$$\vec{F} = -\int_0^\delta v_\infty \rho v_\infty\, bdy|x = 0 + \int_0^\delta \rho V(y) bdy\,|x = L$$

$$m = \rho v_\infty bdy$$

$$\rho v(y) bdy|x = L$$

$$\vec{F} = -\int_0^\delta v_\infty \rho v(y) bdy\Big|x = L + \int_0^\delta \rho V(y) bdy\,|x = L$$

$\vec{F} = -D$, where D = drag

$$D = \rho b \int_0^\delta v(y)\,[v_\infty - v(y)] dy|x = L$$

$$D \propto [v_\infty - v(y)]$$

Velocity deficit created by the B.L. is the physical origin of the drag force!

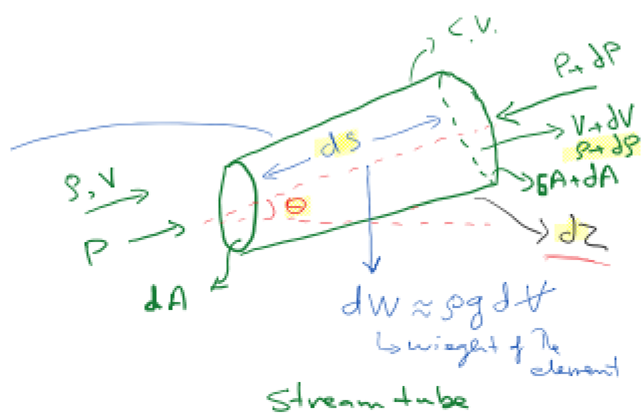
Von Karman defined this relationship in 1921!

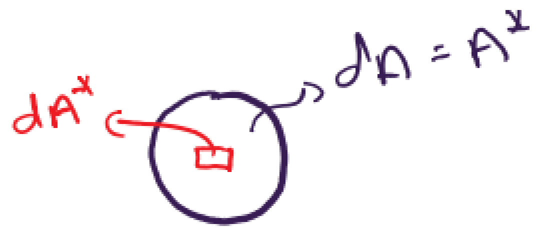
Our Goals for Today (lect. 6)

1. A very fast review of derivation of Bernoulli eq. with emphasis on non-steady form of the equation
2. Invitation to self-study notes for Sec. 3.6 and 3.7 (angular momentum and Energy formulations of RTT)
3. Starting Chapter 4 and introducing differential forms of conservation of mass & momentum

Sec. 3.5 Bernoulli Equation

Will not consider shear stress or the wall, So ONLY <u>frictionless flow</u>





Writing RTT for momentum Equation (2):

$$d\vec{F} = \frac{d}{dt}\int_{cv} \vec{v}\rho d\forall^* + \int_{cs} \vec{v}\,\rho\vec{v}\cdot\vec{n}dA^*$$

An element of an infinite section:

- $A^* = dA$
- $s^* = ds$
- $d\forall \approx dAds$

$$d\vec{F_s} = \frac{d}{dt}\int_{s} v_s\rho dAds^* - \int_{cs} v_s\rho v_s dA^* + \int_{cs} (v_s + dv_s)\rho(v_s + dv_s)dA^*$$

$d\vec{F_s}$ – along the streamline

ds^* - integrate over starred variable

dA^* - integrate over starred variable

$$dF_s = \frac{d}{dt}[v_s\rho dAds] - v_s\rho v_s dA + (v_s + dv_s)\rho(v_s + dv_s)(dA + \delta A)$$

$dAds$ → constant in time

$v_s\rho v_s dA$ → $\dot{m}$

$\rho(v_s + dv_s)(dA + \delta A)$ → $d\dot{m}$

$$dFs = \frac{\partial v_s}{\partial t}\rho dAds + \frac{\partial \rho}{\partial t}v_s dA\partial s - v_s d\dot{m}_{in} + v_s d\dot{m}_{out} + dv_s d\dot{m}_{out}$$

$-v_s d\dot{m}_{in} + v_s d\dot{m}_{out}$ → $+v_s(d\dot{m}_{out} - d\dot{m}_{in})$

Where $(d\dot{m}_{out} - d\dot{m}_{in})$ → $d\dot{m}^*$

$dv_s d\dot{m}_{out} \approx dv_s d\dot{m}$

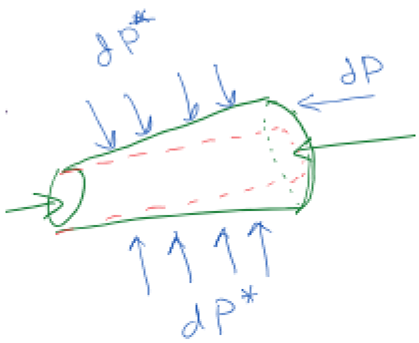
$$df_s \approx \frac{\partial v_s}{\partial t}\rho dAds + \frac{\partial \rho}{\partial t} v_s dAds + v_s d\dot{m}^* + dv_s d\dot{m}$$

Due to mass conservation, these terms cancel each other, see Eq. (1) : $\frac{\partial \rho}{\partial t} v_s dAds, v_s d\dot{m}^*$

On the other hand, forces on stream tube element are pressure & gravity so,

$$dF_s = -\rho g dAdS sin\theta + dP\delta A - dP(dA + \delta A)$$

$dP\delta A - dP(dA + \delta A)$ → gauge pressure only; for a small angled tube, approx. dPdA



$$dF_s \approx -\rho g dAdz - dPdA$$

$$-\rho g dAdz - dPdA = \frac{\partial v_s}{\partial t}\rho dAds + d\dot{m}dv_s$$

$$\frac{\partial v_s}{\partial t} dS + \frac{dP}{\rho} + v_s dv_s + gdz = 0 \quad (15)$$

Equation (15) is unsteady frictionless along a stream tube (to write for a streamline, we put stream on a diet!)

If we consider flow is steady state, then $\frac{\partial v_s}{\partial t} = 0$

If we consider fluid to be incompressible (ρ = constant), Then I can simply write $dP = P_2 - P_1$ → decoupling P & ρ

So, integrating between 2 points, 1&2:

$$\frac{P_2 - P_1}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = 0 \quad (16)$$

NOTE: In deriving Equations 15 & 16, we used conservation of mass & momentum but NOT conservation of Energy! → so, if heat or work are added/removed from c.v., Then equations (15&16) are not applicable. Do NOT use!

Observation:

From 16: $\frac{P_1}{\rho} + \frac{1}{2}v_1^2 + gz_1 = constant$

$$\frac{P_1}{\rho g} + \frac{1}{2}\frac{v_1^2}{g} + z_1 = H$$

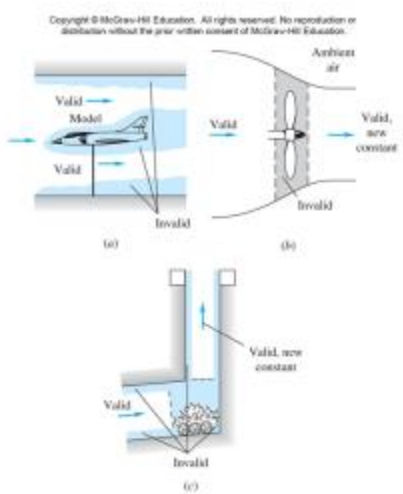
$\frac{P_1}{\rho g}$ → pressure head

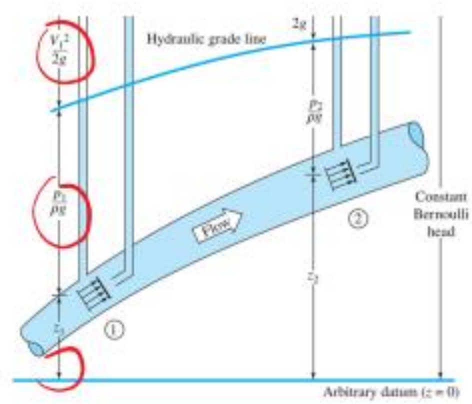
$\frac{v_1^2}{g}$ → dynamic head (velocity head)

z_1 → static head

H → Head

See Figure 3.13





2. If the change in Z is negligible, then Equation 16:

$$P_1 + \frac{1}{2}\rho v_1^2 = constant = P_o$$

P_1 → static pressure

$\frac{1}{2}\rho v_1^2$ → dynamic pressure

P_o→ Total pressure (stagnation pressure)

Look at the exp. 3.16 in the textbook.

Sec. 3.6 – Angular Momentum Theorem:

In systems like pumps and turbines <u>where flow rotates</u>, the linear may not be very useful for analysis. Also, anywhere that there is a <u>misalignment between fluid flow and force line of action</u>.

Note a fluid is deformable unlike a solid, so angular momentum should be written on elemental basis:

So,

$$\vec{H}_o = \int_{sys} (\vec{r} \times \vec{v}) dm \quad (17)$$

Rotation about point "o"

$$\beta = \frac{d\vec{H}_o}{dm} = \vec{r} \times \vec{v}$$

From RTT

$$\left.\frac{d\vec{H}_o}{dt}\right| sys = \frac{\partial}{\partial t} \int_{cv} (\vec{r} \times \vec{v}) \rho d\forall + \int_{cs} (\vec{r} \times \vec{v}) \rho (\vec{v} \cdot \vec{n}) dA$$

From mechanisms, we know: $\frac{dH_o}{dt} = \sum M_o = \sum(\vec{r} \times \vec{F})$

M_o → summation of moments for all forces (gravity, pressure, etc.) around "o"

$$\sum M_o = \frac{\partial}{\partial t} \int_{cv} (\vec{r} \times \vec{v}) \rho d\forall + \int_{cv} (\vec{r} \times \vec{v}) \rho (\vec{v} \cdot \vec{n}) dA \quad (18)$$

Most problems can be treated as 1D intel/outlet and analyzed in steady conditions.

So:

$$\sum M_o = \sum (\vec{r} \times \vec{v})_{out}\, \dot{m}_{out} - \sum (\vec{r} \times \vec{v})_{in}\, \dot{m}_{in} \quad (19)$$

Otherwise, one must use a differential approach (see Chapter 4) and use computers to solve equations numerically.

Section 3.7 – Energy Equation

This is our 4th law to consider:

Energy (e) has several forms: $e_{internal}(\propto U), e_{kinetic}\left(\propto \frac{1}{2}mv^2\right), e_{potential}(\propto mgz)$, and many other chem., elec., …

Usually, we deal with the 1st 3 types:

- Work (w) can usually have 3 forms:

- W_{shaft}
- $W_{pressure} \rightarrow \dot{w}_P = \frac{dw_P}{dt} = P((\vec{v}\cdot\vec{n})dA)$
- $W_{viscosity} \rightarrow \dot{w}_v = \frac{dw_v}{dt} = -\tau\vec{v}dA$

- The heat (Q) has normally 3 forms: $Q_{conv.}, Q_{cond.}, Q_{rad}$, but in this course we mainly deal with isothermal flows (no details for heat transfer)

  Using RTT with $\beta = \hat{u} + \frac{1}{2}v^2 + gz$

  $$\frac{dE}{dt}\bigg| sys = \dot{Q} - \dot{w}_s - \dot{w}_v = \frac{\partial}{\partial t}\int_{cv}\left(\hat{u} + \frac{1}{2}v^2 + gz\right)\rho d\forall + \int_{cs}(h + \frac{1}{2}v^2 + gz)\,\rho(\vec{v}\cdot\vec{n})dA$$

  Where h = enthalpy = $\frac{P}{\rho} + \hat{u}$ → pressure work brought from LHS to RHS

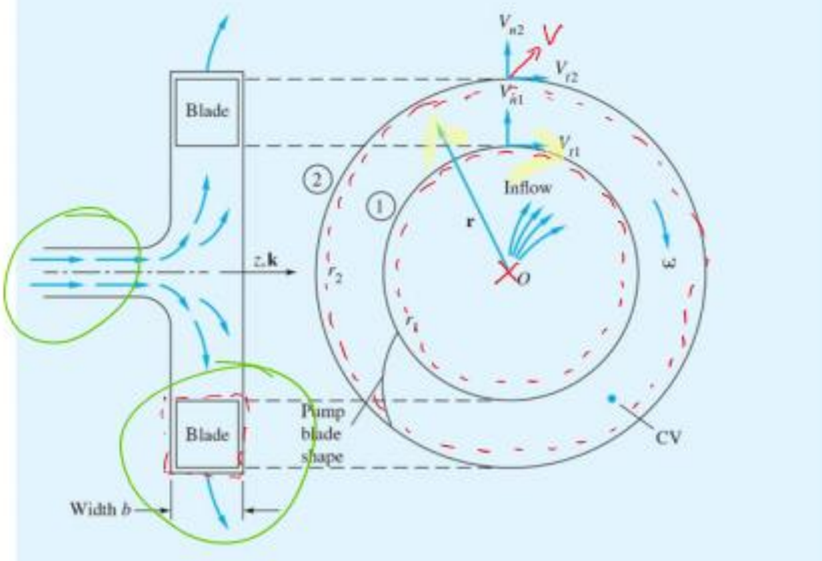
  For steady, 1D flow:

$$\dot{Q} - \dot{w}_s - \dot{w}_v = \sum\left(h + \frac{1}{2}v^2 + gz\right)_{out}\dot{m}_{out} - \sum(h + \frac{1}{2}v_m^2 gz)_{in}\dot{m}_{in}$$

Evaluate $\left(h + \frac{1}{2}v^2 + gz\right)$ *and* $(h + \frac{1}{2}v_m^2 gz)$ as an average over the in/out area

EXAMPLE 3.18

Figure 3.15 shows a schematic of a centrifugal pump. The fluid enters axially and passes through the pump blades, which rotate at angular velocity ω; the velocity of the fluid is changed from V_1 to V_2 and its pressure from p_1 to p_2. (*a*) Find an expression for the torque T_o that must be applied to these blades to maintain this flow. (*b*) The power supplied to the pump would be $P = \omega T_o$. To illustrate numerically, suppose $r_1 = 0.2$ m, $r_2 = 0.5$ m, and $b = 0.15$ m. Let the pump rotate at 600 r/min and deliver water at 2.5 m³/s with a density of 1000 kg/m³. Compute the torque and power supplied.



1. Pay attention to the how c.v. is selected to make the analysis a simple flow in → flow out problem.
2. No need to consider pressure as live action passes through "O", so momentum is generated
3. The same as 2 is true for normal component of velocity vector
4. Assume 1D flow due to defined c.v., then use Eq. 18

$$\sum M_o = T_o = (\vec{r}_2 \times \vec{v}_2)\dot{m}_{out} - (\vec{r}_1 \times \vec{v}_1)\dot{m}_{in}$$

T_o → Torque

Steady flow: $\dot{m}_{in} = \rho v_{n_1} \cdot 2\pi r_1 b = \dot{m}_{out} = \rho v_{n_2}^2 \pi r_2 b = \rho Q$

Q → Flow rate

$\vec{r}_2 \times \vec{v}_2 = r_2 v_{t2} sin90^{\circ} \hat{k}$ → where sin90 = 1; clockwise

$\vec{r}_1 \times \vec{v}_1 = r_1 v_{t1} k$

From (1): $T_o = \rho Q(r_2 v_{t2} - r_1 v_{t1})\hat{k}$ → clockwise

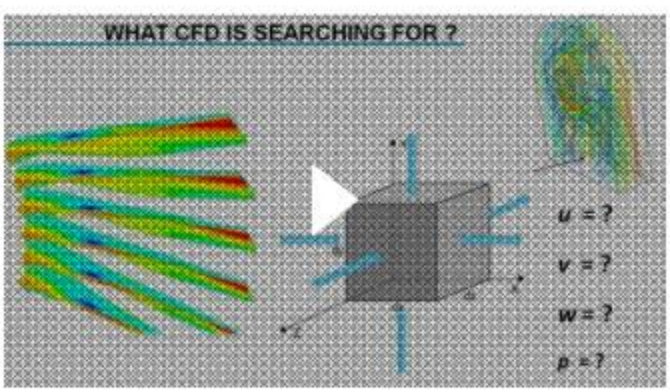
Knowing that $v_{t1} = wr_1$ & $v_{t2} = wr_2$

$T_o = \rho Q w(r_2^2 - r_1^2)$ → clockwise; Euler equation of pump

Do part (b) on your own.

Chapter 4

Watch this video Introduction to Computational Fluid Dynamics (CFD)



Chapter 4: Differential Approach to fluid flow

Sec 4.1

Some math reminders!

Remember $\vec{V} = v(x, y, z, t)$

$$\vec{a} = \frac{d\vec{V}}{dt} = \hat{\imath}\frac{du}{dt} + \hat{\jmath}\frac{dv}{dt} + \hat{k}\frac{dw}{dt} \quad (1)$$

But u = u (x, y, z, t)

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dy}{dt} + \frac{\partial u}{\partial t}\frac{dz}{dt} + \frac{\partial u}{\partial t} \quad (2)$$

Where $u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}$

Aside: $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ (Operator!)

$$a_x = \frac{\partial u}{\partial t}(V \cdot \nabla)u \quad (3)$$

$$V \rightarrow u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

Same way a_y and a_z can be found

Finally: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}$

$\frac{\partial \vec{V}}{\partial t}$ → local term

$(\vec{V} \cdot \nabla)\vec{V}$ → convective term due to e.g., change in geometry. Note: particles of fluid have relative motion w.r.t. one another unlike solids.

So even a steady flow, can have acceleration because of convective term (e.g., steady flow through a converging nozzle)

How about Pressure?

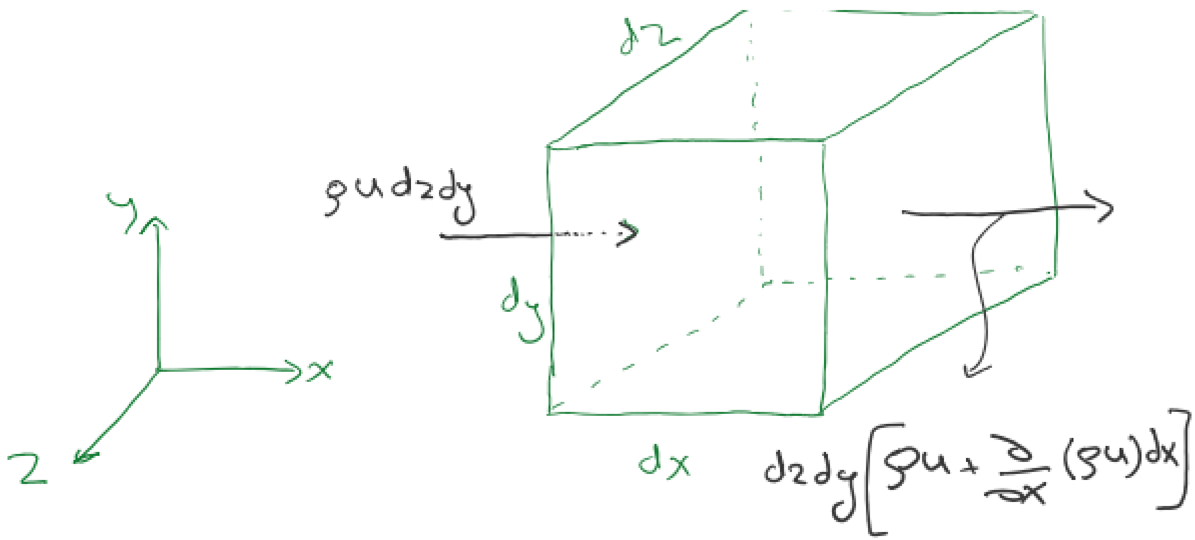
P = P (x, y, z, t), so in general:

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + (V \cdot \nabla)P \quad (4)$$

Aside: In an isotropic flow: $P = P_x = P_y = P_z$

Let's write on 4 laws to do differential analysis for flows:

Sec 4.2 Conservation of Mass



NOTE: All other pair of sides will also have mass flow into/out of them, but for graphical clarity not shown

Writing similarity for z & y directions AND considering the mass charge over time within the element (i.e., $\frac{\partial \rho}{\partial t} dxdydz$) that must be balanced to zero (to satisfy conservation of mass):

$$\frac{\partial \rho}{\partial t}\cancel{dxdydz} + \frac{\partial}{\partial x}(\rho u)\cancel{dxdydz} + \frac{\partial}{\partial y}(\rho v)\cancel{dxdydz} + \frac{\partial}{\partial z}(\rho w)\cancel{dxdydz} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \quad (5)$$

If the flow is incompressible, regardless of whether or not it is steady

$$\nabla(\rho \vec{V}) = 0 \quad (6)$$

Example: An inviscid flow through a 60˚ angle elbow. Here are the flowing velocity components

$$u = a(x^2 - y^2)$$

$$w = b$$

Where a & b are constants

If flow is incompressible, what is the velocity component v?

Solution: The flow everywhere should satisfy conservation of mass, momentum, etc. Looking at conservation of momentum, i.e., Equation 5 or 6, one can find v knowing u & w, so:

$$\frac{\partial}{\partial x}(ax^2 - ay^2) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial z}(b) = 0$$

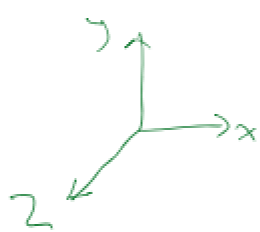
$$\frac{\partial v}{\partial y} = -2ax$$

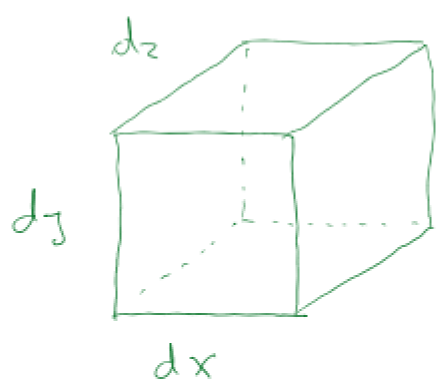
$$v = \int -2axdy = -2axy + f(x, z, t)$$

Where f (x, z, t) → arbitrary function

ASIDE: If the Mach number of flow is below 0.3, then any flow (gas or liquid) can be considered incompressible; so far air flow below ≈ 300km/hr in normal atmosphere is considered incompressible.

Let's do the 2nd law in Fluid Mechanics: Conservation of Momentum



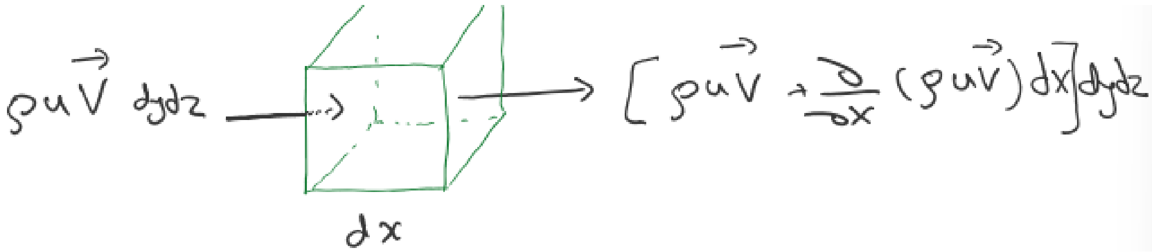


What type of forces act on this element?

1. Change in momentum in any x, y, or z direction <u>over the element</u>
2. Body force (e.g., gravity, electrical, magnetic, etc.)
3. Surface Forces
    a. Hydrostatic pressure (P)
    b. Viscos stresses (τ_{ij})

1 Momentum Flux

Consider the x-direction

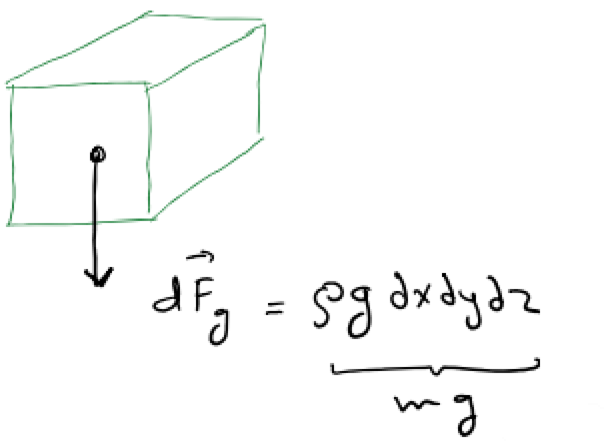


Not momentum flux in s-direction: $\frac{\partial}{\partial x}(\rho u \vec{V}) dxdydz$

Same can be done for the other two direction (i.e., y & z)

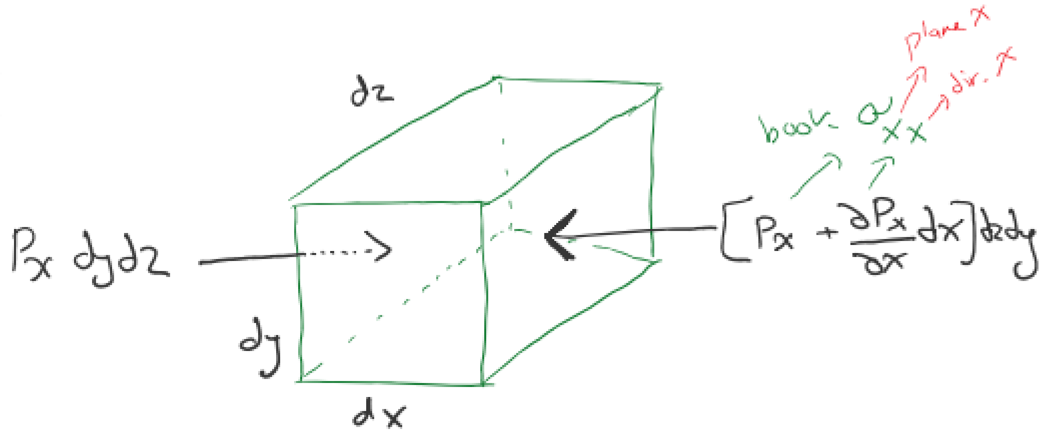
2 Gravity

Only in y-direction



3a Hydrostatic Pressure

Consider x-direction



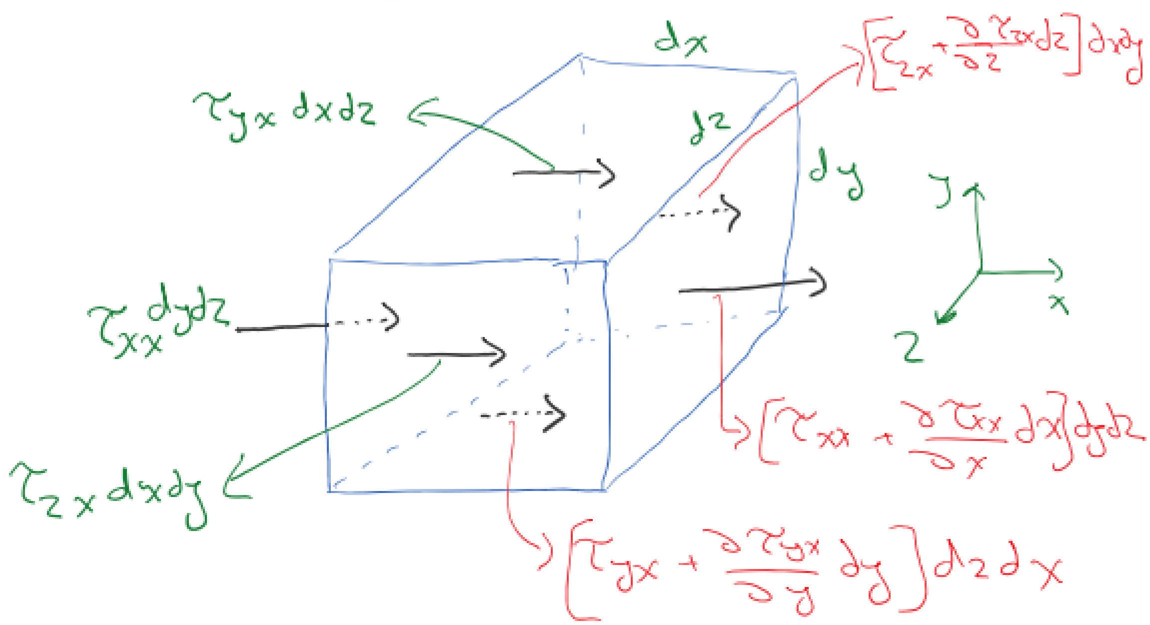
Net force = $-\frac{\partial P_x}{\partial x}dxdydz = \frac{\partial P}{\partial x}dxdydz$ (isotropic flow)

Negative direction can be seen by arrow in element.

The same treatment can be done for other directions.

3b Viscos Stresses

Consider x-direction



$\tau_{xx}dydz$ → normal stress due to shear forces

Net force due to shear in x-direction = $[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \tau_{yx}}{\partial y}]dxdydz$

Same for other directions → a viscos stress term!

Consider 2nd law of Newton: $\vec{F} = m\vec{a}$

Let's write it in the x-direction:

$$\frac{\cancel{Gravity\ force}(zero\ in\ x - direction) + Pressure + Viscos}{Unit\ Volume\ (dxdydz)} = density\ \times acceleration(Eq.3)$$

$$-\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\frac{\partial u}{\partial t} + \rho(v \cdot \nabla)u\ (7)$$

Where, $-\frac{\partial P}{\partial x}$ → hydro-static pressure and $\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$ is viscos.

In the y-direction:

$$\rho g - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\frac{\partial v}{\partial t} + \rho(\vec{v} \cdot \nabla)v\ (8)$$

Where, ρg → due gravity

In the z-direction:

$$-\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\frac{\partial w}{\partial t} + \rho(\vec{v} \cdot \nabla)w\ (9)$$

Equations (7-9) are general momentum (linear) equations in 3D

<u>For inviscid flow:</u>

$$-\frac{\partial P}{\partial x} = \rho\frac{\partial u}{\partial t} + (\vec{v} \cdot \nabla)u$$

$$\rho g - \frac{\partial P}{\partial y} = \rho\frac{\partial v}{\partial t} + (\vec{v} \cdot \nabla)v$$

$$-\frac{\partial P}{\partial z} = \rho\frac{\partial w}{\partial t} + (\vec{v} \cdot \nabla)w\ \ (10)$$

Euler Equations for fluid flow

Equations (7-9) can be recasted for Newtonian fluids by relating shear to the rate of change of velocity in space (x, y, z), using the concept of viscosity.

The 3D treatment of viscosity for an incompressible fluid is:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), etc.$$

Substituting above relationships for τ into equations 7-9, one has:

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{\partial u}{\partial t} + \rho(\vec{v}\cdot\nabla)u$$

$$\rho g - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{\partial v}{\partial t} + \rho(\vec{v}\cdot\nabla)v$$

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{\partial w}{\partial t} + \rho(\vec{v}\cdot\nabla)w \ (11)$$

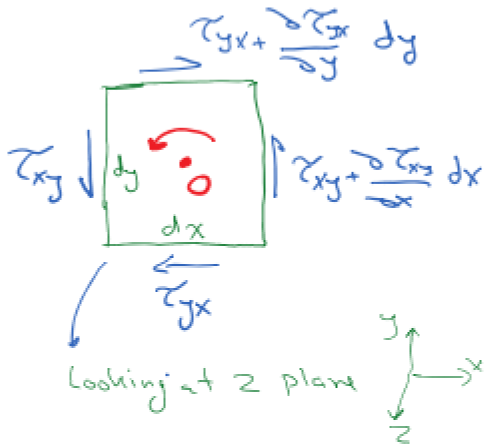
Where, $\rho(\vec{v}\cdot\nabla)u$, $\rho(\vec{v}\cdot\nabla)v$, and $\rho(\vec{v}\cdot\nabla)w$ are the convective term & it is a non-linear differential equation.

- ➔ RHS if equations are the inertial terms

Equation (11) is called <u>Navier-Stokes Equation</u> for <u>incompressible</u> fluids.

Sec 4.4 Differential Equations of Angular Momentum (3rd flow)

Let's consider the notation around centroid of an element (O, i.e., where z-axis crosses the x-y plane)



If I write the equation for moment calculation, I find that:

$$\tau_{xy} \approx \tau_{yx}$$

For rotation around y or x axes, one finds:

$$\tau_{xz} \approx \tau_{zx} \,\&\, \tau_{yz} \approx \tau_{zy}$$

The above means that there is no differential equation form for conservation of angular momentum → one should use the integral forms given in chapter 3.

Note 1: P, τ_{xz}, τ_{xz}, and τ_{xz} all pass through the centroid of the element (O) so they have momentum around O.

Note 2: Fluid similar to solids experiences symmetric shear stresses.

Sec. 4.4 Differential Equations for Conservation of Energy (4th Law)

It can be shown that the conservation of energy equation for an element will be equation (12):

$$\dot{Q} - \dot{w}_v = \left(\rho \frac{de}{dt} + \vec{v} \cdot \nabla P + \rho \nabla \cdot \vec{v}\right) dxdydz \quad (12)$$

$\frac{de}{dt}$ → defined in chapter 3

Note: There is no infinitesimal shaft work (a mechanical shaft cannot be the size of an element)

- ➔ Need to find differential forms of $\dot{Q}$ and $\dot{w}_v$

1. Heat Transfer to the Element
   To deal with $\dot{Q}$, we only consider conduction (vast application only have conduction)

Fourier's Law for Conduction

$\dot{q}_x = -k\frac{\partial T}{\partial x}$ or in general (3D)