25/11/2025

AI 46, Linear Algebra, session 3, Mansoura.

Review

Fundamental spaces of a Matrix.

ex.

$$A_{3\times 5} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 6 & 7 \\ 0 & 2 & 2 & 9 & 9 \end{bmatrix}$$

RREF

$$\begin{array}{l} \\ \div 2 \\ \div 2 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & \cancel{2}1 & \cancel{2}1 & \cancel{2}1 \\ 0 & \cancel{2}1 & \cancel{2}1 & \cancel{9}\,4.5 & \cancel{9}\,4.5 \end{bmatrix}$$

$$r_2 \times -2 + r_1 \quad \begin{bmatrix} 1 & \cancel{2}0 & \cancel{3}1 & \cancel{4}-5 & \cancel{5}-4 \\ 0 & 1 & 1 & 4.5 & 4.5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$r_3 \times -1 + r_1 \quad \begin{bmatrix} 1 & 0 & \cancel{1}0 & \cancel{-5}-6 & \cancel{-4}-5 \\ 0 & 1 & \cancel{1}0 & \cancel{4.5}\,3.5 & \cancel{4.5}\,3.5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad r_3 \times -1 + r_2$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & -6 & -5 \\ 0 & 1 & 0 & 3.5 & 3.5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mathbb{R}^5$$

rank $= r = 3$

$\dim(C(A)) = \dim(R(A)) = r = 3$

$\mathbb{R}^3$

Null space

Vectors $\vec{x} \in N(A)$

if $A\vec{x} = \vec{0}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\left.\begin{bmatrix} \leftarrow & \vec{r}_{A_1} & \rightarrow \\ \leftarrow & \vec{r}_{A_2} & \rightarrow \\ \leftarrow & \vec{r}_{A_3} & \rightarrow \end{bmatrix}\right\}3}_{5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\vec{y}_{3\times1} = A_{3\times5} \; \vec{x}_{5\times1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \vec{r}_{A_1} \cdot \vec{x} \\ \vec{r}_{A_2} \cdot \vec{x} \\ \vec{r}_{A_3} \cdot \vec{x} \end{bmatrix}$$

row picture of matrix multiplication

Solving for Null space

$\vec{x}\,;\quad A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & -5 \\ 0 & 1 & 0 & 3.5 & 3.5 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

free variables

$$x_1 - 6x_4 - 5x_5 = 0$$

$$x_2 + 3.5x_4 + 3.5x_5 = 0$$

$$x_3 + \cancel{3.5}\, x_4 + x_5 = 0$$

let $x_4 = a$, $x_5 = b$

$\Rightarrow x_1 = 6a + 5b$

$x_2 = -3.5a - 3.5b$

$x_3 = -a - b$

$$\begin{bmatrix} 6a + 5b \\ -3.5a - 3.5b \\ -a - b \\ a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 6a \\ -3.5a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 5b \\ -3.5b \\ -b \\ 0 \\ b \end{bmatrix}}_{\text{basis of null space}}$$

Matrix A

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ c_{A_1} & c_{A_2} & \cdots & c_{A_n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} \longleftarrow & r_{A_1} & \longrightarrow \\ \longleftarrow & r_{A_2} & \longrightarrow \\ & \vdots & \\ \longleftarrow & r_{A_m} & \longrightarrow \end{bmatrix}$$

Grey scale picture

$$\begin{bmatrix} \times & \times & \times & \cdots \\ \vdots & & & \\ & & & \end{bmatrix}$$

Matrix as a transformation

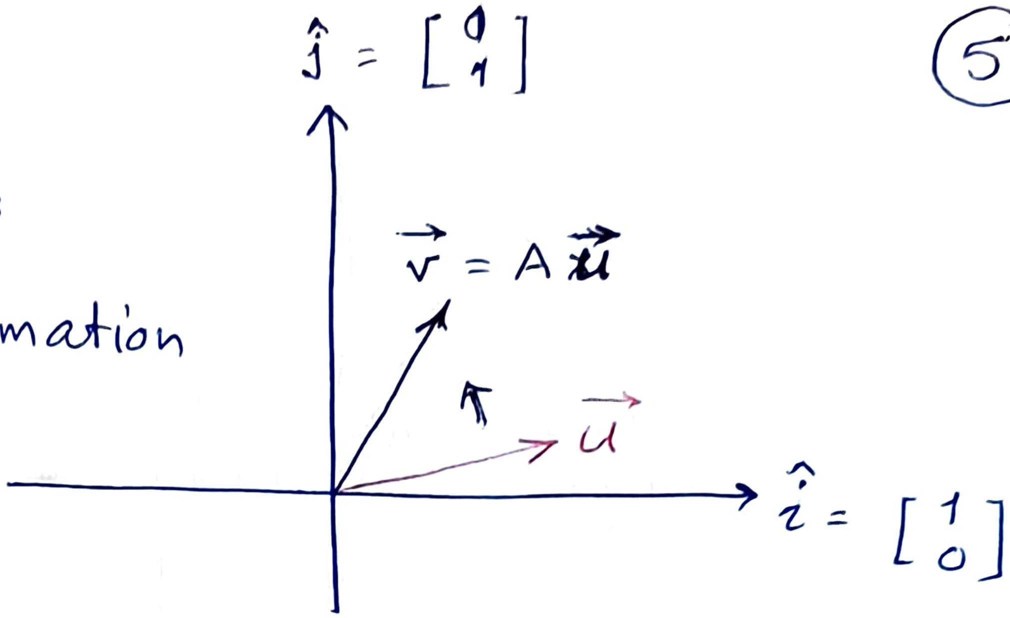
$$\vec{x} \longrightarrow \boxed{T_A} \longrightarrow \vec{y}$$

$$\Rightarrow \quad \vec{y} = A\vec{x} \quad ; \quad \vec{y} = T_A(\vec{x})$$

$$\vec{y} = ABC\vec{x} \iff \vec{y} = T_A\left(T_B\left(T_C(\vec{x})\right)\right)$$

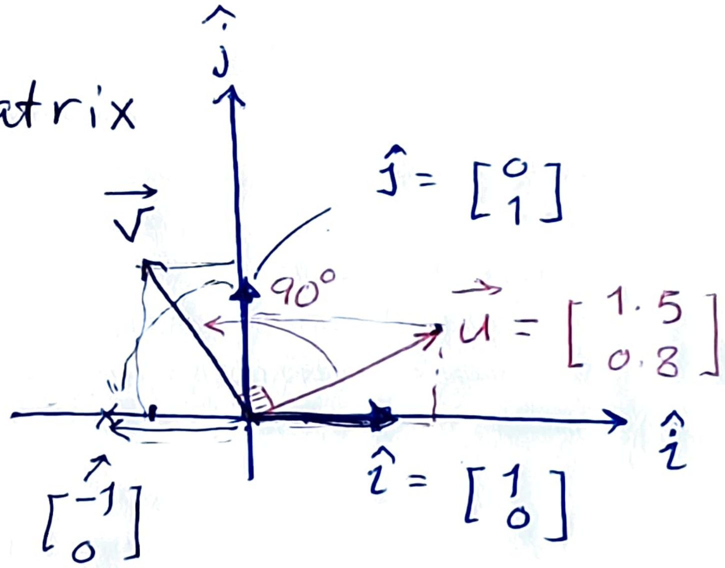
$$AB \neq BA$$

Matrix as a tranformation



<u>examples</u> ;

1) rotation matrix



$$\vec{v} = A\,\vec{u}$$

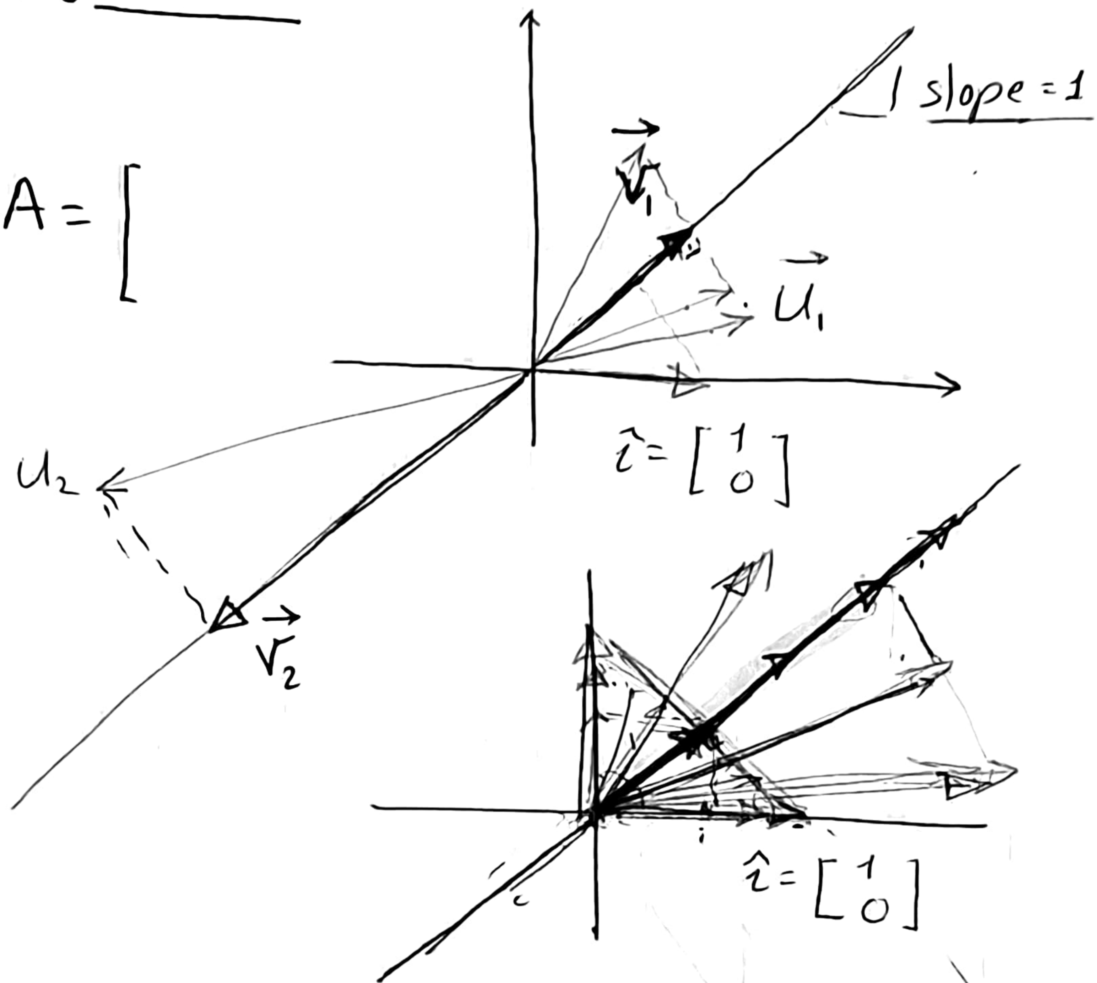
$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

90° CCW Rotation matrix

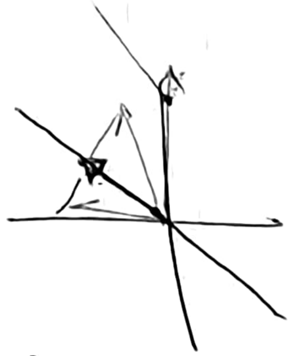
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0.8 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.5 \end{bmatrix}$$

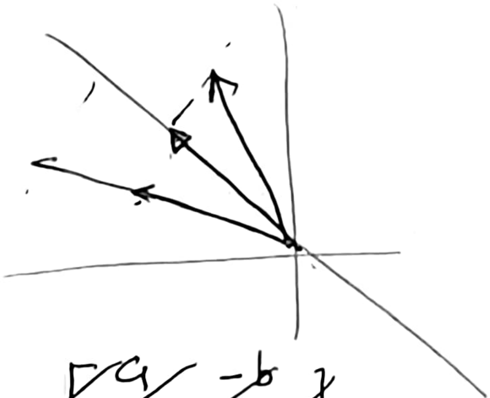
Projection

$A = [$



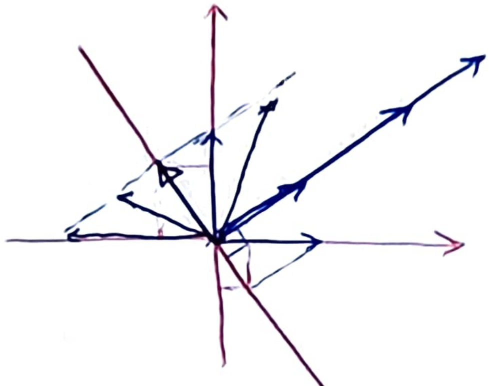
$$\underline{A} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$





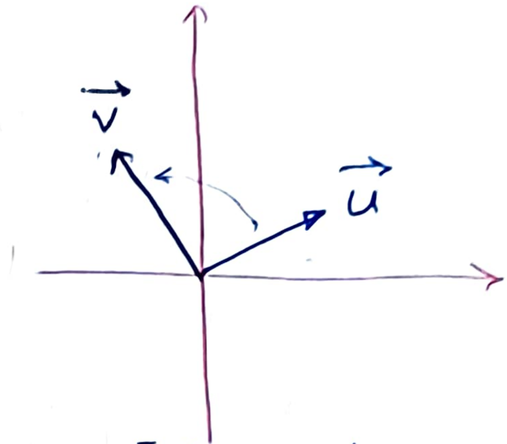
$$\begin{bmatrix} a & -b \\ -b & c \end{bmatrix}$$

Invertibility of a matrix



$A = \cancel{\alpha}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

projection



$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

rotation

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relate to math, multiplicative inverse of a variable x is $\cancel{\frac{1}{x}}$ x^{-1}

$$x\,x^{-1} = 1 \qquad x^{-1}\,x = 1$$

→ define inverse of a square matrix A to be A^{-1} ;

$$A_{n\times n} A^{-1}_{n\times n} = I_{n\times n} \qquad A^{-1}_{n\times n} A_{n\times n} = I_{n\times n}$$

$$I_{n\times n} \equiv I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

→ main diagonal has 1's

→ all other elements $= 0$

→ if inverse exists.

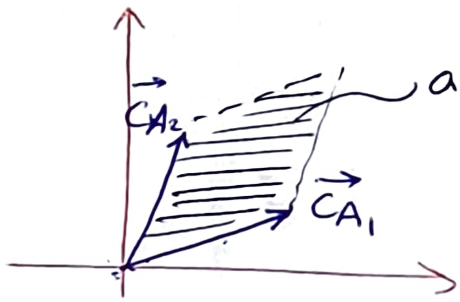
$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

1. det(A) ; determinant of A ; $|A|$

$$\det(A_{2\times2}) = \left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right|$$

$$= a_{11} \times a_{22} - a_{12} \times a_{21}$$

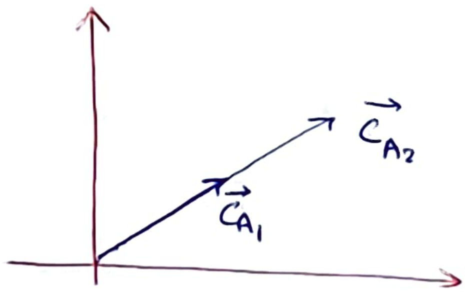
recall;
rank(A) = # of independent columns



what if the two columns are dependent?

$$\Rightarrow \vec{C}_{A_1} = \beta \vec{C}_{A_2}$$

determenent = 0
if rank $(A_{n\times n}) < n$



if ~~matrix~~ if square matrix $A_{n\times n}$ has a rank $r < n \Rightarrow \det(A) = 0$
" not full rank "

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$$\rightarrow \det(A_{3\times 3}) = \begin{array}{c} \\ \\ 3 \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & \\ \end{array} \\ \begin{bmatrix} {}^{+}a_{11} & {}^{-}a_{12} & {}^{+}a_{13} \\ {}^{-}a_{21} & {}^{+}a_{22} & {}^{-}a_{23} \\ {}^{+}a_{31} & {}^{-}a_{32} & {}^{+}a_{33} \end{bmatrix} \end{array}$$

$$(-1)^{3+2} = (-1)^5$$

$$A^{-1} = \frac{1}{\det(A)} \, \underline{\text{adj}(A)}$$

adjoint or adjugate

$\Rightarrow \text{adj}(A) = C^T$

$\Rightarrow$ C : cofactor matrix

$$C = \left((-1)^{i+j} M_{ij}\right)_{1 \le i \le n,\ 1 \le j \le n}$$

; i : row number

j : column number

M : Minors

M_{ij} : determinant of small matrix after crossing out row # i, column # j

~~adj (A)~~ =

Finding Matrix Inverse using Gaussian Elimination

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 0 \times 0 - (-1 \times 1) = +1$$

$$C = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

$$\frac{C^T}{\mathrm{Det}(A)} = \frac{1}{1} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

$$\left[\underbrace{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}}_{A} \;\middle|\; \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I}\right]$$

Augment $A_{n \times n}$ with $I_{n \times n}$

Gauss-Jordan elimination

$\Downarrow$

RREF

$\Rightarrow$

$$\left[\; I \;\middle|\; A^{-1} \;\right]$$

$$(KA)^{-1} = \frac{1}{K} A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(A) = \frac{1}{\det(A^{-1})}$$