Lecture 7 Notes:

$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\sigma u}{\sigma y} + \frac{\sigma v}{\sigma x}\right), \tau_{xz} = \tau_{zx} = \mu\left(\frac{\sigma u}{\sigma z} + \frac{\sigma w}{\sigma x}\right)$ etc. etc.

Substituting above relationships for τ into Eqs. 7-9, one has:

$$-\frac{\partial \rho}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{\partial u}{\partial t} + \rho(\vec{v}.\nabla)u$$

$$\rho g - \frac{\partial \rho}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{\partial v}{\partial t} + \rho(\vec{v}.\nabla)v$$

$$\rho g - \frac{\partial \rho}{\partial y} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{\partial w}{\partial t} + \rho(\vec{v}.\nabla)w$$

- *Underlined In Blue:* Convectine term & it is non-linear diff. Eq.
- *Underlined In Red:inertial terms*
- Eq(11) is called Navier-Stokes Eq for incompressible fluids
    - Eq(11) is usually solved by CFD Techniques (Computational Fluid Dynamics) due to non-linearity – only a few analytical solutions exists

Our Goals for Today (lect. 7)

1. Cont'd Chapter 4 and introducing differential form of conservation of Angular momentum & energy
2. Solving sample problems using differential forms of conservation equations
3. Intro. to irrotational flow & vorticity (time permitting)

Sec. 4.4 – Diff. Eq.s of Angular Momentum (3rd law)

Lets consider the notation around centroid of an element (0, i.e. where Z axis crosses the x-y plane

If I write the quation for moment cqlculation I find that

$\tau_{xy} \approx \tau_{yx}$

For notation around y or x axes, one finds:

$\tau_{xz} \approx \tau_{zx}$ & $\tau_{yz} \approx \tau_{zy}$

The above means that there is no diff. Req. Form for conservativation of angular momentum → one should use the integral form given in chapter 3

Note 1: $\rho, \tau_{xx}, \tau_{yy}, \tau_{zz}$ all pan through the centroid of the element(0) so they have no moment around 0.

Note 2: Fluid similar to solids experfiences symmetric shear stressors.

Sec 4.4. Diff. Eq. For Conservation of Energy (4th law)

It can be shown that the conservation of energy eq. For an element will be eq.(12):

$$\dot{Q} - \dot{w}_v = \left(\rho\frac{\mathrm{de}}{\mathrm{dt}} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v}\right)dxdy\,\mathrm{dz}$$

Note: There is no infinitesimal shaft wor (a mechanical shaft cannot be the size of an element)

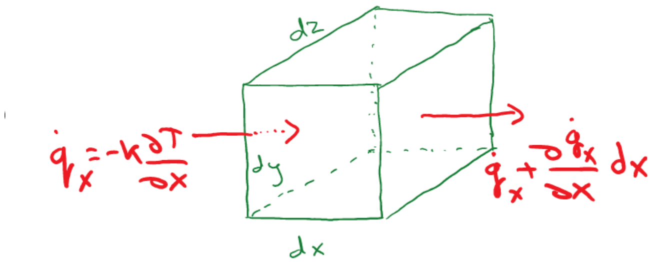
→ Need to find differential forms of $\dot{Q}$ & $\dot{w}_v$

1- Heat transfer to the element to deal with $\dot{Q}$, we only consider conduction (vast application only have conduction)

Fourier’s Law for Conduction

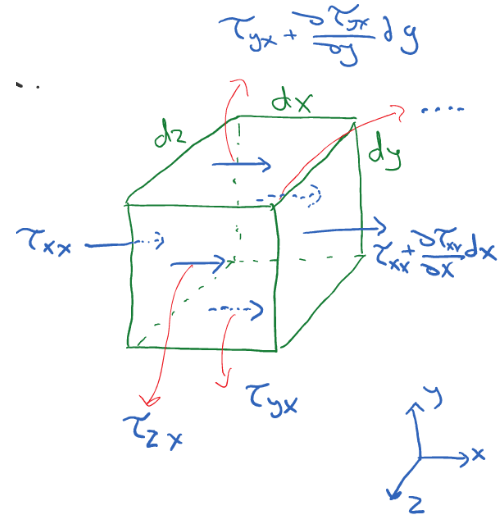
$\dot{q}_x = -k\frac{\partial T}{\partial x}$ or in general (3D)

k = coefficient of thermal conductivity $\hat{q} = -k\nabla\hat{T}$



Note: internal elememtal energy is ignored (e.g. chem. Reaction)
Th net tranfer to the element in the x-dir.

$\} = \dot{q}_x \mathrm{dydz} - [\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx]dydz = -\frac{\partial \dot{q}_x}{\partial x} dxdydz \rightarrow$
convetion from heat transfer



Considering the conduction in all 3 directions:

$\dot{Q} = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dxdydz = -\nabla \hat{q} dxdydz$ (13)

2- Finding Diff/ Eq. For Viscos Work ($\dot{w}_v$)

$\dot{w}_v$ = shear force x area x corresponding vel.

So viscos work (or viscos disipation) in the x-dir:

$\dot{w}_x = -[\tau_{xx} u + \tau_{yx} v + \tau_{zx} w] dydz$

Viscos wotrk out is: $\dot{w}_x + \frac{\partial \dot{w}_x}{\partial x} dx$

Considering all 3 directions:

$$\dot{w}_v = -\left[\frac{\partial}{\partial x}\ (u\tau_{xx} + v\tau_{yx} + w\tau_{zx}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})\right] dxdydz$$

In Compact form:

- $\dot{w}_v = -\nabla(\vec{v}.\tau_{ij}) dxdydx$ (14)
- Practical form of Eq(14) where viscos losses are seperate is as follows:
- $\dot{w}_v = -[\vec{v}.(\nabla\, \tau_{ij}) + \varphi] dxdydx$ (15)
- $\varphi \rightarrow$ Viscos-dissipation term (essentially this is flow "losses"

  Where

  $$\varphi = \mu[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2]$$

  - Only for Newtonian fluids
  - φ is always positive (due to all terms having powers of 2), which makes senses, as losses in a flow must be positie to comply with 2nd law of Thermodynamics

Sub in Eqs. (13) & (15) into eq(12), cons. Of energy, and after simplification (note: $e = \hat{u} + \frac{1}{2}v^2 + gz$), and utilizing the conservation of linear momentum:

$\rho \frac{d\hat{u}}{dt} + \rho(\nabla \cdot \vec{v}) = \nabla(k\nabla T) + \varphi (16)$

Eq(16) is the general form of conservation of energy for a flow

If we consider the flow in comprenible and $d\hat{u} = C_v dT, \mu\ and\ k\ \approx const.$

$\rho C_v \frac{d\hat{u}}{dt} + \rho(\nabla \cdot \vec{v}) = \nabla(k\nabla T) + \varphi (1)$

dT→ Total diff. On T(x,y,z,t)

φ→ dissipation term (if the flow is very very slow, it can be ignored)

How to solve above equations with 5 unknowns (i.e. $\rho, \vec{v}, \hat{u}, and\ T$) when we have 3 eqs.?

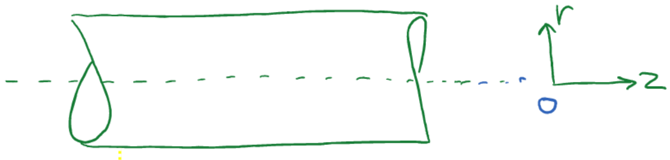
Read Sec 4.6 of the book to learn about various BC

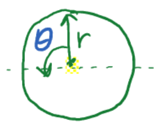
- Eq. of state (idel gas law)
- Inlet/outlet info. V, p
- Solid wall→ no slip no. Temp. Jump
- Free interface: vertical vel (w) equity & mech. Force balance at the interface (τ) ; or thermal conditions, e.g. equal temp. or equal thermal flux.

Sec 4.10 (For now we will skip Sec 4.7-4.9):

Read sec 4.10 for analytically solving some special cases here we deal with the example of a Laminin flow within a pipe.

Ex: What is the vel. Profile in the pipe shown. Assume the flow is laminin, and fully developed. Neglect the gravity and assume the flow is steady and incompressible. As this is a straight pipe flow can be considered axisymmetric.





pipe cross section

So!

- What co-ordinate system to use?
  Cyl.
- What does axial flow means?
  A flow that is fully developed is an axial flow $v_r = v_\theta = 0$
- What does axial symmetry mean?
  $$\frac{\partial}{\partial\theta} = 0$$
- What does it mean that flow is incompressible $\frac{\partial\rho}{\partial\uparrow} = 0$ & ρ=constant

Let's start with continuity (cons. Of mass): $(\nabla \cdot \vec{v}) = 0$ in cyl. Coordinates (see Sec 4.2) has the form

$$\cancel{\frac{\partial\rho}{\partial t}} + \frac{1}{r}\frac{\partial}{\partial r}(r\cancel{\rho v_r}) + \frac{1}{r}\cancel{\frac{\partial}{\partial\theta}(\rho v_\theta)} + \frac{\partial}{\partial z}(\rho v_z) = 0$$

1st term→ incompressible

v_r and v_θ → Zero, since flow was fully developed, i.e. arial flow

$\rho\frac{\partial v_z}{\partial z} = 0 => \frac{\partial v_z}{\partial z} = 0\ or\ v_z = v_z(r)$ where t-steady, θ axisymmetric, z – fully developed

Writing N.S. eq. in Cyl. Co-ordinate r-component:

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}}\partial\cancelto{0}{\frac{v_\theta^2}{r}} + v_z\underbrace{\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial v_r}{\partial r} + \cancel{\rho g_r} + \mu[\frac{1}{r}\frac{\partial}{\partial r}(r\cancel{\frac{\partial v_r}{\partial r}} - \cancel{\frac{v_r}{r^2}} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}$$

ρg_r=0 no gravity

$\left(-\frac{\partial \rho}{\partial r} = 0 => \frac{\partial \rho}{\partial r} = 0 = \rho = \rho(2)\right)$

$\rho -> (t, r, \theta, z)$

No flow in θ dire => no eqs. In θ dire. since $\frac{\partial}{\partial \theta} = 0$

Z-dir:

$$\rho\left[\cancel{\frac{\partial v_z}{\partial t}} + v_r \cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z \cancel{\frac{\partial v_z}{\partial z}}\right] = -\frac{\partial \rho}{\partial z} + \cancel{\rho g_z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)$$

$\frac{\partial v_z}{\partial t}$→ steady flow

ρg_r=0 no gravity

$$\frac{\partial \rho}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

Integrated w.r.f. r twice:

$$v_z = \frac{\partial \rho}{\partial z}\frac{r^2}{4\mu} + C_1 In(r) + C_2$$

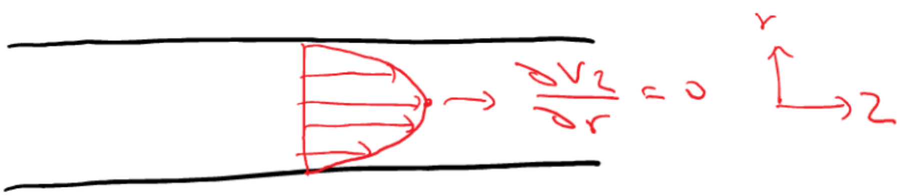
$In(r)$→ to avoid singularity ar r=0 → v_z = finite value→ $\frac{\partial v_z}{\partial r}|$ =0. R=0 –>centaline of pipr, ∂r →2nd BC

1st BC no slip at r=R, R=pipe radius

Using the 2 BC

$$v_z = \left(-\frac{\partial \rho}{\partial z}\right)\frac{1}{4\mu} + (R^2 - r^2)$$

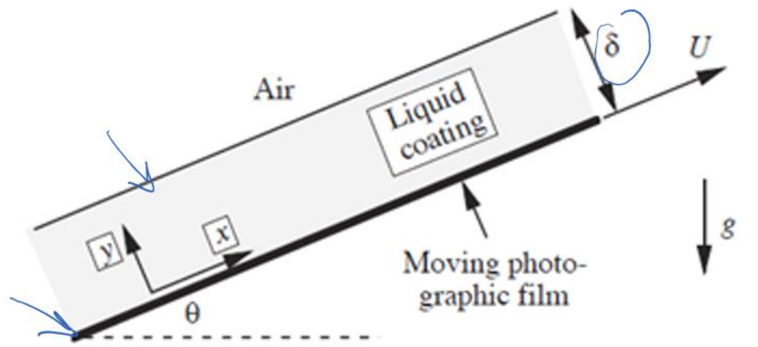
Eq. of a paraboloid



Note: Knowing vel. Profile allows one to find:

1. Flow Rate (Q), i.e. $Q = \int V_z \, dA$ where $dA = 2\pi r \, dr$
2. Shear at the wall (T_{wall}): $T_{wall} = \mu\left|\frac{\partial v_z}{\partial r}\right|_{r=R} = \frac{R}{2}\frac{\Delta p}{L}$; L= length of pipe segment Δp – prev. Drop along z-dir., for pipe length L.

Fig. E6.3.1 shows a coating experiment involving a flat photographic film that is being pulled up from a processing bath by rollers with a steady velocity U at an angle θ to the horizontal. As the film leaves the bath, it entrains some liquid, and in this particular experiment it has reached the stage where: (a) the velocity of the liquid in contact with the film is $V_x = U \; at \; y = 0$, (b) the thickness of the liquid is constant at a value ∂, and (c) there is no net flow being pulled up by the film as is falling back by gravity). (Clearly, if the film were to retain a permanent coating, a net upwards flow of liquid would be needed.)



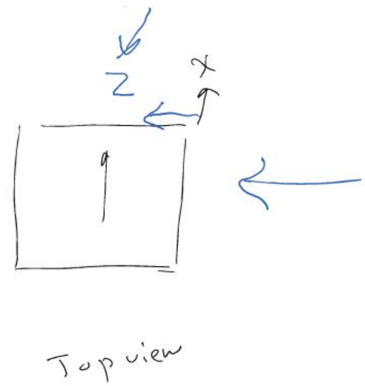


Fig. E6.3.1 Liquid coating on a photographic film.

Note: Since there is no temp change (energy or work in/out), then we don't concern ourselves with the energy eq. (also because we're not concerned with losses (φ term ins eq.)

Assumptions:

- Newtonian fluid, steady flow, p=constant
- No variation of vel. In 2 dir. => 2D flow
- No rotation => no worries for conservation angular momentum => continuity linear momentum

  Continuity- Linear 2D Momentum: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = \frac{-\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{-\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g_y$$

- Note: Orientation being x or y axis for ρg_x and ρg_y
- Note: looking from top and side views of the system +> u>>v or w, so we can neglect v & w or v=w=0

Momentum:

$$\frac{\partial P}{\partial x} - Pg\sin\theta = \mu\frac{\partial^2 u}{\partial y^2} \qquad g_x = g\sin\theta$$

$$\frac{\partial P}{\partial y} = +pg\cos\theta \quad g_y = g\cos\theta$$

Continuity $\frac{\partial u}{\partial x} = 0$ *note v=0*

BC to solve

1. y=0, u=U
2. all free surface shear force(T_{xy}) is zero, mechanical balance:
   $$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{2y} = O \quad at\ y = \delta$$
3. Since we have a flat interface with the dir. At $y = \delta$=> P=0; gauge at the moment

Solve above PDF with the 3 BC:

Answer:

$$u = U - \frac{\rho g\sin\theta}{\mu}y\left(\delta - \frac{y}{2}\right)$$

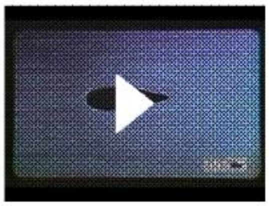
$$\delta = \sqrt{\frac{3u\mu}{\rho g \sin\theta}}$$

Note: N.S eqs. Are complicated, so to find analytical solution, either if the phys. Y problem allows, simplification, e.g. prob. Above, or use of math tech should be done to find solution Otherwise CFD is needed

Irrotation flow and vorticity

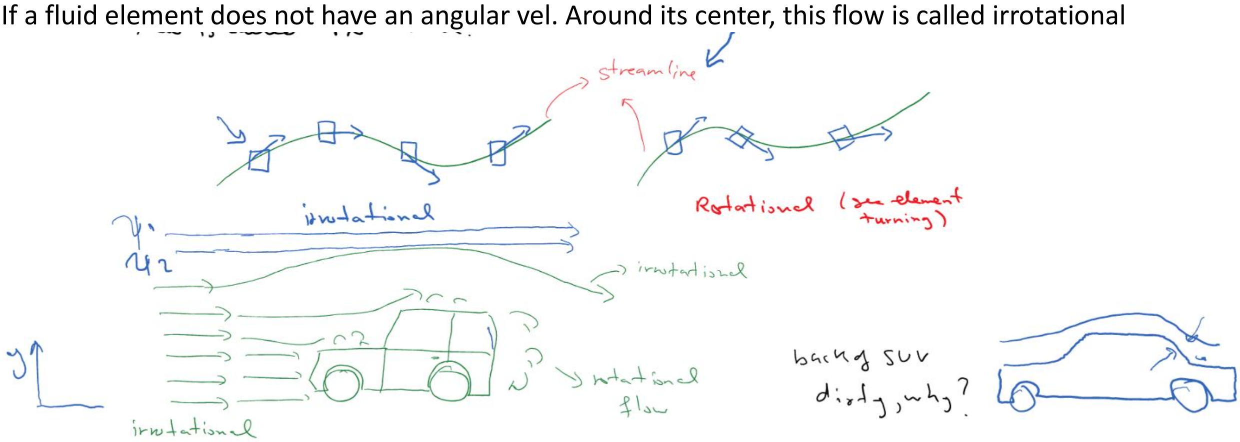
These concepts are math. Tech. to help with finding analytical sol. To POE of NS; see videos

Rotational & irrotational flows

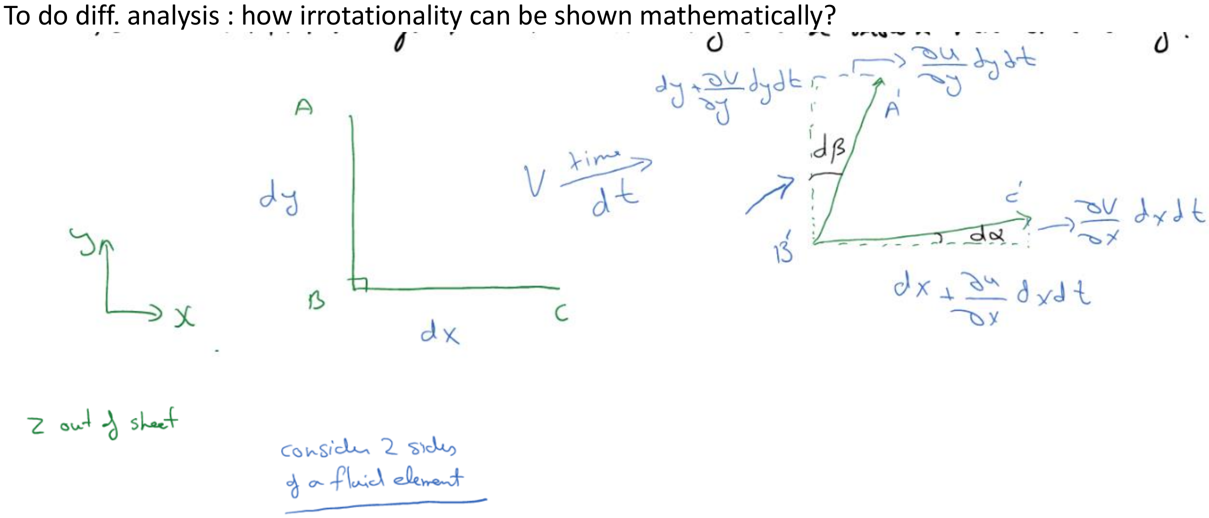


Curl - Grad, Div and Curl (3/3)

If a fluid element does not have an angular vel. Around its center, this flow is called irrotational



To do diff. analysis : how irrotationality can be shown mathematically?



Avg. rotational vel. Of element around 2 axis (ω_z):

$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$ – the sign of operation is the direction of notation 9see fig. above)

By trigonometry, it can be shown

$$\frac{d\alpha}{dt}=\frac{\partial v}{\partial x}\ \&\ \frac{dP}{dt}=\frac{\partial u}{\partial y}$$

Then

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

For rotation around x & y it can be shown:

$$\omega_x=\frac{1}{2}\left(\frac{\partial \omega}{\partial y}-\frac{\partial v}{\partial z}\right)\quad \&\ w_y=\frac{1}{2}1\frac{\partial u}{\partial z}-\frac{\partial \omega}{x}$$

$$\vec{\omega}=i\omega_x+j\omega_y+k\omega_z=\frac{1}{2}\nabla\times\vec{v}=\frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & \omega \end{vmatrix}$$

$\nabla\times\vec{v}$→ curl $\vec{v}$

Vorticity-> $\zeta=2\vec{\omega}=\nabla\times\vec{v}$

If $\nabla\times\vec{v}=0$ or zero vorticity, then flow is irrotational (18)

What is this good for?

We have a math tool to solve for flow field (get v, u, P …)

- This tool is best suited for 2D flows where there is no work or heat involved since there is no work or heat, I can be concerned with conservation of man & momentum:

For an incompressible flow, where flow is steady:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{(i)}$$

$$\rho\hat{g}-\nabla P+\mu\nabla^2\vec{v} \quad \text{(ii)}$$

Consider a function ψ exists ($\psi(x,y)$ that one can write:

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)=0 \text{ (iii)}$$

Then by inspection of eq.(iii) vis. a. vis. eq(i):

$$u=\frac{\partial\psi}{\partial y}\ \&\ v=\frac{\partial\psi}{\partial x} \text{ (19)}$$

If one taken the curl of momentum Eq. (i.e. ∇x[Eq, (ii)]), and use ψ *definition from eq. 19, then:*

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)=\frac{\mu}{\rho}\nabla^2(\nabla^2\psi) \text{ (iv)}$$

Eq (iv) is a 4th order PDE that needs 4BC

Aside from driving Eq. (iv) $\begin{cases}\nabla x(\nabla F)=0\\ \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\nabla^2\psi\end{cases}$

Special Case!

Irrotational flow => ∇x $\vec{v}$=0 → in 2D $\quad -k\nabla^2\psi=0$

Eq iv) now simplified to a 2nd order PDE, which needs 2 BC only

BC$\begin{cases}\psi=const(at\ body)\\ \psi=U_\infty y+const\end{cases}$ (at far field _-> very much away from the object)

Interestingly, ψ represents stream lines in a physical sense, hence it is called a stream function

Definition of stream line $\frac{dy}{dx}=\frac{v}{n}$→ satisfy $\psi\ u\,dy-v\,dx=0$

It can show that$Q_{1\to2}=\Psi_2-\psi_1$ with Q as flow rate and $\Psi_2-\psi_1$ scalar

OR $\dot{m}_{1\to2}=\psi_2-\psi_1$ if we write the above same formulation but keep the ρ in the eq. (i)

Therefore the concept of stream function(ψ) works for both compressible and incompressible flows, but not for 3D or unsteady flow

Ex. If a stream function exists for vel. Field as follows, find it, plot it, and interpret it.
$u = a(x^2 - y^2), v = -2axy$

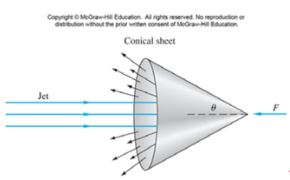
Looking at vel. Functions → flow is 2D and it's independent of time -> steady => chance ρ exists

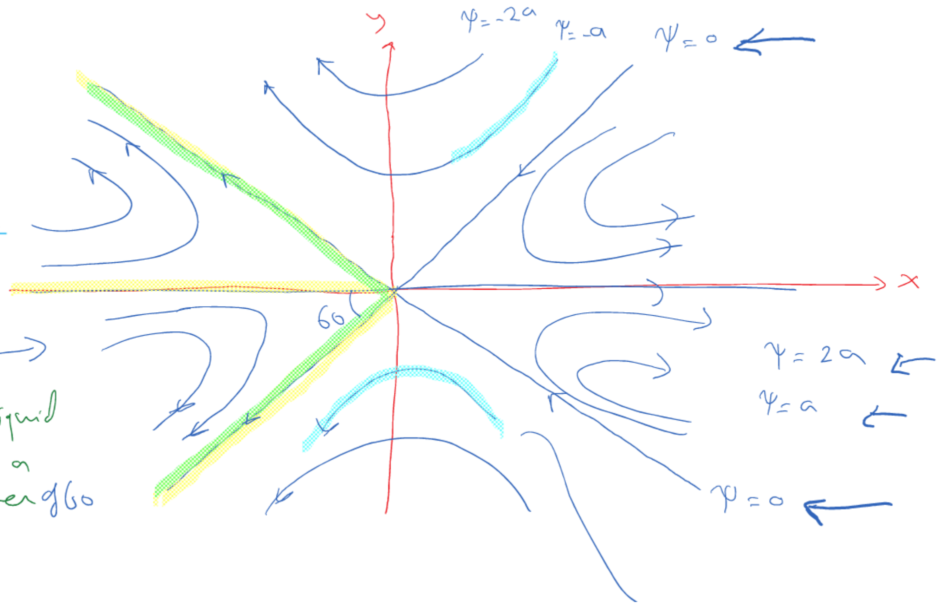
$$u = \frac{\partial\psi}{\partial y} = ax^2 - ay^2 \Rightarrow \Psi = ax^2y - \frac{a}{3}y^3 + f(x)$$

To find $f(x)$ $\frac{\partial\psi}{\partial y} = 2axy - 0 + f(x)$ -> $f(x) = 0 \rightarrow f = constant$

$$v\left(= -\frac{\partial\psi}{\partial x}\right)$$

$$\psi = \partial\left(x^2y - \frac{y^3}{3}\right) + C$$

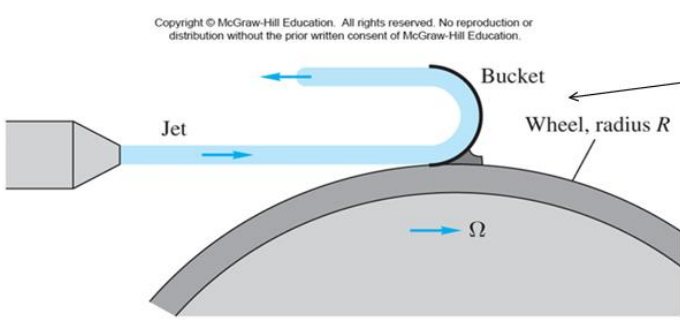


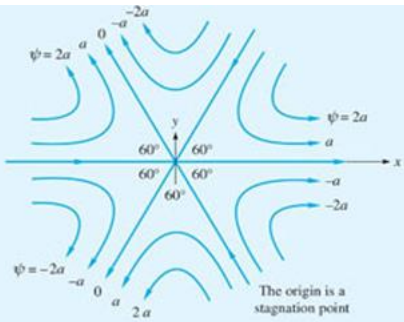


jet →

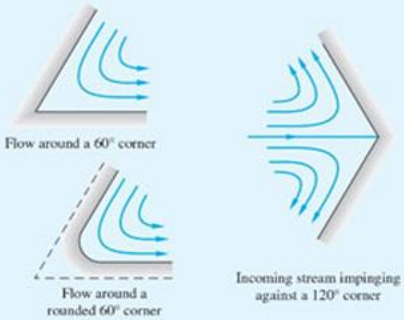
jet of liquid impacting a corner of 60

→ Can represent a curved vane seen in turbine (water)





E4.7*a*



E4.7*b*

By allowing the flow to slip as a frictionless approximation, we could let any given streamline be a body shape. Some examples are shown in Fig. E4.7*b*.