2/12/2025 

→ Linear Algebra for Data Science, AI46
session 5 , Mans.

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→ Review
→ Diagonalization
→ Eigen decomposition
→ Singular Value Decomposition (SVD)
→ Principal Component Analysis (PCA)

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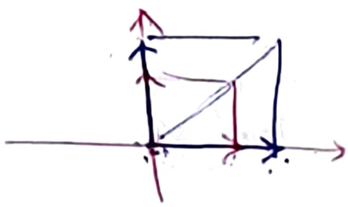
→ Algebraic Multiplicity of an eigenvalue ($\lambda_i$): # of repeated values of an eigenvalue.

ex.

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$



eig.: $\det(\lambda I - A) = 0$

$$(\lambda - 1.5)(\lambda - 1.5) = 0$$

$$\Rightarrow \lambda_1 = 1.5 \quad , \lambda_2 = 1.5$$

matrix A has an eigenvalue 1.5 with algebraic multiplicity of 2

→ Geometric Multiplicity of an eigen vector. ②

for $\lambda = 1.5$

$$\underbrace{(\lambda I - A)}\vec{V}_A = \vec{0}$$

$N(\_\_)$ →

Null space

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} - \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$v_1, v_2$ are both free variable.

ex $v_1 = K_1$ $v_2 = K_2$

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

let $K_1 = 1$, $K_2 = 0$

$$\vec{v}_{A_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftarrow$$

let $K_2 = 0$, $K_1 = 1$

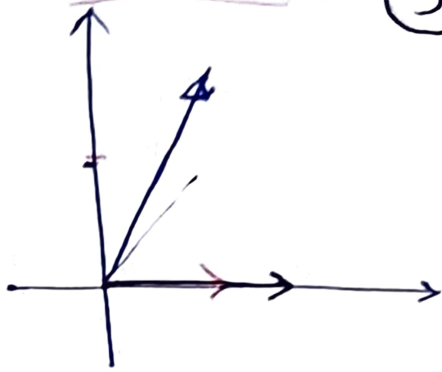
$$\vec{v}_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftarrow$$

Geometric multiplicity is dimensionality of eigen vector space (# of eigen vectors) associated with a given $\lambda$.

In this ex. Geometric Multiplicity = 2

# Defective Matrices (not diagonalizable)





ex

$$A = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}$$

$\lambda_i$ : $\det(\lambda I - A) = 0$

Characteristic equation $\Longrightarrow$

$$\begin{vmatrix} \lambda - 1.5 & 1 \\ 0 & \lambda - 1.5 \end{vmatrix} = 0$$

$$(\lambda - 1.5)(\lambda - 1.5) - 1 \times 0 = 0$$

$$(\lambda - 1.5)^2 = 0$$

$$\lambda_1 = 1.5 \quad , \quad \lambda_2 = 1.5$$

$\lambda = 1.5$ , with algebraic multiplicity $= 2$

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eigenvectors

$$\lambda I - A$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} - \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Geometric Multiplicity $= 1$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{A_1} \\ v_{A_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_{A_2} = 0$ $\qquad$ $v_{A_1}$ is a free variable

$v_{A_1} = K$

$\Rightarrow \vec{v}_{A_1} = \begin{bmatrix} K \\ 0 \end{bmatrix}$ , let $K = 1$

# Decomposition of A

$A = BCD$



## I- Diagonalization

Review

→ Identity Matrix $I_{3\times 3} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — diagonal

→ diagonal Matrix $D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$ — main diagonal

Scaling "scalar" Matrix $= KI = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix}$ — main diagonal

→ Diagonalization

$A_{n\times n} = B_{n\times n} D_{n\times n} C_{n\times n}$

matrix — diagonal matrix — matrix

$A_{n\times n} = P_{n\times n} D_{n\times n} P^{-1}_{n\times n}$

① is every square matrix diagonalizable? No.

② if so, how to find P, D?

③ why?

# eigendecomposition



if $A$ is diaagonalizable

$\Rightarrow$ $\underset{n \times n}{A} = \underset{n\times n}{Q} \, \underset{n\times n}{\Lambda} \, \underset{n \times n}{Q^{-1}}$

square matrix

$A$ $a_{ij}$

- $\Lambda$ : diagonal matrix that has eigenvalues of $A$ in its main diagonal

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix}$$

$$Q = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ v_{A_1} & v_{A_2} & \cdots & v_{A_n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$Q^{-1} = \mathrm{inv}(Q)$

$$A = Q \Lambda Q^{-1} \Leftrightarrow \Lambda = Q^{-1} A Q$$

$$\boxed{A = P D P^{-1} \Leftrightarrow D = P^{-1} A P}$$

given a matrix $A$ ⑥

how to find $A^K$ ?

$$A^K = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdots \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

if <u>A is diagonalizable</u> $A = PDP^{-1}$

$$\Rightarrow A^K = (PDP^{-1})^K$$

$$= (PD\cancel{P^{-1})(P}D\cancel{P^{-1})(P}DP^{-1}) \cdots (PDP^{-1})$$

$$I$$

$$= P D^K P^{-1}$$

$$= [P] \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}^K P^{-1}$$

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$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & \ddots & d_n \end{bmatrix}^K = \begin{bmatrix} d_{11}^K & 0 & 0 & - \\ & d_{22}^K & & \\ & & \ddots & d_{nn}^K \end{bmatrix}$$

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# Dimensionality Reduction

$\Rightarrow$ Rank Reduction (Reduced Rank Matrix)

**ex**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{23} & a_{23} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\approx \underbrace{\begin{bmatrix} \uparrow & \uparrow & \cancel{\uparrow} \\ v_{A_1} & v_{A_2} & \cancel{v_{A_3}} \\ \downarrow & \downarrow & \cancel{\downarrow} \end{bmatrix}}_{Q} \begin{bmatrix} 10 & & \\ & 6 & \\ & & \underbrace{\cancel{0.5}}_{\to 0} \end{bmatrix} \underbrace{\begin{bmatrix} \text{———} \\ \text{———} \\ \cancel{\text{———}} \end{bmatrix}}_{Q^{-1}}$$

$$A \approx \begin{bmatrix} \uparrow & \uparrow & 0 \\ v_{A_1} & v_{A_2} & 0 \\ \downarrow & \downarrow & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \text{———} \\ \text{———} \\ 0 \quad 0 \quad 0 \end{bmatrix}$$

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## Outer Product vs. Inner "dot" Product

$\vec{v}_1 \otimes \vec{v}_2$

$= \vec{v}_1 \vec{v}_2^{\,T} = \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \begin{bmatrix} - v_2 - \end{bmatrix}$

$\vec{v}_1 \otimes \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$\begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix} \cdot \begin{bmatrix} | \\ \vec{v}_2 \\ | \end{bmatrix} = \text{scalar}$

$\vec{v}_1^{\,T} \vec{v}_2 = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$

$= \vec{v}_1 \cdot \vec{v}_2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v}_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \times 3 + 2 \times 4 = 11$

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ v_{A_1} & v_{A_2} & v_{A_3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \leftarrow \vec{u}_{q_1} \rightarrow \\ \leftarrow \vec{u}_{q_2} \rightarrow \\ \leftarrow \vec{u}_{q_3} \rightarrow \end{bmatrix}$$

$$= \underbrace{\lambda_1 \begin{bmatrix} \uparrow \\ v_{A_1} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow u_{q_1} \rightarrow \end{bmatrix}}_{A_1} + \underbrace{\lambda_2 \begin{bmatrix} \uparrow \\ v_{A_2} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow \vec{u}_{q_2} \rightarrow \end{bmatrix}}_{A_2}$$

$$+ \cancel{\lambda_3 \begin{bmatrix} \leftarrow \vec{v}_{A_3} \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ u_{q_3} \\ \downarrow \end{bmatrix}}$$

$$+ \lambda_3 \begin{bmatrix} \uparrow \\ v_{A} \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow u_{q_3} \rightarrow \end{bmatrix}$$

$[A_1]$

# ⇒ Singular Value Decomposition (SVD) 9

$$A_{m\times n} = U_{m\times m}\ \Sigma_{m\times n}\ V^T_{n\times n}$$

$U$ : orthogonal matrix
$V$ : " "

starting with $(A^TA)_{n\times n}$
square matrix

→ eigenvectors $(A^TA)$ ⇝ Right singular vectors
→ eigen values $(A^TA)$ ⇝ $\sigma(A)$ singular values

then ________ $(AA^T)_{m\times m}$
square matrix

eigenvectors $(AA^T)$ ⇝ left singular vectors

non-zero eigen values $(AA^T)$ = non-zero eigen values $(A^TA)$

$$\Sigma_{m\times n} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \\ & & & \sigma_n \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_n & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \uparrow & \uparrow & \\ \text{Right singular} & \text{vectors} & \\ \downarrow & \downarrow & \end{bmatrix}$$

$$U = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \text{Left} & \text{singular} & \text{vectors} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

ex $X = \begin{bmatrix} \uparrow & \uparrow \\ \hat{v}_1 & \hat{v}_2 \\ \downarrow & \downarrow \end{bmatrix}$ $\quad \hat{v}_1 \perp \hat{v}_2$

$\hat{v}_1 \cdot \hat{v}_2 = 0$

$$X^T = \begin{bmatrix} \leftarrow \hat{v}_1 \rightarrow \\ \leftarrow \hat{v}_2 \rightarrow \end{bmatrix}$$

$$X^T X = \cancel{\begin{bmatrix} \uparrow & \uparrow \\ \hat{v}_1 & \hat{v}_2 \\ \downarrow & \downarrow \end{bmatrix}} \begin{bmatrix} \leftarrow \hat{v}_1 \rightarrow \\ \leftarrow \hat{v}_2 \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \hat{v}_1 & \hat{v}_2 \\ \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \hat{v}_1 \cdot \hat{v}_1 & \hat{v}_1 \cdot \hat{v}_2 \\ \hat{v}_2 \cdot \hat{v}_1 & \hat{v}_2 \cdot \hat{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

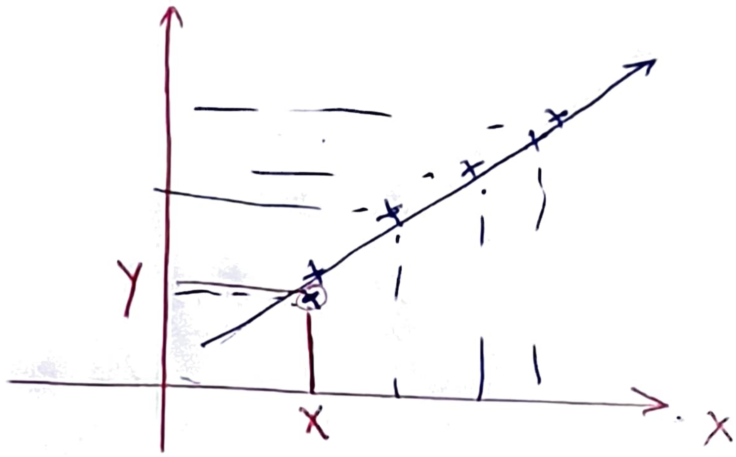
$$X^T X = I \Rightarrow X^T = X^{-1}$$

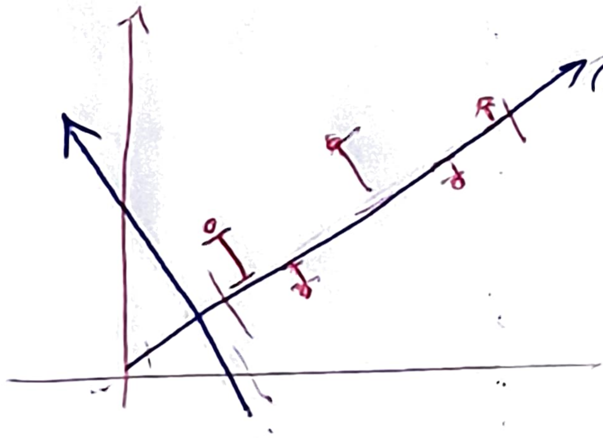
if ~~ort~~ X is orthogonal Matrix ⇒ column vectors are unit vectors & column vectors are orthogonal to each other.

(column vectors are orthonormal)

# Principal Component Analysis

## dimensinality Reduction



PC1
Pricipal Component # 1

$\vec{v}_1 \in \mathbb{R}^2$

1 Dimension

note PCA vs. LS solutions