1. Pseudoinverse matrices. Skeletonization. Singular value decomposition (SVD)

Problem 1. Skeletonization and pseudoinverse matrix

Find the pseudoinverse matrix to matrix A using skeletonization

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Solution: Let's start with skeletonization and then find the pseudoinverse matrix by the formula

$$A^+ = G^*(GG^*)^{-1}(F^*F)^{-1}F^*,$$

$$A = \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{matrix}}}^{F} & \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \overbrace{\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{matrix}}}^{G} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix}.$$

Let's check

$$F \cdot G = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Correct. Now we need to find the pseudoinverse matrix

$$G^*(GG^*)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix},$$

$$(F^*F)^{-1}F^* = \left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \tag{1}$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

All things considered, we can obtain pseudoinverse matrix A^+

$$A^+ = \frac{1}{36} \begin{bmatrix} 5 & -2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 13 & -4 & 0 \\ -4 & 4 & 0 \\ 5 & 4 & 0 \end{bmatrix}.$$

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Note

Matrix (GG^*) is called Gramm matrix and contains results of scalar products.

Definition: SVD

Singular Value Decomposition of matrix $A \in M_{m\times n}(\mathbb{C})$ is a decomposition of a kind

$$A = U\Sigma V^*,$$

where $U \in M_{m\times m}(\mathbb{C})$ and $V \in M_{n\times n}(\mathbb{C})$ are unitary matrices

$$U^*U = UU^* = UU^{-1} = I$$
$$V^*V = VV^* = VV^{-1} = I$$

and $\Sigma \in M_{m\times n}(\mathbb{C})$ is a diagonal matrix of a kind

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & \sigma_r \\ & & \mathbf{0} & \end{bmatrix},$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ – singular values and $r = \operatorname{rank} A$.

Theorem: How to find SVD

The eigenvectors of A^*A make up the columns of V , the eigenvectors of AA^* make up the columns of U, the singular values in Σ are square roots of eigenvalues from AA^* or A^*A.

Theorem: How to find pseudoinverse using SVD

Let $U\Sigma V^*$ be the SVD of a matrix $A \in M_{m\times n}(\mathbb{C})$, then

$$A^+ = U\Sigma^+ V^*.$$

Problem 2. Singular Value Decomposition

Find singular value decomposition of a matrix

$$A = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

<u>Solution:</u> Let's start with finding $\Sigma \in M_{2\times 2}(\mathbb{R})$

$$A^*A = \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 25 \\ 25 & 25 \end{bmatrix}.$$

Next step is to find eigenvalues and eigenvectors for the matrix A^*A.

$$(A^*A - \lambda_2 I)\vec{v}_2 = 0 \Leftrightarrow \det\left(\begin{bmatrix} 25-\lambda & 25 \\ 25 & 25-\lambda \end{bmatrix}\right) = 0 \Leftrightarrow \lambda^2 - 50\lambda = 0 \Leftrightarrow \lambda_1 = 50,\ \lambda_2 = 0.$$

Hence singular values are equal to $\sigma_1 = \sqrt{50},\ \sigma_2 = 0$ and

$$\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let's find $V \in M_{2\times 2}(\mathbb{R})$. We calculate eigenvalues of previously obtained matrix A^*A.

$$(A^*A - \lambda_1 I)\vec{v}_1 = 0 \Leftrightarrow \begin{bmatrix} -25 & 25 \\ 25 & -25 \end{bmatrix}\vec{v}_1 = 0.$$

One of the solutions is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We need to normalize it, so $\vec{v}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly by solving $(A^*A-\lambda_2 I)\vec{v}_2 - 0$, we get $\vec{v}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Combining $\vec{v}_1$ and $\vec{v}_2$, we get

$$V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let's find matrix $U \in M_{3\times 3}(\mathbb{R})$, which is constructed of a new matrix AA^* eigenvectors.

$$AA^* = \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 4 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & 32 \end{bmatrix}.$$

We calculate the new matrix AA^* eigenvectors

$$(AA^* - \lambda_1 I)\vec{u}_1 = 0 \Leftrightarrow \begin{bmatrix} -32 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & -18 \end{bmatrix}\vec{u}_1 = 0.$$

Solving by Gaussian elimination

$$\left[\begin{array}{ccc|c} -4 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

Similarly by solving $(AA^* - \lambda_2 I)\vec{u} = 0$, we get

$$\left[\begin{array}{ccc|c} 18 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 0 & 32 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{1}{5}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}.$$

All things considered, the result of singular value decomposition is the following

$$A = U\Sigma V^*,$$

where

$$U = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

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Note

We could find more simple decomposition of the matrix A by omitting all calculations with zero singular values. Such decomposition is called compact version of the SVD

$$A = U_r \Sigma_r V_r^*, \quad r = \operatorname{rank} A.$$

For example, matrix A from the previous problem has following compact SVD

$$A = U_r \Sigma_r V_r^*,$$

where

$$U_r = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \Sigma_r = \begin{bmatrix} \sqrt{50} \end{bmatrix}, V_r = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$