MATH 1A, Complete Lecture Notes

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Contents

Part I

Limit

Lecture 1

Sets and Functions

1.1 Sets

Often, in life and science one considers collections of objects and rules describing how objects of different nature correspond to each other. For example, all the people were born, so to each human we can assign a number — the year of birth.

In mathematical terms, we deal with two sets here. The first set consists of all humans ever born, the second one is formed by all integers.

A *set* is a collection of objects of any nature. These objects are *points* or *elements* of the set.

The standard notation is $x \in A$ or $A \ni x$ for point a x belonging to a set A.

If each element of the set A belongs to the set B, we say that A is a *subset* of B. If A is a subset of B and at the same time B is a subset of A, then the sets A and B are said to be *equal*.

Notice that if two sets are equal by the last definition, then these sets are essentially the same set — they consist of exactly the same elements.

The standard notation is $A \subset B$ for A subset of B, $A = B$ for equal sets A and B. In Calculus, we deal with sets consisting of numbers. The most important of them are

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers,

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers,

$\mathbb{Q} = \left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is the set of rational numbers,

$\mathbb{R}$ is the set of real numbers.

1.2 Functions

Consider two sets A called the *domain* and B called the *codomain*. A function f assigns to each element $x \in A$ exactly one element $f(x) \in B$ called the *value* of the function f at the point x.

Usually, one writes $f : A \to B$ or $A \xrightarrow{f} B$ for a function f from the set A to the set B.

If K is a subset of A, then the set $f(K)$ of all values $f(x)$ for all $x \in K$ is *image* of the set K. The image of the whole domain $f(A)$ is called the *range* of the function f.

It is important to understand the difference between the codomain and the range. First, we define a domain and a codomain. Second, we construct a function. Third, the function has got the range — the collection of all values at all points of the domain. Usually, the range is less than the codomain.

Example 1 Let A be the collection of patients in a hospital and let $K \subset A$ be the subset of alive patients. Consider a function $f : A \to \mathbb{R}$ that equals a patient's temperature. Then

$$\begin{array}{rcl} f(A) & \subset & [-272, +\infty), \\ f(K) & \subset & [34, 42]. \end{array} \tag{1.1}$$

Example 2 Let A be the set of all men and let B be the set of all women. Suppose f assigns a girlfriend to any man. Then f is not a function, because some men can have several girlfriends and some certainly do not have one.

Example 3 Let T be the set of all triangles of the plane. Then the area of a triangle is a function $S : T \to \mathbb{R}_+$, where by $\mathbb{R}_+$ we mean the set of positive real numbers $(0, +\infty)$.

In this course, we usually deal with functions $A \to \mathbb{R}$, where $A \subset \mathbb{R}$. An important case is

A function $\mathbb{N} \to \mathbb{R}$ is called a *sequence*. If $A = \{n_0, n_0 + 1, n_0 + 2, n_0 + 3, \dots\}$, then a function $A \to \mathbb{R}$ is also a sequence.

For sequences, one writes a_n instead of $f(n)$ and $\{a_n\}$, $\{a_n\}_{n=n_0}^{\infty}$, $\{a_1, a_2, a_3, \dots\}$ etc. instead of f.

1.3 How to define a function

The first way to give a function is simply a *verbal* one. The description just by words has been used in Section 1.2. Here, the domain and the codomain are to be given verbally as well.

The second way to define a function is to describe its domain and codomain by words and provide a *table of values* like

Jon Brown	1978
Jane Smith	1985
Nick White	1930

Further, it is possible to present a real-valued functions visually by drawing the *graph* of a function (see figure 1.1). Here, the domain is found automatically from the picture.

In calculus we deal with functions such that the both domain and codomain consist of real numbers. Usually, we describe them *algebraically*, by a formula. There are the following basic cases:

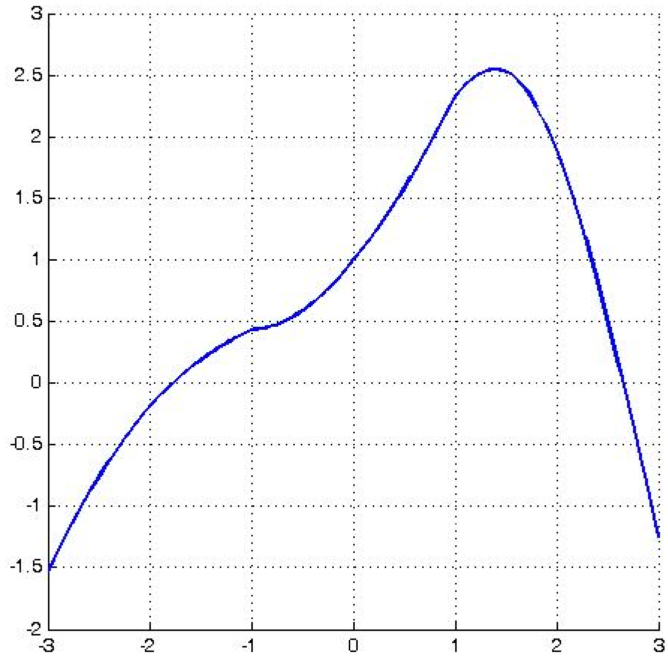


Figure 1.1: The graph of a function

- *Explicit* formulae like

$$\begin{aligned} f(x) &= \sin(1+x^3), \\ g(y) &= \frac{1+y}{1-y}, \\ a_n &= 2^n. \end{aligned} \tag{1.2}$$

- *Implicit* formulae like

$$\begin{aligned} f(x) + (f(x))^3 &= x, \\ \sin g(t) &= t, \ -\pi/2 \le g(t) \le \pi/2. \end{aligned} \tag{1.3}$$

- *Recurrent* formulae for a sequence, like

$$\begin{aligned} & a_1 = 2, \ a_{n+1} = 2a_n, \\ & a_1 = 1, \ a_2 = 2, \ a_{n+1} = a_n + a_{n-1}. \end{aligned} \tag{1.4}$$

Example 4 $f(x) = 1 - x^2$. This expression is defined for any number $x \in \mathbb{R}$, so the domain is $\mathbb{R}$. Further, $x^2 \geq 0$, so $1 - x^2 \leq 1$. Therefore the image is the interval $(-\infty, 1]$.

Example 5 $f(x) = \sqrt{1 - x^2}$. This expression is defined only when $1 - x^2 \geq 0$, so x must belong to the interval $[-1, 1]$. Thus the domain is $[-1, 1]$. Since $1 - x^2 \leq 1$, we have $\sqrt{1 - x^2} \leq 1$. Thus the image is the interval $[0, 1]$.

Example 6 $\sin f(x) = x$, $-\pi/2 \leq f(x) \leq \pi/2$. Here, we know that $-1 \leq \sin \alpha \leq 1$ for any α, so $x \in [-1, 1]$ and the domain is $[-1, 1]$. Here, the image is the whole interval $[-\pi/2, \pi/2]$.

Example 7 $a_n = \dfrac{1}{5 - n}$. Here, if $n = 5$, then we have to divide by 0, which is impossible. Therefore the domain is the set of natural numbers excluding 5; one usually denotes it by

$$\mathbb{N} \setminus \{5\}. \tag{1.5}$$

Formally speaking, it's not a sequence, but it becomes one once we restrict the domain onto the set $\{6, 7, 8, 9, \dots\}$.

Example 8 $f(x) = \sqrt{\dfrac{x - 1}{x^2 + 1}}$. First let's find the domain. The denominator can't vanish, because $x^2 + 1 \geq 1 > 0$. The root exists whenever $x - 1 \geq 0$, so the domain is $[1, \infty)$.

Let's find the image, which is the set of all y such that the equation $f(x) = y$ has a solution. First of all notice that a square root is non-negative, therefore $y \geq 0$. Further,

$$\begin{aligned} y &= \sqrt{\frac{x - 1}{x^2 + 1}}, \\ y^2 &= \frac{x - 1}{x^2 + 1}, \\ (x^2 + 1)y^2 &= x - 1, \\ (y^2)x^2 - x + (y^2 + 1) &= 0, \end{aligned} \tag{1.6}$$

which is the quadratic equation in x. A quadratic equation is solvable if and only if its discriminant is non-negative:

$$\begin{aligned}
1-4y^2(y^2+1) &\geq 0, \\
-4y^4-4y^2+1 &\geq 0, \\
-(2y^2+1)^2+2 &\geq 0, \\
(2y^2+1)^2 &\leq 2, \\
2y^2+1 &\leq \sqrt{2}, \\
y^2 &\leq \frac{\sqrt{2}-1}{2}, \\
y &\leq \sqrt{\frac{\sqrt{2}-1}{2}},
\end{aligned} \tag{1.7}$$

therefore the image is the interval $\left[0, \sqrt{\frac{\sqrt{2}-1}{2}}\right]$

1.4 Properties of functions

A function $f : A \to \mathbb{R}$ is *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$.
A function $f : A \to \mathbb{R}$ is *decreasing* if $f(x) \geq f(y)$ whenever $x \leq y$.
A function f is *monotonic* if it is either increasing or decreasing.

We'll write $f \uparrow$ for an increasing function and $f \downarrow$ for a decreasing one.

Example 9 e^x, $\ln x$, x^2 are increasing on the interval $(0, +\infty)$

Example 10 $\arccos x$ is decreasing on the interval $[-1, 1]$

Example 11 x^2 is not monotonic on the interval $(-\infty, +\infty)$

A function is *bounded* if there is $M > 0$ such that $|f(x)| \leq M$ for any x.
A function is *locally bounded* at a point x_0 if there are some numbers $\delta, M > 0$ such that $|f(x)| \leq M$ for any $x \in [x_0 - \delta, x_0 + \delta]$.

Example 12 $\sin x$, $\cos x$ are bounded

Example 13 x^2 is unbounded

Example 14 x^2 is bounded on the interval $[-1, 1]$ since $x^2 \leq 1$ whenever $x \in [-1, 1]$.

A function $f(x)$ is *odd* if it satisfies

$$f(x) + f(-x) = 0 \tag{1.8}$$

for all x, *even* if it satisfies

$$f(x) = f(-x) \tag{1.9}$$

for all x and *periodic* if it satisfies

$$f(x + T) = f(x) \tag{1.10}$$

for any x and certain $T \in \mathbb{R}$ called a *period*.

Example 15 $\sin x$ and $\cos x$ are both periodic with period 2π, $\sin x$ is odd, $\cos x$ is even.

Example 16 x^2 is even, x^3 is odd. Moreover, a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is odd whenever $0 = a_0 = a_2 = a_4 = \cdots$. Similarly, it is even whenever $0 = a_1 = a_3 = a_5 = \cdots$.

Example 17 e^x is neither odd, nor even.

Example 18 Given any function $f(x)$, the function $g(x) = f(|x|)$ is even.

Lecture 2

Limit of a Sequence

2.1 How to state the negation

Recall that a sequence $\{a_n\}$ is called bounded if there is a number $M > 0$ such that for any n, we have $|a_n| \leq M$. Using symbols we can write it as

$$\exists M > 0 : \qquad \forall n \qquad |a_n| \leq M \tag{2.1}$$

Suppose that we need to check that some sequence is not bounded. To do so, we need to state the negation of this definition in some way more clever than just 'there is no number $M > 0$'. It's easy though. We simply reverse the symbols:

$$\forall M > 0 : \qquad \exists n \qquad |a_n| > M \tag{2.2}$$

Thus we obtain the following definition:

> We say that a sequence $\{a_n\}$ is unbounded if for any $M > 0$ there is an index n such that $|a_n| > M$.

Example 19 Let $a_n = \ln n$. Then for $M > 0$ we can take $n > e^M$, so $|a_n| = \ln n > M$.

Example 20 Let $a_n = n^2 - n$. Then for $M > 0$, we need $n^2 - n > M$, so $n > \dfrac{1 + \sqrt{1 + 4M}}{2}$

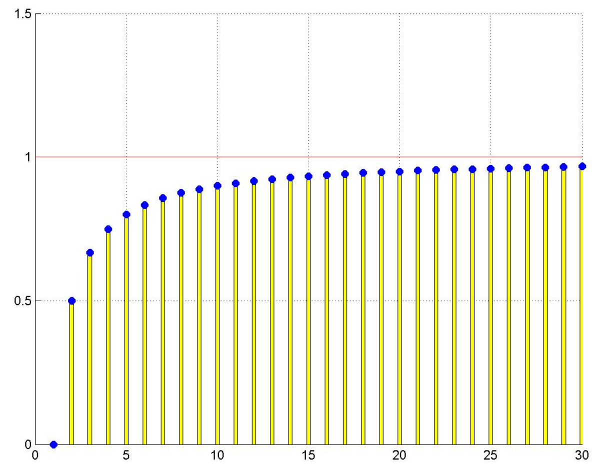


Figure 2.1: The sequence $a_n = 1 - \dfrac{1}{n}$

In a similar manner, a sequence is *increasing* if for each n we have $a_{n+1} \geq a_n$. Thus a sequence is *non-increasing* if there is n such that $a_{n+1} < a_n$

A sequence is *monotonic* if $a_{n+1} \geq a_n$ for each n or $a_{n+1} \leq a_n$ for each n. Thus a sequence is *non-monotonic* if there is n_1 such that $a_{n_1+1} > a_{n_1}$ and there is n_2 such that $a_{n_2+1} < a_{n_2}$.

2.2 Limit of an increasing sequence

Given an increasing sequence $\{a_n\}$, it can either grow to infinity or be bounded. In the latter case, it naturally approaches some number $L \in \mathbb{R}$. This number is the smallest possible upper bound. For example, the sequence $a_n = 1 - \dfrac{1}{n}$ approaches the value 1 as shown in Figure 2.1

If a number L is the smallest upper bound for a sequence $\{a_n\}$, it means that no number smaller than L is. In other words, for arbitrary $\varepsilon > 0$ (no matter how small it is), $L - \varepsilon$ is not an upper bound for the sequence $\{a_n\}$. Thus we can state the following definition

Given an increasing sequence $\{a_n\}$, we say that it *converges* to a number L or that L is its *limit* if

1. For any n, we have $a_n \leq L$
2. For any $\varepsilon > 0$, there is an index n such that $a_n > L - \varepsilon$.

The standard notation is $\lim\limits_{n\to\infty} a_n = L$ for L to be the limit of the sequence $\{a_n\}$.

Example 21 Let us prove that $\lim\limits_{n\to\infty} \dfrac{-1}{n} = 0$. We need to check that for any $\varepsilon > 0$ there is n such that $\dfrac{-1}{n} > -\varepsilon$, that is, $n > \dfrac{1}{\varepsilon}$.

2.3 Limit of an arbitrary sequence

We see that an increasing sequence has limit L if no matter how small the gap is, some term a_n is within this gap from the number L. Generally, we say that an arbitrary sequence $\{a_n\}$ approaches some value L if no matter how small the gap is, all terms of the sequence lie within this gap from the number L starting at some place.

In other words, we have the following definition:

A sequence $\{a_n\}$ *converges* to a number L or has *limit* L if for any $\varepsilon > 0$ (no matter how small it is) there is N such that for any $n > N$, we have $L - \varepsilon < a_n < L + \varepsilon$.

Notice that the last condition can be also written as $x \in (L - \varepsilon, L + \varepsilon)$ or as $|x - L| < \varepsilon$.

Example 22 Let us prove that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. Given $\varepsilon > 0$, we need to have $\frac{1}{n^2} < \varepsilon$, that is, $n > \frac{1}{\sqrt{\varepsilon}}$. Now if we put $N = \frac{1}{\sqrt{\varepsilon}}$, then the conditions is fulfilled for $n > N$.

Example 23 Let us prove that $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$. We need to have $\left|\frac{\cos n}{n}\right| < \varepsilon$. Since $-1 \le \cos x \le 1$, we have $\left|\frac{\cos n}{n}\right| < \frac{1}{n}$, which is true for $n > N = \frac{1}{\varepsilon}$.

The definition can be modified. For example,

Lemma 1 *A sequence $\{a_n\}$ has limit L if and only if for any $\varepsilon > 0$ there is N such that for $n \ge N$, we have $L - \varepsilon \le a_n \le L + \varepsilon$.*

Proof. Suppose that for any $\varepsilon > 0$, there is $N_1(\varepsilon)$ such that for $n \ge N_1(\varepsilon)$, we have $|a_n - L| \le \varepsilon$. Put $N_2(\varepsilon) = N_1(\varepsilon/2)$. Then for $n > N_2$, we also have $n \ge N_2$, therefore $|a_n - L| \le \varepsilon/2 < \varepsilon$, so by definition, we see that $\lim\limits_{n\to\infty} a_n = L$ ⌣̈

Lecture 3

Linearity of the Limit and Squeeze Theorem

3.1 How the limit definition works

Example 24 Let's check that $\lim_{n\to\infty} a = a$ (here we mean the constant sequence $a_1 = a_2 = a_3 = \cdots = a \in \mathbb{R}$).

By definition, for any $\varepsilon > 0$ we are supposed to find N such that for any $n > N$ we have $|a_n - a| < \varepsilon$. Since $a_n = a$, we have $|a_n - a| = 0$, which is always smaller than any positive ε. Thus $N = 1$ fits (actually, any N does).

Example 25 Let's check that $\lim_{n\to\infty} e^{-n} = 0$.

Given $\varepsilon > 0$, we are supposed to find N such that $|a_n - 0| < \varepsilon$ for $n > N$. Substituting $a_n = e^{-n}$, we get the inequality $e^{-n} < \varepsilon$ to establish. Taking the logarithm, we see that it is equivalent to $n > \log \frac{1}{\varepsilon}$. By putting $N = \log \frac{1}{\varepsilon}$, we complete the job.

Example 26 Let's check that $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$.

Given $\varepsilon > 0$, we are supposed to find N such that $|a_n - 1| < \varepsilon$ for $n > N$. Substituting the actual formula for a_n, we obtain

$$\left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1} < \varepsilon$$

This inequality is equivalent to $n > \dfrac{2}{\varepsilon} - 1$. Putting $N = \dfrac{2}{\varepsilon} - 1$, we complete the job.

Example 27 Let's check that $\lim\limits_{n\to\infty} \dfrac{n - \log n}{n + \sqrt{n}} = 1$.

Given $\varepsilon > 0$, we are supposed to find N such that $|a_n - 1| < \varepsilon$ for $n > N$. Substituting the actual formula for a_n, we obtain

$$\left|\frac{n - \log n}{n + \sqrt{n}} - 1\right| = \frac{\log n + \sqrt{n}}{n + \sqrt{n}} < \varepsilon$$

It's impossible to solve this inequality, but we don't need to. It's enough just to estimate it from above. For example, since $\log n < \sqrt{n}$ and $\sqrt{n} > 0$, we have

$$\frac{\log n + \sqrt{n}}{n + \sqrt{n}} < \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}}$$

Let's try to make the last expression smaller than ε. The inequality $\dfrac{2}{\sqrt{n}} < \varepsilon$ is equivalent to $n > \dfrac{4}{\varepsilon^2}$.

Finally, put $N = \dfrac{4}{\varepsilon^2}$. If $n > N$, then

$$\left|\frac{n - \log n}{n + \sqrt{n}} - 1\right| = \frac{\log n + \sqrt{n}}{n + \sqrt{n}} < \frac{2}{\sqrt{n}} < \varepsilon,$$

which is required.

3.2 Simple limit properties

The following proposition is not a theorem to be proved. In fact, it is called *completeness* of the real numbers and can be considered as a part of a definition of the real numbers.

Statement 2 *A monotonic bounded sequence $\{a_n\}$ always has a limit L. If $\{a_n\}$ is increasing, then L is the least upper bound. If $\{a_n\}$ is decreasing, then L is the greatest lower bound.*

Now let's prove some theorems.

Theorem 3 *A converging sequence is bounded.*

Proof. Assume that $\lim_{n\to\infty} a_n = L$. Thus there is N such that for any $n > N$, we have $L - 1 < a_n < L + 1$. Put

$$\begin{aligned} m &= \min(a_1, a_2, \ldots, a_N, L - 1), \\ M &= \max(a_1, a_2, \ldots, a_N, L + 1) \end{aligned} \tag{3.1}$$

We see that for any n, we have $m \le a_n \le M$, which means that the sequence is bounded. $\ddot\smile$

Lemma 4 *Removing the first term of a sequence does not change the limit. In other words, assume that $\lim_{n\to\infty} a_n = L$ and that $b_n = a_{n+1}$, $n \ge 1$. Then $\lim_{n\to\infty} b_n = L$.*

Proof. We are given that for any $\varepsilon > 0$, there is $N_1(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for $n > N_1(\varepsilon)$. We need to estimate the expression $|b_n - L| = |a_{n+1} - L|$, which is smaller than ε for $n+1 > N_1(\varepsilon)$, that is, $n > N_1(\varepsilon) - 1$. Thus put $N_2(\varepsilon) = N_1(\varepsilon) - 1$. Finally, for $n > N_2$ we have $|b_n - L| < \varepsilon$, which concludes the proof. $\ddot\smile$

Theorem 5 *Changing any finite number of a sequence's terms does not change the limit.*

Proof. In a similar way to Lemma 4, we can prove that adding any number to the beginning of a sequence does not change the limit.

Now in order to change a finite number of the sequence's terms, we first remove several and then add several new ones. On each step, the limit remains the same. $\ddot\smile$

Theorem 6 *The limit is unique. Specifically, if we know that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.*

Proof. Assume that $L \ne M$. For $\varepsilon = \dfrac{|L - M|}{2}$, there must be numbers N_1 and N_2 such that

1. for $n > N_1$, we have $|a_n - L| < \frac{|L-M|}{2}$,
2. for $n > N_2$, we have $|a_n - M| < \frac{|L-M|}{2}$,

Thus for $n > \max(N_1, N_2)$, we have

$$|L - M| = |L - a_n + a_n - M| \le |L - a_n| + |a_n - M| < \frac{|L-M|}{2} + \frac{|L-M|}{2} = |L - M|,$$

which cannot be. This contradiction completes the proof. ⌣̈

3.3 Linearity of the limit

Lemma 7 *Assume that* $\lim\limits_{n\to\infty} a_n = L$ *and* a *is some number. Then* $\lim\limits_{n\to\infty} aa_n = aL$.

Proof. If $a = 0$, then everything is Ok. Consider $a \neq 0$.

We know that for any $\varepsilon > 0$ there is $N_1(\varepsilon)$ such that for any $n > N_1(\varepsilon)$, we have $|a_n - L| < \varepsilon$.

We need to fulfil $|aa_n - aL| < \varepsilon$. We have $|aa_n - aL| = |a||a_n - L|$. In order for it to be smaller than ε, we need $|a_n - L| < \frac{\varepsilon}{|a|}$. Thus put $N_2(\varepsilon) = N_1\left(\frac{\varepsilon}{|a|}\right)$.

Now we see that for $n > N_2(\varepsilon)$, we have $|aa_n - aL| < \varepsilon$, which completes the proof. ⌣̈

Lemma 8 *Assume that* $\lim\limits_{n\to\infty} a_n = L$ *and* $\lim\limits_{n\to\infty} b_n = M$. *Then* $\lim\limits_{n\to\infty}(a_n + b_n) = L + M$

Proof. We are given that

1. For any $\varepsilon > 0$ there is $N_1(\varepsilon)$ such that for $n > N_1(\varepsilon)$, we have $|a_n - L| < \varepsilon$,
2. For any $\varepsilon > 0$ there is $N_2(\varepsilon)$ such that for $n > N_2(\varepsilon)$, we have $|b_n - M| < \varepsilon$.

We need to fulfil $|a_n + b_n - L - M| < \varepsilon$. Let's transform this expression. We have

$$|a_n + b_n - L - M| = |a_n - L + b_n - M| \le |a_n - L| + |b_n - M|, \tag{3.2}$$

which contains summands we know what to do with. Specifically, we can make both $|a_n - L|$ and $|b_n - M|$ smaller than $\frac{\varepsilon}{2}$ for their sum to be smaller than ε.

Thus let's put

$$N(\varepsilon) = \max\left(N_1\left(\frac{\varepsilon}{2}\right), N_2\left(\frac{\varepsilon}{2}\right)\right). \tag{3.3}$$

Then $n > N(\varepsilon)$ implies both $|a_n - L| < \dfrac{\varepsilon}{2}$ and $|b_n - M| < \dfrac{\varepsilon}{2}$, which by inequality (3.2) leads to the desired statement

$$|a_n + b_n - L - M| < \varepsilon \tag{3.4}$$

Lemma is proved. $\ddot\smile$

Together, these two lemmas imply

Theorem 9 *Given two converging sequences $\{a_n\}$ and $\{b_n\}$ and two numbers a and b, we have*

$$\lim_{n\to\infty}(aa_n + bb_n) = a\lim_{n\to\infty} a_n + b\lim_{n\to\infty} b_n \tag{3.5}$$

In mathematics, conditions of this kind are referred to as linearity.

3.4 Squeeze Theorem

Theorem 10 (Squeeze Theorem) *Assume that*

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,
2. *there is some N such that for each $n > N$, we have $a_n \leq b_n \leq c_n$.*

Then $\lim_{n\to\infty} b_n = L$.

Proof. First, by applying Theorem 5, we may suppose that $a_n \leq b_n \leq c_n$ for each n. Now we know that for each $\varepsilon > 0$ there are $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that

1. $a_n \in (L - \varepsilon, L + \varepsilon)$ for $n > N_1$
2. $c_n \in (L - \varepsilon, L + \varepsilon)$ for $n > N_2$

We need to establish that $b_n \in (L - \varepsilon, L + \varepsilon)$ for n large enough. Let's take $n > \max(N_1, N_2)$.

Notice that $a_n \in (L - \varepsilon, L + \varepsilon)$ and $c_n \in (L - \varepsilon, L + \varepsilon)$ imply that $(a_n, c_n) \subset (L - \varepsilon, L + \varepsilon)$. Thus we have $b_n \in (a_n, c_n) \subset (L - \varepsilon, L + \varepsilon)$ for $n > \max(N_1, N_2)$. Putting $N(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon))$ completes the proof. $\ddot\smile$

3.5 Limits and inequalities

Theorem 11 *Assume that for each n, we have $a_n \le a$ and that $\lim_{n\to\infty} a_n = L$. Then $L \le a$.*

Proof. Assume that $L > a$. Let's take $\varepsilon = L - a$. By the limit's definitions, there is N such that for any $n > N$, we have $a_n \in (L-\varepsilon, L+\varepsilon)$. In particular, $a_n > L-\varepsilon = a$, which is a contradiction to the theorem's assumption. $\ddot\smile$

Remark 12 *In a similar manner, if $a_n \ge a$, then $\lim_{n\to\infty} a_n \ge a$.*

Lecture 4

Limit Laws

4.1 Subsequences

Given a sequence, one can select some of its elements to form a new sequence.

> If $\{a_n\}$ is an arbitrary sequence and $\{n_k\}$ is a strictly increasing (meaning that $n_1 < n_2 < n_3 < \cdots$) sequence of natural numbers,
>
> $$\{b_k = a_{n_k}\}_{k=1}^{\infty} \tag{4.1}$$
>
> is called a *subsequence* of $\{a_n\}$.

We'll write $\{b_k\} \subset \{a_n\}$ for a subsequence.

Example 28 $a_n = (-1)^n$. Taking $b_k = a_{2k} = (-1)^{2k} = 1$ and $c_k = a_{2k+1} = (-1)^{2k+1} = -1$, we obtain two constant subsequences.

Example 29 The sequence given by $a_n = \log n$, subsequence given by $b_k = a_{k^3} = 3 \log k$.

Example 30 The sequence given by a recurrence relation $a_1 = 1$, $a_{n+1} = \sin a_n$. Subsequence $b_k = a_{k+1}$ also satisfies $b_k = \sin a_k$.

Theorem 13 *If a sequence has a limit, then any its subsequence has the same limit.*

Proof. Consider a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = L$ and a subsequence $\{a_{n_k}\} \subset \{a_n\}$. We know that for any $\varepsilon > 0$ there is $N(\varepsilon)$ such that for any $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon$. Obviously, $n_k > k$, so if $k > N$, we have in particular $n_k > N$ and therefore $|a_{n_k} - L| < \varepsilon$. It completes the proof. ⌣̈

Example 31 The sequence $a_n = (-1)^n$ has no limit. Indeed, two its subsequences $a_{2k} \equiv 1$ and $a_{2k+1} \equiv -1$ converge to different numbers.

Theorem 14 *Assume that*

1. $\{b_k\} \subset \{a_n\}$ *and* $\{c_k\} \subset \{a_n\}$,
2. $\{a_n\} = \{b_k\} \cup \{c_k\}$ *meaning that any element of the sequence* $\{a_n\}$ *belongs to at least one of the subsequences* $\{b_k\}$ *and* $\{c_k\}$
3. $\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = L$

Then $\lim_{n\to\infty} a_n = L$

Proof. We know that given any $\varepsilon > 0$,

1. There is K_1 such that for any $k > K_1$, we have $|b_k - L| < \varepsilon$
2. There is K_2 such that for any $k > K_2$, we have $|c_k - L| < \varepsilon$

Both indices K_1 and K_2 correspond to some N_1 and N_2 in the initial sequence $\{a_n\}$. Put $N(\varepsilon) = \max(N_1, N_2)$. Assume that $n > N$ and consider a_n. The element a_n is at the same time either b_k with $k > K_1$ or c_k with $k > K_2$, so we have $|a_n - L| < \varepsilon$. ⌣̈

4.2 Limit of the product

Theorem 15 *For any converging sequences $\{a_n\}$ and $\{b_n\}$, we have*

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) \tag{4.2}$$

Proof. Assume that $\lim_{n\to\infty} a_n = L$ and that $\lim_{n\to\infty} b_n = M$. Thus for each $\varepsilon > 0$, there are numbers $N_1(\varepsilon)$ and $N_2(\varepsilon)$ such that

1. For $n > N_1(\varepsilon)$, we have $|a_n - L| < \varepsilon$
2. For $n > N_2(\varepsilon)$, we have $|b_n - M| < \varepsilon$

Given $\varepsilon > 0$, we need to find $N(\varepsilon)$ to satisfy the inequality

$$|a_n b_n - LM| < \varepsilon \tag{4.3}$$

for $n > N(\varepsilon)$.

Let's do a trick. We have,

$$|a_n b_n - LM| = |a_n b_n - a_n M + a_n M - LM| \le |a_n| \cdot |b_n - M| + |M| \cdot |a_n - L| \tag{4.4}$$

Further, since the sequence $\{a_n\}$ converges, it must be also bounded, so there is some constant B such that $|a_n| \le B$. Using it, we obtain

$$|a_n b_n - LM| \le |M| \cdot |a_n - L| + B \cdot |b_n - M| \tag{4.5}$$

Now putting $N(\varepsilon) = \max\left(N_1\left(\frac{\varepsilon}{2|M|}\right), N_2\left(\frac{\varepsilon}{2B}\right)\right)$ completes the proof. ☺

In a similar manner, one can prove the following

Theorem 16 *Assume that*

1. $\lim_{n\to\infty} a_n = L$,
2. $\lim_{n\to\infty} b_n = M \neq 0$.

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \tag{4.6}$$

or the following

Theorem 17 *Assume that* $\lim_{n\to\infty} a_n = L \geq 0$. *Then* $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Example 32 Let's find $\lim_{n\to\infty} \frac{n+1}{n-1}$. We have,

$$\frac{n+1}{n-1} = \frac{1+\frac{1}{n}}{1-\frac{1}{n}}, \tag{4.7}$$

so the limit is $\frac{1+0}{1-0} = 1$.

The whole set of these statements about the limit is called *Limit Laws*.

4.3 Non-converging sequences

Also, the limit laws cover some cases involving a non-converging sequence

Theorem 18 *Given two sequences* $\{a_n\}$ *and* $\{b_n\}$*, assume that*

- $\{a_n\}$ *does not converge*
- $\{b_n\}$ *converges*

Then the sequence $\{a_n + b_n\}$ *does not converge.*

Proof. Assume the opposite: the sequence $c_n = a_n + b_n$ converges. Then for the sequence $a_n = c_n - b_n$, we must have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n - \lim_{n\to\infty} b_n, \tag{4.8}$$

which contradicts the assumption of the theorem. ⌣̈

In a similar manner, we have

Theorem 19 *Given two sequences* $\{a_n\}$ *and* $\{b_n\}$*, assume that*

- $\{a_n\}$ *does not converge*

- *$\{b_n\}$ has limit $L \neq 0$*

Then the sequence $\{a_n b_n\}$ does not converge.

or

Theorem 20 *Given two sequences $\{a_n\}$ and $\{b_n\}$, assume that*

- *$\{a_n\}$ does not converge*
- *$\{b_n\}$ has limit $L \neq 0$*

Then the sequence $\left\{\frac{a_n}{b_n}\right\}$ does not converge.

4.4 How to calculate the limit using the limit laws

Example 33 Let's find $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$. We have,

$$\sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}, \tag{4.9}$$

for which we have

$$0 < \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}, \tag{4.10}$$

so the limit is 0 by the squeeze theorem.

Example 34 Let's find $\lim_{n\to\infty}\frac{2n+\sin(e^n)}{n+\sqrt{n}}$. We have,

$$\frac{2n+\sin(e^n)}{n+\sqrt{n}} = \frac{2+\frac{\sin(e^n)}{n}}{1+\frac{1}{\sqrt{n}}} \to \frac{2+0}{1+0} = 2 \tag{4.11}$$

Lecture 5

Limit of a Function

5.1 Limit of a function

Consider a function f. Roughly speaking, if $f(x)$ can be made close to some number l by taking the value of the variable x close to a, but not equal to a, then we say that l is f's limit at the point a.

We have already defined the limit of a sequence. In the same manner, one can define the limit of a function.

> A function f approaches a *limit* l near a point a means: for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that, for all x satisfying $0 < |x - a| < \delta(\varepsilon)$, we have $|f(x) - l| < \varepsilon$.

The standard notation is $\lim\limits_{x \to a} f(x) = l$ or $f(x) \overset{x \to a}{\to} l$ for a function $f(x)$ having limit l as x tends to a.

Let's clarify. In order to show that a function has a limit, we do as follows. First, a (small) number $\varepsilon > 0$ is given. Then we write down the inequality

$$|f(x) - l| < \varepsilon \tag{5.1}$$

and try to solve it. If it's impossible to solve, then we try to simplify it by estimating $|f(x) - l|$ from above.

Example 35 Let's check that $\lim_{x\to 2} 17x = 34$. We have $f(x) = 17x$, $a = 2$, $l = 34$. Thus $|f(x) - l| = |17x - 34| = 17|x - 2|$ must be made smaller than ε. We have

$$17|x-2| < \varepsilon \Leftrightarrow |x-2| < \frac{\varepsilon}{17} \tag{5.2}$$

Put $\delta(\varepsilon) = \frac{\varepsilon}{17}$. Then all the above calculations show that inequality $|x - 2| < \delta(\varepsilon)$ implies $|f(x) - l| < \varepsilon$.

Example 36 Let's check that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$. Thus we have $f(x) = x\sin\frac{1}{x}$, $a = 0$, $l = 0$. Given $\varepsilon > 0$, the desired inequality is

$$|f(x) - l| = \left| x \sin\frac{1}{x} \right| < \varepsilon. \tag{5.3}$$

This inequality seems unsolvable, but we don't need to solve it. Instead, we notice that

$$\left| x \sin\frac{1}{x} \right| = |x| \cdot \left| \sin\frac{1}{x} \right| \leq |x| = |x - a| \tag{5.4}$$

Thus let's put $\delta(\varepsilon) = \varepsilon$. Now if $0 < |x - a| < \delta(\varepsilon)$, by the above argumentation, it follows that $|f(x) - l| < \varepsilon$.

5.2 Equivalent definitions

Just like we defined a bounded function in four equivalent ways, there are different possibilities to define the limit. For example,

> A function f approaches a *limit* l near a point a means: for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that, for all $x \neq a$ satisfying $a - \delta < x < a + \delta$, we have $l - \varepsilon < f(x) < l + \varepsilon$.

A function f approaches a *limit* l near a point a means: for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that, for all $x \in (a - \delta, a + \delta) \setminus \{a\}$ (this notation means the interval $(a - \delta, a + \delta)$ excluding the point a) we have $f(x) \in (l - \varepsilon, l + \varepsilon)$.

A function f approaches a *limit* l near a point a means: for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that the whole graph of the function $f(x)$ on the interval $(a - \delta, a + \delta)$ excluding point a lies between the lines $y = l - \varepsilon$ and $y = l + \varepsilon$.

A function f approaches a *limit* l near a point a means: for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that, for all $x \neq a$ satisfying $a - \delta \leq x \leq a + \delta$, we have $l - \varepsilon \leq f(x) \leq l + \varepsilon$.

A function f approaches a *limit* l near a point a means: for any open interval I containing the point l there is an open interval J containing the point a such that for any $x \in J \setminus \{a\}$, we have $f(x) \in I$.

All these definitions are equivalent. For example, let's prove

Lemma 21 *The limit of a function $f(x)$ equals l as x tends to a if and only if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that, for all $x \neq a$ satisfying $a - \delta \leq x \leq a + \delta$, we have $l - \varepsilon \leq f(x) \leq l + \varepsilon$.*

Proof. First, assume that $\lim_{x \to a} f(x) = l$. By definition, it means that for any $\varepsilon > 0$ there is $\delta_1(\varepsilon) > 0$ such that for all x satisfying $0 < |x - a| < \delta_1(\varepsilon)$, we have $|f(x) - l| < \varepsilon$. Put $\delta_2(\varepsilon) = \dfrac{\delta_1(\varepsilon)}{2}$. Assume that $x \neq a$ satisfies $a - \delta_2 \leq x \leq a + \delta_2$. Then x also satisfies $0 < |x - a| < \delta_1$, which implies the inequality $|f(x) - l| < \varepsilon$ and therefore $l - \varepsilon \leq f(x) \leq l + \varepsilon$.

Conversely, assume that for each $\varepsilon > 0$, there is $\delta_2(\varepsilon)$ such that for all $x \neq a$ satisfying $a - \delta \leq x \leq a + \delta$, we have $l - \varepsilon \leq f(x) \leq l + \varepsilon$. Put $\delta_1(\varepsilon) = \delta_2\left(\dfrac{\varepsilon}{2}\right)$. Suppose now that $0 < |x - a| < \delta_2$. Therefore we have $0 < |x - a| \leq \delta_1\left(\dfrac{\varepsilon}{2}\right)$, which is the same as $x \neq a$ and $a - \delta_1\left(\dfrac{\varepsilon}{2}\right) \leq x \leq a + \delta_1\left(\dfrac{\varepsilon}{2}\right)$. Hence we must have $l - \dfrac{\varepsilon}{2} \leq f(x) \leq l + \dfrac{\varepsilon}{2}$, which implies $l - \varepsilon < f(x) < l + \varepsilon$, which is the same as $|f(x) - l| < \varepsilon$. $\ddot\smile$

5.3 Limit laws

The following statements about the limit of a function are similar to those about the limit of a sequence:

Theorem 22 *If $\lim_{x \to a} f(x) = l \in \mathbb{R}$, then the function f is locally bounded at the point a.*

Theorem 23 *Given two functions $f(x)$ and $g(x)$ and a real number $a \in \mathbb{R}$, assume that*

1. *There is some open interval I containing a such that for any $x \in I$, we have $f(x) = g(x)$,*
2. *$\lim_{x \to a} f(x) = l$.*

Then $\lim_{x \to a} g(x) = l$.

Theorem 24 (Uniqueness) *Let f be a function. If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$, then $l = m$.*

Theorem 25 (Linearity) *Given two functions $f(x)$ and $g(x)$ and two numbers α and β, assume that*

1. $\lim_{x \to a} f(x) = l$,

2. $\lim_{x \to a} g(x) = m$.

Then $\lim_{x \to a}(\alpha f(x) + \beta g(x)) = \alpha l + \beta m$.

Theorem 26 (Squeeze Theorem) *Given three functions* $f(x)$, $g(x)$, $h(x)$, *assume that*

1. $f(x) \leq g(x) \leq h(x)$ *for all* x,

2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$.

Then $\lim_{x \to a} g(x) = l$.

Theorem 27 *Given two functions* $f(x)$ *and* $g(x)$, *assume that*

1. $f(x) \leq g(x)$ *for all* x,

2. $\lim_{x \to a} f(x) = l$,

3. $\lim_{x \to a} g(x) = m$.

Then $l \leq m$.

Theorem 28 (Product Rule) *Assume that* $\lim_{x \to a} f(x) = l$ *and* $\lim_{x \to a} g(x) = m$. *Then* $\lim_{x \to a} (f(x)g(x)) = lm$

Theorem 29 (Ratio Rule) *Assume that* $\lim_{x \to a} f(x) = l$ *and* $\lim_{x \to a} g(x) = m \neq 0$. *Then* $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$.

Theorem 30 (Root Rule) *Assume that* $\lim_{x \to a} f(x) = l \geq 0$. *Then* $\lim_{x \to a} \sqrt{f(x)} = \sqrt{l}$.

5.4 How to find the limit using limit laws

Example 37 Let's find $\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$. First of all, let's notice that the value of the function at $x = 2$ does not matter, so we may assume $x \neq 2$, which allows us to simplify the function as

$$\frac{x^2 - x - 2}{x - 2} = x + 1 \tag{5.5}$$

Since (obviously) $\lim_{x\to 2} x = 2$ and $\lim_{x\to 2} 1 = 1$, we have, by the sum law,

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2}(x + 1) = 3. \tag{5.6}$$

Example 38 Let's find $\lim_{x\to 0} e^{\sin(\cot x)} x^5$. First, sine the sine is between -1 and 1, we have $e^{-1} \leq e^{\sin(\cot x)} \leq e$. Therefore,

$$e^{-1}x^5 \leq e^{\sin(\cot x)} x^5 \leq ex^5. \tag{5.7}$$

Since $\lim_{x\to 0} x = 0$, by the product law, it follows that $\lim_{x\to 0} x^5 = 0^5 = 0$, and, by linearity, we have $\lim_{x\to 0} ex^5 = e \cdot 0 = 0$ and $\lim_{x\to 0} e^{-1}x^5 = e^{-1} \cdot 0 = 0$. Finally, by the squeeze theorem, we get

$$\lim_{x\to 0} e^{\sin(\cot x)} x^5 = 0. \tag{5.8}$$

Finally, let's mention a very important statement that was proved in class:

Theorem 31 *The following equalities hold:*

$$\lim_{x\to 0} \sin x = 0, \qquad \lim_{x\to 0} \cos x = 1, \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{5.9}$$

Lecture 6

Continuity

6.1 Continuous functions

One of the main reasons to introduce the concept of limit is defining continuous functions and, further, derivatives and integrals. Naively, one can say that a function is continuous if its graph is a solid line so it can be drawn without lifting the pen from the paper. However, in math one considers functions continuous not necessarily on a whole interval (when, indeed, the graph is a solid line), but at a single point.

A function $f(x)$ is *continuous* at a point $a \in \mathbb{R}$ if

$$\lim_{x \to a} f(x) = f(a). \tag{6.1}$$

If a function is continuous at any point of an interval I, then we say that it is continuous on the interval I.

The set of functions continuous at a point a is denoted by $C(a)$. Thus the notation $f \in C(a)$ means that the function $f(x)$ is continuous at the point $a \in \mathbb{R}$. Similarly, by $f \in C(I)$ one denotes the set of functions continuous on an interval $I \subset \mathbb{R}$.

Lemma 32 *1. A function $f(x)$ is continuous at $a \in \mathbb{R}$ if and only if for some $l \in \mathbb{R}$ the following statement holds: for any $\varepsilon > 0$ there is $\delta > 0$ such that for any x satisfying $|x - a| < \delta$, we have $|f(x) - l| < \varepsilon$*

2. *In such an occasion, we have $l = f(a)$.*

Proof. Left as an exercise. ⌣̈

Example 39 A polynomial function is continuous everywhere.

6.2 One-to-one functions

Let $f : I \to \mathbb{R}$ be a function defined on some interval $I \subset \mathbb{R}$. It is called *one-to-one* or *injective* if for $x_1 \neq x_2$, we always have $f(x_1) \neq f(x_2)$

Equivalently, we could say that $f : I \to \mathbb{R}$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Let $f : I \to \mathbb{R}$ be some function. We say that $g : f(I) \to \mathbb{R}$ is the *inverse* function to f if both

1. For each $x \in I$, we have $g(f(x)) = x$,
2. For each $y \in f(I)$, we have $f(g(y)) = y$.

The inverse of a function f is denoted by f^{-1}.

Theorem 33 *$f : I \to \mathbb{R}$ is one-to-one if and only if f^{-1} exists.*

Proof. We have to prove two implications:

$\Rightarrow$ Let f be one-to-one. Consider any number $y \in f(I)$. By the definition of a one-to-one function, there is exactly one number x such that $f(x) = y$. Put $g(y) = x$. Thus $f(g(y)) = f(x) = y$ and $g(f(x)) = g(y) = x$ which, by definition, means $g = f^{-1}$.

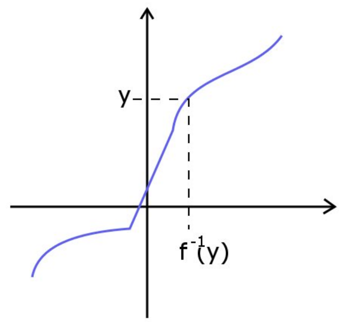


Figure 6.1: An increasing continuous function

$\Leftarrow$ Assume that there is the inverse function f^{-1}. Consider $x_1, x_2 \in I$ such that $f(x_1) = f(x_2)$. Hence we have $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$. By definition of f^{-1}, we get $x_1 = x_2$, which means that f is one-to-one.

Equivalence is proved. ⌣̈

Example 40 Any strictly monotonic function is one-to-one.

6.3 Monotonic continuous function

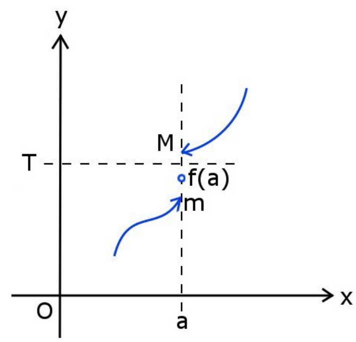
Looking at the graph of some monotonic continuous function (figure 6.3), it's quite natural to assume that there is an inverse.

Theorem 34 *Let $f : I \to \mathbb{R}$. Then any two conditions of the following list imply the third one of them:*

1. $f \in C(I)$.
2. *f is strictly monotonic.*
3. *$f(I)$ is an interval and $f^{-1} \in C(f(I))$.*

We give just an idea of why the statement is true. A rigorous proof would require some more advanced technique than we have for the moment. The theorem, in fact, has three statements:

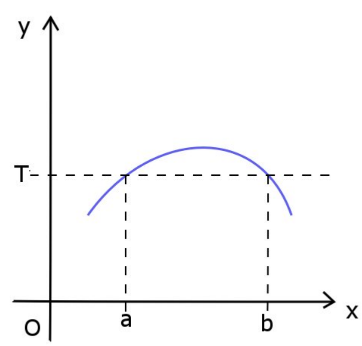
2, 3 ⇒ 1 Suppose that the function $f(x)$ is increasing, but discontinuous at some point $x \in I$. Then the graph looks like



and some point $T \in (m, M)$ has no pre-image (that is, for no $x \in I$, we would have $f(x) = T$). Contradiction with the existence of the inverse.

1, 2 ⇒ 3 Suppose now that the function is continuous and increasing. If some T has no pre-image, then the function can't be continuous (see the same picture above).

1, 3 ⇒ 2 If the function f is continuous, but not monotonic, then there should be a local maximum or local minimum:



Then there can't be any inverse, because some points have two pre-images like on the picture $T = f(a) = f(b)$.

Example 41 Since $x^n : [0, \infty) \to \mathbb{R}$ is continuous strictly increasing, $\sqrt[n]{x}$ must be also continuous strictly increasing.

Example 42 If we manage to prove that e^x is continuous, then $\log x$ is also established to be continuous.

Example 43 Trigonometric and inverse trigonometric functions are continuous (was in class).

Lecture 7

Continuity Laws. Symbols 'o' and 'O'. Taylor's Expansion.

7.1 Composition Rule

Let's introduce one more Limit Law. In the same manner as Sum, Product, Ratio Rules, we call it the Composition Rule.

Theorem 35 *Given two functions $f(y)$ and $g(x)$, assume that*

1. $\lim_{x\to a} g(x) = l$,
2. $f \in C(l)$.

Then we have $\lim_{x\to a} f(g(x)) = f(l)$.

Proof. Let's write down carefully all the definitions involved:

1st step Given $\varepsilon > 0$, there is $\delta_1(\varepsilon) > 0$ such that for $|y - l| < \delta_1$, we have $|f(y) - f(l)| < \varepsilon$

2nd step Also, there is $\delta_2(\varepsilon) > 0$ such that for $0 < |x-a| < \delta_2$, we have $|g(x)-l| < \delta_1$.

3rd step Since $|g(x) - l| < \delta_1$, we have $|f(g(x)) - f(l)| < \varepsilon$

4th step Together, for each $\varepsilon > 0$ we've found $\delta_2 = \delta_2(\delta_1(\varepsilon))$ such that for $0 < |x - a| < \delta_2$, we have $|f(g(x)) - f(l)| < \varepsilon$.

Theorem is proved. $\ddot{\smile}$

In the same manner, for sequences we would have

Theorem 36 *Given a function $f(x)$ and a sequence $\{a_n\}$, assume that*

- $\lim_{n\to\infty} a_n = L$,
- $f \in C(L)$.

Then we have $\lim_{n\to\infty} f(a_n) = f(L)$.

7.2 Continuity Laws

Since the limit and the continuity are very similar notions, we can state Limit Laws in the language of continuous functions. Summarizing this idea, we get

Theorem 37 *Given two continuous functions, their sum, difference, product, ratio, composition are also continuous.*

7.3 Elementary functions

Elementary functions are those obtained from

1. constants,
2. variables,
3. exponential, logarithm, trigonometric, inverse trigonometric functions

by addition, subtraction, multiplication, division, power, and composition.

In other words, elementary functions are those defined by a single formula.

Theorem 38 *Any elementary function is continuous on its natural domain.*

Proof. Since a constant function and the function $f(x) = x$ are obviously continuous and for sum, difference, product, ratio, and the composition the statement follows from Theorem 37, it suffices to prove that

1. Exponential, logarithm, trigonometric, inverse trigonometric functions are continuous.

2. The power of two continuous functions is continuous.

At the moment we postpone the proof about the exponential and the logarithmic functions because we don't actually know what they are.

Let's check that $\lim\limits_{x\to a} \sin x = \sin a$. We have

$$|\sin x - \sin a| = \left|2\cos\frac{x+a}{2}\sin\frac{x-a}{2}\right| \leq |x-a|,$$

where $|\cos\alpha| \leq 1$ and $|\sin\alpha| \leq |\alpha|$. By the squeeze theorem, it follows that $\lim\limits_{x\to a} |\sin x - \sin a| = 0$, which is equivalent to the desired limit.

In a similar manner, one can check that the cosine is continuous. Tangent and cotangent are continuous as the ratio of the sine and the cosine. Inverse trigonometric functions are continuous as inverse functions by Theorem 3 from Lecture 6.

Finally, for the power we have $f(x)^{g(x)} = e^{g(x)\log f(x)}$, so it must be also continuous. $\smile$

7.4 Symbols O and o

Given functions $f(x)$, $g(x)$ and a number a, one says that *f is o of g* as x tends to a if $f(x) = \alpha(x)g(x)$ for some function $\alpha(x)$ satisfying $\lim\limits_{x\to a}\alpha(x) = 0$.

One simply writes $f = o(g)(x \to a)$ for f being o of g as x tends to a. In the same manner, for sequences $a_n = o(b_n)(n \to \infty)$ means that $a_n = b_n c_n$ where $\lim\limits_{n\to\infty} c_n = 0$.

The definition can be re-written using $\varepsilon - \delta$ stuff as

Given two functions f and g and a number a, the notation $f = o(g)(x \to a)$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x satisfying $0 < |x - a| < \delta$, we have $|f(x)| \le \varepsilon |g(x)|$.

Changing this definition a bit, we obtain another useful notion:

Given two functions f and g and a number a, the notation $f = O(g)(x \to a)$ means that there exist $C > 0$ and $\delta > 0$ such that for any x, $0 < |x - a| < \delta$ implies $|f(x)| \le C|g(x)|$.

In other words,

Given functions $f(x)$ and $g(x)$, one says that

$$f = O(g)(x \to a) \tag{7.1}$$

if $f(x) = \alpha(x)g(x)$ for some function $\alpha(x)$ locally bounded at the point a.
In a similar manner, for sequences $\{a_n\}$ and $\{b_n\}$, one says that $a_n = O(b_n)(n \to \infty)$ if $a_n = b_n c_n$ for some bounded sequence $\{c_n\}$.

Example 44 $x^a = o(x^b)(x \to 0)$ for $a > b$.

Example 45 $n^a = o(n^b)(n \to \infty)$ for $a < b$.

Example 46 $\sin x = x + o(x)(x \to 0)$.

Example 47 $\sqrt{1 + x} = 1 + \dfrac{x}{2} + o(x)(x \to 0)$.

Corollary 39 *1.* $f = o(1)(x \to a)$ *if and only if* $\lim_{x\to a} f(x) = 0$*,*

2. $f = O(1)(x \to a)$ *if and only if* f *is locally bounded at* a*,*

3. $f = l + o(1)(x \to a)$ *if and only if* $\lim_{x\to a} f(x) = l$*.*

and similar statements hold for sequences.

7.5 Properties of O and o

Actually, the symbols o and O are something like the signs $<$ and $\leq$ for functions. The following properties are quite easy to proof:

Theorem 40 *1. Any function is* O *of itself, that is,* $f = O(f)$

2. Constant 0 *is* o *of any function, that is,* $0 = o(f)$

3. If $g = o(f)$*, then also* $g = O(f)$*. This is usually written as* $o(f) = O(f)$*, but it's not actually an equality since it works only one way.*

4. If $g = o(f)$*, then* $Cg = o(f)$ *for any constant* $C \in \mathbb{R}$*. This is usually written as* $C \cdot o(f) = o(f)$*.*

5. If $g = o(f)$ *and* $h = o(f)$*, then* $g + h = o(f)$*. This is written as* $o(f) + o(f) = o(f)$*, where one has to realize that all three os in the formula are different functions.*

6. If $g = O(f)$*, then* $Cg = O(f)$ *for any constant* $C \in \mathbb{R}$*. This is usually written as* $C \cdot O(f) = O(f)$*.*

7. If $g = O(f)$ *and* $h = O(f)$*, then* $h + g = O(f)$*. This is written as* $O(f) + O(f) = O(f)$*, where one has to realize that all the three Os in the formula are different functions.*

8. If $h_1 = o(f)$ *and* $h_2 = O(g)$*, then* $h_1 h_2 = o(fg)$*. This can be written as* $o(f) \cdot O(g) = o(fg)$

9. If $g = o(f)$ *and* $h = O(g)$*, then* $h = o(f)$*. This can be written as* $O(o(f)) = o(f)$

10. If $g = O(f)$ *and* $h = o(g)$*, then* $h = o(f)$*. This can be written as* $o(O(h)) = o(h)$

11. If $g = O(f)$ and $h = O(g)$, then $h = O(f)$. This can be written as $O(O(h)) = O(h)$

Proof. Just write the definitions. $\ddot{\smile}$

Example 48 Let's find $\lim_{x\to 0} \dfrac{\sin x}{1 - \sqrt{1+x}}$. We have as $x \to 0$, the following expansion

$$\frac{\sin x}{1-\sqrt{1+x}} = \frac{x + o(x)}{1 - 1 - \frac{x}{2} - o(x)} = \frac{x + o(x)}{-\frac{x}{2} + o(x)} = \frac{1 + o(1)}{-\frac{1}{2} + o(1)}, \tag{7.2}$$

so the limit is -2

Example 49 Let's find $\lim_{n\to\infty} 2^n \sin \dfrac{\pi}{2^n}$. Since as $n \to \infty$, we have $\dfrac{\pi}{2^n} \to 0$, so

$$2^n \sin\frac{\pi}{2^n} = 2^n\left(\frac{\pi}{2^n} + o\left(\frac{\pi}{2^n}\right)\right) = \pi + o(1), \tag{7.3}$$

which means that the limit is π.

Corollary 41 *For the actual calculation, we need the following statements that can be easily deduced from Theorem 40:*

1. *$Ao(f) + Bo(f) = o(f)$ for any constants A and B*
2. *$o(Cf) = o(f)$ for any constant C*
3. *$o(f) \cdot g = o(fg)$ for any function g*
4. *If $f = o(g)$, then $o(f) + o(g) = o(g)$*

Notice that equalities with o and O work in one direction. For example, you can replace $o(f) \cdot g$ with $o(fg)$, but not conversely.

7.6 Taylor's expansion

Actually, the following simple formulae hold:

1. $(1+x)^\alpha = 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2}x^2+\cdots+\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n)(x \to 0)$

2. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n)(x \to 0)$

3. $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n+1}}{n}x^n + o(x^n)(x \to 0)$

4. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + o(x^{2n+2})(x \to 0)$

5. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n+1})(x \to 0)$

We'll prove them later. Now we just memorize and use them.

Example 50 Let's find the limit $\lim_{x\to 0} \frac{e^x - \log(1 + x) - 1}{1 - \cos x}$. We have as $x \to 0$,

$$\frac{e^x - \log(1 + x) - 1}{1 - \cos x} = \frac{1 + x + \frac{x^2}{2} + o(x^3) - (x - \frac{x^2}{2} + o(x^3)) - 1}{1 - (1 - \frac{x^2}{2} + o(x^3))} \tag{7.4}$$

By Theorem 40, we can replace $o(x^3) - o(x^3)$ with just single $+o(x^3)$. Thus our expression equals

$$\frac{x^2 + o(x^3)}{\frac{x^2}{2} + o(x^3)} = \frac{2 + o(x)}{1 + o(x)} \tag{7.5}$$

since by Theorem 40 we can do cancellation inside o. Finally, we have $o(x) = o(o(1)) = o(1)$, which has limit 0, so the total limit is

$$\frac{2 + 0}{1 + 0} = 2 \tag{7.6}$$

Lecture 8

Tricks to Find a Limit. Intermediate Value Theorem.

8.1 Algebraic tricks to find limits

8.1.1 Power

Whenever having a power of two functions, we simplify it as follows:

$$f(x)^{g(x)} = e^{g(x)\log f(x)} \tag{8.1}$$

Example 51 Let's find $\lim_{x\to 0}(1-5x)^{\frac{1}{x}}$. We have,

$$(1-5x)^{\frac{1}{x}} = e^{\frac{\log(1-5x)}{x}} \tag{8.2}$$

Further,

$$\frac{\log(1-5x)}{x} = \frac{-5x+o(5x)}{x} = \frac{-5x+o(x)}{x} = -5+o(1) \tag{8.3}$$

Thus, the limit is e^{-5}.

8.1.2 Limit at $a \neq 0$

If we have the limit as x tends to some number $a \neq 0$, then we just introduce a new variable h in by putting $x = a + h$, so x tends to a means h tends to 0.

Example 52 Let's find $\lim_{x\to\pi/2} \frac{\cos x}{x-\frac{\pi}{2}}$. Put $x = \frac{\pi}{2} + h$. Then

$$\frac{\cos x}{x-\frac{\pi}{2}} = \frac{\cos\left(\frac{\pi}{2}+h\right)}{h} = \frac{-\sin h}{h} = \frac{-h+o(h^2)}{h}(h\to 0) = -1+o(h)(h\to 0), \quad (8.4)$$

so the limit is -1.

8.1.3 Power of some expression

We know how to deal with $(1+x)^\alpha$ as $x\to 0$. If we have something else inside the power, then we factor out whatever needed to get $(1+x)^\alpha$.

Example 53 Let's find $\lim_{x\to 0} \frac{\sqrt{2+3x}-\sqrt{2-7x}}{x}$. We have,

$$\frac{\sqrt{2+3x}-\sqrt{2-7x}}{x} = \frac{\sqrt{2}\sqrt{1+\frac{3}{2}x}-\sqrt{2}\sqrt{1-\frac{7}{2}x}}{x} =$$

$$\sqrt{2}\frac{\left(1+\frac{3}{2}x\right)^{1/2}-\left(1-\frac{7}{2}x\right)^{1/2}}{x} = \sqrt{2}\frac{1+\frac{3}{4}x+o(x)-(1-\frac{7}{4}x+o(x))}{x} = \quad (8.5)$$

$$\sqrt{2}\left(\frac{3}{4}+\frac{7}{4}\right)+o(1) = \frac{5}{\sqrt{2}}+o(1),$$

so the limit is $\frac{5}{\sqrt{2}}$

8.1.4 Limit at infinity

If we are finding the limit of a sequence, then instead of using powers of x as $x\to 0$ we use powers of $\frac{1}{n}$ as $n\to\infty$.

Example 54 Let's find $\lim_{n\to\infty} n^6\left(\sqrt[7]{1+n^7}-n\right)$. We have,

$$\begin{gathered} n^6\left(\sqrt[7]{1+n^7}-n\right)=\left[n\left(1+\frac{1}{n^7}\right)^{1/7}-n\right]n^6 \\ \left[n\left(1+\frac{1}{7n^7}+o\left(\frac{1}{n^7}\right)\right)-n\right]n^6=\left[\frac{1}{7n^6}+o\left(\frac{1}{n^6}\right)\right]n^6=\frac{1}{7}+o(1) \end{gathered} \tag{8.6}$$

so the limit is $\frac{1}{7}$.

8.2 Functions continuous on a whole interval

The following three statements are intuitively obvious once you look at the graph of a continuous function, but the proofs involve some advanced notions from the year 3 course of Real Analysis.

8.2.1 Intermediate Value Theorem

Theorem 42 *Assume that $f \in C[a,b]$ and $f(a)\cdot f(b) \leq 0$. Then there is at least one point $x_0 \in [a,b]$ such that $f(x_0)=0$.*

8.2.2 Continuous function is bounded

Theorem 43 *Assume that $f : [a,b] \to \mathbb{R}$ is continuous on the whole interval $[a,b]$. Then f is bounded, that is, there are some constants m and M such that for each $x \in [a,b]$, we have $m \leq f(x) \leq M$*

Notice that here f is not locally, but globally bounded on the whole interval!

8.2.3 Maximum and minimum are reached

Theorem 44 *Assume that $f : [a,b] \in \mathbb{R}$ is continuous on the whole interval $[a,b]$. Then there are some points $x_m, x_M \in [a,b]$ such that for any $x \in [a,b]$, we have $f(x_m) \leq f(x) \leq f(x_M)$*

Lecture 9

Finding an Arbitrary Limit

9.1 Finding limits using o and O

9.1.1 Compositions

Theorem 45 *The following statements also hold*

1. *If $g = o(h)$ and $f = o(g+h)$, then $f = o(h)$. It can be written as $o(h+o(h)) = o(h)$.*

2. *If $g = O(h)$ and $f = o(g+h)$, then $f = o(h)$. It can be written as $o(h+O(h)) = o(h)$.*

Proof.

1. By the basic rules from Lecture 7, we have $h + o(h) = O(h) + o(O(h)) = O(h) + O(h) = O(h)$ and $o(O(h)) = o(h)$

2. By the basic rules from the previous lecture, we have $h + O(h) = O(h) + O(h) = O(h)$ and $o(O(h)) = o(h)$.

   ☺

Example 55 Let's find $\lim\limits_{x\to 0} \dfrac{\sin(\sin x) - x}{x^3}$. First, we have

$$\sin(\sin x) = \sin\left(x - \frac{x^3}{6} + o(x^3)\right) = \\ \left(x - \frac{x^3}{6} + o(x^3)\right) - \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3 + o\left(x - \frac{x^3}{6} + o(x^3)\right)^3 \tag{9.1}$$

Noticing that $x^m = o(x^3)$ for any $m > 3$, we see that $\left(x - \frac{x^3}{3} + o(x^3)\right)^3 = x^3 + o(x^3)$ because everything except x^3 has degree at least 4. Substituting it to the original expression, we see that

$$\sin(\sin x) = x - \frac{x^3}{6} - \frac{x^3}{6} + o(x^3) = x - \frac{x^3}{3} + o(x^3) \tag{9.2}$$

Finally,

$$\frac{\sin(\sin x) - x}{x^3} = -\frac{1}{3} + o(1), \tag{9.3}$$

so the answer is $-\frac{1}{3}$.

9.1.2 Fractions

The division can be treated as taking power -1 and multiplying by it.

For example, let's find an expression for $\tan x$. We have, as x tends to 0,

$$\tan x = \sin x \cdot (\cos x)^{-1} = \left(x - \frac{x^3}{6} + o(x^4)\right) \cdot \left(1 - \frac{x^2}{2} + o(x^3)\right)^{-1} =$$

$$\left(x - \frac{x^3}{6} + o(x^4)\right) \cdot \left(1 - \left(-\frac{x^2}{2} + o(x^3)\right) + \left(-\frac{x^2}{2} + o(x^3)\right)^2 + o\left(-\frac{x^2}{2} + o(x^3)\right)^2\right) =$$

$$\left(x - \frac{x^3}{6} + o(x^4)\right) \cdot \left(1 + \frac{x^2}{2} + o(x^3)\right) = x + \frac{x^3}{2} - \frac{x^3}{6} + o(x^4) =$$

$$x + \frac{x^3}{3} + o(x^4) \tag{9.4}$$

Now we can use this result in actual calculations!

9.2 Infinite limit

A notion similar to the one of an unbounded function is the function of infinite limit. We know that one cannot divide by zero because the result is infinity. This allows us to introduce the following definition.

One says that a function $f(x)$ has *infinite* limit as x tends to $a \in \mathbb{R}$ if

$$\lim_{x \to a} \frac{1}{f(x)} = 0 \tag{9.5}$$

In the same way, we say that the limit of a sequence $\{a_n\}$ is infinite if

$$\lim_{n \to \infty} \frac{1}{a_n} = 0 \tag{9.6}$$

One writes $\lim_{x \to a} f(x) = \infty$ or $\lim_{n \to \infty} a_n = \infty$ for infinite limit of a function and of a sequence respectively.

Example 56 Since negative power is the same as division by positive power, we have

$$\lim_{x \to 0} x^a = \begin{cases} 0, & a > 0 \\ \infty, & a < 0 \end{cases} \tag{9.7}$$

and

$$\lim_{n \to \infty} n^a = \begin{cases} 0, & a < 0 \\ \infty, & a > 0 \end{cases} \tag{9.8}$$

9.3 One-hand limits

Also, sometimes one needs to consider the limit when the variable approaches some value from above or from below. For example. if the variable t represents time that tends to 0, then since the time cannot be negative, we don't consider $t < 0$ at all.

One says that a function f has *right-hand* limit l as x tends to a if $\lim_{x\to a} g(x) = l$, where

$$g(x) = \begin{cases} f(x), & x > a \\ \text{undefined}, & x \le a \end{cases} \tag{9.9}$$

In a similar manner, a function f has *left-hand* limit l as x tends to a if $\lim_{x\to a} h(x) = l$, where

$$h(x) = \begin{cases} f(x), & x < a \\ \text{undefined}, & x \ge a \end{cases} \tag{9.10}$$

Usual notations are $\lim_{x\to a+} f(x) = l$ and $\lim_{x\to a-} f(x) = l$ for right-hand and left-hand limits respectively.

Example 57 Consider the function $f(x) = \frac{|x|}{x}$. Getting rid of the modulus, we obtain

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{9.11}$$

Therefore,

$$\lim_{x\to 0+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 0-} f(x) = -1 \tag{9.12}$$

9.4 Checking that a function does not have a limit

Theorem 46 *Assume that* $\lim_{x\to a} f(x) = l$. *Then for any sequence* $\{a_n\}$ *satisfying*

1. *For each n, we have* $a_n \ne a$,
2. $\lim_{n\to\infty} a_n = a$,

we have

$$\lim_{n\to\infty} f(a_n) = l \tag{9.13}$$

Proof. Let's write down the definitions carefully. First, for any $\varepsilon > 0$ there is $\delta > 0$ such that for $0 < |x - a| < \delta$, we have $|f(x) - l| < \varepsilon$.

Further, since the sequence $\{a_n\}$ converges to a, we can find N such that for $n > N$, we have $|a_n - a| < \delta$, which, since $a_n \neq a$, implies $|f(a_n) - l| < \varepsilon$. The proof is complete. $\ddot\smile$

Assume now that a function $f(x)$ does not have a limit as x tends to a. Logically, there are three possibilities:

9.4.1 Infinite limit

Theorem 47 *If* $\lim\limits_{x\to a} f(x) = \infty$*, then* f *doesn't have a limit as* $x \to a$*.*

Proof. Suppose that, conversely, $\lim\limits_{x\to a} f(x) = l$. If $l \neq 0$, then, by the Ratio Rule, $\lim\limits_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{l} \neq 0$, which contradicts the theorem's assumption.

Thus $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0$, which means that for $\varepsilon = 1$ for x close enough to 0, we must have

$$|f(x)| < 1 \qquad \text{and} \qquad \frac{1}{|f(x)|} < 1 \tag{9.14}$$

at the same moment. It is impossible, so this contradiction concludes the proof. $\ddot\smile$

This can be checked by calculating $\lim\limits_{x\to a} \dfrac{1}{f(x)}$ directly.

Example 58 Let's check that $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^4} = \infty$. We have,

$$\frac{x^4}{\sin x - x} = \frac{x^4}{x - \left(x - \frac{x^3}{6} + o(x^3)\right)} = \frac{x^4}{\frac{x^3}{6} + o(x^3)} = \frac{x}{1 + o(1)}, \tag{9.15}$$

so the limit is, indeed, 0

9.4.2 Different one-hand limits

Theorem 48 *For any function $f(x)$ and a number $a \in \mathbb{R}$, we have $\lim_{x\to a} f(x) = l$ if and only if both $\lim_{x\to a+} f(x) = l$ and $\lim_{x\to a-} f(x) = l$ are fulfilled.*

Proof. Recall that the functions

$$g(x) = \begin{cases} f(x), & x > a \\ \text{undefined}, & x \le a \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} f(x), & x < a \\ \text{undefined}, & x \ge a \end{cases} \tag{9.16}$$

are used in the definitions of the right-hand and the left-hand limits. Now the theorem consists of the two statements.

First, assuming that f has limit a we need to check that both g and h have limit l. Here we just write down the definitions and see that we can take exactly the same δ for g and h as for f.

Second, assuming that both g and h have limit l, we need to check that f has limit l. We are given that for any $\varepsilon > 0$ there are $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ such that

1. For $0 < |x-a| < \delta_1$, we have $|g(x)-l| < \varepsilon$. But $g(x)$ is defined only for $x > a$, where it equals $f(x)$. Thus here we actually see that for $a < x < a + \delta_1$, we have $|f(x) - l| < \varepsilon$.

2. For $0 < |x-a| < \delta_2$, we have $|h(x)-l| < \varepsilon$. But $h(x)$ is defined only for $x < a$, where it equals $f(x)$. Thus here we actually see that for $a - \delta_2 < x < a$, we have $|f(x) - l| < \varepsilon$.

Taking $\delta(\varepsilon) = \min(\delta_1, \delta_2)$ completes the proof. ⌣̈

Example 59 Let's check that $\lim_{x\to 0} \frac{\sin|x|}{x}$ does not exist. We have,

$$\frac{\sin|x|}{x} = \frac{|x| + o(x)}{x} = \frac{|x|}{x} + o(1), \tag{9.17}$$

where the one-hand limits of the function $\frac{|x|}{x}$ have been already found to be $+1$ and -1. Since they are different, the limit of the original function does not exist.

9.4.3 Totally no limit

This is the trickiest case. Assume that a function $f(x)$ has no infinite limit as $x \to a$ and neither it has one-hand limits. We can detect such a case by finding two sequences $\{a_n\}$ and $\{b_n\}$ both converging to a such that $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$.

Example 60 Let's check that the function $\cos\frac{1}{x}$ has not limit as $x \to 0$. Since the cosine is between -1 and 1, let's try to find sequences where the value of the function is, respectively, -1 and 1.

First,

$$\cos\frac{1}{x} = 1 \Leftrightarrow \frac{1}{x} = 2\pi n \Leftrightarrow x = \frac{1}{2\pi n} \tag{9.18}$$

Second,

$$\cos\frac{1}{x} = -1 \Leftrightarrow \frac{1}{x} = \pi(2n+1) \Leftrightarrow x = \frac{1}{\pi(2n+1)} \tag{9.19}$$

Now putting

$$a_n = \frac{1}{2\pi n} \qquad \text{and} \qquad b_n = \frac{1}{\pi(2n+1)}, \tag{9.20}$$

we see that

1. $a_n \neq 0$,
2. $b_n \neq 0$,
3. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,
4. $\lim_{n\to\infty} f(a_n) = 1 \neq -1 = \lim_{n\to\infty} f(b_n) = 0$,

which implies that the limit does not exist.

9.5 Limit Laws for functions not having a limit

9.5.1 Sum Rule

Theorem 49 *Given two functions $f(x)$ and $g(x)$ such that $\lim_{x\to a} f(x) = l \in \mathbb{R}$, we have*

1. *If $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}(f(x) + g(x)) = \infty$.*

2. *If $g(x)$ has different one-hand limits, then so does $f(x) + g(x)$.*

3. *If $g(x)$ has totally no limit, then so does $f(x) + g(x)$.*

9.5.2 Product Rule

Theorem 50 *Given two functions $f(x)$ and $g(x)$ such that $\lim_{x\to a} f(x) = l \neq 0$, we have*

1. *If $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = \infty$.*

2. *If $g(x)$ has different one-hand limits, then so does $f(x)g(x)$.*

3. *If $g(x)$ has totally no limit, then so does $f(x)g(x)$.*

9.5.3 Ratio Rule

Theorem 51 *Given two functions $f(x)$ and $g(x)$ such that $\lim_{x\to a} f(x) = l \neq 0$, we have*

1. *If $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$.*

2. *If $g(x)$ has different one-hand limits, then so does $\frac{f(x)}{g(x)}$.*

3. *If $g(x)$ has totally no limit, then so does $\frac{f(x)}{g(x)}$.*

Example 61 Consider $\lim_{x\to 0} \frac{e^x - 1}{x} \cos \frac{1}{x}$. We have

$$\frac{e^x - 1}{x} \cos \frac{1}{x} = \frac{1 + x + o(x) - 1}{x} \cos \frac{1}{x} = (1 + o(1)) \cos \frac{1}{x}, \tag{9.21}$$

so it's a product of an expression of limit 1 and an expression of totally no limit. Thus it has totally no limit.

Example 62 Consider $\lim_{x\to 0} \frac{e^x - 1 - \sin x}{x} \cos^3 \frac{1}{x^7}$. We have then

$$\begin{gathered} \frac{e^x - 1 - \sin x}{x} \cos^3 \frac{1}{x^7} = \\ \frac{1 + x + o(x) - 1 - (x + o(x))}{x} \cos^3 \frac{1}{x^7} = o(1) \cdot O(1) = o(1), \end{gathered} \tag{9.22}$$

so the limit is 0. Here we replaced $\cos^3$ with $O(1)$ because $O(1)$ represents any bounded function, which $\cos^3$ is.

Part II

Derivative

Lecture 10

Definition. Rules of Calculating.

10.1 Derivative

Imagine, for example, that you travel by a car. The average speed is the distance travelled divided by the time spent. But during your journey, sometimes you stay on a traffic light and sometimes you drive by a highway, so your actual current speed can be quite different. Particularly, if it's too high, you may be fined.

What's an instant speed? Let $s(t)$ be the distance travelled from the beginning of the journey until the time moment t. Then the distance covered in the time interval $[t, t+h]$ for some small h is $s(t+h) - s(t)$. Thus the average speed is

$$v_{avg} = \frac{s(t+h) - s(t)}{h} \tag{10.1}$$

The average speed on a very small interval is close to the instant speed, thus we can say that the instant speed at a moment t is

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \tag{10.2}$$

Given a function $f(x)$, it's *derivative* at a point a is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{10.3}$$

If the derivative exists, then we say that the function f is *differentiable* at the point a.

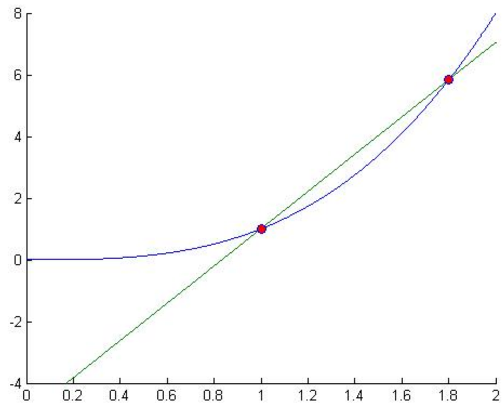


Figure 10.1: A chord through two points on a graph

Standard notations are $f'(a)$ for the derivative at the point a and $f \in D(a)$ for the function f being differentiable at the point a.

Remark 52 *By putting $x = a + h$, we can define the derivative to be*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{10.4}$$

Also, involving o-notation, we can say that $f'(a)$ equals the number D satisfying

$$f(a + h) = f(a) + Dh + o(h)(h \to 0) \tag{10.5}$$

Another motivation appears if we try to find an equation of the tangent line to a function's graph.

Let $f(x)$ be a function. Fix a point a and consider $a + h$ with h approaching 0. Let's draw the straight line passing through the points corresponding to $x = a$ and $x = a + h$ on f's graph (see Figure 10.1). The equation of this straight line is, obviously,

$$l_{chord}(x) = \frac{f(a + h) - f(a)}{h}(x - a) + f(a) \tag{10.6}$$

When h tends to 0 this chord seems to be approaching the tangent line.

But what is the tangent line, actually? In fact, we have just justified the following definition:

Given a function $f \in D(a)$, the straight line defined by the equation

$$l_{tangent}(x) = f'(a) \cdot (x - a) + f(a) \tag{10.7}$$

is called the *tangent* line to the graph of the function $f(x)$ at the point $x = a$.

10.2 Continuity of a differentiable function

Theorem 53 *If a function f is differentiable at a point a, then f is also continuous at the point a. Briefly, we can write it as*

$$D(a) \subset C(a) \tag{10.8}$$

Proof. Assume that $f \in D(a)$. It means that

$$f(a + h) = f(a) + f'(a)h + o(h)(h \to 0) \tag{10.9}$$

Taking the limit of this expression as $h \to 0$, we obtain

$$\lim_{h\to 0} f(a + h) = f(a), \tag{10.10}$$

which is just the definition of a continuous function. ⌣̈

Notice that sometimes a function is continuous, but not differentiable. Basically, there are three cases of such a situation.

10.2.1 Infinite limit

First, $\lim_{h\to 0} \dfrac{f(a + h) - f(a)}{h}$ might happen to be infinite. Graphically, it means that the tangent line exists, but is vertical (see Figure 10.2).

Example 63 For the function $f(x) = \sqrt[3]{x}$, the derivative is $f'(x) = \dfrac{1}{3x^{2/3}}$, which is infinite at $x = 0$.

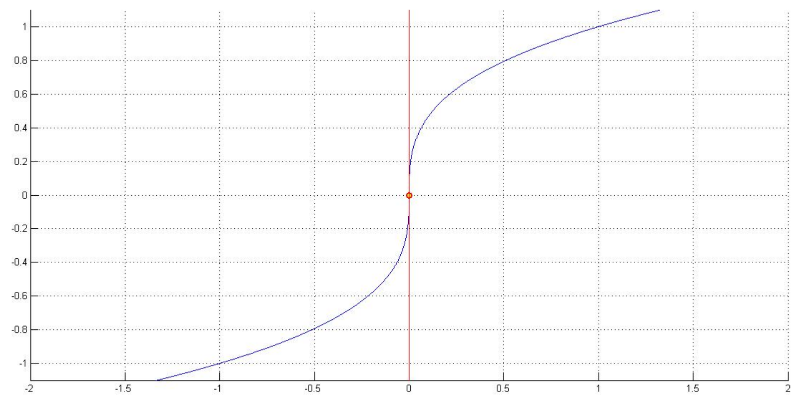


Figure 10.2: The graph of the function $f(x) = \sqrt[3]{x}$

10.2.2 Different one-hand limits

Second, the expression $\dfrac{f(a+h) - f(a)}{h}$ might have different one-hand limits as $h \to 0$. Graphically, it looks like the curve has an angle at $x = a$ as shown in Figure 10.3.

Example 64 Let's take $f(x) = \sin|x|$. By definition, $f'(0) = \lim_{h\to 0} \dfrac{\sin|h|}{h}$, which has different one-hand limits.

10.2.3 Totally no limit

Finally, $\dfrac{f(a+h) - f(x)}{h}$ might have totally no limit as $h \to 0$.

Example 65 Consider

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{10.11}$$

This function is clearly continuous (the graph is shown in Figure 10.4). However, by definition, $f'(0) = \lim_{h\to 0} \sin \dfrac{1}{h}$, which is not defined.

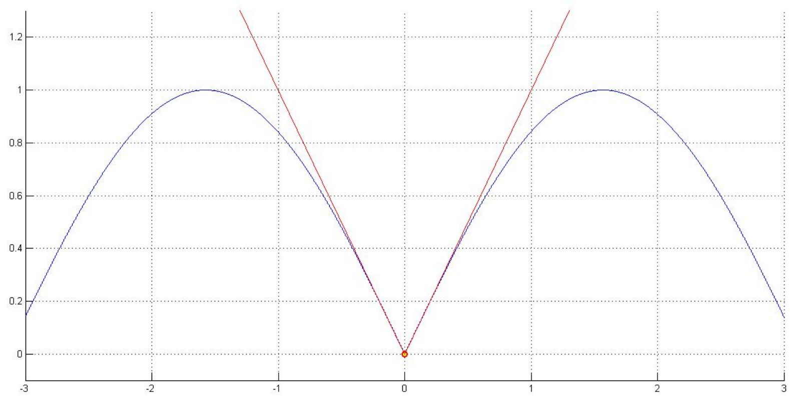


Figure 10.3: The graph of the function $f(x) = \sin|x|$

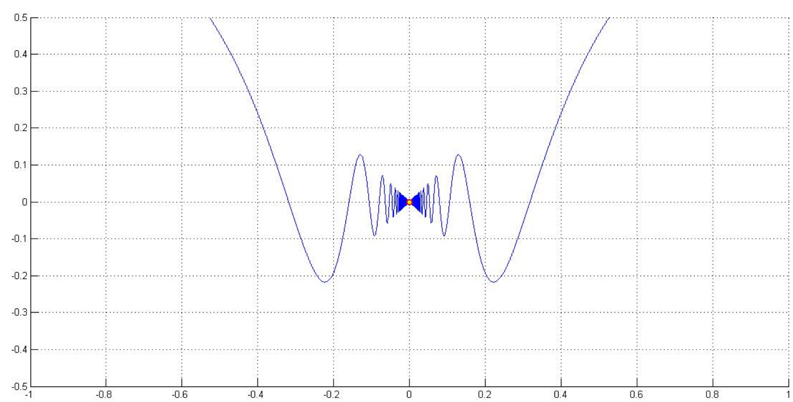


Figure 10.4: The graph of the function $f(x) = x \sin \dfrac{1}{x}$

10.3 How to find the derivative

Theorem 54 *Given two functions f and g, we have*

Sum rule *For any real constants c, d,*

$$(cf + dg)'(x) = cf'(x) + dg'(x) \tag{10.12}$$

Product rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{10.13}$$

Ratio rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \tag{10.14}$$

Chain rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x) \tag{10.15}$$

Proof. Let's prove one of these cases; the others are quite similar. For example, for the product we have

$$\begin{gathered}(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \\ \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = \\ \lim_{h\to 0} g(x+h)\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h} = \\ f'(x)g(x) + f(x)g'(x)\end{gathered} \tag{10.16}$$

$\ddot\smile$

Further, we need to differentiate all basic elementary functions.

Theorem 55 *The following formulae hold*

$$\begin{array}{lll} C' = 0 & (x^\alpha)' = \alpha x^{\alpha-1} & (e^x)' = e^x \\ \ln' x = \dfrac{1}{x} & \sin' x = \cos x & \cos' x = -\sin x \\ \arcsin' x = \dfrac{1}{\sqrt{1-x^2}} & \arccos' x = -\dfrac{1}{\sqrt{1-x^2}} & \arctan' x = \dfrac{1}{1+x^2} \end{array} \tag{10.17}$$

Proof. Here we are not able to do everything because we don't yet know what the exponential and the logarithmic functions are.

1. First, if $f(x) = C$ is a constant, then

$$f'(a) = \lim_{h\to 0} \frac{C - C}{h} = 0 \tag{10.18}$$

Done.

2. Further, if $f(x) = x^n$, $n \in \mathbb{N}$ is a positive natural power, then

$$f'(a) = \lim_{h\to 0} \frac{(a+h)^n - a^n}{h} = \lim_{h\to 0} \frac{(a^n + na^{n-1}h + \cdots) - a^n}{h} = na^{n-1} \tag{10.19}$$

Done

3. Further, is $f(x) = x^{-n}$ for $n \in \mathbb{N}$, $n \in Z$, then $f(x) = \dfrac{1}{x^n}$ and we differentiate it using the Ratio Rule.

4. Further, consider $y = f(x) = x^{1/n}$, $n \in \mathbb{Z}$. Put $g(y) = y^n$, that is, $g(f(x)) = x$. Let's differentiate it by the Chain Rule:

$$1 = [g(f(x))]' = g'(y) \cdot f'(x) = ny^{n-1} f'(x) = nx^{\frac{n-1}{n}} \cdot f'(x), \tag{10.20}$$

which gives us

$$f'(x) = \frac{1}{n} x^{-\frac{n-1}{n}} = \frac{1}{n} x^{\frac{1}{n}-1} \tag{10.21}$$

Done

5. Further, consider $f(x) = x^{m/n}$. This can be found using the Chain Rule as the function is the composition $(x^{1/n})^m$. It'll give us the formula

$$(x^\alpha)' = \alpha x^{\alpha-1} \tag{10.22}$$

for all $\alpha \in \mathbb{Q}$ (rational). For irrational α we cannot yet prove this formula because we don't actually know what the irrational power is.

6. Consider $f(x) = \sin x$. We have

$$f'(a) = \lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h} = \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos\left(a + \frac{h}{2}\right)}{h} = \cos a \tag{10.23}$$

because of the known limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{10.24}$$

taken for $x = \dfrac{h}{2}$.

7. Further, for $f(x) = \cos x$, we do the same as for $\sin x$.

8. Further, for $f(x) = \tan x$, we use the Ratio Rule since $\tan x = \dfrac{\sin x}{\cos x}$

9. Further, if $y = f(x) = \arcsin x$, then $\sin y = x$. By the chain rule, we have $y' \cos y = 1$ or $y' = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1 - x^2}}$.

10. For $f(x) = \arccos x$ we do the same as for $\arcsin x$.

11. Finally, consider $y = f(x) = \arctan x$. Thus $\tan y = x$. By the chain rule, $y' / \cos^2 y = 1$. Thus $y' = \cos^2 y = \cos^2 \arctan x$. Obviously,

$$\tan^2 \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1 - \cos^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha} - 1. \tag{10.25}$$

Therefore,

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} \tag{10.26}$$

Finally,

$$y' = \cos^2 \arctan x = \frac{1}{1 + \tan^2 \arctan x} = \frac{1}{1 + x^2}. \tag{10.27}$$

Thus we checked everything. $\ddot\smile$

Lecture 11

Optimization. Rolle's and Mean Value Theorems. L'Hôpital's Rule.

11.1 Local maxima and minima

Consider a natural problem: suppose we know that some process is described by a function and we need to find its maximal value. For example, if a businessman knows how the income depends on the price, it's crucial to find the price that maximizes the income.

Given a function $f(x)$, a point x_M is called its *local maximum* if there exists some open interval $(a, b) \ni x_M$ such that for any $x \in (a, b)$, if $f(x)$ is defined, then we have

$$f(x) \leq f(x_M). \tag{11.1}$$

Similarly, a point x_m is called a *local minimum* if there exists some open interval $(a, b) \ni x_m$ such that for any $x \in (a, b)$, if $f(x)$ is defined, we have

$$f(x) \geq f(x_m). \tag{11.2}$$

Example 66 Let $f(x) = \sin|x| : \mathbb{R} \to \mathbb{R}$ (see Figure 11.1). Then $x = 0$ is a local minimum because $f(0) = 0$ and we have $\sin|x| \geq 0$ for $|x| < \pi$.

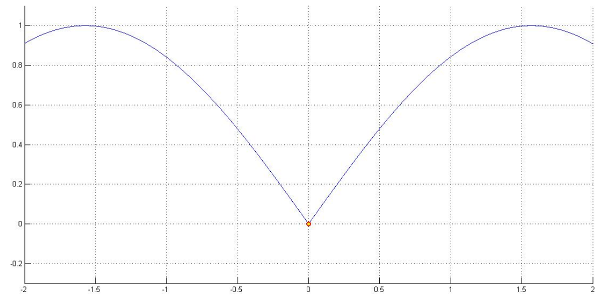


Figure 11.1: The graph of the function $f(x) = \sin|x|$

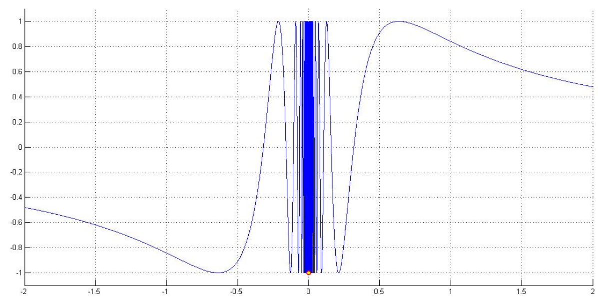


Figure 11.2: The graph of $f(x) = \sin\dfrac{1}{x}$ for $x \neq 0$ and $f(0) = -1$

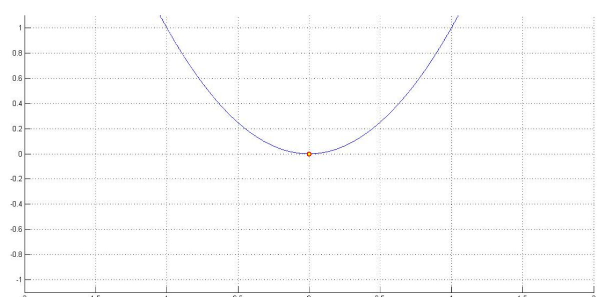


Figure 11.3: The graph of the function $f(x) = x^2$

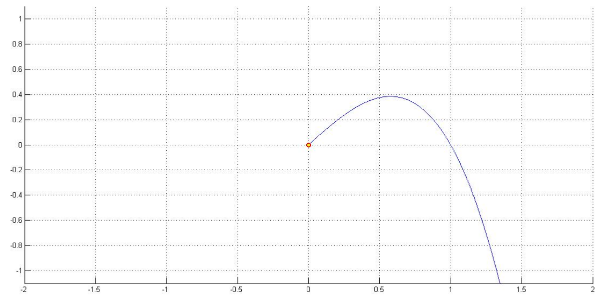


Figure 11.4: The graph of the function $f(x) = x - x^3 : [0, +\infty) \to \mathbb{R}$

Example 67 Let

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ \\ -1, & x = 0 \end{cases} : \mathbb{R} \to \mathbb{R}$$

(see Figure 11.2). Then $x = 0$ is a local minimum because $f(0) = -1$ and we have $f(x) \geq -1$ for all $x \in \mathbb{R}$.

Example 68 Let $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$ (see Figure 11.3). Then $x = 0$ is a local minimum because $f(0) = 0$ and we have $x^2 \geq 0$ for all $x \in \mathbb{R}$.

Example 69 Let $f(x) = x - x^3 : [0, +\infty) \to \mathbb{R}$ (see Figure 11.4). Then $x = 0$ is a local minimum because $f(0) = 0$, for $0 < x < 1$, we have $f(x) > 0$ and for $x < 0$ the function is not even defined.

Now if we are able to find all the points of local maxima and minima, then we can choose one of them carrying the largest value of the function and the smallest value of the function.

Theorem 56 (Fermat) *If a function $f(x)$ is differentiable at some point a and a is a local maximum or local minimum, then $f'(a) = 0$.*

Proof. Assume that a is a local minimum (the proof for maximum is similar). Then for any h small enough, we have $f(a + h) \geq f(a)$. Thus,

$$\frac{f(a+h) - f(a)}{h} \geq 0 \text{ for } h > 0 \qquad \text{and} \qquad \frac{f(a+h) - f(a)}{h} \leq 0 \text{ for } h < 0 \tag{11.3}$$

Since the derivative exists, we have

$$\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0-} \frac{f(a+h)-f(a)}{h} = f'(a). \tag{11.4}$$

By (11.3), the right-hand limit is non-negative while the left-hand limit is non-positive. Thus the only possibility for it is to be zero. ☺

Notice that this theorem is quite natural geometrically. It means just that the tangent line is horizontal at a local maximum or local minimum.

11.2 One-dimensional optimization

Given a function $f(x) : I \to \mathbb{R}$ defined on some interval $I \subset \mathbb{R}$, in order to calculate its largest or smallest value, we do as follows.

First, find all points where $f(x)$ is non-differentiable. They include

- Endpoints of the interval I,
- Points where $f(x)$ is discontinuous,
- Points where $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \infty$,
- Points where $\lim_{h\to 0+} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0-} \frac{f(a+h)-f(a)}{h}$,
- Points where $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist at all.

Second, find all the roots of the derivative, that is, solutions to the equation $f'(x) = 0$.

Finally, evaluate $f(x)$ at each point of the list and pick up the largest or the smallest value. Here, if the interval I is infinite, we might have to find limits $\lim_{x\to -\infty} f(x)$ or $\lim_{x\to +\infty} f(x)$.

Example 70 Let us find the smallest and the largest values of the function $f(x) = \frac{x^3}{3} - x^2 - 3x$ on the interval $[0, 5]$. The functions is continuous and differentiable everywhere and the derivative equals

$$f'(x) = x^2 - 2x - 3 \tag{11.5}$$

Thus we need to solve the equation $x^2 - 2x - 3 = 0$, which gives us the roots $x = -1$ and $x = 3$. Since $x = -1$ does not belong to $[0, 5]$, we consider only $x = 3$ and the endpoints of the given interval $x = 0$ and $x = 5$. We have

$$f(0) = 0, \qquad f(3) = -9, \qquad f(5) = \frac{5}{3} \tag{11.6}$$

Thus the smallest value is -9 and the largest one is $\frac{5}{3}$.

Example 71 Let us find the smallest and the largest values of the function $f(x) = \frac{x^3}{3} - x^2 - 3x$ on the interval $[0, +\infty)$. First, let's determine the function's behavior at infinity. We have

$$f(x) = \frac{x^3}{3} \cdot \left(1 - \frac{3}{x} - \frac{1}{x^2}\right), \tag{11.7}$$

which is increasing for x large enough and has infinite limit, so it there is no maximal value — the function can be made arbitrary large. The smallest value must be the same as in the first example, that is, -9.

11.3 Critical points

Fermat's theorem shows the significance of a point where the derivative vanishes or does not exists. However, it is important to understand that sometimes the derivative is zero, but the point is neither local maximum or local minimum.

Example 72 The function $f(x) = x^3$ is strictly increasing, so there cannot be any local maximum or minimum, though $f'(0) = 0$.

> A *critical point* of a function $f : A \to \mathbb{R}$ is a number $a \in A$ such that either $f'(a) = 0$ or $f'(a)$ does not exist. The number $f(a)$ is called a *critical value*.

Example 73 $f(x) = \sqrt[3]{x} - x$. Here,

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - 1. \tag{11.8}$$

Let's find all the critical points. First of all, the derivative does not exist at $x = 0$. Further, if $f'(x) = 0$, then

$$x^{-\frac{2}{3}} = 3 \Leftrightarrow x^{-2} = 27 \Leftrightarrow x^2 = \frac{1}{27} \Leftrightarrow x = \pm\frac{1}{3\sqrt{3}}. \tag{11.9}$$

Thus there are three critical points.

11.4 Rolle's and Mean Value theorems

Theorem 57 (Rolle) *Given a function $f(x)$, suppose that*

1. *f is continuous on a closed interval $[a, b]$,*
2. *f is differentiable on the open interval (a, b),*
3. *$f(a) = f(b)$.*

Then there is point $x \in (a, b)$ such that $f'(x) = 0$.

Proof. Since f is continuous, there is a maximum and a minimum. If both of them are the endpoints of the interval, then since $f(a) = f(b)$, we must have $f \equiv f(a) = f(b)$. The derivative of the constant function is zero at any point $x \in (a, b)$.

If a maximum or a minimum lies inside the interval, then by Fermat's theorem, the derivative there is zero. $\ddot\smile$

Theorem 58 (Mean Value) *Given a function $f(x)$, suppose that*

1. *f is continuous on a closed interval $[a, b]$,*
2. *f is differentiable on the open interval (a, b).*

Then there is a number $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \tag{11.10}$$

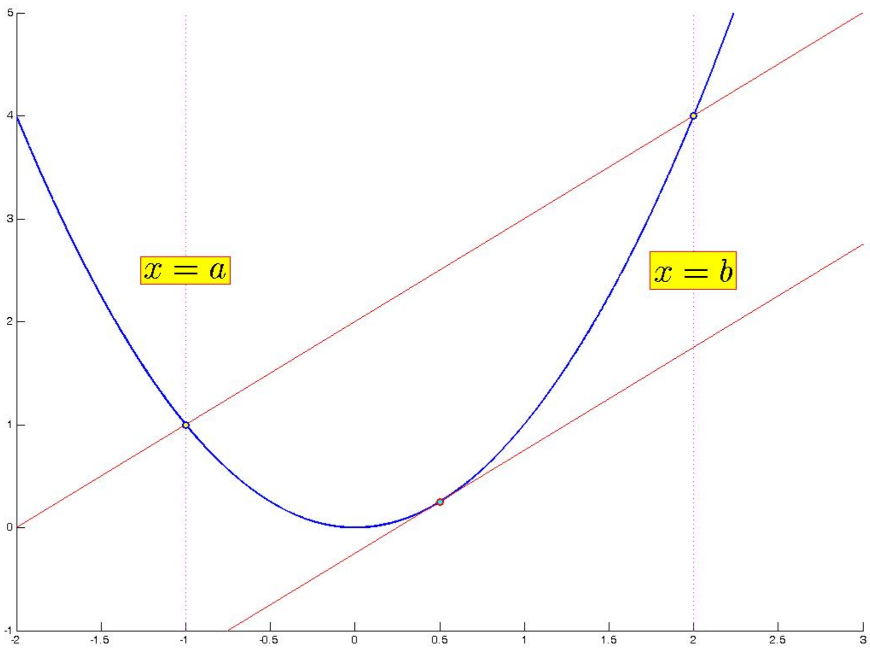


Figure 11.5: Mean value

Proof. Let

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{11.11}$$

Obviously, $h(a) = h(b) = f(a)$. By Rolle's theorem, there exists a point $x \in (a, b)$ such that $h'(x) = 0$. Thus,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} = 0, \tag{11.12}$$

which is the desired formula ☺

Notice that this theorem is natural geometrically. It means that there is a point where the tangent line is parallel to the chord passing through the endpoints of the interval as shown in Figure 11.5.

Corollary 59 *If $f'(x) = 0$ on an interval, then f is constant on the interval.*

Corollary 60 *If $f' = g'$ on an interval, then $f = g + C$, where C is some constant.*

11.5 Critical points and monotonic functions

Theorem 61 *Suppose that a function $f(x)$ is differentiable on an interval (a, b). If $f' > 0$, then f is increasing. If $f' < 0$, then f is decreasing.*

Proof. Let $x, y \in (a, b)$, $x < y$. By the Mean Value Theorem, there is some number $p \in (x, y)$ such that

$$f'(p) = \frac{f(y) - f(x)}{y - x}. \tag{11.13}$$

Now we see that $f'(p) > 0$ implies $f(x) < f(y)$. Similarly, $f'(p) < 0$ implies $f(x) > f(y)$. $\ddot{\smile}$

Remark 62 *Let's notice that conversely, if a function f is increasing, then $f' \geq 0$; if f is decreasing, then $f' \leq 0$ (it's absolutely obvious from the definition of the derivative). Here, the inequalities cannot be replaced by strict ones as the example of the function $f(x) = x^3$ shows.*

Example 74 $f(x) = \sqrt[3]{x} - x$. As we already know, there are three critical points:

$$x = -\frac{1}{3\sqrt{3}}, 0, \frac{1}{3\sqrt{3}}. \tag{11.14}$$

Are they local minima, maxima, or neither? Let's find the sign of the derivative on each interval between them. Since the derivative

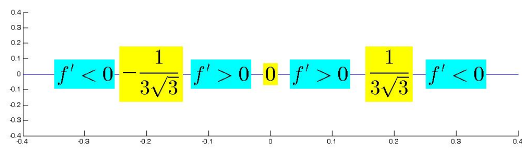
$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - 1 \tag{11.15}$$

is elementary function and therefore continuous where defined, to find the sign it's enough to calculate the value at some single point.

We have

$$\begin{aligned}
&-1 \in (-\infty, -\frac{1}{3\sqrt{3}}), && f'(-1) = -\frac{2}{3} < 0\\
&-0.001 \in (-\frac{1}{3\sqrt{3}}, 0), && f'(-0.001) = 100/3 - 1 > 0\\
&0.001 \in (0, \frac{1}{3\sqrt{3}}), && f'(0.001) = 100/3 - 1 > 0\\
&1 \in (\frac{1}{3\sqrt{3}}, +\infty), && f'(1) = -\frac{2}{3} < 0
\end{aligned} \tag{11.16}$$

Thus we have the following signs for the derivative:



Therefore $-\dfrac{1}{3\sqrt{3}}$ is the local minimum, $\dfrac{1}{3\sqrt{3}}$ is the local maximum, and 0 is neither of the both.

11.6 Cauchy's Theorem and l'Hôpital's rule

Theorem 63 (Cauchy's Mean Value) *Given functions $f(x)$ and $g(x)$, suppose that*

1. *f and g are continuous on a closed interval $[a, b]$,*

2. *f and g are differentiable on the open interval (a, b),*

Then there is $x \in (a, b)$ such that

$$g'(x)[f(b) - f(a)] = f'(x)[g(b) - g(a)]. \tag{11.17}$$

Proof. Let

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]. \tag{11.18}$$

Then $h(a) = h(b) = f(a)g(b) - g(a)f(b)$. By Rolle's theorem, there is $x \in (a, b)$ such that

$$h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)] = 0. \tag{11.19}$$

$\ddot{\smile}$

Notice that if $g'(x) \neq 0$ and $g(a) \neq g(b)$, then the equation in the theorem can be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}, \tag{11.20}$$

which is a more traditional notation for it.

The immediate corollary of this statement is the following

Theorem 64 (L'Hôpital's Rule) *Given two functions $f(x)$ and $g(x)$ differentiable on an open interval $I \ni a$, assume that*

1. *There is no point $x \in I$ such that $f'(x) = g'(x) = 0$,*
2. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$
3. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ *is finite or infinite*

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{11.21}$$

Proof. First, set $f(a) = g(a) = 0$. Then f and g become continuous at a. Now by the Cauchy Mean Value theorem, we have

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)} \tag{11.22}$$

for some $\alpha_x \in (a, x)$. Obviously, α_x approaches a as x approaches a. Therefore, the two limits coincide. ⌣̈

Corollary 65 *In a similar manner and under similar assumptions, we have*

1. *If* $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty$, *then* $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$.
2. *If* $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = 0$, *then* $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to +\infty} \dfrac{f'(x)}{g'(x)}$.
3. *If* $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = \infty$, *then* $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to +\infty} \dfrac{f'(x)}{g'(x)}$.

Now we'll apply it to find a few important limits. Also, later we'll use l'Hôpital's Rule to prove Taylor's Formula.

Theorem 66 *The following equalities hold for any numbers $n > 0$ and $a > 0$ no matter how small a is and how big n is*

1. $\lim\limits_{x \to +\infty} x^n e^{-ax} = 0$,

2. $\lim_{x\to+\infty} \frac{\log^n x}{x^a} = 0$,

3. $\lim_{x\to 0} x^a \log^n x = 0$

Proof. First, let's notice that in all the three cases it's enough to prove the formula for $n \in \mathbb{N}$ because we can apply the Squeeze Theorem for non-integer $n > 0$ afterwards. Thus we assume n to be integer.

For the first formula, we have by the Limit Laws

$$\lim_{x\to+\infty} x^n e^{-ax} = \left(\lim_{x\to+\infty} \frac{x}{e^{\frac{a}{n}x}}\right)^n, \tag{11.23}$$

where by l'Hôpital's Rule,

$$\lim_{x\to+\infty} \frac{x}{e^{\frac{a}{n}x}} = \lim_{x\to+\infty} \frac{1}{\frac{a}{n}e^{\frac{a}{n}x}} = 0. \tag{11.24}$$

The second expression can be done in a similar manner. Finally, for the third one, we have

$$x^a \log^n x = \left(x^{\frac{a}{n}} \log x\right)^n = \left(\frac{x^{\frac{a}{n}}}{1/\log x}\right)^n, \tag{11.25}$$

where by l'Hôpital's Rule, we have

$$\lim_{x\to 0} \frac{x^{\frac{a}{n}}}{1/\log x} = \lim_{x\to 0} \frac{\frac{a}{n}x^{\frac{a}{n}-1}}{1/x} = \lim_{x\to 0} \frac{a}{n}x^{\frac{a}{n}} = 0 \tag{11.26}$$

The theorem is proved. ⌣̈

Example 75 Let's find $\lim_{n\to\infty} \sqrt[n]{n}$. We have

$$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{\log n}{n}} \to e^0 = 1 \tag{11.27}$$

Thus the answer is 1.

Lecture 12

Convex Functions. Continuity of a Derivative.

12.1 Convexity and concavity

Let's look at Figure 12.1. Here there are graphs of two functions. The both are increasing, but the graphs look different: the red one is like waves while the other one is something convex.

A function $f(x)$ is called *concave upward* (or *convex downward* or just *convex*) on an interval I if any line segment joining two points of the function's graph lies above this graph as shown in Figure 12.2.
A function $f(x)$ is called *concave downward* (or just *concave*, or *convex upward*) on an interval I if any line segment joining two points of the function's graph lies below this graph as shown in Figure 12.3.

Lemma 67 *A function $f(x)$ is concave upward if and only if for any $a < x < b$ we have*

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a} \tag{12.1}$$

Also, $f(x)$ is concave downward if and only if for any $a < x < b$ we have

$$\frac{f(x) - f(a)}{x - a} > \frac{f(b) - f(a)}{b - a} \tag{12.2}$$

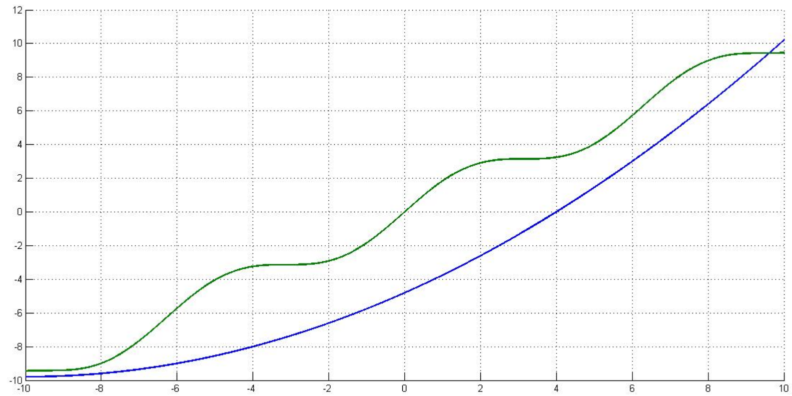


Figure 12.1: Convex and non-convex function

Proof. Let's prove it for a function concave upward (for concave downward everything is similar). Given $a < x < b$, the equation of the straight line passing through the points $x = a$ and $x = b$ on f's graph is

$$l(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \tag{12.3}$$

By definition, f is concave downward if and only if $f(x) < l(x)$, which is

$$f(x) < \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \tag{12.4}$$

After an easy simplification, we obtain the inequality from the lemma's statement. $\ddot{\smile}$

If a function is differentiable, then the concavity upward means, actually, that the tangent lies below the graph. Similarly, the concavity downward means that the tangent lies above the graph Thus if the tangent intersects the graph, then it must be some special point: concavity upward is being replaced with concavity downward or vice versa.

Given a function $f(x)$ defined on some open interval $I \ni a$, one says that a is an *inflection point* if f is neither concave downward nor concave upward on any interval $(s - \delta, s + \delta)$ no matter how small $\delta > 0$ is.

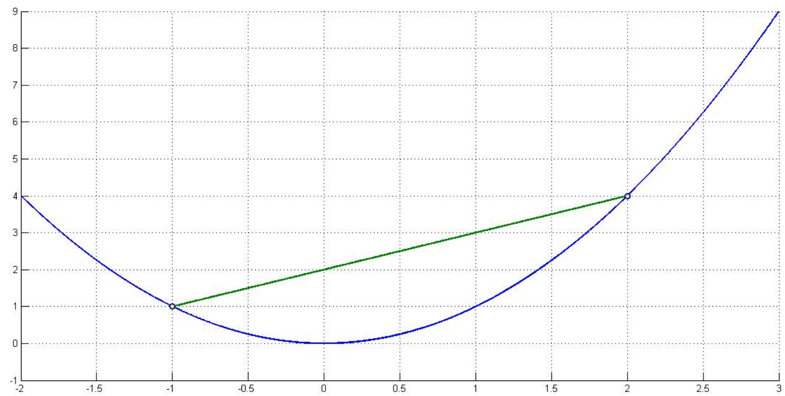


Figure 12.2: A function concave upward

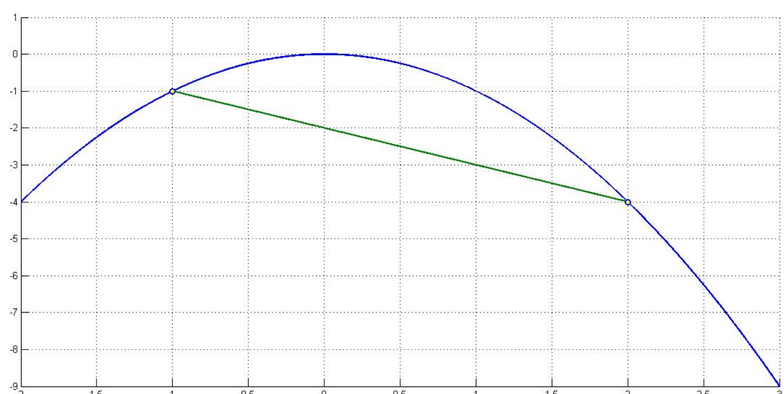


Figure 12.3: A function concave downward

Given a function $f(x)$, its *second derivative* is the derivative of $f'(x)$. More generally, its nth derivative is

$$\underbrace{((f')' \cdots)'}_{n \text{ differentiations}} \tag{12.5}$$

Standard notations are

- f'' for the second derivative,
- f''' for the third derivative,
- f^{iv} for the fourth derivative,
- $f^{(n)}$ for the nth derivative,
- $f \in D^{(n)}(a)$ for a function differentiable n times at $x = a$,
- $f \in C^{(n)}(a)$ for a function whose nth derivative is continuous at $x = a$.

It is possible to prove theorems related to second derivative in the same manner as Rolle's and Mean Value theorems are related to the first derivative. Here, we just briefly state the main result.

Theorem 68 *Given a function f twice differentiable on some open interval $I \ni a$, the following statements hold*

1. *If x is an inflection point, then $f''(a) = 0$,*
2. *$f'' > 0$ implies that f is concave upward,*
3. *$f'' < 0$ implies that f is concave downward,*
4. *$f'(a) = 0$ and $f''(a) > 0$ imply that x is a local minimum,*
5. *$f'(a) = 0$ and $f''(a) < 0$ imply that x is a local maximum.*

12.2 Continuity and differentiability

For elementary functions, the derivative, if exists, is always continuous. Generally, it's not true.

Example 76 Consider the function

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{12.6}$$

Is it differentiable at $x = 0$? Let's apply the definition:

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = 0 \tag{12.7}$$

Yes, it is. Is f' continuous at $x = 0$? Let's check. For $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \tag{12.8}$$

which has no limit as $x \to 0$, so it is discontinuous.

Example 77 In a similar manner, consider the function

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{12.9}$$

We have,

$$f'(0) = \lim_{h\to 0} \frac{h^3 \sin \frac{1}{h} - 0}{h} = 0 \tag{12.10}$$

For $x \neq 0$, we have

$$f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}, \tag{12.11}$$

so $\lim\limits_{x\to 0} f'(x) = 0 = f'(0)$. Thus here $f \in C^{(1)}(0)$. Is this function twice differentiable at $x = 0$? Let's check. By definition,

$$f''(0) = \lim_{h\to 0} \frac{2h^2 \sin \frac{1}{h} - h \cos \frac{1}{h}}{h} = \lim_{h\to 0} \left[2h \sin \frac{1}{h} - \cos \frac{1}{h} \right], \tag{12.12}$$

which is not defined. Therefore, $f \notin D^{(2)}(0)$.

Part III

Integral

Lecture 13

Definition and Basic Properties of the Integral

13.1 Area of a figure

Given an figure bounded with some curve, how can we find its area? If it's a rectangle, then the area equals the width times the height. In a simplest non-trivial case, it can bounded by a curve $y = f(x)$ and the straight line $y = 0$ like Figure 13.1.

The idea here is to approximate this area with a collection of rectangular strips as shown in Figure 13.2. Here we do as follows. First, divide the given interval $[a, b]$ into n equal subintervals. Specifically, put

$$a_i = a + \frac{b-a}{n} i, \tag{13.1}$$

so we have subintervals $[a_{i-1}, a_i]$ for $i = 1, 2, \ldots, n$. Let's choose a point on each subinterval, say $x_i^* \in [a_{i-1}, a_i]$. Then ith rectangle has width $\dfrac{b-a}{n}$ and height $f(x_i^*)$. Thus the total area equals

$$\frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*). \tag{13.2}$$

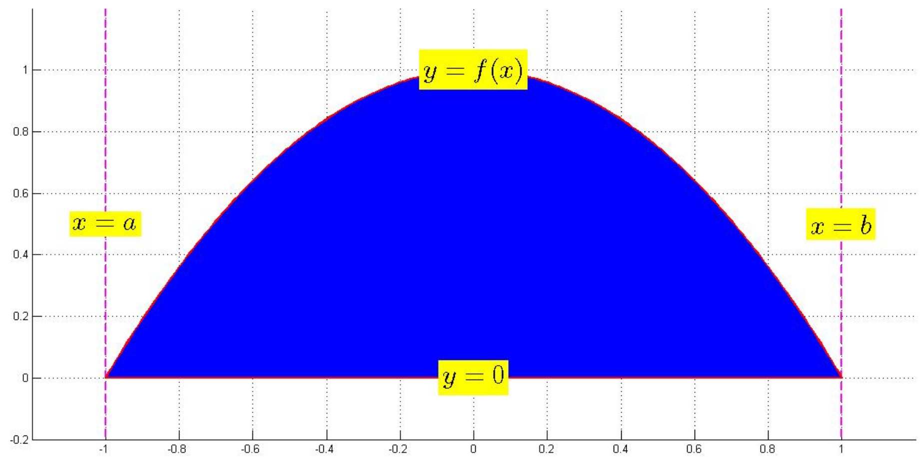


Figure 13.1: Area bounded by the curve $y = 1 - x^2$

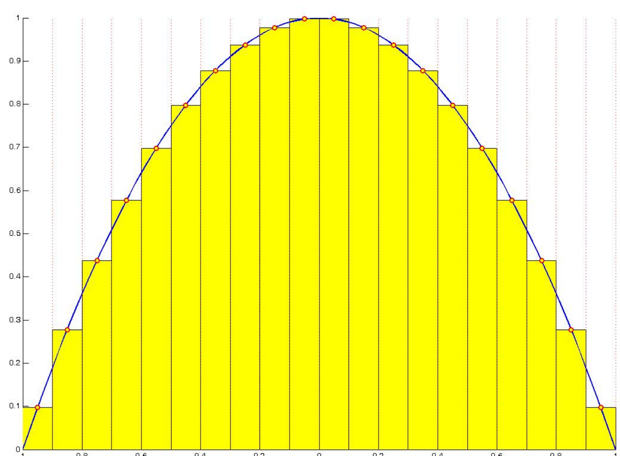


Figure 13.2: Area bounded by the curve $y = 1-x^2$ approximated with a collection of strips, where $x_i^* = \dfrac{a_i + a_{i-1}}{2}$

Given a function $f \in C[a, b]$, a *Riemann sum* is

$$\frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*), \tag{13.3}$$

where $n \in \mathbb{N}$ is some number, points $a_i = a + \dfrac{b-a}{n} i$ form a *partition* of the interval $[a, b]$, and $x_i^* \in [a_{i-1}, a_i]$ are *sample points*.
The limit of the Riemann sums

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*), \tag{13.4}$$

is called the *Riemann integral* of the function $f(x)$ from a to b.

The notation is $\displaystyle\int_a^b f(x)dx$ for the integral of $f(x)$ on the interval $[a, b]$.

Also, we set

$$\begin{aligned} &\int_a^a f(x)dx = 0, \\ &\int_a^b f(x)dx = -\int_b^a f(x)dx, \quad a > b \end{aligned} \tag{13.5}$$

13.2 The Riemann integral is well-defined

Here, the proof lies beyond our course. Some clarification is made in Spivak's textbook. Also, everybody is welcome to take Real Analysis I, II, and III ;)

Theorem 69 *Assume that $f \in C[a,b]$. No matter what sample points $x_i^* \in [a_{i-1}, a_i]$ we take, the limit*

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*) \tag{13.6}$$

exists and is the same for any choice of sample points.

Some possible choices for x_i^* could be

1. $x_i = a_{i-1}$, that is, the left end-point of the interval $[a_{i-1}, a_i]$,
2. $x_i = a_i$, that is, the right end-point of the interval $[a_{i-1}, a_i]$,
3. $x_i = \dfrac{a_{i-1} + a_i}{2}$, that is, the mid-point of the interval $[a_{i-1}, a_i]$,
4. x_i is the point where $f(x)$ reaches the maximum on $[a_{i-1}, a_i]$,
5. x_i is the point where $f(x)$ reaches the minimum on $[a_{i-1}, a_i]$.

Also, notice that the variable of integration does not matter. For example,

$$\int_a^b f(x)dx = \int_a^b f(\zeta)d\zeta \tag{13.7}$$

However, it better not to mix the limits of integration and the variable of integration. A notation like

$$\int_a^x f(x)dx \tag{13.8}$$

would cause a mess.

13.3 Linearity

Theorem 70 *Given two functions $f, g \in C[a,b]$ and two numbers $c, d \in \mathbb{R}$, we have*

$$\int_a^b [cf(x) + dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx \tag{13.9}$$

Proof. By definition,

$$\int_a^b [cf(x) + dg(x)]dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} [cf(x_i^*) + dg(x_i^*)] \tag{13.10}$$

Due to the linearity of the limit, it equals

$$c \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*) + d \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} g(x_i^*), \tag{13.11}$$

which is, by definition, the sum of integrals

$$c \int_a^b f(x)dx + d \int_a^b g(x)dx \tag{13.12}$$

Proof is complete $\ddot\smile$

13.4 Inequalities

Theorem 71 *Assume that $m \le f(x) \le M$ for any $x \in [a, b]$. Then*

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a) \tag{13.13}$$

Proof. Since the inequality $m \le f(x) \le M$ holds always, in particular, it holds at any sample point, that is, we have $m \le f(x_i^*) \le M$. Thus for the Riemann sums we have

$$m(b-a) = \frac{b-a}{n} \sum_{i=1}^{n} m \le \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*) \le \frac{b-a}{n} \sum_{i=1}^{n} M = M(b-a) \tag{13.14}$$

Taking the limit as $n \to \infty$, we get the inequality (13.13) $\ddot\smile$

13.5 Additivity of the Riemann integral

Theorem 72 *Also, the following formula holds:*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \tag{13.15}$$

Proof. This statement is a more tricky one, so we don't give a complete proof to it.

Assume that $a < b < c$; for any other ordering of the triple a, b, c the reasoning would be similar. Consider the interval $[a, c] = [a, b] \cup [b, c]$. Given a collection of sample points $x_i^* \in [a_{i-1}, a_i]$, the Riemann sum is

$$\frac{c-a}{n} \sum_{i=1}^{n} f(x_i^*). \tag{13.16}$$

Notice that some sample points belong to the interval $[a, b]$ while the rest lies on $[b, c]$. Let's assume that

$$a_0, a_1, \ldots, a_k \in [a, b], \qquad a_{k+1}, \ldots, a_n \in [b, c] \tag{13.17}$$

Let's notice that k is proportional to the length of the interval $[a, b]$ because the sample points are distributed uniformly over an interval. In other words, we have

$$\lim_{n\to\infty} \frac{k}{n} = \frac{b-a}{c-a} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n-k}{n} = \frac{c-b}{c-a} \tag{13.18}$$

Thus we have as n tends to infinity,

$$\begin{gathered}\frac{c-a}{n} \sum_{i=1}^{n} f(x_i^*) = \frac{c-a}{n} \sum_{i=1}^{k} f(x_i^*) + \frac{c-a}{n} \sum_{i=k+1}^{n} f(x_i^*) = \\ \underbrace{\frac{c-a}{b-a} \cdot \frac{k}{n}}_{\to 1} \cdot \underbrace{\left[\frac{b-a}{k} \sum_{i=1}^{k} f(x_i^*) \right]}_{\to \int_a^b f(x)dx} + \underbrace{\frac{c-a}{c-b} \cdot \frac{n-k}{n}}_{\to 1} \cdot \underbrace{\left[\frac{c-b}{n-k} \sum_{i=k+1}^{n} f(x_i^*) \right]}_{\to \int_b^c f(x)dx}\end{gathered} \tag{13.19}$$

The proof is complete ⌣̈

The additivity allows us to update the definition of the integral.

Assume that a function $f : [a,b] \to \mathbb{R}$ has only finitely many points of discontinuity, say $x_1, \ldots, x_k \in [a,b]$ and for each of them the one-hand limits exist and are different, that is,

$$\lim_{x \to x_i -} f(x) \neq \lim_{x \to x_i +} f(x) \tag{13.20}$$

Then we put

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_k}^{b} f(x)dx \tag{13.21}$$

13.6 Continuity of the Riemann integral

Theorem 73 *Given some function $f \in C[a,b]$, construct a new function $F(x)$ defined by*

$$F(x) = \int_a^x f(t)dt \tag{13.22}$$

Then $F \in C[a,b]$

Proof. We need to check that $\lim\limits_{h \to 0} F(x+h) = F(x)$ or, equivalently, that $\lim\limits_{h \to 0}[F(x+h) - F(x)] = 0$. First, let's notice that $f(x)$, being continuous, must be also bounded, that is,

$$m \leq f(x) \leq M \tag{13.23}$$

for any $x \in [a,b]$.

By the additivity of the integral, we have

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt, \tag{13.24}$$

for which we have the inequality

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh \tag{13.25}$$

The Squeeze Theorem gets us the desired limit $\smile$

Lecture 14

Fundamental Theorem of Calculus

14.1 Fundamental Theorem of Calculus

Theorem 74 (Fundamental Theorem of Calculus) *Assume that* $f \in C[a,b]$ *and a function* $F(x)$ *is given as*

$$F(x) = \int_a^x f(t)dt : [a,b] \to \mathbb{R} \tag{14.1}$$

Then

$$F'(x) = f(x) \tag{14.2}$$

for any $x \in (a,b)$

Proof. By definition of the derivative,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)d(t) \tag{14.3}$$

By definition of a continuous function, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x - \delta < t < x + \delta$, we have $f(x) - \varepsilon < f(t) < f(x) + \varepsilon$. Taking any $0 \leq h < \delta$, by the theorem about integrating inequalities, we get

$$h(f(x) - \varepsilon) < \int_x^{x+h} f(t)dt < h(f(x) + \varepsilon). \tag{14.4}$$

Dividing by h, obtain

$$f(x) - \varepsilon < \frac{1}{h} \int_x^{x+h} f(t)dt < f(x) + \varepsilon, \tag{14.5}$$

which is half of the desired limit. Specifically, we have just proved that

$$\lim_{h\to 0+} \frac{F(x+h) - F(x)}{h} = f(x) \tag{14.6}$$

The limit as $h \to 0-$ can be proved in the same way

Example 78 Let's consider a function

$$f(x) = \int_1^{\log x} e^{-\sin t} dt : (0, +\infty) \to \mathbb{R} \tag{14.7}$$

It's pretty unlikely to find an elementary formula for this function, though we can analyze the function nevertheless. Chain Rule together with the Fundamental Theorem of Calculus give

$$f'(x) = \frac{e^{-\sin(\log x)}}{x} \tag{14.8}$$

For example, it is always positive, so $f(x)$ is increasing. Further, we can differentiate it one more time to see where it is concave upward and where downward to graph it.

14.2 Newton-Leibnitz' Formula

Given a function $f(x)$, any function satisfying

$$F' = f \tag{14.9}$$

is called an *anti-derivative (primitive function, indefinite integral)* to the function f.

Let us recall the following statement we already deduced from the Mean Value Theorem:

Lemma 75 *If F and G are both anti-derivatives to a function f, then $F(x) \equiv G(x) + C$ for some constant C.*

The following fact is actually an immediate corollary of the Fundamental Theorem of Calculus:

Theorem 76 (Newton-Leibnitz' Formula) *If F is an anti-derivative to a continuous function f, then*

$$\int_a^b f(x)dx = F(b) - F(a) \tag{14.10}$$

Proof. By the Fundamental Theorem of Calculus,

$$G(x) = \int_a^x f(x)dx \tag{14.11}$$

is an anti-derivative. Since F is also an anti-derivative, we have

$$F(x) \equiv G(x) + C \tag{14.12}$$

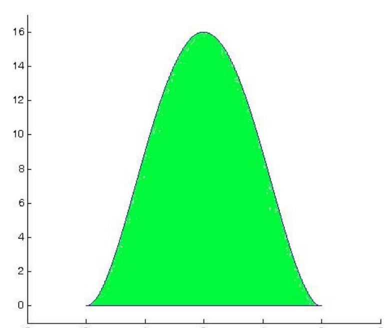
for some constant C. Now we have

$$F(b) - F(a) = G(b) + C - G(a) - C = \int_a^b f(x)dx - \int_a^a f(x)dx = \int_a^b f(x)dx \tag{14.13}$$

The proof is complete ⌣̈

The standard notation for an anti-derivative of a function $f(x)$ is $\int f(x)dx$.

Example 79 Let's calculate the area of a figure bounded by the curves $y = (x-2)^2(x+2)^2$ and $y = 0$:



Since the curves $y = (x-2)^2(x+2)^2$ and $y = 0$ intersect at points $(-2, 0)$ and $(2, 0)$, the area equals the integral

$$\int_{-2}^{2}(x-2)^2(x+2)^2dx = \int_{-2}^{2}(x^2-4)^2dx = \int_{-2}^{2}(x^4-8x^2+16)dx \tag{14.14}$$

Now we obviously have

$$\left(\frac{x^5}{5} - \frac{8}{3}x^3 + 16x\right)' = x^4 - 8x^2 + 16 \tag{14.15}$$

Therefore the function $F(x) = \frac{x^5}{5} - \frac{8}{3}x^3 + 16x$ is an anti-derivative to the function $f(x) = x^4 - 8x^2 + 16$. Thus,

$$\int_{-2}^{2}(x-2)^2(x+2)^2dx = F(2) - F(-2) = 2\left(\frac{32}{5} - \frac{8\cdot 8}{3} + 16\cdot 2\right) \tag{14.16}$$

Example 80 Let's evaluate the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}. \tag{14.17}$$

By definition of the integral, we have

$$\int_0^1 \sqrt{x}dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}} \tag{14.18}$$

Obviously, the function $\frac{2}{3}x^{\frac{3}{2}}$ is an anti-derivative to $\sqrt{x}$. Thus,

$$\int_0^1 \sqrt{x}dx = \frac{2}{3}1^{\frac{3}{2}} - \frac{2}{3}0^{\frac{3}{2}} = \frac{2}{3}, \tag{14.19}$$

so the answer is $\frac{2}{3}$.

Lecture 15

Applications of Integration. Techniques of Integration.

15.1 Application of integration in physics

Assume that we need to pump out some liquid from a tank. For example, the tank is a cylinder of radius $r = 0.4$ and length $l = 0.5$ lying on a side. How much work do we do when pumping out a liquid of density ρ through a hole on top as shown in Figure 15.1?

Generally, given a body of mass m, the work needed to lift it onto height h is mgh. Let us divide the whole liquid into n layers as shown in Figure 15.2. Then we can find the work needed to lift each particular layer up to the hole, through which we pump out.

Let h be the vertical distance from a particular point in the tank to the top and let $s(h)$ be the area of the cross-section on the height h. Assume that a sample height h_i^* lies in ith layer. If we consider the ith layer to have a rectangular shape, then its volume is $\frac{r}{n}s(h_i^*)$, thus the work needed to lift the ith layer is

$$\rho \cdot \frac{2r}{n} s(h_i^*) \cdot g \cdot h_i^*, \tag{15.1}$$

so the total work is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{2r}{n} s(h_i^*) \cdot g \cdot h_i^* = 2\rho g \int_0^{2r} hs(h)dh \tag{15.2}$$

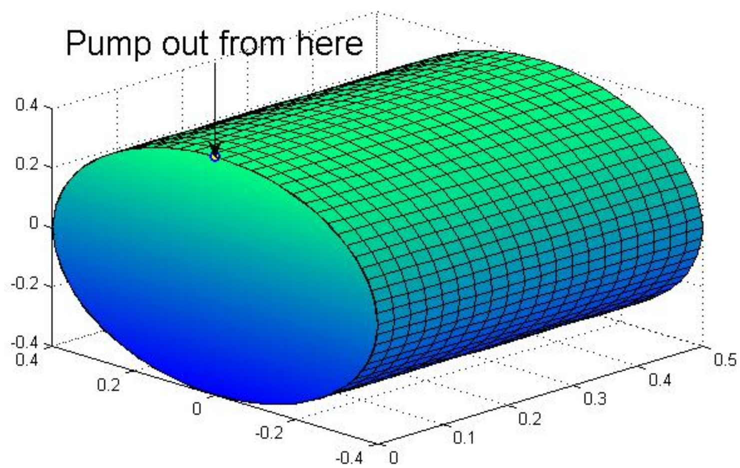


Figure 15.1: Cylindrical fuel tank

Finally, we clearly have

$$s(h) = 2l\sqrt{r^2 - (r-h)^2} \tag{15.3}$$

Thus the total work is

$$4l\rho g \int_0^{2r} h\sqrt{r^2 - (r-h)^2}. \tag{15.4}$$

15.2 Techniques of anti-differentiation

Now we see that finding an anti-derivative is an important problem. Unfortunately, this is much harder that finding a derivative. In particular, an anti-derivative of an elementary function is not necessary elementary.

First of all, we have the following obvious list of anti-derivatives:

1. $\displaystyle\int x^a dx = \frac{x^{a+1}}{a+1} + C$ for $a \neq -1$
2. $\displaystyle\int \frac{dx}{x} = \ln x + C,\ x > 0$

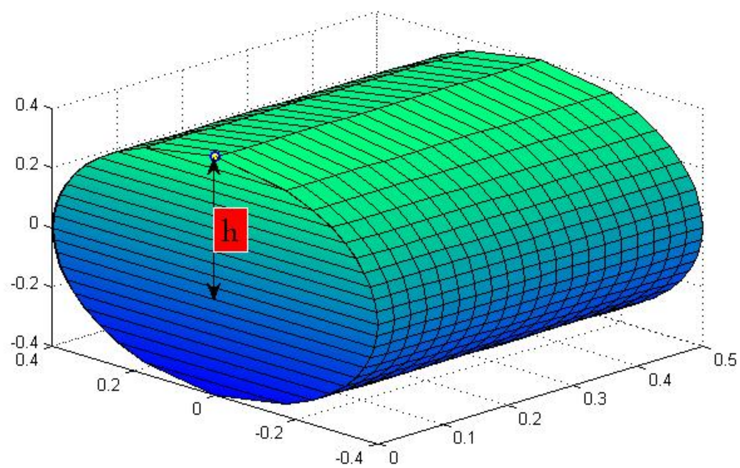


Figure 15.2: Cylindrical fuel tank cut into horizontal layers

3. $\displaystyle\int e^x dx = e^x + C$

4. $\displaystyle\int \sin x dx = -\cos x + C$

5. $\displaystyle\int \cos x dx = \sin x + C$

6. $\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C$

7. $\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x + C$

8. $\displaystyle\int \frac{dx}{1 + x^2} = \arctan x + C$

9. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$

Further, the rules of differentiation give us the following rules of integration.

Sum rule From the sum rule for differentiation, we obtain

$$\int (\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx \tag{15.5}$$

for any $\alpha, \beta \in \mathbb{R}$.

Integration by parts From the product rule for differentiation, we obtain

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{15.6}$$

Substitution rule From the chain rule for differentiation, we obtain

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{15.7}$$

for $u = g(x)$

15.3 Examples

Sum rule

$$\int \frac{x^2 dx}{1+x^2} = \int \frac{1+x^2-1}{1+x^2}dx = \int dx - \int \frac{dx}{1+x^2} = x - \arctan x \tag{15.8}$$

Integration by parts

$$\int \ln x dx = \int 1 \cdot \ln x dx = x \ln x - \int \frac{x dx}{x} = x \ln x - x \tag{15.9}$$

Integration by parts several times

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx = x^2 e^x - 2xe^x + 2\int e^x dx = e^x(x^2 - 2x + 2) \tag{15.10}$$

Integration by parts with additional calculations Suppose we need to calculate $\int e^x \sin x dx$. We have

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx \tag{15.11}$$

From this equation, we have

$$\int e^x \sin x dx = \frac{e^x \sin x - e^x \cos x}{2} \tag{15.12}$$

Linear substitution

$$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)(ax+b)'dx = \frac{1}{a}\int f(u)du, \quad u = ax+b \tag{15.13}$$

For example,

$$\int \frac{dx}{2x-5} = \frac{1}{2}\int \frac{(2x-5)'dx}{2x-5} = \frac{1}{2}\ln|2x-5| \tag{15.14}$$

or

$$\int a^x dx = \int e^{x\ln a}dx = \frac{1}{\ln a}\int e^{x\ln a}(x\ln a)'dx = \frac{e^{x\ln a}}{\ln a} = \frac{a^x}{\ln a} \tag{15.15}$$

Trigonometric substitutions

$$\int \tan x dx = \int \frac{\sin x dx}{\cos x} = -\int \frac{\cos' x dx}{\cos x} = -\ln|\cos x| \tag{15.16}$$

Power substitutions

$$\int \frac{dx}{(1+x)\sqrt{x}} = 2\int \frac{\sqrt{x}'dx}{1+\sqrt{x}^2} = 2\arctan\sqrt{x} \tag{15.17}$$

All together

$$\int \frac{\ln x dx}{x\sqrt{1+\ln x}} = \int \frac{\ln x(\ln' x)dx}{\sqrt{1+\ln x}} = \int \frac{udu}{\sqrt{1+u}}, \tag{15.18}$$

where $u = \ln x$. Integrating by parts, we have

$$\int \frac{udu}{\sqrt{1+u}} = 2u\sqrt{1+u} - 2\int \sqrt{1+u}du = 2u\sqrt{1+u} - \frac{4}{3}(1+u)^{\frac{3}{2}} \tag{15.19}$$

Finally,

$$\int \frac{\ln x dx}{x\sqrt{1+\ln x}} = 2\ln x\sqrt{1+\ln x} - \frac{4}{3}(1+\ln x)^{\frac{3}{2}} \tag{15.20}$$

15.4 Quasi-polynomials

Although generally it's impossible to integrate any elementary function in terms of elementary, one can describe some large natural classes of integrable elementary functions. First of all, it's possible to find an anti-derivative of any polynomial.

A *quasi-monomial* of degree n is $x^n e^{ax}\sin(bx)$ or $x^n e^{ax}\cos(bx)$, a *quasi-polynomial* is simply a sum of quasi-monomials.

First, let's integrate a quasi-monomial of degree 0 by parts:

$$\begin{gathered}\int e^{ax}\cos(bx)dx = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)dx = \\ \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) - \frac{b^2}{a^2}\int e^{ax}\cos(bx)dx\end{gathered} \tag{15.21}$$

Therefore,

$$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos(bx)dx = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) \tag{15.22}$$

Similarly, one can find the anti-derivative

$$\int e^{ax}\sin(bx)dx \tag{15.23}$$

Thus a quasi-monomial of degree 0 can be integrated in elementary functions, the anti-derivative is again a quasi-polynomial of degree 0.

Conversely, a derivative of a quasi-monomial of degree 0 is again a quasi-monomial of degree 0. This makes us able to reduce the degree of a quasi-polynomial being integrated. Indeed, if $Q_n(x) = x^n Q_0'(x)$ is a quasi-monomial of degree n, where Q_0 is some quasi-polynomial of degree 0, then integrating by parts, we obtain

$$\int Q_n(x)dx = \int x^n Q_0'(x)dx = x^n Q_0(x) - n\int x^{n-1}Q_0(x)dx \tag{15.24}$$

Repeating this trick n times, one can find the anti-derivative of any quasi-polynomial. Let's notice that the anti-derivative of a quasi-polynomial is a quasi-polynomial of either the same degree or of degree greater by 1.

Lecture 16

More Techniques of Integration

16.1 Trigonometric functions

Suppose we have a trigonometric polynomial, e.g., the sum of terms like

$$\cos^m x \sin^n x \tag{16.1}$$

How to integrate this?

Obviously, we have

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \end{aligned} \tag{16.2}$$

Thus if we deal with an expression

$$\cos^m x \sin^n x, \tag{16.3}$$

where the both m and n are even, we can simply reduce its degree replacing x with $2x$. If the both m and n here are odd, then we can do it anyway recalling that

$$\sin x \cos x = \frac{\sin 2x}{2} \tag{16.4}$$

Example 81

$$\int \cos^3 x \sin^3 x dx = \int \frac{\sin 2x}{2} \frac{1 - \cos^2 2x}{4} dx = \tag{16.5}$$

If n is even and m is odd (or vice versa), then we simply introduce a new variable $u = \sin x$ or $v = \cos x$.

Continuing the example, we obtain

$$= \frac{1}{8}\int \sin 2x dx - \frac{1}{8}\int \sin 2x \cos^2 2x dx = -\frac{1}{16}\cos 2x + \frac{1}{48}\cos^3 2x \qquad (16.6)$$

16.2 More about substitution

In fact, a substitution in an anti-derivative can be done in two ways. First, in the expression of the kind $\int g(f(x))f'(x)dx$ one can substitute $u = f(x)$ so that

$$\int g(f(x))f'(x)dx = \int g(u)du. \qquad (16.7)$$

Another way is the opposite one: in the expression of the kind $\int f(x)dx$ substitute $x = u(t)$, so it is

$$\int f(x)dx = \int f(u(t))u'(t)dt \qquad (16.8)$$

16.3 Calculating areas

Due to the very definition, the integral can be used to calculate an area. We know that if $f \geq 0$ on some interval $[a, b]$, then the area under the graph of the function equals

$$\int_a^b f(x)dx. \qquad (16.9)$$

Dealing with some more complicated figure, one has to split the area into several pieces such that

- each piece is described by a certain integral
- the total area is the sum or difference of the pieces

Example 82 Let's calculate the area $A(a,b)$ under the function $f(x) = a - b\ln(1+x)$ for some positive parameters a, b. The figure is defined by

$$0 \le y \le a - b\ln(1+x), \tag{16.10}$$

so the area is simply the integral from 0 to the point x such that $a - b\ln(1+x) = 0$, so $\ln(1+x) = a/b$, therefore $x = e^{a/b} - 1$. Thus the area equals

$$A(a,b) = \int_0^{e^{a/b}-1} (a - b\ln(1+x))dx = ax - \frac{b}{1+x}\bigg|_0^{e^{a/b}-1} \tag{16.11}$$

Example 83 Let's calculate the area A between the graphs of the functions $f = x/2$ and $g = x(1-x)$. The figure is defined by

$$\frac{x}{2} \le y \le x(1-x). \tag{16.12}$$

Obviously, this area is the difference between the area under the graph of the function $g(x) = x(1-x)$ and the area under the graph of the function $f(x) = x/2$. Thus,

$$A = \int_0^{1/2} x(1-x)dx - \int_0^{1/2} \frac{xdx}{2} = \frac{x^2}{4} - \frac{x^3}{3}\bigg|_0^{1/2} = \frac{1}{16} - \frac{1}{24} = \frac{1}{48} \tag{16.13}$$

Example 84 Let's calculate the area A inside the curve $x^{2/3} + y^{2/3} = 1$. First of all, we see that if we replace x by $-x$ or y by $-y$, the equation does not change, so the figure is symmetric. Hence we can calculate the area only for $x > 0, y > 0$ and multiply by 4. For $x > 0, y > 0$, the curve is defined by the equation

$$y = \left(1 - x^{2/3}\right)^{3/2} \tag{16.14}$$

Thus our area is

$$A = 4\int_0^1 \left(1 - x^{2/3}\right)^{3/2} dx \tag{16.15}$$

Substituting $x = t^3$, we obtain

$$A = 4\int_0^1 \left(1 - t^2\right)^{3/2} 3t^2 dt = 12\int_0^1 t^2(1-t^2)\sqrt{1-t^2}dt, \tag{16.16}$$

which can be done by a trigonometric substitution $t = \sin u$. Thus,

$$A = 12\int_0^{\pi/2} \sin^2 u \cos^2 u \cos u \cos u du = 12\int_0^{\pi/2} \sin^2 u \cos^4 u du, \qquad (16.17)$$

which is standard.

Lecture 17

Integrating a Rational Function

17.1 Integrating a simple fraction

Another big class of functions integrable in elementary terms is the set of rational functions. A rational function is

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{17.1}$$

The idea how to find an anti-derivative to any such a function is to partition this complicated formula into simple bricks, so called partial fractions. A partial fraction is one of the following list

1. Dx^k
2. $\dfrac{C}{(x-c)^k}$
3. $\dfrac{Ax+B}{(x^2+px+q)^k}$, where the function x^2+px+q has no real root.

First of all, let's explain how one integrates these simple fractions. The first two are obvious, the third one can be reduced by obvious substitution to a linear combination of

$$\frac{xdx}{(1+x^2)^n} \tag{17.2}$$

and

$$\frac{dx}{(1+x^2)^n} \tag{17.3}$$

The first of them can be done by the obvious substitution $u = x^2$, the second one for $n > 1$ can be integrated by parts in the following way:

$$\int 1 \cdot \frac{dx}{(1+x^2)^{n-1}} = \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{x^2 dx}{(1+x^2)^n} =$$

$$\frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{1+x^2-1}{(1+x^2)^n}dx = \tag{17.4}$$

$$\frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{dx}{(1+x^2)^{n-1}} - 2(n-1)\int \frac{dx}{(1+x^2)^n}$$

Simplifying this equality, we obtain:

$$2(n-1)\int \frac{dx}{(1+x^2)^n} = \frac{x}{(1+x^2)^{n-1}} + (2n-3)\int \frac{dx}{(1+x^2)^{n-1}}, \tag{17.5}$$

which gives us a way to reduce the power step by step and finally, for $n = 1$ everything is led to

$$\int \frac{dx}{1+x^2} = \arctan x \tag{17.6}$$

Thus to integrate a rational function

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0}, \tag{17.7}$$

it's enough to express it as a sum of partial fractions.

17.2 Properties of polynomials

Division To divide a polynomial $P(x)$ by a polynomial $Q(x)$ means to find an expression

$$P(x) = q(x)Q(x) + r(x), \tag{17.8}$$

where $\deg r(x) < \deg Q(x)$. Just like for integers, the polynomial $q(x)$ is called the *quotient*, the polynomial $r(x)$ is the *remainder*. This can be done with long division:

$$\begin{array}{r|l} & x \quad -3 \\ \hline x+5 & x^2 + 2x - 12 \\ & \underline{x^2 + 5x} \\ & \quad -3x - 12 \\ & \quad \underline{-3x - 15} \\ & \qquad\qquad 3 \end{array} \tag{17.9}$$

Factorization Each polynomial $Q(x)$ can be factorized:

$$Q(x) = A(x - x_1)(x - x_2)\cdots(x - x_n), \tag{17.10}$$

where $x_1, \ldots, x_n$ are the roots. Some of the roots are complex, but complex roots occur only only in couples $a - bi$, $a + bi$. For each couple, we have

$$(x - a + bi)(x - a - bi) = (x - a)^2 + b^2,$$

so totally any polynomial is a product of linear and quadratic polynomials, each quadratic multiplier has no real root. The quantity of each multiplier in the product is called the *multiplicity*.

Guessing a root Dealing with polynomial over $\mathbb{Z}$ (it means that all the coefficients are integers), one can guess an integer root. Indeed, if $Q(x) = 0$, then

$$a_0 = -a_n x^n - a_{n-1}x^{n-1} - \cdots - a_1 x, \tag{17.11}$$

which is divisible by x. Thus we get the statement:

Lemma 77 *If all the coefficients of a polynomial $Q(x)$ are integers and the root x is integer, then x divides the constant term.*

Example 85 Sometimes factorizing a polynomial is easy:

$$x^3 - x^2 + x - 1 = x^2(x - 1) + (x - 1) = (x - 1)(x^2 + 1), \tag{17.12}$$

where $x^2 + 1$ has no real root.

Example 86 Sometimes, one has to guess a root:

$$Q(x) = x^4 - x^3 + x^2 - 11x + 10 \tag{17.13}$$

Let's try to guess the roots. Trying the divisors of 10, we get two real roots: $x = 1, 2$. Thus there must be multipliers $(x - 1)(x - 2)$. Dividing by $Q(x)$ by $(x - 1)(x - 2) = x^2 - 3x + 2$, we get

$$Q(x) = (x - 1)(x - 2)(x^2 + 2x + 5), \tag{17.14}$$

where $x^2 + 2x + 5$ has no real root, so this is already our decomposition.

Example 87 Sometimes, one needs to guess a root first and then to apply the standard formula:

$$Q(x) = x^3 - 10x^2 + 21x - 10 \tag{17.15}$$

First, let's try to guess a root. Trying $Q(1) = 1 - 10 + 21 - 10 = 2$ does not satisfy us. Trying $Q(2) = 8 - 40 + 42 - 10 = 0$ shows us that there must be a multiplier $(x - 2)$. Dividing $Q(x)$ by $(x - 2)$, we get

$$x^3 - 10x^2 + 21x - 10 = (x - 2)(x^2 - 8x + 5) \tag{17.16}$$

Now let's try to find the roots of $x^2 - 8x + 5$.

$$\begin{gathered} x^2 - 8x + 5 = 0 \\ x = \frac{8 \pm \sqrt{64 - 20}}{2} = 4 \pm \sqrt{11} \end{gathered} \tag{17.17}$$

Thus,

$$x^3 - 10x^2 + 21x - 10 = (x - 2)(x - 4 - \sqrt{11})(x - 4 + \sqrt{11}) \tag{17.18}$$

17.3 Integrating a rational function

Suppose we have a rational function

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{17.19}$$

A partial fraction is one of the following list

1. Dx^k
2. $\dfrac{C}{(x - c)^k}$
3. $\dfrac{Ax + B}{(x^2 + px + q)^k}$, where the function $x^2 + px + q$ has no real root.

We already integrated the partial fractions in the previous lecture. Thus in order to find an anti-derivative to an arbitrary rational function, it's enough to represent it as a sum of partial fractions:

Step 1: First, we divide $P(x)$ by $Q(x)$:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{q(x)Q(x) + r(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}, \tag{17.20}$$

where the polynomial $q(x)$ is easy to integrate and the degree of $r(x)$ is less than that of $Q(x)$.

Step 2: Second, we factorize $Q(x)$ into the product of linear and quadratic multipliers:

$$Q(x) = A\,\underbrace{(x - c_1)^{l_1} \cdots (x - c_\alpha)^{l_\alpha}}_{\text{linear multipliers}}\underbrace{((x - a_1)^2 + b_1^2)^{q_1} \cdots ((x - a_\beta)^2 + b_\beta^2)^{q_\beta}}_{\text{quadratic multipliers}} \tag{17.21}$$

Step 3: Now each multiplier $(x - c)^l$ in $Q(x)$, gives us partial fractions

$$\frac{C_1}{x - c}, \frac{C_2}{(x - c)^2}, \ldots, \frac{C_l}{(x - c)^l} \tag{17.22}$$

Similarly, each multiplier $((x - a)^2 + b^2)^q$ gives us partial fractions

$$\frac{A_1 x + B_1}{(x - a)^2 + b^2}, \frac{A_2 x + B_2}{((x - a)^2 + b^2)^2}, \ldots, \frac{A_q x + B_q}{((x - a)^2 + b^2)^q} \tag{17.23}$$

Example 88

$$\frac{2x + 1}{x^4 - x^3 + x^2 - 11x + 10} = \frac{C_1}{x - 1} + \frac{C_2}{x - 2} + \frac{Ax + B}{x^2 + 2x + 5}, \tag{17.24}$$

where the coefficients A, B, C_1, C_2 are to be found:

$$\begin{gathered}\frac{C_1}{x - 1} + \frac{C_2}{x - 2} + \frac{Ax + B}{x^2 + 2x + 5} = \\ \frac{C_1(x^3 + x - 10) + C_2(x^3 + x^2 + 3x - 5) + (Ax + B)(x^2 - 3x + 2)}{x^4 - x^3 + x^2 - 11x + 10} = \\ \frac{(C_1 + C_2 + A)x^3 + (C_2 + B - 3A)x^2}{x^4 - x^3 + x^2 - 11x + 10} + \\ \frac{(C_1 + 3C_2 + 2A - 3B)x + (-10C_1 - 5C_2 + 2B)}{x^4 - x^3 + x^2 - 11x + 10},\end{gathered} \tag{17.25}$$

where we know the nominator to equal $2x + 1$. Thus,

$$\begin{cases} \quad C_1 + C_2 + A = 0 \\ \quad C_2 - 3A + B = 0 \\ \quad C_1 + 3C_2 + 2A - 3B = 2 \\ -10C_1 - 5C_2 + 2B = 1 \end{cases} \tag{17.26}$$

Let us solve this system of equations:

$$\begin{cases} C_1 + C_2 + A = 0 \\ C_2 - 3A + B = 0 \\ 2C_2 + A - 3B = 2 \\ 5C_2 + 10A + 2B = 1 \end{cases} \tag{17.27}$$

$$\begin{cases} C_1 + C_2 + A = 0 \\ C_2 - 3A + B = 0 \\ 7A - 5B = 2 \\ 25A - 3B = 1 \end{cases} \tag{17.28}$$

$$\begin{cases} C_1 + C_2 + A = 0 \\ C_2 - 3A + B = 0 \\ 7A - 5B = 2 \\ -104A = 1 \end{cases} \tag{17.29}$$

Thus we can solve the system and find A, B, C_1, C_2.

Example 89 Let's find an anti-derivative of the function

$$f(x) = \frac{x^6 + 2x^4 + x^2 + x + 1}{x^5 + 2x^3 + x} \tag{17.30}$$

First step:

$$f(x) = \frac{x(x^5 + 2x^3 + x) + x + 1}{x^5 + 2x^3 + x} = x + \frac{x + 1}{x^5 + 2x^3 + x} \tag{17.31}$$

Second step:

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2 + 1)^2 \tag{17.32}$$

Third step:

$$\frac{x+1}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2} \tag{17.33}$$

We have:

$$\frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2} =$$

$$\frac{a(x^4+2x^2+1) + x(bx+c)(x^2+1) + x(dx+e)}{x(x^2+1)^2} = \tag{17.34}$$

$$\frac{(a+b)x^4 + cx^3 + (2a+b+d)x^2 + (c+e)x + a}{x(x^2+1)^2}$$

Thus we obtain the system

$$\begin{cases} a + b = 0 \\ c = 0 \\ 2a + b + d = 0 \\ c + e = 1 \\ a = 1 \end{cases} \tag{17.35}$$

Therefore, the answer is

$$a = 1,\ b = -1,\ c = 0,\ d = -1,\ e = 1 \tag{17.36}$$

Concluding everything above, we obtain

$$f(x) = x + \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}, \tag{17.37}$$

where the only non-trivial summand is the last one. Let's use our formula:

$$2\int \frac{dx}{(x^2+1)^2} = \frac{x}{1+x^2} + \int \frac{dx}{x^2+1} = \frac{x}{1+x^2} + \arctan x \tag{17.38}$$

Thus we obtain

$$\int f(x)dx = \frac{x^2}{2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + \frac{1}{2(1+x^2)} + \frac{x}{2(1+x^2)} + \frac{\arctan x}{2} \tag{17.39}$$

Lecture 18

Integrating Trigonometric and Irrational Functions

18.1 Ratio of trigonometric functions

Let's briefly explain how to integrate the ratio of two trigonometric polynomials. Sometimes there is an obvious substitution to obtain a usual rational function. For example:

$$\int \frac{\cos x}{1+\sin^3 x}dx = \int \frac{\sin' x dx}{1+\sin^3 x} = \int \frac{du}{1+u^3} \tag{18.1}$$

for $u = \sin x$ can be done by the standard method for rational functions.

If it is not so clear, one can always apply a universal substitution

$$u = \tan \frac{x}{2} \tag{18.2}$$

Indeed,

$$\begin{aligned}
\cos x &= 2\cos^2 \frac{x}{2} - 1 = \frac{2}{1+\tan^2 \frac{x}{2}} - 1 = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \\
\sin x &= 2\sin \frac{x}{2} \cos \frac{x}{2} = 2\tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \\
\tan' \frac{x}{2} &= \frac{1}{2\cos^2 \frac{x}{2}} = \frac{1+\tan^2 \frac{x}{2}}{2}
\end{aligned} \tag{18.3}$$

For example,

$$\int \frac{\sin^3 x + \cos x}{\sin x - \cos x} dx = \int \frac{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)^3 + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} - \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} \frac{2\tan'\frac{x}{2}dx}{1+\tan^2\frac{x}{2}} \tag{18.4}$$

18.2 Irrational functions

Sometimes, a function with roots and powers can be integrated using a trigonometric substitution. Here is the manual.

18.2.1 Treating $\sqrt{1-x^2}$

If we see an expression of the kind $\sqrt{1-x^2}$ inside the formula to be integrated, the idea is to do the following substitution

$$\begin{aligned} x &= \sin u, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2} \\ dx &= \cos u du, \\ \sqrt{1-x^2} &= \cos u \end{aligned} \tag{18.5}$$

18.2.2 Treating $\sqrt{x^2+1}$

If we see an expression of the kind $\sqrt{x^2+1}$ inside the formula to be integrated, the idea is to do the following substitution

$$\begin{aligned} x &= \tan u, \quad -\frac{-\pi}{2} < u < \frac{\pi}{2} \\ dx &= \sec^2 u du, \\ \sqrt{x^2+1} &= \sec u \end{aligned} \tag{18.6}$$

18.2.3 Treating $\sqrt{x^2-1}$

If we see an expression of the kind $\sqrt{x^2-1}$ inside the formula to be integrated, the idea is to do the following substitution

$$
\begin{aligned}
x &= \sec u, \quad -\frac{\pi}{2} < u < \frac{\pi}{2} \\
dx &= \sin u \sec^2 u du, \\
\sqrt{x^2 - 1} &= \tan u
\end{aligned} \tag{18.7}
$$

Example 90 Let's find the anti-derivative

$$\int \sqrt{1 - x^2} dx \tag{18.8}$$

After the substitution $x = \sin u$, we have

$$\int \sqrt{1 - x^2} dx = \int \cos^2 u du = \int \frac{1 + \cos 2u}{2} du = \frac{u}{2} + \frac{\sin 2u}{4} \tag{18.9}$$

Therefore,

$$\int \sqrt{1 - x^2} dx = \frac{\arcsin x}{2} + \frac{\sin(2 \arcsin x)}{4} \tag{18.10}$$

Notice that sometimes an irrational function cannot be integrated in elementary terms at all!

18.3 Calculating the definite integral

Instead of finding an anti-derivative using integration by parts and substitution first and then applying the Newton-Leibnitz Formula

$$\int_a^b f(x) dx = F(b) - F(a), \tag{18.11}$$

one can do as follows.

Theorem 78 (Integration by Parts) *Assume that both functions $f(x)$ and $g(x)$ have continuous derivative. Then*

$$\int_a^b f(x) g'(x) dx = f(x) g(x) \Big|_a^b - \int_a^b f'(x) g(x) dx \tag{18.12}$$

Theorem 79 (Substitution) *Assume that $f(x)$ is continuous on an interval $[a,b]$ and assume that another function $u(t)$ has continuous derivative on the interval $[\alpha,\beta]$, where $a = u(\alpha)$ and $b = u(\beta)$. Then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(u(t))u'(t)dt \tag{18.13}$$

Example 91 Let's find $\int_0^1 \sqrt{1-x^2}dx$. Substitute $x = \sin t$. Since $\sin 0 = 0$ and $\sin\frac{\pi}{2} = 1$, we have

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \cos^2 t dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt = \frac{\pi}{4} \tag{18.14}$$

Part IV

Series

Lecture 19

Convergence. Limit Test. Integral Test.

19.1 Series

Given a sequence $\{a_n\}_{n=1}^{\infty}$, a *series* is the infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots \tag{19.1}$$

For any natural number n, the nth *partial sum* is

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \tag{19.2}$$

A series is *convergent* if the sequence of partial sums has a finite limit $L = \lim_{n\to\infty} S_n$, otherwise the series is called *divergent*. The *sum* of a convergent series is simply this limit of partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to+\infty} S_n \tag{19.3}$$

If the sequence $\{a_n\}$ begins not with a_1, but with a_0, a_2, or any a_k, the definition of the series is just the same; the sequence of partial sums begins with S_0, S_2, or any S_k then.

N.B. Here, there is a certain mess with the notation. Actually, the expression

$$\sum_{n=1}^{\infty} a_n \tag{19.4}$$

means the both series (the whole object, the sequence) and, if the series converges, its sum (the number). In the latter case one just writes

$$\sum_{n=1}^{\infty} a_n = L, \tag{19.5}$$

but usually the actual value L is not known, it can only be proved to exist. If the limit of the sequence of partial sums is infinite, then one can write

$$\sum_{n=1}^{\infty} a_n = \pm\infty, \tag{19.6}$$

but sometimes there is no $\lim S_n$ at all.

It might be possible to calculate the partial sums and establish whether the series is convergent or not:

Example 92 If $a_n = x^n$, $n = 0, 1, 2, \ldots$, for some number $x \in \mathbb{R}$, then

$$S_n = 1 + x + x^2 + \cdots + x^n = \begin{cases} n+1, & x = 1 \\ \dfrac{1 - x^{n+1}}{1 - x}, & x \neq 1 \end{cases} \tag{19.7}$$

which has limit for $-1 < x < 1$, does not have a limit for $x \leq -1$ and tends to $+\infty$ for $x \geq 1$. Thus,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{19.8}$$

for $-1 < x < 1$ and the series is divergent for $|x| \geq 1$.

Example 93 If $a_n = \sin\dfrac{2n+1}{2n(n+1)}\sin\dfrac{1}{2n(n+1)}$, then we can examine the series $\sum a_n$ in the following way. First, try to transform the expressions in the sines to the sums of partial fractions:

$$a_n = \sin\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)\cdot\sin\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right) \tag{19.9}$$

Now recalling the formula

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}, \tag{19.10}$$

we see that

$$a_n = \frac{1}{2}\left(\cos\frac{1}{n+1}-\cos\frac{1}{n}\right) \tag{19.11}$$

Thus,

$$\begin{aligned} S_n = \frac{1}{2}\Bigg(&\underbrace{\cos\frac{1}{2}-\cos\frac{1}{1}}_{a_1}+\underbrace{\cos\frac{1}{3}-\cos\frac{1}{2}}_{a_2}+\underbrace{\cos\frac{1}{4}-\cos\frac{1}{3}}_{a_3} \\ &+\cdots+\underbrace{\cos\frac{1}{n}-\cos\frac{1}{n-1}}_{a_{n-1}}+\underbrace{\cos\frac{1}{n+1}-\cos\frac{1}{n}}_{a_n}\Bigg) = \\ = \frac{1}{2}&\left(\cos\frac{1}{n+1}-\cos 1\right)\longrightarrow\frac{1-\cos 1}{2}\ (n\to\infty) \end{aligned} \tag{19.12}$$

Thus,

$$\sum_{n=1}^{\infty}\sin\frac{2n+1}{2n(n+1)}\sin\frac{1}{2n(n+1)} = \frac{1-\cos 1}{2} \tag{19.13}$$

Example 94 Let $a_n = (-1)^n$. Then, on the one hand,

$$\sum_{n=0}^{\infty}a_n = 1-1+1-1+\cdots = (1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0 \tag{19.14}$$

On the other hand, if the sum of this series is x, then

$$\begin{aligned} x &= 1-1+1-1+1-1+1-1+\cdots \\ x-1 &= 0-1+1-1+1-1+1-1+\cdots \\ -x &= \quad -1+1-1+1-1+1-1+\cdots \end{aligned} \tag{19.15}$$

Thus, $x-1=-x$, so $x=1/2$. This example shows that one cannot treat a series just like a finite sum.

19.2 Linearity

Given two sequences $\{a_n\}$ and $\{b_n\}$, put $c_n = a_n + b_n$ or, more generally,

$$c_n = \alpha a_n + \beta b_n \tag{19.16}$$

for some real numbers α, β. Then, obviously,

$$\sum_{i=1}^{n} c_i = \alpha \sum_{i=1}^{n} a_n + \beta \sum_{i=1}^{n} b_n \tag{19.17}$$

Due to the Sum Law for sequences, the following theorem holds:

Theorem 80 (Sum Law for Series) *Suppose, the series $\sum a_n$ and $\sum b_n$ are convergent. Then for any constants α and β the series*

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) \tag{19.18}$$

is convergent as well and

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{i=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n \tag{19.19}$$

Example 95

$$\sum_{n=0}^{\infty} \frac{1+2^{n+1}}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + 2\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-1/3} + \frac{2}{1-2/3} = \frac{15}{2} \tag{19.20}$$

19.3 Limit Test

Theorem 81 (Limit Test) *If a series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$*. In other words, if the sequence* $\{a_n\}$ *does not have limit* 0*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Assume that $\sum_{n=1}^{\infty} a_n = A$. By definition, it means that $\lim_{n\to\infty} S_n = A$, where $S_n = a_1 + a_n + \cdots + a_n$. Obviously, $a_n = S_n - S_{n-1}$. By the linearity of the limit, we have then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = A - A = 0. \tag{19.21}$$

Theorem is proved. $\ddot\smile$

Example 96 $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ diverges because the limit of the sequence is 1, which is not 0.

Example 97 For the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the Limit Test does not give an answer because in both cases the limit of the sequence is 0, so the series might converge and might diverge.

19.4 Improper integrals

We have to explain one useful notion here.

An *improper integral*

$$\int_a^{+\infty} f(x)dx \tag{19.22}$$

is convergent if

$$\lim_{b\to+\infty} \int_a^b f(x)dx \tag{19.23}$$

exists. Otherwise, it is divergent. If the integral is convergent, we simply put

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx \tag{19.24}$$

Example 98

$$\int_1^{+\infty} \frac{1}{x^2}\cos\frac{1}{x}dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\cos\frac{1}{x}dx = -\lim_{b\to\infty} \sin\frac{1}{x}\bigg|_1^b = \sin 1 \tag{19.25}$$

Example 99

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\bigg|_1^b = \lim_{b\to+\infty} \ln b = +\infty \tag{19.26}$$

19.5 Series of positive terms

Recall that, for monotonic sequences, to say that a sequence has a finite limit is equivalent to say that it is bounded. Notice that a series has positive terms if and only if its sequence of partial sums is growing. Thus we obtain the following statement

Lemma 82 *Assume that* $a_n \geq 0$ *for any* n *large enough. Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the sequence of its partial sums is bounded.*

Actually, this lemma is just reformulated Statement 1 from Lecture 3. In the same manner, for improper integrals we have

Lemma 83 *Assume that* $f(x) \geq 0$ *for any* x *large enough. Then the improper integral* $\int_a^{\infty} f(x)dx$ *converges if and only if the function* $F(x) = \int_a^x f(t)dt$ *is bounded.*

19.6 Integral Test

Theorem 84 (Integral Test) *Suppose a function* $f : [0, +\infty) \to \mathbb{R}$ *is decreasing and has limit* 0 *at* $+\infty$. *Then the series*

$$\sum_{n=0}^{n} f(n) \tag{19.27}$$

converges if and only if the improper integral

$$\int_0^{+\infty} f(x)dx \tag{19.28}$$

does.

Proof. Since the function is decreasing and tending to 0, it must be non-negative. Therefore the integral is just the area under the graph. Further, as shown in Figure 19.1, we have the inequality

$$f(1) + \cdots + f(n) \leq \int_0^n f(x)dx \leq f(0) + f(1) + \cdots + f(n), \tag{19.29}$$

which together with Lemmas 82 and 83 completes the proof. ⌣̈

Example 100 Let $f(x) = \dfrac{1}{x}$, which is decreasing for $x > 0$ and has limit 0 at $+\infty$. Thus we can apply the Integral Test. For the improper integral, we have

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \ln b = +\infty. \tag{19.30}$$

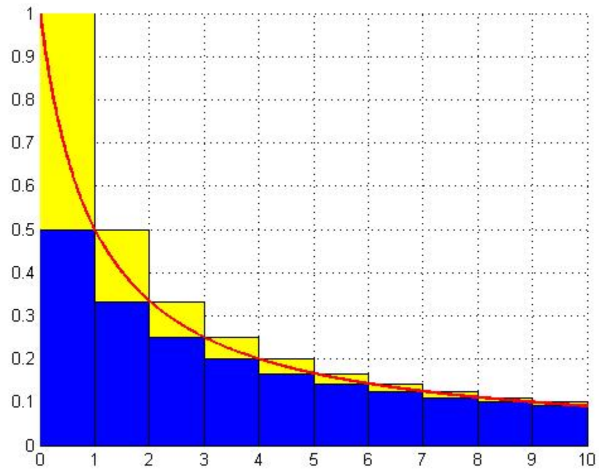


Figure 19.1: The area under the graph exceeds the sum of blue rectangles and is less than the sum of yellow and blue rectangles

Since the integral diverges, we conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as well.

Example 101 Let $f(x) = \frac{1}{x^2}$, which is decreasing for $x > 0$ and has limit 0 at $+\infty$. Thus we can apply the Integral Test. For the improper integral, we have

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to+\infty} \left(1 - \frac{1}{b}\right) = 1. \tag{19.31}$$

Since the integral converges, we conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as well.

Lecture 20

Comparison Test

Recall that, for a series of non-negative elements $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ converges if and only the sequence of partial sums

$$S_n = a_1 + a_2 + \cdots + a_n \tag{20.1}$$

is bounded.

Lemma 85 (Basic Comparison Test) *Suppose that $a_n \leq b_n \geq 0$ for any n. Then*

- *If the series $\sum b_n$ converges, then so does $\sum a_n$.*
- *If the series $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof. Obviously, for partial sums we have

$$S_n = a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n = R_n \tag{20.2}$$

Thus, if the sequence $\{R_n\}$ is bounded, then so is $\{S_n\}$. Similarly, if the sequence $\{S_n\}$ is unbounded, then so is $\{R_n\}$. ☺

Example 102 Let's check that the series $\sum_{n=6}^{\infty} \frac{1}{\sqrt{n-5}}$ diverges. Obviously, we have

$$\frac{1}{\sqrt{n-5}} > \frac{1}{\sqrt{n}} > \frac{1}{n} \tag{20.3}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ has been already proved to diverge by the Integral Test. Thus the series $\sum_{n=6}^{\infty} \frac{1}{\sqrt{n-5}}$ is also divergent.

Theorem 86 (General Comparison Test) *Assume that $a_n \geq 0$ and $b_n \geq 0$ for any n. Besides, suppose that $a_n = O(b_n)(n \to \infty)$. Then*

- *If the series $\sum b_n$ converges, then so does $\sum a_n$.*
- *If the series $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof. The notation $a_n = O(b_n)$ means that $a_n \leq Cb_n$ for some constant $C > 0$. By the Sum Law, multiplying by a non-zero constant does not affect convergence. Thus the General Comparison Test immediately follows from the Basic one (Lemma 85). $\ddot\smile$

Example 103 Let's consider the series

$$\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right) \tag{20.4}$$

We have

$$a_n = 1 - \cos \frac{1}{n} = 1 - \left(1 - O\left(\frac{1}{n^2}\right)\right) = O\left(\frac{1}{n^2}\right), \tag{20.5}$$

so the series is convergent.

Example 104 Let's consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}} \tag{20.6}$$

We have

$$a_n = (n^2+1)^{-1/2} = \frac{1}{n}\left(1 + \frac{1}{n^2}\right)^{-1/2} = \frac{1}{n}\left(1 + O\left(\frac{1}{n^2}\right)\right) = \frac{1}{n} + O\left(\frac{1}{n^3}\right) \tag{20.7}$$

so our series is the sum of the divergent $1/n$ and a convergent one $O(1/n^3)$, so it is divergent.

Theorem 87 (Limit Comparison Test) *Assume that $a_n \geq 0$ and $b_n \geq 0$ for any n and that*

$$\infty \neq \lim_{n\to\infty} \frac{a_n}{b_n} = L > 0. \tag{20.8}$$

Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Proof. Due to the General Comparison Test, it is enough to show that at the same time we have both $a_n = O(b_n)(n \to \infty)$ and $b_n = O(a_n)(n \to \infty)$.

The assumption $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = L + o(1)(n \to \infty)$, which is $a_n = Lb_n + o(b_n)$. Since o is also O, we have $a_n = Lb_n + o(b_n) = O(b_n) + O(b_n) = O(b_n)$.

In the same manner, the equation $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$ implies $b_n = O(a_n)$.

Example 105 Let's check again that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges. Let's compare it to the series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have,

$$\frac{1/\sqrt{n^2+1}}{1/n} = \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1(n \to \infty) \tag{20.9}$$

Since the limit is non-zero, both of the series diverge.

Generally, if evaluating the limit does not require advanced technologies like Taylor's expansion or l'Hôpital's Rule, then applying the Limit Comparison Test might be a bit easier then the General one.

Lecture 21

Ration and Root Tests. Tests for Conditional Convergence.

21.1 Ratio and Root Tests

The following two statements are quite similar.

Theorem 88 (Ratio Test) *Suppose that all $a_n \geq 0$ and there is*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c. \tag{21.1}$$

If $c > 1$, then the series $\sum a_n$ diverges. If $c < 1$, then the series $\sum a_n$ converges.

Proof. First, let $c < 1$. Then there is $\varepsilon > 0$ such that

$$0 < c - \varepsilon < c < c + \varepsilon < 1 \tag{21.2}$$

Further, for any number $n > M$ we have

$$c - \varepsilon < \frac{a_{n+1}}{a_n} < c + \varepsilon \tag{21.3}$$

Taking the product of two such inequalities, we get

$$(c-\varepsilon)^2 < \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} < (c+\varepsilon)^2 \tag{21.4}$$

Similarly, taking the product of m such inequalities, we get

$$(c-\varepsilon)^m < \frac{a_{n+m}}{a_n} < (c+\varepsilon)^m \tag{21.5}$$

In particular, for any $m > 0$, we have

$$a_{M+1}(c-\varepsilon)^m < a_{M+m+1} < a_{M+1}(c+\varepsilon)^m \tag{21.6}$$

In other words, $a_n = O(c+\varepsilon)^n (n \to \infty)$, so the series $\sum a_n$ converges.

Similarly, if $c > 1$, then we find $\varepsilon > 0$ such that $1 < c - \varepsilon$ and

$$(c-\varepsilon)^n = O(a_n)(n \to \infty) \tag{21.7}$$

so the original series is divergent. $\ddot\smile$

The following statement is similar, so we don't prove it:

Theorem 89 (Root Test) *Suppose that all $a_n \geq 0$ and there is*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = c. \tag{21.8}$$

If $c > 1$, then the series $\sum a_n$ diverges. If $c < 1$, then the series $\sum a_n$ converges.

Example 106 Consider the series

$$\sum_{n=1}^{\infty} \frac{100^n}{n!} \tag{21.9}$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{100^{n+1} n!}{100^n (n+1)!} = \frac{100}{n+1} \to 0, \tag{21.10}$$

so the series is convergent.

Example 107 Consider the series

$$\sum_{n=0}^{\infty} \frac{n}{2^n} \tag{21.11}$$

Then

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n}}{2} \to \frac{1}{2}, \tag{21.12}$$

so the series is convergent.

Sometimes, the limit from the root test does not exist, but the series is obviously convergent. For example, if we add some zeros to a convergent series like:

$$\left\{0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \cdots\right\}, \tag{21.13}$$

then we obtain a convergent series as well, but the original root test cannot be applied, since the sequence $\{\sqrt[n]{a_n}\}$ is now the following:

$$\left\{\sqrt[1]{0}, \frac{1}{\sqrt{2}}, \sqrt[3]{0}, \frac{1}{\sqrt[4]{4}}, \sqrt[5]{0}, \frac{1}{\sqrt[6]{8}}, 0, \cdots\right\} = \left\{0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots\right\}, \tag{21.14}$$

so it has no limit.

Fortunately, the same argumentation as in the original root test can prove the more general

Theorem 90 (Upgraded Root Test) *Assume that $a_n \geq 0$ for any n.*

- *If there exist $c < 1$ and $M > 0$ such that*

$$\sqrt[n]{a_n} \leq c \tag{21.15}$$

  for any $n > M$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

- *If there exist $c > 1$ and $M > 0$ such that*

$$\sqrt[n]{a_n} \geq c \tag{21.16}$$

  for any $n > M$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 108 Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{3 + (-1)^n}{5}\right)^n \tag{21.17}$$

Then

$$\sqrt[n]{a_n} = \frac{3 + (-1)^n}{5} \leq \frac{4}{5} < 1, \tag{21.18}$$

so the series is convergent.

21.2 Absolute and conditional convergence

- A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values
$$\sum_{n=1}^{\infty} |a_n| \tag{21.19}$$
is convergent.
- If $\sum a_n$ is convergent, but $\sum |a_n|$ is not, then the series $\sum a_n$ is called *conditionally convergent*.

Theorem 91 *If a series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, then it converges.*

Proof. We are given that $\sum_{n=1}^{\infty} |a_n|$ converges.

First of all, let's notice that $0 \le a_n + |a_n| \le 2|a_n|$. Therefore, by the Comparison Test, the series
$$\sum_{n=1}^{\infty} (a_n + |a_n|) \tag{21.20}$$
also converges. Finally,
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|, \tag{21.21}$$
so it also converges as the difference of convergent series $\ddot\smile$

21.3 Dirichlet's Test

We've been considering tests for series of non-negative terms or, in other words, for absolute convergence. How about conditional convergence? Does this notion make sense? Is there any conditionally convergent series?

Example 109 Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{21.22}$$

Then

$$S_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = \sum_{i=1}^{n} \frac{1}{2i(2i-1)}, \tag{21.23}$$

which has a limit since

$$\frac{1}{2n(2n-1)} = O\left(\frac{1}{n^2}\right) \tag{21.24}$$

Thus $\lim S_{2n} = L$. Obviously, $S_{2n+1} = S_{2n} + 1/(2n+1)$, so $\lim S_{2n+1} = L$ as well. Therefore the original series conditionally converges to L.

The most important test to detect convergence that can be only conditional is

Theorem 92 (Dirichlet's Test) *Given two sequences $\{a_n\}$ and $\{b_n\}$, suppose that*

1. *Partial sums $\sum_{k=1}^{n} a_k$ are bounded,*
2. *$\{b_n\}$ is decreasing,*
3. $\lim_{n\to\infty} b_n = 0$

Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Example 110 Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \tag{21.25}$$

Put

$$a_n = (-1)^n \qquad \text{and} \qquad b_n = \frac{1}{\sqrt{n}} \tag{21.26}$$

Then a partial sum $\displaystyle\sum_{k=1}^{n} a_k$ is 0 for even n and -1 for odd n, so

$$-1 \leq \sum_{i=1}^{n} a_n \leq 0, \tag{21.27}$$

and the sequence $\{b_n\}$ is decreasing and tending to 0. By the Dirichlet Test, the series is convergent.

Example 111 Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n} \tag{21.28}$$

First of all, let's try to estimate partial sums $\displaystyle\sum_{k=1}^{n} \cos k$. We have,

$$\sum_{k=1}^{n} \cos k = \frac{1}{\sin \frac{1}{2}} \sum_{k=1}^{n} \sin \frac{1}{2} \cos k = \frac{1}{2 \sin \frac{1}{2}} \sum_{k=1}^{n} \left[\sin\left(k + \frac{1}{2}\right) - \sin\left(k - \frac{1}{2}\right) \right] \tag{21.29}$$

Now, let's look at this sum more carefully:

$$\sum_{k=1}^{n} \left[\sin\left(k + \frac{1}{2}\right) - \sin\left(k - \frac{1}{2}\right) \right] = - \sin \frac{1}{2} + \sin \frac{3}{2} - \sin \frac{3}{2} + \sin \frac{5}{2} - \cdots \tag{21.30}$$

so everything except the first and the last terms cancel. Thus,

$$\sum_{k=1}^{n} \cos k = \frac{- \sin \frac{1}{2} + \sin\left(n + \frac{1}{2}\right)}{2 \sin \frac{1}{2}}, \tag{21.31}$$

which is obviously bounded. Further, $b_n = \dfrac{1}{n}$ decreases and has limit 0. Therefore, the series converges by Dirichlet's Test.

21.4 Alternating Series Test and Abel's Test

The following theorem is just a particular case of more general Dirichlet's Test:

Theorem 93 (Alternating Series Test) *Given a sequence* $\{b_n\}$*, assume that*

1. $\{b_n\}$ *is decreasing,*
2. $\lim_{n\to\infty} b_n = 0.$

Then the series $\sum_{n=1}^{\infty}(-1)^n b_n$ *converges.*

Usually, the following theorem is not useful, but sometimes it can simplify things a bit:

Theorem 94 (Abel's Test) *Given two sequences* $\{a_n\}$ *and* $\{b_n\}$*, suppose that*

1. *The series* $\sum_{n=1}^{\infty} a_n$ *converges*
2. $\{b_n\}$ *is monotonic,*
3. $\lim_{n\to\infty} b_n = L$ *is some finite number.*

Then the series $\sum_{n=1}^{\infty} a_n b_n$ *converges.*

Example 112 Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{\sqrt{n}} \tag{21.32}$$

Put

$$a_n = \frac{(-1)^n}{\sqrt{n}} \quad \text{and} \quad b_n = \arctan n \tag{21.33}$$

Then the series $\sum a_n$ is convergent and the sequence $\{b_n\}$ is growing and has finite limit $\pi/2$, so the original series is convergent by the Abel Test.

Example 113 Let's consider the same series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{\sqrt{n}} \tag{21.34}$$

again and try to do it using the following Taylor expansion for arctan:

$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + o\left(\frac{1}{x}\right) (x \to \infty) \tag{21.35}$$

Thus

$$\frac{(-1)^n \arctan n}{\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}} \left(\frac{\pi}{2} - \frac{1}{n} + o\left(\frac{1}{n}\right)\right) = \frac{\pi(-1)^n}{2\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right), \tag{21.36}$$

which is convergent as a sum of (conditionally) convergent and (absolutely) convergent.

Example 114 Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan \sqrt[37]{n}}{\sqrt[20]{n}} \tag{21.37}$$

Just like in the second example, one can do it by Abel's and Dirichlet's Tests together, but Taylor's expansion is already hardly applicable.

21.5 General strategy

Asked to define whether a series $\sum_{n=1}^{\infty} a_n$ absolutely converges (usually it just means that all $a_n \geq 0$), you

Limit Test, Taylor's Expansion: Do the Limit Test. If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent and you stop. How do you calculate the limit? If a_n is given by an elementary formula, just find the Taylor expansion for a_n. Usually, you will get something like

$$a_n = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \tag{21.38}$$

and diagnose divergence or

$$a_n = O\left(\frac{1}{n\sqrt{n}}\right) \tag{21.39}$$

and you report convergence by the Comparison Test.

Ratio Test, Root Test, Integral Test: Try the Ratio Test, the Root Test, or the Integral Test if they are easy to apply. Usually, a_n convenient for the Ratio Test contains long products like

$$a_n = \left(\frac{1}{1} - \frac{1}{2}\right) \cdot \left(\frac{1}{2} - \frac{1}{3}\right) \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right) \tag{21.40}$$

Similarly, a_n good for the Root Test is something with nth power, for example

$$a_n = \left(1 + \frac{1}{n}\right)^{-n^2} \tag{21.41}$$

Something good for the Integral Test is a function you know how to integrate. However, here you must remember that the Integral Test is only for monotonic functions, so

$$f(x) = \frac{\sin \ln x}{x} \tag{21.42}$$

cannot be done by the Integral Test.

Comparison Test: Try to apply the Comparison Test by estimating a_n. Usually, here the inequality

$$n! < n^n \tag{21.43}$$

might be useful (if you're on this stage, it means that you failed to find the Taylor Expansion, so a_n contains some non-elementary formulas like factorials).

If you are supposed to test a series for conditional convergence,

Limit Test, Taylor's Expansion: First, you do the same as for absolute convergence. Try to find the Taylor expansion for the both a_n and $|a_n|$. Usually, you will get something like

$$a_n = \frac{(-1)^n}{n} + O\left(\frac{1}{n^2}\right), \qquad |a_n| = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \tag{21.44}$$

and report conditional convergence by the Dirichlet Test.

Dirichlet's and Abel's Tests: In most interesting cases you'll have to think how your a_n can be simplified for the Dirichlet and Abel Tests. Here it's good to remember that partial sums

$$\sum_{k=1}^{n} \sin kc, \quad \sum_{k=1}^{n} \cos kc, \quad \sum_{k=1}^{n} (-1)^k c \tag{21.45}$$

are bounded for any constant c

Lecture 22

Power Series

22.1 Series of functions

Suppose $u_1(x), u_2(x), u_3(x), \dots$ are functions of a variable x. Thus the *series of functions*

$$u(x) = \sum_{n=1}^{\infty} u_n(x) \tag{22.1}$$

is a function of x, its domain is the set of x such that the series $\sum_{n=1}^{\infty} u_n(x)$ converges.

Let's consider several examples.

Exponential function: The series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \tag{22.2}$$

converges for any x, which can be easily checked by the Ratio Test. In fact, the sum of this series is just the exponential function $\exp(x) = e^x$.

Riemann's ζ-function: The series

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{22.3}$$

converges for any $x > 1$.

Fourier series: Given some coefficients $\{a_n, b_n\}$, one can consider a series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{22.4}$$

Sometimes, the domain of this series can be found by the Dirichlet Test

22.2 Power series

Among all functions, probably the most convenient to work with are polynomials, i.e., expressions like

$$1 + x^2,\ 13 - x - x^3,\ 2 + \sqrt{2}x^5, \ldots \tag{22.5}$$

Unfortunately, there are too few of them. For example, $\sin'' x = -\sin x$, but there does not exist any polynomial $f(x) \neq 0$ satisfying $f''(x) = -f(x)$. Indeed, the differentiation decreases the degree of the polynomial, so $\deg f'' = \deg f - 2$.

Let's check if it's possible to define something similar to a polynomial, but possessing the property $f'' = -f$. A polynomial is a finite sum $f(x) = \sum_{n=0}^{k} c_n x^n$. Let's replace this sum with an infinite one $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and try to find coefficients c_n such that $f''(x) = f(x)$ in some natural sense. We have,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n, \\ f'(x) &= \sum_{n=1}^{\infty} n c_n x^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \end{aligned} \tag{22.6}$$

Two polynomial are equal if and only if their coefficients are equal. Lets try to find c_n such that the coefficients of $-f$ are equal to the coefficients of f'':

$$-c_{n-2} = n(n-1)c_n, \tag{22.7}$$

which gives us a way to calculate all c_n if c_0 and c_1 are given:

$$\begin{aligned} c_2 &= c_0/(2 \cdot 1), \\ c_3 &= c_1/(3 \cdot 2), \\ c_4 &= c_2/(4 \cdot 3), \\ c_5 &= c_3/(5 \cdot 4), \\ &\vdots \end{aligned} \tag{22.8}$$

Thus it's really possible to define an infinite polynomial similar to the sine.

A *power series* in x is a series of functions

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_0 + \sum_{n=1}^{\infty} c_n x^n \tag{22.9}$$

where $c_0, c_1, c_2, \cdots \in \mathbb{R}$ are some constants and x is a variable.

Example 115 The sum of the geometric progression:

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \tag{22.10}$$

The domain of this series is already known to be the open interval $(-1, 1)$.

Lecture 23

Functions Defined by Power Series

23.1 Radius and interval of convergence

Let's think about a question: given a power series, what is its domain? Where does it converge? Consider some

Example 116 For the geometric progression $1 + x + x^2 + x^3 + \cdots$, the domain is the interval $(-1, 1)$.

Example 117 A polynomial $\sum_{n=0}^{k} c_n x^n$ is a series such that $c_n = 0$ for $n > k$. The domain is the set of real numbers, i.e., the infinite interval $(-\infty, \infty)$.

Example 118 Consider the series $\sum_{n=1}^{\infty} n!x^n$. If $x \neq 0$, then $n!$ grows faster than any exponential x^n decreases, so $\lim n!x^n = \infty$ and the series diverges. Thus the domain is just the point 0, which can be considered as the interval $[0, 0]$.

We see that the domain is an interval in any case. Let's prove it carefully.

Lemma 95 *Suppose the series $f(x) = \sum c_n x^n$ is convergent for some $x_1 \in \mathbb{R}$. Then the series is absolutely convergent for any $x \in (-|x_1|, |x_1|)$.*

Proof. We are given that $\sum c_n x_1^n$ is convergent. Therefore the sequence $\{c_n x_1^n\}$ converges (to 0) and, particularly, bounded, so $|c_n x_1^n| < M$ for any n. Assume now that $0 < |x| < |x_1|$. We have

$$|c_n x^n| = \left| c_n x_1^n \left(\frac{x}{x_1} \right)^n \right| < M \left| \frac{x}{x_1} \right|^n . \tag{23.1}$$

Thus $\sum |c_n x^n|$ is convergent by the comparison test, since the series $\sum |x'/x_1|^n$ is convergent whenever $|x'/x| < 1$. ⌣̈

Theorem 96 *Given a power series $f(x) = \sum c_n x^n$, one and only one of the following conditions is fulfilled*

1. *The series is convergent for any $x \in \mathbb{R}$.*
2. *The series is convergent only for $x = 0$.*
3. *There exists a real number $R > 0$ such that the series is convergent for any $x \in (-R, R)$ and divergent for any $x \notin [-R, +R]$.*

Proof. Suppose the opposite: the series satisfies none of these conditions. Therefore, there exists a couple of numbers $0 \neq x_1, x_2 \in \mathbb{R}$ such that

1. $f(x_1)$ is divergent, so 1st condition fails,
2. $f(x_2)$ is convergent, so 2nd condition fails,
3. $0 < |x_1| < |x_2|$, so 3rd condition fails.

Obviously, it's a contradiction to lemma 95. ⌣̈

Given a power series $f(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n$, its domain I is called *the interval of convergence*. By theorem 96, the only possibilities for the set I are

1. $I = \mathbb{R}$, in this case put $R = +\infty$,
2. $I = \{0\}$, in this case put $R = 0$,
3. $0 < R < +\infty$:
   (a) $I = (-R, R)$,
   (b) $I = (-R, R]$,
   (c) $I = [-R, R)$,
   (d) $I = [-R, R]$.

Anyway, $R \in [0, +\infty]$ is called *the radius of convergence*.

23.2 How to calculate the radius of convergence

Theorem 97 (Root and Ratio Rules) *Given a power series $f(x) = \sum c_n x^n$, its radius of convergence R is defined by*

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|c_n|}}, \tag{23.2}$$

where the value R can be either a non-negative real number, or infinity, or by

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|, \tag{23.3}$$

if these limits exist.

Proof. We prove only the first expression, the second one is similar.

Actually, we need to show that the series $f(x)$ is convergent for $|x| < R$ and divergent for $|x| > R$.

Put $a_n = |c_n x^n|$. Let's apply the Root Test to the series $\sum a_n$. Obviously,

$$\lim \sqrt[n]{|a_n|} = \lim \sqrt[n]{|c_n|} \cdot |x| = \lim \sqrt[n]{|c_n|} \cdot R \cdot \frac{|x|}{R} = \frac{|x|}{R}, \tag{23.4}$$

so the series is convergent when $|x|/R < 1$ and divergent when $|x|/R > 1$. ⌣̈

However, it might happen that $\lim \sqrt[n]{c_n}$ and $\lim(c_n/c_{n+1})$ do not exist. For example, the series

$$f(x) = 1 + \sum_{n=1}^{\infty} x^{2n} \tag{23.5}$$

have the even-indexed coefficients to be all ones and the odd-indexed to be all zeros, so the sequence of $\sqrt[n]{c_n}$ looks like

$$\{1, 0, 1, 0, 1, 0, 1, \dots\}, \tag{23.6}$$

which has no limit. At the same time, $f(x) = g(x^2)$, where

$$g(x) = 1 + \sum_{n=1}^{\infty} x^n, \tag{23.7}$$

so $f(x)$ converges whenever $g(x^2)$ exists that is $x^2 \in (-1, 1)$, so $x \in (-1, 1)$. Thus the radius of convergence of the series $f(x)$ is 1. The Upgraded Root Test can be applied to prove the general formula for the radius of convergence. First, let's state a useful definition

Given a sequence $\{a_n\}$, its *upper limit* (or *limit superior*)

$$\limsup_{n\to\infty} a_n \tag{23.8}$$

is the maximal possible limit of all its subsequences. If there are no convergent subsequences or if there are subsequences of infinite limit, we put

$$\limsup_{n\to\infty} a_n = +\infty \tag{23.9}$$

Theorem 98 (Upgraded Root Rule) *Given a power series $f(x) = \sum c_n x^n$, its radius of convergence R is defined by*

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}}, \tag{23.10}$$

Example 119 Consider a polynomial

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n. \tag{23.11}$$

Actually, it's a power series with $c_k = 0$ for $k > n$. Obviously, $\lim \sqrt[n]{|c_n|} = 0$, so $R = \infty$, which is natural, because a polynomial $f(x)$ is defined for any value $x \in \mathbb{R}$.

Example 120 Consider the series

$$1 + \frac{x^3}{3} + \frac{x^6}{6} + \cdots + \frac{x^{3n}}{3n} + \cdots \tag{23.12}$$

We have $c_n = 1/n$ for $n = 3k$ and $c_n = 0$ otherwise, so the sequence $\{\sqrt[n]{c_n}\}$ looks like

$$1, 0, 0, \frac{1}{\sqrt[3]{3}}, 0, 0, \frac{1}{\sqrt[6]{6}}, 0, 0, \frac{1}{\sqrt[9]{9}}, \cdots \tag{23.13}$$

The limit of a subsequence can be either 0 for a subsequence consisting of zeros or 1 for the subsequence

$$a_n = c_{3n} = 1/\sqrt[3n]{3n} \tag{23.14}$$

Thus the radius of convergence is 1.

Let's determine the interval of convergence. If $x = 1$, our series is $\sum 1/3n$, which is divergent. If $x = -1$, our series turns out to be $\sum(-1)^n/3n$, which is conditionally convergent by the alternating series test. We obtain that the interval of convergence is $[-1, 1)$.

23.3 Continuity, differentiability and integrability

Here, we have to act without proofs because they involve a notion of uniform convergence, which lies beyond our course. First, notice that given a power series

$$c_0 + \sum_{n=1}^{\infty} c_n x^n, \tag{23.15}$$

its formal derivative

$$\sum_{n=1}^{\infty} nc_n x^{n-1} = c_1 + \sum_{n=1}^{\infty} c_{n+1}(n+1)x^n, \tag{23.16}$$

has the same radius of convergence since $\lim\limits_{n\to\infty} \sqrt[n]{n+1} = 1$. This is not accidental:

Theorem 99 *For any sequence $\{c_n\}$, the function*

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n \tag{23.17}$$

satisfies the following properties:

1. *It is continuous on the interval of convergence.*
2. *It is differentiable on the open interval $(-R, R)$ (where R is the radius of convergence) and*

$$f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} = c_1 + \sum_{n=1}^{\infty} c_{n+1}(n+1)x^n \tag{23.18}$$

3. Its anti-derivative is

$$F(x) = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} x^n \tag{23.19}$$

Example 121 Consider the series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \tag{23.20}$$

We have

$$f'(x) = \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n!} = 1 + \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = f(x) \tag{23.21}$$

Therefore, $f = Ce^x$ for some constant C. Substituting zero, we have

$$1 = Ce^0, \tag{23.22}$$

so $C = 1$ and $f = e^x$.

Example 122 Consider the series

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} 2nx^{2n-1} = 2x - 4x^3 + 6x^5 - 8x^7 + \cdots \tag{23.23}$$

Integrating it term-by-term, we obtain

$$\begin{aligned} \int_0^x f(t)dt &= \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n} = x^2 - x^4 + x^6 - x^8 + \cdots \\ &= x^2(1 - x^2 + x^4 - x^6 + \cdots) = \frac{x^2}{1+x^2} \end{aligned} \tag{23.24}$$

since it's just the sum of the geometric series. Therefore,

$$f(x) = \frac{d}{dx} \int_0^x f(t)dt = \frac{2x}{(1+x^2)^2} \tag{23.25}$$

Part V

Appendix A: Taylor's Formula

23.4 First order Taylor's formula

First, recall that a function $f(x)$ is differentiable at the point 0 if

$$f(x) = f(0) + Dx + o(x)(x \to 0), \tag{23.26}$$

where $D = f'(0)$ is called the derivative of the function $f(x)$ at the point $x = 0$. Moving $f(0)$ to the right part of this equation and dividing by x, we obtain

$$\frac{f(x) - f(0)}{x} = f'(0) + o(1), \tag{23.27}$$

which means

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}. \tag{23.28}$$

Given a function $f \in D(a)$, its *first-order Taylor's formula* is

$$f(a + h) = f(a) + f'(a)h + o(h)(h \to 0) \tag{23.29}$$

Here, the linear polynomial $D(h) = f'(a)h$ called the *differential*.

Example 123 $(1 + x)^{\alpha} = 1 + \alpha x + o(x)(x \to 0)$,

Example 124 $e^x = 1 + x + o(x)(x \to 0)$,

Example 125 $\ln(1 + x) = x + o(x)(x \to 0)$,

Example 126 $\sin x = x + o(x)(x \to 0)$,

Example 127 $\tan x = x + o(x)(x \to 0)$,

Example 128 $\cos x = 1 + o(x)(x \to 0)$.

23.5 Second order Taylor's formula

Sometimes, first-order Taylor's formula is not enough, for example:

$$\lim_{x\to 0}\frac{1-\cos x}{\ln(1+x^2)} = \lim_{x\to 0}\frac{o(x)}{o(x^2)} = ? \tag{23.30}$$

In a similar manner to the first-order one, a second-order Taylor's formula at $x = a$ is an equation

$$f(a+h) = a + bh + ch^2 + o(h^2)(h \to 0) \tag{23.31}$$

Example 129 Let's construct a second-order Taylor's formula for the cosine, that is,

$$\cos x = 1 + cx^2 + o(x^2) \tag{23.32}$$

This equality is the same as

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} = c \tag{23.33}$$

Calculating it using l'Hospital's Rule, we obtain $c = -\frac{1}{2}$

Generally, if in the equation $f(x) = a + bx + cx^2 + o(x^2)(x \to 0)$, we substitute $x^2 = o(x)$, we get $f(x) = a + bx + o(x)$, which means $a = f(0)$ and $b = f'(0)$. Thus we obtain $f(x) = f(0) + f'(0)x + cx^2 + o(x^2)(x \to 0)$. Dividing by x^2, we get

$$c = \lim_{x\to 0}\frac{f(x) - f(0) - f'(0)x}{x^2} \tag{23.34}$$

23.6 General Taylor's formula

Fortunately, it's possible to construct the nth order Taylor formula. In fact, if a function f is differentiable n times on some interval containing a point a, then there is a polynomial T_n such that

$$f(a+h) = T_n(h) + o(h^n)(h \to 0) \tag{23.35}$$

Here, $T_1(h) = f(a) + f'(a)h$. Since the coefficients of the first-order Taylor polynomial are expressed in terms of $f(a)$, $f'(a)$, it's natural to seek $T_n(h)$ as a polynomial of degree n in h with coefficients expressed in terms of $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$.

First, let's consider the case when f is a polynomial. For example, if

$$f(x) = 2x^3 - 5x^2 + 2x - 3, \tag{23.36}$$

then $-3 = f(0)$, $2 = f'(0)$, $-5 = f''(0)/2$, $2 = f'''(0)/6$.

Generally,

Lemma 100 *Suppose*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{23.37}$$

Then for any k we have

$$a_k = \frac{f^{(k)}(0)}{k!} \tag{23.38}$$

Proof. Differentiating k times the polynomial f cancels all the terms of degree $< k$, so

$$f^{(k)}(x) = k \cdot (k-1) \cdots 1 a_k + (k+1) \cdot k \cdots 2 a_{k+1} x + \cdots \tag{23.39}$$

Therefore,

$$f^{(k)}(0) = k! a_k, \tag{23.40}$$

which is the original statement. $\ddot\smile$

Corollary 101 *For any polynomial $f(x)$ of degree n, we have*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{23.41}$$

Therefore it's natural for any function $f(x)$ to try

$$T_n(h) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} h^k \tag{23.42}$$

as a Taylor polynomial.

Lemma 102 *Let f be a function n times differentiable on some open interval $I \ni a$. Suppose*

$$T(h) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} h^k. \tag{23.43}$$

Then

$$T^{(i)}(0) = f^{(i)}(a) \tag{23.44}$$

for any $i = 0, 1, \ldots, n$.

Proof. Differentiating a polynomial k times just like in the previous lemma, we obtain the statement. $\ddot\smile$

Now we may try to prove the following general fact

Theorem 103 *Suppose that a function f is n times differentiable on an interval $(a-\delta, a+\delta)$ for some $\delta > 0$ and suppose that the nth derivative $f^{(n)}$ is continuous at the point a. Then*

$$f(a+h) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}h^k + o(h^n)(h \to 0) \tag{23.45}$$

Proof. By the definition of $o(h^n)$, we need to prove that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}h^k}{h^n} = 0 \tag{23.46}$$

Applying L'Hôspital's rule n times, we get

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}h^k}{h^n} = \lim_{h\to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{n!} = 0, \tag{23.47}$$

since $f^{(n)}$ was supposed to be continuous. $\ddot\smile$

Corollary 104 *Suppose that a function f is $n+1$ times differentiable on an interval $(a-\delta, a+\delta)$ for some $\delta > 0$ and suppose that the $(n+1)$th derivative $f^{(n+1)}$ is continuous at the point a. Then*

$$f(a+h) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}h^k + O(h^{n+1})(h \to 0) \tag{23.48}$$

Proof. Indeed, by the theorem about the Taylor formula,

$$f(a+h) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n+1)}(a)}{(n+1)!}h^{n+1} + o(h^{n+1})(h \to 0), \tag{23.49}$$

where $h^{n+1} = O(h^{n+1})$, $o(h^{n+1}) = O(h^{n+1})$ and thus $\frac{f^{(n+1)}(a)}{(n+1)!}h^{n+1} + o(h^{n+1}) = O(h^{n+1})$. $\ddot\smile$

It's good to memorize the following important cases of Taylor's formula:

1. $(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+O(x^{n+1})(x \to 0)$

2. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + O(x^{n+1})(x \to 0)$

3. $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n+1}}{n}x^n + O(x^{n+1})(x \to 0)$

4. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + O(x^{2n+3})(x \to 0)$

5. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + O(x^{2n+2})(x \to 0)$

23.7 Taylor's formula is not the same as Taylor's series

Let's consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{23.50}$$

Try to differentiate it a few times. Notice, that for $x \neq 0$ its derivative of any order looks like

$$f^{(k)}(x) = R_k(x)e^{-1/x^2}, \tag{23.51}$$

where $R_k(x)$ is some rational function. Taking the limit at $x = 0$, we obtain

$$\lim_{x\to 0} R_k(x)e^{-1/x^2} = \lim_{x\to\infty} R_k\left(\frac{1}{x}\right)e^{-x^2} = 0 \tag{23.52}$$

because the exponential function decreases faster than any polynomial.

Therefore we have generally

$$f^{(k)}(x) = \begin{cases} R_k(x)e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{23.53}$$

In particular, it means that $f^{(k)}(0) = 0$ for any k, which means that Taylor's series of this function is constant zero. But the function itself it not zero. On other words, the sum of this function's Taylor's series does not equal to the function itself. However, it does not contradict Taylor's formula because it just says that

$$e^{-1/x^2} = o(x^n)(x \to 0) \tag{23.54}$$

for any n.

Part VI

Appendix B: Transcendental Functions

23.8 Exponential function defined on rationales

An exponential function $f(x) = a^x$ was defined only for rational x at high school. First, for $x = m \in \mathbb{N}$, we have

$$x^m = \underbrace{x \cdot x \cdot \ldots \cdot x}_{m \text{ times}} \tag{23.55}$$

Then, obviously, we have $x^{m+n} = x^m x^n$. Further, the definition $a^{p/q} = \sqrt[q]{a^p}$ is designed to obtain a function $f : \mathbb{Q} \to \mathbb{R}$ such that

(A) $f(0) = 1$,

(B) $f(1) = a$

(C) $f(x + y) = f(x)f(y)$

Theorem 105 *The conditions (A)-(C) define $f : \mathbb{Q} \to \mathbb{R}$ uniquely.*

Proof. If $x = m \in \mathbb{N}$, we have

$$f(m) = f(\underbrace{1 + 1 + \cdots + 1}_{m \text{ times}}) = \underbrace{f(1) \cdot f(1) \cdot \ldots \cdot f(1)}_{m \text{ times}} = a^m \tag{23.56}$$

If $x = -m$, we have

$$1 = f(0) = f(m - m) = f(m)f(-m) = a^m f(-m) \Longrightarrow f(-m) = \frac{1}{a^m} \tag{23.57}$$

If $x = 1/q$, we have

$$a = f(1) = f\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \left(f\left(\frac{1}{q}\right)\right)^q \Longrightarrow f\left(\frac{1}{q}\right) = \sqrt[q]{a} \tag{23.58}$$

Finally, if $x = p/q$, we have

$$f\left(\frac{p}{q}\right) = f\left(\underbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}_{p \text{ times}}\right) = \left(f\left(\frac{1}{q}\right)\right)^p \Longrightarrow f\left(\frac{p}{q}\right) = \sqrt[q]{a^p} \tag{23.59}$$

Thus (A)-(C) imply that $f(p/q) = \sqrt[q]{a^p}$. $\smile$

Our goal is to define a function $f(x) : \mathbb{R} \to \mathbb{R}$ such that (A) and (C) hold and, besides $f'(0) = 1$. After that we put $e = f(1)$.

23.9 Rigorous definition of logarithm and exponential functions

Given a number $x > 0$, its *logarithm* is

$$\ln x = \int_1^x \frac{dt}{t} \tag{23.60}$$

The *exponential* function e^x is the inverse to the logarithm.

Let's prove some obvious properties of these functions

Theorem 106 *To emphasize that exponential and logarithm functions are inverse to each other, let's write the statement in two columns as*

The logarithm function satisfies	*The exponential function satisfies*
1. $\ln' x = \frac{1}{x} > 0$	*1.* $(e^x)' = e^x > 0$
2. $\ln 1 = 0$	*2.* $e^0 = 1$
3. $\ln(xy) = \ln x + \ln y$	*3.* $e^{x+y} = e^x \cdot e^y$

Proof.

1. From the Fundamental Theorem of Calculus, we have $\ln' x = \dfrac{1}{x} > 0$, so function is increasing. Therefore the inverse function e^x is also increasing. For it's derivative, we have $\ln(e^x) = x$, so

$$1 = [\ln(e^x)]' = \frac{(e^x)'}{e^x}. \tag{23.61}$$

   It must positive because the range of the inverse function is the domain of the original one, which in our case means that $e^x > 0$.

2. Obvious.

3. First, by substitution $u = xt$, we get
$$\int_1^y \frac{dt}{t} = \int_x^{xy} \frac{du}{u} \tag{23.62}$$
Further,
$$\ln x + \ln y = \int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{xy} \frac{dt}{t} = \ln(xy) \tag{23.63}$$
Thus the logarithm sends product to sum. Therefore the exponential function, being logarithm's inverse, sends sum to product.

☺

Theorem 107 *The domain of* $\ln x$ *is* $(0, +\infty)$*, the range is* $(-\infty, +\infty)$*. Respectively, the domain of* e^x *is* $(-\infty, +\infty)$*, the range is* $(0, +\infty)$*.*

Proof. The domain of the logarithm is given in its definition. Further, since logarithm is differentiable and therefore continuous, it is enough to prove that in can take arbitrary large positive and arbitrary large negative value.

First, let's prove that logarithm is unbounded from above. For any $N > 0$ we need to find $M > 0$ such that $\ln x > N$ for some x. Let's consider the logarithm $\ln 2^m$ for $m \in \mathbb{N}$. By Theorem 106, $\ln 2^m = m \ln 2$. If $m > N/\ln 2$, then $\ln 2^m = m \ln 2 > N$.

Second, since $\ln \dfrac{1}{x} = \ln 1 - \ln x = -\ln x$, the logarithm function is not bounded from below either.

We see that, indeed, the domain of $\ln x$ is $(0, +\infty)$, the range is $(-\infty, +\infty)$. Finally, for the inverse function e^x the domain and the range are swapped. ☺

The constant e is
$$e = e^1 \tag{23.64}$$
(it's not a tautology because at the moment the notation e^x means the inverse function to the logarithm, we don't know yet that it is, indeed, e to the power x).
The *general exponential function* is
$$a^x = e^{x \ln a} \tag{23.65}$$
where $0 < a \neq 1$. Only now we see that e^x is, indeed, e to the power x :)

Remark 108 *We can now use Taylor's formula to prove that*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{23.66}$$

Historically, this equation was the definition of Euler's number (discovered by Jacob Bernoulli :).

Exercise 1 *Prove that the domain of* a^x *is* $\mathbb{R}$*, the range is* $(0, +\infty)$.

Exercise 2 *Prove that* a^x *is growing for* $a > 1$ *and decreasing for* $a < 1$.

Exercise 3 *Prove that* a^x *is the usual power operation for rational* x.

23.10 An important differential equation

We already know the derivative of the exponential function is the function itself. It's interesting that the converse is also true!

Theorem 109 *If a differentiable function satisfies*

$$f'(x) = f(x), \tag{23.67}$$

then $f(x) = ce^x$ *for some constant* c.

Proof. Let $g(x) = f(x)e^{-x}$. Then

$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} = 0 \tag{23.68}$$

Therefore $g(x) = c$ — some constant. $\ddot\smile$

23.11 Hyperbolic functions

Trying to present e^x as the sum of an even and an odd function, we get

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}. \tag{23.69}$$

Notice that $\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$. Since the curve $u^2 - v^2 = 1$ is called the *hyperbola*, we introduce a definition

The function $\cosh x = \dfrac{e^x + e^{-x}}{2}$ is called the *hyperbolic cosine*.
The function $\sinh x = \dfrac{e^x - e^{-x}}{2}$ is called the *hyperbolic sine*.
The function $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is called the *hyperbolic tangent*.

The main hyperbolic identity is

$$\cosh^2 x - \sinh^2 x = 1, \tag{23.70}$$

which is similar to the trigonometric one $\cos^2 x + \sin^2 x = 1$. Generally, hyperbolic functions are quite similar to trigonometric functions (may be that's why there are also called sine, cosine and tangent). Compare for example:

$$\begin{array}{rcl} \cosh 2x = \cosh^2 x + \sinh^2 x & \text{and} & \cos 2x = \cos^2 x - \sin 2x \\ \sinh 2x = 2\cosh x \sinh x & \text{and} & \sin 2x = 2\cos x \sin x \end{array} \tag{23.71}$$

It's easy to check that these functions are defined, continuous, and differentiable for any real number x. For their derivatives, we have

$$\cosh' x = \sinh x, \qquad \sinh' x = \cosh x, \qquad \tanh' x = \frac{1}{\cosh^2 x} \tag{23.72}$$

Finally, the range of $\cosh x$ is $[1, +\infty)$, the range of $\sinh x$ is $\mathbb{R}$, the range of $\tanh x$ is $(-1, 1)$.

23.12 Inverse hyperbolic functions

For the hyperbolic sine, we have $\sinh' x = \cosh x > 0$. Therefore it monotonically increasing on the whole real line. Since its image is $\mathbb{R}$, we see that there exists the inverse function

$$\sinh^{-1} y : \mathbb{R} \to \mathbb{R}, \tag{23.73}$$

which is also increasing on the whole real line.

Recall that, generally, given a function $y = f(x)$, we have $x = f^{-1}(y)$. Differentiating this equality by the chain rule, we get

$$\left(f^{-1}\right)'(y) \cdot f'(x) = 1 \tag{23.74}$$

Exercise 4 *Prove that*

$$\left(\sinh^{-1}\right)'(y) = \frac{1}{\sqrt{y^2+1}} \tag{23.75}$$

Exercise 5 *Observe that the equality* $\sinh^{-1} y = x$ *means that* $y = \dfrac{e^x - e^{-x}}{2}$*, which is the same as*

$$e^{2x} - 2ye^x - 1 = 0 \tag{23.76}$$

Find an elementary formula for $\sinh^{-1} y$ *from this equation.*

Exercise 6 *Using the substitution* $x = \sinh t$ *to find an elementary formula for the anti-derivative* $\displaystyle\int \sqrt{x^2+1}\,dx$.

Exercise 7 *Define the inverse functions* $\cosh^{-1}$ *and* $\tanh^{-1}$*. Find their domains and ranges. Differentiate them. Find elementary formulae for* $\cosh^{-1}$ *and* $\tanh^{-1}$*. Think how one can use substitutions* $x = \cosh t$ *and* $x = \tanh t$ *in integrals.*