Linear Algebra for Data Science

Session 2

Linear Algebra for Data Science

Session 2 Contents

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- Vector Space and Subspaces
- Linear Span
- Linear Independence
- Basis of Vector Space
- Basis and Orthonormal Basis (more on this later)
- The Fundamental Subspaces: Row Space, Column Space, Null Space

1. Linear Combinations

Linear Combinations of Vectors

- If one vector is equal to the sum of scalar multiples of other vectors, it is said to be a linear combination of the other vectors.

- Given the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in \mathbb{R}^m$, we can combine them to produce a “**linear combinations**” of the vectors:

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n, \qquad where\ c_1 \to c_n \in \mathbb{R}$$

- For example, suppose **a** = 2**b** + 3**c**,
- The vector **a** is a linear combination of the vectors **b** and **c**.

$$\underset{\mathbf{a}}{\begin{bmatrix} 11 \\ 16 \end{bmatrix}} = 2\underset{\mathbf{b}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + 3\underset{\mathbf{c}}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 2*1 + 3*3 \\ 2*2 + 3*4 \end{bmatrix}$$

Linear Combinations of Two Vectors, Special Cases

- Given the linear combinations of the vectors $\boldsymbol{v}\ and\ \boldsymbol{w}$:

$$c\boldsymbol{v} + d\boldsymbol{w}$$

- Four special linear combinations are: sum, difference, zero, and scalar multiple $c\boldsymbol{v}$:

- $1\boldsymbol{v} + 1\boldsymbol{w} = \text{sum of vectors}$
- $1\boldsymbol{v} - 1\boldsymbol{w} = \text{difference of vectors}$
- $0\boldsymbol{v} + 0\boldsymbol{w} =$ zero vector
- $\mathrm{c}\boldsymbol{v} + 0\boldsymbol{w} =$ vector $c\boldsymbol{v}$ in the direction of $\boldsymbol{v}$

Linear Combinations of Three Vectors

- Consider three (independent) vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, and the scalars $c, d, e \in \mathbb{R}$, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$

1. The combinations $c\boldsymbol{u}$ fill a ***line through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$***.***
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane through*** $(\mathbf{0}, \mathbf{0}, \mathbf{0})$***.***
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***3D space.***

- This is the typical situation: **Line**, then **plane**, then **space**.

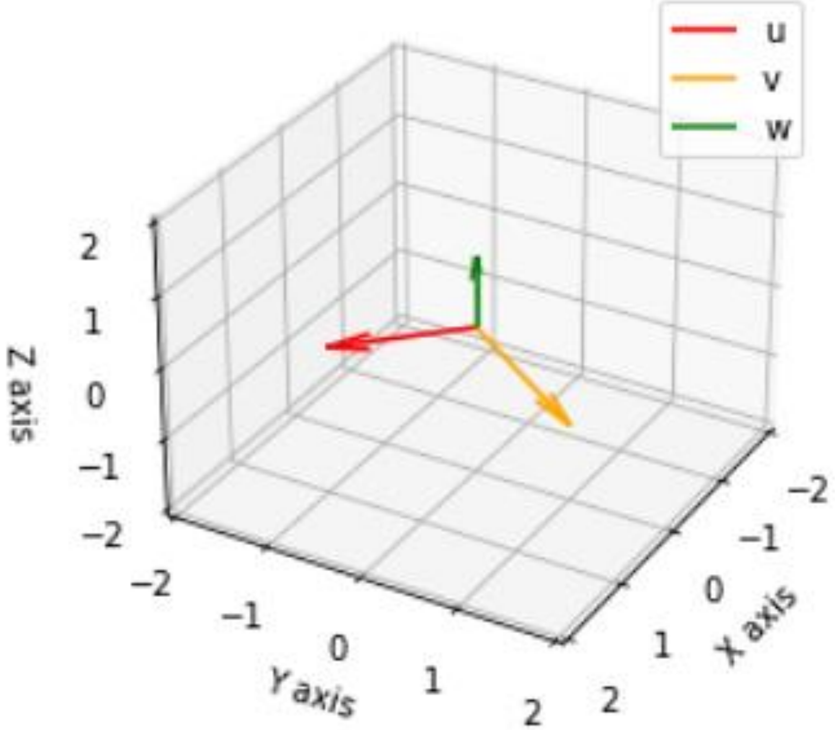
- Note that other possibilities may exist:
- When $\boldsymbol{v} = c\boldsymbol{u}$, the vector $\boldsymbol{v}$ is on the line of $\boldsymbol{u}$. The combinations of $\boldsymbol{u}, \boldsymbol{v}$ will not go outside that $c\boldsymbol{u}$ line.
- When $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$, the third vector $\boldsymbol{w}$ is in the plane of the first two. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \mathbf{w}$ will not go outside that $\boldsymbol{u}\boldsymbol{v}$ plane.

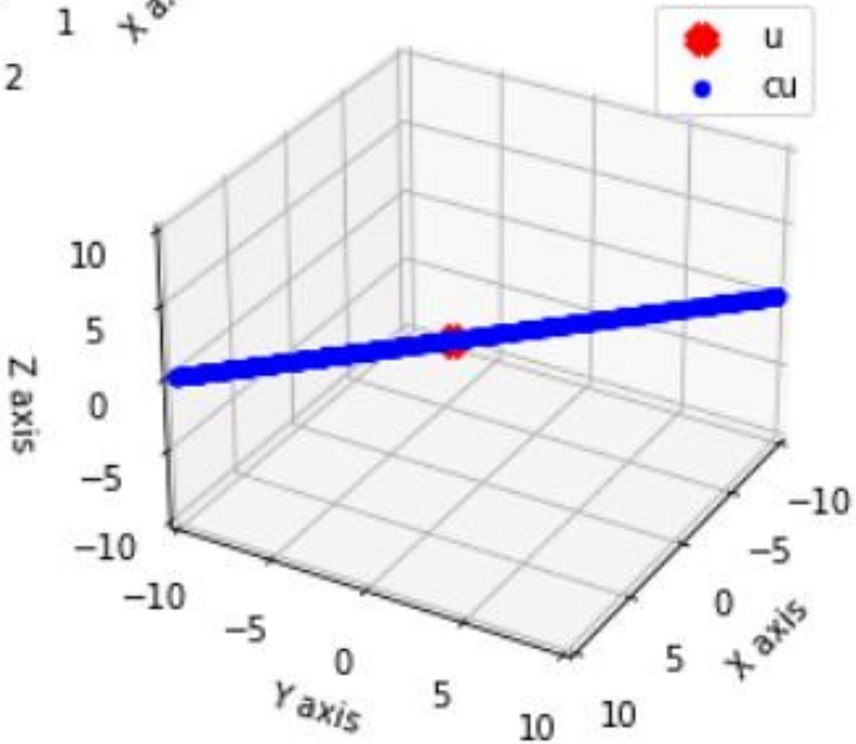
Linear Combinations Illustration (python)

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Visualize $c\boldsymbol{u} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}$

- (Sketch the vector $c\boldsymbol{u}$ by changing the value of the constant c)



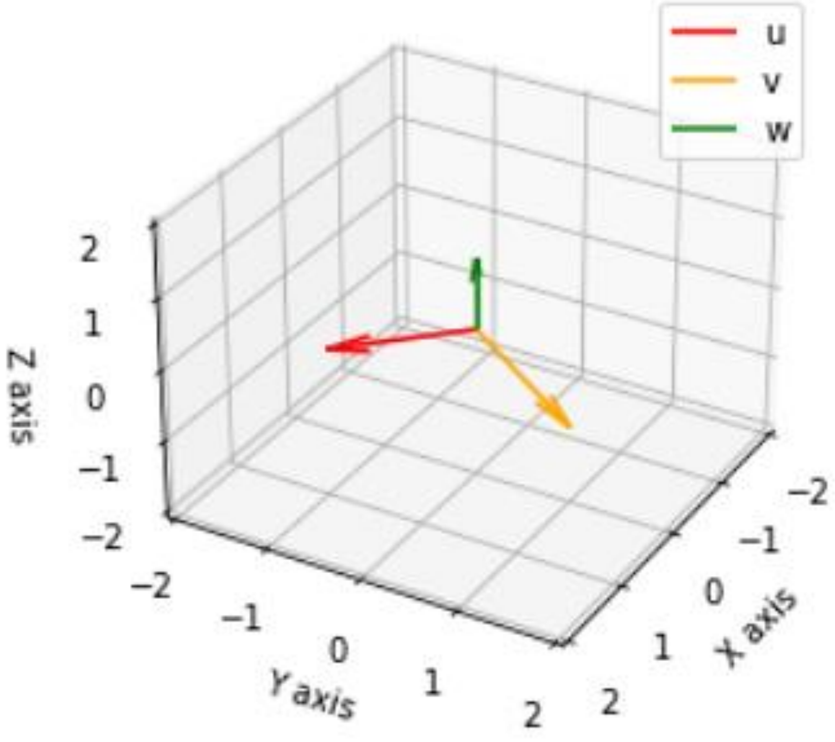


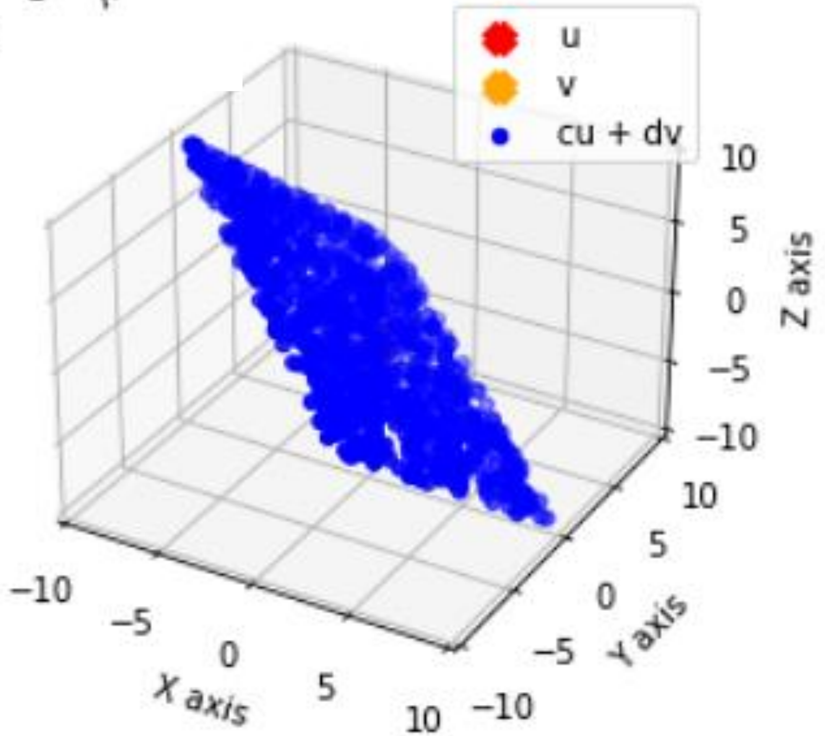
Linear Combinations Illustration (python)

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Visualize $c\boldsymbol{u} + d\boldsymbol{v} = \begin{bmatrix} c \\ -c + d \\ -d \end{bmatrix}$

- (visualize the linear combination $c\boldsymbol{u} + d\boldsymbol{v}$ by changing the values of both c and d)



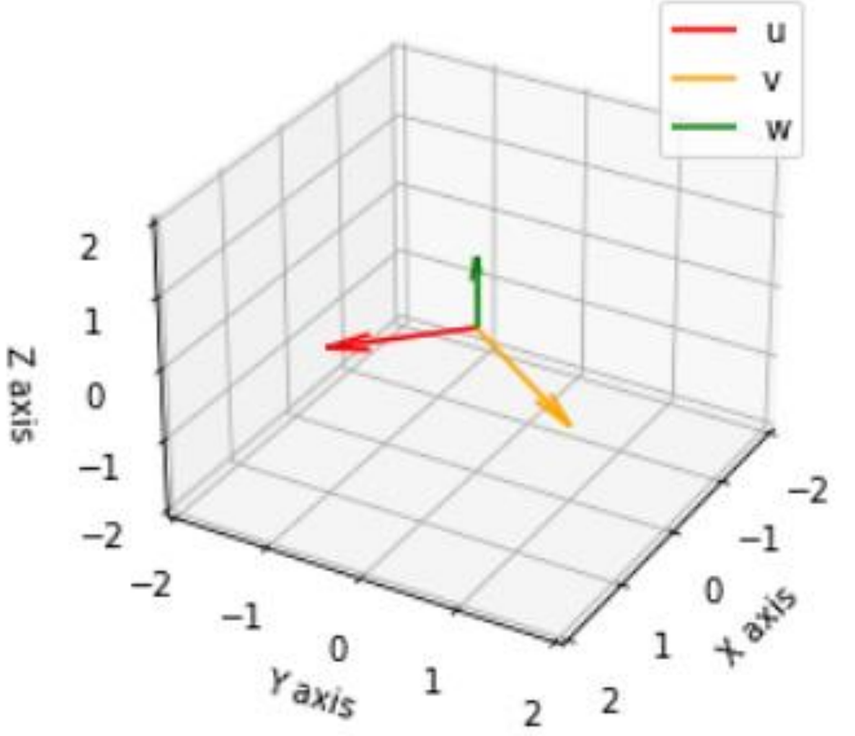


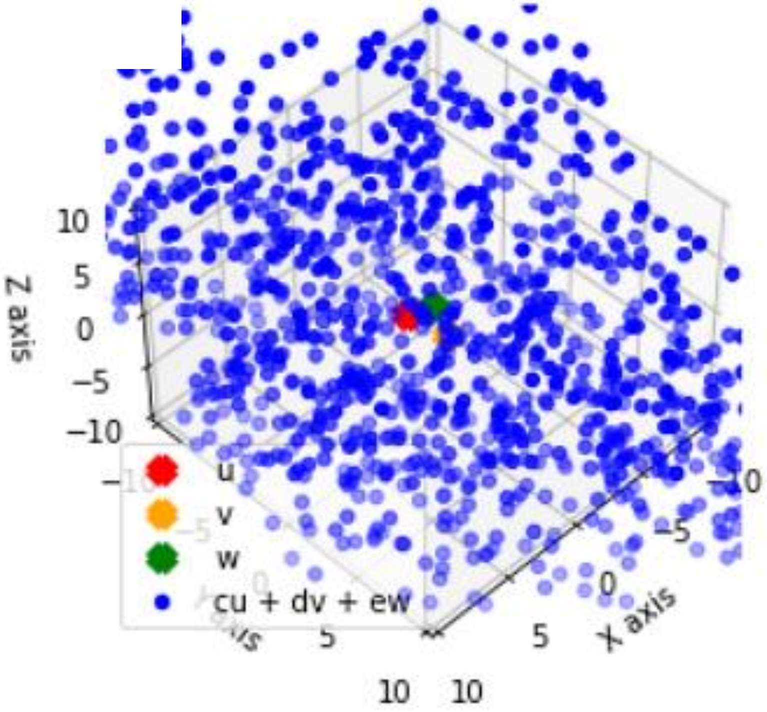
Linear Combinations Illustration (python)

Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- Visualize $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \begin{bmatrix} c \\ -c + d \\ -d + e \end{bmatrix}$

- (Visualize the linear combination$c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ by changing the values of the constants c, d, and e)





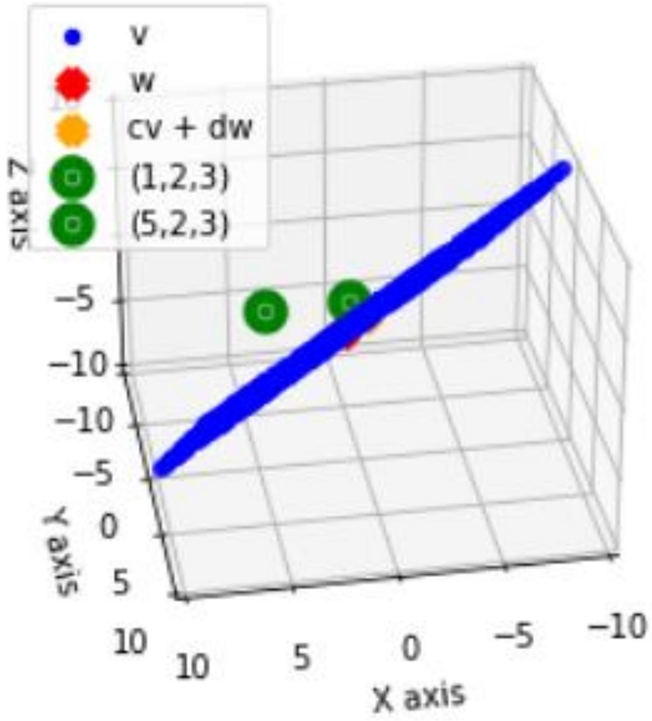
Linear Combinations, Example

- The linear combinations of $\boldsymbol{v} = (1,1,0)\ and\ \boldsymbol{w} = (0,1,1)$ fill a plane in $\mathbb{R}^3$. Describe that plane. Find a vector that is not a combination of $\boldsymbol{v}\ and\ \boldsymbol{w}$-not on the plane.

- $Combinations\ c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix}$ fill a plane.

- Four vectors in this plane are $(0,0,0)$ and $(2,3,1)$ and $(5,7,2)$ and $(\pi, 2\pi, \pi)$. The second component $c + d$ is always the sum of the first and third components.

- $(1,2,3)$ is not in the plane because $2 \neq 1 + 3$



2. Vector Spaces and Subspaces

Vector Spaces and Subspaces (Definitions)

- A **Vector space** is a set of vectors that is closed under linear combinations (all linear combinations of elements stay in the vector space).
- The vector spaces $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ contain vectors with $1,2,3, \ldots$, n components, respectively.
- A **subspace** of a vector space is a non-empty subset that satisfies the requirement of a vector space, i.e., that all the linear combinations of elements stay in the subspace.
- A plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it's contained in.
- <u>The smallest subspace contains only the zero vector.</u>

Vector Spaces and Subspaces

- If $\mathcal{V}$ is a subset of $\mathbb{R}^n$, to be a valid subspace it must satisfy three conditions:

1. $\mathcal{V}$ contains the zero vector: $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \in \mathcal{V}$
2. If $\boldsymbol{x} \in \mathcal{V}$, then $\mathrm{c}\boldsymbol{x} \in \mathcal{V}$. Closed under scalar multiplication.
3. If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, then $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{V}$. Closed under addition.

For example: Here is a list of all the possible subspaces of $\mathbb{R}^3$:

(**L**) Any line through $(0,0,0)$.

(**P**) Any plane through $(0,0,0)$.

($\mathbb{R}^3$) The whole space is a subspace (of itself).

(**Z**) The single vector $(0,0,0)$.

3. Vector Span (Linear Span)

Vector Span

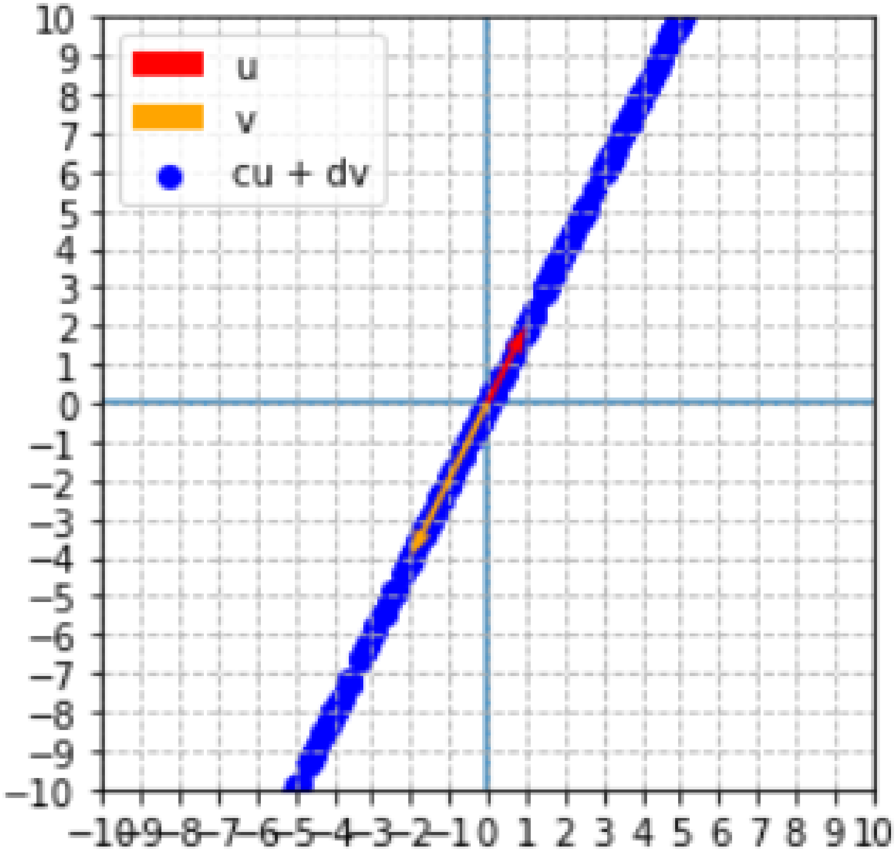
The **span** of a set $\mathcal{S}$ of vectors, denoted by span($\mathcal{S}$), is the smallest linear subspace that contains the set. It can be characterized as the set of linear combinations of elements of $\mathcal{S}$.

$$span(\mathcal{S}) = \{\textstyle\sum_{i=1}^{n} c_i \boldsymbol{v}_i \,|\, n \in \mathbb{N}, \boldsymbol{v}_i \in \mathcal{S}, c_i \in \mathbb{R}\}$$

$$\textstyle\sum_{i=1}^{n} c_i \boldsymbol{v}_i = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$$

The **span** of a set of vectors is the set of all linear combinations of the vectors. For example, if $v^1 = [11,\ 5,\ -7,\ 0]^T$ and $v^2 = [2,\ 13,\ 0,\ -7]^T$ then the span of v^1 and v^2 is the set of all vectors of the form sv^1+tv^2 for some scalars s and t.

Linear Span, Example 1



- Given the vectors $\boldsymbol{u} = \begin{bmatrix}1\\2\end{bmatrix}, \boldsymbol{v} = \begin{bmatrix}-2\\-4\end{bmatrix}$, find the span$(\boldsymbol{u}, \boldsymbol{v})$.

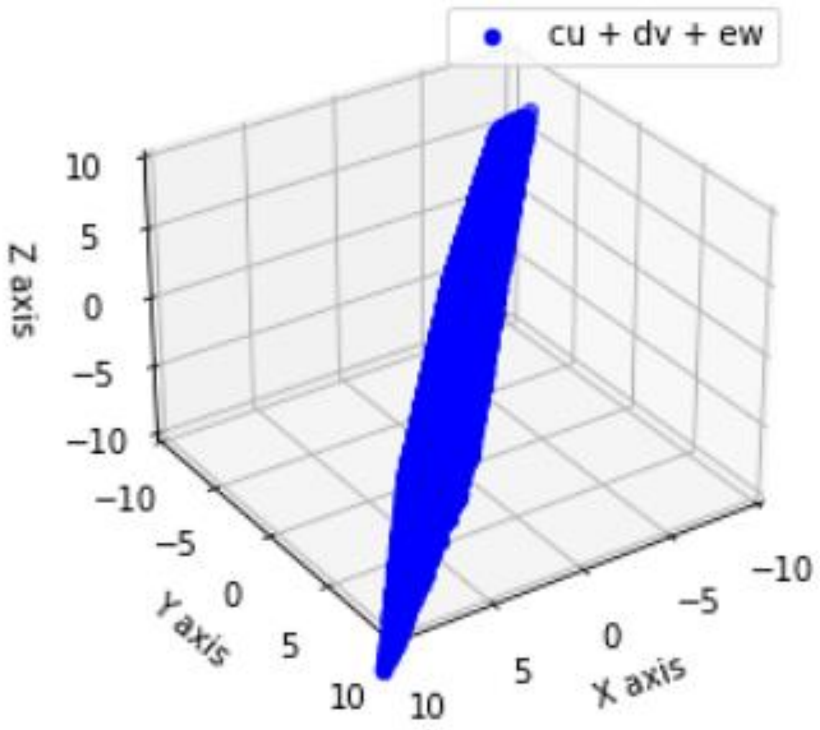
- Span$(\boldsymbol{u}, \boldsymbol{v})$ =

$$c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}-2\\-4\end{bmatrix} = c\begin{bmatrix}1\\2\end{bmatrix} - 2d\begin{bmatrix}1\\2\end{bmatrix} = (c-2d)\begin{bmatrix}1\\2\end{bmatrix} = e\begin{bmatrix}1\\2\end{bmatrix},$$

where $c, d, e \in \mathbb{R}$.

- All linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ are on the line $e\begin{bmatrix}1\\2\end{bmatrix}$.

- We can't represent most of the vectors $\in \mathbb{R}^2$ by a linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$, we say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are colinear.



Linear Span, Example 2

- Given the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$,
- what is the span$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$?
- span$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ is a plane in $\mathbb{R}^3$. Why?
- We can't represent most of the vectors $\in \mathbb{R}^3$ by a linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\boldsymbol{w}$.

Linear Span, Example 3

- Given the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, what is the span($\boldsymbol{u}$), span($\boldsymbol{u}, \boldsymbol{v}$), and span($\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$)?

- Span($\boldsymbol{u}$) $= c \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c \\ 3c \end{bmatrix}$, where $c \in \mathbb{R}$
- Span($\boldsymbol{u}, \boldsymbol{v}$) $= c \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 3c + 2d \end{bmatrix}$, where $c, d \in \mathbb{R}$
- Span($\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$) $= c \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 2 \end{bmatrix} + e \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} c + 3d \\ 3c + 2d + 4e \end{bmatrix}$, where $c, d, e \in \mathbb{R}$

- Span($\boldsymbol{u}$) $= c\boldsymbol{u}$ Line in $\mathbb{R}^2$
- Span($\boldsymbol{u}, \boldsymbol{v}$) $= \mathbb{R}^2$. You can represent any vector in $\mathbb{R}^2$ with a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.
- Span($\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$) $= \mathbb{R}^2$

Vector Span, Example 4

Let $V = \{(2,3),(1,2)\}$. Show whether or not the vector $(19,3) \in \operatorname{span}(V)$.

By the definition of a vector existing within the span of V, we must find scalars c_1 and c_2 such that:

$$c_1(2,3) + c_2(1,2) = (19,3)$$

If we cannot such scalars, then $(19,3) \notin \operatorname{span}(V)$ and similarly, if we can find such scalars, then $(19,3) \in \operatorname{span}(V)$. We thus obtain the following system of linear equations:

$$\begin{aligned} 2c_1 + c_2 &= 19 \\ 3c_1 + 2c_2 &= 3 \end{aligned}$$

When we reduce the augment matrix of this system to reduced row echelon form, we get that:

$$\begin{aligned} c_1 + 0c_2 &= 35 \\ 0c_1 + c_2 &= -51 \end{aligned}$$

Therefore, we have found a set of scalars c_1, c_2 which satisfy our condition, and therefore, $(19,3) \in \operatorname{span}(V)$ since $35(2,3) + -51(1,2) = (19,3)$.

4. Linear Independence

Linear Independence of Vectors

- If $\boldsymbol{v_1} = 5\boldsymbol{v_2} + 7\boldsymbol{v_3}$, then $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are not linearly independent since at least one of them can be expressed as the sum of other vectors.

- **Definition:** The sequence of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ is linearly independent if the only combination that gives the zero vector is $0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_n$.

- If $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n = 0$ and not all $a_i = 0$, then the vectors are not linearly independent.

- **Definition:** The columns of A are linearly independent when the only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$. ***No other combination $A\boldsymbol{x}$ of the columns gives the zero vector.***

- I.e., the columns are independent when the nullspace $N(A)$ contains only the zero vector.

Linear Independence of Vectors

- Given a set of vectors, the following method can be used to check whether they are linearly independent or not.

- $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n = 0$ can be written as

$$[v_1 v_2 \ldots. v_n]\begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} = 0 \; where \;\; v_i \in \mathbb{R}^{m\times 1} \; \forall i \in \{1,2,\ldots,n\}, \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$$

- Solving for $[a_1 a_2 \ldots a_n]^T$, if the only solution we get is the zero vector, then the set of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ is said to be linearly independent.

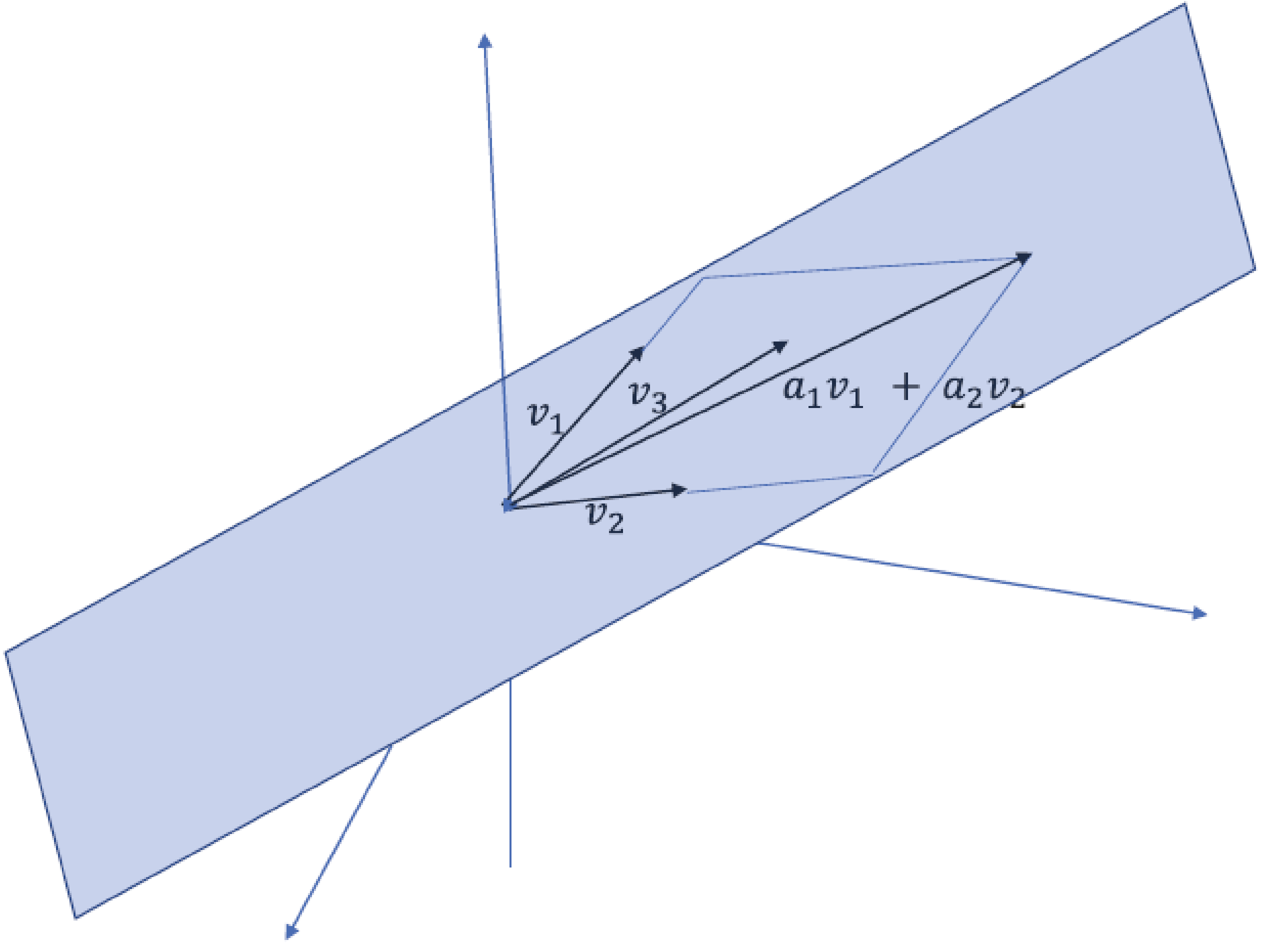
Span and Linear Independence of Vectors

- If a set of *n* vectors $\boldsymbol{v_i} \in \mathbb{R}^{n\times 1}$ is linearly independent, then these vectors span the whole *n*-dimensional space
- by taking linear combinations of the *n* vectors, one can produce all possible vectors in the *n*-dimensional space.
- If the *n* vectors are not linearly independent, they span only a subspace within the *n*-dimensional space.

$$[v_1 v_2 \,....\, v_n]\begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} = 0 \; where \;\; v_i \in \mathbb{R}^{m\times 1} \; \forall i \in \{1,2,\ldots,n\}, \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$$

Span and Linear Independence of Vectors, Example

- To illustrate this, take vectors in the 3D space.
- 3 vectors that are not linearly independent span only a subspace (plane) within the 3-dimensional space.



Span and Linear Independence of Vectors, Example 2

- If we have a vector $\boldsymbol{v_1} = [1\ 2\ 3]^T$, we can span only one dimension in the three-dimensional space because all the vectors that can be formed with this vector would have the same direction as that of $\boldsymbol{v_1}$, with the magnitude being determined by the scaler multiplier. In other words, each vector would be of the form $a_1\boldsymbol{v_1}$.

- Take another vector $\boldsymbol{v_2} = [5\ 9\ 7]^T$, whose direction is not the same as that of $\boldsymbol{v_1}$. The span of the two vectors *Span*$(\boldsymbol{v_1}, \boldsymbol{v_2})$ is the linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

- With these two vectors, we can form any vector of the form $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2}$ that lies in the plane of the two vectors.

- Basically, we will span a two-dimensional subspace within the three-dimensional space.

Linear Independence and Span, Illustration

- If three vectors in $\mathbb{R}^3$ are not in the same plane, they are independent.
- No combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ in the figure gives a zero except $0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + 0\boldsymbol{v}_3$.
- If three vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are in the same plane, they are dependent.

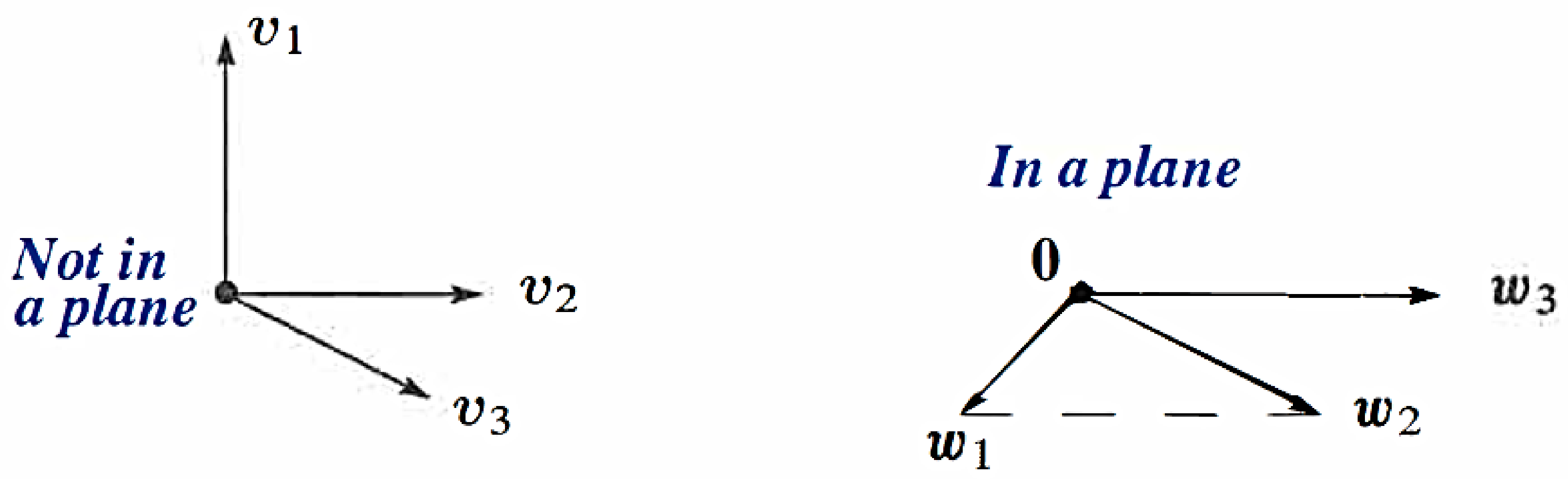


Figure 3.4: Independent vectors v_1, v_2, v_3. Only $0v_1 + 0v_2 + 0v_3$ gives the vector $\mathbf{0}$. Dependent vectors w_1, w_2, w_3. The combination $w_1 - w_2 + w_3$ is $(0, 0, 0)$.

Linear Independence

- We call a set of vectors a **linearly dependent set** when at least one of the vectors of the set can be represented by a linear combination of the other vectors in the set.

- Any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

- A matrix $A \in \mathbb{R}^{m \times n}$ with $m < n$ has dependent columns: At least $n - m$.

- If set $\mathcal{S}$ of n vectors is a linearly independent set, then $span(\mathcal{S}) = n - dimension\ subspace$.

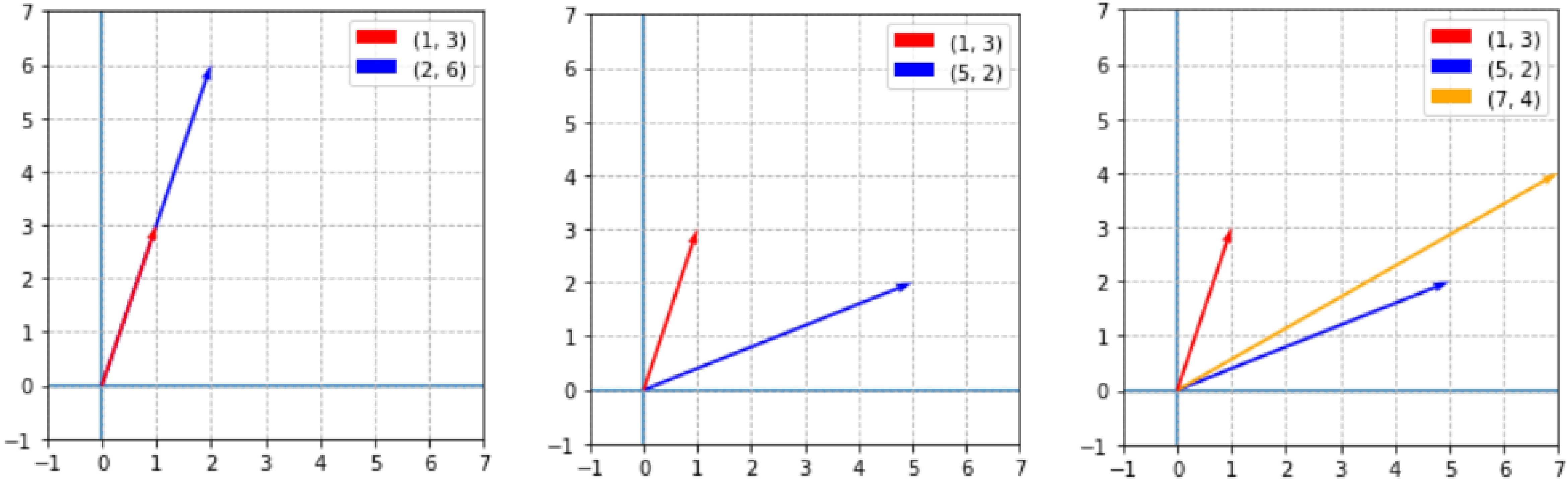
- If set $\mathcal{S}$ of n vectors $\in \mathbb{R}^n$ is a linearly independent set, then $span(\mathcal{S}) = \mathbb{R}^n$.

Linear Independence, Examples

$\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}2\\6\end{bmatrix}\right\}$ are not linearly independent.

$\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\right\}$ are linearly independent.

$\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix},\begin{bmatrix}7\\4\end{bmatrix}\right\}$ are not linearly independent.

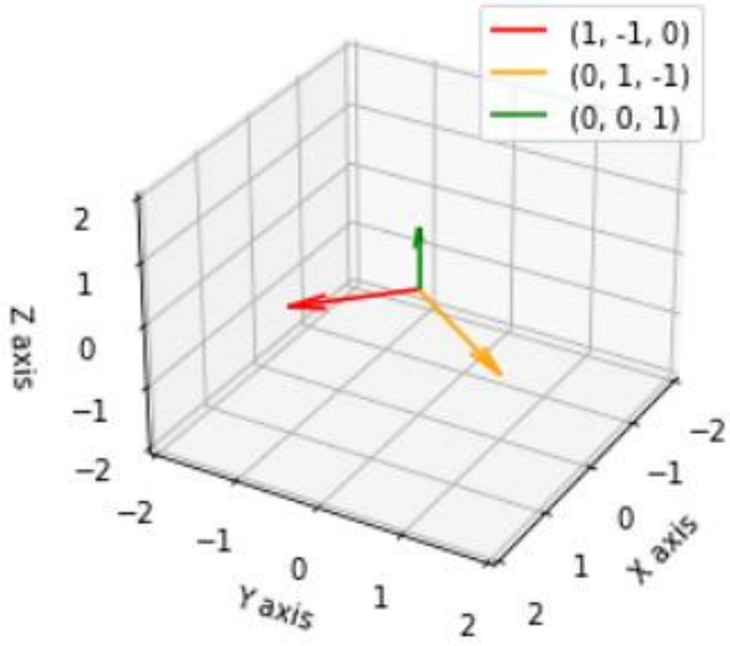


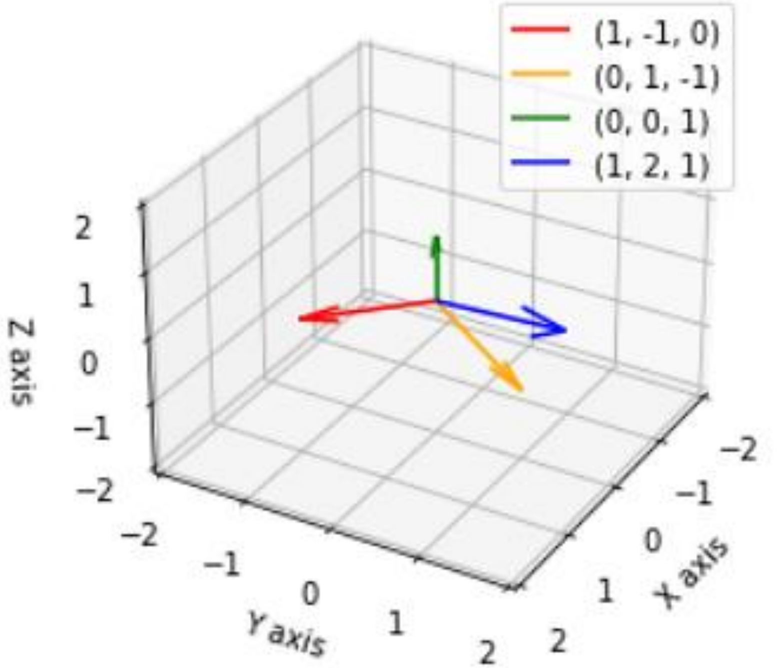
Linear Independence, Examples, cont'd

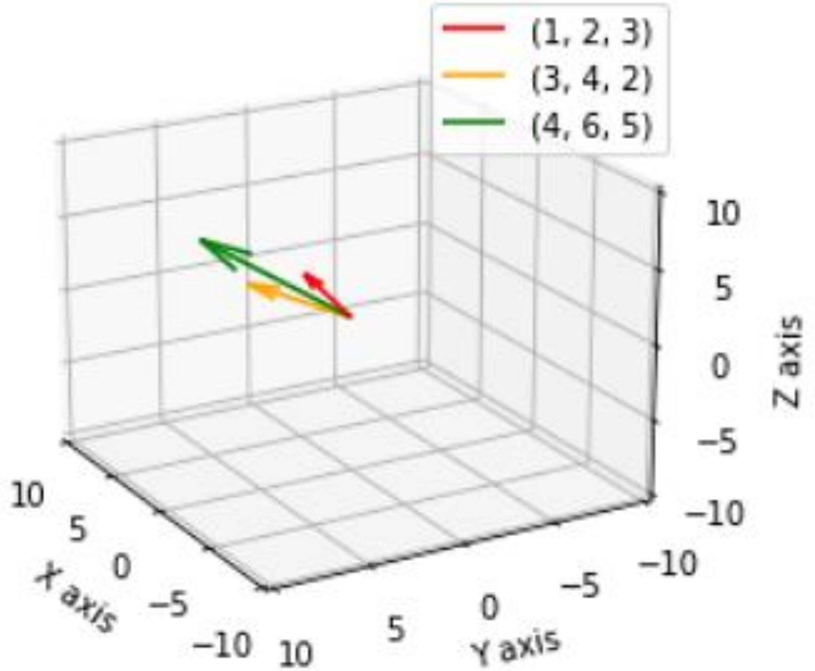
$\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ are linearly independent.

$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\4\\2\end{bmatrix}, \begin{bmatrix}4\\6\\5\end{bmatrix}\right\}$ are not linearly independent.

$\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$ are not linearly independent.







5. Basis of Vector Space

Basis

- Why are we interested in the concept of basis?
- How the concept of basis of a vector space is based on span and linear combination of vectors?
- The link between matrices and linear transformations?
- Definition and dimension of a basis of a vector space.

Basis of Vector Space

- **Definition:** A basis for a vector space is a sequence of vectors with two properties: The basis vectors are linearly independent, and they span the space.
- The basis is **the minimum set** of vectors that span the space.
- For a given vector $\boldsymbol{v}$, there is only one way to write $\boldsymbol{v}$ as a combination of the basis vectors.
- A vector space V does not have a unique basis. There are infinite number of bases.
- All different bases for V have the same number of vectors. This number is the **dimension** of V.
- **Definition:** The **dimension** of a space is the number of vectors in every basis.

Basis of Vector Space

- The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbb{R}^2$.
- The basis vectors $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are independent. They span $\mathbb{R}^2$.
- The columns of 3×3 identity matrix are the standard basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.
- The columns of $n \times n$ identity matrix give the "standard basis" for $\mathbb{R}^n$.
- The columns of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ are basis for $\mathbb{R}^3$, independent columns.
- The columns of $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ are not basis, dependent columns.

The fundamental Subspaces

The column space $C(A)$

The null space $N(A)$

The row space $R(A)$

Column Space of a Matrix

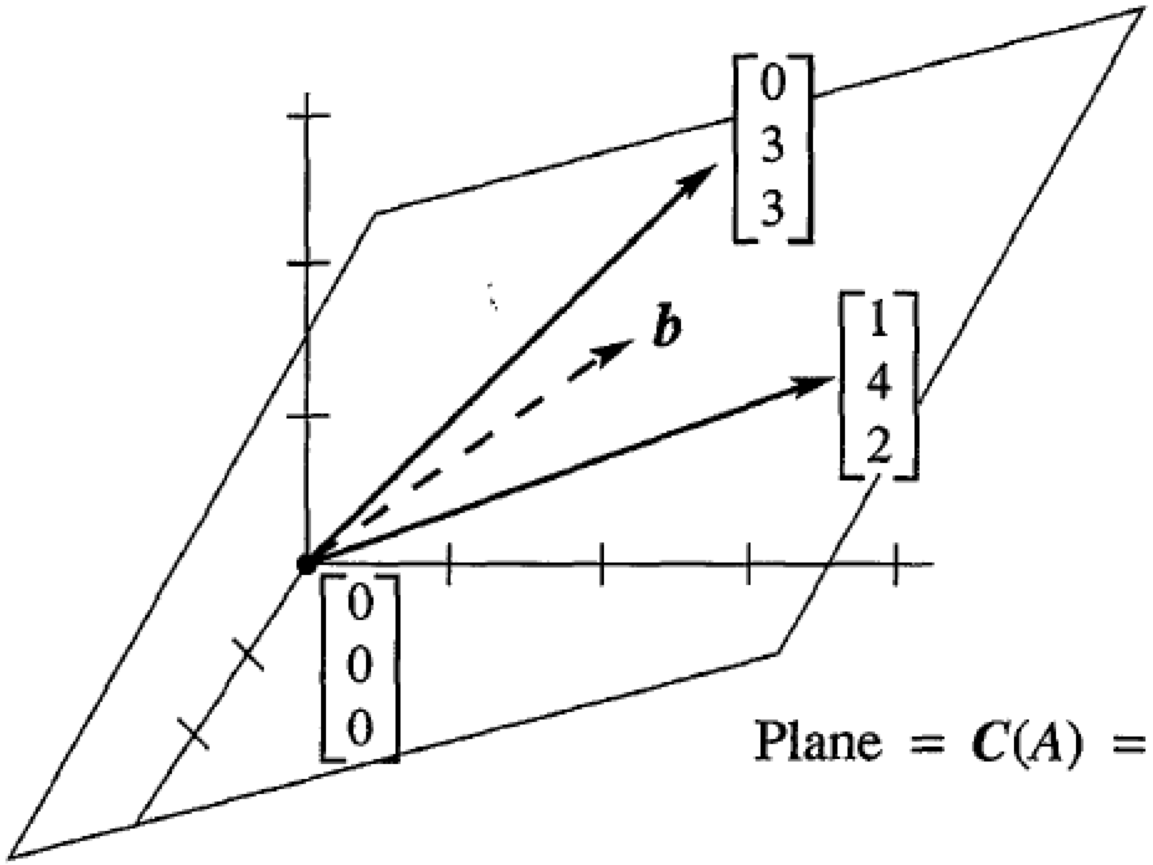
The $column\ space\ C(A)$ of a real $m \times n$ matrix A is a subspace of $\mathbb{R}^m$ and contains all the linear combinations of the columns of A.

- $C(A) = \{\boldsymbol{b} | \boldsymbol{b} = A\boldsymbol{x}\ and\ \boldsymbol{x} \in \mathbb{R}^n\}$

- A system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is solvable if and only if $\boldsymbol{b}$ is in the column space of A.

Column Space Example

$$Ax \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{which is} \quad x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Plane $= C(A) =$ all vectors Ax

Null Space of a Matrix

The null space $N(A)$ of a real $m \times n$ matrix A is a subspace of $\mathbb{R}^n$ and contains all vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{0}$.

- $N(A) = \{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} = \boldsymbol{0}\}$
- The null space $N(A)$ always contains the zero vector $\boldsymbol{0}$.

- If $\boldsymbol{x}_p$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, then so is every $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ where $\boldsymbol{x}_n$ is any vector in the null space $N(A)$.

Null Space, Example 1

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular!

Solution Apply elimination to the linear equations $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \quad \to \quad \begin{aligned} x_1 + 2x_2 &= 0 \\ \mathbf{0} &= \mathbf{0} \end{aligned}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $\boldsymbol{N}(A)$. It contains all solutions (x_1, x_2).

To describe this line of solutions, here is an efficient way. Choose one point on the line (one "***special solution***"). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. The special solution $\boldsymbol{s}$ is $(-2, 1)$:

Special solution The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Null Space, Example 2

Example 3 Describe the nullspaces of these three matrices A, B, C:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $A\boldsymbol{x} = \mathbf{0}$ has only the zero solution $\boldsymbol{x} = \mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $\boldsymbol{x} = \mathbf{0}$ in $\mathbf{R}^2$. This comes from elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}.$$

A is invertible. There are no special solutions. All columns of this A have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $B\boldsymbol{x} = \mathbf{0}$ again require $\boldsymbol{x} = \mathbf{0}$. The last two equations would also force $\boldsymbol{x} = \mathbf{0}$. When we add extra equations, the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors $\boldsymbol{x}$ in the nullspace.

Null Space, Example 2, cont'd

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector $\boldsymbol{x}$ has *four* components. Elimination will produce pivots in the first two columns of C, but the last two columns are "free". They don't have pivots:

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \textbf{pivot columns} \quad \textbf{free columns} \end{array}$$

For the free variables x_3 and x_4, we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are determined by the

equation $U\boldsymbol{x} = \boldsymbol{0}$. We get two special solutions in the nullspace of C (which is also the nullspace of U). The special solutions are $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$:

$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \quad \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \quad \text{variables} \end{array}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U! We can continue to make this matrix simpler, in two ways:

Reduced form R

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \text{ becomes } R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \\ \uparrow \quad \uparrow \qquad\qquad \\ \text{now the pivot columns contain } I \end{array}.$$

Row Space

- The row space of a matrix A is the set of linear combinations of the rows of A.

- The row space *R(A)* is the orthogonal complement of the null space *N(A)*. This means that for all vectors $\boldsymbol{v} \in R(A)$ and all vectors $\boldsymbol{w} \in N(A)$, we have $\mathbf{v} \cdot \mathbf{w} = 0$.

- Together, the null space and the row space form the domain of the transformation = $N(A) \oplus R(A)$, where $\oplus$ stands for orthogonal direct sum.

Row Space, Example

Consider the following matrix and its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 7 & 6 \\ 3 & 9 & 9 & 10 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} \mathbf{1} & 3 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}.$$

The reduced row echelon form of the matrix A contains three pivots. The locations of the pivots will play an important role in the following steps.

The vectors $\{(1,3,0,0),(0,0,1,0),(0,0,0,1)\}$ form a basis for $\mathcal{R}(A)$.

Questions for the instructor?