Dual Numbers Algebra

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1 Definition

A dual number is a number in the form $a + b\epsilon$ where $a, b \in \mathbb{R}$ and ϵ is a nilpotent matrix such that $\epsilon \neq 0$ and $\epsilon^2 = 0$.And For any function $f(x)$ that has the Taylor series around $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \sum_{k=2}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

we have that:

$$f(a + b\epsilon) = f(a) + f'(a)b\epsilon$$

2 Closure under Addition/Subtraction

For any two dual numbers $a + b\epsilon$ and $c + d\epsilon$, $(a + b\epsilon) + (c + d\epsilon)$ is also a dual number and has the value $(a + b) + (c + d)\epsilon$.

Proof. Simply by gathering the real components and the dual components into isolated terms. □

3 Closure under Multiplication

For any two dual numbers $a + b\epsilon$ and $c + d\epsilon$, $(a + b\epsilon)(c + d\epsilon)$ is also a dual number and has the value $ac + (ad + bc)\epsilon$.

Proof.

$$\begin{aligned}(a + b\epsilon)(c + d\epsilon) &= ac + ad\epsilon + bc\epsilon + bd\epsilon^2 \\ &= ac + (ad + bc)\epsilon\end{aligned}$$

□

4 Closure under Division

For any two dual numbers $a + b\epsilon, c + d\epsilon$ and $c \neq 0$, $\frac{a+b\epsilon}{c+d\epsilon}$ is also a dual number and has the value $\frac{a}{c} + \frac{bc-ad}{c^2}\epsilon$.

Proof.

$$\begin{aligned}\frac{a + b\epsilon}{c + d\epsilon} &= \frac{(a + b\epsilon)(c - d\epsilon)}{(c + d\epsilon)(c - d\epsilon)} = \frac{ac + (bc - ad)\epsilon}{c^2} \\ &= \frac{a}{c} + \frac{bc - ad}{c^2}\epsilon\end{aligned}$$

□

5 Closure under Exponentiation

5.1 Non-zero dual component in base

For any two Dual numbers $a + b\epsilon, c + d\epsilon$ and $b \neq 0$, $(a + b\epsilon)^{c+d\epsilon}$ is also a dual number and has the value $a^c + a^{c-1}\left(\frac{ad}{b}\ln a + c\right)b\epsilon$

Proof. let $\zeta : \mathbb{R} \to \mathbb{R}$ be a mapping for which:

$$\zeta(x) = \frac{d}{b}x - (\frac{ad}{b} + c)$$

such that $\zeta(a + b\epsilon) = c + d\epsilon$.

Also let $w : \mathbb{R} \to \mathbb{R}$ be a mapping for which:

$$w(x) = x^{\zeta(x)}$$

such that $w(a + b\epsilon) = (a + b\epsilon)^{c+d\epsilon}$.

From the definition of dual numbers:

$$w(a + b\epsilon) = w(a) + w'(a)b\epsilon$$

$$\because w'(x) = x^{\zeta(x)-1}\left(x\zeta'(x)\ln x + \zeta(x)\right), \zeta'(x) = \frac{d}{b}$$

$$\therefore w(a + b\epsilon) = (a + b\epsilon)^{c+d\epsilon} = a^c + a^{c-1}\left(\frac{ad}{b}\ln a + c\right)b\epsilon$$

□

5.2 zero dual component in base

For any two Dual numbers $a + 0.\epsilon, c + d\epsilon$ and $(a + 0.\epsilon)^{c+d\epsilon}$, or equivalently $a^{c+d\epsilon}$ is also a dual number and has the value $a^c + a^c b \ln a\epsilon$

Proof. let $w : \mathbb{R} \to$ be a mapping for which:

$$w(x) = a^x$$

such that $w(c + d\epsilon) = a^{c+d\epsilon}$
From the definition of dual numbers:

$$w(c + d\epsilon) = w(c) + w'(c)b\epsilon$$

$$\because w'(x) = a^x \ln x$$

$$\therefore w(c + d\epsilon) = a^c + a^c b \ln a\epsilon$$

□

6 Some Functions of Dual Numbers

6.1 Natural Logarithm

For any dual number $a + b\epsilon$ where $a \neq 0$, $\ln(a + b\epsilon)$ is also a dual number and has the value $\ln a + \frac{b}{a}\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned}\ln(a + b\epsilon) &= \ln a + (\ln a)'b\epsilon \\ &= \ln a + \frac{b}{a}\epsilon\end{aligned}$$

□

6.2 The exp Function

For any dual number $a + b\epsilon$, $\exp(a + b\epsilon)$ is also a dual number and has the value $\exp(a) + \exp(a)\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned}\exp(a + b\epsilon) &= \exp(a) + (\exp(a))'b\epsilon \\ &= \exp(a) + \exp(a)b\epsilon\end{aligned}$$

□

6.3 The $\sin$ Function

For any dual number $a+b\epsilon$, $\sin(a+b\epsilon)$ is also a dual number and has the value $\sin(a)+\cos(a)\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned} \sin(a+b\epsilon) &= \sin(a) + (\sin(a))'b\epsilon \\ &= \sin(a) + \cos(a)b\epsilon \end{aligned}$$

□

6.4 The $\cos$ Function

For any dual number $a+b\epsilon$, $\cos(a+b\epsilon)$ is also a dual number and has the value $\cos(a)-\sin(a)\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned} \cos(a+b\epsilon) &= \cos(a) + (\cos(a))'b\epsilon \\ &= \cos(a) - \sin(a)b\epsilon \end{aligned}$$

□

6.5 The $\tan$ Function

For any dual number $a+b\epsilon$, $\tan(a+b\epsilon)$ is also a dual number and has the value $\tan(a) + \sec^2(a)\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned} \tan(a+b\epsilon) &= \tan(a) + (\tan(a))'b\epsilon \\ &= \tan(a) + \sec^2(a)b\epsilon \end{aligned}$$

□

6.6 The Square Root Function

For any dual number $a+b\epsilon$, $\sqrt{a+b\epsilon}$ is also a dual number and has the value $\sqrt{a} + \frac{1}{2\sqrt{a}}\epsilon$

Proof. From the definition of dual numbers:

$$\begin{aligned} \sqrt{a+b\epsilon} &= \sqrt{a} + (\sqrt{a})'b\epsilon \\ &= \sqrt{a} + \frac{1}{2\sqrt{a}}\epsilon \end{aligned}$$

□