Chapter 1

Rappel des lois de la mecanique Newtonienne

1.1 Enonces des lois de Newton

- loi d'inertie
  Une particule isolee , sur la quelle n'agit aucune force exterieur , rest au repos ou conserve un mouvement recteligne uniform

$$\boxed{\overrightarrow{F} = \overrightarrow{0} \Longleftrightarrow \overrightarrow{v} = \overrightarrow{cte}}$$

- loi fondamentale de la dynamique
  - $\overrightarrow{F} = m\overrightarrow{a}$
    * $\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt}$
    * $\overrightarrow{v} = \frac{d\overrightarrow{r}}{dt}$
    * $\overrightarrow{r} = \overrightarrow{OM}$ (O est l origin , M est la point ou F agit)
  - $\overrightarrow{F} = \frac{d\overrightarrow{P}}{dt}$(avec $\overrightarrow{P} = m\overrightarrow{v}$ quantite de mouvement)

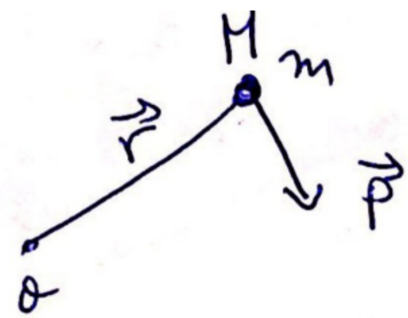
- principe de la action et de la reaction $\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$

1.2 Loi de conservaton pour un point materiel

- Conservation de la quantite de mouvement
  si $\vec{F} = 0$, $\vec{F} = \frac{d\vec{P}}{dt} = \vec{0} \implies \vec{P} = \overrightarrow{cte}$

- Conservation du moment angulair

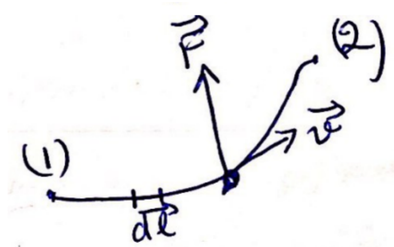
$\overrightarrow{L_o} = \vec{r} \wedge \vec{p}$
$\frac{d\overrightarrow{L_o}}{dt} = \vec{r} \wedge \vec{F}$
si $\vec{F} = \vec{0} \implies L = \overrightarrow{cte}$
si $\vec{F}$ est porte par $\vec{r} \implies \vec{L} = \overrightarrow{cte}$



- Conservation de lenergie mecanique total

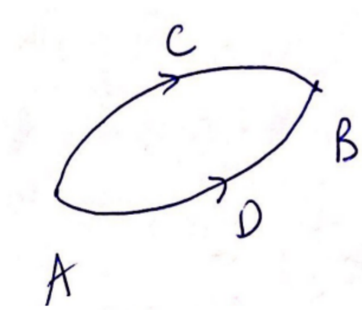
  - Theorem d energie cinetique

$\Delta E_c = W$, $W = \int \vec{F} d\vec{l}$



  - Forces conservatives

si $W_{ACB} = W_{ADB} \implies$ les forces exterieur sont conservatives

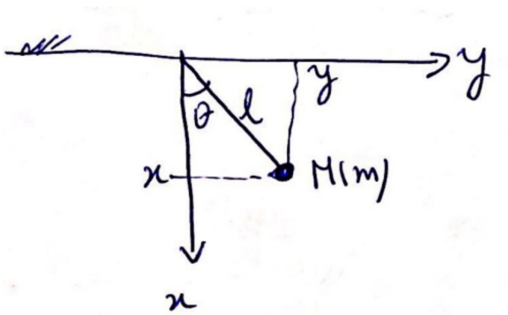


  - Une condition necessaire et suffisant pour que W_{AB} soit independant du chemin est que $\vec{F}$ derive d un potentielle
    $\vec{F} = -\overrightarrow{grad}(U)$ (U : energie potentielle)

1.3 Contraintes et coordonees generalisees

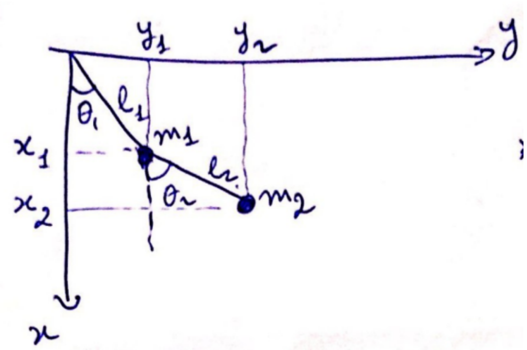
Les contraintes du systeme introduisent des dependances entre les coordonee
les contraintes sont par exemple des hypothese de rigidite , limitation de sont cadre d evolution ...
exemple :

- pendule simple



$$\boxed{x^2 + y^2 = l^2}$$

- pendule simple



$$\boxed{x_1^2 + y_1^2 = l_1^2}$$
$$\boxed{(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2}$$

System de N particule

- Aucun contraint (independante) $\Longrightarrow$ 3N coordone (3N degree de liberte)
- K contrainte $\Longrightarrow$ 3N -k coordonees independant

Chapter 2

Formalise de lagrange

2.1 Holonomic systeme

A system in which one can deduce the state of a system by knowing only information about the change of positions of the components of the system over time

2.2 Calcule differentielle

Soit f une fonctino de N variables $f = f(r_1 \ldots, r_N)$

- la differentielle totale de f est : $df = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} dr_i$

- la derive de f par rapport a lune de ses variable (r_j) :

$$\frac{df}{dr_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{dr_i}{dr_j}$$

- si tout les variable sont independant

$$\frac{df}{dr_j} = \frac{\partial f}{\partial r_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{dr_i}{dr_j} = \sum_{i=1}^{N} \frac{\partial f}{\partial r_i} \frac{\partial r_i}{\partial r_j}$$

2.3 Equation generale de lagrange

On considere un system holonome de N particule , d degre du liberte , avex $\overrightarrow{F_\alpha}$ est la force appliquee sur la particule α

2.3.1 lenergie cinetique est :

$T = \sum_{\alpha=1}^{N} T_\alpha = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \overrightarrow{\dot{r}_\alpha}^2$

- $\overrightarrow{r_\alpha} = \overrightarrow{r_\alpha}(q_1, q_2, q_3 \dots q_d, t)$
- $\alpha = 1, 2, 3 \dots N$
- q_i: coordone generalise
- $i = 1, 2, 3 \dots d$
- $\overrightarrow{\dot{r}} = \frac{d\overrightarrow{r_\alpha}}{dt}$

$T_\alpha = T_\alpha(\underbrace{q_1, q_2 \dots q_d}_{\text{position}}, \underbrace{\dot{q_1}, \dot{q_2} \dots \dot{q_d}}_{\text{vitess}}, \underbrace{t}_{\text{temp}})$

2.3.2 les forces generalise associe a q_i

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_i}}\right) - \frac{\partial T}{\partial q_i}}$$

<u>Preuve :</u>

- on a :$dT = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} d\dot{r_\alpha}$ (differentielle totale de T) $\implies \frac{dT}{d\dot{q_i}} = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} \frac{d\dot{r_\alpha}}{d\dot{q_i}}$
- Puisque les variable sont independant alors :
  $\frac{dT}{d\dot{q_i}} = \frac{\partial T}{\partial \dot{q_i}} = \sum_{\alpha=1}^{N} \frac{\partial T}{\partial \dot{r_\alpha}} \frac{\partial \dot{r_\alpha}}{\partial \dot{q_i}} \implies \frac{\partial T}{\partial \dot{q_i}} = \frac{\partial}{\partial \dot{q_i}} (\sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha (\overrightarrow{\dot{r_\alpha}})^2) = \frac{1}{2} (\sum_{\alpha=1}^{N} m_\alpha \frac{\partial}{\partial \dot{q_i}} (\overrightarrow{\dot{r_\alpha}})^2)$
  $= \frac{1}{2} \times 2 \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r_\alpha}} \frac{\partial \overrightarrow{\dot{r_\alpha}}}{\partial \dot{q_i}} \implies \boxed{\frac{\partial T}{\partial \dot{q_i}} = \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r_\alpha}} \frac{\partial \overrightarrow{\dot{r_\alpha}}}{\partial \dot{q_i}}}$

- Chercher $\overrightarrow{\dot{r}_\alpha}$, ona :

  - $\overrightarrow{r_\alpha} = \overrightarrow{r_\alpha}(q_1, q_2 \ldots q_d, t)$
  - $\overrightarrow{r_\alpha} = \frac{d((\overrightarrow{r_\alpha}))}{dt}$

$d\overrightarrow{r_\alpha} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_1} dq_1 + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_2} dq_2 + \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_d} dq_d + \frac{\partial \overrightarrow{r_\alpha}}{\partial t} dt$
$\frac{d\overrightarrow{r_\alpha}}{dt} = \overrightarrow{\dot{r}_\alpha} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_2} \frac{dq_2}{dt} + \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_d} \frac{dq_d}{dt} + \frac{\partial \overrightarrow{r_\alpha}}{\partial t}$
$\frac{d\overrightarrow{r_\alpha}}{dt} = \overrightarrow{\dot{r}_\alpha} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_1} \frac{\partial q_1}{\partial t} + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_2} \frac{\partial q_2}{\partial t} + \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_d} \frac{\partial q_d}{\partial t} + \frac{\partial \overrightarrow{r_\alpha}}{\partial t}$

$$\boxed{\overrightarrow{\dot{r}_\alpha} = \sum_{i=1}^{d} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \dot{q}_i + \frac{\partial \overrightarrow{r_\alpha}}{\partial t}} \text{ avec } \alpha = 1, 2 \ldots N$$

- demontrer que $\frac{\partial \overrightarrow{\dot{r}_\alpha}}{\partial \dot{q}_i} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i}$
$\overrightarrow{\dot{r}_\alpha} = \sum_{i=1}^{d} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \dot{q}_i + \frac{\partial r_\alpha}{\partial t} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_1} \dot{q}_1 + \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \dot{q}_i + \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_d} \dot{q}_d + \frac{\partial \overrightarrow{r_\alpha}}{\partial t}(q_i, t)$
$\frac{\partial}{\partial \dot{q}} \left(\frac{\partial \vec{r}}{\partial t} \right) = 0$ car $\frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{r}}{\partial t}(q_i, t)$
$\frac{\partial \overrightarrow{\dot{r}_\alpha}}{\partial \dot{q}_i} = 0 + 0 + 0 \ldots + \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \frac{\partial \dot{q}_i}{\partial \dot{q}_i} + \ldots + 0$
$= \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \implies \boxed{\frac{\partial \overrightarrow{\dot{r}_\alpha}}{\partial \dot{q}_i} = \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i}}$

- alors $\frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r}_\alpha} \frac{\partial \overrightarrow{\dot{r}_\alpha}}{\partial \dot{q}_i} \implies \boxed{\frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r}_\alpha} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i}}$

- Calcule de $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right)$

  - $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left(\sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r}_\alpha} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\ddot{r}_\alpha} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} + \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{\dot{r}_\alpha} \underbrace{\frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right)}$

  - Calcule de $\frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right)$
  $d \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) dq_i + \frac{\partial}{\partial t} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) dt$
  $\frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) \dot{q}_i + \frac{\partial}{\partial t} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) \implies \frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \sum_{i=1}^{d} \frac{\partial}{\partial q_i} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) \dot{q}_i + \frac{\partial}{\partial q_i} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial t} \right)$
  $\frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left(\sum_{i=1}^{d} \frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial t} \overrightarrow{r_\alpha} \right) = \frac{\partial}{\partial q_i} \overrightarrow{\dot{r}_\alpha} = \frac{\partial}{\partial q_i} \frac{d}{dt} (\overrightarrow{r_\alpha}) \implies \boxed{\frac{d}{dt} \left(\frac{\partial \overrightarrow{r_\alpha}}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \overrightarrow{\dot{r}_\alpha}}$

- Verifier que $\sum_{\alpha=1}^{N} m_\alpha \vec{\ddot{r}_\alpha} \frac{\partial \vec{r_\alpha}}{\partial q_i} = \frac{\partial T}{\partial q_i}$

  on a $T = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}_\alpha}^2 \implies \frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} m_\alpha \frac{\partial}{\partial q_i}(\vec{\dot{r}_\alpha})^2 \implies \boxed{\frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} m_\alpha \vec{\dot{r}_\alpha} \frac{\partial \vec{\dot{r}_\alpha}}{\partial q_i}}$

alors

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i} + \sum_{\alpha=1}^{N} m_\alpha \vec{\ddot{r}_\alpha} \frac{\partial}{\partial q_i} \vec{r_\alpha}}$$

- on utilise la loi de Newton : $\vec{F_\alpha} = m_\alpha \vec{\ddot{r}_\alpha} \implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i} + \sum_{\alpha=1}^{N} \vec{F_\alpha} \frac{\partial \vec{r_\alpha}}{\partial q_i}$
  avec $\sum_{\alpha=1}^{N} \vec{F_\alpha} \frac{\partial \vec{r_\alpha}}{\partial q_i} = Q_i$

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}}$$

2.4 Formalisme de lagrange : cas d un system conservatif

- Si les forces $\vec{F_\alpha}$ sont conservatif , On suppose que toutes les forces agissant sur ce system derivent d'ume mem energie potentielle U
  avex U depend de position $U = U(\vec{r_1}, \vec{r_2} \ldots, \vec{r_N})$ et $\vec{F_\alpha} = -\overrightarrow{grad} U$
  Donc , on a $\boxed{\sum_{\alpha=1}^{N} \vec{F_\alpha} d\vec{r_\alpha} = -dU}$

- On a l equation de lagrange $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \sum_{\alpha=1}^{N} \vec{F_\alpha} \frac{\partial \vec{r_\alpha}}{\partial q_i} = Q_i$
  avec $Q_i = \sum_{\alpha=1}^{N} \vec{F_\alpha} \frac{\partial \vec{r_\alpha}}{\partial q_i} = \frac{-dU}{dq_i} \implies \boxed{Q_i = \frac{-\partial U}{\partial q_i}}$(si les forces sont conservatives)

- L'equation devien $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{-\partial U}{\partial q_i} \implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{-\partial}{\partial q_i}(T - U) = 0$

- U depend seulment du position alors $\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}(T - U)$
  l equation devien $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}(T - U)\right) - \frac{\partial}{\partial q_i}(T - U) = 0$

- On introdui la fonction de lagrange (lagrangien)

$$\boxed{L(qi, \dot{q}_i, t) = T - U}$$

T depend de vitess $\dot{q}_i$ et U depend de position q_i

- L equation devien

$$\text{Equation Euler-lagrange : } \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$