Low-Rank Matrix Completion via Deep Neural Networks

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Northeastern University

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Matrix Completion Problem Statement

- Given a partially observed matrix $M \in \mathbb{R}^{m\times n}$, where only a subset of entries $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ are known, the goal is to recover the full matrix by estimating the missing entries.
- This is typically done under the assumption that the true underlying matrix is *low-rank*, i.e.,

$$\mathrm{rank}(M) \ll \min(m, n).$$

Our Problem

- In this project, we restrict our attention to square matrices $M \in \mathbb{R}^{n\times n}$.
- Number of independent matrix elements in a square matrix of size n and rank r is $2nr - r^2$.
- We mask $m < n^2 - 2nr + r^2$ entries from the matrix and have the neural network predict them.

Examples

2 × 2 example (rank 1):

$$\begin{pmatrix} 0.6 & -0.2 \\ 1.2 & * \end{pmatrix} \to \begin{pmatrix} 0.6 & -0.2 \\ 1.2 & -0.4 \end{pmatrix}$$

4 × 4 example (rank 2):

$$\begin{pmatrix} -0.303 & 0.388 & * & -0.64 \\ -4.413 & -2.146 & -0.12 & 0.776 \\ * & 0.362 & 1.1 & -0.307 \\ 1.375 & 0.96 & 1.717 & * \end{pmatrix} \to \begin{pmatrix} -0.303 & 0.388 & 3.082 & -0.64 \\ -4.413 & -2.146 & -0.12 & 0.776 \\ 0.356 & 0.362 & 1.1 & -0.307 \\ 1.375 & 0.96 & 1.717 & -0.619 \end{pmatrix}$$

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We generate random matrices A of size (n, n) and rank r as follows:

- Generate two matrices U and V of sizes (n, r)
- $A = UV^T$
- The matrix A will have rank r.

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Proof.

Left as an exercise to the reader. $\square$

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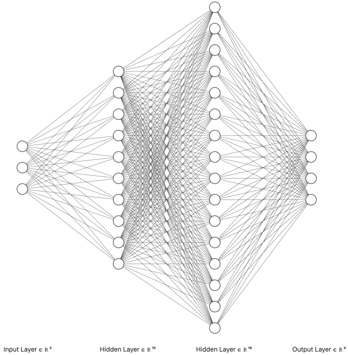
Architecture: Feed-Forward Neural Network, with two hidden layers and ReLU activations.[1]


Input Layer $\in \mathbb{R}^3$
Hidden Layer $\in \mathbb{R}^{10}$
Hidden Layer $\in \mathbb{R}^{16}$
Output Layer $\in \mathbb{R}^4$

$$\text{Proposed number of neurons} = \mathcal{O}(n^2)$$

[1] Addition of the second hidden layer significantly improved the model.

Model Summary:

- ADAM optimizer with initial learning rate $\alpha = 0.001$
- Training epochs = 10,000
- Mini-batch Gradient Descent with batch size = 64
- 90-10 Data split (training/testing)

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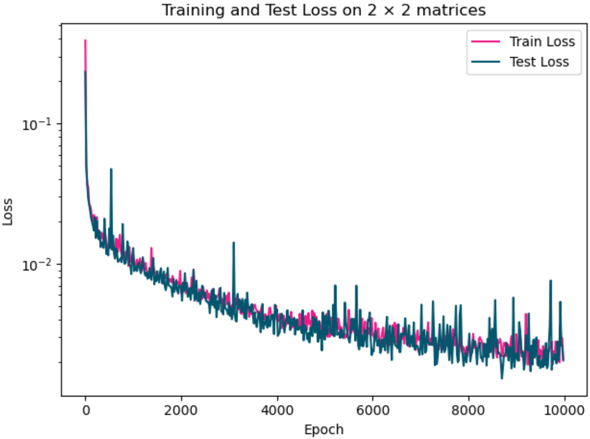
Performance on 2 × 2 matrices



Performance on 2×2 matrices

Input:

$$\begin{pmatrix} -0.0233782 & -0.0575182 \\ -0.12784362 & * \end{pmatrix}$$

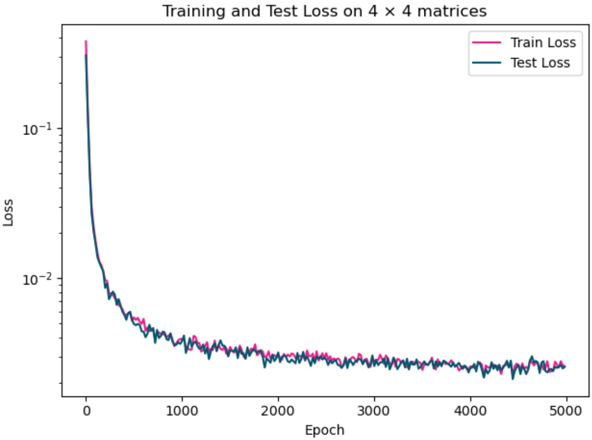
Ground truth:

$$\begin{pmatrix} -0.0233782 & -0.0575182 \\ -0.12784362 & -0.31453815 \end{pmatrix}$$

Prediction:

$$\begin{pmatrix} -0.02139766 & -0.05180938 \\ -0.13365555 & -0.30243173 \end{pmatrix}$$

Performance on 4×4 matrices



Performance on 4 × 4 matrices

Input:

$$\begin{pmatrix} -0.784 & -0.07 & -0.167 & 0.333 \\ -1.185 & 0.579 & 0.841 & * \\ * & * & -2.152 & -0.031 \\ 1.413 & -0.628 & -0.904 & -0.565 \end{pmatrix}$$

Ground truth:

$$\begin{pmatrix} -0.784 & -0.07 & -0.167 & 0.333 \\ -1.185 & 0.579 & 0.841 & 0.472 \\ 0.221 & -1.358 & -2.152 & -0.031 \\ 1.413 & -0.628 & -0.904 & -0.565 \end{pmatrix}$$

Prediction:

$$\begin{pmatrix} -0.766 & -0.074 & -0.143 & 0.313 \\ -1.187 & 0.603 & 0.854 & 0.388 \\ 1.625 & -0.602 & -2.103 & -0.028 \\ 1.378 & -0.627 & -0.876 & -0.572 \end{pmatrix}$$

Performance on 4 × 4 matrices

But how good are these predictions?

Analysis of 2×2 Results

Recall that

$$\begin{aligned} M &= UV^T && (1) \\ &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix} && (2) \end{aligned}$$

Since $u_2 \sim \mathcal{N}(0,1)$ and $v_2 \sim \mathcal{N}(0,1)$, the distribution of the element $u_2 v_2$ follows

$$P_Z = \frac{1}{\pi} K_0(|Z|) \qquad (3)$$

where K_0 is the modified Bessel function of the second kind of order zero, **which also has variance = 1**.

Distribution of elements in $n \times n$ matrix (rank r)

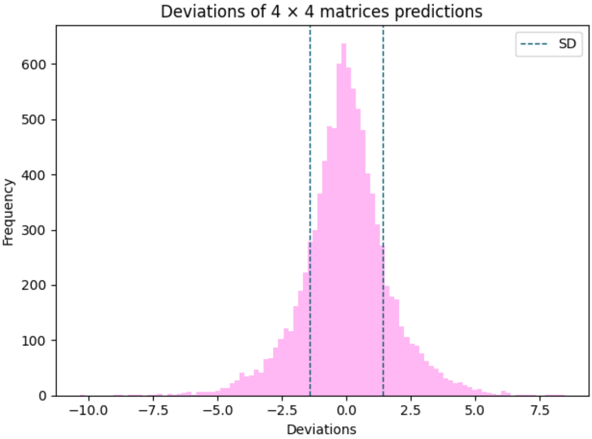
For a general $n \times n$ matrix of rank r, we have

$$M_{ij} = \sum_{k=1}^{r} U_{ik} V_{jk} \tag{4}$$

Since U_{ik} and $V_{jk} \sim \mathcal{N}(0,1)$, M_{ij} is a sum of r independent random variables with variance 1, therefore

$$\mathrm{Var}[P(M_{ij})] = r \tag{5}$$

Distribution of elements in $n \times n$ matrix



68.4 % lie within 1 σ from the mean![2]

[2] Thanks Aria for the idea!

Going forward..

- General $n \times m$ matrix completion problem
- Varying the masking with iterations
- Varying input size

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Thank You!

Questions?