CSCI 2110 Data Structures and Algorithms

Module 2: Introduction to Algorithm Complexity

Learning Objectives/Topics

- *What is algorithm time complexity? Why should we care?*
- *How do you express algorithm time complexity in terms of basic operations?*
- *Define the standard measure of algorithm complexity – the order of complexity or big O.*
- *Study practical examples of typical big O's and know simple rules for deriving big O.*
- *Know other types of complexity, namely, Big-Omega, Big-Theta and Little-O and their relation to Big-O.*
- *Distinguish between average case, worst-case and best-case running time complexities.*

What is algorithm time complexity?

- *Algorithm time complexity is just a measure of how fast your algorithm/program runs.*

- *Thus it is a measure of the efficiency of the algorithm.*

- *This efficiency can be expressed by the speed or the running time of the algorithm (that is, of the program implementing the algorithm).*

Why should we care?

- *Understanding algorithm time complexity can make a world of difference in software design.*
- *We can compare algorithms and choose the right algorithm for the right task.*
- *Some simple examples:*
  - *Bubble sort algorithm on a million records → 1 billion steps.*
  - *Quick sort algorithm on a million records → 20 million steps!*
  - *Linear search algorithm on a million records → 1 million steps.*
  - *Binary search algorithm on a million records → 20 steps!*

Time Complexity vs. Space Complexity

- *In the previous examples, we are concerned about the runtime efficiency or the time complexity of the algorithm.*
- *Another factor that can also determine the efficiency of the algorithm is the space complexity.*
- *Space complexity is a measure of the memory required by the algorithm.*
- *Principles behind concepts of understanding time and space complexity are similar.*
- *We will focus on time complexity.*

We need to measure the run time – why not use the "wall clock" approach?

- *Suppose that we run a competition in this class to test who has built the fastest spell checker algorithm.*
- *Manvi says:" My spell checker took 2.5 minutes to complete its task".*
- *Derek announces:" My spell checker tookonly 1.5 minutes"*
- *Mohamad shouts:" My spell checker took 50 seconds".*
- *Yiyang pipes in: "My spell checker took 40 seconds!"*
- *Megan quietly texts: "My spell checker just took 10 seconds."*
- *This is called the" wall clock" approach to measuring time complexity→ looking at the absolute time the program took to run.*

The “wall clock” approach is not reliable …

- *This is not a reliable measure because …*
- *…many factors cloud the actual efficiency of the algorithm, for example,*
  - *CPU Speed and OS*
  - *System environment (how many other processes were running simultaneously?)*
  - *Programming language and platform*
  - ***Differences in the document size (the size of the input)***

What is a better approach?

- *We need to compare algorithms independent of the peripheral issues such as CPU speed, etc.*
- *Hence a better approach is to count the number of basic operations in the algorithm for a specific input size.*
- *The running time will be proportional to this count.*
- *What are the basic operations of an algorithm?*
  - *Additions/subtractions*
  - *Multiplications/divisions*
  - *Comparisons*
  - *Assignment (copy) operations, etc.*

We will sneak in an approximation….

- *We will say that every basic operation such as*
  - *Addition/ Subtraction*
  - *Multiplication/Division/Modulus*
  - *Comparison*
  - *Assignment (copy)*
  - *etc.*

  takes the same amount of time.
- *We'll see later that in the long run, this approximation is valid.*
- *In some cases, we may not even count all the basic operations, but just some dominant operations.*
- *Again, in the long run this approximation will be valid.*

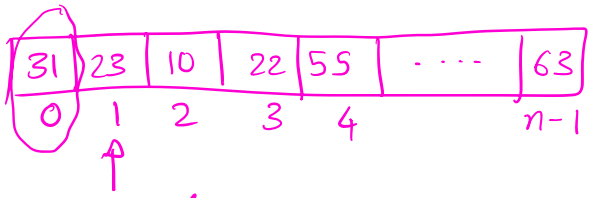
What about the input data size?

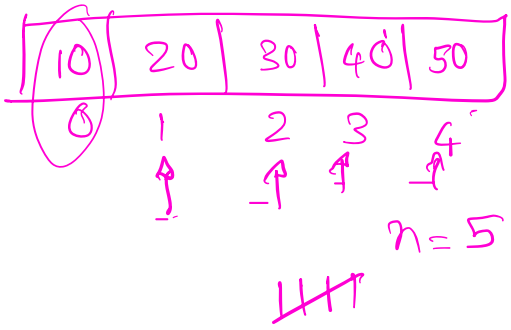
- *This is perhaps the most important parameter in comparing algorithms.*
- *If we say that an algorithm A runs faster than algorithm B, we need to make sure that they are running the same input data size.*
- *The input data could be, for example, the size of the array to be processed, number of characters in a file, number of records in a database, etc.*
- *If we increase the input size, will algorithm A still be faster than B?*
- *Therefore, we need to express the number of basic operations in terms of the input data size.*

Let's work through some examples to determine the running times in terms of basic operations.

Example 1: Algorithm to find the largest integer in an array of n integers.

```
public static int findLargest(int[] arr)
{
    int largest = arr[0];
    int index = 1;
    while (index < arr.length)
    {
        if (arr[index] > largest)
            largest = arr[index];
        index++;
    }
    return largest;
}
```





Let t be the time for one basic operation.

Total run time of the algorithm (that is, the above code)

$= t + t + nt + (n-1)t + (n-1)t + (n-1)t + t$

(1) (2) (3) (4) (5) (6) (7)

(5) ↑ worst case

$= 3t + nt + nt - t + nt - t + nt - t$

$= 3t + 4nt - 3t = 4nt$

Run time of the algorithm = 4nt

or Run time is proportional to 4n

or simply, **Run time is 4n**

4 * x * x * x * x

Example 2: Algorithm to evaluate a polynomial

Write a program to evaluate

$P(x) = 4x^4 + 2x^3 - 5x^2 + 10x - 5$

Given some value of x, say $x = 2$

Number of mults = $4 + 3 + 2 + 1 + 0$

Number of adds/subs = 4

Let's generalize this (n = degree of polynomial)

$P(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + mx + q$

Number of mults = $n + (n-1) + (n-2) + \ldots + 1 + 0$

$= \frac{n(n+1)}{2}$ (from sum of series formula)

Number of adds/subs = n

Total number of operations = $\frac{n(n+1)}{2} + n = \frac{n^2}{2} + \frac{n}{2} + n$

$= 0.5n^2 + 0.5n + n = 0.5n^2 + 1.5n$

Run time is $0.5n^2 + 1.5n$

Example 3: Algorithm to find the weighted average of n items

Determine the average response time for visiting n websites, given the response time for each website and the number of times that website is visited.

Website	#times	Avg. Response Time
1	m_1	f_1
2	m_2	f_2
⋮	⋮	⋮
n	m_n	f_n

Weighted average = $\frac{(m_1 * f_1 + m_2 * f_2 + m_3 * f_3 + \ldots + m_n * f_n)}{(m_1 + m_2 + m_3 + \ldots + m_n)}$

Run time of the program = n mults + $(n-1)$ adds + $(n-1)$ adds + 1 div

$= n + (n-1) + (n-1) + 1 = 3n - 1$

Run time = $3n - 1$

Ok, let's compare algorithms….

- *Suppose that we have three algorithms with the following run times (or run times proportional to):*
  - *Algorithm A: 5000 n + 1000*
  - *Algorithm B: 200 n^2 + 500 n*
  - *Algorithm C: 1.1^n*
- *Which algorithm is the best?*
- *From the calculations for n=10, n=100, n=1000, and n=10,000 we conclude that Algorithm A performs the best in the "long run" or for "large enough values of n".*
- *This is refered to as the asymptotic complexity*
- *WE COMPARE ALGORITHMS FOR LARGE VALUES OF THE INPUT SIZE OR IN OTHER WORDS, BASED ON THEIR ASYMPTOTIC COMPLEXITIES*

Run times for different values of n
(smaller the number, better the algorithm)

	n = 10	n = 100	n = 1000	n = 10,000
A	51000	501,000	5,001,000	50,001,000
B	25000	2,050,000	25,000,000	20,005,000,000
C	3	13780.19	2.5×10^{41}	A VERY, VERY, VERY, VERY, VERY, VERY LARGE NUMBER!

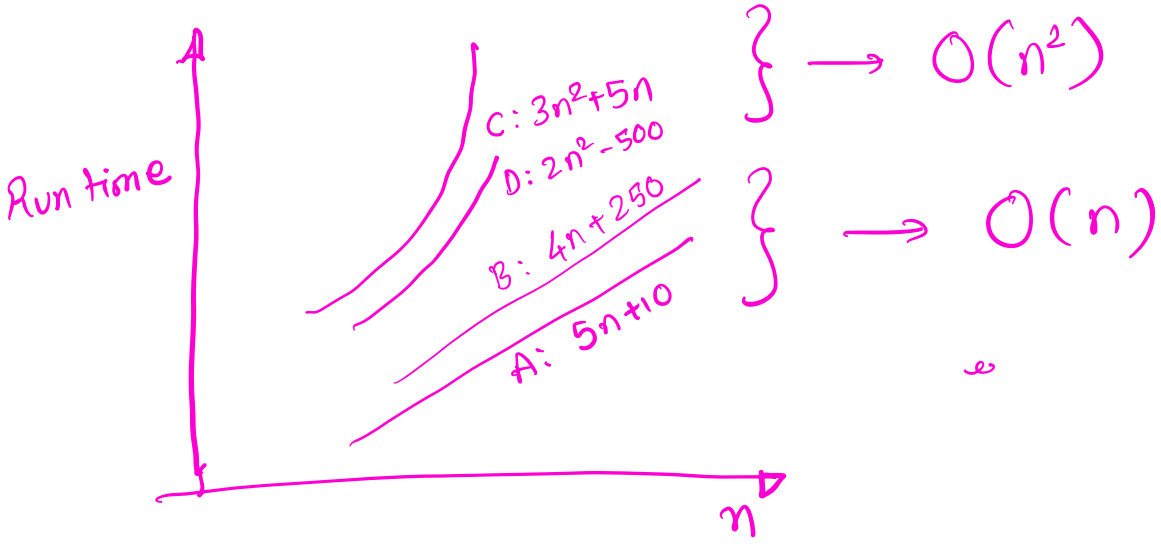
Now comes the Big O notation….

- *Compare the following algorithms:*

- *Algorithm A: 5 n + 10*
- *Algorithm B: 4 n + 250*
- *Algorithm C: $3n^2 + 5n$*
- *Algorithm D: $2 n^2 - 500$*

- *We say that Algorithms A and B are in the "same league" for large values of n.*
- *Similarly Algorithms C and D are in the "same league" for large values of n.*
- *The Big O notation classifies algorithms into the "same league".*
- *In other words, we say that the complexity of algorithms A and B is O(n) and the complexity of algorithms C and D is $O(n^2)$.*

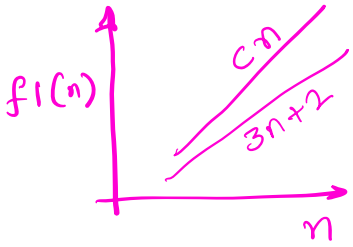
→ same breed of race horses.



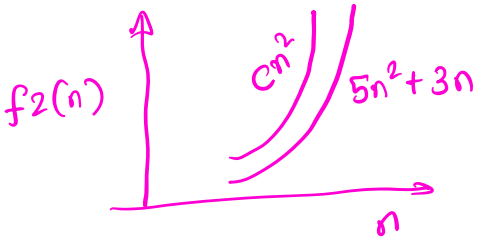
NOW A LITTLE MATH....

FORMAL DEFINITION OF BIG O OR THE O( ) NOTATION

Why is a function $f1(n) = 3n+2$ said to be $O(n)$?
Why is a function $f2(n) = 5n^2+3n$ said to be $O(n^2)$?
The basis is in the growth of these functions for large values of n.

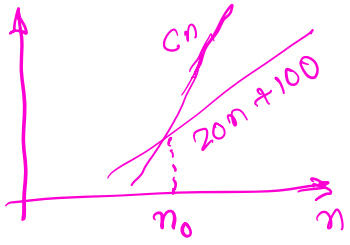


$3n+2$ is always bounded by cn, where c is some constant. That is, we will always be able to find such a value c.



$5n^2+3n$ is always bounded by cn^2, where c is some other constant.

What if the function looks as follows:



Even in this case, $20n+100$ is bounded by cn for $n >= n_0$

Let's call $3n+2$ as $f(n)$ and cn as $O(g(n))$

A function $f(n)$ is said to be of the order $g(n)$, written as $O(g(n))$, if there exist positive constants c and n_0 such that $f(n) <= c\,g(n)$ for $n >= n_0$

Deriving the Big O is easy!

- *Replace all additive constants in the run time with the constant 1.*
- *Retain only the highest order term in this modified run time.*
- *If the highest order term is 1, then the order is O(1).*
- *If the highest order term is not 1, remove the constant (if any) that multiplies the term.*
- *You are left with the order.*

Examples

Derive the order of complexities (Big O) for the following functions (that is, run times expressed in terms of the input size n)

1. $f(n) = 3n + 25 \rightarrow 3n + 1 \rightarrow 3n \rightarrow n$ $\boxed{O(n)}$

2. $25n^3 + 2n + 5 \rightarrow 25n^3 + 2n + 1 \rightarrow 25n^3 \rightarrow n^3$ $\boxed{O(n^3)}$

3. $\frac{n^2}{2} + 2000n + 1 \rightarrow \frac{n^2}{2} \rightarrow n^2$ $\boxed{O(n^2)}$

4. $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

   from sum of series formula

   * Must express as a polynomial * $\rightarrow \boxed{O(n^2)}$

5. $(3n+1)(n+2) = 3n^2 + n + 6n + 2$
   $= 3n^2 + 7n + 2 \rightarrow n^2$ $\boxed{O(n^2)}$

EXAMPLES FOR DERIVING THE BIG O (cont'd.)

6. $\frac{10n + 5n}{3n} + 4 = \frac{10n}{3n} + \frac{5n}{3n} + 4 \rightarrow O(1)$

7. $3\log_2 n + 50n + 30 \rightarrow 50n \rightarrow n \quad O(n)$

8. $250n + 10n\log_2 n + 25$
$\rightarrow 250n + 10n\log_2 n + 1 \rightarrow n\log_2 n \quad O(n\log_2 n)$

9. $n\log_2 n^3 + \log_2 n^4 + n^2 \quad \log_2 n^4 \rightarrow 4\log_2 n$
$= 3n\log_2 n + 4\log_2 n + n^2 \rightarrow n^2 \quad O(n^2)$

10. $12 \times 2^n + 500 + n^{10} \rightarrow O(2^n)$

11. $\frac{200}{n} + \frac{50}{\sqrt{n}} + 500 \rightarrow O(1)$

12. $\frac{200}{n} + \frac{50}{n^2} \rightarrow O\left(\frac{1}{n}\right)$

ORDER TABLE

$$\frac{1}{n^2} < \frac{1}{n} < 1 < \log_2 n < \sqrt{n} < n < n\log n < n\sqrt{n} < n^2 < n^3 < 2^n$$

PRACTICAL EXAMPLES

O(1): Constant Time Complexity

(Means that the algorithm requires the same fixed number of steps irrespective of the size of the task)

- Bunch of arithmetic statements
- Bunch of assignment statements
- Accessing the ith element in an array of size n
  num = a[i];

O(n): LInear Time Complexity

Find the largest integer in an array of size n.

Run time = $4n \longrightarrow O(n)$

O(n^2): Quadratic Time Complexity

Evaluation of a polynomial of degree n

Run time = $0.5n^2 + 1.5n \longrightarrow O(n^2)$

O(log n): Logarithmic Time Complexity

Binary search of a sorted array of size $n \rightarrow O(\log_2 n)$

$2^{10} = 1024 \rightarrow \log_2 1024 = 10$. We can search a sorted array of size 1024 in just 10 steps.

O(n log n): En Log En Time Complexity

Quick Sort, Heap Sort } Sort an array of n items.

O(n^3): Cubic Time Complexity

Multiplication of two $n \times n$ matrices $\rightarrow O(n^3)$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} x1 & x2 & x3 \\ x4 & x5 & x6 \\ x7 & x8 & x9 \end{bmatrix}$$

$x1 = aj + bm + cp$

operations to derive $x1$ = 3 mults & 2 adds

operations to derive all outputs = $3 \times 3 \times (3+2)$

∴ # operations to multiply 2 $n \times n$ matrices = $n \times n \times (n + n - 1) \longrightarrow O(n^3)$

O(k^n) Exponential Time Complexity

"Brute Force" or exhaustive search through all possible combinations.

Example: Breaking a secret key stored as a n-bit binary number.

Encryption: Plaintext $\xrightarrow{\text{secret key}}$ Ciphertext

Decryption: Ciphertext $\xrightarrow{\text{secret key}}$ Plaintext

Caeser Cipher: HEY (Plaintext) $\xrightarrow{\text{secret key } K=2}$ JGA

Many strong encryption algorithms like AES, 3DES, etc. exist?

Problem: Knowing the ciphertext & guessing the plaintext, how can a hacker get the secret key?

Solution: Brute Force search → check all possible combinations of the n-bit secret key.

n = 3: 000, 001, 010, 011, 100, 101, 110, 111 (8) = 2^3

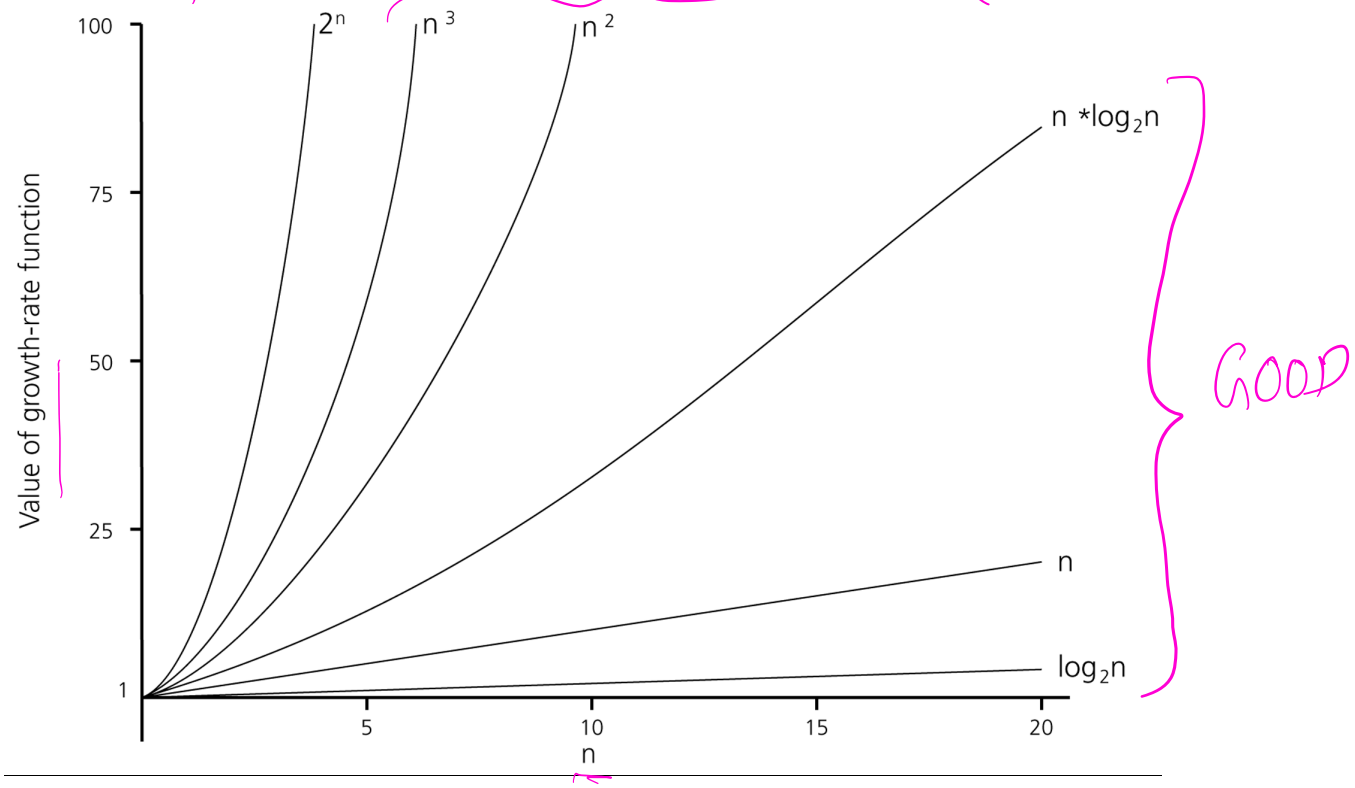
n = 4: 0000, 0001, ... , 1111 (16) = 2^4

n = 5: 00000, 00001, ... , 11111 (32) = 2^5

For an n-bit key, brute force search takes $O(2^n)$ complexity

COMPARISON OF GROWTH RATES

(b)



(a)

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Example of an algorithm with O(n!) complexity:
Travelling Salesperson Problem: Given n cities and their inter-city distances, find the shortest path that starts at a city, visits each city exactly once and returns to the starting city.

ESTIMATION PROBLEMS WITH BIG O

1. An algorithm takes 1 ms to run when the input size is 1000. Approximately, how long will the algorithm take to process an input size of 5000 if it has the following orders of complexities:

   a. Linear
   b. Quadratic
   c. nlogn

Given the data size, estimate the time

a) Linear $O(n)$

Answer: 5 ms

# Steps	Time
1000	1 ms
5000	x ?

$x = \frac{5000 \times 1}{1000} = 5$

b) Quadratic $O(n^2)$

Answer = 25 ms

# steps	Time
1000×1000	1 ms
5000×5000	x ?

$x = \frac{5000 \times 5000 \times 1}{1000 \times 1000} = 25$

c) $n \log n$

Answer = 6.16 ms

#steps	Time
1000 log 1000	1 ms
5000 log 5000	x ?

$x = \frac{5000 \log 5000 \times 1}{1000 \log 1000} = 6.16$

2. Algorithm A takes 10 ms to solve a problem of size 1000. Algorithm B takes 100 ms to solve a problem of size 10,000. Algorithm A's complexity is quadratic while Algorithm B's complexity is cubic. How large a problem can each solve in 1 second?

1 sec = 1000 ms

Given the time, estimate the data size

Algorithm A

$O(n^2)$

Answer: 10000

Time	# Steps
10	1000×1000
1000	$x * x$

$\therefore x^2 = \frac{1000 \times 1000 \times 1000}{10}$

$= 1000 \times 1000 \times 100$

$\therefore x = \sqrt{1000 \times 1000 \times 100} = 10,000$

Algorithm B

$O(n^3)$

Answer: 21544

Time	# Steps
100	$10000 \times 10000 \times 10000$
1000	$x * x * x$

$x^3 = \frac{10000 \times 10000 \times 10000 \times 1000}{100}$

$x = \sqrt[3]{10000 \times 10000 \times 1000 \times 10}$

$= 21544$

3. Software packages A and B spend exactly $T_A=c_An^2$ and $T_B = c_Bn^3+500$ milliseconds to process n data items, respectively, where c_A and c_B are some constants. During a test, A takes 1000 milliseconds and B takes 600 milliseconds to process n=100 data items. Which package is better for processing 10 data items? Which package is better for processing 1000 data items? (Show steps).

First find C_A:

$T_A = C_A n^2$

$1000 = C_A \times 100 \times 100$

$\therefore C_A = \frac{1000}{100 \times 100}$

$= 0.1$

Next find C_B:

$T_B = C_B n^3 + 500$

$600 = C_B \times 100 \times 100 \times 100 + 500$

$100 = C_B \times 100 \times 100 \times 100$

$\therefore C_B = \frac{100}{100 \times 100 \times 100}$

$= 0.0001$

$n = 10$

A is better

Alg. A

$T_A = C_A n^2$

$= 0.1 \times 10 \times 10$

$= 10$ ms

Alg. B

$T_B = C_B n^3 + 500$

$= 0.0001 \times 10^3 + 500$

$= 500.1$ ms

$n = 1000$

A is better

Alg. A

$T_A = C_A n^2$

$= 0.1 \times 1000^2$

$= 100{,}000$ ms

Alg. B

$T_B = C_B n^3 + 500$

$= 0.0001 \times 1000^3 + 500$

$= 100{,}500$ ms

OTHER ORDERS OR COMPLEXITY

Although we will use the Big O primarily as a measure of algorithm complexity in this course, there are three other types of complexity related to Big O.

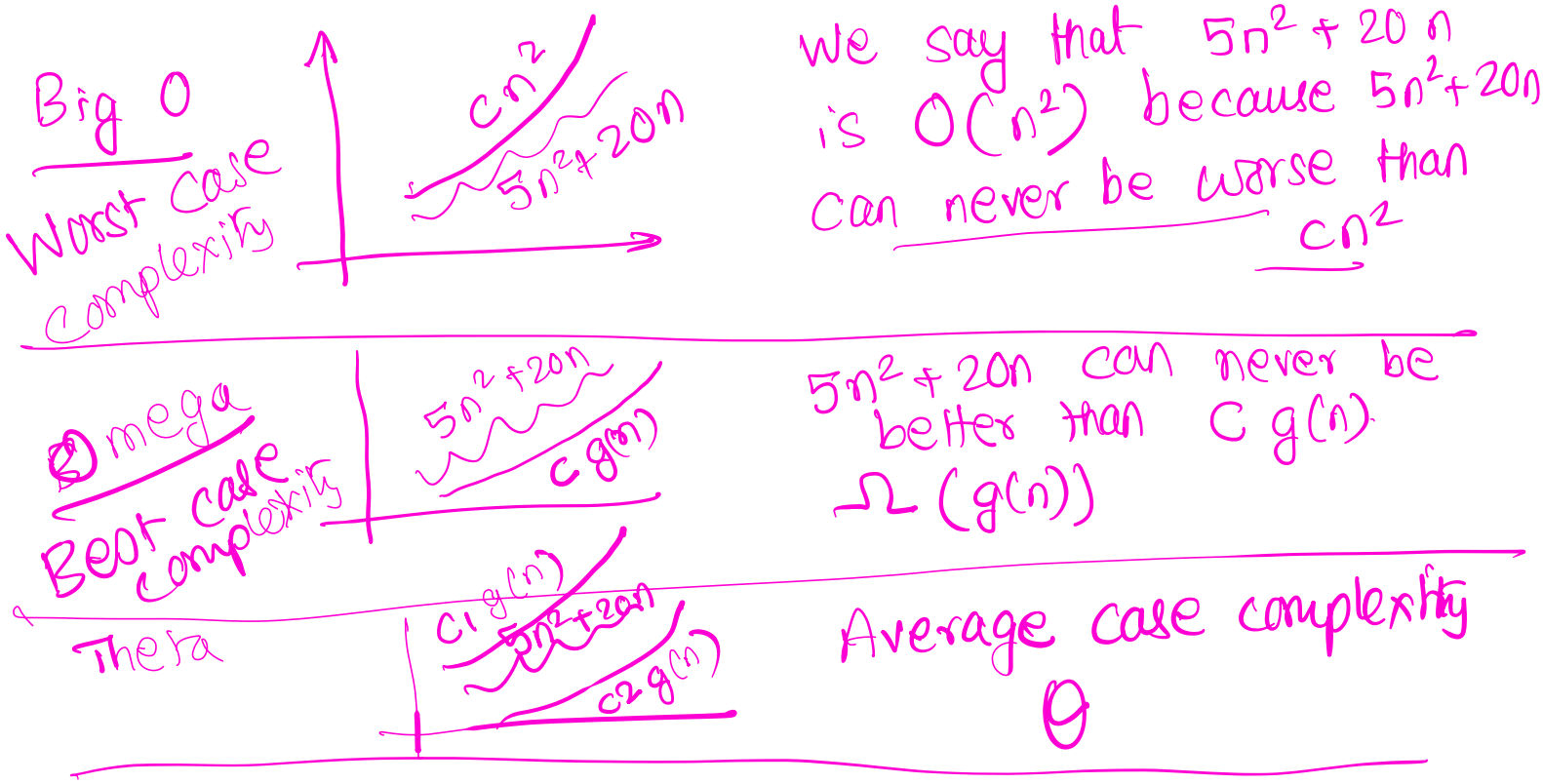
We redefine Big-O to place it in context.

Big O: A growth function T(N) is O(F(N)) if there are positive constants c and N_0 such that T(N) <= cF(N) when N>=N_0 .

Big Omega: A growth function T(N) is Ω(F(N)) if there are positive constants c and N_0 such that T(N) >= cF(N) when N>=N_0 .

Big Theta: A growth function T(N) is Θ(F(N)) if and only if T(N) is O(F(N) and T(N) is Ω(F(N)).

Little-O: A growth function T(N) is o(F(N)) if and only if T(N) is O(F(N) and T(N) is is not Θ(F(N)).



ORDER ARITHMETIC

Let runtime of an algorithm A be $f1(n) = O(g1(n))$

" " B be $f2(n) = O(g2(n))$

What is the order of complexity of an algorithm whose run time is

a) $f1(n) + f2(n)$? $MAX(O(g1(n)), O(g2(n)))$

b) $K * f1(n)$? $O(g1(n))$

c) $f1(n) * f2(n)$? $O(g1(n) * g2(n))$

Alg A	Alg B
$f1(n)$	$f2(n)$
$O(n)$	$O(n \log n)$

Then an algorithm with runtime $f1(n) + f2(n)$

$= O(n \log n)$

Another algorithm with runtime $f1(n) * f2(n)$

$= O(n^2 \log n)$

SOME SIMPLE RULES OF THUMB

How to look at code and quickly derive the complexity

For each of the following code segments, derive the order of complexity

1.
```
for (int i=1; i<=n; i++)
{
// some set of p statements
// where p is a constant
}
```

iterations = n

O(n) # operations = p * n

2.
```
for (int i=1; i<=n; i++)
{
        for( j=1; j<=n; j++)
        {
                // p statements
        }
}
```

iterations = n^2

$O(n^2)$

3. Code 1 followed by Code 2 in sequence → $n + n^2$ → $O(n^2)$

4.
```
for (int i=1; i<=1000; i++){
        for (j=1; j<=n; j++){
                // set of p statements
        }
}
```

{ → 1000 × n → O(n)

5.
```
for (int i=1; i<=n; i++)
{
        for (int j=1; j<=100; j++) { // set of p statements}
        for (int k=1; k<=n; k++) { // set of q statements}
}
```

} n (100 + n) iterations

= 100 n + n^2

→ $O(n^2)$

6.
```
for (int i=1; i<=n; i=2)
{
        //set of p statements
}
```

i<n; i=2*i)

→ $O(\log_2 n)$

n	# iterations
1 (2^0)	0
2 (2^1)	1
4 (2^2)	2
8 (2^3)	3
16 (2^4)	4

Best case, worst case and average case complexity

- The best-case running time of an algorithm is the running time under ideal or best conditions.
- The worst-case running time of an algorithm offers a guarantee that the running time will never be worse than it.
- The average running time of an algorithm is the expected running time on the average.
- If there is no reference to worst-case or average complexity order, it usually means worst-case.

Find the largest integer in an array of integers (size of the array is n) - Revisited

```
public static int findLargest(int [] a){
  int largest = a[0];                          t
  int i = 1;                                   t
  while (i<a.length)                           nt
  {
      if (a[i] > largest)                      (n-1)t
          largest = a[i];                      (n-1)t
      i++;                                     (n-1)t
  }
  return largest;                              t
}
```

10	20	30	40	50	60

Worst case = $3t + nt + 3(n-1)t$

$\rightarrow O(n)$

60	50	40	20	30	10

Best case = $3t + nt + 2(n-1)t$

$\rightarrow O(n)$

10	30	40	70	60	50	20

Average case = $3t + nt + 2(n-1)t + \frac{(n-1)t}{2}$

$\rightarrow O(n)$

Summary

- The most reliable way to measure the run time of an algorithm is to count the number of basic operations it performs.
- Express the count of the basic operations of an algorithm as a function of the input size n.
- Then express the function in terms of the Big O.
- Big O refers to the asymptotic growth of the function for large values of n.
- This is the measure that we use to compare algorithms.

Summary

- Algorithm run time complexity can be best case, worst case or average case.
- Default algorithm time complexity refers to worst case.
- Deriving Big O is easy!
- Some typical run time orders are: O(1) (constant), O(log n) (logarithmic), O(n) (linear),

O(n log n), $O(n^2)$ (quadratic), $O(n^3)$ (cubic), $O(k^n)$ (exponential).