HW4

Problem 1:

System: $\dot{X} = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ I_n \end{bmatrix} u$

stabilize to origin $\leftarrow 0$, $(q - q_d)$ $\leftarrow$ $\dot{q} - \dot{q}_d \rightarrow 0$ set Point tracking

The feedback controller $u = -K_P q - K_D \dot{q} = -[K_P \; K_D] X$

This is a set Point tracking PD controller where the desired goal is the origin.

The State vector $X = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$ where n is the number of the generalized coordinates.

So in the general case

$$\dot{X} = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ I_n \end{bmatrix} [-K_P, -K_D] X$$

$$\therefore (A - BK) X = \begin{bmatrix} 0 & I \\ -K_P & -K_D \end{bmatrix} X$$

For the system, to be stable $\rightarrow$ to stabilize to the origin in this case. $(A - BK)$ has to stable

Real Parts of Eigenvalues have to strictly less than zero.

$$\det(\lambda I - (A - BK))$$

$$\det(\lambda I - (A - BK)) = \det\left(\begin{bmatrix} \lambda & -I \\ +K_P & \lambda + K_D \end{bmatrix}\right) = 0$$

For the simplest case where $n = 1$, $I = 1$.

$$\det\left(\begin{bmatrix} \lambda & -1 \\ K_P & \lambda + K_D \end{bmatrix}\right) = \underbrace{\lambda^2 + \lambda K_D + K_P}_{\text{Characteristic Polynomial of the system}} = 0$$

$$\lambda_{1,2} = \frac{-K_D \pm \sqrt{K_D^2 - 4K_P}}{2}$$

$$\lambda_{1,2} = -\frac{K_D}{2} \pm \frac{\sqrt{K_D^2 - 4K_P}}{2}$$

Since K_P and K_D are Positive definite

∴ The real Part of the eigenvalues of the $(A - BK)$ matrix will always be Less than Zero ⟶ the system is Stable and in this case where $q_d = 0$, The system will stabilize to the origin.

Problem 2:

2D Planar Robotic arm

$$M\ddot{q} + C\dot{q} = \tau \longrightarrow \text{Dynamic model of the system.}$$

in this system, we have two generalized coordinates in a second order system, so Defining the state variable to be

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\dot{\vec{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \left[M^{-1}(\tau - C\dot{q})\right]_{2\times 1} \end{bmatrix}$$

$$\left[M^{-1}(\tau - C\dot{q})\right] \rightarrow \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Rewriting it in the standard state space form:

$$\dot{\vec{X}} = AX + Bu$$

given that $u = \begin{bmatrix} -K_{P1}e_1 - K_{d1}\dot{e}_1 \\ -K_{P2}e_2 - K_{d2}\dot{e}_2 \end{bmatrix}$

$e_1 = q_1 - q_1^{\to 0} = q_1$, $\dot{e}_1 = \dot{q}_1 - \dot{q}_1^{\to 0} = \dot{q}_1$, $e_2 = q_2 - q_2^{\to 0} = q_2$, $\dot{e}_2 = \dot{q}_2 - \dot{q}_2^{\to 0} = \dot{q}_2$

$$\dot{\vec{X}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \left[M^{-1}C\right] \\ 0 & 0 & \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \left[M^{-1}\right] \\ \end{bmatrix}_{2\times 2} \begin{bmatrix} -K_{P1} & 0 & -K_{D1} & 0 \\ 0 & -K_{P2} & 0 & -K_{D1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\left[M^{-1}C\right]_{2\times 2}$

- K_{P1}, K_{d1} are controller gains for Joint 1.
- K_{P2}, K_{d2} are controller gains for Joint 2.

Select gains to be Positive definite

Based on the state space form, Matlab ode45 computation has been done and yielded the response plotted in the upcoming pages.

The following code has been developed based on the planar Arm code provided for HW4 in addition to adding an animation part for animating the robotic arm responding to the PD controller:

The PlanarArm implementation code:

```
%This code has been developed by: Mostafa Atalla%
%HW:4 for RBE502 Robotics Control Course by: Prof. Jie Fu%

clc
clear all;
close all;

%% Initialization %%
x0=[1 0.4 0.3 0.1];          %Setting initial conditions for the state vector
xf = [0,0, 0, 0];            %Final state desired
kp=[250 0;0 250];            %Proportional gain vector
kd=[160 0;0 160];            %Dervative gain vector
tf=10;                       %Final time

%% Implement the PD control for set point tracking %%
options = odeset('RelTol',1e-4,'AbsTol',[1e-4, 1e-4, 1e-4, 1e-4]);
[T,X] = ode45(@(t,x) odefcn(t,x,xf,kp,kd),[0 tf],x0, options);

%% Plotting theta1,theta2 under the PD controller %%
figure('Name', 'Theta1, Theta2 under PD SetPoint Control')
movegui(1,'northeast')
plot(T, (X(:,1)*(180/pi)),'r-');
hold on
plot(T, (X(:,2)*(180/pi)),'r--');
hold on
title('Theta1, Theta2 under PD SetPoint Control');
xlabel('TIME')
ylabel('THETA (Degrees)')
legend('Theta1','Theta2')
grid on



%% Animation of the TWO Link Manipulator %%

%Manipulator Parameters%
I1=10;  I2 = 10; m1=5; r1=.5; m2=5; r2=.5; l1=1; l2=1;

%Initialization of Tip point of every Link%
[k j]=size(X);
pointl1=zeros(k,2); %End Point of Link1
pointl2=zeros(k,2); %End Point of Link2

%Create Figure for the Animation%
figure('Name', 'Animation of the 2 Link Manpiulator under PD Control')
movegui(2,'northwest')

%Computing the Tip Point positions corresponding to theta1, theta2%
for ii=1:k;

%Adjusting Figure for every iteration%
clf
xlabel('X - POSITION')
ylabel('Y - POSITION')
grid on
title('TWO LINK MANIPULATOR ANIMATION UNDER PD CONTROL');
axis([floor(min(pointl2(:,1))) 2.2 floor(min(pointl2(:,2))) 3]);
```

```
axis square

%Computing the points%
pointl1(ii,1) =  l1*cos(X(ii,1)) ;     %Computing x positions of end point of link 1
pointl1(ii,2) = l1*sin(X(ii,1));       %Computing y positions of end point of link 1
pointl2(ii,1) = pointl1(ii,1) + (l2*cos(X(ii,1)+X(ii,2))); %Computing x positions of end
point of link 2
pointl2(ii,2) = pointl1(ii,2)+(l2*sin(X(ii,1)+X(ii,2)));   %Computing y positions of end
point of link 2

%Plotting the links%
line([0,pointl1(ii,1)],[0,pointl1(ii,2)],'linewidth',2,'color','black');
line([pointl1(ii,1),pointl2(ii,1)],[pointl1(ii,2),pointl2(ii,2)],'linewidth',2,'color','b
lue');

hold on

plot(0,0,'o','markersize',7)
plot(pointl1(ii,1),pointl1(ii,2),'o','markersize',7)
plot(pointl2(ii,1),pointl2(ii,2),'o','markersize',7)
plot(pointl2(:,1),pointl2(:,2),'-')
pause(.07)


end
```

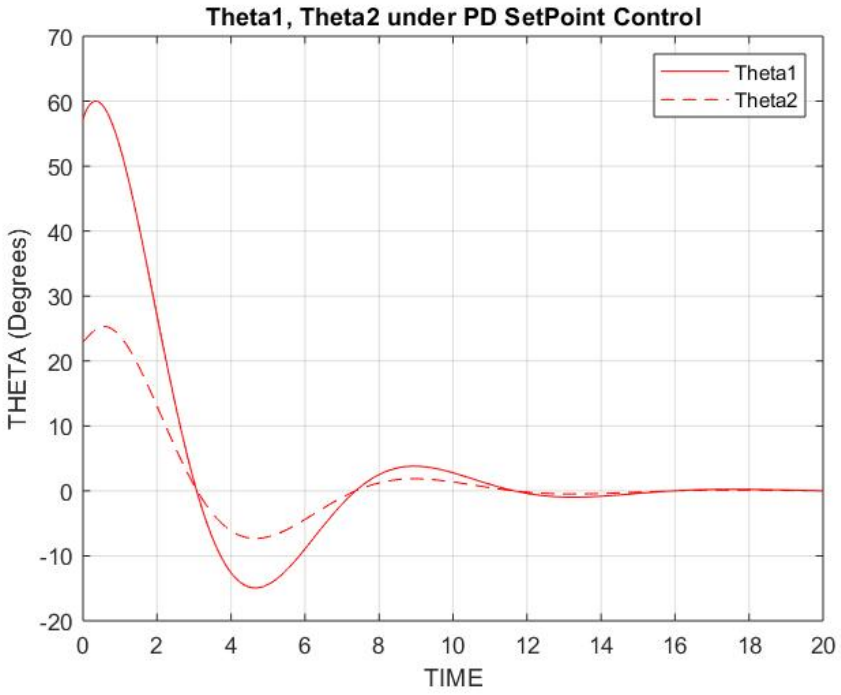
The ODE Function code:

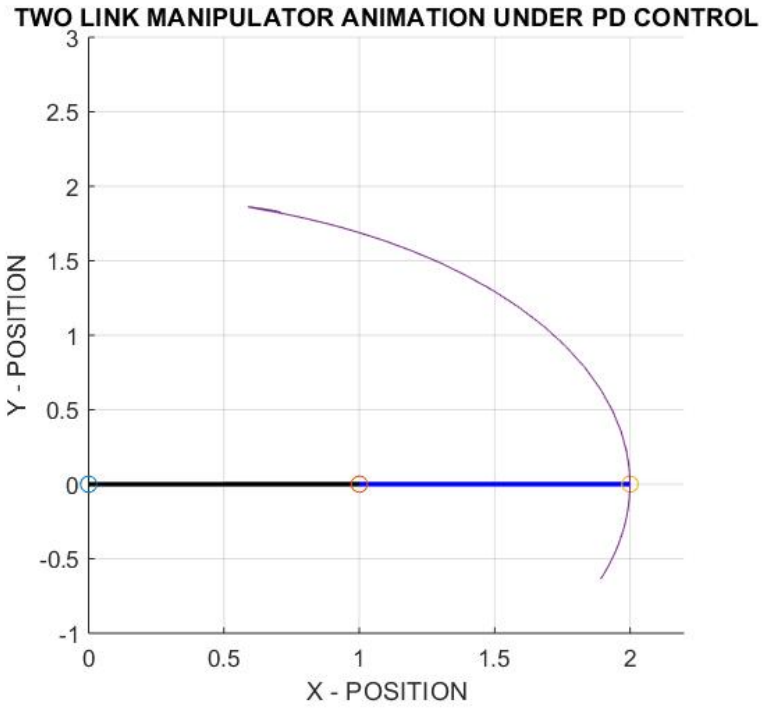
```
%This code has been developed by: Mostafa Atalla%
%HW:4 for RBE502 Robotics Control Course by: Prof. Jie Fu%
%% Defining the 1st order ode to be solved by the ode45%%
function dxdt=odefcn(t,x,xf,kp,kd)

%Manipulator Parameters%

I1=10;  I2 = 10; m1=5; r1=.5; m2=5; r2=.5; l1=1; l2=1;
a = I1+I2+m1*r1^2+ m2*(l1^2+ r2^2);
b = m2*l1*r2;
d = I2+ m2*r2^2;

%Inertia and Coriolos Matrices Computation%

M = [a+2*b*cos(x(2)), d+b*cos(x(2)); d+b*cos(x(2)), d] %Inertia Matrix
C = [-b*sin(x(2))*x(4), -b*sin(x(2))*(x(3)+x(4)); b*sin(x(2))*x(3),0] %Coriolos Materix
invM = inv(M);
invMC= inv(M)*C;

%Setting PD Controller for each Joint input Torque - Set Point Tracking%
%General PD Controller u=-Kp(e)-Kd*e_dot

e=[x(1)-xf(1);x(2)-xf(2)];          %position error vector
e_dot=[x(3)-xf(3);x(4)-xf(4)];      %Velocity error vector
u=-kp*e-kd*e_dot;                   %Controller


%State Space Representation%
%4 first order ode%

dxdt=zeros(4,1);   %initialization of first order ode vector%
dxdt(1)=x(3);
dxdt(2)=x(4);
dxdt(3)= invMC(1,:)*[x(3);x(4)]+invM(1,:)*u;
dxdt(4)= invMC(2,:)*[x(3);x(4)]+invM(2,:)*u;
end
```

Results:

Given this initialization values:

```
x0=[1 0.4 0.3 0.1];          %Setting initial conditions for the state vector
xf = [0,0, 0, 0];            %Final state desired
kp=[300 0;0 300];            %Proportional gain vector
kd=[150 0;0 150];            %Derivative gain vector
```

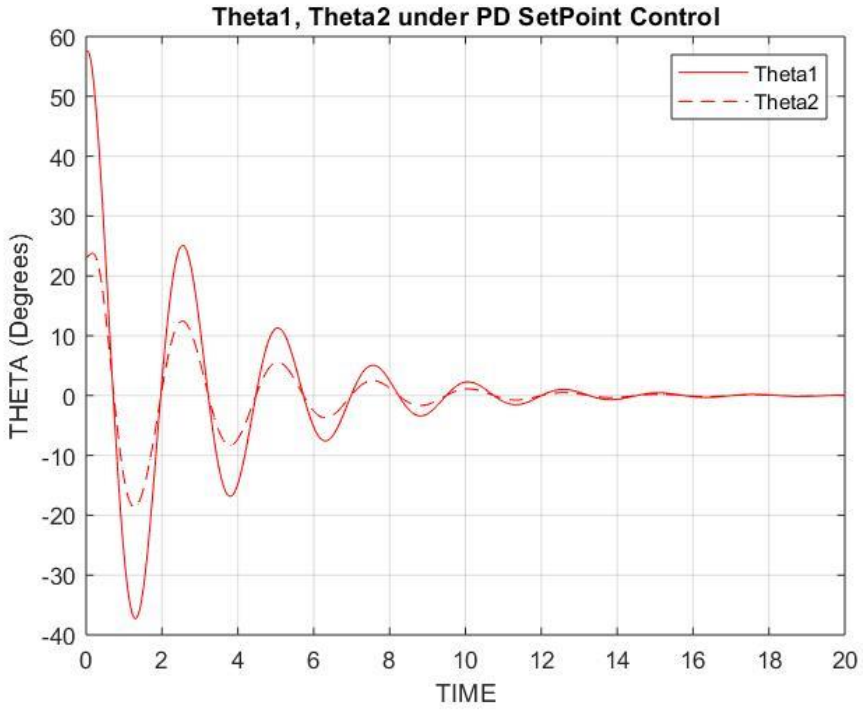


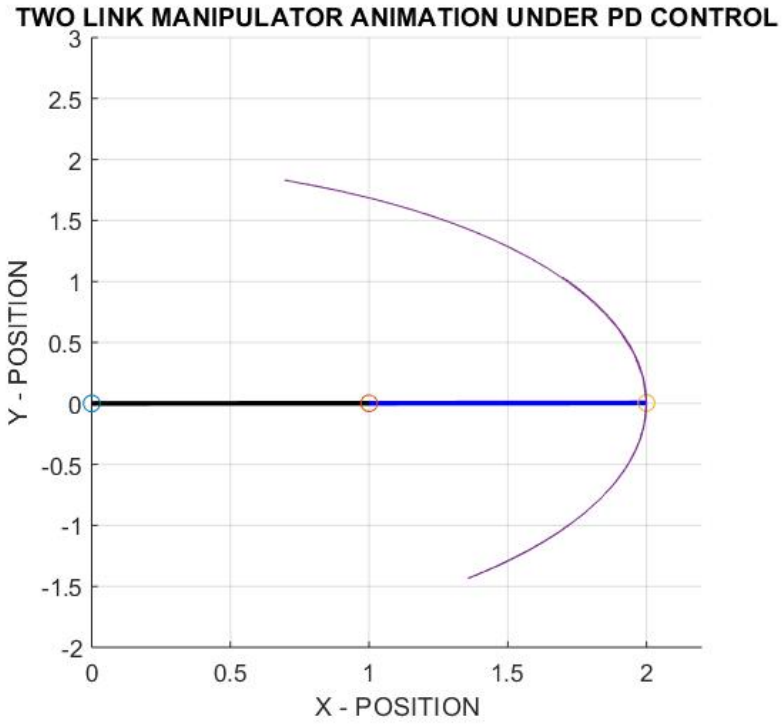


Increasing the proportional gain values to be 250 to make the response faster:

```
kp=[250 0;0 250];              %Proportional gain vector
kd=[25 0;0 25];                %Derivative gain vector
```

The response becomes faster but with a lot of oscillations > Increase Kd





Increasing the derivative gain values to be 160 to make the response faster:

```
kp=[250 0;0 250];              %Proportional gain vector
kd=[160 0;0 160];              %Derivative gain vector
```

The oscillations have been eliminated.