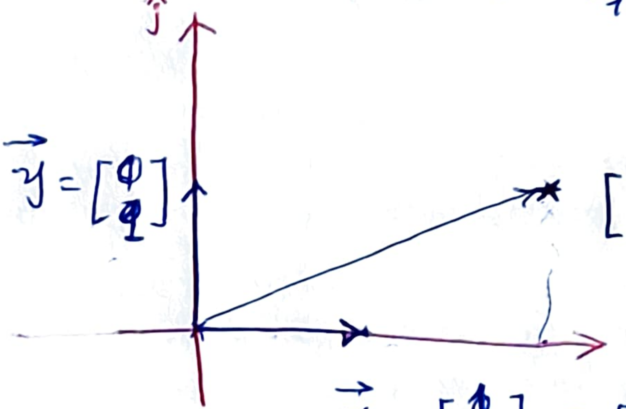
22/11/2025

LA , AI46 , Mansoura , session 2

1 linear combination of vectors

$c_1 \vec{x} + c_2 \vec{y}$ = linear combination of $\vec{x}$ and $\vec{y}$

$\sum_{i=1}^{n} c_i \vec{v}_i$ = linear combination of vectors $\vec{v}_i$



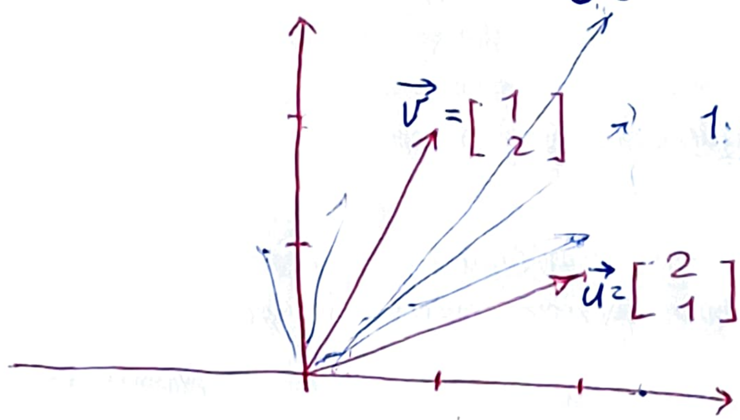
$\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= 2\vec{x} + 1\vec{y}$

$\hat{i}$, $\hat{j}$

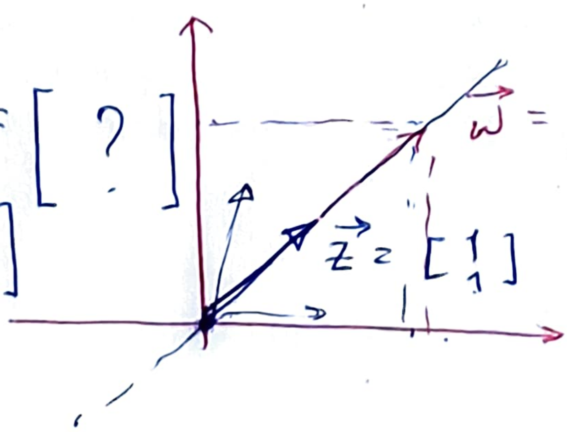
$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0 \ 1]^T = (0, 1)$



$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\rightarrow$ $1.5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2.5 \\ 3.5 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$m\vec{z} + n\vec{w} = \begin{bmatrix} ? \end{bmatrix}$

$= \begin{bmatrix} m \\ m \end{bmatrix} + \begin{bmatrix} 2n \\ 2n \end{bmatrix}$

$= \begin{bmatrix} m+2n \\ m+2n \end{bmatrix}$

$\vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = c\vec{z}$

$= 2\vec{z}$

$\vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{w}$ is a linear combination of $\vec{z}$

ex.

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

construct a matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

$-r_1 + r_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$-r_2 + r_3 \to r_3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-r_1 \times 2 + r_3 \Rightarrow r_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & +2 & +2 \end{bmatrix}$$

$r_2 \times -1 + r_1 \Rightarrow r_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-r_2 \times +2 + r_3 \Rightarrow r_3$

$$\vec{z} = 2\vec{x} + 2\vec{y}$$

ex. $\vec{z} = c_1 \vec{x} + c_2 \vec{y}$ $\Leftarrow$ dependent

$$\underline{c_1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 0$$

$$\Rightarrow \boxed{\begin{aligned} c_1 + 0 + 2 &= 0 \\ c_1 + c_2 + 4 &= 0 \\ c_2 + 2 &= 0 \end{aligned}}$$

dependent vectors

$\Leftrightarrow$ one (or more of them) can be expressed as a linear combination of the others.

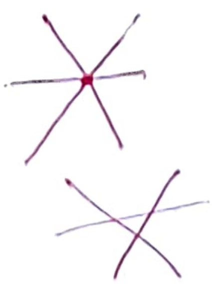
Independent vectors $\Leftrightarrow \vec{z} \neq c_1 \vec{x} + c_2 \vec{y}$

$c_1 \vec{x} + \vec{c_2} y + c_3 \vec{z} = 0$, only if

$c_1 = c_2 = c_3 = 0$

"trivial solution"

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



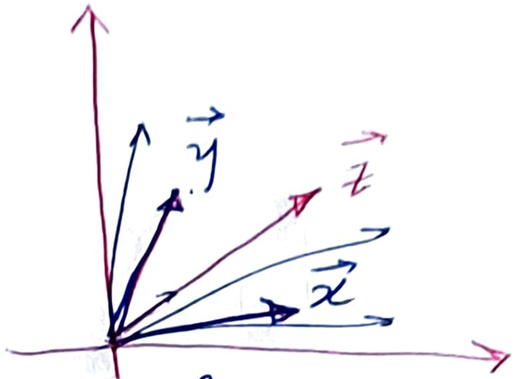
RREF

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + 2 = 0 \Rightarrow c_1 = -2$

$c_2 + 2 = 0 \qquad c_2 = -2$

Linear span;



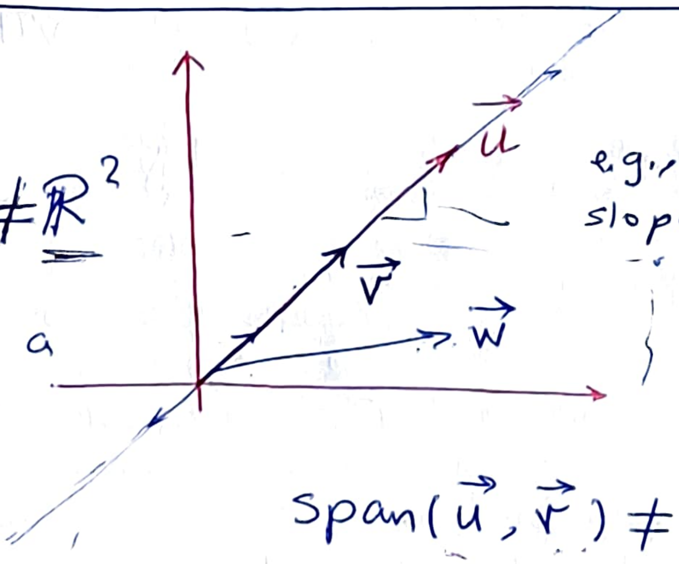
ex

→ $\text{span}(\vec{x}, \vec{y}) : \mathbb{R}^2 = \text{span}(\vec{x}, \vec{y}, \vec{z}) : \mathbb{R}^2$

→ $\vec{x}, \vec{y}$ span $\mathbb{R}^2$

→ any vector $\in \mathbb{R}^2$ can be represented as a linear combination of $\vec{x}, \vec{y}$

- $\text{span}(\vec{u}, \vec{v}) \neq \mathbb{R}^2$
- $\text{span}(\vec{u}, \vec{v})$ is a subspace of $\mathbb{R}^2$



e.g., slope = 1

$\text{span}(\vec{u}, \vec{v}) \neq \text{span}(\vec{u}, \vec{v}, \vec{w})$

Basis of a vector space

→ minimal set of vectors that span the vector space (independent)

⇒ $\mathbb{R}^n$: basis of $\mathbb{R}^n$ has n indep. vectors

Fundamental subspaces

$C(A)$: column space of A

$R(A)$: row space

$N(A)$: Null space

$C(A)$: → space contains all column vectors of matrix A, and their linear combinations.

→ span of column vectors of A

$$\vec{Y}_{m\times 1} = A_{m\times n}\,\vec{X}_{n\times 1}$$

$$\vec{y}_{m\times 1} = \underset{\substack{\swarrow \qquad \searrow \\ \#\text{of rows} \qquad \#\text{of columns}}}{A_{m\times n}}\;\vec{x}_{n\times 1}$$

$\vec{x} \in \mathbb{R}^n$

$\vec{y} \in \mathbb{R}^m$

$A \in \mathbb{R}^{m\times n}$

e.g. given a system of linear equations

$$A\vec{x} = \vec{b}$$

is solvable iff $\vec{b} \in C(A)$

let $A_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \end{bmatrix}$ $\Rightarrow$

$r_1 \times -2 + r_2$

RREF

ROW

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$$

$0.5 r_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$r_2 \times -3 + r_1$

RREF(A) $\Leftarrow$ $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

row

$\rightarrow$ indep. columns in RREF

columns that have pivots

rank = number of independent rows

= number of independent colums.

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix} \right)$$

$$\boxed{\vec{y}_{2\times 1} = A_{2\times 3}\, \vec{x}_{3\times 1}}$$

Row space of A

$R(A)$

① $\underset{2\times3}{A} \Rightarrow RREF(A)$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R(A)$: span of independent row vectors

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Null space $N(A)$

space that contains the solution set of $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⇓

$$\begin{bmatrix} 1 & \boxed{\begin{matrix} 2 \\ 0 \end{matrix}} & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \boxed{x_2} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

no pivot

Free Variable

$\rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$\underline{x_3 = 0}$

let $x_2 = K$

$\Rightarrow$ solution set ;

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2K \\ K \\ 0 \end{bmatrix}$$

$$N(A) = \text{span}\left(\begin{bmatrix} -2K \\ K \\ 0 \end{bmatrix} \right)$$

e.g., let $K = -1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times 2 + 2\times -1 + 3\times 0 \\ 2\times 2 - 4\times 1 + 8\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$