Problems on Group Theory

1. Let G be a group of order pq, where p and q are distinct primes. prove that G is abelian.

Statement : If $|G|=pq$ with ~~wit~~ distinct primes p, q, then G is abelian.

↳ False.

why False : By sylow theory the sylow-q subgroup is normal, so, G is a semi-direct product $P \rtimes Q$. If $p \nmid (q-1)$ the semidirect product is forced to be direct (so abelian), but when $p | (q-1)$ nontrivial semidirect products exits and can be nonabelian.
Example : $|G| = 6$ gives S_3, which is nonabelian.

2. prove that if G is a group of order p^2, where p is prime, then G is abelian if and only if it has $p+1$ subgroups of order p.

Statement : If $|G| = p^2$ (p prime), then G is abelian iff it has $p+1$ subgroups of order p.

⇒ True.

why True: The only groups of order p^2 are C_{p^2} (cyclic, one subgroup of order p) and $C_p \times C_p$ (elementary abelian, exactly $p+1$ subgroups of order p). Thus having $p+1$ subgroups of order p characterizes the abelian $C_p \times C_p$ case.

3. Let G be a finite group and H be a proper subgroup of G. prove that the union of all conjugates of H cannot be equal to G.

Statement: for finite G and proper $H \leq G$, the union of all conjugates of H cannot equal G.

⇒ True

why True: Let the conjugates be $H_1, \ldots, H_k$. Each conjugate intersects another in a proper subset, and counting shows $|\bigcup_i H_i| \le k(|H|-1)+1 \le [G:H](|H|-1)+1 < |G|$. So the union is strictly smaller than G.

4. Let G be a group and N be a normal subgroup of G. If G/N is cyclic and N is cyclic, prove that G is abelian.

Statement: If $N \trianglelefteq G$, N cyclic and G/N cyclic, then G is abelian.

$\leadsto$ False.

Why False: Take the dihedral group D_8 (order 8). Its rotation subgroup $N \cong C_4$ is cyclic and normal, and $D_8/N \cong C_2$ is cyclic, yet D_8 is nonabelian. (A correct stronger condition: if N and G/N are cyclic and their orders are coprime, then G is cyclic hence abelian.)

5. Prove that in any group G, the set of elements of finite order form a subgroup of G.

Statement : In any group G, the set of elements of finite order forms a subgroup.

⇒ False.

Why False : In general nonabelian groups torsion elements need not be closed under multiplication.

Example : the infinite dihedral group D_∞ has many reflections of order 2; product of two reflections is a translation of infinite order. So torsion elements donot form a subgroup in general. (They do form a subgroup in every abelian group — the torsion subgroup)

6. Let G be a finite group and p be the smallest prime dividing $|G|$. prove that any subgroup of index p in G is normal.

Statement : Let G be finite and let p be the smallest prime dividing $|G|$. Any subgroup of index p in G is normal.

$\Rightarrow$ True

why True: Let H have index p. The action of G on cosets gives $\varphi: G \to S_p$. By minimality of p the image $\varphi(G)$ must have order either 1 or p; transitivity forces order p. But a subgroup of of S_p of order p fixes a coset so, the kernel of the action equals H. Hence H is normal.

7. Let G be a group and $a, b \in G$. prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$.

Statement: If $a^4 = b^2$ and $ab = ba$ then $(ab)^6 = e$.

$\Rightarrow$ False.

why false: If a and b commute the $(ab)^6 = a^6 b^6$. From $b^2 = a^4$ we get $b^6 = (b^2)^3 = a^{12}$, so $(ab)^6 = a^{18}$. There is no reason $a^{18} = e$ in general. counterexample: in the infinite cyclic group $\langle g \rangle$ take $a = g$, $b = g^2$. Then $a^4 = g^4 = b^2$, they commute, but $(ab)^6 = g^{18} \neq e$. The claim needs extra hypotheses (e.g. finite orders ~~of~~ forcing $a^{18} = e$) to hold.

8. Let G be a group and H be a subgroup of G. prove that if $[G:H]=n$, then for any $x \in G$, $x^n \in H$.

Statement: If $[G:H]=n$ then for any $x \in G$, $x^n \in H$.

$\Rightarrow$ False.

why False: The general true statement is $x^{n!} \in H$ for all $x \in G$. Reason: the permutation action of G on there n cosets gives $\phi : G \to S_n$; the order of $\phi(x)$ divides $n!$, so $x^{n!} \in \ker \phi = \bigcap_{g \in G} gHg^{-1} \subseteq H$. The exponent n is not sufficient in general.

9. Let G be a finite group and p be a prime number. If G has exactly one subgroup of order p^k for each $k \le n$, where p^n divides $|G|$, prove that G has a normal sylow p-subgroup.

statement : If G has exactly one subgroup of order p^k for each $k \leq n$ (and p^n $\| |G|$), then G has a normal sylow p- subgroup.

$\Rightarrow$ True.

why True: Let P be the (unique) subgroup of order p^n. Any conjugate of P has the same order p^n, hence must equal P by uniqueness. Thus P is normal and is the sylow p-subgroup.

10. Let G be a finite group and H be a subgroup of G. prove that if $|G| = p$ where p is ~~normal~~ prime and p does not ~~the~~ divide m, and $|H| = p^n$, then H is normal ~~is~~ in G.

statement; If $|G| = p^n m$ with p prime and and $p \nmid m$, and if $H \leq G$ with $|H| = p^n$, then H is normal in G

⇒ True

Why True: A subgroup of order p^n is a Sylow p- subgroup. By Sylow theorem the number n_p of such subgroups divides m and satisfies $n_p \equiv 1 \pmod p$. since $p \nmid m$, the only divisor of m congruent to 1 (mod p) is 1, so $n_p = 1$. uniqueness implies normality.