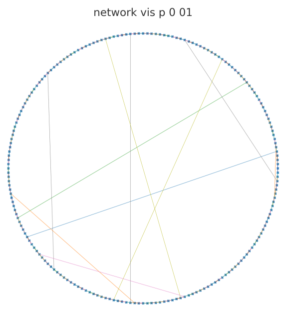
(e) $p = 0.1$



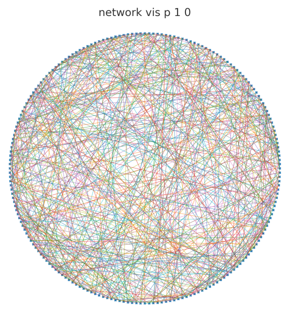
(f) $p = 1.0$




(g) Network: $p = 0.0$




(h) Network: $p = 0.01$




(i) Network: $p = 1.0$

Figure 4: Complete analysis of network properties vs rewiring probability $p$: (a,b) network metrics, (c-f) degree distributions, (g-i) network visualizations.

# References

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