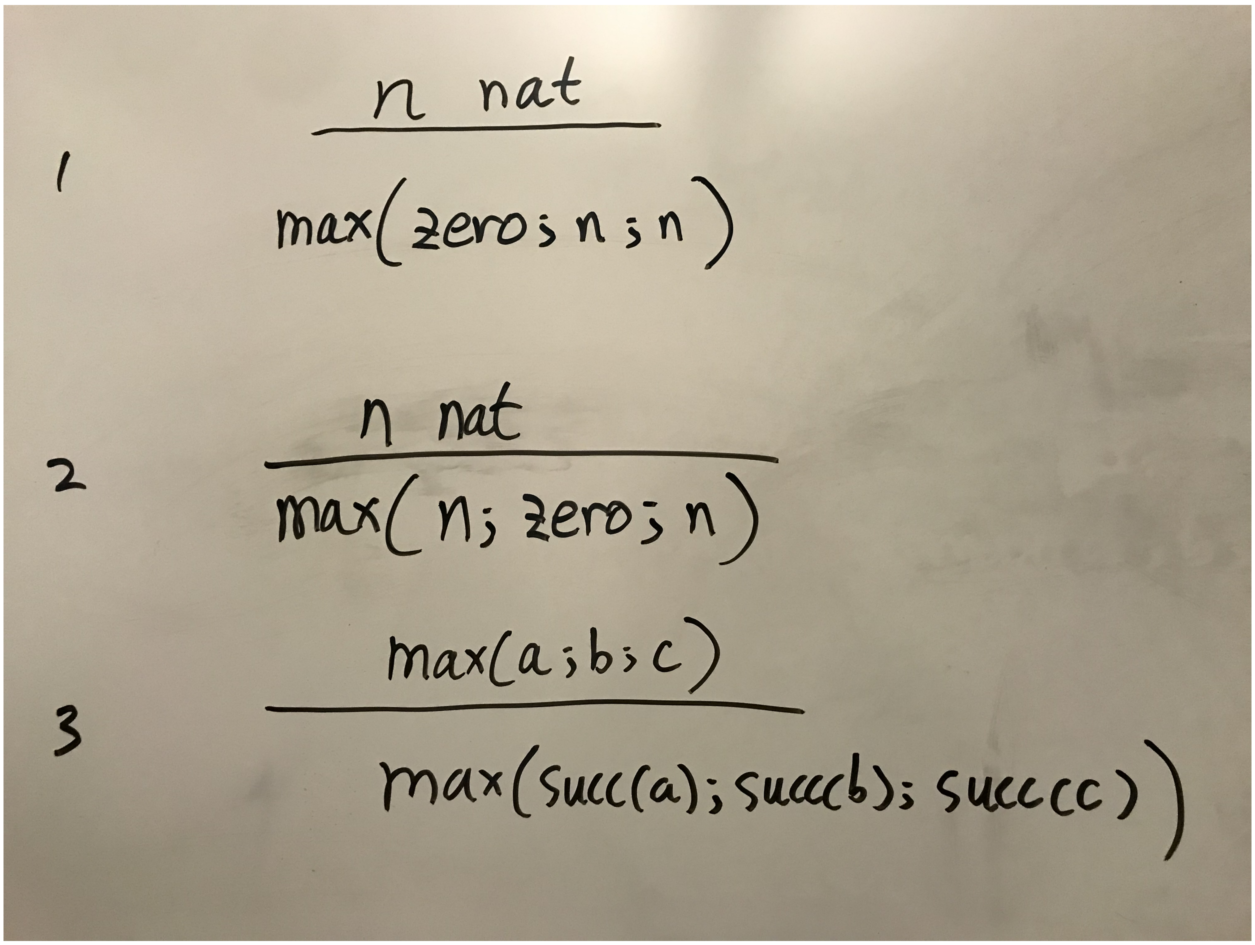
Exercise 2.1-2.6

2.1

Inductive Definition:



Proof. The proof decomposes into two parts:

1. (Existence) If `a nat` and `b nat` then there exists `c nat` such that `max(a;b;c)`
2. (Uniqueness) If `max(a;b;c)` and `max(a;b;c')` then `c is c'`

For existence, Let $\mathcal{P}(a, b)$ be the proposition by fixing `a nat` and `b nat` then there exists `c` such that `max(a;b;c)`.

   1. We are to show $\mathcal{P}(zero, b)$. Then let c be b.

2. We are to show $\mathcal{P}(a, zero)$. Then let c be a.

3. Assume $\mathcal{P}(a, b)$ and let c be such that `max(a;b;c)`. Then take c' to be $succ(c)$. Then we have `max(succ(a); succ(b); succ(c))`.

Uniqueness is similar so we will skip

2.2

To show function `hgt` is well-defined, we need to show that `t tree` then there exists an unique `n nat` such that `hgt(t; n)`.

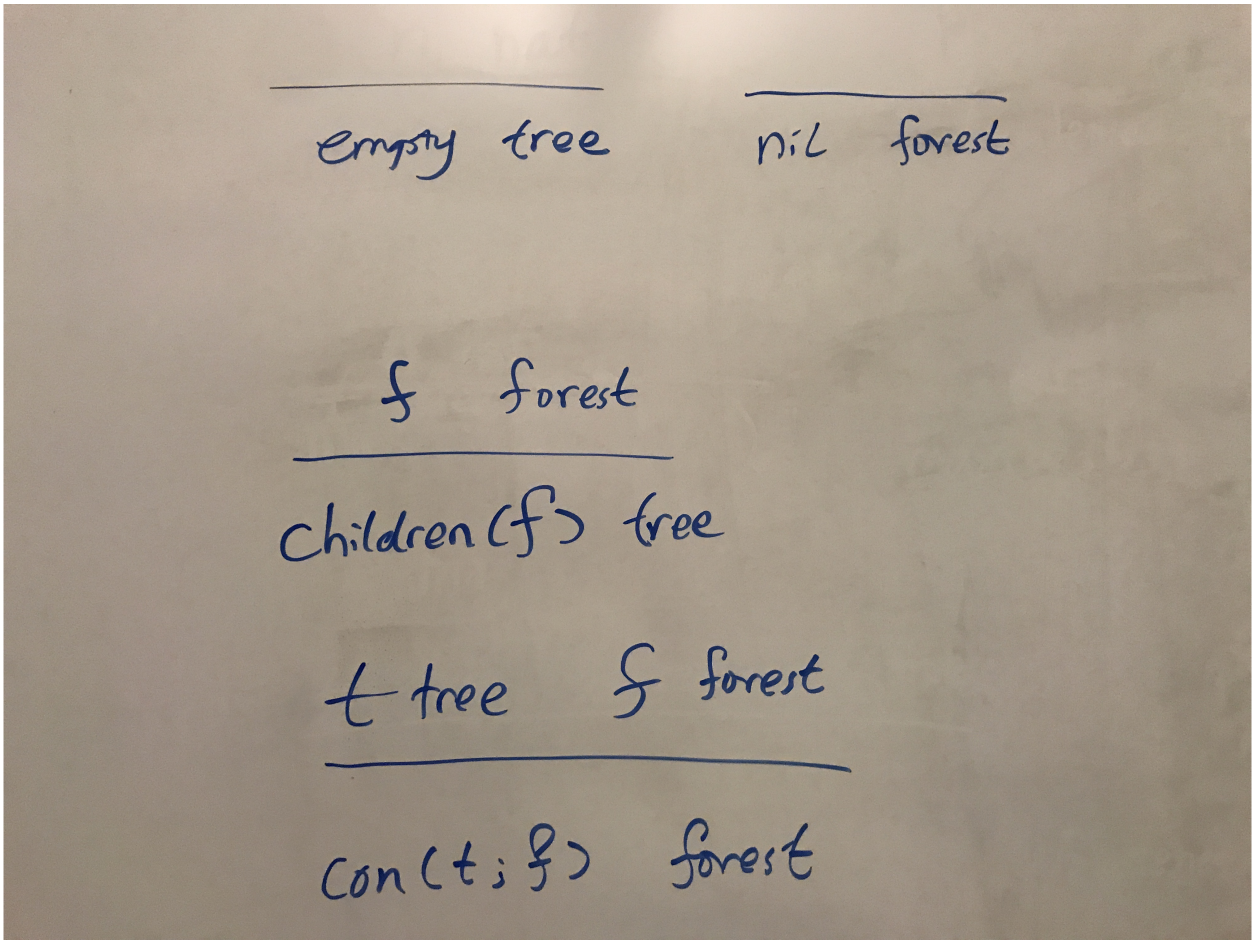
We show by structural induction on $P(t)$, which states that there exists `n nat` such that `hgt(t ;n)`.

1. We are to show $P(empty)$, then let `n` be `zero`.
2. We assume $P(t1)$, and $P(t1)$, and we want to show $P(node(t1; t2))$, Let `n1` be such that `hgt(t1; n1)` and `n2` be `hgt(t2;n2)`. Then we let `n` be `max(n1;n2;n)` **proved by 2.1**. Then we have `hgt(node(t1;t2;);n)`.

For uniqueness we want to show that if `hgt(t;n1)` and `hgt(t;n2)` , then `n1 is n2`.

1. If `t` is `empty` then `n1 is n2`
2. If `t is node(t1;t2)` then this is proved by the uniqueness of `max`

2.3


empty tree
nil forest
f forest
children(f) tree
t tree
f forest
con(t; f) forest

2.4 ???

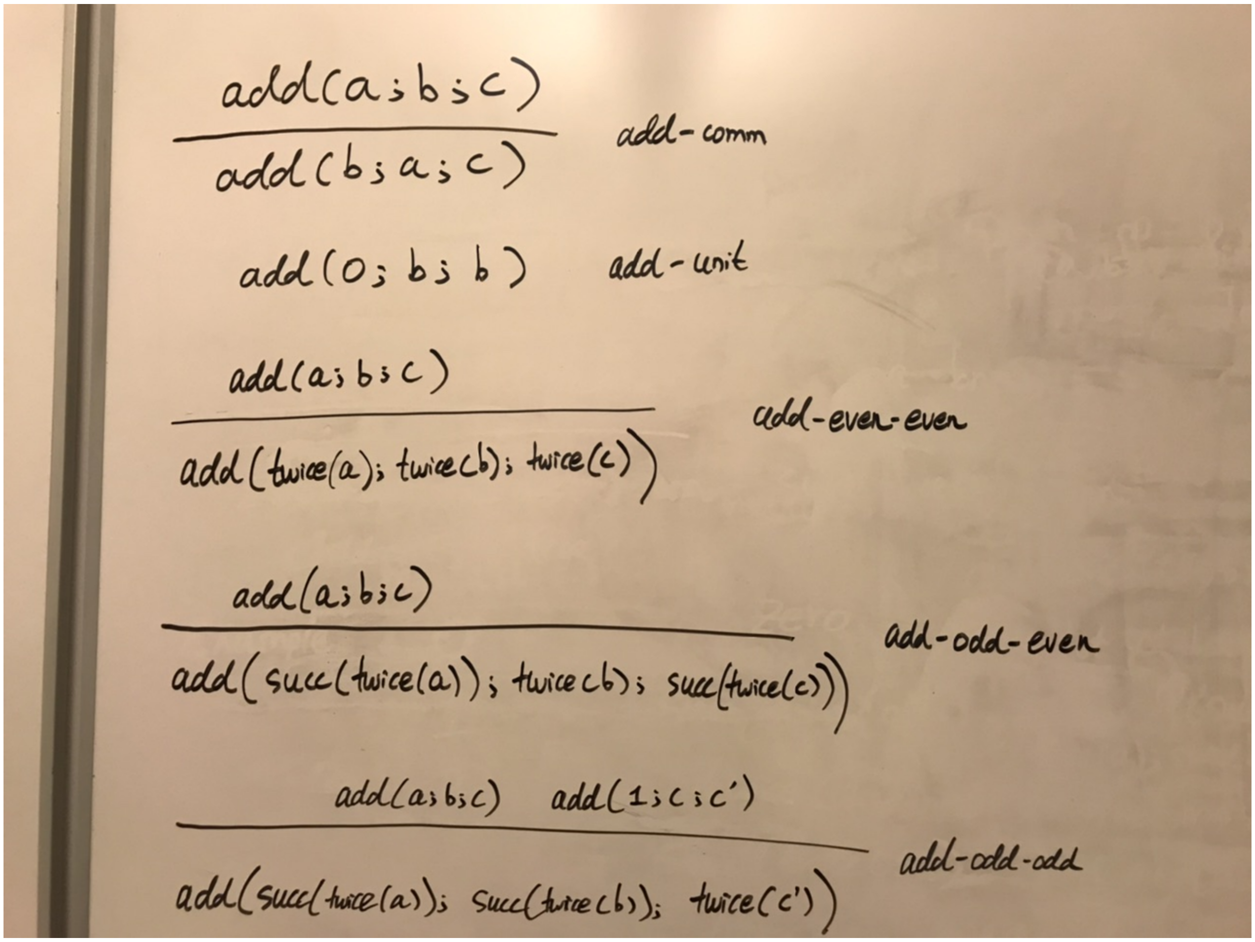
2.5

$$\frac{}{\text{zero bnn}} \qquad \frac{a \ \text{bnn}}{\text{twice}(a) \ \text{bnn}} \qquad \frac{a \ \text{bnn}}{\text{succ}(\text{twice}(a)) \ \text{bnn}}$$

example:

0	zero
1	succ(twice(0))
2	twice(1)
3	succ(twice(1))
4	twice(2)
5	succ(twice(2))
6	twice(3)

2.6


$$\frac{\mathrm{add}(a;b;c)}{\mathrm{add}(b;a;c)}\ \text{add-comm}$$
$$\mathrm{add}(0;b;b)\ \text{add-unit}$$
$$\frac{\mathrm{add}(a;b;c)}{\mathrm{add}(\mathrm{twice}(a);\mathrm{twice}(b);\mathrm{twice}(c))}\ \text{add-even-even}$$
$$\frac{\mathrm{add}(a;b;c)}{\mathrm{add}(\mathrm{succ}(\mathrm{twice}(a));\mathrm{twice}(b);\mathrm{succ}(\mathrm{twice}(c)))}\ \text{add-odd-even}$$
$$\frac{\mathrm{add}(a;b;c)\quad \mathrm{add}(1;c;c')}{\mathrm{add}(\mathrm{succ}(\mathrm{twice}(a));\mathrm{succ}(\mathrm{twice}(b));\mathrm{twice}(c'))}\ \text{add-odd-odd}$$