29/11/2025

LA , AI 46 , Mansoura, session 4

1- Gram-Schmidt Orthonormalization

2- eigen vectors & eigenvalues

→ eigen decomposition

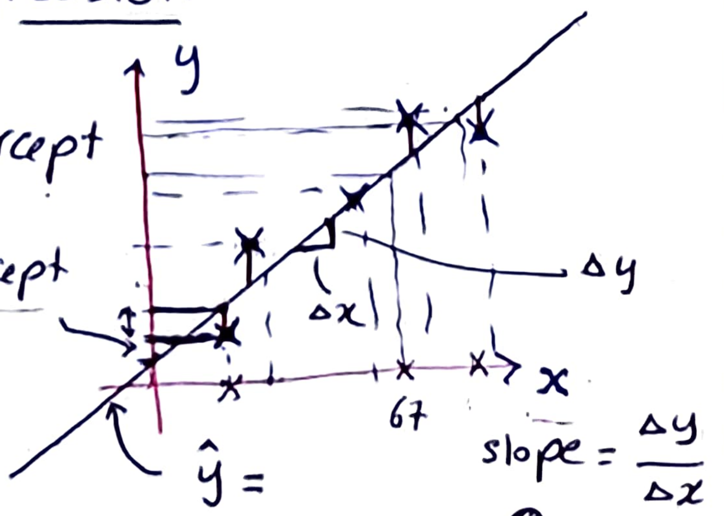
Review

→ Generalized Matrix Inverse

→ Pseudo-Inverse

example Linear Regression

$\hat{y}$ = slope * x + intercept



intercept
θ_0
α
a

slope = $\frac{\Delta y}{\Delta x}$
θ_1
β
b

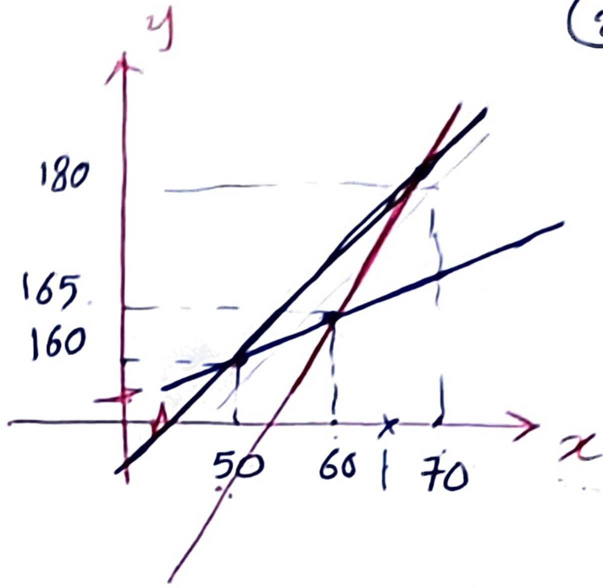
$\hat{y} = \alpha + \beta x$

$\theta_0 + \theta_1 x$

$y = \hat{y} + \epsilon$

$y = \alpha + \beta x + \epsilon$

$$y_i = \alpha + \beta x_i$$



$$160 = \alpha + \beta \times 50 \quad ①$$

$$165 = \alpha + \beta \times 60 \quad ②$$

$$180 = \alpha + \beta \times 70 \quad ③$$

$$1\alpha + 50\beta = 160 \quad ①$$

$$1\alpha + 60\beta = 165 \quad ②$$

$$1\alpha + 70\beta = 180 \quad ③$$

$$\underset{3\times 2}{A}\ \underset{2\times 1}{\vec{\theta}} = \underset{3\times 1}{\vec{y}} \Leftrightarrow \begin{bmatrix} 1 & 50 \\ 1 & 60 \\ 1 & 70 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\vec{\theta}} = \begin{bmatrix} 160 \\ 165 \\ 180 \end{bmatrix}$$

Review

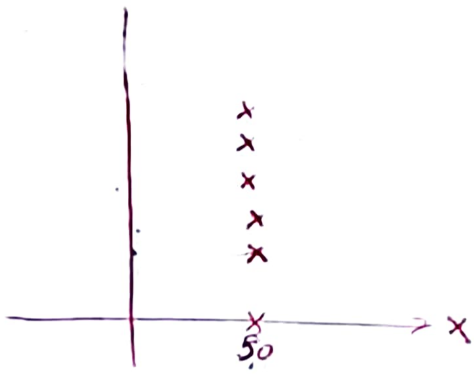
for square matrix A

$$\underbrace{\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\vec{\theta}} = \underbrace{\begin{bmatrix} \; \end{bmatrix}}_{\vec{y}} \Rightarrow A\vec{\theta} = \vec{y}$$

$$A^{-1} A \vec{\theta} = A^{-1} \vec{y}$$

$$I \vec{\theta} = A^{-1} \vec{y}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{\theta} = A^{-1} \vec{y}$$



$$\underbrace{(A^T)_{2\times 3} A_{3\times 2}} \vec{\theta}_{2\times 1} = A^T \vec{y}_{3\times 1}$$

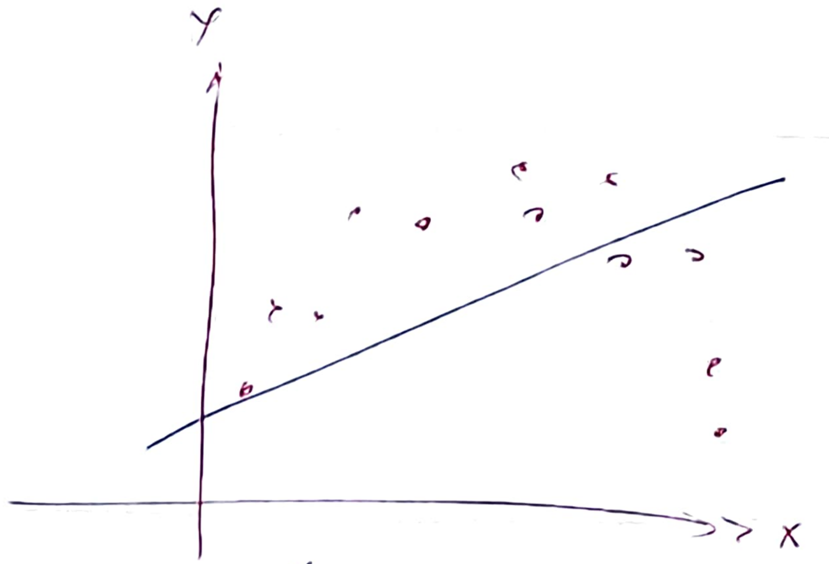
$$\underbrace{(A^TA)^{-1}}\underbrace{(A^TA)_{2\times 2}} \vec{\theta}_{2\times 1} = (A^TA)^{-1} A^T_{2\times 3} \vec{y}_{3\times 1}$$

$$I \vec{\theta}_{2\times 1} = (A^TA)^{-1} A^T \vec{y}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{\theta} = \underbrace{(A^TA)^{-1} A^T} \vec{y}$$

generalized Inverse $\Leftrightarrow$ Pseudo Inverse of Matrix A

$\Leftrightarrow$ Moore - Penrose Inverse



Gram-Schmidt Orthonormalization

Orthogonalization + normalization

step 1 step 2

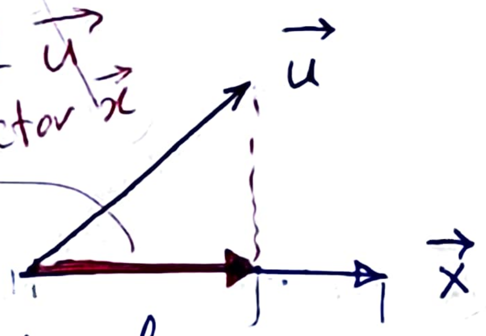
basis of a vector space

basis (independent but not orthogonal)

orthogonal basis

Projection of $\vec{u}$ on the vector $\vec{x}$

$Proj_{\vec{x}}(\vec{u})$



→ Component of a vector along another vector?

$$Proj_{\vec{x}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\right)\vec{x}$$

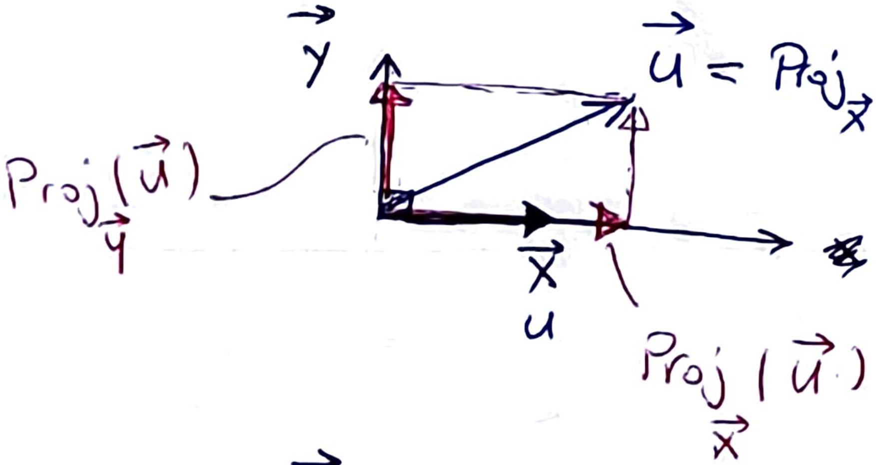
ratio

$$\vec{u}\cdot\vec{x} = \|u\|\,\|x\|\cos\theta$$

$$\vec{x}\cdot\vec{x} = \|x\|\,\|x\|\underbrace{\cos\theta}_{1} = \|x\|^2$$

$$\mathrm{Proj}_{\vec{x}}(\vec{u}) = \frac{\|u\|\,\cancel{\|x\|}\,\cos\theta}{\|x\|\,\cancel{\|x\|}} = \left(\frac{\|u\|\,\cos\theta}{\|x\|}\right)\vec{x}$$

ratio between projection of ~~vector~~ lenght $\|u\|$ and $\|x\|$.



$$\vec{u} = \mathrm{Proj}_{\vec{x}}(\vec{u}) + \mathrm{Proj}_{\vec{y}}(\vec{u})$$

$$\underbrace{\vec{u} - \mathrm{Proj}_{\vec{x}}(\vec{u})} = \underbrace{\mathrm{Proj}_{\vec{y}}(\vec{u})}$$

orthogonal

Starting with non-orthogonal basis

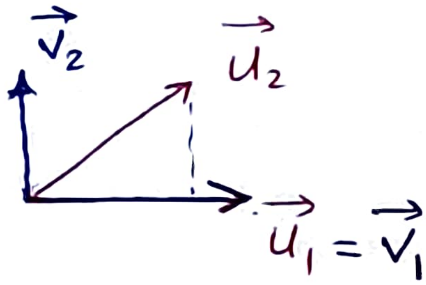
$\ast\ \underline{\vec{u}_1}, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n$

basis for $\mathbb{R}^n$

$\Rightarrow$ create an orthogonal set of basis $\Rightarrow \vec{v}_1, \vec{v}_2, \text{----}, \vec{v}_n$

step ① :

$$\vec{v}_1 = \vec{u}_1$$



$$\vec{v}_2 = \vec{u}_2 - \text{Proj}_{\vec{v}_1}(u_2)$$

$$\vec{v}_3 = \underline{\vec{u}_3} - \text{proj}_{\vec{v}_1}(\vec{u}_3) - \text{proj}_{\vec{v}_2}(u_3)$$

$$\vdots$$

$$\vec{v}_n = \vec{u}_n - \text{Proj}_{\vec{v}_1}(\vec{u}_n) - \text{proj}_{v_2}(\vec{u}_n) - \dots - \text{proj}_{\vec{v}_{n-1}}(\vec{u}_n)$$

$\hookrightarrow$ orthogonal set of basis of $\mathbb{R}^n$

② Review <u>normalization</u> ; normal basis vectors.

$\Rightarrow$ norm $= 1$

e.g., standard unit vectors $\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}} \in \mathbb{R}^3$

"standard basis" of $\mathbb{R}^3$

$$\|\hat{i}\| = \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| = 1 \quad \|\hat{j}\| = \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\| = 1 \quad \|\hat{k}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| = 1$$

Step 2 normalization

$$\hat{e}_1 = \frac{\vec{V}_1}{\|\vec{V}_1\|}$$

$$\hat{e}_2 = \frac{\vec{V}_2}{\|\vec{V}_2\|}$$

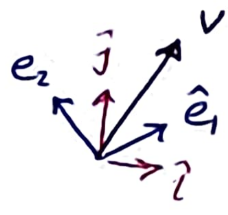
$$\vdots$$

$$\hat{e}_n = \frac{\vec{V}_n}{\|\vec{V}_n\|}$$

$\rightarrow$ orthogonal + normal } orthonormal

Change of Basis Matrix in $\mathbb{R}^2$

basis $\hat{e}_1, \hat{e}_2 \Leftrightarrow \hat{\imath}, \hat{\jmath}$



eg., $\vec{V} = v_1 \hat{e}_1 + v_2 \hat{e}_2 = v_x \hat{\imath} + v_y \hat{\jmath}$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \hat{\imath}\cdot\hat{e}_1 & \hat{\imath}\cdot\hat{e}_2 \\ \hat{\jmath}\cdot\hat{e}_1 & \hat{\jmath}\cdot\hat{e}_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \hat{\imath}\cdot\hat{e}_1 & \hat{\imath}\cdot\hat{e}_2 \\ \hat{\jmath}\cdot\hat{e}_1 & \hat{\jmath}\cdot\hat{e}_2 \end{bmatrix}^{-1} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

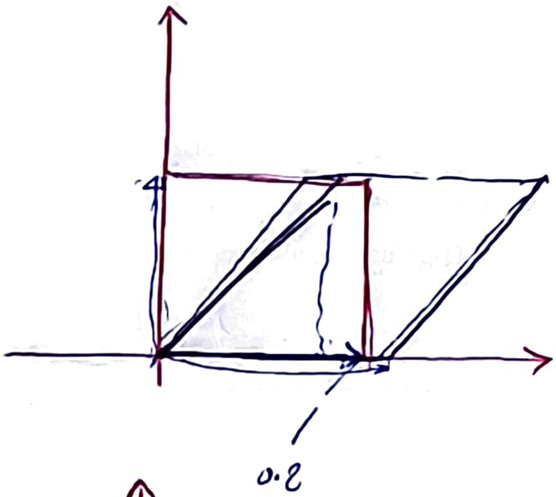
eigen values & eigen vectors

for square Matrices

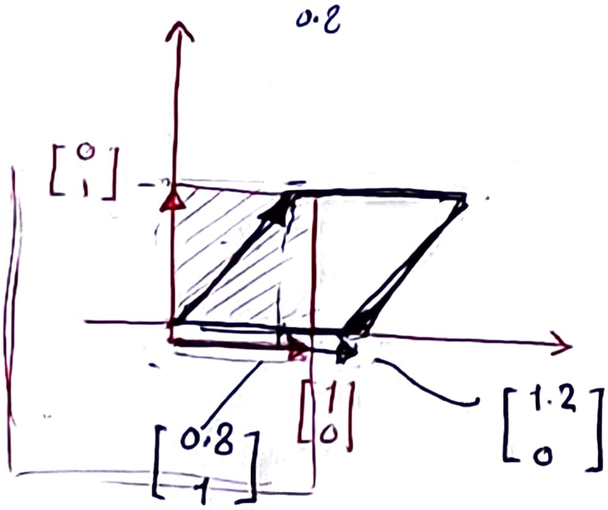
Review ⇒ Matrix A as a tranformation

example

$$A = \begin{bmatrix} 1.2 & 0.8 \\ 0 & 1 \end{bmatrix}$$



if $\vec{v}$ is an eigenvector of the Matrix A



$$\Rightarrow A\vec{v} = \lambda\vec{v} \rightsquigarrow (1)$$

λ: eigenvalue associated with eigenvector $\vec{v}$

solving (1) we can find eigenvalues & eigenvectors

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\cancel{(A - \lambda I)\vec{v} = \vec{0}} \Rightarrow \underline{\det}(A - \lambda I) = 0$$

λ ✓

ex solving for eig. vectors & λ's of $A = \begin{bmatrix} 1.2 & 0.8 \\ 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\left| \begin{bmatrix} 1.2 & 0.8 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 1.2-\lambda & 0.8 \\ 0 & 1-\lambda \end{vmatrix}$$

$$(1.2-\lambda)(1-\lambda) - \underbrace{\cancel{0.8 \times 0}}_{\text{zero}} = 0$$

$$(\lambda - 1.2)(\lambda - 1) = 0$$

$$\lambda_1 = 1.2 \qquad \lambda_2 = 1 \quad \leftarrow \text{eigenvalues}$$

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solving for eigenvectors ;

$$(A - \lambda I)\vec{V} = \vec{0}$$

for $\lambda_1 = 1.2 \Rightarrow \begin{bmatrix} 1.2 & 0.8 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0.8 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0v_1 + 0.8v_2 = 0$

$0v_1 - 0.2v_2 = 0$

$\Rightarrow \underline{v_2 = 0}$, $\underline{v_1}$: free variable for $\lambda_1 = 1.2 \Rightarrow \vec{v}_{A_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

~~Repeat~~ Repeat for $\lambda_2 = 1$

$$\begin{bmatrix} 1.2-1 & 0.8 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 & 0.8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0.2\, v_1 + 0.8\, v_2 = 0$$

free variable

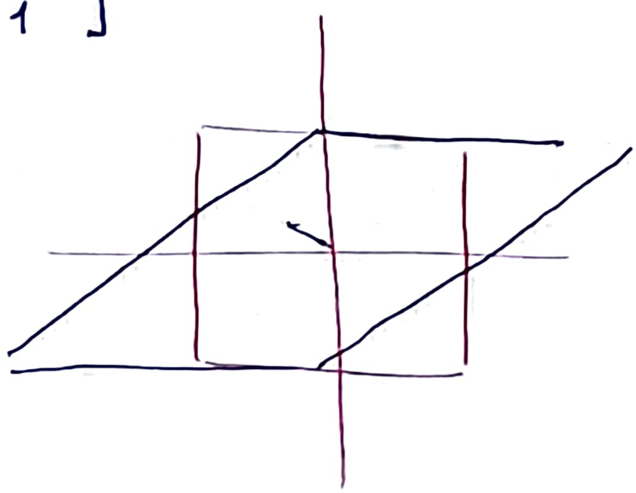
$$0.2\, v_1 = -0.8\, v_2$$

let $v_2 = k$

$$v_1 = -4\, v_2$$

$$v_2 = k \;,\; v_1 = -4\,k$$

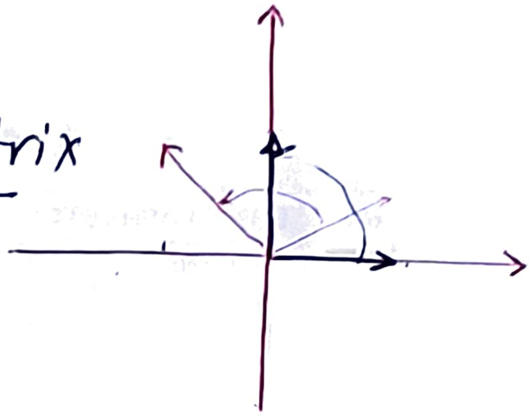
$$\vec{v}_{A_2} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$



ex 2

90° ccw rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\det(A - \lambda I) = 0$$

$$\equiv \det(\lambda I - A) = 0$$

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$(-\lambda)^2 - (-1 \times 1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \sqrt{-1} \doteq \pm i$$

$$\lambda_1 = +i \quad , \quad \lambda_2 = -i$$

→ eigen decomposition

→ diagonalization

$$\underset{n\times n}{A} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & d_n \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

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trace (A) = sum of diagonal elements

$$\text{trace}(A) = tr(A)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & & \\ & & & a_{nn} \end{bmatrix}$$

$$= \sum_{i}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

why A^TA is always invertible.