وزارة الإتصالات
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Linear Algebra for Data Science

AI TALENTS HUB

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ÉCOLE D'INGÉNIEURS EN INFORMATIQUE

Session 3

Linear Algebra for Data Science

Outline – Session 3

- Linear Transformation
- Matrix Rank
- Matrix Determinant
- The Properties of the Determinant
- Matrix Inverse
- Orthonormal and Non-Orthonormal Space

Review - fundamental spaces

The Fundamental Spaces, Review

Given an MxN matrix **A**, we have the following four fundamental spaces:

FUNDAMENTAL THEOREM OF LINEAR ALGEBRA, PART I

The **column space** of **A** is denoted $C(\mathbf{A})$.
Its dimension is the rank r.

The **nullspace** of **A** is denoted $N(\mathbf{A})$.
Its dimension is $N - r$.

The **rowspace** of **A** is spanned by the rows of **A** and equals the column space $C(\mathbf{A}^T)$ of $\mathbf{A}^T$.
Its dimension is r.

The **left nullspace** of **A** is the nullspace of $\mathbf{A}^T$ and equals the nullspace $N(\mathbf{A}^T)$ of $\mathbf{A}^T$.
Its dimension is $M - r$.

Subspaces of $\mathbb{R}^N$.

Subspaces of $\mathbb{R}^M$.

Matrix rank

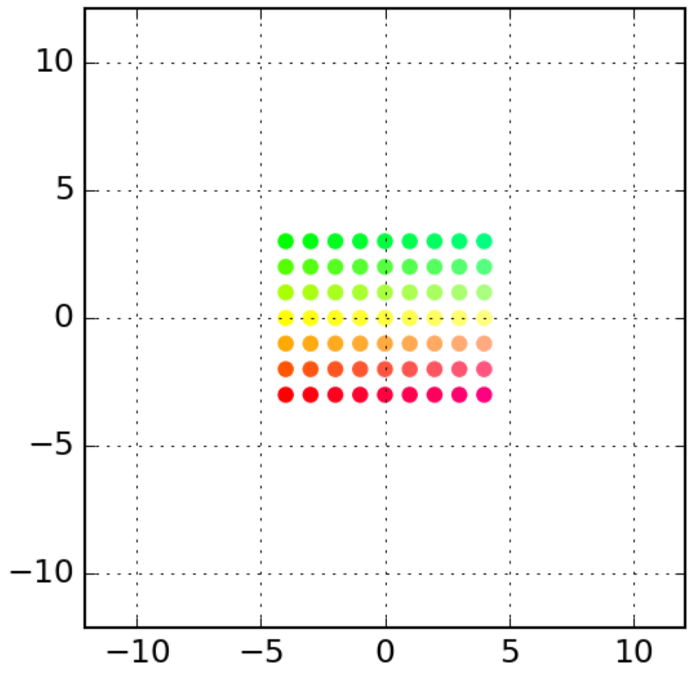
Rank (A) = $\dim(C(A))$ = $\dim(R(A))$ = number of inpendent rows = number of independent columns

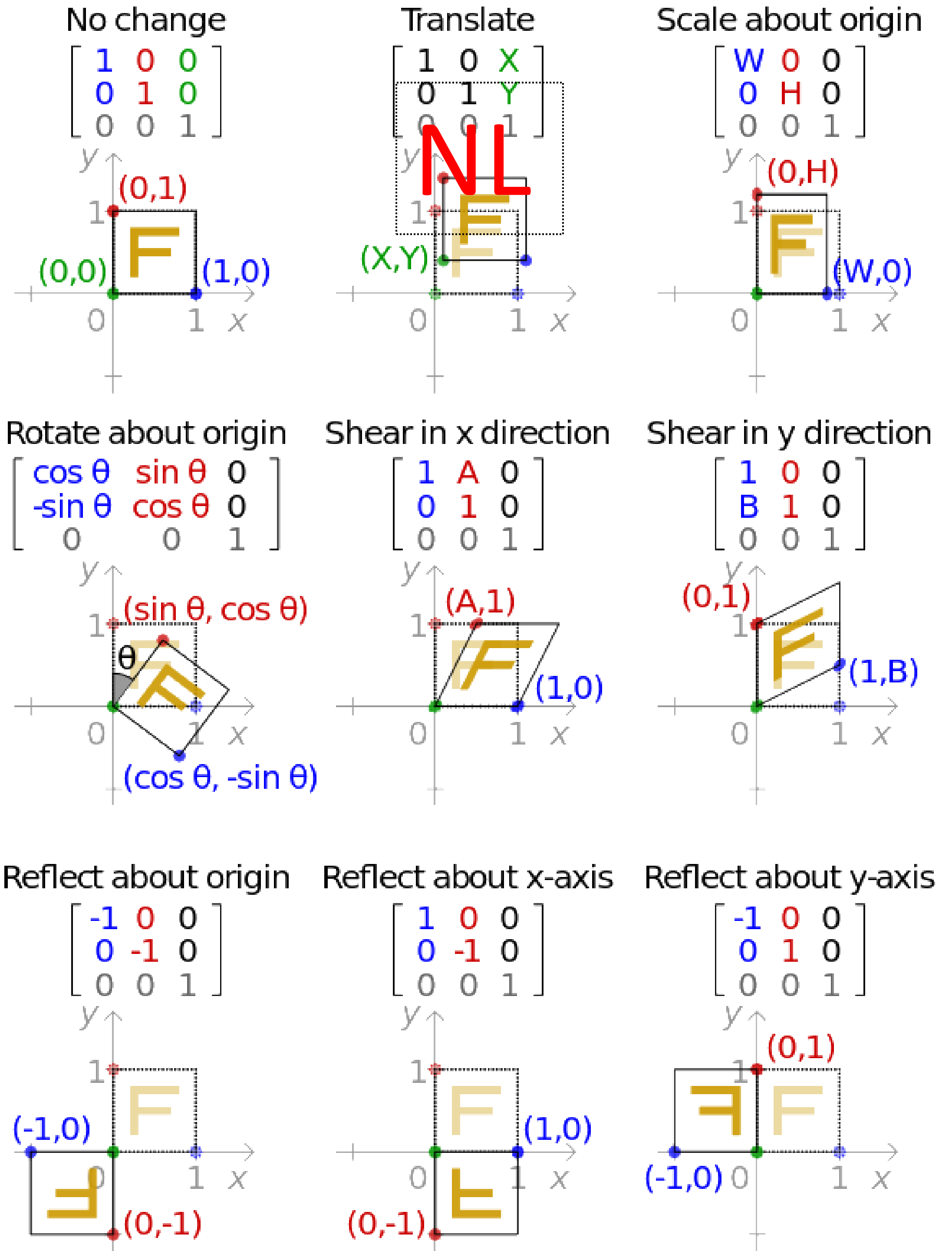
Linear Transformations

And relation to invertibility

Remember

- Matrices can represent:
  - Linear equations.
  - Group of vectors
  - Linear Transformations.
  - Images.
  - Graphs.





Definition of a linear transformation

- A linear transformation (or a linear map) is a function T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ that carries elements of the vector space $\mathbb{R}^n$ (called the domain) to the vector space $\mathbb{R}^m$ (called the codomain), and satisfies the following properties:
- T(**x** + **y**) = T(**x**) + T(**y**)
- T(a**x**) = aT(**x**)
- for any vectors **x**, **y** $\in \mathbb{R}^n$ and any scalar a $\in \mathbb{R}^1$
- If each component of f(x) is a number times one of the components of x, then f is a linear transformation.

Note that order matters: TB(TA(**x**))= B A **x**

Linear and non-linear transformations, Examples

- The function $f(x,y,z) = (3x-y, 3z, 0, z-2x)$ is a linear transformation.

- $h(x,y,z) = (3x-y, 3xz, 0, z-2x)$ is not a linear transformation, (h has a nonlinear component $3xz$).

- $g(x,y,z) = (3x-y, 3z+2, 0, z-2x)$ is not a linear transformation, (the second component has the term +2 that is a constant that doesn't contain any components of our input vector (x,y,z)).

Transformation Matrices

- Matrices give us a powerful systematic way to describe a wide variety of transformations (e.g., rotations, reflections, dilations, ….)
- A transformation matrix takes a vector **v** and multiplies it on the left by the matrix M.

- If **v** is the position vector of the point (x,y), then, the transformed position vector is

$$T_M(\mathbf{v}) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

or equivalently, $T_M(x,y) = (ax+by, cx+dy)$.

Example: Rotation

Example: We want to create a matrix $\mathbf{A}_\alpha$ that performs a rotation by an angle α.

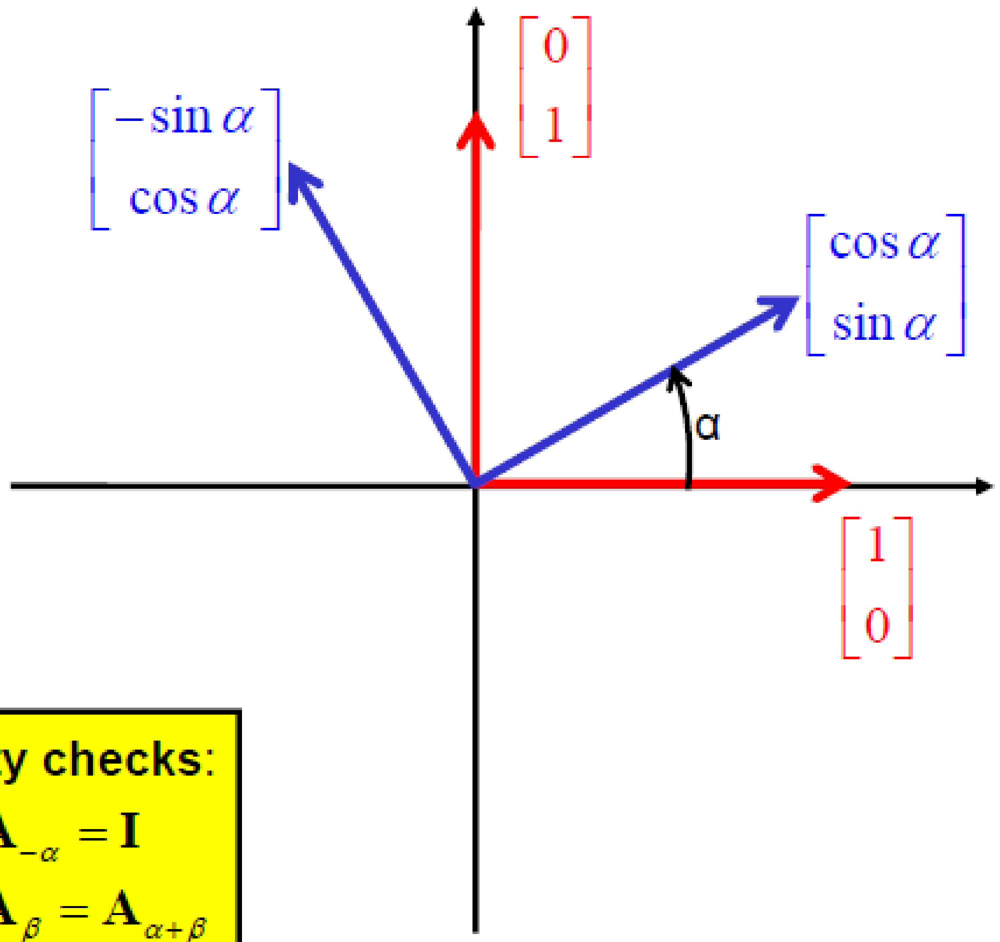
First determine what hapens to the basis vectors.

$$\mathbf{A}_\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$\mathbf{A}_\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$

This gives us the two columns of $\mathbf{A}_\alpha$ directly:

$$\mathbf{A}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$



Sanity checks:

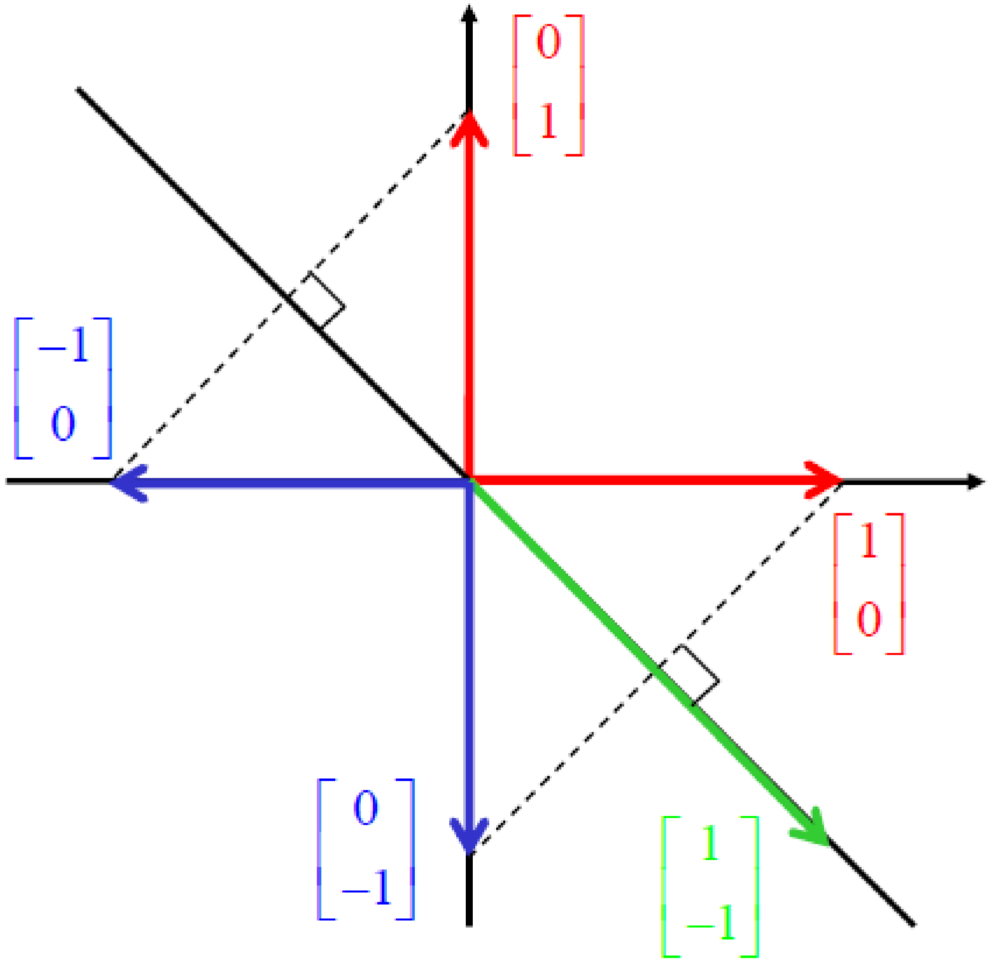
$$\mathbf{A}_\alpha \mathbf{A}_{-\alpha} = \mathbf{I}$$

$$\mathbf{A}_\alpha \mathbf{A}_\beta = \mathbf{A}_{\alpha+\beta}$$

Example: Reflection

Example: We want to create a matrix **A** that reflects vectors onto in the space (line) spanned by the vector [1 -1]^T.

First determine what hapens to the basis vectors.



This gives us the two columns of **A** directly:

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Sanity check:

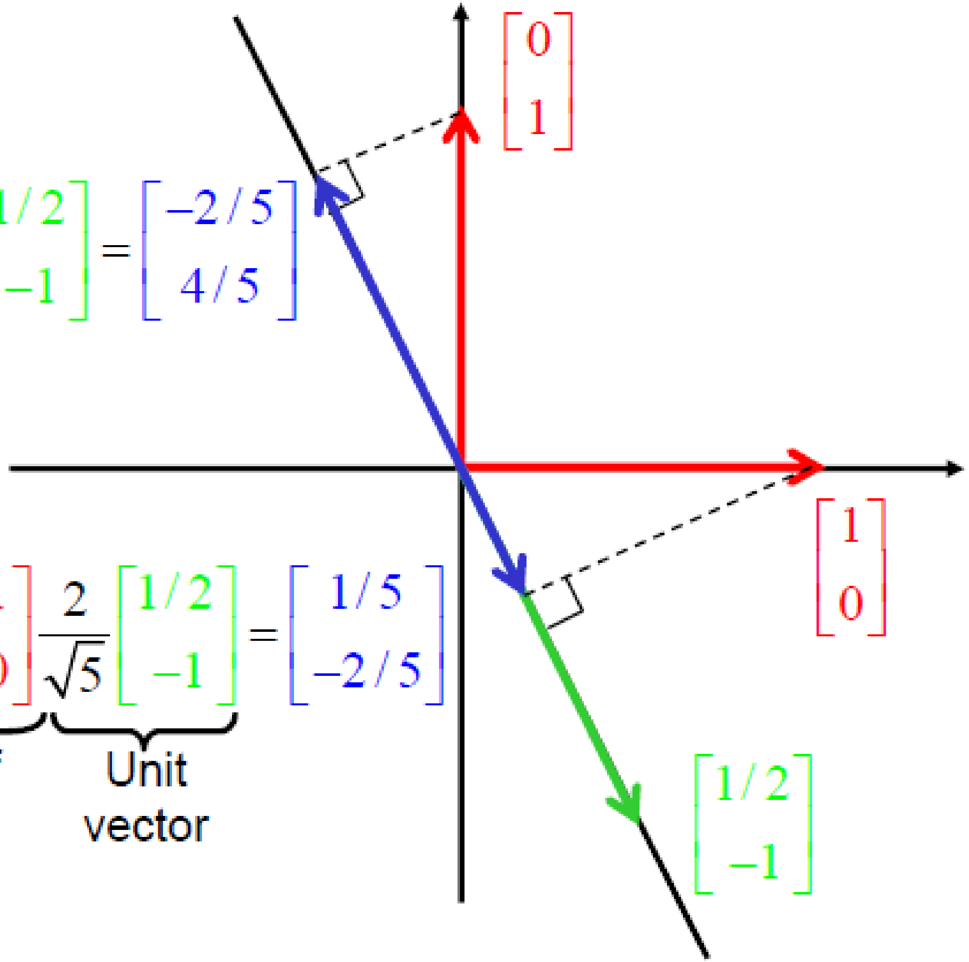
$$\mathbf{A}^2 = \mathbf{I}$$

Example: Projection

Example: We want to create a matrix **A** that projects vectors onto the space (line) spanned by the vector [1/2 -1]^T.

First determine what hapens to the basis vectors.

$$\frac{2}{\sqrt{5}}\begin{bmatrix}1/2\\-1\end{bmatrix}^T\begin{bmatrix}0\\1\end{bmatrix}\frac{2}{\sqrt{5}}\begin{bmatrix}1/2\\-1\end{bmatrix}=\begin{bmatrix}-2/5\\4/5\end{bmatrix}$$



This gives us the two columns of **A** directly:

$$\mathbf{A}=\begin{bmatrix}1/5 & -2/5\\-2/5 & 4/5\end{bmatrix}$$

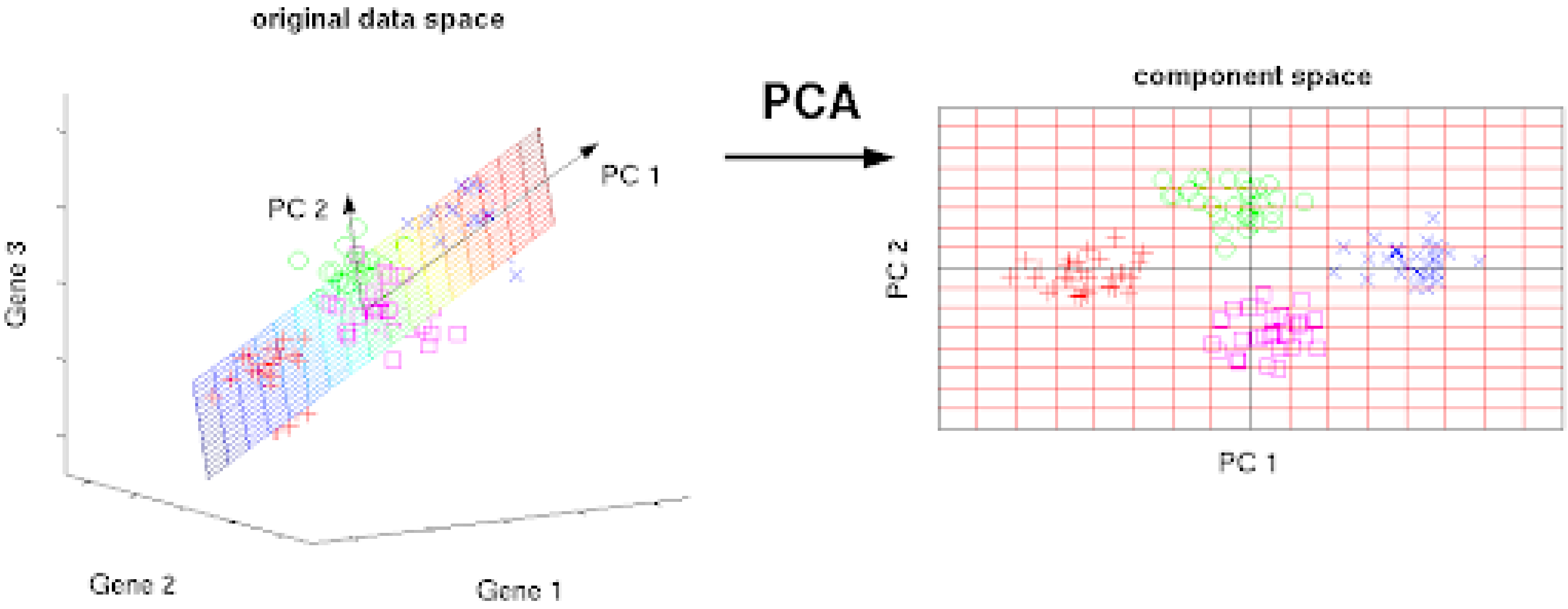
$$\underbrace{\frac{2}{\sqrt{5}}\begin{bmatrix}1/2\\-1\end{bmatrix}^T\begin{bmatrix}1\\0\end{bmatrix}}_{\text{"Length" of projection}}\underbrace{\frac{2}{\sqrt{5}}\begin{bmatrix}1/2\\-1\end{bmatrix}}_{\text{Unit vector}}=\begin{bmatrix}1/5\\-2/5\end{bmatrix}$$

Sanity check:

$$\mathbf{A}^k=\mathbf{A}$$

Example for the applications of Linear transformations in machine learning

- Principal component analysis (PCA) lets us reduce the dimensionality of our data.
- PCA can be viewed as a linear transformation



Inverse of a Matrix

If we can find a transformation that reverses the effect of the original transformation, then the matrix is invertible

$$Ax = v \qquad x = A^{-1}v$$

A: Transformation matrix

x: Input vector

v: Output vector

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A

Transformation matrix

90 Counterclockwise

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A^{-1}

Inverse Transformation matrix

90 Clockwise

Matrix determinant

Matrix determinant

- Many of the main uses for matrices involve calculating ***the determinant***.
- Denoted as det(A) or $|A|$.
- Defined only for ***square matrices***, i.e., n×n matrices.
- Can be viewed as a function whose input is a square matrix and whose output is a number.

- Specific properties of the determinants make them useful for different applications like solving the linear system of equations, checking the invertibility of a matrix, finding the area and volume of geometric shapes, ….

Determinant of a 2x2 matrix

- The determinant of a 2^{nd} order square matrix is represented and evaluated as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

2nd Order Determinant

Determinant of a 3x3 matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- The expansion of determinant |A| in terms of the first row is:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Similarly, we can expand the determinant |A| in terms of the second column as:

$$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$
$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Determinant of a matrix, general formula

- The determinant of a square matrix is represented and evaluated using any one row or any one column

$$\sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

$$\sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$$

Where the **Minors** and **Cofactors** are defined by:

$$M_{i,j} = \det\left((A_{p,q})_{p \neq i, q \neq j} \right)$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

C_{ij}
$A_{p,q}$ is the small matrix remaining after crossing out the ith row and jth column.

This can be used to recursively calculate the determinant of any *N*x*N* matrix. After *N*-1 steps we arrive at the scalar case.

Minors and Cofactors, Example

To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

To compute the minor $M_{2,3}$ and the cofactor $C_{2,3}$, we find the determinant of the above matrix with row 2 and column 3 removed.

$$M_{2,3} = \det \begin{bmatrix} 1 & 4 & \Box \\ \Box & \Box & \Box \\ -1 & 9 & \Box \end{bmatrix} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = 9 - (-4) = 13$$

So the cofactor of the (2,3) entry is

$$C_{2,3} = (-1)^{2+3}(M_{2,3}) = -13.$$

https://en.wikipedia.org/wiki/Minor_(linear_algebra)

Determinant of a matrix, example 1

- Note that the determinant calculated using an expansion in terms of any row or column is the same.
- The trick for reducing the computation effort while manually calculating the determinant is to select the row or column having the maximum number of zeros.
- **<u>Example,</u>** expansion in terms of the second column.

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 0 & 4 \\ 4 & 2 & 7 \end{vmatrix} = -3\begin{vmatrix} 2 & 4 \\ 4 & 7 \end{vmatrix} + 0 - 2\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= -3(2 \times 7 - 4 \times 4) - 2(1 \times 4 - 5 \times 2)$$

$$= -3(14 - 16) - 2(4 - 10)$$

$$= 18$$

Determinant of a matrix, example 2

- Expansion in terms of the first row

$$\begin{vmatrix} 5 & 3 & 58 \\ -4 & 23 & 11 \\ 34 & 2 & -67 \end{vmatrix} = 5\begin{vmatrix} 23 & 11 \\ 2 & -67 \end{vmatrix} - 3\begin{vmatrix} -4 & 11 \\ 34 & -67 \end{vmatrix} + 58\begin{vmatrix} -4 & 23 \\ 34 & 2 \end{vmatrix}$$

$$= 5\left[23 \times (-67) - 11 \times 2\right] - 3\left[(-4) \times (-67) - 11 \times 34\right] + 58\left[(-4) \times 2 - 23 \times 34\right]$$

$$= 5(-1541 - 22) - 3(268 - 374) + 58(-8 - 782)$$

$$= -53317$$

Determinant of a 4x4 matrix

- We can use the Laplace's expansion for nth order determinant in a similar way as the 3rd order determinant.
- We should further expand the cofactors in the first expansion until the second-order (2 x 2) cofactor is reached.

$$\begin{vmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 0 & 1 & 3 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} + 0 - 2\begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{vmatrix} + 0$$

$$(Expand\,by\,Col.\ 2)\quad (Expand\,by\,Row\,1)$$

$$= -1\left(-2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 0 - 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}\right) - 2\left(2\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + 0\right)$$

$$= -1[-2(3\times 3 - 1\times 2) - 1(1\times 1 - 3\times 3)] - 2[2(2\times 3 - 3\times 1) - 3(1\times 3 - 3\times 2)]$$

$$= -1[(-2)\times 7 - 1\times(-8)] - 2[2\times 3 - 3\times(-3)]$$

$$= -1(-14+8) - 2(6+9)$$

$$= -24$$

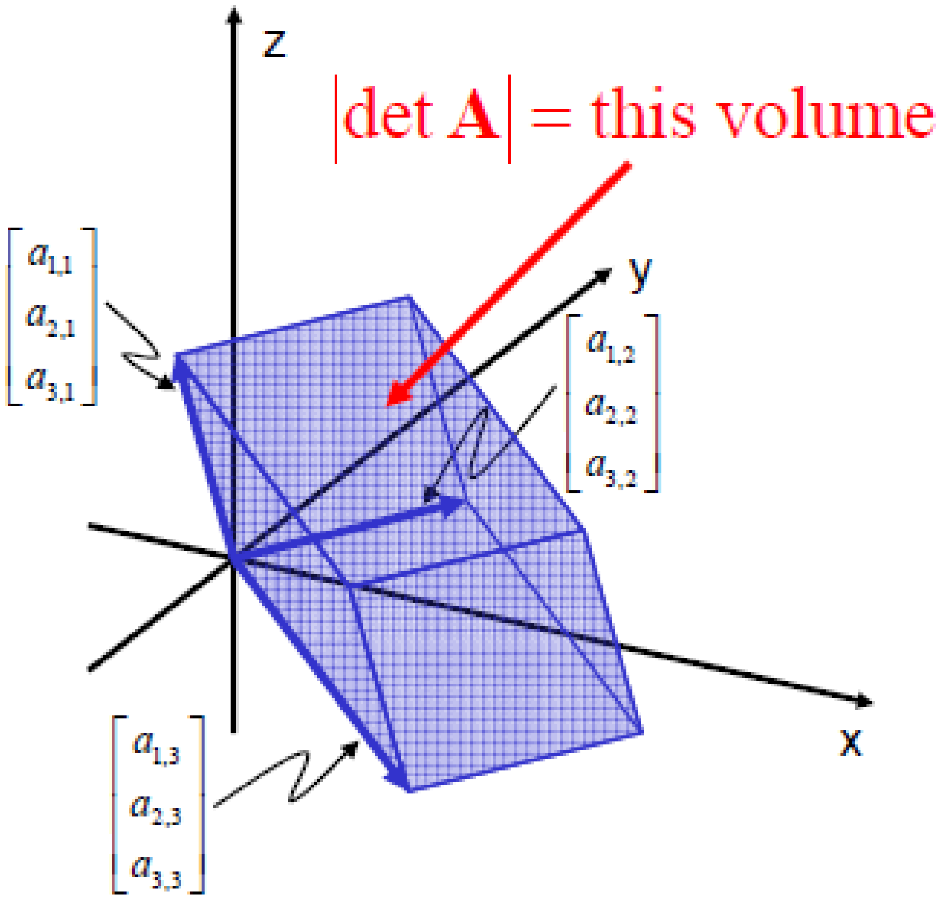
Properties of the determinant

Determinant of a matrix and its rank

The determinant of an *N*x*N* matrix **A** can be interpreted as the volume of a parallelepiped in $\mathbf{R}^N$ where the edges come from the columns (or rows) of **A**.

- If the determinant = 0 → the matrix is not full rank
- Remember: Rank (A) = dim(*C*(*A*)) = dim(*R*(*A*)) = number of impendent rows = number of independent columns

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$



Determinant of a matrix and linear independence

Suppose that

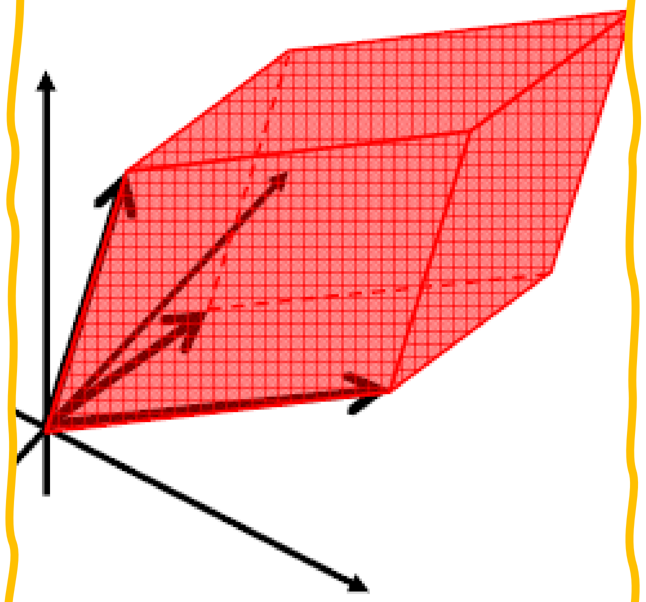
$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_N\mathbf{a}_N = 0$$

only happens when

$$\alpha_1 = \alpha_2 = \ldots = \alpha_N = 0.$$

Then the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ... $\mathbf{a}_N$ are ***lineraly independent***.

If any α_k's are nonzero, the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ... $\mathbf{a}_N$ are ***lineraly dependent***. At least one vector is a combination of the others.



These three vectors are linearly independent if the determinant of the matrix they form as columns is nonzero.

Properties of the determinant (1)

- **Multiplication of the Determinants**
- The product of two n^{th} order determinants is also a determinant of the order n.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|B| = \begin{vmatrix} l & m & n \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$|A| \times |B| = \begin{vmatrix} al+bm+cn & ap+bq+cr & ax+by+cz \\ dl+em+fn & dp+eq+fr & dx+ey+fz \\ gl+hm+in & gp+hq+ir & gx+hy+iz \end{vmatrix}$$

Properties of the determinant (2)

Interchanging the rows with columns (transpose of a matrix) does not alter the value of the determinant.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A^T| = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$\Longrightarrow \quad |A^T| = |A|$$

Properties of the determinant (corollary)

Corollary: If a line (row or column) is shifted by k places, then the determinant of the resulting matrix is $|A'|=(-1)^k\ |A|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A'| = \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$\implies |A'| = -|A|$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A'| = \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$$

$$\implies |A'| = (-1)^2|A|$$

$$\implies |A'| = |A|$$

Properties of the determinant (3)

If a line (row or column) of a determinant is multiplied by a constant value, then the resulting determinant can be evaluated by multiplying the original determinant by the same constant value.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A'| = \begin{vmatrix} a & b & c \\ pd & pe & pf \\ g & h & i \end{vmatrix}$$

$$\implies |A'| = p|A|$$

Properties of the determinant (4)

If any line (row or column) of the determinant has each element written as a sum of t terms, then the determinant can be written as the sum of t determinants.

$$|A| = \begin{vmatrix} a & b & c+j-m \\ d & e & f+k-n \\ g & h & i+l-o \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix} - \begin{vmatrix} a & b & m \\ d & e & n \\ g & h & o \end{vmatrix}$$

Properties of the determinant (5)

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The value of the determinant remains the same if a line (row or column) is added by multiples of one or more parallel lines. We can prove this property using the corollary of the 4th property and the 5th property.

$$|A'| = \begin{vmatrix} a+pb-qc & b & c \\ d+pe-qf & e & f \\ g+ph-qi & h & i \end{vmatrix}$$

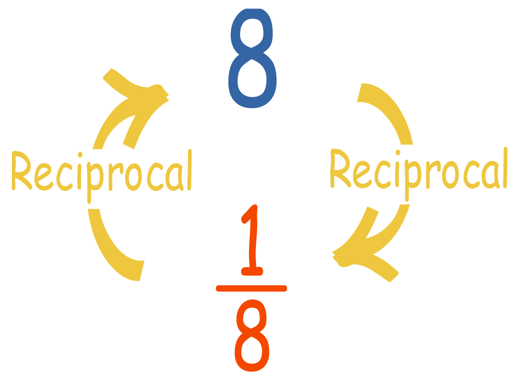
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} pb & b & c \\ pe & e & f \\ ph & h & i \end{vmatrix} - \begin{vmatrix} qc & b & c \\ qf & e & f \\ qi & h & i \end{vmatrix}$$

$$\Longrightarrow \ |A'| = |A| + 0 + 0 = |A|$$

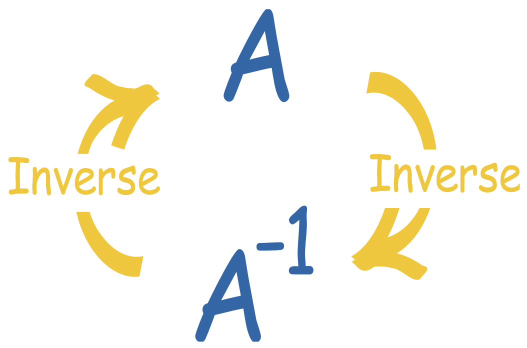
Inverse of a Matrix

Inverse of a matrix, Intuition

Reciprocal of a Number



Inverse of matrices



Inverse of a Matrix, reversing the transformation

$$Ax = v \qquad x = A^{-1}v$$

A: Transformation matrix

x: Input vector

v: Output vector

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A

Transformation matrix

90 Counterclockwise

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A^{-1}

Inverse Transformation matrix

90 Clockwise

Inverse of a Matrix

$$A\,A^{-1} = I$$

A: Transformation matrix

A^{-1}: Inverse

I: Identity matrix

$$A\,A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Basis vector still in the original positions
- **Identity matrix**: is the transformation that does nothing

Inverse of a Square Matrix, Review

- When a matrix is multiplied times its inverse the result is the Identity matrix of the vector space. I.e., the matrix inverse reverses its effect

$$A^{-1}A = AA^{-1} = I$$

$$A\,A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Basis vector still in the original positions
- Identity matrix: is the transformation that does nothing

$$Ax = v \qquad x = A^{-1}v$$

A: Transformation matrix

x: Input vector

v: Output vector

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A

Transformation matrix

90 Counterclockwise

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A^{-1}

Inverse Transformation matrix

90 Clockwise

Transformations shall be discussed today

Inverse of a Square Matrix, Review

- The adjugate of **A** is the transpose of the cofactor matrix **C** of **A**

If $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A}).$$

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}}$$

$$\mathbf{C} = \left((-1)^{i+j}\mathbf{M}_{ij}\right)_{1 \le i,j \le n}$$

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}} = \left((-1)^{i+j}\mathbf{M}_{ji}\right)_{1 \le i,j \le n}$$

Inverse of a Square Matrix, some properties

- Note that:

$$(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1}$$
$$(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$$

If $\mathbf{A}$ is an invertible matrix, then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}).$$

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}}$$

$$\mathbf{C} = \left((-1)^{i+j}\mathbf{M}_{ij}\right)_{1\le i,j\le n}$$

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^{\mathsf{T}} = \left((-1)^{i+j}\mathbf{M}_{ji}\right)_{1\le i,j\le n}$$

Inverse of a matrix, solving a system of equations example

$$\begin{aligned} x &+ y &+ z &= 6 \\ & \quad 2y &+ 5z &= -4 \\ 2x &+ 5y &- z &= 27 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & -1 \end{bmatrix}^{-1} = \frac{1}{-21} \begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$x = 5,$

$y = 3,$

$z = -2$

https://www.mathsisfun.com/algebra/systems-linear-equations-matrices.html

Inverse of a Matrix, Gaussian Elimination

A. Using row operations

One approach for computing the inverse is to use the Gauss–Jordan elimination procedure. Start by creating an array containing the entries of the matrix A on the left side and the identity matrix on the right side:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right].$$

Now we perform the Gauss-Jordan elimination procedure on this array.

1) The first row operation is to subtract three times the first row from the second row: $R_2 \leftarrow R_2 - 3R_1$. We obtain:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & -3 & 1 \end{array}\right].$$

2) The second row operation is divide the second row by 3: $R_2 \leftarrow \frac{1}{3}R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right].$$

3) The third row operation is $R_1 \leftarrow R_1 - 2R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -\frac{2}{3} \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right].$$

The array is now in reduced row echelon form (RREF). The inverse matrix appears on the right side of the array.

“Linear Algebra Explained in four Pages”: https://courses.engr.illinois.edu/ece498rc3/fa2016/material/linearAlgebra_4pgs.pdf

Singular (non-invertible) Matrix

- When the determinant of a matrix is zero, i.e., |A|=0, then that matrix is called as a **Singular Matrix.**

- The matrix with a non-zero determinant is called the **Non-singular Matrix**.

- All the singular matrices are **Non-invertible Matrices**, i.e., it is not possible to take an inverse of a matrix

E.g., the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 4 & 6 \end{bmatrix}$

is singular.

Why?

Existence of an Inverse

For an $n \times n$ matrix A, the following statements are equivalent:

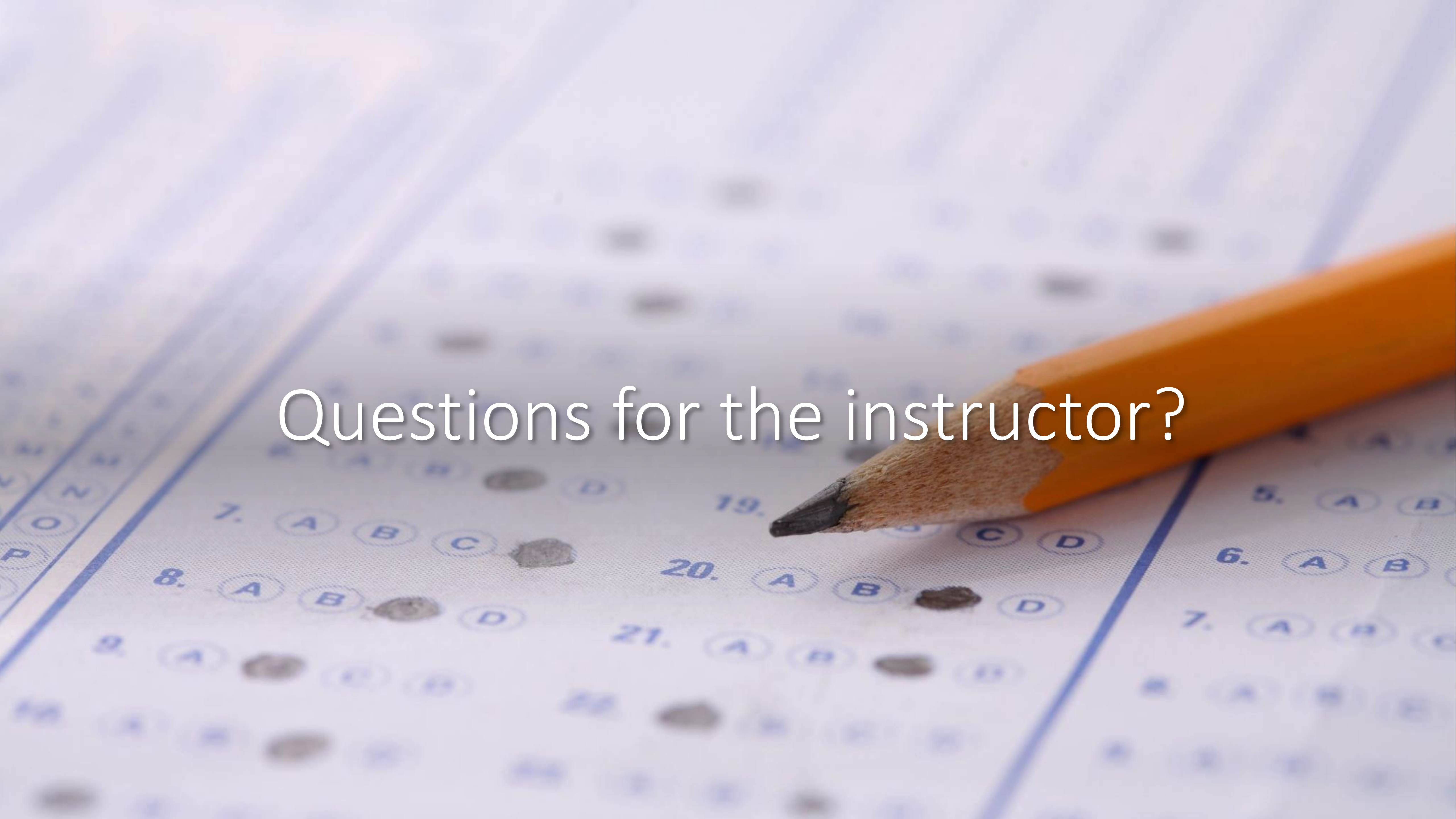
1. A is invertible (non-singular)
2. The RREF of A is the $n \times n$ identity matrix
3. The rank of the matrix is 'n'
4. The row space of A is $\mathbb{R}^n$
5. The column space of A is $\mathbb{R}^n$
6. A doesn't have a null space (only the zero vector $N(\mathrm{A}) = \{\mathbf{0}\}$)
7. The determinant of A is nonzero


Thank You!

Python-based Examples

- Kindly check the provided notebooks.

Questions for the instructor?



Resources

- Books and Articles
  - “Linear Algebra Explained in four Pages”: https://courses.engr.illinois.edu/ece498rc3/fa2016/material/linearAlgebra_4pgs.pdf
  - G. Strang, “Linear Algebra and Its Applications“ Fourth edition, Brooks/Cole
  - Odd K. Moon and Wynn C. Stirling, “Mathematical Methods and Algorithms for Signal Processing”, Prentice Hall 1999
- Online Courses and Slides
  - Linear Algebra, MIT Open Coursware, Prof. Gilbert Strang, http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/index.htm *
  - Linear Algebra for Wireless Communication, Ove Edfors, Lund University: http://www.eit.lth.se/index.php?ciuid=384&coursepage=1300&L=1

Notes

- Determinants and linear transformations.
- The relationship between determinants and the area of a parallelogram.

References

- https://mathinsight.org/linear_transformation_definition_euclidean
- https://textbooks.math.gatech.edu/ila/linear-transformations.html
- https://www.codeformech.com/determinant-linear-algebra-using-python/