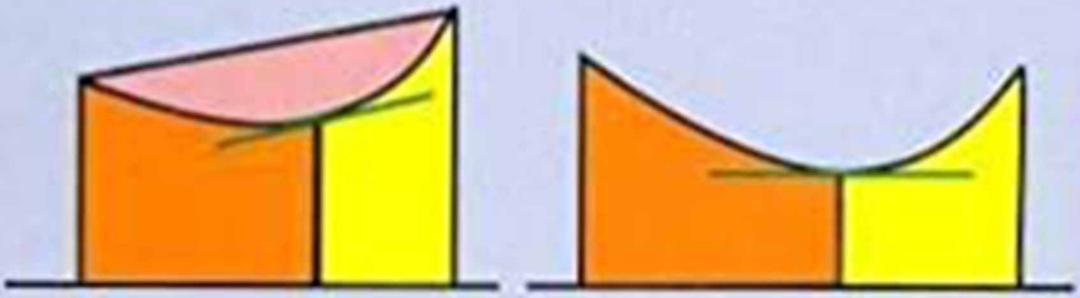
Introduction to Real Analysis

Revised 6th edition



S. K. Mapa

Introduction to
REAL ANALYSIS
(FOR DEGREE HONOURS COURSE)

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PREFACE

This is a thoroughly revised edition. A new chapter on 'Improper integrals' has been added. Additional materials have been incorporated in some chapters in order to stimulate the reader's interest. Some more examples have been fully worked out throughout the chapters in order to help the reader develop skill in working out exercises.

The mistakes and misprints that crept in the previous edition have been eliminated.

The author likes to appreciate suggestions given by his esteemed colleagues and by his inquisitive students for the improvement of the book.

The author conveys his sincere thanks to SARAT BOOK DISTRIBUTORS and the printer for the care and co-operation rendered by them in course of publication of the book.

CALCUTTA
July, 2006

S. K. Ma

PREFACE TO THE FIRST EDITION

This is an elementary treatise covering only a part of Real Analysis and it is designed to serve as a text book for undergraduate students of Mathematics Honours.

Keeping in mind that the volume is meant for the beginners of the subject the discussion of each topic has been supplemented by a good number of examples so that the reader can learn and understand the basic principles of Analysis.

There are many standard texts on Real Analysis. The author expresses his indebtedness to the authors of some of these texts which have been consulted during the preparation of this beginners' volume. A bibliography of such texts is given at the end.

The author likes to convey his sincere thanks to the publisher, the composer and the printer for the care and co-operation rendered by them in the process of publication of the book.

Any suggestion from the readers for the improvement of the book will be highly appreciated.

CALCUTTA
1.10.1996

S. K. MAPA

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1. SET THEORY

1.1. Introduction.

The concept of a 'set' is basic in all branches of Mathematics. In vie of the importance of set theory we present here a brief account of it. W take an entirely naive and intuitive view of a set.

For our purpose, a set is a well defined *collection* (or aggregate) c distinct objects (which are also called elements or points).

A set is usually denoted by capital letters $A, B, X, \ldots$, and an elemer of a set is denoted by small letters $a, b, x, \ldots$ When x is an element of set of A, it is expressed by the symbol $x \in A$ (read as x belongs to A and when x is not an element of a set A, it is expressed by the symb $x \notin A$ (read as x does not belong to A).

A set may be described by listing all its elements, usually betwee braces $\{\cdots\}$. Thus the set of all natural numbers less that 5 is $\{1, 2, 3, 4$ There is no significance in the order in which the elements are liste Therefore $\{2, 3, 1, 4\}$ and $\{1, 2, 3, 4\}$ describe the same set.

Another way of describing a set is $\{x : P(x)\}$ where $P(x)$ is tl proposition about x and $\{x : P(x)\}$ is the collection of those elements for which $P(x)$ is true. For example, the set $\{x : x$ is an even positi integer$\}$ is the set $\{2, 4, 6, \ldots \ldots\}$.

Throughout this text we use accepted notations for some familiar se of numbers.

$\mathbb{N} = \{1, 2, 3, \ldots \ldots\}$, the set of all natural numbers,
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots \cdots\}$, the set of all integers,
$\mathbb{Q} = \{\frac{p}{q} : p$ and q are integers, $q \neq 0\}$, the set of all rational numbe
$\mathbb{R} =$ the set of all real numbers.

1.2. Subsets.

Let A and B be two sets. If $x \in A \Rightarrow x \in B$ then A is said to l a *subset* of B. This means that each element of A is an element of This is expressed by $A \subset B$ or by $B \supset A$. In this case B is said to be *superset* of A.

We conceive of the existence of a set which contains no element. This is called the *null set* or *empty set* and is denoted by ϕ. For logical consistency, $\phi \subset A$ for every set A.

For every set A, $A \subset A$. Also $\phi \subset A$ for every set A. A and ϕ are said to be the *improper subsets* of A. Any other subset of A is called a *proper subset* of A.

For example, the set $\mathbb{Z}$ is a proper subset of the set $\mathbb{Q}$.

Two sets A and B are said to be *equal* if $A \subset B$ and $B \subset A$.

Definition. A set S is said to be a *finite set* if either it is empty or it contains a finite number of elements ; otherwise it is said be an *infinite set*.

1.3. Algebraic operations on sets.

In this section we shall discuss several ways of combining different sets. For this purpose we shall consider several sets, in a particular discussion, as subsets of a single fixed set, called the *universal set* in relation to its subsets. A universal set is generally denoted by U.

Let U be the universal set and $A, B, C, \ldots$ be subsets of U. We define the following operations on the class of all subsets of U.

(a) Union. The *union (or join)* of the subsets A and B is a subset of U, denoted by $A \cup B$ and is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\} .$$

Therefore $A \cup B$ is the set of all those elements which belong to A or to B or to both. It follows that $A \subset A \cup B, B \subset A \cup B$.

(b) Intersection. The *intersection (or meet)* of two subsets A and B is a subset of U, denoted by $A \cap B$ and is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Therefore $A \cap B$ is the set of all those elements which belong to both A and B. It follows that $A \cap B \subset A, A \cap B \subset B$.

Definition. If two subsets A and B have no common element then $A \cap B = \phi$ and A and B are said to be *disjoint*.

Example.

1. Let $A = \{1,2,3,4,5\}, B = \{3,5,7,9,11\}, C = \{2,4,6,8,10\}, D = \{2,6,10\}$ be subsets of the set $\mathbb{N}$.

Then $A \cup B = \{1,2,3,4,5,7,9,11\}, A \cap B = \{3,5,\}, A \cap C = \{2,4\}, B \cap C = \phi$, $C \cup D = C$, $C \cap D = D$.

B and C are disjoint subsets of $\mathbb{N}$. D is a proper subset of C.

Properties.

1. Consistency property. The three relations $A \subset B, A \cup B = B$ and $A \cap B = A$ are equivalent, i.e., one of these implies the other two.

1a. $A \cup \phi = A$, $A \cap \phi = \phi$. This follows from 1 by taking $\phi \subset A$.

1b. $A \cup U = U$, $A \cap U = A$. This follows from 1 by taking $A \subset U$.

1c. $A \cup A = A$, $A \cap A = A$. *(Idempotent property).* This follows from 1 by taking $A \subset A$.

1d. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$. *(Absorptive property).*This follows from 1 by taking $A \cap B \subset A \subset A \cup B$.

2. $A \cup B = B \cup A$, $A \cap B = B \cap A$. *(Commutative property).*

3. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$. *(Associative property).*

4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(Distributive property).*

(c) Complementation. The *complement* of a subset A is a subset of U, denoted by A' (or A^c) and is defined by

$$A' = \{x \in U : x \notin A\}.$$

A' contains all those elements of U which are not in A.

Properties.

5. $A \cup A' = U, A \cap A' = \phi$ for any subset A.

6. $(A')' = A$ for any subset A.

7. De Morgan's Laws. $(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$.

(d) Difference. The *difference* of two subsets A and B is a subset of U, denoted by $A - B$ and is defined by

$$A - B = \{x \in A : x \notin B\}.$$

$A - B$ is a subset of A and is the set of those elements of A which are not in B. $A - B$ is the relative complement of B in A. The difference $A - B$ can be expressed as $A - B = A \cap B'$.

Examples (continued).

2. Let $U = \{1, 2, 3, \ldots, 10\}$ be the universal set and $A = \{1, 3, 5, 7, 9\}$. Then $A' = \{2, 4, 6, 8, 10\}$.

3. Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8, 10\}$ be subsets of the set $\mathbb{N}$. Then $A - B = \{1, 3, 5\}, B - A = \{8, 10\}$.

(e) Symmetric Difference. The *symmetric difference* of two subsets A and B is a subset of U, denoted by $A \triangle B$ and is defined by

$$A \triangle B = (A - B) \cup (B - A).$$

$A \triangle B$ is the set of all those elements which belong either to A or to B but not to both.

Example (continued).

4. Let $A = \{1, 2, 3, 4\}, B = \{2, 4, 6, 8\}$ be subsets of the set $\mathbb{N}$. Then $A \triangle B = \{1, 3, 6, 8\}$.

Properties.

8. $A \triangle A = \phi$ for all subsets $A \subset U$.

9. $A \triangle B = B \triangle A$.

10. $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

1.4. Family of sets.

We have defined a set as a collection of its elements. If the elements be the subsets of a universal set then we have a *set of sets* or a *family of sets*.

Examples.

1. Let X be a non-empty set. The collection of all subsets of X is a family of sets. This family is called the *power set* of X and is denoted by $P(X)$.

If $X = \{1, 2, 3\}$ then $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$.

If X contains n elements then $P(x)$ contains 2^n elements.

2. Let I be the finite set $\{1, 2, \ldots, n\}$ and $\mathcal{F}$ be the family of n sets $A_1, A_2, \ldots, A_n$. $\mathcal{F}$ is expressed as $\{A_\alpha : \alpha \in I\}$. I is called the *index set*. The elements of $\mathcal{F}$ are said to be indexed by the index set I.

3. Let $I = \mathbb{N}$ and $\mathcal{F}$ be the family of sets $A_1, A_2, \ldots \ldots$ $\mathcal{F}$ is expressed as $\{A_n : n \in \mathbb{N}\}$. Here $\mathbb{N}$ is the index set.

4. Let $\wedge$ be an arbitrary set. The family of sets $\mathcal{F} = \{A_\alpha : \alpha \in \wedge\}$ is the collection of sets A_α, for each $\alpha \in \wedge$. Here $\wedge$ is the index set.

5. For each $n \in \mathbb{N}$, let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$. Then we have a family of sets $\{I_n : n \in \mathbb{N}\}$. The union of the family is denoted by $\bigcup_{n \in \mathbb{N}} I_n$ and the intersection of the family is denoted by $\bigcap_{n \in \mathbb{N}} I_n$.

Here $\bigcup_{n \in \mathbb{N}} I_n = \{x \in \mathbb{R} : 0 < x < 1$ and $\bigcap_{n \in \mathbb{N}} I_n = \phi$.

1.5. Cartesian product of sets.

Let A and B be non-empty sets. The *Cartesian product* of A and l denoted by $A \times B$, is the set defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

$A \times B$ is the set of all ordered pairs (a, b), the first element of the pa being an element of A and the second being an element of b.

Let $A_1, A_2, \ldots, A_n$ be a finite collection of non-empty sets. The Cart sian product of the collection, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the s defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i, i = 1, 2, \ldots, n\}.$$

In particular if $A_1 = A_2 = \cdots = A_n = A$, then the Cartesian pro uct of the collection, denoted by A^n, is the set of all ordered n tupl $\{(a_1, a_2, \ldots, a_n) : a_i \in A, i = 1, 2, \ldots, n\}$.

Examples.

1. Let $A = \{1, 2, 3\}, B = \{2, 4\}$.
   Then $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$;
   $B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$.

2. Let $A = \mathbb{Z}, B = \mathbb{Z}$. Then $A \times B$ is the set of all ordered pairs integers.

3. Let $A = \mathbb{R}, B = \mathbb{R}, C = \mathbb{R}$. Then $A \times B \times C$ is denoted by $\mathbb{R}^3$ and is the set of all ordered triplets $\{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$.

4. $\mathbb{R}^n$ is the set of all ordered n-tuples $\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$.

1.6. Relation on a set.

Let A and B be two non-empty sets. Intuitively, a relation ρ *betwee* A and B is a *rule* that associates some or all the elements of A with sor element or elements of B.

Definition. Let A and B be two non-empty sets. A relation ρ betwee A and B is a subset of $A \times B$. If the ordered pair $(a, b) \in \rho$ then th element a of the set A is said to be related to the element b in B by th relation ρ. If $(a, b) \in (A \times B) - \rho$, then a is said to be *not related* to b t the relation ρ.

Example.

1. Let $A = \{2, 3, 4, 5\}, B = \{4, 6, 8, 9\}$. A relation ρ between A and B defined by specifying that an element a in A is related to an element in B if a is a divisor of b.

Then $\rho = \{(2,4),(2,6),(2,8),(3,6),(3,9),(4,4),(4,8)\}$. $(2,5) \in A \times B$ but $(2,5) \notin \rho$ since 2 is not a divisor of 5. 2 in A is related to three elements in B, 3 in A is related to two elements in B, 4 in A is related to two elements in B, 5 in A is related to no element in B.

Definition. Let A be a non-empty set. A relation ρ on A is a subset of $A \times A$.

If (a, b) be an element of $A \times A$ and $(a, b) \in \rho$, then a is said to be related to b by the relation ρ. This is expressed by $a \ \rho \ b$.

If (p, q) be an element of $A \times A$ and $(p, q) \notin \rho$, then p is said to be not related to q by the relation ρ. This is expressed by $p \ \bar{\rho} \ q$.

Let X be a non-empty set. A relation ρ on X is said to be *reflexive* if $a \ \rho \ a$ holds for all $a \in X$.

ρ is said to be *symmetric* if $a \ \rho \ b \Rightarrow b \ \rho \ a$ for $a, \ b \in X$.

ρ is said to be *anti-symmetric* if $a \ \rho \ b$ and $b \ \rho \ a \Rightarrow a = b$ for $a, \ b \in X$.

ρ is said to be *transitive* if $a \ \rho \ b$ and $b \ \rho \ c \Rightarrow a \ \rho \ c$ for $a, \ b, \ c \in X$.

Examples.

1. Let a relation ρ be defined on $\mathbb{Z}$ by "$a \ \rho \ b$ if and only if $a - b$ is even" for $a, b \in \mathbb{Z}$. Then ρ is reflexive, symmetric and transitive, but not anti-symmetric.

2. Let a relation ρ be defined on $\mathbb{N}$ by "$a \ \rho \ b$ if and only if a is a divisor of b". Then ρ is reflexive, anti-symmetric and transitive, but not symmetric.

1.7. Order relation on a set.

Definition. Partially ordered set.

A relation ρ on a non-empty set X is said to be a *partial order relation* if ρ is reflexive, anti-symmetric and transitive.

A set X equipped with a partial order relation ρ is said to be a *partially ordered set* (or a *poset*) and it is denoted by (X, ρ).

Note. A partial order relation is commonly denoted by the symbol $\leq$ or $\geq$ and read in usual manner. Thus $a \leq b$ is read as "a is less than or equal to b". A partially ordered set X with a partial order $\leq$ is denoted by $(X, \leq)$.

Examples (continued).

3. $(\mathbb{N}, \leq)$ is a poset where "$a \leq b$ means a is a divisor of b", for $a, b \in \mathbb{N}$.

4. Let $\mathcal{F}$ be a family of sets. Then the set inclusion $\subset$ is a partial order on $\mathcal{F}$ and $(\mathcal{F}, \subset)$ is a poset.

Definition.

A relation ρ on a set X is said to be a *strict order relation* if it is anti-symmetric and transitive and for which $(a, a) \notin \rho$ for all $a \in X$.

If a partial order be denoted by $\leq$, then the corresponding strict order is denoted by $<$.

Two elements a and b in a partially ordered set are said to be *comparable* if one of them is related to the other, i.e., one of the relations $a \leq b$, $b \leq a$ must hold. In a partially ordered set there may exist elements a and b which are not comparable. For example, in Ex.3, the integers 4 and 6 are not comparable, because neither is a divisor of the other.

If a partial order relation satisfies a fourth condition that 'any two elements are comparable' then it is called a *total order relation*.

Definition. Ordered set.

A partial order $\leq$ on a set X is said to be a *linear order* (or a *total order*) if any two elements of X be comparable, i.e., for $a, b \in X$, either $a \leq b$ or $b \leq a$.

This property is known as the *law of dichotomy* and is expressed by saying that the elements of the set X are comparable under $\leq$.

Translating in terms of strict order $<$, it says that for all $a, b \in X$, either $a < b$, or $a = b$, or $a > b$. This property is known as the *law of trichotomy*.

A set X together with a linear order $\leq$ defined on it is said to be a *linearly ordered set*, or a *totally ordered set*.

In a poset $(X, \leq)$ a subset C which is linearly ordered under the given order relation $\leq$ on X is called a *chain*.

A linearly ordered set $(X, \leq)$ is said to be a *well ordered set* if every non-empty subset S of X has a *least* element. A least element in the subset S is an element a in S such that $a \leq s$ for each element s in S.

Examples (continued).

5. $(\mathbb{N}, \leq)$ is a linearly ordered set where $a \leq b$ has its usual meaning. It is also a well-ordered set.

6. In $(\mathbb{N}, \leq)$ of Ex.3, the subset $\{1, 2, 4, 8, \ldots\}$ is a chain.

Definition.

Let $(X, \leq)$ be an ordered set. Let $S \subset X$.

An element $u \in X$ is said to be an *upper bound* for S if $s \leq u$ for each $s \in S$. An element $l \in X$ is said to be a *lower bound* for S if $l \leq s$ for each $s \in S$.

A subset S of X is said to be *bounded* above if S has an upper bound.

A subset S of X is said to be *bounded* below if S has a lower bound.

In an ordered set $(X, \leq)$ the empty set ϕ is bounded above and bounded below. Every element of X is an upper bound of ϕ and also every element of X is a lower bound of ϕ.

Let $(X, \leq)$ be an ordered set and S be a subset of X bounded above. S is said to have a *least upper bound* (or a *supremum*) (in X) if there exists an upper bound x^* of S such that $x^* \leq u$ for every upper bound u of S.

Let $(X, \leq)$ be an ordered set and S be a subset of X bounded below. S is said to have a *greatest lower bound* (or an *infimum*) (in X) if there exists a lower bound x_* of S such that $l \leq x_*$ for every lower bound l of S.

Theorem 1.7.1. In an ordered set $(X, \leq)$ if a subset S has a supremum x^*, then x^* is unique.

Proof. If possible, let x^*, x_1^* be two suprema of S.

As x^* is a supremum and x_1^* is an upper bound of S, $x^* \leq x_1^*$.

As x_1^* is a supremum and x^* is an upper bound of S, $x_1^* \leq x^*$.

It follows that $x_1^* = x^*$.

Theorem 1.7.2. In an ordered set $(X, \leq)$ if a subset S has an infimum x_*, then x_* is unique.

Similar proof.

Theorem 1.7.3. In an ordered set $(X, \leq)$ the following statements are equivalent:

(a) Every non-empty subset S that is bounded above, has a supremum.

(b) Every non-empty subset S that is bounded below, has an infimum.

Proof. We prove that (a) implies (b).

Let S be a non-empty subset of X, bounded below. Let l_0 be a lower bound of S. Let $T = \{l : l \in X \text{ and } l \text{ is a lower bound of } S\}$. Then T is a non-empty subset of X because $l_0 \in T$. Moreover $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above. Thus T is a non-empty subset of X, bounded above.

By (a), T has a supremum. Let $\sup T = L$.

Then (i) $t \leq L$ for every $t \in T$, since L is an upper bound of T;

and (ii) since every $s \in S$ is an upper bound of T and $L = \sup T, L \leq s$ for every $s \in S$.

(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. Consequently, $L = \inf S$.

We prove that (b) implies (a).

Let S be a non-empty subset of X, bounded above.

Let u_0 be an upper bound of S. Let $T = \{u : u \in X$ and u is an upper bound of $S\}$. Then T is a non-empty subset of X because $u_0 \in T$

Moreover $x \in T$ and $s \in S \Rightarrow x \geq s$. This shows that T is bounded below. Thus T is a non-empty subset of X bounded below.

By (b) T has an infimum.

Let $\inf T = U$. Then (i) $U \leq t$ for every $t \in T$, since V is a lower bound of T; and (ii) since every $s \in S$ is a lower bound of T and $U = \inf T, U \geq s$ for every $s \in S$.

(ii) shows that U is an upper bound of S and (i) shows that $U \leq$ any upper bound of S.

Consequently, $U = \sup S$.

Definition. An ordered set $(X, \leq)$ is said to be *order complete* if every *non-empty* subset of X which has an upper bound, has a least upper bound (a supremum), or equivalently, every *non-empty* subset of X which has a lower bound, has a greatest lower bound (an infimum).

Note. In a later chapter we shall see that the set $\mathbb{R}$ is order complete but the set $\mathbb{Q}$ is not.

1.8. Function.

Let A and B be two non-empty sets. A *function* f from A to B is a *rule of correspondence* that assigns to each element x in A, *a unique* y in B.

A is said to be the *domain* of f and B, the *co-domain* of f. We say that f is a function or a mapping from A to B and we write $f : A \longrightarrow B$

The unique element y in B that corresponds to x in A is said to be *the image* of x under f and is denoted by $f(x)$. x is said to be a *pre-image* (or an inverse image) of $f(x)$.

The set $\{f(x) : x \in A\}$ is said to be the *range* of f and is denoted by $f(A)$.

The set $\{x : f(x) = y\}$ is said to be the *pre-image set* (or the *inverse image set*) of y and is denoted by $f^{-1}(y)$.

This is to emphasize that the inverse image set of an element y in B may be a void set, or a singleton set, or a set containing more than one elements.

Definitions.

A function $f : A \to B$ is said to be *injective* (or one-to-one) if $x_1 \neq x_2$ in $A \Rightarrow f(x_1) \neq f(x_2)$ in B.

A function $f : A \to B$ is said to be *surjective* if $f(A) = B$.

A function $f : A \to B$ is said to be *bijective* if f is injective as well as surjective.

If $f : A \to B$ is injective then each element of B has at most one pre-image.

If $f : A \to B$ is surjective then each element of B has at least one pre-image.

Therefore if $f : A \to B$ is bijective then each element of B has exactly one pre-image. In this case the pre-image set of each element y in B, i.e., $f^{-1}(y)$ is a singleton set.

If $f : A \to B$ is bijective, each element in A has a definite image in B and each element in B has a definte pre-image in A. Thus f sets up a one-to-one correspondence between the elements of A ana B.

Examples.

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x, x \in \mathbb{Z}$. f is injective but not surjective. The range set is $2\mathbb{Z}$ (the set of all even integers) and it is a proper subset of the co-domain set $\mathbb{Z}$.

Here $f(0) = 0$, $f(1) = 2$, $f(2) = 4$, ...

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|, x \in \mathbb{Z}$. f is not injective, since $f(1) = f(-1) = 1$. f is not surjective, since -1 in the co-domain set has no pre-image.

Here $f^{-1}(1) = \{1, -1\}$, $f^{-1}(-1) = \phi$, ...

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1, x \in \mathbb{Z}$. f is injective as well as surjective. Therefore f is bijective.

Equality of functions.

Two functions f and g are said to be *equal* if

(i) f and g have the same domain D, and (ii) $f(x) = g(x)$ for all $x \in D$.

Example (continued).

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|, x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x, x \geq 0$ Then $f = g$.

$$= -x, x < 0.$$

1.9. Equipotent sets. Enumerable set.

Let A and B be subsets of a universal set $P(X)$, the power set of a non-empty set X. A is said to be *equipotent* with B if there exists a bijective mapping f from A to B. We write $A \sim B$.

The relation of equipotence on the set $P(X)$, is an equivalence relation, because

(i) for any subset $A \in P(X)$, the identity mapping $i : A \to A$ is a bijective mapping and therefore $A \sim A$ for all $A \in P(X)$.

(ii) If $A \sim B$ for $A, B \in P(X)$, then there exists a bijective mapping $f : A \to B$ and this ensures the existence of the bijective mapping $f^{-1} : B \to A$, proving that $B \sim A$.

(iii) If $A \sim B$ and $B \sim C$ for $A, B, C \in P(X)$, then there exist bijective mappings $f : A \to B$ and $g : B \to C$ and this ensures the existence of the bijective mapping $g.f : A \to C$, proving that $A \sim C$.

In consequence of this equivalence relation on $P(X)$, the set $P(X)$ is partitioned into classes of equipotent sets.

The sets belonging to the same equipotence class are said to have the same *potency* or the same *cardinal number*.

The cardinal number assigned to the equipotence class of finite sets each with n elements is n. The cardinal number assigned to the null set ϕ is 0.

The cardinal number of an infinite set is said to be a *transfinite cardinal number*. The cardinal number of the set $\mathbb{N}$ is denoted by d.

Definition. A set A is said to be *enumerable* (or denumerable) if A is equipotent with $\mathbb{N}$. A set which is either finite or enumerable is said to be a *countable* set. Enumerable sets are also called *countably infinite* sets.

When a set A is enumerable, there is a bijective mapping $f : \mathbb{N} \to A$. Corresponding to each $n \in \mathbb{N}$ there is a unique element $f(n)$ in A as the image of n. Thus the elements of A can be described as $f(1), f(2), \ldots, f(n), \ldots$, or as $a_1, a_2, \ldots, a_n, \ldots$, showing that the elements of A are indexed by the set $\mathbb{N}$.

Examples.

1. The cardinal number of the set $\{1, 2, 3, \ldots, 10\}$ is 10.

2. The cardinal number of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is d, because S is equipotent with $\mathbb{N}$. S is an enumerable set.

That there exist sets larger than the enumerable sets (i.e., the sets with cadinal number greater than d) has been established by George Cantor, a German mathematician, in his remakable theorem.

Theorem 1.9.1. Cantor's theorem.

If A be a non-empty set, there is no surjection $\phi : A \rightarrow P(A)$, where $P(A)$ is the power set of A.

Proof. Let $a \in A$. Let $f : A \rightarrow P(A)$ be a surjection. Then $f(a)$ is an element of $P(A)$, i.e., $f(a)$ is a subset of A. The element a may or may not belong to the subset $f(a)$.

Let $S = \{a \in A : a \notin f(a)\}$. Since S is a subset of A, $S \in P(A)$ and therefore there exists an element $a_0 \in A$ such that $f(a_0) = S$.

We must have either $a_0 \in S$ or $a_0 \notin S$.

$a_0 \in S \Rightarrow a_0 \notin f(a_0)$ (by definition of S), i.e., $a_0 \notin S$, a contradiction.
$a_0 \notin S \Rightarrow a_0 \in f(a_0)$ (by definition of S), i.e., $a_0 \in S$, a contradiction.
Therefore f does not exist and the proof is complete.

The theorem asserts the existence of larger and still larger sets, i.e., the sets with greater and still greater cardinal numbers.

We shall prove in a subsequent article that the set $\mathbb{R}$ is non enumerable.

Exercises 1

1. If A, B, C be subsets of $\mathbb{R}$, prove that

(i) $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$;

(ii) $(A \cap B \cap C) \cup (A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A \cap B' \cap C') = A$;

(iii) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

2. Find $\bigcup\limits_{n=1}^{\infty} I_n$ and $\bigcap\limits_{n=1}^{\infty} I_n$, where for each $n \in \mathbb{N}$,

(i) $I_n = \left\{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}\right\}$, (ii) $I_n = \left\{x \in \mathbb{R} : -1 + \frac{1}{n} \leq x \leq 2 - \frac{1}{n}\right\}$.

3. Let S be the set of all positive divisors of 30. Prove that $(S, \leq)$ is a poset where $a \leq b$ means a is a divisor of b, for $a, b \in S$.

4. Prove that the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are neither injective nor surjective.

(i) $f(x) = \frac{x}{x^2+1}, x \in \mathbb{R}$, (ii) $f(x) = \frac{|x|}{|x|+1}, x \in \mathbb{R}$, (iii) $f(x) = [x], x \in \mathbb{R}$.

5. Are the two functions f and g equal? Give reasons.

(i) $f : D \rightarrow \mathbb{R}$ defined by $f(x) = \sin x - \cos x, x \in D$,
$g : D \rightarrow \mathbb{R}$ defined by $g(x) = \sqrt{1 - \sin 2x}, x \in D$;
and $D = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}\}$.

(ii) $f : D \rightarrow \mathbb{R}$ defined by $f(x) = 2\tan^{-1} x, x \in D$,
$g : D \rightarrow \mathbb{R}$ defined by $g(x) = \tan^{-1} \frac{2x}{1-x^2}, x \in D$;
and $D = \{x \in \mathbb{R} : x > 1\}$.

2.1. Natural numbers.

The natural numbers are $1, 2, 3, \ldots$ The set of all natural numbers denoted by $\mathbb{N}$.

We assume familiarity with the algebraic operations of addition a multiplication on the set $\mathbb{N}$ and also with the linear order relation $<$ $\mathbb{N}$ defined by "$a < b$ if $a, b \in \mathbb{N}$ and a is less than b".

We discuss the following fundamental properties of $\mathbb{N}$.

1. Well ordering property.
2. Principle of induction.

2.1.1. Well ordering property. Every non-empty subset of $\mathbb{N}$ ha least element.

This means that if S is a non-empty subset of $\mathbb{N}$ then there is element m in S such that $m \leq s$ for all $s \in S$. In particular, $\mathbb{N}$ itself the least element 1.

Proof. Let S be a non-empty subset of $\mathbb{N}$. Let k be an element o Then k is a natural number.

We define a subset T by $T = \{x \in S : x \leq k\}$. Then T is a non-em subset of $\{1, 2, \ldots, k\}$.

Clearly, T is a finite subset of $\mathbb{N}$ and therefore it has a least elem say m. Then $1 \leq m \leq k$.

We now show that m is the least element of S. Let s be any elem of S.

If $s > k$ then the inequality $m \leq k$ implies $m < s$.

If $s \leq k$ then $s \in T$; and m being the least element of T, we $m \leq s$.

Thus m is the least element of S. This completes the proof.

2.1.2. Principle of induction. Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and

(ii) if $k \in S$ then $k + 1 \in S$.

Then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. Then by the well ordering property of $\mathbb{N}$, the non-empty subset T has a least element, say m.

Since $1 \in S$ and 1 is the least element of $\mathbb{N}, m > 1$.

Hence $m - 1$ is a natural number and $m - 1 \notin T$. So $m - 1 \in S$.

But by (ii) $m - 1 \in S \Rightarrow (m - 1) + 1 \in S$, i.e., $m \in S$.

This contradicts that m is the least element in T. Therefore our assumption is wrong and $T = \phi$.

Therefore $S = \mathbb{N}$. This completes the proof.

Theorem 2.1.3. Let $P(n)$ be a statement involving a natural number n.

If (i) $P(1)$ is true, and

(ii) $P(k + 1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S be the set of those natural numbers for which the statement $P(n)$ is true.

Then S has the properties (a) $1 \in S$ by (i)

(b) $k \in S \Rightarrow k + 1 \in S$ by (ii).

By the principle of induction $S = \mathbb{N}$.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$. This completes the proof.

Note. Let a statement $P(n)$ satisfies the conditions

(i) for some $m \in \mathbb{N}, P(m)$ is true (m being the least possible)

and (ii) $P(k)$ is true $\Rightarrow P(k + 1)$ is true for all $k \geq m$.

Then $P(n)$ is true for all natural numbers $\geq m$.

Worked Examples.

1. Prove that for each $n \in \mathbb{N}, 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

The statement is true for $n = 1$, because $1 = \frac{1(1+1)}{2}$.

Let the statement be true for some natural number k.

Then $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ and therefore

$(1 + 2 + 3 + \cdots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$

or, $1 + 2 + 3 + \cdots + (k + 1) = \frac{(k+1)(k+2)}{2}$.

This shows that the statement is true for the natural number $k + 1$ if it is true for k. By the principle of induction, the statement is true for all natural numbers.

2. Prove that for each $n \geq 2, (n + 1)! > 2^n$.

The inequality holds for $n = 2$ since $(2 + 1)! > 2^2$.

Let the inequality hold for some natural number $k \geq 2$.
Then $(k+1)! > 2^k$

$$\begin{aligned} \text{and } (k+2)! &= (k+2)(k+1)! \\ &> 2.2^k \text{ since } k+2 > 2 \\ \text{or, } (k+2)! &> 2^{k+1}. \end{aligned}$$

This shows that if the inequality holds for $k(\geq 2)$ then it also holds for $k+1$.

By the principle of induction, the inequality holds for all natural numbers ≥ 2.

[Note that the inequality does not hold for $n = 1$.]

2.1.4. Second principle of induction.

Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and (ii) if $\{1, 2, 3, \ldots, k\} \subset S$ then $k+1 \in S$.
Then $S = \mathbb{N}$.

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. Then T will have a least element, say m, by the well ordering property of $\mathbb{N}$. Since $1 \in S, 1 \notin T$.

As m is the least element in T and $1 \notin T, m > 1$.
By the choice of m, all natural numbers less than m belong to S.
That is, $1, 2, \ldots, m-1$ all belong to S.

Then by (ii) $m \in S$ and consequently, $m \notin T$, a contradiction.
It follows that $T = \phi$ and therefore $S = \mathbb{N}$.

This completes the proof.

Worked Example (continued).

3. Prove that for all $n \in \mathbb{N}, (3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer.

Let $P(n)$ be the statement "$(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer ".
$P(1)$ is true since $(3+\sqrt{5})^1 + (3-\sqrt{5})^1 = 6$, an even integer.

Let us assume that $P(n)$ is true for $n = 1, 2, \ldots, k$.

$(3+\sqrt{5})^{(k+1)} + (3-\sqrt{5})^{(k+1)}$
$= a^{k+1} + b^{k+1}$ where $a = 3+\sqrt{5}, b = 3-\sqrt{5}$
$= (a^k + b^k)(a+b) - (a^{k-1} + b^{k-1})ab$
$= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1})$.

It is an even integer, since $a^k + b^k$ and $a^{k-1} + b^{k-1}$ are even integers.
Hence $P(k+1)$ is true whenever $P(n)$ is true for $n = 1, 2, \ldots, k$.

By the second principle of induction, $P(n)$ is true for all natural numbers.

2.2. Integers.

Addition and multiplication are defined on the set $\mathbb{N}$. Subtraction is not defined on the set $\mathbb{N}$ in the sense that if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ that $a - b$ is not always an element of $\mathbb{N}$. The set $\mathbb{N}$ is enlarged by the inclusion of 0 and the numbers of the form $-n$ (called *negative* of n) for all $n \in \mathbb{N}$. The new set is called the set of *all integers* and is denoted by $\mathbb{Z}$.

$$\mathbb{Z} = \{0, 1, 2, 3, \ldots, -1, -2, -3, \ldots\}.$$

On $\mathbb{Z}$, subtraction is defined as the inverse of additon. For each $a \in \mathbb{Z}, -a \in \mathbb{Z}$. If $a \in \mathbb{Z}, b \in \mathbb{Z}$ then $a - b$ is defined by $a + (-b)$ and $a - b \in \mathbb{Z}$.

Multiplication is defined on $\mathbb{Z}$. But division, the inverse operation of multiplication, is not defined on $\mathbb{Z}$ in the sense that if $a \in \mathbb{Z}, b \in \mathbb{Z}$ then $\frac{a}{b}$ is not always an element of $\mathbb{Z}$. If the set $\mathbb{Z}$ could be enlarged by the inclusion of all numbers of the form $\frac{a}{b}$ where $a \in \mathbb{Z}, b \in \mathbb{Z}$ then the new enlarged set might be rich anough to allow division as the inverse operation of multiplication. But if $a \in \mathbb{Z}$ and $b = 0$ then there is no number of the form $\frac{a}{0}$ and therefore the enlargement of the set $\mathbb{Z}$ by inclusion of all numbers of the form $\frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{Z}$ cannot be completed.

2.3. Rational numbers.

A *rational number* is of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$.

Evidently every integer is a rational number. The set $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$.

We now describe some fundamental properties of the set $\mathbb{Q}$.

1. Algebraic properties of $\mathbb{Q}$.
2. Order properties of $\mathbb{Q}$.
3. Density property of $\mathbb{Q}$.

2.3.1. Algebraic properties of $\mathbb{Q}$.

Addition and multiplication are defined on the set $\mathbb{Q}$ satisfying the following properties:

A1. $a + b \in \mathbb{Q}$ for all $a, b \in \mathbb{Q}$,

A2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Q}$,

A3. there exists an element 0 in $\mathbb{Q}$ (called the zero element) such that $a + 0 = a$ for all $a \in \mathbb{Q}$,

A4. for each a in $\mathbb{Q}$ there exists an element $-a$ in $\mathbb{Q}$ such that $a + (-a) = 0$,

A5. $a + b = b + a$ for all $a, b \in \mathbb{Q}$, M1. $a.b \in \mathbb{Q}$ for all $a, b \in \mathbb{Q}$,

M2. $(a.b).c = a.(b.c)$ for all $a, b, c \in \mathbb{Q}$,

M3. there exists an element 1 in $\mathbb{Q}$ (called the unity) such that $a.1 =$ for all $a \in \mathbb{Q}$.

M4. for each element $a \neq 0$ in $\mathbb{Q}$ there exists an element $\frac{1}{a}$ in $\mathbb{Q}$ su that $a.\frac{1}{a} = 1$,

M5. $a.b = b.a$ for all $a, b \in \mathbb{Q}$,

D. $a.(b + c) = a.b + a.c$ for all $a, b, c \in \mathbb{Q}$.

$-a$ is the additive inverse of a. It is called the *negative* of a. $\frac{1}{a}$ the multiplicative inverse of a. It is also called the *reciprocal* of a. T reciprocal of a exists provided $a \neq 0$. The zero element 0 and the un 1 are unique elements.

A2 states that addition is associative on $\mathbb{Q}$. A5 states that additi is commutative on $\mathbb{Q}$. M2 states that multiplication is associative $\mathbb{Q}$. M5 states that multiplication is commutative on $\mathbb{Q}$. D states distributive property. Multiplication is distributive over addition.

The set $\mathbb{Q}$ is said to form a *field* under addition and multiplicatio

2.3.2. Order properties of $\mathbb{Q}$.

On the set $\mathbb{Q}$, a linear order relation $<$ is defined by "$a < b$ if $\mathbb{Q}, b \in \mathbb{Q}$ and a is less than b "and it satisfies the following condition:

O1. If $a, b \in \mathbb{Q}$ then exactly one of the following statements holds $a < b$, or $a = b$, or $b < a$; (law of trichotomy)

O2. $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbb{Q}$; (transitivity)

O3. $a < b \Rightarrow a + c < b + c$ for $a, b, c \in \mathbb{Q}$;

O4. $a < b$ and $0 < c \Rightarrow ac < bc$ for $a, b, c \in \mathbb{Q}$.

Note 1. $a < b$ is equivalently expressed as $b > a$ (b is greater than

The law of trichotomy states that a rational number a is one of following : $a < 0, a = 0, 0 < a$. i.e., $a < 0, a = 0, a > 0$.

A rational number a is said to be *positive* if $a > 0$ and is said to *negative* if $a < 0$.

2. If $a, b, c \in \mathbb{Q}$ and $a < c$, $c < b$ both hold, we write $a < c < b$. We that c lies between a and b.

3. The field $\mathbb{Q}$ together with the order relation defined on $\mathbb{Q}$ satisfy O1-O4 becomes an *ordered field*.

2.3.3. Density property of $\mathbb{Q}$.

If x and y be any two rational numbers and $x < y$, there exists a rational number r such that $x < r < y$. That is, between any two rational numbers there exists a rational number.

$$\begin{aligned} x < y \quad \Rightarrow \quad & x + y < y + y, \text{ by } O3 \\ \Rightarrow \quad & \tfrac{1}{2}(x+y) < \tfrac{1}{2}(2y), \text{ by } O4 \\ \text{i.e.,} \quad & \tfrac{1}{2}(x+y) < y. \end{aligned}$$

$$\begin{aligned} \text{Again, } x < y \quad \Rightarrow \quad & x + x < x + y, \text{ by } O3 \\ \Rightarrow \quad & \tfrac{1}{2}(2x) < \tfrac{1}{2}(x+y), \text{ by } O4 \\ \text{i.e.,} \quad & x < \tfrac{1}{2}(x+y). \end{aligned}$$

Therefore we have $x < \frac{1}{2}(x+y) < y$. Then $r = \frac{1}{2}(x+y)$.

We observe that between two rational numbers x and y (where $x < y$) there exists another rational number $\frac{1}{2}(x+y)$. Again between x and $\frac{1}{2}(x+y)$ (since $x < \frac{1}{2}(x+y)$) there exists another rational number and the process can be continued indefinitely.

We say that between any two rational numbers x and y (where $x < y$) there exist infinitely many rational numbers. This is expressed by saying that the set $\mathbb{Q}$ is dense and this property of $\mathbb{Q}$ is called the *density property* of $\mathbb{Q}$.

Because of this density property of $\mathbb{Q}$, between any two rational numbers x and y we can interpolate infinitely many rational numbers.

2.3.4. Geometrical representation of rational numbers.

Rational numbers can be represented by points on a straight line. Let $X'X$ be a directed line. We take a point O on the line. O divides the line into two parts. The part to the right of O is called the positive side and the part to the left of O is called the negative side.

Let us take a point A to the right of O. Let O represent the rational number *zero* and A represent the rational number *one*. Taking the distance OA as the unit distance on some chosen scale, each rational number can be represented by a unique point on the line. First of all, the positive integers $2, 3, \cdots$ are represented by the points $A_2, A_3, \cdots$ lying to the right of O where $OA_2 = 2OA, OA_3 = 3OA, \cdots$ and the negative integers $-1, -2, \cdots$ are represented by the points $A_1', A_2', \cdots$ lying to the left O such that $OA_1' = OA, OA_2' = 2OA, \cdots$

To represent a positive rational number r of the form $\frac{p}{q}$ where p, q are positive integers, we measure p times the distance OA to the right of O and get a point B and then measure the qth part of the distance OB to the right of O to the get the point P. P represents the rational number

r. If r be a negative rational number $(-s)$ then the point P' to the left of O (where $OP' = OP$ and P represents s) represents r.

Thus every rational number can be made to correspond to a point on the line. If a point that corresponds to a rational number be called a rational point then we observe that between any two rational points there lie infinitely many rational points. If all the rational numbers be plotted as points on the line it appears that the whole line is covered by rational points i.e., the whole line is composed of only rational points.

A little further examination will show that such a view point is not tenable.

If we take a point D to the right of O such that OD is the length of the diagonal of the square on the side OA, then D is not a rational point as can be established by the following example.

Example. There does not exist a rational number r such that $r^2 = 2$.

If possible, let p and q are integers such that $(p/q)^2 = 2$. It may be assumed that p and q have no common factor other than 1.

Now $(p/q)^2 = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even, because if p be odd then p^2 is also odd.

Let $p = 2m$, where m is an integer.
Then $p^2 = 2q^2 \Rightarrow 2m^2 = q^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even.

Thus we arrive at a contradiction to the assumption that p and q have no common factor other than 1.

Therefore there is no rational number whose square is 2.

The point D, therefore corresponds to a new type of number, called an *irrational number*.

The next theorem shows the existence of many irrational numbers.

Theorem 2.3.5. Let m be a non-square positive integer. There does not exist a rational number r such that $r^2 = m$.

Proof. Since m is a non-square positive integer, there exist two consecutive square integers λ^2 and $(\lambda+1)^2$ such that $\lambda^2 < m < (\lambda+1)^2$.

If possible, let $r = p/q$ (where p and q are positive integers prime to each other) be such that $r^2 = m$.

Then $\lambda^2 < (p/q)^2 < (\lambda+1)^2$
or, $\lambda < p/q < \lambda + 1$
or, $\lambda q < p < \lambda q + q$
or, $0 < p - \lambda q < q$ ... ... (i)

$$\begin{aligned} m(p-\lambda q)^2 &= mp^2 - 2\lambda mpq + \lambda^2 mq^2 \\ &= m^2q^2 - 2\lambda mpq + \lambda^2 p^2, \text{ since } mq^2 = p^2 \\ &= (mq-\lambda p)^2. \end{aligned}$$

So $m = (\frac{mq-\lambda p}{p-\lambda q})^2$.

Thus $m = (p/q)^2 \Rightarrow m = (\frac{mq-\lambda p}{p-\lambda q})^2$.

Since p and q are prime to each other and $(p/q)^2, (\frac{mq-\lambda p}{p-\lambda q})^2$ are two representations of m, we must have $p - \lambda q > q$ and this contradicts (i).

So our assumption that $r^2 = m$ is wrong and the theorem is done.

2.4. Real numbers.

The set containing all rational as well as irrational numbers is called the set of all *real numbers*. The set of all real numbers is denoted by $\mathbb{R}$.

We now describe some fundamental properties of the set $\mathbb{R}$.

1. Algebraic properties of $\mathbb{R}$.

2. Order properties of $\mathbb{R}$.

3. Completeness property of $\mathbb{R}$.

4. Archimedean property of $\mathbb{R}$.

5. Density property of $\mathbb{R}$.

2.4.1. Algebraic properties of $\mathbb{R}$.

Addition and multiplication are defined on the set $\mathbb{R}$ satisfying the following properties :

A1. $a + b \in \mathbb{R}$ for all a, b in $\mathbb{R}$;

A2. $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$;

A3. there exists an element 0 in $\mathbb{R}$ (called the zero element) such that $a + 0 = a$ for all a in $\mathbb{R}$;

A4. for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$;

A5. $a + b = b + a$ for all a, b in $\mathbb{R}$;

M1. $a.b \in \mathbb{R}$ for all a, b in $\mathbb{R}$;

M2. $(a.b).c = a.(b.c)$ for all a, b, c in $\mathbb{R}$;

M3. there exists an element 1 in $\mathbb{R}$ (called the unity) such that $a.1 = a$ for alll a in $\mathbb{R}$;

M4. for each element $a \neq 0$ in $\mathbb{R}$ there exists an element $\frac{1}{a}$ in $\mathbb{R}$, such that $a.\frac{1}{a} = 1$;

M5. $a.b = b.a$ for all a, b in $\mathbb{R}$;

D. $a.(b+c) = a.b + a.c$ for all a, b, c in $\mathbb{R}$.

$-a$ is the additive inverse of a: It is also called the *negative* of a. $1/$ is the multiplicative inverse a. It is also called the *reciprocal* of a.

The reciprocal of a exists provided $a \neq 0$.

The zero element 0 and the unity 1 are unique.

$\mathbb{R}$ is said to form a *field* under the operations- addition and multipl cation.

Addition and multiplication are both commutative and associative c the set $\mathbb{R}$. Multiplication is distributive over addition.

Theorem 2.4.2. Let $a, b, c \in \mathbb{R}$. Then

(i) $a + b = a + c$ implies $b = c$ (cancellation law for addition);

(ii) $a \neq 0$ and $a.b = a.c$ implies $b = c$ (cancellation law for multip cation).

Proof. (i) $a + b = a + c$.

$-a \in \mathbb{R}$, since $a \in \mathbb{R}$. Therefore $-a + (a+b) = -a + (a+c)$

or, $(-a+a) + b = (-a+a) + c$, by A2

or, $0 + b = 0 + c$, by A4

or, $b = c$.

(ii) $a.b = a.c$.

$\frac{1}{a} \in \mathbb{R}$, since $a \neq 0$. Therefore $(\frac{1}{a}).(a.b) = (\frac{1}{a}).(a.c)$

or, $(\frac{1}{a}.a).b = (\frac{1}{a}.a).c$, by M2

or, $1.b = 1.c$, by M4

or, $b = c$.

Theorem 2.4.3. Let $a \in \mathbb{R}$. Then

(i) $a.0 = 0$,

(ii) $(-1).a = -a$,

(iii) $-(-a) = a$,

(iv) $1/(1/a) = a$, provided $a \neq 0$.

Proof. (i) We have $0 + 0 = 0$ in $\mathbb{R}$.

Then $a.(0+0) = a.0$

or, $a.0 + a.0 = a.0$, by D

$-(a.0) \in \mathbb{R}$. Therefore $-(a.0) + [a.0 + a.0] = (-a.0) + a.0$

or, $[-(a.0) + a.0] + a.0 = 0$, by A2 and A4

or, $0 + a.0 = 0$, by A4

or, $a.0 = 0$, by A3.

(ii) We have $1 + (-1) = 0$ in $\mathbb{R}$.

Then $[1 + (-1)].a = 0$

or, $a+(-1).a=0$

$-a\in\mathbb{R}$. Therefore $-a+[a+(-1).a]=-a+0$

or, $(-a+a)+(-1).a=-a$, by A2 and A3

or, $0+(-1).a=-a$, by A4

or, $(-1).a=-a$, by A3.

(iii) We have $a+(-a)=0$, by A4.

Since $-a\in\mathbb{R}$, $-a+\{-(-a)\}=0$, by A4.

Therefore $-a+a=-a+\{-(-a)\}$.

or, $a=-(-a)$, by cancellation law for addition.

(iv) Since $a\neq 0, \frac{1}{a}\in\mathbb{R}$ and $a.(\frac{1}{a})=1$.

$a.\frac{1}{a}=1\Rightarrow\frac{1}{a}\neq 0$, because $\frac{1}{a}=0\Rightarrow 1=0$.

Since $\frac{1}{a}\neq 0, 1/(1/a)\in\mathbb{R}$ and $\frac{1}{a}.\{1/(1/a)\}=1$.

Therefore $\frac{1}{a}.a=\frac{1}{a}.\{1/(1/a)\}$.

Since $\frac{1}{a}\neq 0$, $a=1/(1/a)$, by cancellation law for multiplication.

Theorem 2.4.4. Let $a,b,c\in\mathbb{R}$. Then $a.b=0$ implies $a=0$, or $b=0$.

Proof. Let $a\neq 0$. Then $\frac{1}{a}\in\mathbb{R}$ and $\frac{1}{a}.a=1$.

$a.b=0\Rightarrow\frac{1}{a}.(ab)=\frac{1}{a}.0\Rightarrow(\frac{1}{a}.a).b=0\Rightarrow b=0$.

Therefore $a\neq 0\Rightarrow b=0$.

Contrapositively, $b\neq 0\Rightarrow a=0$.

Therefore either $a=0$ or $b=0$.

Theorem 2.4.5. Let $a,b\in\mathbb{R}$. Then

(i) $a.(-b)=(-a).b=-(a.b)$,

(ii) $(-a).(-b)=a.b$.

Proof. We have $b+(-b)=0$ in $\mathbb{R}$.

Therefore $a.[b+(-b)]=a.0$.

or, $a.b+a.(-b)=0$, by D and theorem 2.4.3 (i)

$-(a.b)\in\mathbb{R}$. Therefore $-(a.b)+[a.b+a.(-b)]=-(a.b)$.

or, $[-(a.b)+a.b]+a.(-b)=-(a.b)$, by A2

or, $0+a.(-b)=-(a.b)$, by A4

or, $a.(-b)=-(a.b)$, by A3.

Again $-a+a=0$.

Therefore $[-a+a].b=0.b$.

Proceeding similarly, we can prove $(-a).b=-(a.b)$.

Therefore $a.(-b)=(-a).b=-(a.b)$.

(ii) Let $p=-a$. Then $p\in\mathbb{R}$.

$(-a).(-b)=p.(-b)=-(p.b)$, by (i)

$=-[(-a).b]=-(-(a.b))=a.b$, by theorem 2.4.3 (iii).

2.4.6. Order properties of $\mathbb{R}$.

On the set $\mathbb{R}$, a linear order relation $<$ is defined by "$a < b$ if $a \in \mathbb{R}, b \in \mathbb{R}$ and a is less than b" and it satisfies the following conditions :

O1. If $a, b \in \mathbb{R}$, then exactly one of the following statements holds – $a < b$, or $a = b$, or $b < a$ (law of trichotomy);

O2. $a < b$ and $b < c \Rightarrow a < c$ for $a, b, c \in \mathbb{R}$ (transitivity);

O3. $a < b \Rightarrow a + c < b + c$ for $a, b, c \in \mathbb{R}$;

O4. $a < b$ and $0 < c \Rightarrow ac < bc$ for $a, b, c \in \mathbb{R}$.

Note. $a < b$ is equivalently expressed as $b > a$ (b is greater than a).

The law of trichotomy states that a real number a is one of the following : $a < 0$, $a = 0$, $0 < a$. i.e., $a < 0$, $a = 0$, $a > 0$.

a is said to be a *positive* real number if $a > 0$.

a is said to be a *negative* real number if $a < 0$.

We use the symbol $a \geq 0$ to mean that a is either positive or zero; $a \leq 0$ to mean that a is either negative or zero.

If $a, b, c \in \mathbb{R}$ and $a < c$, $c < b$ both hold, we write $a < c < b$ and say that c lies between a and b.

Note. The field $\mathbb{R}$ together with the order relation defined on $\mathbb{R}$ satisfying O1-O4 becomes an *ordered field.*

Remark. On a field $(F, +, .)$, in general, an order relation is defined with the help of a *positive set* in F. A subset P of F is called a positive set if

(1) $a \in P$, $b \in P \Rightarrow a + b \in P$ and $a.b \in P$,

(2) if $c \in F$ then exactly one of the following statements holds-

$$c \in P,\ c = 0,\ -c \in P.$$

The positive set P is used to define an order $<$ in F.

Definition. If $a, b \in F$, then $a < b$ (a is less than b) if and only if $b - a \in P$.

$a < b$ is same as $b > a$ (b is greater than a).

From definition it follows that $a > 0$ if and only if $a - 0 \in P$, i.e., $a \in P$.

The order properties O1-O4 can be deduced from the above definition.

O1. Let $a, b \in F$. Then $a - b \in F$.

Therefore by (2) exactly one of the following statements holds-

$a - b \in P$, $a - b = 0$, $-(a - b) \in P$

i.e., $a - b > 0$, $a - b = 0$, $-(a - b) > 0$

i.e., $b < a$, $b = a$, $a < b$.

O2. Let $a < b$ and $b < c$.
Then $b - a \in P$ and $c - b \in P$ and by (1), $(b - a) + (c - b) \in P$
or, $c - a \in P$, i.e., $a < c$.

O3. Let $a, b, c \in F$ and $a < b$. Then $b - a \in P$.
Therefore $(b + c) - (a + c) \in P$, i.e., $a + c < b + c$.

O4. Let $a < b$ and $c > 0$. Then $b - a \in P, c \in P$.
By (1), $(b - a)c \in P$
or, $bc - ac \in P$, i.e., $ac < bc$.

The field $\mathbb{R}$ is an ordered field. The positive set in $\mathbb{R}$ is called the set of all positive real numbers and is denoted by $\mathbb{R}^+$.

Theorem 2.4.7. Let $a \in \mathbb{R}$. Then

(i) $a > 0 \Rightarrow -a < 0$;

(ii) $a < 0 \Rightarrow -a > 0$.

Proof. (i) $a \in \mathbb{R}$ and $a + (-a) = 0$, by A4.
By the law of trichotomy, either $-a < 0$ or $-a = 0$, or $-a >$
Let $-a > 0$.

$-a > 0, a \in \mathbb{R} \Rightarrow -a + a > a$, by O3
$\Rightarrow 0 > a$, a contradiction.

Let $-a = 0$. Then $a + (-a) = a + 0 = a$,
and also $a + (-a) = 0$, by A4.

Therefore $a = 0$, a contradiction.

We conclude that $-a < 0$.

(ii) Similar proof.

Theorem 2.4.8. Let $a, b \in \mathbb{R}$. Then

(i) $a > 0, b > 0 \Rightarrow a + b > 0$,

(ii) $a < 0, b < 0 \Rightarrow a + b < 0$,

(iii) $a > 0, b > 0 \Rightarrow ab > 0$,

(iv) $a < 0, b < 0 \Rightarrow ab > 0$,

(v) $a > 0, b < 0 \Rightarrow ab < 0$.

Proof. (i) $a > 0$ and $b \in \mathbb{R} \Rightarrow a + b > b$, by O
$a + b > b$ and $b > 0 \Rightarrow a + b > 0$, by O2

(ii) Similar proof.

(iii) $a > 0, b > 0 \Rightarrow a.b > 0.b$, by O4
i.e., $ab > 0$.

$$\begin{aligned}
\text{(iv)} a < 0, b < 0 \quad &\Rightarrow \quad a < 0, -b > 0 \\
&\Rightarrow \quad a.(-b) < 0(-b), \text{ by O4} \\
&\Rightarrow \quad -ab < 0 \\
&\Rightarrow \quad -(-ab) > 0, \text{ by Theorem 2.4.7 (ii)} \\
&\Rightarrow \quad ab > 0, \text{ by Theorem 2.4.3 (iii).}
\end{aligned}$$

(v) Similar proof.

Theorem 2.4.9. Let $a, b, c, d \in \mathbb{R}$ and $a > b$, $c > d$. Then $a + c > b + d$

Proof. $a > b$ and $c \in \mathbb{R} \Rightarrow a + c > b + c$, by O3
$c > d$ and $b \in \mathbb{R} \Rightarrow b + c > b + d$, by O3
$a + c > b + c$ and $b + c > b + d \Rightarrow a + c > b + d$, by O2.

Corollary. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > b_i$ for $i = 1, 2, \ldots, n$.
Then $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

Theorem 2.4.10. Let $a, b, c, d \in \mathbb{R}$ and $a > 0, b > 0, c > 0, d > 0$. The $a > b, c > d \Rightarrow ac > bd$.

Proof. $a > b$ and $c > 0 \Rightarrow ac > bc$, by O4
$c > d$ and $b > 0 \Rightarrow bc > bd$, by O4
$ac > bc$ and $bc > bd \Rightarrow ac > bd$, by O2.

Corollary 1. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > 0, b_i > 0$ fo $i = 1, 2, \ldots, n$.
Then $a_i > b_i \Rightarrow a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$.

Corollary 2. Let $a, b \in \mathbb{R}$ and $a > b > 0$. Then $a^n > b^n$ for all $n \in \mathbb{N}$.

Theorem 2.4.11. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

Proof. Since $a \neq 0$, either $a < 0$ or $a > 0$, by O1.

Case I. Let $a < 0$. Then $-a > 0$, by Theorem 2.4.7 (ii)
By O4, $a. - a < 0. - a$. Therefore $-a^2 < 0$.
This implies $a^2 > 0$, by Theorem 2.4.7 (ii)

Case II. Let $a > 0$.
By O4, $a.a > a.0$. Therefore $a^2 > 0$.

Combining the cases, we have $a^2 > 0$ for all $a \neq 0$.

Corollary. $1 > 0$, since $1 = 1.1 = 1^2$.

Theorem 2.4.12. Let $a \in \mathbb{R}$. Then
(i) $a > 0 \Rightarrow \frac{1}{a} > 0$, (ii) $a < 0 \Rightarrow \frac{1}{a} < 0$.

Proof left to the reader.

Theorem 2.4.13. $n > 0$ for all $n \in \mathbb{N}$.

Proof. The statement holds for $n = 1$, since $1 > 0$.

Let us assume that the statement holds for $n = k$, where $k \in \mathbb{N}$. Then $k > 0$. $k > 0$ and $1 > 0 \Rightarrow k + 1 > 0$, by Theorem 2.4.8.

This shows that the statement holds for $k + 1$ if it holds for k. By the principle of induction, the statement holds for all $n \in \mathbb{N}$.

Deduction. For all $n \in \mathbb{N}$, $\frac{1}{n} > 0$.

Theorem 2.4.14. Let $a, b \in \mathbb{R}$. Then $a < b \Rightarrow a < \frac{a+b}{2} < b$.

Proof. $a < b \Rightarrow a + a < a + b$
$\Rightarrow 2a < a + b$
$\Rightarrow \frac{1}{2}.2a < \frac{1}{2}(a + b)$, since $\frac{1}{2} \in \mathbb{R}$ and $\frac{1}{2} > 0$
$\Rightarrow a < \frac{a+b}{2}$.

Also $a < b \Rightarrow a + b < b + b$
$\Rightarrow a + b < 2b$
$\Rightarrow \frac{1}{2}(a + b) < \frac{1}{2}.2b$, since $\frac{1}{2} \in \mathbb{R}$ and $\frac{1}{2} > 0$
$\Rightarrow \frac{a+b}{2} < b$.

Therefore $a < \frac{1}{2}(a + b) < b$.

Corollary. There is no least positive real number.

If possible, let a be the least positive real number. Then $a > 0$. $0 < a \Rightarrow 0 < \frac{1}{2}a < a$ by the theorem.

This shows that $\frac{1}{2}a$ is a positive real number and $\frac{1}{2}a < a$ indicates that a is not the least positive real number.

It follows that there is no least positive real number.

2.4.15. Absolute value.

Let $a \in \mathbb{R}$. The absolute value of a, denoted by $|a|$, is defined by

$$\begin{aligned} |a| &= a, \text{ if } a > 0 \\ &= 0, \text{ if } a = 0 \\ &= -a, \text{ if } a < 0. \end{aligned}$$

For example, $|3| = 3, |-2| = 2, |0| = 0$.

It follows from definition that $|a|$ is a non-negative real number. $|a| = 0$ if and only if $a = 0$.

Theorem 2.4.16.

(i) $|-a| = |a|$ for all $a \in \mathbb{R}$;

(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$;

(iii) if $a, c \in \mathbb{R}$ and $c > 0$, then $|a| < c \Leftrightarrow -c < a < c$;

(iv)$-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

Proof. **(i)**
Case I. Let $a > 0$. Then $-a < 0$ and $| -a | = -(-a) = a = | a |$.

Case II. Let $a < 0$. Then $-a > 0$ and $| -a | = -a = | a |$.

Case III. Let $a = 0$. Then $-a = 0$ and $| -a | = 0 = | a |$.

It follows that $| - a| = |a|$.

(ii) Case I. Let one or both of a, b be 0. Then $ab = 0$.
In this case $|ab| = 0$ and $|a||b| = 0$. Therefore $|ab| = |a||b|$.

Case II. Let $a > 0, b > 0$. Then $ab > 0$ and $|ab| = ab$, $|a| = a$, $|b| = b$. Therefore $|ab| = |a||b|$.

Case III. Let $a < 0, b > 0$. Then $ab < 0$ and $|ab| = -ab$, $|a| = -a$, $|b| = b$. Therefore $|ab| = |a||b|$.

Case IV. Let $a > 0, b < 0$.
Similar proof.

Case V. Let $a < 0, b < 0$. Then $ab > 0$ and $| ab | = ab, | a | = -a$, $| b | = -b$. Therefore $|ab = |a||b|$.

Combining the cases, we have $|ab| = |a||b|$.

Deduction. $| a^2 | = | a |^2$ for all $a \in \mathbb{R}$.

(iii) Let $|a| < c$. Then if $a \geq 0$, $a < c$ and if $a < 0$, $-a < c$ and this implies $-c < a$. Therefore $|a| < c \Rightarrow -c < a < c$.

Conversely, let $c > 0$ and $-c < a < c$.
Then we have $a < c$, $0 < c$ and $-a < c$.
Combining, we have $|a| < c$.

Corollary. If $c \in \mathbb{R}$ and $c > 0$ then $| a | \leq c \Leftrightarrow -c \leq a \leq c$.

(iv) Let $a > 0$. Then $- | a | < 0$ and $a = | a |$.
Therefore $- | a | < a = | a |$.
Let $a = 0$. Then $- | a | = a = | a |$.
Let $a < 0$. Then $a = - | a |$ and $a < | a |$.
Therefore $- | a | = a < | a |$.
Combining the cases, we have $-|a| \leq a \leq |a|$.

Theorem 2.4.17. (Triangle inequality)

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$.

Then $-(|a|+|b|) \leq a+b \leq |a|+|b|$
This implies $|a+b| \leq |a|+|b|$, since $-c \leq a \leq c \Rightarrow |a| \leq c$.

Corollary 1. $|a-b| \leq |a|+|b|$ for all $a, b \in \mathbb{R}$.

Proof. Replacing b by $-b$ in the triangle inequality we get the inequality.

Corollary 2. $||a|-|b|| \leq |a-b|$.

Proof. $|a|=|a-b+b| \leq |a-b|+|b|$.
or, $|a|-|b| \leq |a-b|$.
Again $|b|=|b-a+a| \leq |b-a|+|a|$
or, $|b|-|a| \leq |b-a|=|a-b|$.
So we have $-|a-b| \leq |a|-|b| \leq |a-b|$.
This implies $||a|-|b|| \leq |a-b|$, since $-c \leq a \leq c \Rightarrow |a| \leq c$.

Corollary 3. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$.
Then $|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$.

Theorem 2.4.18. If $a, b \in \mathbb{R}$,

$$\max\{a, b\}=\tfrac{1}{2}\{a+b+|a-b|\},$$
$$\min\{a, b\}=\tfrac{1}{2}\{a+b-|a-b|\}.$$

Proof.
$$\begin{aligned}\max\{a, b\} &= a \text{ if } a>b\\ &= b \text{ if } a<b\\ &= \tfrac{1}{2}(a+b) \text{ if } a=b.\end{aligned}$$
$$\begin{aligned}\min\{a, b\} &= b \text{ if } a>b\\ &= a \text{ if } a<b\\ &= \tfrac{1}{2}(a+b) \text{ if } a=b.\end{aligned}$$

It follows that

if $a>b$ $\quad \max\{a, b\}-\min\{a, b\}=a-b=|a-b|$,

if $a<b$ $\quad \max\{a, b\}-\min\{a, b\}=b-a=|a-b|$,

if $a=b$ $\quad \max\{a, b\}-\min\{a, b\}=0=|a-b|$.

Also if $a>b$ $\quad \max\{a, b\}+\min\{a, b\}=a+b$,

if $a<b$ $\quad \max\{a, b\}+\min\{a, b\}=b+a$,

if $a=b$ $\quad \max\{a, b\}+\min\{a, b\}=\frac{1}{2}(a+b)+\frac{1}{2}(a-b)=a+b$.

Therefore we have $\max\{a, b\}+\min\{a, b\}=a+b$ for $a, b \in \mathbb{R}$;

$\max\{a, b\}-\min\{a, b\}=|a-b|$ for $a, b \in \mathbb{R}$.

Consequently, $\max\{a, b\}=\frac{1}{2}\{a+b+|a-b|\}$;

$\min\{a, b\}=\frac{1}{2}\{a+b-|a-b|\}$ for all $a, b \in \mathbb{R}$.

Worked Examples.

1. Solve the equation $\mid \frac{x+2}{2x-1} \mid = 3$.

$\mid \frac{x+2}{2x-1} \mid = 3 \Rightarrow \frac{x+2}{2x-1} = \pm 3$.
$\frac{x+2}{2x-1} = 3 \Rightarrow x + 2 = 6x - 3 \Rightarrow x = 1$
$\frac{x+2}{2x-1} = -3 \Rightarrow x + 2 = -6x + 3 \Rightarrow x = \frac{1}{7}$.
Therefore $x = 1, \frac{1}{7}$.

2. Find the solution set of the inequality $\mid \frac{x+3}{2x-6} \mid \leq 1$.

The solution set is the union of two sets S_1 and S_2 where
$S_1 = \{x : 2x - 6 > 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\}$
$S_2 = \{x : 2x - 6 < 0 \text{ and } -1 \leq \frac{x+3}{2x-6} \leq 1\}$.
If $2x - 6 > 0$, then
$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow -2x + 6 \leq x + 3 \leq 2x - 6 \quad \ldots \ldots \text{(i)}$
If $2x - 6 < 0$, then
$-1 \leq \frac{x+3}{2x-6} \leq 1 \Leftrightarrow 2x - 6 \leq x + 3 \leq -2x + 6 \quad \ldots \ldots \text{(ii)}$

From (i) $x > 3$ and $x \geq 1$ and $x \geq 9$ simultaneously.
From (ii) $x < 3$ and $x \leq 9$ and $x \leq 1$ simultaneously.

Therefore $S_1 = \{x \in \mathbb{R} : x \geq 9\}$ and $S_2 = \{x \in \mathbb{R} : x \leq 1\}$.
So the solution set is $\{x \in \mathbb{R} : x \geq 9\} \cup \{x \in \mathbb{R} : x \leq 1\}$.

2.4.19. Completeness property of $\mathbb{R}$.

Definition. Let S be a subset of $\mathbb{R}$. A real number u is said to be a *upper bound* of S if $x \in S \Rightarrow x \leq u$. A real number l is said to be a *lower bound* of S if $x \in S \Rightarrow x \geq l$.

Let S be a subset of $\mathbb{R}$. S is said to be *bounded above* if S has a upper bound. S is said to be *bounded below* if S has a lower bound.

In other words, a set $S \subset \mathbb{R}$ is said to be bounded above if there exis a real number u such that $x \in S \Rightarrow x \leq u$; S is said to be bounded belc if there exists a real number l such that $x \in S \Rightarrow x \geq l$.

S is said to be a *bounded set* if S be bounded above as well as bound below.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. S is bounded above, 1 being an upper bour S is bounded below, 0 being a lower bound.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 2\}$. S is bounded above, 2 being an upp bound. S is bounded below, 1 being a lower bound.

3. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. S is bounded above, 2 being an upper bound. S is bounded below, 1 being a lower bound.

4. Let $S = \phi$. Every real number x is an upper bound of the set S. Every real number x is a lower bound of the set S. S is a bounded set.

Definition. Let S be a subset of $\mathbb{R}$. If S be bounded above, then an upper bound of S is said to be the *supremum* of S (or *the least upper bound* of S) if it is less than every other upper bound of S. If S be bounded below then a lower bound of S is said to be the *infimum* of S (or *the greatest lower bound* of S) if it is greater than every other lower bound of S.

If a set $S \subset \mathbb{R}$ be bounded above then S has an upper bound. If u be an upper bound of S then obviously each of $u+1, u+2, \cdots$ is an upper bound of S. Therefore for a set S bounded above, there exist infinitely many upper bounds.

It is not possible to ascertain if S has a least upper bound. It is a deeper property of the set $\mathbb{R}$ that if S be a *non-empty* subset of $\mathbb{R}$, bounded above, then the set of all upper bounds of S has a least element. We shall take this property of $\mathbb{R}$ as an axiom, called "the least upper bound axiom". This property is also called *the supremum property of* $\mathbb{R}$.

Statement of the property.

Every *non-empty* subset of $\mathbb{R}$ that is bounded above has a *least upper bound* (or a supremum).

A similar approach can be made in respect of a non-empty subset of $\mathbb{R}$ that is bounded below and we can obtain the greatest lower bound property of $\mathbb{R}$, or *the infimum property of* $\mathbb{R}$ in the following form.

Every *non-empty* subset of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or an infimum).

We can establish that these two properties are equivalent in the sense that one of these implies the other. However, we assume the supremum property of $\mathbb{R}$ as an axiom and call it *the completeness property* of $\mathbb{R}$ and treat the other property (the infimum property) as a theorem.

For a non-empty set S, bounded above, the supremum of S is denoted by $\sup S$. $\sup S$ may or may not belong to S. For a non-empty set S, bounded below, the infimum of S is denoted by $\inf S$. $\inf S$ may or may not belong to S.

If S happens to be a non-empty finite set, then $\sup S$ and $\inf S$ both exist and belong to S. They are said to be the *maximum* and the *minimum* of S respectively and are denoted by $\max S$ and $\min S$.

Theorem 2.4.20. Let S be a non-empty subset of $\mathbb{R}$, bounded below.

Then S has an infimum.

Proof. Let l_o be a lower bound of S. Let $T = \{l \in \mathbb{R} : l$ is a lower bound of $S\}$. Then T is a non-empty subset of $\mathbb{R}$ because $l_o \in T$.

Moreover, $x \in T$ and $s \in S \Rightarrow x \leq s$. This shows that T is bounded above.

Thus T is a non-empty subset of $\mathbb{R}$, bounded above. By the supremum property of $\mathbb{R}, T$ has a supremum. Let $\sup T = L$.

Then (i) $t \leq L$ for every $t \in T$, since L is an upper bound of T.
and (ii) since every $s \in S$ is an upper bound of T and $L = \sup T$,
$L \leq s$ for every $s \in S$.

(ii) shows that L is a lower bound of S and (i) shows that $L \geq$ any lower bound of S. Consequently, $L = \inf S$.

Therefore S has an infimum and the proof is complete.

An ordered field is said to be a *complete ordered field* if the completeness property (i.e., the supremum property, or the infimum property) holds in it. Thus $\mathbb{R}$ is a complete ordered field.

The ordered field $\mathbb{Q}$ of all rational numbers does not have the supremum property. For example, the set $S = \{1, 1 + \frac{1}{1!}, 1 + \frac{1}{1!} + \frac{1}{2!}, \cdots\}$ which is a subset of $\mathbb{Q}$, is bounded above, because each element of the set is less than 3. But there is no rational number which is the supremum of the set S. (The supremum of the set is e, an irrational number.)

It is this completeness property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and that transforms $\mathbb{R}$ from an algebraic system into a structure rich in abundant materials of analysis.

Theorem 2.4.21. Let S be a non-empty subset of $\mathbb{R}$, bounded above. An upper bound u of S is the supremum of S if and only if for each positive ϵ there exists an element s in S such that $u - \epsilon < s \leq u$.

Proof. Let $u = \sup S$. Let us choose $\epsilon > 0$. Then $u - \epsilon$ is not an upper bound of S. Therefore there exists at least one element of S, say s, such that $s > u - \epsilon$.
Since $u = \sup S$ and $s \in S$, we have $s \leq u$. Consequently, $u - \epsilon < s \leq u$.

Conversely, let u be an upper bound of S such that for each chosen $\epsilon > 0$, there is an element, say s, of S such that $u - \epsilon < s < u$.

We prove that u is the least upper bound of S, i.e., no upper bound of S is less than u.

If possible, let u_0 be an upper bound of S such that $u_0 < u$.

Let $\epsilon = \frac{1}{2}(u - u_0)$. Then $\epsilon > 0$ and $u - \epsilon = u_0 + \epsilon$.

By the stated condition, there exists an element in S, say s', such that $u - \epsilon < s' \leq u$.

or, $s' > u_0 + \epsilon$ and this shows that u_0 can not be an upper bound of S. Hence u is the least upper bound of S.

2.4.22. Properties of the supremum and the infimum.

Let S be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup S$ exists. Let $M = \sup S$. Then $M \in \mathbb{R}$ and M satisfies the following conditions :

(i) $x \in S \Rightarrow x \leq M$, and

(ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $M - \epsilon < y \leq M$.

Let S be a non-empty subset of $\mathbb{R}$, bounded below. Then $\inf S$ exists. Let $m = \inf S$. Then $m \in \mathbb{R}$ and m satisfies the following conditions :

(i) $x \in S \Rightarrow x \geq m$, and

(ii) for each $\epsilon > 0$, there exists an element $y(\epsilon)$ in S such that $m \leq y < m + \epsilon$.

Note. The symbol $y(\epsilon)$ indicates dependence of y on the choice of ϵ.

Worked Examples (continued).

3. Prove that the set $\mathbb{N}$ is not bounded above.

The set $\mathbb{N}$ is a non-empty subset of $\mathbb{R}$, since $1 \in \mathbb{N}$.

Let $\mathbb{N}$ be bounded above. Then $\mathbb{N}$ being a non-empty subset of $\mathbb{R}$ bounded above, $\sup \mathbb{N}$ exists by the supremum property of $\mathbb{R}$. Let $u = \sup \mathbb{N}$. Then (i) $x \in \mathbb{N} \Rightarrow x \leq u$, and

(ii) for each $\epsilon > 0$ there exists an element, say y in $\mathbb{N}$ such that $u - \epsilon < y \leq u$.

Let us choose $\epsilon = 1$. Then there exists an element k in $\mathbb{N}$ such that $u - 1 < k \leq u$. $u - 1 < k \Rightarrow k + 1 > u$.

Since k is a natural number, $k + 1$ is also a natural number. $k + 1 > u$ implies that u is not an upper bound of $\mathbb{N}$.

Thus we arrive at a contradiction. So our assumption that $\mathbb{N}$ is bounded above is wrong. Hence the set $\mathbb{N}$ is not bounded above.

4. Let S be a non-empty subset of $\mathbb{R}$, bounded above and $T = \{-x : x \in S\}$. Prove that the set T is bounded below and $\inf T = -\sup S$.

$\sup S$ exists. Let $u = \sup S$. Then $x \in S \Rightarrow x \leq u$.

Let $y \in T$. Then $-y \in S$ and therefore $-y \leq u$, i.e., $y \geq -u$. This implies that $-u$ is a lower bound of T. Therefore the set T is bounded below.

Let us choose $\epsilon > 0$. Since $u = \sup S$, there exists an element p in S such that $u - \epsilon < p \leq u$. Therefore $-u \leq -p < -u + \epsilon$. ... (i)

Let $q = -p$. then $q \in T$.

(i) shows that for a pre-assigned positive ϵ there exists an element q in T such that $-u \leq q < -u + \epsilon$.

This proves that $-u = \inf T$. Therefore $\inf T = -\sup S$.

5. Let S be a non-empty bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$. Prove that the set $T = \{| x - y |: x \in S, y \in S\}$ is bounded above and $\sup T = M - m$.

$x \in S \Rightarrow m \leq x \leq M$, $y \in S \Rightarrow m \leq y \leq M$.

Therefore $m - M \leq x - y \leq M - m$, i.e., $| x - y | \leq M - m$.

This shows that the set T is bounded above, $M - m$ being an upper bound.

Let $a \in S$. Then $| a - a | \in T$ showing that T is non-empty. By the supremum property of $\mathbb{R}$, $\sup T$ exists.

We now prove that no real number less than $M - m$ is an upper bound of T.

If possible, let $p < M - m$ be an upper bound of T.

Let $(M - m) - p = 2\epsilon$. Then $\epsilon > 0$ and $p + \epsilon = M - m - \epsilon$.

Since $\sup S = M$, there exists an element $x \in S$ such that

$$M - \tfrac{\epsilon}{2} < x \leq M.$$

Since $\inf S = m$, there exists an element $y \in S$ such that

$$m \leq y < m + \tfrac{\epsilon}{2}.$$

Now $x - y > M - m - \epsilon$, i.e., $x - y > p + \epsilon$.

This shows that p is not an upper bound of T.

Therefore no real number less than $M - m$ is an upper bound of T. That is, $\sup T = M - m$.

6. Let A, B be bounded subsets of $\mathbb{R}$ such that $x \in A$, $y \in B \Rightarrow x \leq y$. Prove that $\sup A \leq \inf B$.

Since A, B are non-empty bounded subsets of $\mathbb{R}$, $\sup A$, $\inf B$ exist. Let $\sup A = a^*$, $\inf B = b_*$.

Let $b \in B$. Then $x \in A \Rightarrow x \leq b$. This shows that b is an upper bound of A. Since $\sup A = a^*$ and b is an upper bound of A it follows that $a^* \leq b$.

Now $a^* \leq b$ for all $b \in B$. Therefore a^* is a lower bound of B. Since $\inf B = b_*$ and a^* is a lower bound of B it follows that $a^* \leq b_*$, i.e. $\sup A \leq \inf B$.

7. Let S be the subset of $\mathbb{Q}$ defined by $S = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$. Show that S is a non-empty subset of $\mathbb{Q}$ bounded above but $\sup S$ does not belong to $\mathbb{Q}$.

S is non-empty, since $1 \in S$. S is bounded above, since 2 is an upper bound of the set.

If possible, let $\sup S \in \mathbb{Q}$ and $\sup S = u$. Then $u > 0$ and $u \in \mathbb{Q}$.

By the law of trichotomy, exactly one of the following holds :

$u^2 > 2, u^2 = 2, u^2 < 2$.

Case 1. Let $u^2 > 2$. Then $u^2 - 2 > 0$.

Let us take another rational number $r = \frac{4+3u}{3+2u}$. Then $r > 0$.

$u - r = u - \frac{4+3u}{3+2u} = \frac{2(u^2-2)}{3+2u} > 0$. Therefore $0 < r < u$ ... ... (i)

$r^2 - 2 = \frac{(4+3u)^2 - 2(3+2u)^2}{(3+2u)^2} = \frac{u^2-2}{(3+2u)^2} > 0$.

Therefore $r > 0$ and $r^2 > 2$ ... ...(ii)

(ii) shows that r is an upper bound of S and (i) shows that u is not the supremum of S. This is a contradiction to the assumption that $u = \sup S$. Therefore $u^2 \not> 2$.

Case II. $u^2 = 2$. We have seen that there exists no rational number r such that $r^2 = 2$. Therefore $u^2 \neq 2$.

Case III. $u^2 < 2$.

Let us take again the rational number $r = \frac{4+3u}{3+2u}$. Then $r > 0$ and $r - u = \frac{2(2-u^2)}{3+2u} > 0$. Therefore $0 < u < r$ ... ... (iii)

$2 - r^2 = \frac{2-u^2}{(3+2u)^2} > 0$. Therefore $r > 0$ and $r^2 < 2$ ... ... (iv)

(iv) shows that $r \in S$.

From (iii) it follows that u belongs to S and u is less than an element r of S. Therefore u is not the supremum of S, a contradiction. Therefore $u^2 \not< 2$.

None of the three possibilities provided by the law of trichotomy can hold. Hence our assumption that $\sup S$ is a rational number is wrong. Therefore no rational number can be the supremum of S.

Note 1. This example shows that the supremum property which is an important property of $\mathbb{R}$ is not satisfied in the subset $\mathbb{Q}$ of $\mathbb{R}$.

2. If we regard this set S as a subset of $\mathbb{R}$, then by the supremum property of $\mathbb{R}$, $\sup S$ exists as a real number.

2.4.23. Archimedean property of $\mathbb{R}$.

If $x, y \in \mathbb{R}$ and $x > 0, y > 0$, then there exists a natural number n such that $ny > x$.

Proof. If possible, let there exist no natural number n for which $ny > x$. Then for every natural number k, $ky \leq x$.

Thus the set $S = \{ky : k \in \mathbb{N}\}$ is bounded above, x being an upper bound. S is non-empty because $y \in S$.

By the supremum property of $\mathbb{R}$, $\sup S$ exists. Let $\sup S = b$.

Then $ky \leq b$ for all $k \in \mathbb{N}$.

$b - y < b$ since $y > 0$. This shows that $b - y$ is not an upper bound of S and therefore there exists a natural number p such that $b - y < py \leq b$. This implies $(p+1)y > b$ ... ... (i)

But $p \in \mathbb{N} \Rightarrow p + 1 \in \mathbb{N}$ and therefore $(p+1)y \in S$.

(i) shows that b is not the supremum of S, a contradiction.

Therefore our assumption is wrong and the existence of a natural number n satisfying $ny > x$ is proved.

Important deductions.

(i) If $x \in \mathbb{R}$, then there exists a natural number n such that $n > x$.

Case 1. $x > 0$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$ and hence the existence is proved.

Case 2. $x \leq 0$. Then $n = 1$.

(ii) If $x \in \mathbb{R}$ and $x > 0$, then there exists a natural number n such that $0 < \frac{1}{n} < x$.

Taking $y = 1$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $nx > 1$.

Since n is a natural number, $n > 0$ and therefore $\frac{1}{n} > 0$ and also $\frac{1}{n} < x$. Therefore we have $0 < \frac{1}{n} < x$.

(iii) If $x \in \mathbb{R}$ and $x > 0$, there exists a natural number m such that $m - 1 \leq x < m$.

Taking $y = 1$ and $x > 0$, by Archimedean property of $\mathbb{R}$ there exists a natural number n such that $n.1 > x$, i.e., $n > x$.

Let $S = \{k \in \mathbb{N} : k > x\}$. Then S is non-empty subset of $\mathbb{N}$, since $n \in S$. By the well ordering property of the set $\mathbb{N}$, S has a least element, say m. Since $m \in S, m > x$.

As m is the least element in S, $m-1 \not> x$, i.e., $m-1 \leq x$.
Hence $m-1 \leq x < m$.

(iv) If $x \in \mathbb{R}$, then there exists an integer m such that $m-1 \leq x < m$.

Case 1. $x > 0$.
This is (iii)

Case 2. $x = 0$.
In this case $m = 1$.

Case 3. $x < 0$.
First we assume that x is not a negative integer.

Then $-x > 0$. By case 1, there exists a natural number m' such that $m' - 1 < -x < m'$.

$-x < m' \Rightarrow -m' < x$ and $m' - 1 < -x \Rightarrow x < -m' + 1$.

Therefore $-m' < x < -m' + 1$.

Let $m = -m' + 1$. Since m' is a natural number, m is an integer ≤ 0. So we have $m - 1 < x < m$.

If however, x is a negative integer, then $x = m - 1$.

Combining, we have $m - 1 \leq x < m$.

Note. An ordered field is called an *Archimedean ordered field* if the Archimedean property holds in it. Thus $\mathbb{R}$ is an Archimedean ordered field. $\mathbb{Q}$ is also an Archimedean ordered field. But $\mathbb{Q}$ is not a complete Archimedean ordered field, while $\mathbb{R}$ is so.

Worked Examples (continued).

8. Show that there exists a unique positive real number x such that $x^2 = 2$.

Let $S = \{s \in \mathbb{R} : s \geq 0 \text{ and } s^2 < 2\}$. S is a non-empty subset of $\mathbb{R}$, since $0 \in S$. S is bounded above, 2 being an upper bound.

By the supremum property of $\mathbb{R}$, $\sup S$ exists. Let $x = \sup S$. Clearly, $x > 0$. $1 \in S$ and 1 is not an upper bound of S and therefore $x > 1$ also. We shall prove that $x^2 = 2$.

If not, let $x^2 > 2$. Then $\frac{x^2-2}{2x} > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < \frac{x^2-2}{2x}$. Therefore $\frac{2x}{m} < x^2 - 2$.

$$\begin{aligned} (x - \tfrac{1}{m})^2 &= x^2 - \tfrac{2x}{m} + \tfrac{1}{m^2} \\ &> x^2 - \tfrac{2x}{m} > 2. \end{aligned}$$

$x - \frac{1}{m} > 0$, since $x > 1$. $(x - \frac{1}{m})^2 > 2$ shows that $x - \frac{1}{m}$ is an upper bound of S which contradicts that $\sup S = x$.

Therefore $x^2 \not> 2$ ... ... (i)

Let $x^2 < 2$. Then $2 - x^2 > 0$ and $\frac{2-x^2}{2x+1} > 0$

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \frac{2-x^2}{2x+1}$, i.e., $\frac{1}{k}(2x+1) < 2 - x^2$.

$$\begin{aligned} (x + \tfrac{1}{k})^2 &= x^2 + \tfrac{1}{k}(2x + \tfrac{1}{k}) \\ &\leq x^2 + \tfrac{1}{k}(2x + 1) < 2. \end{aligned}$$

This shows that $x + \frac{1}{k} \in S$ and as $x + \frac{1}{k} > x$, x fails to be the supremum of S.

Therefore $x^2 \not< 2$ ... ... (ii)

From (i) and (ii) $x \in \mathbb{R}$ and $x^2 = 2$.

We prove that x is unique.

Let us assume that there exists another real number y such that $y > 0$ and $y^2 = 2$.

Now $x > 0$ and $x^2 = 2$; $y > 0$ and $y^2 = 2$.

This implies $x^2 = y^2$.

Let $x > y$. Then $x > 0, x > y \Rightarrow x^2 > xy$
and $y > 0, x > y \Rightarrow xy > y^2$.

It follows that $x^2 > y^2$, a contradiction.

Let $x < y$. Then $x > 0, x < y \Rightarrow x^2 < xy$
and $y > 0, x < y \Rightarrow xy < y^2$.

It follows that $x^2 < y^2$, a contradiction. Consequently, $x = y$.

This proves that x is a unique positive real number such that $x^2 = 2$.

Note. x is denoted by $\sqrt{2}$.$\sqrt{2}$ is therefore an irrational real number.

9. If n be a positive integer ≥ 2 and a be a positive real number, show that there exists a unique positive real number x such that $x^n = a$.

Let $S = \{s \in \mathbb{R} : s > 0 \text{ and } s^n < a\}$.

Let $t = \frac{a}{1+a}$. Then $0 < t < 1$ and also $0 < t < a$.

This implies $t^n < t < a$.

$t > 0$ and $t^n < a \Rightarrow t \in S$, proving that S is non-empty.

Let $u = 1 + a$. Then $u > 1$ and $u > a$.

This implies $u^n > u > a$.

Since $u^n > a$ and $u > 0, u$ is an upper bound of S.

Thus S is a non-empty subset of $\mathbb{R}$, bounded above and hence $\sup S$ exists.

Let $x = \sup S$. Clearly, $x > 0$. We prove that $x^n = a$.

If not, either $x^n > a$ or $x^n < a$. (by the law of trichotomy)

Case 1. Let $x^n > a$. Then $\frac{x^n - a}{(1+x)^n - x^n} > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number m such that $0 < \frac{1}{m} < \frac{x^n - a}{(1+x)^n - x^n}$

or, $x^n - a > \frac{1}{m}[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n}]$.

$$\begin{aligned} (x - \tfrac{1}{m})^n &= x^n - \tbinom{n}{1}x^{n-1}.\tfrac{1}{m} + \cdots + (-1)^n\tbinom{n}{n}.\tfrac{1}{m^n} \\ &> x^n - \tfrac{1}{m}[\tbinom{n}{1}x^{n-1} + \tbinom{n}{2}x^{n-2} + \cdots + \tbinom{n}{n}] \\ &> x^n - (x^n - a) = a. \end{aligned}$$

This shows that $x - \frac{1}{m}$ is an upper bound of S and this contradicts that $x = \sup S$.

Case 2. Let $x^n < a$. Then $\frac{a - x^n}{(1+x)^n - x^n} > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \frac{a - x^n}{(1+x)^n - x^n}$

or, $a - x^n > \frac{1}{k}[\binom{n}{1}x^{n-1}.\frac{1}{k} + \binom{n}{2}.x^{n-2} + \cdots + \binom{n}{n}]$

$$\begin{aligned} (x + \tfrac{1}{k})^n &= x^n + \tbinom{n}{1}x^{n-1}.\tfrac{1}{k} + \tbinom{n}{2}.x^{n-2}\tfrac{1}{k^2} + \cdots + \tbinom{n}{n}.\tfrac{1}{k^n} \\ &< x^n + \tfrac{1}{k}[\tbinom{n}{1}x^{n-1} + \tbinom{n}{2}x^{n-2} + \cdots + \tbinom{n}{n}] \\ &< x^n + (a - x^n) = a. \end{aligned}$$

This shows that $x + \frac{1}{k} \in S$ and this contradicts that $x = \sup S$.

In view of the cases 1 and 2, we have $x^n = a$.

We prove that x is unique.

If possible, let $y \neq x$ and $y^n = a$.

Now $y > 0, x > 0$ and $y \neq x \Rightarrow y^n \neq x^n$.

Therefore $y^n \neq a$ and x is unique.

Note. This unique positive real number is denoted by $\sqrt[n]{a}$.

2.4.24. Density property of $\mathbb{R}$.

1. If x, y are real numbers with $x < y$ then there exists a rational number r such that $x < r < y$.

2. If x, y are real numbers with $x < y$ then there exists an irrational number s such that $x < s < y$.

Proof. **1.** $y - x > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number n such that $0 < \frac{1}{n} < y - x$

or, $ny - nx > 1$

or, $nx + 1 < ny$. ... ... (i)

$nx \in \mathbb{R}$. By deduction (iv) of Archimedean property, there exists an integer m such that $m - 1 \leq nx < m$. ... ... (ii)

$m - 1 \leq nx \Rightarrow nx + 1 \geq m$.

Therefore $m \leq nx + 1 < ny$ from (i). Also $nx < m$ from (ii).

Therefore $nx < m < ny$

or, $x < \frac{m}{n} < y$.

Since m is an integer and n is a natural number, $\frac{m}{n}$ is a rational number.

Let $r = \frac{m}{n}$. Then the rational number r is such that $x < r < y$.

2. $\sqrt{2}x, \sqrt{2}y$ are real numbers and $\sqrt{2}x < \sqrt{2}y$.

By Density property 1, there exists a rational number r such that $\sqrt{2}x < r < \sqrt{2}y$. Without loss of generality, we assume $r \neq 0$.

Then $x < \frac{r}{\sqrt{2}} < y$.

Let $s = \frac{r}{\sqrt{2}}$. Then s is an irrational number satisfying $x < s < y$.

2.4.25. Geometrical representation of real numbers.

The real numbers can be represented by points on a straight line. Let $X'X$ be a directed line. We take a point O on the line. O divides the line into two parts. The part to the right of O is called the positive side, the part to the left of O is called the negative side. Let us take a point A to the right of O.

Let O represent the real number *zero* and A represent the real number *one*. Taking the distance OA as the unit distance on some chosen scale, each real number can be represented by a unique point on the line; a positive real number by a point lying to the right of O and a negative real number by a point lying to the left of O. A point that represents a rational number is called a rational point and a point that represents an irrational number is called an irrational point. By the density property of $\mathbb{R}$, between any two points on the line there lie infinitely many rational points as well as infinitely many irrational points.

Having a complete representation of the set $\mathbb{R}$ as points on the line, the question comes -"Does there exist any other point on the line that does not correspond to a real number?" The answer to the question is provided by Cantor-Dedekind axiom which states that there is a one-to-one correspondence between the set of all points on a line and the set of all real numbers.

Therefore each point on the line corresponds to only one real number and conversely, each real number is represented by only one point on the line.

Note. It will be convenient for us to suppose that a straight line is composed of points which correspond to all the numbers in the set $\mathbb{R}$. The points on the line can be looked upon as images of the numbers in

$\mathbb{R}$. In view of the one-to-one correspondence between the two sets (the set of points on the line and the set of numbers in $\mathbb{R}$) we shall use the word "a point" for "a real number" and vice versa.

Definition. The aggregate of all real numbers is called the *arithmetical continuum* and the aggregate of all points on a straight line is called the *linear continuum.*

2.4.26. Extended set of real numbers.

It is often convenient to extend the set $\mathbb{R}$ by the addition of two elements ∞ and $-\infty$. This enlarged set is called the *extended set of real numbers* and is often denoted by $\mathbb{R}^*$.

In the extended set $\mathbb{R}^*$ we define –

$$\begin{aligned} \text{for all } x \in \mathbb{R}, x + \infty &= \infty + x = \infty \\ x + (-\infty) &= (-\infty) + x = -\infty; \end{aligned}$$

$$\begin{aligned} \text{for all } x > 0, x.\infty &= \infty.x = \infty \quad \text{and} \\ x.(-\infty) &= (-\infty).x = -\infty; \end{aligned}$$

$$\begin{aligned} \text{for all } x < 0, x.\infty &= \infty.x = -\infty \quad \text{and} \\ x.(-\infty) &= -\infty.x = \infty; \end{aligned}$$

$$\infty + \infty = \infty, (-\infty) + (-\infty) = -\infty$$
$$\infty.\infty = \infty, \quad (-\infty).\infty = \infty.(-\infty) = -\infty, \quad (-\infty).(-\infty) = \infty.$$

$\infty + (-\infty)$, $(-\infty) + \infty$, $0.\infty$, $\infty.0$, $0. - \infty$, $-\infty.0$ are not defined.

$\mathbb{R}^*$ is not a field. It is not even an algebraic system since addition and multiplication are not fully defined on the set.

The order relation in $\mathbb{R}^*$ satisfies the inequality $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

If x be a positive real number, then $0 < x < \infty$.

If x be a negative real number, then $-\infty < x < 0$.

If S be a non-empty subset of $\mathbb{R}$ having no upper bound, we define $\sup S = \infty$. If S be a non-empty subset of $\mathbb{R}$ having no lower bound, we define $\inf S = -\infty$.

Therefore for every non-empty subset S in $\mathbb{R}, \sup S$ and $\inf S$ both exist in $\mathbb{R}^*$ and $\inf S \leq \sup S$.

If S be the empty set, we have $\sup S = -\infty$ and $\inf S = \infty$.

The advantage of the extended set of real numbers is that we can speak of $\sup S$ and $\inf S$ of any type of subset S of $\mathbb{R}$.

Exercises 2

1. Use the principle of induction to prove that

(i) if $x > -1, (1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$;

(ii) $3^{2^n} - 1$ divisible by 2^{n+2} for all $n \in \mathbb{N}$;

(iii) if $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2+u_n}$ for all $n \geq 1$, then $u_n < 2$ for al $n \in \mathbb{N}$;

(iv) $u_{n+2} + u_n = 4u_{n+1}$ for all $n \in \mathbb{N}$, where $u_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

2. Show that there does not exist a rational number r such that $r^2 = 5$.

3. Show that the following numbers are irrational numbers.

(i) $1+\sqrt{2}+\sqrt{3}$, (ii) $1-\sqrt{2}+\sqrt{3}$, (iii) $1+\sqrt{2}-\sqrt{3}$, (iv) $1-\sqrt{2}-\sqrt{3}$.

4. Show that $\log_{10} n$ is not a rational number if n is any integer not a powe of 10.

5. Let $a, b \in \mathbb{R}$ and $ab > 0$. Prove that either $a > 0$ and $b > 0$, or $a < 0$ anc $b < 0$.

6. If $a, b \in \mathbb{R}$ and $0 \leq a - b < \epsilon$ holds for every positive ϵ, prove that $a = b$.

7. If $a \in \mathbb{R}$ and $0 \leq a < \frac{1}{n}$ for every natural number n, prove that $a = 0$.

8. Find the solution set of the inequality

(i) $\frac{3x}{2x-1} < 3$, (ii) $\frac{x+2}{x-1} < 4$, (iii) $\frac{4x}{2x-3} > \frac{1}{2} + \frac{3x}{2x+3}$,

(iv) $\left| \frac{x+3}{6-5x} \right| \leq 2$, (v) $\left| \frac{2x-5}{x-6} \right| < 3$.

9. Prove that $|x| + |y| + |z| \leq |x+y-z| + |y+z-x| + |z+x-y|$ for all $x, y, z \in \mathbb{R}$.

10. Find $\sup A$ and $\inf A$, where

(i) $A = \{x \in \mathbb{R} : x^2 < 1\}$, (ii) $A = \{x \in \mathbb{R} : 3x^2 + 8x - 3 < 0\}$,

(iii) $A = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$, (iv) $A = \{\frac{n+(-1)^n}{n} : n \in \mathbb{N}\}$.

11. If y be a positive real number show that there exists a natural number m such that $0 < 1/2^m < y$.

12. Let S be a bounded subset of $\mathbb{R}$ and T be a non-empty subset of S. Prove that $\inf S \leq \inf T \leq \sup T \leq \sup S$.

13. Let S and T be two non-empty bounded subsets of $\mathbb{R}$ and $U = \{x+y : x \in S, y \in T\}$. Prove that $\sup U = \sup S + \sup T$, $\inf U = \inf S + \inf T$.

14. Let S be a non-empty subset of $\mathbb{R}$, bounded below and $T = \{-x : x \in S\}$. Prove that the set T is bounded above and $\sup T = -\inf S$.

15. Let S be a bounded subset of $\mathbb{R}$ with $\sup S = M$ and $\inf S = m$. Prove that the set $T = \{x - y : x \in S, y \in S\}$ is a bounded set and $\sup T = M - m$, $\inf T = m - M$.

16. A and B are non-empty bounded subsets of $\mathbb{R}$. Prove that

(i) $\sup A \cup B = \max\{\sup A, \sup B\}$, (ii) $\inf A \cup B = \min\{\inf A, \inf B\}$.

17. Let S be a non-empty subset of $\mathbb{R}$ bounded below. A lower bound l of S is such that for each natural number n there exists an element s_n in S satisfying $s_n < l + \frac{1}{n}$. Prove that $l = \inf S$.

18. Show that the subset $S = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$ is a non-empty subset of $\mathbb{Q}$, bounded below; but $\inf S$ does not belong to $\mathbb{Q}$.

Hint. Assume $\inf S = l \in \mathbb{Q}$. Take $r = \frac{4-3l}{3+2l} \in \mathbb{Q}$ and show that either $r < l$ if $l^2 > 2$, or $0 < l < r$ if $l^2 < 2$.]

how that there exists a unique positive real number x such that $x^2 = 5$

3.1. Intervals.

Let $a, b \in \mathbb{R}$ and $a < b$.

The subset $\{x \in \mathbb{R} : a < x < b\}$ is said to be an *open interval.* The points a and b are called the *end points* of the interval. a and b are not points in the open interval. This open interval is denoted by (a, b).

The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is said to be a *closed interval.* The end points a and b are points in the closed interval. This closed interval is denoted by $[a, b]$.

The subsets $\{x \in \mathbb{R} : a < x \leq b\}$ and $\{x \in \mathbb{R} : a \leq x < b\}$ are said to be *half open* (or *half closed*) intervals. One of the end points is a point in the interval. These half open intervals are denoted by $(a, b]$ and $[a, b)$ respectively.

The subset $\{x \in \mathbb{R} : x > a\}$ is an *infinite open interval.* This is denoted by (a, ∞).

The subset $\{x \in \mathbb{R} : x \geq a\}$ is an *infinite closed interval.* This is denoted by $[a, \infty)$

The subset $\{x \in \mathbb{R} : x < a\}$ is an *infinite open interval.* This is denoted by $(-\infty, a)$.

The subset $\{x \in \mathbb{R} : x \leq a\}$ is an *infinite closed interval.* This is denoted by $(-\infty, a]$.

When both the end points of an interval belong to $\mathbb{R}$, the interval is said to be a *bounded interval.*

Therefore the intervals $(a, b), [a, b], (a, b], [a, b)$ are all bounded intervals.

The intervals $(a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$ are *unbounded intervals.*

If $a = b$, the closed interval $[a, a]$ is the singleton set $\{a\}$.

The set $\mathbb{R}$ is also denoted by $(-\infty, \infty)$. This is an unbounded interval without end points.

3.2. Neighbourhood.

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a *neighbourhood* of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly, an open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.

A closed bounded interval containing the point c may not be a neighbourhood of c. For example, $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.

Let $c \in \mathbb{R}$ and $\delta > 0$. The open interval $\boxed{(c - \delta, c + \delta)}$ is said to the δ-neighbourhood of c and is denoted by $\boxed{N(c, \delta)}$. Clearly, the δ-neighbourhood of c is an open interval symmetric about c.

Theorem 3.2.1. Let $c \in \mathbb{R}$. The union of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$. Let $a_3 = \min\{a_1, a_2\}$, $b_3 = \max\{b_1, b_2\}$. Then $(a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

Now $(a_1, b_1) \subset S_1 \cup S_2$ and $(a_2, b_2) \subset S_1 \cup S_2$
$\Rightarrow (a_3, b_3) = (a_1, b_1) \cup (a_2, b_2) \subset S_1 \cup S_2$.
Thus $c \in (a_3, b_3) \subset S_1 \cup S_2$.
This proves that $S_1 \cup S_2$ is a neighbourhood of c.

Hold for nbd

Note. The union of a finite number of neighbourhoods of c is a neighbourhood of c.

ed Arbitrary Union

Theorem 3.2.2. Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$.

Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$. Then $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

$$\begin{aligned} \text{Now } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_1, b_1) \subset S_1 \\ \text{and } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_2, b_2) \subset S_2 \\ &\Rightarrow (a_3, b_3) \subset S_1 \cap S_2. \end{aligned}$$

Thus $c \in (a_3, b_3) \subset S_1 \cap S_2$.
This proves that $S_1 \cap S_2$ is a neighbourhood of c.

Note. The intersection of a finite number of neighbourhoods of a point

c is a neighbourhood of c.

The intersection of an infinite number of neighbourhoods of a point c may not be a neighbourhood of c.

For example, for every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbouhood of 0.

$\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not a neighbourhood of 0.

3.3. Interior point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *interior point* of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

The set of all interior points of S is said to be the *interior* of S and is denoted by int S (or by S^o).

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots \cdots\}$. $\Rightarrow S^{\circ} = \phi$

Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

2. Let $S = \mathbb{N}$.

Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

3. Let $S = \mathbb{Q}$.

Let $x \in \mathbb{Q}$. Every neighbourhood of x contains rational as well as irrational points. So x can not be an interior point of $\mathbb{Q}$. Therefore int $S = \phi$.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Every point of S is an interior point of S. Therefore int $S = S$.

5. Let $S = \mathbb{R}$. Every point of S is an interior point of S. Therefore int $S = S$.

6. Let $S = \phi$. S has no interior point. Therefore int $S = \phi$.

3.4. Open set.

Let $S \subset \mathbb{R}$. S is said to be an *open set* if each point of S is an interior point of S.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$. No point of S is an interior point of S. S is not an open set.

2. Let $S = \mathbb{Z}$. No point of S is an interior point of S. S is not an open set.

3. Let $S = \mathbb{Q}$. No point of S is an interior point of S. S is not an open set.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. S is an open set.

5. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. 1 and 3 belong to S but they are not interior points of S. S is not an open set.

6. Let $S = \mathbb{R}$. Each point of S is an interior point of S. S is an open set.

7. Let $S = \phi$. S contains no point. Therefore the requirement in the definition is vacuously satisfied. S is an open set.

Theorem 3.4.1. Let $S \subset \mathbb{R}$. Then S is an open set if and only if $S =$ int S.

Proof. We prove the theorem for a non-empty set S because if $S = \phi$ then $\phi =$ int ϕ holds and also ϕ is an open set.

Let S be a non-empty open set and let $x \in S$. Then x is an interior point of S.

Thus $x \in S \Rightarrow x \in$ int S. Therefore $S \subset$ int S ... ... (i)

Let $y \in$ int S. Then $y \in S$ by the definition of an interior point.
Thus $y \in$ int $S \Rightarrow y \in S$. Therefore int $S \subset S$ ... ... (ii)
From (i) and (ii) we have $S =$ int S.

Conversely, let S be a non-empty set and $S =$ int S.
Let $x \in S$. Then $x \in$ int S, since $S =$ int S.
Thus every point of S is an interior point of S and therefore S is an open set.
This completes the proof.

Theorem 3.4.2. The union of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.
Let $x \in G_1 \cup G_2$. Then $x \in G_1$ or $x \in G_2$.

Let $x \in G_1$. Since G_1 is open set and $x \in G_1, x$ is an interior point of G_1. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_1$.
$N(x) \subset G_1 \Rightarrow N(x) \subset G_1 \cup G_2$.
This shows that x is an interior point of $G_1 \cup G_2$.
Since x is arbitrary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is an open set.

If however, $x \in G_2$, we can prove in a similar manner that $G_1 \cup G_2$ is an open set. This completes the proof.

Theorem 3.4.3. The intersection of two open sets in $\mathbb{R}$ is an open set.

Proof. Let G_1 and G_2 be two open sets in $\mathbb{R}$.

Case 1. $G_1 \cap G_2 = \phi$. Since ϕ is an open set, $G_1 \cap G_2$ is an open set.

Case 2. $G_1 \cap G_2 \neq \phi$. Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$.

Since G_1 is an open set and $x \in G_1, x$ is an interior point of G_1.

Hence there exists a positive δ_1 such that the neighbourhood $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2, x$ is an interior point of G_2.

Hence there exists a positive δ_2 such that the neighbourhood $N(x, \delta_2) \subset G_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$.

$N(x, \delta) \subset N(x, \delta_1) \subset G_1$ and $N(x, \delta) \subset N(x, \delta_2) \subset G_2$.

Consequently, $N(x, \delta) \subset G_1 \cap G_2$.

This shows that x is an interior point of $G_1 \cap G_2$. Since x is arbitrary, $G_1 \cap G_2$ is an open set and this completes the proof.

Theorem 3.4.4. The union of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cup G_2 \cup \cdots \cup G_m$.

Let $x \in G$. Then x belongs to at least one of the sets, say G_k. Since G_k is an open set and $x \in G_k, x$ is an interior point of G_k. Hence there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_k$.

$N(x) \subset G_k \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set. This completes the proof.

Theorem 3.4.5. The intersection of a finite number of open sets in $\mathbb{R}$ is an open set.

Proof. Let $G_1, G_2, \ldots, G_m$ be m open sets in $\mathbb{R}$.

Let $G = G_1 \cap G_2 \cap \cdots \cap G_m$.

Case 1. $G = \phi$. Then G is an open set, since ϕ is an open set.

Case 2. $G \neq \phi$. Let $x \in G$. Then $x \in G_i$ for each $i = 1, 2, \ldots, m$.

Since G_1 is an open set and $x \in G_1$, there exists a positive δ_1 such that $N(x, \delta_1) \subset G_1$.

Since G_2 is an open set and $x \in G_2$, there exists a positive δ_2 such that $N(x, \delta_2) \subset G_2$.

... ...

Since G_m is an open set and $x \in G_m$, there exists a positive δ_m such that $N(x, \delta_m) \subset G_m$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$.

$$N(x,\delta) \subset N(x,\delta_1) \subset G_1$$
$$N(x,\delta) \subset N(x,\delta_2) \subset G_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$N(x,\delta) \subset N(x,\delta_m) \subset G_m.$$

Consequently, $N(x,\delta) \subset G_1 \cap G_2 \cap \cdots \cap G_m = G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set. This completes the proof.

Theorem 3.4.6. The union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set.

Proof. Let $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be an arbitrary collection of open sets in $\mathbb{R}$. Let $G = \bigcup\limits_{\alpha \in \Lambda} G_\alpha$.

Let $x \in G$. Then x belongs to at least one open set of the collection, say $G_\lambda, (\lambda \in \Lambda)$.

Since G_λ is an open set and $x \in G_\lambda, x$ is an interior point of G_λ.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G_\lambda$. $N(x) \subset G_\lambda \Rightarrow N(x) \subset G$.

This shows that x is an interior point of G. Since x is arbitrary, G is an open set and the proof is complete.

Note. The intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Let us consider the sets G_i where

$$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$$
$$G_2 = \{x \in \mathbb{R} : -\tfrac{1}{2} < x < \tfrac{1}{2}\}$$
$$\ldots \quad \ldots \quad \ldots$$
$$G_n = \{x \in \mathbb{R} : -\tfrac{1}{n} < x < \tfrac{1}{n}\}$$
$$\ldots \quad \ldots \quad \ldots$$

Each G_i is an open set. $\bigcap\limits_{i=1}^{\infty} G_i = \{0\}$. This is not an open set.

Let us consider the sets G_i where

$$G_1 = \{x \in \mathbb{R} : -1 < x < 1\}$$
$$G_2 = \{x \in \mathbb{R} : -2 < x < 2\}$$
$$\ldots \quad \ldots \quad \ldots$$
$$G_n = \{x \in \mathbb{R} : -n < x < n\}$$
$$\ldots \quad \ldots \quad \ldots$$

Each G_i is an open set. $\bigcap\limits_{i=1}^{\infty} G_i = G_1$. This is an open set.

From these two examples we conclude that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

Theorem 3.4.7. Let S be a subset of $\mathbb{R}$. Then int S is an open set.

Proof. **Case 1.** int $S = \phi$. Since ϕ is an open set, int S is an open set.

Case 2. int $S \neq \phi$. Let $x \in$ int S. Then x is an interior point of S. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

Let $y \in N(x)$. Then $N(x)$ is a neighbourhood of y also and since $N(x) \subset S$, y is an interior point of S.

Thus $y \in N(x) \Rightarrow y \in$ int S. Therefore $N(x) \subset$ int S.

This shows that x is an interior point of int S.

Thus $x \in$ int $S \Rightarrow x$ is an interior point of int S.

Therefore int S is an open set. This completes the proof.

Theorem 3.4.8. Let $S \subset \mathbb{R}$. Then int S is the largest open set contained in S.

Proof. By the previous theorem, int S is an open set and int $S \subset S$, by definition.

Let P be any open set contained in S.

Let $x \in P$. Since P is an open set, x is an interior point of P.

Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset P$. But $N(x) \subset P \Rightarrow N(x) \subset S$, since $P \subset S$.

This shows that x is an interior point of S, i.e., $x \in$ int S.

Thus $x \in P \Rightarrow x \in$ int S. Therefore $P \subset$ int S.

Since P is arbitrary, int S is the largest open set contained in S.

Note. int S is the union of all open sets contained in S.

Worked Examples.

1. Prove that an open interval is an open set.

Let I be an open interval. Four cases arise.

Case 1. $I = (a, b)$ for some $a, b \in \mathbb{R}$, with $a < b$.

Let $c \in I$. Then I itself is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 2. $I = (a, \infty)$ for some $a \in \mathbb{R}$.

Let $c \in I$. Then $a < c < \infty$. Let $d \in (c, \infty)$. Then $a < c < d$.

The open interval (a, d) is a neighbourhood of c, say $N(c)$ and $N(c) \subset I$. This shows that c is an interior point of I. Thus every point of I is an interior point of I and therefore I is an open set.

Case 3. $I = (-\infty, a)$ for some $a \in \mathbb{R}$.
Similar proof.

Case 4. $I = (-\infty, \infty)$.
Similar proof.

2. Let $S = (0, 1]$ and $T = \{\frac{1}{n} : n = 1, 2, 3, \ldots\}$. Show that $S - T$ is an open set.

$S - T = (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup \cdots$

$S - T$ is the union of an infinite number of open intervals. Since an open interval is an open set, $S - T$ being the union of an infinite number of open sets is an open set.

We have seen that an open interval is an open set in $\mathbb{R}$ and the union of any collection of open sets is an open set in $\mathbb{R}$. Therefore the union of an arbitrary collection of open intervals is an open set in $\mathbb{R}$.

The following theorem deals with the converse problem and it depicts the structural composition of a bounded open set in $\mathbb{R}$.

Theorem 3.4.9. A non-empty bounded open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

Proof. Let G be a non-empty bounded open set in $\mathbb{R}$. Let $x \in G$. Since G is an open set, there is an element $y_o < x$ and an element $z_o > x$ such that $(y_o, x) \subset G$ and $(x, z_o) \subset G$.

Let $A = \{y : (y, x) \subset G\}$, $B = \{z : (x, z) \subset G\}$.

Then A is a non-empty set, since $y_o \in A$; A is bounded below, since G is bounded below. Let $a = \inf A$.

Similarly, B is a non-empty set bounded above. Let $b = \sup B$.

Then $a < x < b$ and $I_x = (a, b)$ is an open interval containing x. We prove $I_x \subset G$.

Let $w \in I_x$ and $a < x < w < b$.

Since $b = \sup B$, there exists an element $z' \in B$ such that $w < z' \leq b$.

Therefore $(x, z') \subset G$, since $z' \in B$. Therefore $w \in G$.

If however, $w \in I_x$ and $a < w < x < b$, then also $w \in G$.

Thus $w \in I_x \Rightarrow w \in G$ and therefore $I_x \subset G$.

We prove $a \notin G, b \notin G$.

If $b \in G$, then for some positive $\epsilon, (b - \epsilon, b + \epsilon) \subset G$, since G is an open set. Let $\delta < \epsilon$. Then $b < b + \delta < b + \epsilon$ and $b + \delta \in G$ contradicting the definition of b. Therefore $b \notin G$. Similarly, $a \notin G$.

Let $\mathcal{G}$ be the collection of open intervals $\{I_x : x \in G\}$. Let $H = \bigcup\limits_{x \in G} I_x$.

Let $x \in G$. Then $x \in I_x$ and $I_x \subset H$.
Thus $x \in G \Rightarrow x \in H$. Therefore $G \subset H$.
Let $y \in H$. Then $y \in I_y$ and $I_y \subset G$.
Thus $y \in H \Rightarrow y \in G$. Therefore $H \subset G$.
Consequently, $G = H = \bigcup_{x \in G} x$.

We prove that two distinct intervals in the collection $\mathcal{G}$ are disjoint.

Let $(a, b), (c, d)$ be two intervals in this collection with a point p in common.

Then $c < b$ and $a < d$.

Since $c \notin G$, c does not belong to (a, b) and therefore $c \leq a$.

Since $a \notin G$, a does not belong to (c, b) and therefore $a \leq c$.

$c \leq a$ and $a \leq c \Rightarrow a = c$. Similarly $b = d$.

Therefore two distinct intervals of the collection are disjoint.

Thus G is the union of disjoint collection of open intervals $\{I_x : x \in G\}$.

We show that the collection is countable.

Let $\mathcal{G}'$ be the collection $\{I_\alpha : \alpha \in \Lambda\}$ where I_α is an open interval and Λ is the index set.

Let $\lambda \in \Lambda$. Then I_λ is an open interval of the collection $\mathcal{G}'$.

Let $x \in I_\lambda$. Then there exists a positive δ such that $(x - \delta, x + \delta) \subset I_\lambda$. There exists a rational number r_λ such that $x - \delta < r_\lambda < x + \delta$. Therefore $r_\lambda \in \mathbb{Q} \cap I_\lambda$.

Let us define a function $f : \Lambda \to \mathbb{Q}$ that assigns $\lambda (\in \Lambda)$ to $r_\lambda (\in \mathbb{Q})$.

Since I_α's are disjoint, the function f is injective.

Since $\mathbb{Q}$ is an enumerable set and f is an injective function, Λ is at most enumerable. Hence $\mathcal{G}'$ is a countable collection.

This completes the proof.

3.5. Limit point.

Let S be a subset of $\mathbb{R}$. A point p in $\mathbb{R}$ is said to be a *limit point* (or an *accumulation point*, or a *cluster point*) of S if every neighbourhood of p contains a point of S other than p.

Therefore p is a limit point of S if for each positive ϵ,

$$[N(p, \epsilon) - \{p\}] \cap S \neq \phi.$$

$N(p, \epsilon) - \{p\}$ is called the *deleted* ϵ-neighbourhood of p and is denoted by $N'(p, \epsilon)$. $N(p) - \{p\}$ is called the *deleted neighbourhood* of p and is denoted by $N'(p)$.

Therefore p is a limit point of S if every deleted neighbourhood of p contains a point of S.

Note. A limit point of S may or may not belong to S. When we say that a set $S \subset \mathbb{R}$ has a limit point we mean that some real number p is a limit point of S and no assertion is made as to whether p belongs to S or not.

3.6. Isolated point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *isolated point* of S if x is not a limit point of S.

Since x is not a limit point of S, there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap S = \phi$. Since $x \in S, N(x) \cap S = \{x\}$.

Therefore x is an isolated point of S if for some positive $\epsilon, N(x, \epsilon)$ contains *no point* of S other than x.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots\}$.

Every point of S is an isolated point of S.

We prove that 0 is a limit point of S. Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$ there exists a natural number m such that $0 < \frac{1}{m} < \epsilon$.
$\frac{1}{m} \in S$ and $\frac{1}{m} \in N'(0, \epsilon)$. Thus the deleted ϵ- neighbourhood of 0 contains a point of S and this holds for each positive ϵ.

Therefore 0 is a limit point of S.

2. Let $S = \mathbb{Z}$.

Every point of $\mathbb{Z}$ is an isolated point of $\mathbb{Z}$. Therefore no point of $\mathbb{Z}$ is a limit point of $\mathbb{Z}$.

Let $x \in \mathbb{R} - \mathbb{Z}$. Then there exists an integer m such that $m - 1 < x < m$. Let $\epsilon = \min\{|x - m|, |x - \overline{m - 1}|\}$. Then the neighbourhood $N(x, \epsilon)$ of x contains no point of $\mathbb{Z}$ and therefore x can not be a limit point of $\mathbb{Z}$.

3. Let $S = \mathbb{Q}$.

No point of S is an isolated point of S. Every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, since each deleted neighbourhood of x contains a point of $\mathbb{Q}$.

4. Let $S = \mathbb{R}$.

No point of S is an isolated point of S. Every point x of $\mathbb{R}$ is a limit point of $\mathbb{R}$, since each deleted neighbourhood of x contains a point of $\mathbb{R}$.

Theorem 3.6.1. Let $S \subset \mathbb{R}$ and p be a limit point of S. Then every neighbourhood of p contains infinitely many elements of S.

Proof. Let $\epsilon > 0$. Since p is a limit point of S, the deleted neighbourhood $N'(p, \epsilon)$ contains a point of S. That is, $N'(p, \epsilon) \cap S \neq \phi$.

Let $B = N'(p, \epsilon) \cap S$. We prove that B is an infinite set.

If not, let B contain only a finite number of elements of S, say $a_1, a_2, \ldots, a_m$.

Let $\delta_1 = | p - a_1 |, \delta_2 = | p - a_2 |, \cdots, \delta_m = | p - a_m |$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$. Then $\delta > 0$ and $a_i \notin N(p, \delta), i = 1, 2, \ldots, m$.

It follows that $N'(p, \delta) \cap S = \phi$ and this disallows p to be a limit point of S.

Thus B is an infinite set. In other words, $N(p, \epsilon)$ contains infinitely many elements of S. This proves the theorem.

Worked Examples.

1. Show that a finite set has no limit point.

Let S be a finite set and $S = \{x_1, x_2, \ldots, x_m\}$. Let $p \in \mathbb{R}$. p can not be a limit point of S because if p be a limit point of S then every neighbourhood of p must contain infinitely many elements of S, which is an impossibility since S contains only a finite number of elements.

Therefore S has no limit point.

2. Show that the set $\mathbb{N}$ has no limit point.

Let $p \in \mathbb{R}$. Let $\epsilon = \frac{1}{2}$. Then the ϵ-neighbourhood $N(p, \frac{1}{2})$ of p contains at most one natural number and p cannot be a limit point of $\mathbb{N}$, because, in order that p may be a limit point of $\mathbb{N}$, each neighbourhood of p must contain infinitely many elements of $\mathbb{N}$.

It follows that $\mathbb{N}$ has no limit point.

Note. By similar arguments it can be established that the set $\mathbb{Z}$ has no limit point.

3. Let S be a subset of $\mathbb{R}$. Prove that an interior point of S is a limit point of S.

Let x be an interior point of S. Then there exists a positive δ such that the neighbourhood $N(x, \delta)$ of x is entirely contained in S.

Let us choose $\epsilon > 0$.

Case 1. $0 < \epsilon < \delta$.

Then $N(x, \epsilon) \subset N(x, \delta) \subset S$ and therefore $N'(x, \epsilon) \cap S \neq \phi$.

Case 2. $\epsilon \geq \delta$. Then $N(x, \delta) \subset N(x, \epsilon)$.

$N(x, \delta) \subset S$ and $N(x, \delta) \subset N(x, \epsilon) \Rightarrow N(x, \delta) \subset N(x, \epsilon) \cap S$.

Then clearly, $N'(x, \epsilon) \cap S \neq \phi$.

In both the cases $N'(x, \epsilon) \cap S \neq \phi$ and this proves that x is a limit point of S.

Theorem 3.6.2. Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ both exist. Let $s^* = \sup S$ and $s_* = \inf S$. Then $x \in S \Rightarrow s_* \leq x \leq s^*$.

Let H be a subset of $\mathbb{R}$ defined by $H = \{x \in \mathbb{R} : x$ is greater than infinitely many elements of $S\}$.

Then $s^* \in H$ and so H is a non-empty subset of $\mathbb{R}$.

Let $h \in H$. Then h is greater than infinitely many elements of S and therefore $h > s_*$, because no element $\leq s_*$ exceeds infinitely many elements of S.

Thus H is a non-empty subset of $\mathbb{R}$, bounded below, s_* being a lower bound. So $\inf H$ exists.

Let $\inf H = \xi$. We now show that ξ is a limit point of S.

Let us choose $\epsilon > 0$.

Since $\inf H = \xi$, there exists an element y in H such that $\xi \leq y < \xi + \epsilon$.

Since $y \in H$, y exceeds infinitely many elements of S and consequently $\xi + \epsilon$ exceeds infinitely many element of S.

Since ξ is the infimum of H, $\xi - \epsilon$ does not belong to H and so $\xi - \epsilon$ can exceed at most a finite number of elements of S. Thus the neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many elements of S.

This holds for each $\epsilon > 0$. Therefore ξ is a limit point of S.

This completes the proof.

3.7. Derived set.

Let $S \subset \mathbb{R}$. The set of all limit points of S is said to be the *derived set* of S and is denoted by S'.

Examples.

1. Let S be a finite set. Then $S' = \phi$.
2. Let $S = \mathbb{N}$. Then $S' = \phi$.
3. Let $S = \mathbb{Z}$. Then $S' = \phi$.
4. Let $S = \mathbb{Q}$. Then $S' = \mathbb{R}$.
5. Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$.
6. Let $S = \phi$. Then $S' = \phi$.

Theorem 3.7.1. Let A, B be subsets of $\mathbb{R}$ and $A \subset B$. Then $A' \subset B'$.

Proof. **Case 1.** $A' = \phi$. Then $A' \subset B'$.

Case 2. $A' \neq \phi$. Let $p \in A'$. Then p is a limit point point of A.
Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A, say q, other than p.
$q \in A \Rightarrow q \in B$. Therefore $N'(p, \epsilon)$ contains a point q of B.

Since ϵ is arbitrary, p is a limit point of B. Therefore $p \in B'$. Thus $p \in A' \Rightarrow p \in B'$ and therefore $A' \subset B'$.

This completes the proof.

Theorem 3.7.2. Let $A \subset \mathbb{R}$. Then $(A')' \subset A'$.

Proof. **Case 1.** $(A')' = \phi$. Then $(A')' \subset A'$.

Case 2 . $(A')' \neq \phi$. Let $p \in (A')'$. Then p is a limit point of A'.
Let $\epsilon > 0$. Then $N(p, \epsilon)$ contains a point of A', say q, other than p.

Since $q \in A'$, q is a limit point of A. Therefore $N(p, \epsilon)$ being a neighbourhood of q also, contains infinitely many points of A.

Since $N(p, \epsilon)$ contains infinitely many points of A, p is a limit point of A. That is, $p \in A'$.
Thus $p \in (A')' \Rightarrow p \in A'$ and therefore $(A')' \subset A'$.

This completes the proof.

Theorem 3.7.3. Let $A, B \subset \mathbb{R}$. Then $(A \cap B)' \subset A' \cap B'$.

Proof. $A \cap B \subset A \Rightarrow (A \cap B)' \subset A'$, since $A \subset B \Rightarrow A' \subset B'$.
$A \cap B \subset B \Rightarrow (A \cap B)' \subset B'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $(A \cap B)' \subset A' \cap B'$.

Note. $(A \cap B)' \neq A' \cap B'$, in general.
For example, let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $B = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \cdots\}$. Then $A' = \{0\}$, $B' = \{0\}$. $A \cap B = \{0\}$, $A' \cap B' = \{0\}$, but $(A \cap B)' = \phi$.

Corollary. Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cap A_2 \cap \cdots \cap A_m)' \subset A_1' \cap A_2' \cap \cdots \cap A_m'$.

Theorem 3.7.4. Let A and B be subsets of $\mathbb{R}$. Then $(A \cup B)' = A' \cup B'$.

Proof. $A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$
$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$, since $A \subset B \Rightarrow A' \subset B'$.

It follows that $A' \cup B' \subset (A \cup B)'$ ... ... (i)

We now prove that $(A \cup B)' \subset A' \cup B'$.
Let $p \notin A' \cup B'$. Then $p \notin A'$ and $p \notin B'$.

So there exists a positive ϵ_1 such that $N'(p, \epsilon_1) \cap A = \phi$ and there exists a positive ϵ_2 such that $N'(p, \epsilon_2) \cap B = \phi$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $\epsilon > 0$ and $N'(p,\epsilon) \cap A = \phi$, $N'(p,\epsilon) \cap B = \phi$.
Therefore $N'(p,\epsilon) \cap (A \cup B) = [N'(p,\epsilon) \cap A] \cup [N'(p,\epsilon) \cap B] = \phi$.
This disallows p to be a limit point of $A \cup B$. So $p \notin (A \cup B)'$.
Thus $p \notin A' \cup B' \Rightarrow p \notin (A \cup B)'$.

Contrapositively, $p \in (A \cup B)' \Rightarrow p \in A' \cup B'$.
Consequently, $(A \cup B)' \subset A' \cup B'$ ... ... (ii)

From (i) and (ii) it follows that $(A \cup B)' = A' \cup B'$.

This completes the proof.

Corollary. Let $A_1, A_2, \ldots, A_m$ be subsets of $\mathbb{R}$. Then $(A_1 \cup A_2 \cup \cdots \cup A_m)' = A_1' \cup A_2' \cup \cdots \cup A_m'$.

Worked Examples.

1. Let S be a bounded subset of $\mathbb{R}$. Prove that S' (the derived set of S) is bounded.

Case 1. Let S be a finite subset of $\mathbb{R}$. Then $S' = \phi$ and it is bounded.

Case 2. Let S be an infinite subset of $\mathbb{R}$. By Bolzano-Weierstrass theorem, S' is a non-empty subset of $\mathbb{R}$.

Let $\sup S = m^*$. Then $x \in S \Rightarrow x \leq m^*$. Let $c > m^*$. Let us choose $\epsilon = \frac{c-m^*}{2}$. Then $m^* + \epsilon = c - \epsilon$ and the ϵ−neighbourhood $(c-\epsilon, c+\epsilon)$ of c contains no point of S. Therefore c cannot be a limit point of S, i.e., $c \notin S'$.

Thus $c > m^* \Rightarrow c \notin S'$. Contrapositively, $c \in S' \Rightarrow c \leq m^*$. This shows that m^* is an upper bound of S', i.e., S' is bounded above.

Let $\inf S = m_*$ and let $d < m_*$.
By similar arguments, d cannot be a limit point of S, i.e., $d \notin S'$.

Thus $d < m_* \Rightarrow d \notin S'$. Contrapositively, $d \in S' \Rightarrow d \geq m_*$. This shows that m_* is a lower bound of S', i.e., S' is bounded below.
Therefore S' is a bounded subset of $\mathbb{R}$.

2. Let S be a non-empty subset of $\mathbb{R}$ bounded above and $s^* = \sup S$. If s^* does not belong to S prove that s^* is a limit point of S and s^* is the greatest element of S'.

Let $\epsilon > 0$. Since $s^* = \sup S$,
(i) $x \in S \Rightarrow x < s^*$ (since $s^* \notin S$) and
(ii) there is an element y in S such that $s^* - \epsilon < y < s^*$.

We have $s^* - \epsilon < y < s^* < s^* + \epsilon$.

Thus the ϵ-neighbourhood $(s^* - \epsilon, s^* + \epsilon)$ of s^* contains a point y of S other than s^*. Since ϵ is arbitrary, s^* is a limit point of S.

Let $t > s^*$ and $\epsilon = \frac{t-s^*}{2}$. Then $\epsilon > 0$ and $s^* + \epsilon = t - \epsilon$. Since $s^* = \sup S$, no point of S is greater than S^*. Therefore the neighbourhood $(t-\epsilon, t+\epsilon)$ of t contains no point of S and so t is not a limit point of S. Consequently, S^* is the greatest element of S'.

3. Let $S = (a, b)$ an open bounded interval. Prove that $S' = [a, b]$.

Case 1. Let $x \in (a, b)$. Then x is an interior point of S.

By worked Example 4, page 53, x is a limit point of S.

Case 2. Let $x = a$.

Let us choose $\epsilon > 0$.

Let $\delta = \min\{\epsilon, b-a\}$. Then $\delta > 0$ and $a < a + \frac{\delta}{2} < a + \delta \leq a + \epsilon$,

$a < a + \frac{\delta}{2} < a + \delta \leq b$.

$a < a + \frac{\delta}{2} < a + \epsilon \Rightarrow a + \frac{\delta}{2} \in N'(a, \epsilon)$.

$a < a + \frac{\delta}{2} < b \Rightarrow a + \frac{\delta}{2} \in S$.

Therefore $a + \frac{\delta}{2} \in N'(a, \epsilon) \cap S$. As $N'(a, \epsilon) \cap S \neq \phi$, a is a limit point of S.

Case 3. Let $x = b$.

In a similar manner we can prove that b is a limit point of S.

Case 4. Let $x > b$.

Let us choose $\epsilon = \frac{x-b}{2}$. Then $\epsilon > 0$ and $b + \epsilon = x - \epsilon$.

The neighbourhood $(x-\epsilon, x+\epsilon)$ contains no point of S and this proves that x is not a limit point of S.

Case 5. Let $x < a$.

Let us choose $\epsilon = \frac{a-x}{2}$. Then $\epsilon > 0$ and $x + \epsilon = a - \epsilon$.

The neighbourhood $(x-\epsilon, x+\epsilon)$ contains no point of S and this proves that x is not a limit point of S.

From the above cases, we conclude that $S' = [a, b]$.

4. Let $S = [a, b]$, a closed bounded interval, then $S' = S$.

The proof is similar.

5. Find the derived set of the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Here $S \subset (0, 1] = B$, say. Therefore $S' \subset B' = [0, 1]$.

First we prove that 0 is a limit point of S.

Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \epsilon$. Then $-\epsilon < \frac{1}{p} < \epsilon$.

This shows that the ϵ-neighbourhod of 0 contains a point $\frac{1}{p}$ of S, other than 0. So 0 is a limit point of S.

Let $c \in (0,1]$. Then $c > 0$. Let us choose a positive ϵ such that $c - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < c - \epsilon$. Then $\frac{1}{n} < c - \epsilon$ for all $n \geq k$.

It follows that at most a finite number of elements of S lie in the neighbourhood $(c-\epsilon, c+\epsilon)$ of c. So c cannot be a limit point of S.

Thus 0 is the only limit point of S. That is, $S' = \{0\}$.

6. Let $S = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.

(i) Show that 0 is a limit point of S.

(ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.

(iii) Find S' (the derived set of S).

(i) Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exist natural numbers p, q such that $0 < \frac{1}{p} < \frac{\epsilon}{2}$, $0 < \frac{1}{q} < \frac{\epsilon}{3}$.

Then $0 < \frac{1}{p} + \frac{1}{q} < \epsilon$.

This shows that the ϵ-neighbourhood $(-\epsilon, \epsilon)$ of 0 contains a point $\frac{1}{p} + \frac{1}{q}$ of S, other than 0. So 0 is limit point of S.

(ii) Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number s such that $0 < \frac{1}{s} < \epsilon$.

Then $-\epsilon < \frac{1}{s} < \epsilon$

or, $\frac{1}{k} - \epsilon < \frac{1}{k} + \frac{1}{s} < \frac{1}{k} + \epsilon$.

This shows that the ϵ-neighbourhood of $\frac{1}{k}$ contains a point $\frac{1}{k} + \frac{1}{s}$ of S, other than $\frac{1}{k}$. So $\frac{1}{k}$ is a limit point of S.

(iii) Let $T = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $T \subset S'$.

Now $S \subset (0,2]$. This implies $S' \subset \{(0,2]\}' = [0,2]$.

We prove that $T = S'$, i.e., no point of $[0,2] - T$ is a limit point of S.

$[0,2] - T = \{2\} \cup I_1 \cup I_2 \cup \cdots$ where $I_1 = (1,2)$, $I_n = (\frac{1}{n}, \frac{1}{n-1})$, for $n = 2, 3, \ldots$

2 cannot be a limit point of S, because the neighbourhood $(2-\frac{1}{2}, 2+\frac{1}{2})$ of 2 contains no point of S, other than 2.

Let $c \in I_1$. Then $1 < c < 2$. Let us choose $\epsilon > 0$ such that $1 < c - \epsilon < c + \epsilon < 2$.

$c - 1 - \epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - 1 - \epsilon$.

Then $1 + \frac{1}{p} < c - \epsilon < c + \epsilon < 2$.

If $m \geq p$, $\frac{1}{m} + \frac{1}{n} \leq 1 + \frac{1}{p}$ for all $n \geq 1$.
If $n \geq p$, $\frac{1}{m} + \frac{1}{n} \leq 1 + \frac{1}{p}$ for all $m \geq 1$.

It follows that only a finite number of elements of S lie in the neighbourhod $(1 - \epsilon, 1 + \epsilon)$ of c. So c cannot be a limit point of S.

Let $c \in I_k, k = 2, 3, \ldots$. Then $\frac{1}{k} < c < \frac{1}{k-1}$.
Let us choose $\epsilon > 0$ such that $\frac{1}{k} < c - \epsilon < c + \epsilon < \frac{1}{k-1}$.
$c - \frac{1}{k} - \epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - \frac{1}{k} - \epsilon$.
Then $\frac{1}{k} + \frac{1}{p} < c - \epsilon < c + \epsilon < \frac{1}{k-1}$ and $\frac{1}{p} < \frac{1}{k-1} - \frac{1}{k}$, i.e., $p > k$, since $k \geq 2$

If $m \leq k - 1$, $\frac{1}{m} + \frac{1}{n} > \frac{1}{k-1}$ for all $n \geq 1$.
If $m \geq k$, $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{k} + \frac{1}{p}$ for all $n \geq p$.

If $n \leq k - 1$, $\frac{1}{m} + \frac{1}{n} > \frac{1}{k-1}$ for all $m \geq 1$.
If $n \geq k$, $\frac{1}{m} + \frac{1}{n} \leq \frac{1}{k} + \frac{1}{p}$ for all $m \geq p$.

It follows that at most a finite number of elements of S lie in the neighbourhood $(c - \epsilon, c + \epsilon)$ of c. So c cannot be a limit point of S.
Thus $c \in [0, 2] - T \Rightarrow c$ is not a limit point of S.

Hence $S' = T$, i.e., $S' = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

7. Let $S = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
(i) Show that 1 and -1 are limit points of S.
(ii) Find S' (the derived set of S).

(i) Let $A = \{(-1)^{2m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$, $B = \{(-1)^{2m-1} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. Then $S = A \cup B$ and therefore $S' = A' \cup B'$.

$A = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$. We prove that 1 is the only limit point of A.
Let $\epsilon > 0$. By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < \epsilon$. Therefore $-\epsilon < \frac{1}{p} < \epsilon$.
or, $1 - \epsilon < 1 + \frac{1}{p} < 1 + \epsilon$.

This shows that the ϵ-neighbourhood $(1 - \epsilon, 1 + \epsilon)$ of 1 contains a point $1 + \frac{1}{p}$ of A, other than 1. So 1 is a limit point of A.

$A \subset (1, 2]$. This implies $A' \subset [1, 2]$.

2 cannot be a limit point of A, because the neighbourhood $(2-\frac{1}{2}, 2+\frac{1}{2})$ of 2 contains no point of A other than 2.

Let $c \in (1, 2)$. Let us choose a positive ϵ such that $1 < c-\epsilon < c+\epsilon < 2$. Then $c - 1 - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number p such that $0 < \frac{1}{p} < c - 1 - \epsilon$. Then $1 < 1 + \frac{1}{p} < c - \epsilon$.

Therefore for all $n \geq p$, $1 + \frac{1}{n} \leq 1 + \frac{1}{p} < c - \epsilon$. So the neighbourhood $(c - \epsilon, c + \epsilon)$ of c contains at most a finite number of elements of A and hence c cannot be a limit point of A. Therefore $A' = \{1\}$.

$B = \{-1 + \frac{1}{n} : n \in \mathbb{N}\}$. We prove that -1 is the only limit point of B.

Let $\epsilon > 0$. There exists a natural number q such that $0 < \frac{1}{q} < \epsilon$.

Therefore $-\epsilon < \frac{1}{q} < \epsilon$, or, $-1 - \epsilon < -1 + \frac{1}{q} < -1 + \epsilon$.

This shows that the ϵ-neighbourhood $(-1 - \epsilon, -1 + \epsilon)$ of -1 contains a point $-1 + \frac{1}{q}$ of B, other than -1. So -1 is a limit point of B.

$B \subset (-1, 0]$. This implies $B' \subset [-1, 0]$.

0 cannot be a limit point of B, because the neighbourhood $(-\frac{1}{2}, \frac{1}{2})$ of 0 contains no point of B other than 0.

Let $c \in (-1, 0)$. Let us choose a positive ϵ such that $-1 < c - \epsilon < c + \epsilon < 0$. Then $c + 1 - \epsilon > 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number q such that $0 < \frac{1}{q} < c + 1 - \epsilon$. Then $-1 + \frac{1}{q} < c - \epsilon$.

Therefore for all $n \geq q$, $-1 + \frac{1}{n} \leq -1 + \frac{1}{q} < c - \epsilon$.

This shows that the neighbourhood $(c - \epsilon, c + \epsilon)$ of c contains at most a finite number of elements of B and hence c cannot be a limit point of B. Therefore $B' = \{-1\}$.

Therefore $S' = A' \cup B' = \{1, -1\}$.

3.8. Closed set.

Let S be a subset of $\mathbb{R}$. S is said to be a *closed set* if $S' \subset S$. (i.e., if S contains all its limit points.)

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. 0 is a limit point of S. As $0 \notin S$, S is not a closed set.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is a limit point of S. 1 and 3 are also limit points of S but $1 \notin S, 3 \notin S$. Therefore S is not a closed set.

3. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Each point of S is a limit point of S. Here $S' = S$. As $S' \subset S$, S is a closed set.

4. Let $S = \mathbb{Z}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

5. Let $S = \mathbb{N}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

6. Let S be a finite set, say $S = \{a_1, a_2, \ldots, a_m\}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

7. Let $S = \mathbb{Q}$. Let $x \in \mathbb{R}$. Every neighbourhood of x contains infinitely many elements of $\mathbb{Q}$. Therefore $x \in \mathbb{Q}'$. Hence $S' = \mathbb{R}$. Here S' is not a subset of S. S is not a closed set.

Note that the set $\mathbb{Q}$ is neither an open set nor a closed set in $\mathbb{R}$.

8. Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$. So $S' \subset S$ and S is a closed set.

9. Let $S = \phi$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

10. Let S be a subset of $\mathbb{R}$. Then $(S')' \subset S'$, by theorem 3.7.2. Therefore S' is a closed set. It follows that the derived set of S is a closed set.

Theorem 3.8.1. The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$. Let $F = F_1 \cup F_2 \cup \cdots \cup F_m$.

Since F_i is a closd set, $F_i' \subset F_i$ for $i = 1, 2, \ldots, m$.

$F' = (F_1 \cup F_2 \cup \cdots \cup F_m)' = F_1' \cup F_2' \cup \cdots \cup F_m'$.
$F_1' \subset F_1 \Rightarrow F_1' \subset F, F_2' \subset F_2 \Rightarrow F_2' \subset F, \cdots, F_m' \subset F_m \Rightarrow F_m' \subset F$.

Therefore $F_1' \cup F_2' \cup \cdots \cup F_m' \subset F$, i.e., $F' \subset F$.
As $F' \subset F$, F is a closed set and the theorem is done.

Theorem 3.8.2. The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and let $F = F_1 \cap F_2 \cap \cdots \cap F_m$.

Since F_i is a closd set, $F_i' \subset F_i$ for $i = 1, 2, \ldots, m$.

$F' = (F_1 \cap F_2 \cap \cdots \cap F_m)' \subset F_1' \cap F_2' \cap \cdots \cap F_m'$.

$$F_1' \cap F_2' \cap \cdots \cap F_m' \subset F_1' \subset F_1.$$
$$F_1' \cap F_2' \cap \cdots \cap F_m' \subset F_2' \subset F_2.$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$F_1' \cap F_2' \cap \cdots \cap F_m' \subset F_m' \subset F_m.$$

Therefore $F_1' \cap F_2' \cap \cdots \cap F_m' \subset F_1 \cap F_2 \cap \cdots \cap F_m = F$.
It follows that $F' \subset F_1' \cap F_2' \cap \cdots \cap F_m' \subset F$.
As $F' \subset F$, F is a closed set and the theorem is done.

Theorem 3.8.3. The intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $\{F_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set, be a collection of closed sets in $\mathbb{R}$. Then $(F_\alpha)' \subset F_\alpha$ for each $\alpha \in \Lambda$. Let $F = \bigcap\limits_{\alpha\in\Lambda} F_\alpha$.

Case 1. $F' = \phi$. Then obviously $F' \subset F$.

Case 2. $F' \neq \phi$. Let $p \in F'$. Then p is a limit point of F.

Let us choose $\epsilon > 0$. Then $N'(p, \epsilon)$ contains a point, say q, of F. $q \in N'(p, \epsilon) \cap F \Rightarrow q \in N'(p, \epsilon) \cap F_\alpha$ for each $\alpha \in \Lambda$.

This implies p is a limit point of F_α for each $\alpha \in \Lambda$.

Since each F_α is a closed set, $p \in F_\alpha$ for each $\alpha \in \Lambda$.

Hence $p \in \bigcap\limits_{\alpha\in\Lambda} F_\alpha$, i.e., $p \in F$.

Thus $p \in F' \Rightarrow p \in F$ and therefore $F' \subset F$.

This proves that F is a closed set and the theorem is done.

Note. The union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Let us consider the sets F_i, where

$$F_1 = \{x \in \mathbb{R} : -1 \leq x \leq 1\},$$
$$F_2 = \{x \in \mathbb{R} : -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}\},$$
$$\dots \quad \dots \quad \dots$$
$$F_n = \{x \in \mathbb{R} : -\tfrac{1}{n} \leq x \leq \tfrac{1}{n}\},$$
$$\dots \quad \dots \quad \dots$$

Each F_i is a closed set. $\bigcup\limits_{i=1}^{\infty} F_i = F_1$ and this is a closed set.

Let us consider the sets F_i, where

$$F_1 = \{x \in \mathbb{R} : 1 \leq x \leq 2\},$$
$$F_2 = \{x \in \mathbb{R} : \tfrac{1}{2} \leq x \leq 3 - \tfrac{1}{2}\},$$
$$\dots \quad \dots \quad \dots$$
$$F_n = \{x \in \mathbb{R} : \tfrac{1}{n} \leq x \leq 3 - \tfrac{1}{n}\},$$
$$\dots \quad \dots \quad \dots$$

Each F_i is a closed set. $\bigcup\limits_{i=1}^{\infty} F_i = \{x \in \mathbb{R} : 0 < x < 3\}$. This is not a closed set.

These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Theorem 3.8.4. Let G be an open set in $\mathbb{R}$. Then the complement of G (in $\mathbb{R}$) is a closed set in $\mathbb{R}$.

Proof. **Case 1.** $G = \phi$ (an open set in $\mathbb{R}$). The complement of ϕ in $\mathbb{R}$ is $\mathbb{R}$ and $\mathbb{R}$ is a closed set.

Case 2. $G \neq \phi$. Let $x \in G$. Since G is an open set, x is an interior point of G. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset G$.

That is, $N(x) \cap G^c = \phi$ where G^c is the complement of G.

This implies that x is not a limit point of G^c. That is, $x \notin (G^c)'$.

Thus $x \in G \Rightarrow x \notin (G^c)'$.

Contrapositively, $x \in (G^c)' \Rightarrow x \notin G$. i.e., $x \in G^c$.

Therefore $(G^c)' \subset G^c$ and this proves that G^c is a closed set.

This completes the proof.

Theorem 3.8.5. Let F be a closed set in $\mathbb{R}$. Then the complement of F (in $\mathbb{R}$) is an open set in $\mathbb{R}$.

Proof. **Case 1.** $F = \mathbb{R}$ (a closed set). Then the complement of F in $\mathbb{R}$ is ϕ and ϕ is an open set.

Case 2. F is a proper subset of $\mathbb{R}$. Then $F^c \neq \phi$ where F^c is the complement of F.

Let $x \in F^c$. Since F is a closed set and $x \notin F, x$ is not a limit point of F. Therefore there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap F = \phi$.

Since $x \notin F$, $N(x) \cap F = \phi$. That is $N(x) \subset F^c$.

Thus $x \in F^c \Rightarrow N(x) \subset F^c$.

This shows that x is an interior point of F^c. Since x is arbitrary, F^c is an open set and the theorem is done.

Worked Examples.

1. If S be a non-empty closed and bounded subset of $\mathbb{R}$. then prove that sup S and inf S belong to S.

Since S is a non-empty bounded set, by the supremum property of $\mathbb{R}$, sup S and inf S both exist. Let sup $S = m^*$, inf $S = m_*$.

Let us assume that $m^* \notin S$.

Since $\sup S = m^*$, $x \in S \Rightarrow x < m^*$ and for each pre-assigned positive ϵ, there is an element y in S such that $m^* - \epsilon < y < m^*$. Therefore $m^* - \epsilon < y < m^* + \epsilon$.

This shows that $N'(m^*, \epsilon) \cap S \neq \phi$. It follows that m^* is a limit point of S. $m^* \notin S$ and $m^* \in S' \Rightarrow S$ is not a closed set, a contradiction.

Hence $m^* \in S$. In a similar manner it can be proved that $m_* \in S$.

2. Prove that a finite subset of $\mathbb{R}$ is a closed set.

Let $S = \{a_1, a_2, \ldots, a_m\}$ where $a_1 < a_2 < a_3 < \cdots < a_m$. Then S can be expressed as the complement of the union of $m+1$ open intervals $(-\infty, a_1), (a_1, a_2), \cdots, (a_{m-1}, a_m), (a_m, \infty)$.

Since each open interval is an open set and the union of a finite number of open sets is an open set, S is the complement of an open set. Therefore S is a closed set.

3. Prove that a closed and bounded interval is a closed set.

Let $I = [a, b]$ be a closed and bounded interval. Then I can be expressed as $\mathbb{R} - [(-\infty, a) \cup (b, \infty)]$.

$(-\infty, a)$ is an open interval and so it is an open set.

(b, ∞) is an open interval and so it is an open set.

The union $(-\infty, a) \cup (b, \infty)$ is an open set. Therefore I being the complement of an open set, is a closed set.

Note. The closed intervals $(-\infty, a], [a, \infty)$ are closed sets.

3.9. Adherent point.

Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *adherent point* of S if every neighbourhood of x contains a point of S.

It follows that x is an adherent point of S if $N(x, \epsilon) \cap S \neq \phi$ for every $\epsilon > 0$.

The set of all adherent points of S is said to be the *closure* of S and is denoted by $\bar{S}$.

From definition it follows that $S \subset \bar{S}$ for any set $S \subset \mathbb{R}$.

Theorem 3.9.1. Let S be a subset of $\mathbb{R}$. Then $\bar{S} = S \cup S', S'$ being the derived set of S.

Proof. Let $x \in S$. Then every neighbourhood of x contains x, a point of S. Therefore x is an adherent point of S.

Thus $x \in S \Rightarrow x \in \bar{S}$ and therefore $S \subset \bar{S}$ ... ... (i)

Let $x \in S'$. Then x is a limit point of S. Hence every neighbourhood $N(x)$ of x contains a point of S other than x, i.e., $N(x) \cap S \neq \phi$.

Therefore x is an adherent point of S, i.e., $x \in \bar{S}$.

Thus $x \in S' \Rightarrow x \in \bar{S}$ and therefore $S' \subset \bar{S}$ ... ... (ii)

From (i) and (ii) $S \cup S' \subset \bar{S}$ ... ... (iii)

Let $y \notin S \cup S'$. Then $y \notin S$ and $y \notin S'$

Since $y \notin S'$, there exists a neighbourhood $N(y)$ of y such that $N'(y) \cap S = \phi$ and also since $y \notin S$, $N'(y) \cap S = \phi \Rightarrow N(y) \cap S = \phi$.

This shows that y is not an adherent point of S.
Therefore $y \notin S \cup S' \Rightarrow y \notin \bar{S}$.

Contrapositively, $y \in \bar{S} \Rightarrow y \in S \cup S'$ and therefore $\bar{S} \subset S \cup S'$ ... (iv)

From (iii) and (iv) $\bar{S} = S \cup S'$ and this completes the proof.

Theorem 3.9.2. Let S be a subset of $\mathbb{R}$. Then S is a closed set if and only if $S = \bar{S}$.

Proof. Let S be a closed set. Then $S' \subset S$.
$S \cup S' = S$, i.e., $\bar{S} = S$.

Conversely, let S be a subset of $\mathbb{R}$ such that $\bar{S} = S$. Then $S = S \cup S'$.
This implies $S' \subset S$ and therefore S is a closed set.
This completes the proof.

Theorem 3.9.3. Let S be a subset of $\mathbb{R}$. Then $\bar{S}$ is a closed set and it is the smallest closed set containing S.

Proof. $(\bar{S})' = (S \cup S')' = S' \cup (S')'$, since $(A \cup B)' = A' \cup B'$
$= S'$, since $(S')' \subset S'$
$\subset S \cup S' = \bar{S}$.

This proves that $\bar{S}$ is a closed set.

Again, $\bar{S} = S \cup S' \supset S$. Therefore $\bar{S}$ is a closed set containing S.

Let P be any closed set containing S.
$S \subset P$ implies $S' \subset P' \subset P$, since P is a closed set.
$S \subset P$ and $S' \subset P \Rightarrow S \cup S' \subset P$, i.e., $\bar{S} \subset P$.

Since P is arbitrary, $\bar{S}$ is the smallest closed set containing $\bar{S}$.
This completes the proof.

Note. $\bar{S}$ is the intersection of all closed supersets of S.

Worked Example.

1. Let A be a non-empty subset of $\mathbb{R}$ and $d(x, A) = \inf\{|x - y| : y \in A\}$. Prove that $d(x, A) = 0$, if and only if $x \in \bar{A}$.

Let $d(x, A) = 0$. Then $\inf\{| x - y |: y \in A\} = 0$.

Therefore $| x - y | \geq 0$ for all $y \in A$ and for a pre-assigned positive ϵ, there exists a point a in A such that $0 \leq | x - a | < \epsilon$.

This shows that the ϵ-neighbourhood of x contains a point a of A and this holds for each $\epsilon > 0$. Therefore $x \in \bar{A}$.

Conversely, let $x \in \bar{A}$. Then $x \in A \cup A'$.
If $x \in A$ then $d(x, A) = \inf\{|x - y| : y \in A\} = 0$.

If $x \in A'$, then for a chosen positive ϵ, there exists a point $y \in A$ such that $y \in N(x, \epsilon)$. Therefore $0 <| x - y |< \epsilon$ and this holds for each $\epsilon > 0$.

So $\inf\{| x - y |: y \in A\} = 0$.

Theorem 3.9.4. Let A, B be subsets of $\mathbb{R}$ and $A \subset B$. Then $\bar{A} \subset \bar{B}$.

Proof. Since $B \subset \bar{B}$, $A \subset B \Rightarrow A \subset \bar{B}$.
Thus $\bar{B}$ is a closed set containing A. But $\bar{A}$ is the smallest closed set containing A. Therefore $\bar{A} \subset \bar{B}$ and the proof is complete.

Theorem 3.9.5. Let A, B be subsets of $\mathbb{R}$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. $A \subset A \cup B \Rightarrow \bar{A} \subset \overline{A \cup B}$ and $B \subset A \cup B \Rightarrow \bar{B} \subset \overline{A \cup B}$, by Theorem 3.9.4. Therefore $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$... (i)

$A \subset \bar{A} \subset \bar{A} \cup \bar{B}$ and $B \subset \bar{B} \subset \bar{A} \cup \bar{B}$. Therefore $A \cup B \subset \bar{A} \cup \bar{B}$.

Because $\bar{A}$ and $\bar{B}$ are closed sets, $\bar{A} \cup \bar{B}$ is a closed set containing the set $A \cup B$. But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$. Therefore $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$... (ii)

From (i) and (ii) it follows that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
This completes the proof.

Corllary. $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup A_n$.

Theorem 3.9.6. Let A, B be subsets of $\mathbb{R}$. Then $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Proof. $A \cap B \subset A \Rightarrow \overline{A \cap B} \subset \bar{A}$. $A \cap B \subset B \Rightarrow \overline{A \cap B} \subset \bar{B}$.
Therefore $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Note. The equality $\overline{A \cap B} = \bar{A} \cap \bar{B}$ may not hold.
For example, let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $B = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$. Then $\overline{A \cap B} = \phi$, $\bar{A} \cap \bar{B} = \{0\}$. $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Definition. Closed set. A set $S \subset \mathbb{R}$ is said to be a closed set if $\bar{S} \subset S$.

With this definition of a closed set, the theorems 3.8.1 and 3.8.2 will have the following alternative proofs.

Theorem 3.9.7. The union of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and $F = \bigcup_{i=1}^{m} F_i$.

Since F_i is a closed set, $\bar{F}_i \subset F_i$ for $i = 1, 2, \ldots, m$.

$\bar{F} = \bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_m$.

$\bar{F}_i \subset F_i \Rightarrow \bar{F}_i \subset F$ for $i = 1, 2, \ldots, m$.

Therefore $\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_m \subset F$, i.e., $\bar{F} \subset F$, showing that F is a closed set. This completes the proof.

Theorem 3.9.8. The intersection of a finite number of closed sets in $\mathbb{R}$ is a closed set.

Proof. Let $F_1, F_2, \ldots, F_m$ be m closed sets in $\mathbb{R}$ and $F = \bigcap_{i=1}^{m} F_i$.

Since F_i is a closed set, $\bar{F}_i \subset F_i$ for $i = 1, 2, \ldots, m$.

$\bar{F} = \overline{F_1 \cap F_2 \cap \cdots \cap F_m} \subset \bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m$.

$\bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m \subset \bar{F}_i \subset F_i$ for each $i = 1, 2, \ldots, m$ and therefore $\bar{F}_1 \cap \bar{F}_2 \cap \cdots \cap \bar{F}_m \subset F$, i.e., $\bar{F} \subset F$.

Therefore F is a closed set. This completes the proof.

Another definition of a closed set.

A set $S \subset \mathbb{R}$ is said to be a closed set if $\mathbb{R} - S$ is an open set.

Theorem 3.9.9. A set $S \subset \mathbb{R}$ is closed if and only if $S' \subset S$.

Proof. Let S be a closed set in $\mathbb{R}$. Then $\mathbb{R} - S$ is an open set, by definition.

Let $x \in \mathbb{R} - S$. Then x is an interior point of $\mathbb{R} - S$. Therefore there exists a neighbourhood $N(x)$ of x such that $N(x) \subset \mathbb{R} - S$, i.e., $N(x) \cap S = \phi$.

It follows that x is not a limit point of S, i.e., $x \notin S'$.

Thus $x \in \mathbb{R} - S \Rightarrow x \notin S'$.

Contrapositively, $x \in S' \Rightarrow x \notin \mathbb{R} - S$, i.e., $x \in S$.

This proves that $S' \subset S$.

Conversely, let S be a subset of $\mathbb{R}$ such that $S' \subset S$.

Let $x \in \mathbb{R} - S$. Then $x \notin S$ and therefore $x \notin S'$ since $S' \subset S$.

Thus there exists a neighbourhood $N(x)$ of x such that $N(x) \cap S = \phi$ and therefore $N(x) \subset \mathbb{R} - S$.

Thus $x \in \mathbb{R} - S \Rightarrow N(x) \subset \mathbb{R} - S$.

So x is an interior point of $\mathbb{R} - S$, proving that $\mathbb{R} - S$ is an open set. Therefore S is a closed set.

This completes the proof.

Remark. Since $\mathbb{R}$ is an open set , ϕ being the complement of $\mathbb{R}$, is a closed set. Since ϕ is an open set, $\mathbb{R}$ being the complement of ϕ, is a closed set.

Therefore the set $\mathbb{R}$ is both open and closed; the set ϕ is both open and closed in $\mathbb{R}$.

We are now in search of subsets of $\mathbb{R}$ which are both open and closed in $\mathbb{R}$. The next theorem throws light on our search.

Theorem 3.9.10. No non-empty proper subset of $\mathbb{R}$ is both open and closed in $\mathbb{R}$.

Proof. If possible, let S be a non-empty proper subset of $\mathbb{R}$ which is both open and closed. Since S is a proper subset of $\mathbb{R}$, there exists an element c in $\mathbb{R} - S$. Therefore $S \subset (-\infty, c) \cup (c, \infty)$.

Since S is non-empty, at least one of $S \cap (-\infty, c)$ and $S \cap (c, \infty)$ is non-empty.

Let $A = S \cap (-\infty, c) \neq \phi$. A is bounded above, c being an upper bound. Therefore $\sup A$ exists. Let $a = \sup A$. Then $a \leq c$.

For each $\epsilon > 0$, there is an element b in A such that $a - \epsilon < b \leq a$.

$b \in A \Rightarrow b \in S$.

Therefore each ϵ-neighbourhood of a contains a point of S.

So $a \in \bar{S}$ and since S is closed, $a \in S$. Therefore $a < c$.

Since S is open and $a \in S$, for some positive δ, $(a - \delta, a + \delta) \subset S$.

Let $d < \min\{\delta, c - a\}$. Then $a + d < a + \delta$ and $a + d < c$.

Therefore $a + d \in S$ and $a + d \in (-\infty, c)$.

Therefore $a + d \in A$ and this contradicts the definition of a.

Hence our assumption is wrong and the theorem is established.

3.10. Dense set. Perfect set.

Definition. Let S be a subset of $\mathbb{R}$. A subset $T \subset S$ is said to be *dense in* S if $S \subset T'$.

In particular, S is said to be *dense in* $\mathbb{R}$ (or *dense*, or *everywhere dense*) if every point of $\mathbb{R}$ is a limit point of S, or equivalently $S' = \mathbb{R}$.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be *dense-in-itself* if $S \subset S'$.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be a *perfect* set if S is dense-in-itself and closed, i.e., if $S = S'$.

Examples.

1. The set $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\mathbb{Q}' = \mathbb{R}$.
2. Let $S = \{x \in \mathbb{R} : 1 < x < 2\}$. Then $S \subset S'$. S is dense-in-itself.
3. The set $\mathbb{Q}$ is dense-in-itself, since $\mathbb{Q} \subset \mathbb{Q}'$. The set $\mathbb{R}$ is dense-in-itself.
4. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Then S is a perfect set.

Exercises 3

1. Give an example of an infinite set $S \subset \mathbb{R}$ such that
   (i) S has no limit point, (ii) S has only one limit point,
   (iii) S has three limit points, (iv) S has four limit points,
   (v) S is a proper subset of the derived set S'.
2. Let $S = \{m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$. Find the derived set of S.
3. Let $S = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbb{N}\}$.
   (i) Show that -1 and 1 are limit points of S.
   (ii) Find the derived set of S.
4. Let $S = \{\frac{(-1)^m}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
   (i) Show that 0 is a limit point of S.
   (ii) If $k \in \mathbb{N}$, show that $\frac{1}{k}$ is a limit point of S.
   (iii) If $k \in \mathbb{N}$, show that $\frac{-1}{2k-1}$ is a limit point of S.
5. Verify Bolzano-Weierstrass theorem for the set $S \subset \mathbb{R}$.
   (i) $S = \{(-1)^n(1 + \frac{1}{n}) : n \in \mathbb{N}\}$, (ii) $S = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$,
   (iii) $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$, (iv) $S = \{\frac{n-1}{n+1} : n \in \mathbb{N}\}$.
6. Let $S = \{\frac{1}{2^m} + \frac{1}{2^n} : m \in \mathbb{N}, n \in \mathbb{N}\}$.
   (i) Show that 0 is a limit point of S.
   (ii) If $k \in \mathbb{N}$, show that $\frac{1}{2^k}$ is a limit point of S.
   (iii) Find S'.(the derived set of S).
7. Give an example of a set $S \subset \mathbb{R}$ such that
   (i) S is neither open nor closed in $\mathbb{R}$,
   (ii) S is both open and closed in $\mathbb{R}$,
   (iii) S is a proper subset of the derived set S', (iv) $S = S'$.
8. Prove that the set S is an open set, where
   (i) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 < 0\}$, (ii) $S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 > 0\}$,
   (iii) $S = A - B$ where $A = (0, 1)$, $B = \{\frac{1}{2^n} : n \in \mathbb{N}\}$;
   (iv) $S = \{x \in \mathbb{R} : \sin x \neq 0\}$.
9. Examine if the set S is closed in $\mathbb{R}$.
   (i) $S = \{x \in \mathbb{R} : \sin x = 0\}$, (ii) $S = \{x \in \mathbb{R} : \sin \frac{1}{x} = 0\}$,
   (iii) $S = \{\frac{1}{m} + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$, (iv) $S = \{(-1)^m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N}\}$,
   (v) $S = \bigcup_{n=1}^{\infty} I_n$ where $I_n = \{x \in \mathbb{R} : (\frac{1}{3})^n \leq x \leq 1\}$.

10. Let $G \subset \mathbb{R}$ be an open set and $F \subset \mathbb{R}$ be a closed set. Prove that $G - F$ is an open set and $F - G$ is a closed set.

11. Let G be an open set in $\mathbb{R}$ and S be a non-empty finite subset of G. Prove that $G - S$ is an open set.

12. Let G be an open set in $\mathbb{R}$ and S be a subset of $\mathbb{R}$ such that $G \cap S = \phi$. Prove that $G \cap S' = \phi$.

[**Hint.** $G \cap S = \phi \Rightarrow S \subset G^c \Rightarrow S' \subset (G^c)' \Rightarrow S' \subset G^c$(since G^c is closed in $\mathbb{R}$)$\Rightarrow G \cap S' = \phi$.]

13. Let S be a bounded subset of $\mathbb{R}$ and $\sup S = b, \inf S = a$ and $a \neq b$. Prove that $[a, b]$ is the smallest closed interval containing the set S.

14. (i) Let S be a non-empty subset of $\mathbb{R}$ bounded above and $s^* = \sup S$. If $s^* \notin S$, prove that $s^* \in S'$ and $s^* = \sup S'$, S' being the derived set of S.

(ii) Let S be a non-empty subset of $\mathbb{R}$ bounded below and $s_* = \inf S$. If $s_* \notin S$, prove that $s_* \in S'$ and $s_* = \inf S'$, S' being the derived set of S.

15. If S be a non-empty bounded subset of $\mathbb{R}$ prove that $\sup S \in \bar{S}$ and $\inf S \in \bar{S}$.

16. Let $S \subset \mathbb{R}$. Prove that

(i) $(\bar{S})^c = (S^c)^o$, i.e., the complement of the closure of S is the interior of the complement of S;

(ii) $(S^o)^c = (\bar{S^c})$, i.e., the complement of the interior of S is the closure of the complement of S.

17. A set $S \subset \mathbb{R}$ is said to be a *discrete* set if $S' = \phi$ (i.e., if S has no limit point).

A set $S \subset \mathbb{R}$ is said to be an *isolated* set if $S \cap S' = \phi$ (i.e., if each point of S is an isolated point).

(i) Prove that every discrete set is an isolated set, but not conversely.

(ii) Give an example of an infinite discrete set $S \subset \mathbb{R}$.

(iii) Give an example of a bounded discrete set $S \subset \mathbb{R}$.

(iv) Can there be an infinite bounded discrete set $S \subset \mathbb{R}$?

18. Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *boundary point* of S if every neighbourhood $N(x)$ of x contains a point of S and also a point of $\mathbb{R} - S$.

If a boundary point of S is not a point of S prove that it is a limit point of S. Prove that a set $S \subset \mathbb{R}$ is closed if and only if S contains all it boundary points.

3.11. Nested intervals.

If $\{I_n : n \in \mathbb{N}\}$ be a family of intervals such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n\}$ is said to be a family of *nested intervals*.

Examples.

1. Let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$.
Then $I_1 = (0, 1)$, $I_2 = (0, \frac{1}{2})$, $I_3 = (0, \frac{1}{3}), \cdots \cdots$
$I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested open and bounded intervals.

2. Let $I_n = \{x \in \mathbb{R} : x > n\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested open infinite intervals.

3. Let $I_n = \{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals.

4. Let $I_n = \{x \in \mathbb{R} : x \leq \frac{1}{n}\}$.
Then $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$
$\{I_n : n \in \mathbb{N}\}$ is a family of nested closed infinite intervals.

Theorem 3.11.1. (Theorem on nested intervals)

If $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of nested closed and bounded intervals then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Furthermore, if $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, then there is *one and only one point* x such that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Proof. $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \cdots$

Then $a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_3 \leq b_2 \leq b_1$.

The set $A = \{a_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded above, b_1 being an upper bound. By the supremum property of $\mathbb{R}$, $\sup A$ exists. Let $\sup A = x$. Then $a_n \leq x$ for all $n \in \mathbb{N}$.

We now establish that $b_n \geq x$ for all $n \in \mathbb{N}$.

If not, let $b_m < x$ for some $m \in \mathbb{N}$.

Since x is the lub of the set $\{a_1, a_2, a_3, \ldots\}$ and $b_m < x$, there is an element a_k such that $b_m < a_k < x$.

Let $q = \max\{m, k\}$. Then $b_q \leq b_m$ and $a_k \leq a_q$.

Consequently, $b_q \leq b_m < a_k \leq a_q$.

This shows that $b_q < a_q$, a contradiction, since $[a_q, b_q]$ is an interval of the family.

Hence $b_n \geq x$ for all $n \in \mathbb{N}$ and therefore $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$. That is, $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

This proves that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

Second part. If possible, let $x' \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Then $a_n \leq x \leq b_n$, $a_n \leq x' \leq b_n$ for all $n \in \mathbb{N}$.

Therefore $a_n - b_n \leq x - x' \leq b_n - a_n$.

or, $0 \leq | x - x' | \leq b_n - a_n$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0, b_n - a_n \geq 0$ for all $n \in \mathbb{N}$ and there exists an element $b_m - a_m$ of the set (corresponding to some natural number m) such that $0 \leq b_m - a_m < \epsilon$.

Therefore $0 \leq | x - x' | < \epsilon$. Since ϵ is arbitrary, $x = x'$.

This proves that x is unique and the proof is complete.

Note 1. The set $B = \{b_i : i \in \mathbb{N}\}$ is a non-empty subset of $\mathbb{R}$ bounded below. If $\inf B = y$, then $y \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

Note 2. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested open bounded intervals then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} I_n = \phi$.

Note 3. If $\{I_n : n \in \mathbb{N}\}$ be a family of nested closed unbounded intervals then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

For example, if $I_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} I_n = \phi$.

Utilising Nested intervals theorem we now give an alternative proof of Bolzano-Weierstrass theorem (Theorem 3.6.2).

Another proof of Bolzano-Weierstrass theorem.

Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Proof. Let S be a bounded subset of $\mathbb{R}$ containing infinite number of elements. Since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ exist. Let $a_1 = \inf S$, $b_1 = \sup S$.

Then $x \in S \Rightarrow a_1 \leq x \leq b_1$, i.e., $x \in [a_1, b_1]$. Thus S is contained in the closed and bounded interval $I_1 = [a_1, b_1]$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then at least one of the closed intervals $[a_1, c_1], [c_1, b_1]$ must contain infinitely many elements of S. Because, otherwise, S would be a finite set. We take one such subinterval containing infinitely many elements of S and call it $I_2 = [a_2, b_2]$.

$I_2 \subset I_1$ and $| I_2 | = \frac{b_1 - a_1}{2}$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then at least one of the closed intervals $[a_2, c_2], [c_2, b_2]$ must contain infinitely many elements of S. We take one such subinterval containing infinitely many elements of S and call it $I_3 = [a_3, b_3]$.

$I_3 \subset I_2 \subset I_1$ and $| I_3 | = \frac{b_1 - a_1}{2^2}$.

Let $c_3 = \frac{a_3+b_3}{2}$. Continuing in a similar manner we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_1 \supset I_2 \supset I_3 \supset \cdots \cdots$

(ii) $| I_n | = \frac{1}{2^{n-1}}(b_1 - a_1)$, for each $n \in \mathbb{N}$

(iii) I_n contains infinitely many elements of S, for each $n \in \mathbb{N}$.

So $\{I_n : n \in \mathbb{N}\}$ is a family of nested closed and bounded intervals and $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$.

By the nested intervals theorem, there exists precisely one point x such that $\{x\} = \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

We now prove that x is a limit point of S.

Let $\epsilon > 0$. Since $\inf\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, there exists a natural number m such that $0 \leq b_m - a_m < \epsilon$.

Since $x \in I_m$ and $b_m - a_m < \epsilon, I_m \subset N(x, \epsilon)$.

Since I_m contains infinitely many elements of $S, N(x, \epsilon)$ contains infinitely many elements of S and this happens for each $\epsilon > 0$.

Therefore x is a limit point of S.

Thus S has a limit point and the theorem is done.

Theorem 3.11.2. (Cantor's intersection theorem)

Let $F_1, F_2, F_3, \ldots \ldots$ be a countable collection of non-empty closed and bounded subsets of $\mathbb{R}$ such that $F_1 \supset F_2 \supset F_3 \supset \cdots \cdots$

Then the intersection $\bigcap\limits_{i=1}^{\infty} F_i$ is non-empty.

Proof. **Case 1.** Let the collection be a finite collection containing m non-empty closed and bounded subsets $F_1, F_2, \ldots, F_m$ such that $F_1 \supset F_2 \supset \cdots \supset F_m$.

Then obviously, $\bigcap\limits_{i=1}^{m} F_i = F_m$ and this is non-empty by hypothesis.

Case 2. Let the collection be countably infinite. Without loss of generality, we we assume that no two sets of the collection are equal sets, because if there exists some block (or blocks) of equal sets, the inetersection of the whole collection remains same if we replace a block of equal sets by any one set of the block, and re-index the distinct elements of the collection as $F_1, F_2, F_3, \ldots$

Then $F_k - F_{k+1}$ is non-empty for each $k \in \mathbb{N}$.

Since the collection contains infinitely many non-empty closed sets, each F_k contains infinite number of elements , because, if we suppose, on the contrary, that F_j is a finite set containing p elements for some $j \in \mathbb{N}$ then F_{j+p} must be ϕ, a contradiction to hypothesis.

Let us take a point x_1 in $F_1 - F_2, x_2$ in $F_2 - F_3, \ldots, x_k$ in $F_k - F_{k+1}, \ldots \ldots$

Then we obtain an infinite set of points $S = \{x_1, x_2, x_3, \ldots \ldots\}$. Clearly, $S \subset F_1$.

Since F_1 is bounded, S is an infinite bounded subset of $\mathbb{R}$ and by Bolzano-Weierstrass theorem S has a limit point $x \in \mathbb{R}$.

We prove that x is a limit point of each $F_k, k = 1, 2, 3, \ldots \ldots$

Let x be not a limit point of F_m for some $m \in \mathbb{N}$. Then there exists a positive ϵ such that the neighbourhood $N(x, \epsilon)$ of x contains at most a finite number of points of F_m.

That is, $N(x, \epsilon) \cap F_m$ is a finite set.

Since $\{x_m, x_{m+1}, x_{m+2}, \ldots \ldots\} \subset F_m$, $N(x, \epsilon) \cap \{x_m, x_{m+1}, x_{m+2}, \ldots\}$ is a finite set.

Consequently, $N(x, \epsilon) \cap S$ is a finite set.

This disallows x to be a limit point of S. So our assumption that x is not a limit point of F_m is wrong.

Thus x is a limit point of F_m for every $m \in \mathbb{N}$ and since F_m is a closed set, $x \in F_m$ for every $m \in \mathbb{N}$.

That is, $x \in \bigcap\limits_{i=1}^{\infty} F_i$ and this proves that the intersection $\bigcap\limits_{i=1}^{\infty} F_i$ is non-empty. This completes the proof.

Corollary. If I_n be the closed and bounded interval $[a_n, b_n]$ and $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \cdots$ then the intersection $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$ is non-empty.

This is "Nested intervals theorem".

3.12. Decimal representation of a real number.

Let $x \in [0, 1]$. If we divide $[0, 1]$ into 10 equal subintervals, then x lies in at least one of the subintervals $[\frac{a}{10}, \frac{a+1}{10}]$ where a is one of integers $0, 1, 2, \ldots, 9$.

If x be a point of division, then two values of a are possible. We choose one of then and call it a_1. Then

$$\frac{a_1}{10} \leq x \leq \frac{a_1+1}{10}, \text{ where } 0 \leq a_1 \leq 9.$$

The chosen interval $[\frac{a_1}{10}, \frac{a_1+1}{10}]$ is again divided into 10 equal subintervals. Then x lies in at least one of them and

$$\frac{a_1}{10} + \frac{a_2}{10^2} \leq x \leq \frac{a_1}{10} + \frac{a_2+1}{10^2} \text{ where } 0 \leq a_i \leq 9, i = 1, 2.$$

The process is continued and we obtain integers $a_1, a_2, a_3, \cdots$ with $0 \le a_n \le 9$ for all $n \in \mathbb{N}$ such that

$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \le x \le \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n}$ for all $n \in \mathbb{N}$.

We write $x = .a_1 a_2 a_3 \cdots \cdots$ and call it a *decimal representation* of x. *Conversely*, we now show that every decimal of the form $.a_1 a_2 a_3 \cdots \cdots$ is the decimal representation of some real number in $[0, 1]$.

Let $I_1 = [0, 1], I_2 = [\frac{a_1}{10}, \frac{a_1+1}{10}], I_3 = [\frac{a_1}{10} + \frac{a_2}{10^2}, \frac{a_1}{10} + \frac{a_2+1}{10^2}], \cdots,$
$I_{n+1} = [\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}, \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n}], \cdots$

We obtain a family of closed and bounded intervals $\{I_n\}$ satisfying the conditions (i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$ and

(ii) $| I_n |= \frac{1}{10^{n-1}}$.

By the theorem on nested intervals, there exists a unique real number x such that $x \in I_n$ for $n \in \mathbb{N}$. Then

$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} \le x \le \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n+1}{10^n}$ for all $n \in \mathbb{N}$ ... (i)

Therefore $x \in [0, 1]$ and the inequality (i) shows that $0.a_1 a_2 a_3 \cdots$ is the decimal representation of the real number x.

The decimal representation of $x \in (0, 1)$ is unique except when x is a point of subdivision at some stage.

Let n be the least positive integer for which x is a point of subdivision at the nth stage, i.e., x is an end point of the subinterval I_{n+1}.

Then $x = \frac{m}{10^n}$ for some positive integer $m < 10^n$ not divisible by 10.

If we choose a_n such that x is the left end point of the subinterval I_{n+1}, then $\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = x$ and $a_k = 0$ for all $k \ge n + 1$.

In this case $x = .a_1 a_2 \cdots a_n 000 \cdots$

If we choose a_n such that x is the right end point of the subinterval I_{n+1}, then $x = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$ and $a_k = 9$ for all $k \ge n + 1$.

In this case $x = .a_1 a_2 \cdots a_n 999 \cdots$

[For example, if $x = \frac{437}{1000}, x = .437000 \cdots$, or $x = .436999 \cdots$]

The decimal representation of 0 is $0.000 \cdots$

The decimal representation of 1 is $0.999 \cdots$

If $x \ge 1$,then there exists a natural number p such that $p \le x < p + 1$. Therefore $x - p \in [0, 1]$. Then $x = p.a_1 a_2 a_3 \cdots$ where $.a_1 a_2 a_3 \cdots$ is the decimal representation of $x - p$.

A decimal $p.a_1 a_2 a_3 \cdots$ is said to be a **periodic decimal** (or a **recurring decimal**) if there exist natural numbers m and k such that $a_n = a_{n+m}$ for all $n \ge k$. The smallest natural number m with this property is called the **period** of the decimal. In this case the block of digits $a_k a_{k+1} \cdots a_{k+m-1}$ is repeated once the kth digit is reached.

[For example, the decimal $0.235636363\cdots$ is periodic with repeating block 63.]

A decimal $p.a_1a_2a_3\cdots$ is said to be a *terminating decimal* if there exists a natural number k such that $a_n = 0$ for all $n \geq k$. A terminating decimal can be regarded as a periodic decimal with the repeating block 0.

Theorem 3.12.1. A positive real number is rational if and only if its decimal representation is periodic.

Proof. Let x be a positive rational number. Let $x = p/q$ where p, q are natural numbers relatively prime. In the process of long division of p by q, the quotient gives the decimal representation of p/q. Each step in the division process gives a remainder which is an integer r satisfying $0 \leq r \leq q-1$. Therefore after at most q steps, some remainder will occur a second time or the remainder will be zero.

If some remainder recurs then the digits in the quotient will begin to repeat in blocks and we obtain a periodic decimal representation.

If the remainder be zero at some step, then the quotient gives a terminating decimal representation.

Conversely, let $x = p.a_1a_2a_3\cdots$ be a periodic decimal with period m and the period starts from the kth stage.
Then $x = p.a_1a_2\cdots a_n\cdots$ where $a_n = a_{n+m}$ for all $n \geq k$.
$10^{k-1}x = pa_1a_2\cdots a_{k-1}.a_ka_{k+1}\cdots$ and $10^{k+m-1}x =$

$$pa_1a_2\cdots a_{k+m-1}.a_{k+m}a_{k+m+1}\cdots = pa_1a_2\cdots a_{k+m-1}.a_ka_{k+1}\cdots$$

We have $(10^{k+m-1} - 10^{k-1})x = pa_1a_2\cdots a_{k+m-1} - pa_1a_2\cdots a_{k-1}$.

As $pa_1a_2\cdots a_{k+m-1} - pa_1a_2\cdots a_{k-1}$ is an integer and $10^{k+m-1} - 10^{k-1}$ is an integer, x is a rational number, and the theorem is done.

Corollary. A non-terminating non recurring decimal represents a positive irrational number.

3.13. Enumerable set.

Let S be a subset of $\mathbb{R}$. S is said to be *enumerable* (or *denumerable)* if there exists a bijective mapping $f : \mathbb{N} \to S$, i.e., if S and $\mathbb{N}$ are equipotent sets.

A set which is either finite or enumerable is said to be a *countable* (or, an *at most enumerable*) set.

An enumerable set is also called a *countably infinite* set.

If a set S is finite and contains n elements, its elements can be described as $a_1, a_2, \ldots, a_n$, the elements being indexed by the finite set $\{1, 2, \ldots, n\}$.

If S is enumerable, there exists a bijective mapping $f : \mathbb{N} \to S$ and f assigns to each element $n \in \mathbb{N}$ an element $f(n)$ in S. The elements of S can be described as $f(1), f(2), \ldots, f(n), \ldots$, or as $a_1, a_2, \ldots, a_n, \ldots$ showing that the elements are indexed by the set $\mathbb{N}$.

Note. Since an enumerable set is equipotent with the set $\mathbb{N}$, the cardinal number of an enumerable set is d.

Examples.

1. The set $\mathbb{N}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n, n \in \mathbb{N}$ is a bijection.

2. The set $S = \{2, 4, 6, \ldots\}$ is enumerable, because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = 2n, n \in \mathbb{N}$ is a bijection.

3. The set $S = \{1^2, 2^2, 3^2, \ldots\}$ is enumerable because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = n^2, n \in \mathbb{N}$ is a bijection .

4. The set $\mathbb{Z}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{Z}$ by

$$\begin{aligned} f(n) &= \tfrac{1}{2}n, \text{ if } n \text{ be even} \\ &= \tfrac{1}{2}(1-n), \text{ if } n \text{ be odd,} \end{aligned}$$

is a bijection.

Theorem 3.13.1. An infinite subset of an enumerable set is enumerable.

Proof. Let S be an enumerable set and T be an infinite subset of S. Since S is an enumerable set, its element can be described as $a_1, a_2, a_3, \ldots \ldots$

Since T is an infinite subset of S, T contains infinite number of a's and the suffixes of the elements of T form an infinite subset P of $\mathbb{N}$. By the well ordering property of $\mathbb{N}, P$ contains a least element, say μ_1.

$a_{\mu_1} \in T$. Let $T_1 = T - \{a_{\mu_1}\}$. Then T_1 is an infinite subset of S and the suffixes of the elements of T_1 form an infinite subset P_1 of $\mathbb{N}$. Therefore P_1 contains a least element say μ_2. $a_{\mu_2} \in T_1$. Let $T_2 = T - \{a_{\mu_1}, a_{\mu_2}\}$. Then T_2 is an infinite subset of S. Proceeding with similar arguments, we obtain the elements $a_{\mu_3}, a_{\mu_4}, \ldots$

Let us define a mapping $f : \mathbb{N} \to T$ by $f(n) = a_{\mu_n}, n \in \mathbb{N}$. To prove that f is injective, let $p, q \in \mathbb{N}$ and $p < q$.

Then $f(p) \in \{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_{q-1}}\}$ and $f(q) \in T - \{a_{\mu_1}, a_{\mu_2}, \ldots, a_{\mu_{q-1}}\}$.

Therefore $f(p) \neq f(q)$ and this proves that f is injective.

To prove that f is surjective, let $a_r \in T$ for some natural number r.

There are at most $r-1$ elements in T whose suffixes are less than r. So a_r is one of $f(1), f(2), \ldots, f(r)$. Therefore f is surjecive.

Therefore f is a bijection and T is enumerable.

Corollary. A subset of an enumerable set is either finite or enumerable.

Theorem 3.13.2. The union of a finite set and an enumerable set is enumerable.

Proof. Let S be an enumerable set with elements $a_1, a_2, a_3, \ldots \ldots$ and T be a finite set with elements $b_1, b_2, \ldots, b_m$.

Case 1. $S \cap T = \phi$.

Let us define a mapping $f : \mathbb{N} \to S \cup T$ by

$f(i) = b_i, \quad i = 1, 2, \ldots, m$

$f(m+i) = a_i, \quad i = 1, 2, 3, \ldots$

Then f is a bijective mapping. This proves that $S \cup T$ is enumerable.

Case 2. $S \cap T \neq \phi$.

Let $S_1 = S - T$. Then $S_1 \cup T = S \cup T$ and $S_1 \cap T = \phi$.

S_1 is an infinite subset of the enumerable set S and therefore S_1 is enumerable. By case 1, $S_1 \cup T$ is enumerable.

That is, $S \cup T$ is enumerable and the theorem is done.

Theorem 3.13.3. The union of two enumerable sets is enumerable.

Proof. Let S_1, S_2 be two enumerable sets and let

$S_1 = \{a_1, a_2, a_3, \ldots \ldots\}, S_2 = \{b_1, b_2, b_3, \ldots \ldots\}$.

Case 1. $S_1 \cap S_2 = \phi$.

Let us define a mapping $f : \mathbb{N} \to S_1 \cup S_2$ by

$f(n) = a_{(n+1)/2}$, if n be odd

$ = b_{n/2}$, if n be even.

Then f is a bijection and therefore $S_1 \cup S_2$ is enumerable.

Case 2. $S_1 \cap S_2 \neq \phi$

Let $A_1 = S_1, A_2 = S_2 - S_1$. Then $A_1 \cup A_2 = S_1 \cup S_2$ and $A_1 \cap A_2 = \phi$.

A_2 is a subset of S_2. So A_2 is either finite or enumerable. If A_2 is finite then $A_1 \cup A_2$ is the union of an enumerable set and a finite set and therefore it is enumerable.

If A_2 is enumerable, then $A_1 \cup A_2$ is enumerable by Case 1.

Therefore $S_1 \cup S_2$ is enumerable and the proof is complete.

Theorem 3.13.4. The union of an enumerable number of enumerable sets is enumerable.

Proof. Let $S_1, S_2, S_3, \ldots$ be an enumerable family of enumerable sets.

Let $S_1 = \{a_{11}, a_{12}, a_{13}, \ldots, a_{1n}, \ldots\}$
$S_2 = \{a_{21}, a_{22}, a_{23}, \ldots, a_{2n}, \ldots\}$
$\ldots \quad \ldots \quad \ldots$
$S_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots, a_{nn}, \ldots\}$
$\ldots \quad \ldots \quad \ldots$

Case 1. Let $S_i \cap S_j = \phi$ for all i, j.

Let $A = \bigcup_{i=1}^{\infty} S_i$. Each element of A is of the form a_{mn}, where $m, n \in \mathbb{N}$.

Let us define a mapping $f : A \to \mathbb{N}$ by $f(a_{mn}) = 2^m.3^n$.

f is injective because for two distinct elements $a_{mn}, a_{pq} \in A, (m,n) \neq (p,q) \Rightarrow 2^m 3^n \neq 2^p 3^q$.

$f(A)$ is a proper subset of $\mathbb{N}$, because there are elements in $\mathbb{N}$ (for example 5, 7, 11) which have no pre-image in A.

Let $f(A) = N_1$. Then $f : A \to N_1$ is a bijection.

Since N_1 is an infinite subset of $\mathbb{N}$, N_1 is enumerable and since A is equipotent with N_1, A is enumerable.

Case 2. Let the sets $\{S_i\}$ be not pairwise disjoint.

Let us define sets A_i by
$A_1 = S_1, A_2 = S_2 - S_1, A_3 = S_3 - (S_1 \cup S_2), \cdots \cdots,$
$A_k = S_k - (S_1 \cup S_2 \cup \cdots \cup S_{k-1}), \cdots$

Then $A_k \subset S_k$ for all k, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} S_i$ and $A_i \cap A_j = \phi$ for all i, j.

Since $A_k \subset S_k, A_k$ is either finite or enumerable

Therefore $\bigcup_{i=1}^{\infty} A_i$ is enumerable.

This completes the proof.

Examples.

1. The set $\mathbb{Q}$ is enumerable.

Let P be the set of all positive rational numbers, P' be the set of all negative rational numbers. Then $\mathbb{Q} = P \cup P' \cup \{0\}$.

The sets P and P' are equipotent since the mapping $f : P \to P'$ defined by $f(x) = -x, x \in P$ is a bijection.

P can be described as the union $\bigcup_{k=1}^{\infty} A_k$, where

$A_1 = \{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \cdots, \frac{n}{1}, \cdots\}$
$A_2 = \{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots, \frac{n}{2}, \cdots\}$
$\ldots \quad \ldots$
$A_k = \{\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \cdots, \frac{n}{k}, \cdots\}$
$\ldots \quad \ldots$

Each A_i is enumerable. P being the union of an enumerable number of enumerable sets is enumerable.

Since P is equipotent with P', P' is also enumerable.

Therefore $P \cup P'$ is enumerable.

$\mathbb{Q}$ being the union of an enumerable set and the finite set $\{0\}$, $\mathbb{Q}$ is enumerable.

2. The set of all algebraic numbers is enumerable.

[A real number is called an **algebraic number** if it is a root of a polynomial of the form $a_o x^n + a_1 x^{n-1} + \cdots + a_n$ where $a_o, a_1, \ldots, a_n$ are all integers and $a_o \neq 0$.

For example, $\sqrt{2}$ is an algebraic number, since $\sqrt{2}$ is a root of the polynomial $x^2 - 2$. Every rational number $\frac{p}{q}$ is an algebraic number since it is a root of the polynomial $qx - p$.

Not every irrational number is algebraic. e is not an algebraic number, π is not an algebraic number.]

Let $f(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial with integral coefficients. Then every real root of $f(x)$ is an algebraic number.

Let us define the *height* h of the polynomial $f(x)$ by

$h = n + \mid a_o \mid + \mid a_1 \mid + \cdots + \mid a_n \mid$. Then h is a positive integer ≥ 1.

Corresponding to every positive integer h, there exists a finite number of polynomials with integral coefficients with height h. Since each polynomial of degree n has at most n real roots, the number of algebraic numbers corresponding to a positive integer h (as height of a polynomial) is finite.

Therefore for every positive integer h, there corresponds a finite number of algebraic numbers.

The set of all algebraic numbers is the union of an enumerable number of finite sets and therefore it is enumerable.

Theorem 3.13.5. The set $\mathbb{R}$ is not enumerable.

Proof. Let $I = [a, b]$ be a closed and bounded interval.

First we prove that I is non-enumerable.

Let I be enumerable. Then the elements of I can be expressed as $x_1, x_2, x_3, \ldots \ldots$

We divide I into three subintervals $[a, c], [c, d], [d, b]$ by the points $c = \frac{2a+b}{3}, d = \frac{a+2b}{3}$. At least one of these subintervals does not contain x_1. We choose it and call it $I_1 = [a, b]$.

I_1 does not contain x_1 and $\mid I_1 \mid = \frac{b-a}{3}$.

We divide I_1 into three subintervals $[a_1, c_1], [c_1, d_1], [d_1, b_1]$ by the points $c_1 = \frac{2a_1+b_1}{3}, d_1 = \frac{a_1+2b_1}{3}$. At least one of these subintervals does not contain x_2. We choose it and call it $I_2 = [a_2, b_2]$.

I_2 contains none of x_1, x_2; $I_2 \subset I_1 \subset I$ and $\mid I_2 \mid = \frac{b-a}{3^2}$.

We divide I_2 into three subintervals in a similar manner. Continuing this process we obtain a family of closed and bounded intervals $\{I_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$,

(i) I_n contains none of $x_1, x_2, \ldots, x_n$;

(ii) $I_{n+1} \subset I_n$, and

(iii) $\mid I_n \mid = \frac{(b-a)}{3^n}$ and consequently, $\inf\{\mid I_n \mid : n \in \mathbb{N}\} = 0$.

By the theorem on nested intervals (Theorem 3.11.1) there exists one and only one point α such that $\{\alpha\} = \bigcap\limits_{n=1}^{\infty} I_n$. Therefore $\alpha \in [a, b]$.

But by the construction of I_n, $\bigcap\limits_{n=1}^{\infty} I_n$ contains none of $x_1, x_2, x_3, \ldots \ldots$, i.e., $\alpha \notin \{x_1, x_2, x_3, \ldots\}$

Thus α being an element of $[a, b]$ escapes enumeration. So our assumption that $[a, b]$ is enumerable is not tenable and so $[a, b]$ is not enumerable.

Now $[a, b]$ is an infinite subset of $\mathbb{R}$. Therefore $\mathbb{R}$ is not enumerable because every infinite subset of an enumerable set is enumerable.

This completes the proof.

Another proof. Let $I = \{x \in \mathbb{R} : 0 < x < 1\}$.

First we prove that I is non-enumerable. If not, let I be enumerable. Then the elements of the set can be described as $x_1, x_2, x_3, \ldots \ldots$

We consider the decimal representation of each number in I. The integral part of each of them is 0.

Each irrational number in the set has a unique representation as an infinite decimal. Each rational number has either a finite decimal representation, or a recurring decimal representation.

A finite decimal can be expressed as an infinite decimal in two ways - either by using 9's or by using 0's. For example, .23 can be expressed as $.22999\cdots$ or as $.23000\cdots$

If we stick to the recurring decimal representation by using 9's only and ignore the second type (by using 0's) then every real number in I can have a unique non-terminating decimal representation.

$$\begin{aligned} \text{Let } x_1 &= 0.x_{11}x_{12}x_{13}\cdots\cdots \\ x_2 &= 0.x_{21}x_{22}x_{23}\cdots\cdots \\ x_3 &= 0.x_{31}x_{32}x_{33}\cdots\cdots \\ \cdots &\quad\ \ \cdots \quad\ \cdots \text{ where } 0 \le x_{ij} \le 9. \end{aligned}$$

By our assumption every real number in I must have a place in the enumeration $x_1, x_2, x_3, \ldots \ldots$

Let us consider the real number $a = 0.a_1a_2a_3 \cdots \cdots$

where $a_k = 1$ if $x_{kk} \neq 1$
$= 2$ if $x_{kk} = 1$.

Now a is different from each x_i because a differs from x_i in the ith decimal place.

Therefore a does not belong to $\{x_1, x_2, x_3, \ldots \ldots\}$. But $0 < a < 1$.

Thus a belongs to I but does not belong to $\{x_1, x_2, x_3, \ldots \ldots\}$, a contradiction. Therefore I is not enumerable.

Now I is an infinite subset of $\mathbb{R}$. Therefore $\mathbb{R}$ must be non-enumerable, because every infinite subset of an enumerable set is enumerable.

This completes the proof.

Corollary. The set S of all irrational numbers is non-enumerable.

Note. The set $\mathbb{R}$ has a cardinal number different from that of the set $\mathbb{N}$. The cardinal number of the set $\mathbb{R}$ is denoted by c. It is also called the *power* (or, the *potency*) of the continuum.

Worked Examples.

1. Prove that the set of all open intervals having rational end points is enumerable.

Let the set of all rational numbers be enumerated as

$\{x_1, x_2, x_3, \ldots \ldots\}$.

The set of all open intervals having x_1 as the left end point is the set of open intervals of the form (x_1, x_r) such that $x_r > x_1$.

The set $A_1 = \{x_r \in \mathbb{Q} : x_r > x_1\}$ is a proper subset of $\mathbb{Q}$.

Since $\mathbb{Q}$ is enumerable, the set A_1 is at most enumerable. But A_1 is clearly an infinite set so that A_1 is enumerable.

Thus the set of all open intervals having x_1 as the left end point is an enumerable set, say I_1.

The set of all open intervals in question is the set $I_1 \cup I_2 \cup I_3 \cup \cdots$

This being the union of an enumerable collection of enumerable sets, is enumerable.

Thus the set of all open intervals having rational end points is enumerable.

2. Let S be a subset of $\mathbb{R}$ such that no point of S is a cluster point of S. Prove that S is a countable set.

Let $x \in S$. Since x is not a limit point of S, there exists an open interval $I_x = (a_x, b_x)$ containing x such that I_x contains a finite number of points of S.

Let us choose rational numbers r_x, s_x in I_x such that $a_x < r_x < x < s_x < b_x$. Then $J_x = (r_x, s_x)$ is an open interval containing x and having rational end points. Also $J_x \cap S$ being a subset of $I_x \cap S$ contains a finite number of points of S.

The set of *all* open intervals having rational end points being an enumerable set, we can enumerate them as $J_1, J_2, J_3, \cdots \cdots$

Each point of S is contained in some $J_k (k \in \mathbb{N})$ and $J_k \cap S$ is a finite set. Also $S \subset \bigcup_{k=1}^{\infty} (J_k \cap S)$.

Thus S is contained in the countable union of finite sets and therefore S is a countable set.

Exercises 4

1. Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of non-empty closed intervals such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

2. Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of bounded open intervals such that $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} I_n = \phi$.

3. Let $I_n = [a_n, b_n]$ and $I_1 \supset I_2 \supset I_3 \supset \cdots$
   If $\alpha = \sup\{a_n : n \in \mathbb{N}\}, \beta = \inf\{b_n : n \in \mathbb{N}\}$ prove that
   (i) $[\alpha, \beta] = \bigcap_{n=1}^{\infty} I_n$ if $\alpha \neq \beta$, (ii) $\{\alpha\} = \bigcap_{n=1}^{\infty} I_n$ if $\alpha = \beta$.

4. Let S be an enumerable subset and T be a non-enumerable infinite subset of $\mathbb{R}$. Prove that
   (i) $S \cup T$ is non-enumerable, (ii) $S \cap T$ is at most enumerable,
   (iii) $S - T$ is at most enumerable, (iv) $T - S$ is non- enumerable.

6. Prove that $\mathbb{N} \times \mathbb{N}$ is an enumerable set. Deduce that if S be an enumerable set, then $S \times S$ is enumerable.

7. Prove that the set of all circles in the plane having rational radii and centres with rational co-ordinates is enumerable.

8. Prove that the set of all transcendental numbers is a non-enumerable set.
   [A real number is said to be *transcendental* if it is not algebraic.]

9. Show that each of the following sets has the cardinal number c.
   (i) $A = \{x \in \mathbb{R} : 0 < x < \infty\}$, (ii) $B = \{x \in \mathbb{R} : 0 < x < 1\}$.

Hint. (i) The mapping $f : \mathbb{R} \to A$ defined by $f(x) = e^x, x \in \mathbb{R}$ is a bijection.
(ii) The mapping $f : \mathbb{R} \to B$ defined by $f(x) = \frac{e^x}{e^x+1}, x \in \mathbb{R}$ is a bijection.

3.14. Point of condensation.

Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *point of condensation* of S if every neighbourhood of x contains uncountably many points of S.

It follows that every point of condensation of a set S is a limit point of S, but not conversely.

It also follows from the definition that a countable set cannot have a point of condensation.

The set of all points of condensation of a set S is denoted by S_c.

Examples.

1. Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Then every point of S is a point of condensation of S. Here $S_c = S$.

2. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Then $S_c = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Here $S \subset S_c$.

3. Let $S = \mathbb{Q}$. The set S being an enumerable set, $S_c = \phi$.

Theorem 3.14.1. Every uncountable subset S of $\mathbb{R}$ has at least one point of condensation in S.

Proof. Suppose S has no point of condensation. Let $x \in S$. Then x is not a point of condensation of S and there exists an open interval $I_x = (a_x, b_x)$ containing x such that I_x contains a countable number of points of S.

Let us choose rational numbers r_x, s_x in I_x such that $a_x < r_x < x < s_x < b_x$. Then $J_x = (r_x, s_x)$ is an open interval containing x and having rational end points. Also $J_x \cap S$ being a subset of $I_x \cap S$ is a countable set.

The set of *all* open intervals having rational end points being an enumerable set, we can enumerate them as $J_1, J_2, J_3, \ldots \ldots$

Each point of S is contained in some $J_k (k \in \mathbb{N})$. We conclude that $S \subset \bigcup_{k=1}^{\infty} (J_k \cap S)$. Further, each $J_k \cap S$ is countable.

Thus S is a subset of a countable union of countable sets and therefore S is a countable set, a contradiction to the hypothesis. So our assumption is wrong and there is at least one point of condensation in S.

This completes the proof.

Corollary. If no point of S is a condensation point of S then S is a countable set.

Note. This theorem is analogous to Bolzano-Weierstrass theorem but stronger in the sense that the condition of boundedness is not required for the existence of a point of condensation. Moreover, Bolzano-Weierstrass theorem assures the existence of a limit point which may not belong to the set but this theorem assures that some point of condensation is actually contained in the set.

Theorem 3.14.2. For every set $S \subset \mathbb{R}$, $S - S_c$ is a countable set.

Proof. **Case 1.** Let $S - S_c = \phi$. Then $S - S_c$ is a countable set.

Case 2. Let $S - S_c \neq \phi$. Let $x \in S - S_c$. Then x is not a point of condensation of S. So there exists a neighbourhood $N(x)$ of x such that $N(x) \cap S$ is a countable set. Consequently, $N(x) \cap S$ contains only countably many points of $S - S_c$ and as such x cannot be a point of condensation of $S - S_c$.

Thus no point of $S - S_c$ is a point of condensation of $S - S_c$.

By Theorem 3.14.1, $S - S_c$ must be a countable set.

This completes the proof.

Corollary. As $S = (S - S_c) \cup (S \cap S_c)$ it follows from the theorem that if S be an uncountable set then $S \cap S_c$ is uncountable.

That is, if S be an uncountable set, S *contains* uncountably many points of condensation of S.

Theorem 3.14.3. For every set $S \subset \mathbb{R}$, S_c is a closed set.

Proof. **Case 1.** Let $S_c = \phi$. Then S_c is a closed set.

Case 2. Let $S_c \neq \phi$. Let $x \in \bar{S}_c$ and $N(x)$ be a neighbourhood of x. Then $N(x)$ contains at least a point, say y of S_c.

Since $y \in S_c$ and $N(x)$ can be considered as a neighbourhood of y, $N(x)$ contains uncountably many points of S and as such x happens to be a point of condensation of S.

Thus $x \in \bar{S}_c \Rightarrow x \in S_c$ and therefore $\bar{S}_c \subset S_c$.

But by definition, $S_c \subset \bar{S}_c$ and therefore $\bar{S}_c = S_c$. This proves that S_c is a closed set.

3.15. Borel set.

We have seen that the union of an infinite collection of closed sets in $\mathbb{R}$ may not be a closed set in $\mathbb{R}$; the intersection of an infinite collection of open set may not be an open set in $\mathbb{R}$.

If however, the infinite collection be an enumerable collection, then we have a special type of sets.

Definition. The union of an enumerable collection of closed sets in $\mathbb{R}$ is said to be an F_σ **set**.

The intersection of an enumerable collection of open sets in $\mathbb{R}$ is said to be a G_δ **set**.

Examples.

1. $\mathbb{Q}$ is an F_σ set.

Since the set of all rational numbers is enumerable, $\mathbb{Q}$ can be described as the set $\{x_1, x_2, x_3, \ldots \ldots\}$.

Therefore $\mathbb{Q}$ can be expressed as the union of an enumerable collection of closed sets $\{x_1\}, \{x_2\}, \{x_3\}, \cdots \cdots$

So $\mathbb{Q}$ is an F_σ set.

2. A closed and bounded interval $[a, b]$ is a G_δ set.

The interval $[a, b]$ can be expressed as the intersection $\bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$.

For each $n \in \mathbb{N}$, $(a - \frac{1}{n}, b + \frac{1}{n})$ is an open set.

Thus $[a, b]$ is the intersection of an enumerable collection of open sets in $\mathbb{R}$. So $[a, b]$ is a G_δ set.

3. An open bounded interval (a, b) is an F_σ set.

The interval (a, b) can be expressed as the union $\bigcup\limits_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$.

For each $n \in \mathbb{N}$, $[a + \frac{1}{n}, b - \frac{1}{n}]$ is a closed set.

Thus (a, b) is the union of an enumerable collection of closed sets in $\mathbb{R}$. So (a, b) is an F_σ set.

Theorem 3.15.1. (i) The union of a countable collection of F_σ sets is an F_σ set.
(ii) The intersection of a countable collection of G_δ sets is a G_δ set.

Proof. (i) Let $\{F_1, F_2, F_3, \ldots\}$ be a countable collection of F_σ sets in $\mathbb{R}$. Let $F = \bigcup\limits_{i=1}^{\infty} F_i$.

Since F_i is an F_σ set, $F_i = F_{i1} \cup F_{i2} \cup F_{i3} \cup \cdots \cdots$ where each F_{ij} is a closed set in $\mathbb{R}$.

Thus F is the union of a countable collection of enumerable number of closed sets.

It follows that F is the union of an enumerable collection of closed sets in $\mathbb{R}$ and therefore F is an F_σ set.

(ii) Let $\{G_1, G_2, G_3, \ldots \ldots\}$ be a countable collection of G_δ sets in $\mathbb{R}$. Let $G = \bigcup\limits_{i=1}^{\infty} G_i$.

Since each G_i is a G_δ set, $G_i = G_{i1} \cap G_{i2} \cap G_{i3} \cap \cdots \cdots$ where each G_{ij} is an open set in $\mathbb{R}$.

Thus G is the intersection of a countable collection of enumerable number of open sets.

It follows that G is the intersection of an enumerable collection of open sets in $\mathbb{R}$ and therefore G is a G_δ set.

Note. Every open set in $\mathbb{R}$ can be expressed as the union of an enumerable number of closed sets in $\mathbb{R}$.

Every closed set in $\mathbb{R}$ can be expressed as the intersection of an enumerable number of open sets in $\mathbb{R}$.

Theorem 3.15.2. (i) The complement of an F_σ set in $\mathbb{R}$ is a G_δ set.

(ii) The complement of a G_δ set in $\mathbb{R}$ is an F_σ set.

Proof. (i) Let F be an F_σ set and $F = F_1 \cup F_2 \cup F_3 \cup \cdots \cdots$ where each F_i is a closed set in $\mathbb{R}$.

$$\begin{aligned}\mathbb{R} - F &= \mathbb{R} - (F_1 \cup F_2 \cup F_3 \cup \cdots) \\ &= (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) \cap (\mathbb{R} - F_3) \cap \cdots\end{aligned}$$

Each $\mathbb{R} - F_i$ is an open set in $\mathbb{R}$. Therefore $\mathbb{R} - F$ is the intersection of an enumerable collection of open sets.

So $\mathbb{R} - F$ is a G_δ set and the theorem is done.

(ii) similar proof.

Definition. A set that can be obtained as the union and intersection of an enumerable collection of closed sets and open sets in $\mathbb{R}$ is said to be a **Borel set**.

Examples.

1. A G_δ set is a Borel set. An F_σ set is a Borel set.

2. The union of a countable collection of G_δ sets is a Borel set. This is denoted by $G_{\delta\sigma}$.

3. The intersection of a countable collection of F_σ sets is a Borel set. This is denoted by $F_{\sigma\delta}$.

3.16. Cover, open cover.

Let S be a subset of $\mathbb{R}$. A collection C of sets $\{A_\alpha : \alpha \in \Lambda\}$, Λ being the index set, is said to be a *cover* (or a covering) of S if $S \subset \bigcup_{\alpha\in\Lambda} A_\alpha$.

If $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $S \subset \bigcup_{\alpha\in\Lambda} G_\alpha$ then $\mathcal{G}$ is said to be an *open cover* of S.

A family $\mathcal{G}$ of open intervals $\{I_\alpha : \alpha \in \Lambda\}, \Lambda$ being the index set, is said to be an open cover of S if $S \subset \bigcup_{\alpha \in \Lambda} I_\alpha$.

Worked Examples.

1. Let C be the collection of closed intervals $I_n = \{x \in \mathbb{R} : 0 \le x \le n\}$, where $n \in \mathbb{N}$. Show that C is a cover of the set $S = \{x \in \mathbb{R} : x \ge 0\}$.

Let $c \in S$. Then $c \ge 0$. There exists a natural number m such that $m - 1 \le c < m$. [Archimedean property of $\mathbb{R}$, deduction 3].

$m - 1 \le c < m \Rightarrow c \in I_m$. Therefore $c \in S \Rightarrow c \in I_m \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

Hence the collection C is a cover of the set S.

2. Let $\mathcal{G}$ be the collection of open intervals $I_n = \{x \in \mathbb{R} : -n < x < n\}$, where $n \in \mathbb{N}$. Show that the collection $\mathcal{G}$ is an open cover of the set $\mathbb{Q}$.

Let $x \in \mathbb{Q}$. Then $|x| \ge 0$. There exists a natural number k such that $|x| < k$, i.e., $x \in I_k$ and therefore $x \in \bigcup_{n=1}^{\infty} I_n$.

$x \in \mathbb{Q} \Rightarrow x \in \bigcup_{n=1}^{\infty} I_n$ and therefore $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$.

This proves that the collection $\mathcal{G}$ is an open cover of the set $\mathbb{Q}$.

3. Let $\mathcal{G}$ be the family of open intervals $I_n = \{x \in \mathbb{R} : \frac{1}{2^n} < x < 2\}$, $n = 1, 2, 3, \ldots$. Show that $\mathcal{G}$ is an open cover of the set $S = \{x \in \mathbb{R} : 0 < x < 1\}$.

Let $c \in S$. Then $0 < c < 1$. By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < c$.

Since the sequence $\{2^n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number m such that $2^m > k$. Then $0 < \frac{1}{2^m} < \frac{1}{k} < c$.

$0 < c < 1 \Rightarrow 0 < \frac{1}{2^m} < \frac{1}{k} < c < 2$.

$c \in S \Rightarrow 0 < c < 1 \Rightarrow \frac{1}{2^m} < c < 2 \Rightarrow c \in I_m$.

$c \in S \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

This proves that the collection $\mathcal{G}$ is an open cover of the set S.

4. Let $\mathcal{G}$ be the family of open intervals $\{I_n : n \in \mathbb{N}\}$, where $I_n = \{x \in \mathbb{R} : -\frac{1}{n} < x < n\}$. Show that the family $\mathcal{G}$ is an open cover of the set $S = \{x \in \mathbb{R} : x \ge 0\}$.

Let $c \in S$. Then $c \ge 0$. There exists a natural number m such that $m - 1 \le c < m$. [Archimedean property of $\mathbb{R}$, deduction 3].

$m-1 \le c < m \Rightarrow c \in I_m$. Therefore $c \in S \Rightarrow c \in I_m \Rightarrow c \in \bigcup_{n=1}^{\infty} I_n$ and therefore $S \subset \bigcup_{n=1}^{\infty} I_n$.

Hence the collection $\mathcal{G}$ is a cover of the set S.

Sub cover.

Let S be subset of $\mathbb{R}$. Let $\mathcal{C}$ be a collection of sets in $\mathbb{R}$ that covers S. If $\mathcal{C}'$ be a subcollection of $\mathcal{C}$ such that $\mathcal{C}'$ also covers S then $\mathcal{C}'$ is said to be a *sub cover* of the cover $\mathcal{C}$.

If the subcollection $\mathcal{C}'$ contains a finite number of sets of $\mathcal{C}$ and $\mathcal{C}'$ covers S then $\mathcal{C}'$ is said to be a *finite sub cover* of the cover $\mathcal{C}$.

Worked Examples (continued).

5. Show that there is a subcollection of the family $\mathcal{G}$ of Ex.2 that can cover the set $\mathbb{Q}$.

Let $\mathcal{G}' = \{I_{2n} : n \in \mathbb{N}\}$. Then $\mathcal{G}'$ is a sub collection of the family $\mathcal{G}$.

Let $x \in \mathbb{Q}$. Then $|x| \ge 0$. There exists a natural number k such that $|x| < 2k$, i.e., $x \in I_{2k}$ and therefore $x \in \bigcup_{n=1}^{\infty} I_{2n}$.

$x \in \mathbb{Q} \Rightarrow x \in \bigcup_{n=1}^{\infty} I_{2n}$ and therefore $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_{2n}$.

This proves that the collection $\mathcal{G}'$ is an open cover of the set $\mathbb{Q}$. So $\mathcal{G}'$ is a subcover of $\mathcal{G}$.

6. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.2 that can cover the set $\mathbb{Q}$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set S.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $\mathbb{Q} \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then p is a natural number and $I_p = \{x \in \mathbb{R} : -p < x < p\}$.

Now $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \ldots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_P$.

consequently, $\mathbb{Q} \subset I_p$ ... ...(i)

But $p \in \mathbb{Q}$ and $p \notin I_p$. This contradicts (i).

Therefore $\mathcal{G}$ has no finite subcover.

7. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.3 that can cover the interval $I = (0, 1)$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers I.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $I \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then p is a natural number and $I_p = \{x \in \mathbb{R} : -p < x < p\}$.

Now $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \ldots, I_{r_m} \subset I_p$ and therefore $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_P$.

consequently, $I \subset I_p$ ... ...(i)

But $\frac{1}{2^p} \in I$ and $\frac{1}{2^p} \notin I_p$. This contradicts (i).

Therefore there is no finite subcollection of $\mathcal{G}$ that can cover I.

8. Show that there is no finite subcollection of the family $\mathcal{G}$ of Ex.4 that can cover the set $S = x \in \mathbb{R} : x \geq 0$.

Let us assume that there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers the set S.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $r_1, r_2, \ldots, r_m$ are natural numbers and $S \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$ and $q = \min\{r_1, r_2, \ldots, r_m\}$.

Then $I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} = \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$ and $S \subset \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$.

But $p \in S$ and $p \notin \{x \in \mathbb{R} : -\frac{1}{q} < x < p\}$. This shows that $\mathcal{G}'$ cannot cover the set S.

Therefore no finite subcollection of $\mathcal{G}$ can cover S.

Theorem 3.16.1. Heine-Borel theorem.

Let S be a closed and bounded subset of $\mathbb{R}$. Then every open cover of S has a finite sub cover.

Proof. Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda$, Λ being the index set$\}$ in $\mathbb{R}$ such that $\mathcal{G}$ is an open cover of S.

Let us assume that $\mathcal{G}$ contains no finite sub cover. Therefore S is not contained in the union of a finite number of open sets in $\mathcal{G}$.

Since S is bounded there exist two real numbers a_1, b_1 such that $x \in S \Rightarrow a_1 \leq x \leq b_1$. Therefore $S \subset [a_1, b_1]$.

Let $I_1 = [a_1, b_1]$ and let $c_1 = \frac{a_1+b_1}{2}$. Let $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. At least one of the two subsets $S \cap I_1'$ and $S \cap I_1''$ has the property that it must be non-empty and it is not contained in the union of a finite number of open sets in $\mathcal{G}$. For if both of the sets $S \cap I_1'$ and $S \cap I_1''$ be contained in the union of a finite number of open sets in $\mathcal{G}$ then S would be contained in the union of a finite number of open sets in $\mathcal{G}$, a contradiction to the assumption.

If $S \cap I_1'$ be not contained in the union of a finite number of open sets in $\mathcal{G}$, we call $I_2 = I_1'$. If not, we call $I_2 = I_1''$.

We now bisect I_2 into closed subintervals I_2' and I_2'' and at least one of the sets $S \cap I_2'$ and $S \cap I_2''$ has the property and that it is non-empty and it not contained in the union of a finite number of open sets in $\mathcal{G}$. If $S \cap I_2'$ is not contained in the union of a finite number of open sets in $\mathcal{G}$ we call $I_3 = I_2'$. If not, we call $I_3 = I_2''$.

Continuing this process of bisection, we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$;

(ii) for all $n \in \mathbb{N}, S \cap I_n$ is non-empty and is not contained in the union of a finite number of open sets in $\mathcal{G}$;

(iii) $| I_n |= \frac{b_1-a_1}{2^{n-1}}$ and therefore $\lim | I_n |= 0$.

By the nested intervals theorem there exists one and only one point α such that $\alpha \in \bigcap\limits_{n=1}^{\infty} I_n$.

Let us choose $\delta > 0$. There exists a natural number k such that $0 < \frac{b_1-a_1}{2^{k-1}} < \delta$.

As $| I_k |= \frac{b_1-a_1}{2^{k-1}}$, we have $| I_k |< \delta$.

Since $\alpha \in I_k$ and $| I_k |< \delta, I_k \subset (\alpha - \delta, \alpha + \delta)$. Since $I_k \cap S$ is not contained in the union of a finite number of open sets in $\mathcal{G}, I_k$ contains infinite number of elements of S. Consequently, α is a limit point of S. Since S is a closed set, $\alpha \in S$.

Since $\mathcal{G}$ covers S and $\alpha \in S$, there exists an open set $G_\lambda \in \mathcal{G}$ such that $\alpha \in G_\lambda$. Since G_λ is an open set, the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon) \subset G$ for some $\epsilon > 0$.

As $0 < \epsilon$, there exists a natural number m such that $0 < \frac{b_1-a_1}{2^{m-1}} < \epsilon$.

As $| I_m |= \frac{b_1-a_1}{2^{m-1}}$, we have $| I_m |< \epsilon$.

Since $\alpha \in I_m$ and $| I_m |< \epsilon, I_m \subset (\alpha - \epsilon, \alpha + \epsilon)$. That is, $I_m \subset G_\lambda$.

This shows that $S \cap I_m$ is contained in the single open set G_λ of the collection $\mathcal{G}$ but this is contrary to our construction of $\{I_n\}$.

Therefore our assumption that S is not contained in the union of finite number of open sets in $\mathcal{G}$ is not tenable.

Hence there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ covers S. This completes the proof.

Note. In Heine-Borel theorem the hypothesis that the set S is a closed and bounded subset of $\mathbb{R}$ is crucial. The theorem does not hold if S be not closed, or if S be unbounded.

Definition. Let S be a subset of $\mathbb{R}$. S is said to be a **compact set** i every open cover $\mathcal{G}$ of S has a finite subcover. That is, if $\mathcal{G}$ be a family of open sets that covers S then there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

To be explicit, if $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover of $S \subset \mathbb{R}$ then S will be compact if there exists a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_m \in \Lambda$ such that $S \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_m}$.

Heine-Borel theorem states that a closed and bounded subset of $\mathbb{R}$ is compact.

Theorem 3.16.2. (Converse of Heine-Borel theorem)

A compact subset of $\mathbb{R}$ is closed and bounded in $\mathbb{R}$.

Proof. Let K be a compact subset of $\mathbb{R}$. We first prove that K is bounded.

Let $I_n = \{x \in \mathbb{R} : -n < x < n\}$, where $n \in \mathbb{N}$ and $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is a collection of open sets in $\mathbb{R}$.

Clearly, $\mathbb{R} \subset \bigcup_{n=1}^{\infty} I_n$, and therefore $K \subset \bigcup_{n=1}^{\infty} I_n$.

This shows that $\mathcal{G}$ is an open cover of K.

Since K is compact, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$.

Then $I_{r_1} \subset I_p, I_{r_2} \subset I_p, \cdots, I_{r_m} \subset I_p$ and $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p = (-p, p)$.

This shows that K is a bounded subset of $\mathbb{R}$.

We now prove that K is a closed set.

Let $y \in \mathbb{R} - K$. Let us consider the collection of open sets $\{I_n\}$ where

$$I_1 = \{x \in \mathbb{R} : |\, y - x \,| > 1\}$$
$$I_2 = \{x \in \mathbb{R} : |\, y - x \,| > \tfrac{1}{2}\}$$
$$I_3 = \{x \in \mathbb{R} : |\, y - x \,| > \tfrac{1}{3}\}$$
$$\ldots \quad \ldots$$

Clearly, $\bigcup_{n=1}^{\infty} I_n = \mathbb{R} - \{y\}$ and $K \subset \bigcup_{n=1}^{\infty} I_n$.

Let $\mathcal{G} = \{I_n : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is an open cover of K. Since K is compact, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{I_{r_1}, I_{r_2}, \ldots, I_{r_m}\}$. Then $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m}$.

Let $p = \max\{r_1, r_2, \ldots, r_m\}$. Then $I_p \supset I_{r_1}, I_p \supset I_{r_2}, \cdots, I_p \supset I_{r_m}$ and $K \subset I_{r_1} \cup I_{r_2} \cup \cdots \cup I_{r_m} \subset I_p$.

But $I_p = \{x \in \mathbb{R} : |\, y - x \,| > \frac{1}{p}\}$. Let $G = \{x \in \mathbb{R} : |\, y - x \,| < \frac{1}{p}\}$.

Then G is a neighbourhood of y and $G \cap K = \phi$, since $K \subset I_p$.

It follows that y is not a limit point of K.

Thus $y \in \mathbb{R} - K \Rightarrow y \notin K'$ (the derived set of K).
Contrapositively, $y \in K' \Rightarrow y \notin \mathbb{R} - K$, i.e., $y \in K$.
Therefore $K' \subset K$ and this proves that K is a closed set.
This completes the proof.

Theorem 3.16.3. If K be a compact set in $\mathbb{R}$, every infinite subset of K has a limit point in K.

Proof. Let T be an infinite subset of K. Let us suppose that T has no limit point in K.

Let $x \in K$. Then x is not a limit point of T.

Therefore there exists a positive δ_x such that $N'(x, \delta_x) \cap T = \phi$, where $N'(x, \delta_x) = N(x, \delta_x) - \{x\}$.

Let us consider the family $\mathcal{G}$ of neighbourhoods $\{N(x, \delta_x) : x \in K \ \& \ N'(x, \delta_x) \cap T = \phi\}$. Clearly $\mathcal{G}$ is an open cover of K. Since K is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{N(x_1, \delta_{x_1}), N(x_2, \delta_{x_2}), \ldots, N(x_m, \delta_{x_m})\}$.
Then $K \subset N(x_1, \delta_{x_1}) \cup N(x_2, \delta_{x_2}) \cup \cdots \cup N(x_m, \delta_{x_m})$.

As $T \subset K, T \subset N(x_1, \delta_{x_1}) \cup N(x_2, \delta_{x_2}) \cup \cdots \cup N(x_m, \delta_{x_m})$
$= N'(x_1, \delta_{x_1}) \cup N'(x_2, \delta_{x_2}) \cup \cdots \cup N'(x_m, \delta_{x_m}) \cup \{x_1, x_2, \ldots, x_m\}$.

It follows that $T \subset \{x_1, x_2, \ldots, x_m\}$, since $N'(x_i, \delta_{x_i}) \cap T = \phi$, for $i = 1, 2, \ldots, m$.

This shows that T is a finite set, a contradiction.
Thus T has limit point in K and the proof is complete.

Corollary. The set $\mathbb{R}$ is not compact, since the set $\mathbb{Z}$ is an infinite subset of $\mathbb{R}$ having no limit point in $\mathbb{R}$.

Theorem 3.16.4. If K be a subset of $\mathbb{R}$ such that every infinite subset of K has a limit point in K then K is compact.

Proof. First we prove that K is closed.

Let p be a limit point of K. Then for each positive ϵ, $N'(p, \epsilon)$ contains infinitely many elements of K.

Let $\epsilon = 1$. Then $N'(p, 1)$ contains infinitely many elements of K.
Let us choose one such element and call it x_1.

Let $\epsilon = \frac{1}{2}$. Then $N'(p, \frac{1}{2})$ contains infinitely many elements of K.
Let $x_2 (\neq x_1)$ be one such. Then $x_2 \in N'(p, \frac{1}{2})$.
Let $\epsilon = \frac{1}{3}$.

Proceeding in a similar manner, we obtain an infinite subset $S = \{x_1, x_2, x_3, \ldots\}$ of K such that $x_i \in N'(p, \frac{1}{i})$. We now prove that p is the only limit point of S.

Let $N(p,\delta)$ be a neighbourhood of p. Since $0 < \delta$, there exists a natural number m such that $0 < \frac{1}{m} < \delta$.

$N(p,\delta) \supset N(p,\frac{1}{m})$ and $N(p,\frac{1}{m})$ contains each of $x_m, x_{m+1}, \ldots$

Thus $N(p,\delta)$ contains infinitely many elements of S, for each $\delta > 0$. So p is a limit point of S.

Let $q \neq p$. Let $\delta_0 = \frac{1}{2} \mid p - q \mid > 0$.

Then $N(p,\delta_0)$ and $N(q,\delta_0)$ are disjoint neighbourhoods.

There exists a natural number k such that $0 < \frac{1}{k} < \delta_0$.

$N(p,\delta_0) \supset N(p,\frac{1}{k})$ and as $N(p,\frac{1}{k})$ contains each of $x_k, x_{k+1}, \cdots$, $N(q,\delta_0)$ contains at most a finite number of elements of S and therefore q is not a limit point of S.

Hence p is the only limit point of S and by hypothesis, $p \in K$.

Thus $p \in K' \Rightarrow p \in K$. Therefore $K' \subset K$ and K is a closed set.

We prove that K is bounded.

Let us assume that K is unbounded above.

Let us choose an element $x_1 \in K$. Let us choose x_2 in K such that $x_2 > x_1 + 1$. Let us choose x_3 in K such that $x_3 > x_2 + 1, \cdots \cdots$

Thus we obtain an infinite set $\{x_1, x_2, x_3, \ldots \ldots\}$ in K.

But this set has no limit point by the construction of the elements. This contradicts the hypothesis that every infinite subset of K has a limit point in K.

Therefore K is not unbounded above.

Similarly, K is not unbounded below. Therefore K is bounded.

Since K is closed and bounded, K is compact and the proof is complete.

Let K be a subset of $\mathbb{R}$. Then the following statements are equivalent.

(i) K is closed and bounded.

(ii) Every open cover of K has a finite subcover.

(iii) Every infinite subset of K has a limit point in K.

Proof. (i) $\Rightarrow$ (ii) by Heine-Borel theorem. (ii) $\Rightarrow$ (iii) by theorem 3.16.3. (iii) $\Rightarrow$ (i) by theorem 3.16.4.

Therefore the three statements are such that one of them implies the other two. Hence they are equivalent.

Worked Examples (continued).

9. Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.

Let $S = \{a_1, a_2, \dots, a_m\}$ be a finite subset of $\mathbb{R}$. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, be a collection of open sets in $\mathbb{R}$ such that $\mathcal{G}$ covers S.

Since $a_1 \in S$, $a_1 \in G_{\alpha_1}$ of $\mathcal{G}$ for some $\alpha_1 \in \Lambda$.
Since $a_2 \in S$, $a_2 \in G_{\alpha_2}$ of $\mathcal{G}$ for some $\alpha_2 \in \Lambda$.

... ...

Since $a_m \in S$, $a_m \in G_{\alpha_m}$ of $\mathcal{G}$ for some $\alpha_m \in \Lambda$.

Therefore $S \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_m}$.
Let $\mathcal{G}' = \{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_m}\}$. Then $\mathcal{G}'$ is a finite subcollection of $\mathcal{G}$ and $\mathcal{G}'$ covers S.

Since $\mathcal{G}$ is an arbitrary open cover of S, it follows that every open cver of S has a finite subcover. Therefore S is a compact set in $\mathbb{R}$.

10. Let K be a compact subset of $\mathbb{R}$ and $F \subset K$ be a closed subset in $\mathbb{R}$. Prove that F is compact in $\mathbb{R}$.

Since F is closed, $\mathbb{R} - F$ is open.

Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $\mathcal{G}$ is an open cover of F. Let us suppose that $\mathcal{G}$ is not an open cover of K.

Let $\mathcal{G}'$ be the collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$ together with $\mathbb{R} - F$. Clearly, $\mathbb{R} \subset (\bigcup_{\alpha\in\Lambda} G_\alpha) \cup (\mathbb{R} - F)$.

Therefore $K \subset \{\bigcup_{\alpha\in\Lambda} G_\alpha\} \cup (\mathbb{R} - F)$, i.e., $\mathcal{G}'$ is an open cover of K.

Since K is compact there exists a finite subcollection $\mathcal{G}''$ of $\mathcal{G}'$ such that $\mathcal{G}''$ also covers K.

Let $\mathcal{G}'' = \{G_{r_1}, G_{r_2}, \dots, G_{r_m}, \mathbb{R} - F\}$, $r_i \in \Lambda$.

$\mathcal{G}''$ must contain $\mathbb{R} - F$ because K cannot be contained in the union $G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m}$.

Therefore $K \subset G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m} \cup (\mathbb{R} - F)$ and as $F \subset K, F \subset G_{r_1} \cup G_{r_2} \cup \cdots \cup G_{r_m}$.

Let $\mathcal{G}''' = \{G_{r_1}, G_{r_2}, \dots, G_{r_m}\}$. Then $\mathcal{G}'''$ is a finite subcollection of $\mathcal{G}$ such that $\mathcal{G}'''$ also covers F. So F is compact.

11. Let K be a non-empty compact set in $\mathbb{R}$. Show that K has a least element.

Let us assume that K has no least element. For each $a \in K$, let $G_a = \{x \in \mathbb{R} : x > a\}$. Then G_a is an open interval.

Let us consider the family of open intervals $\mathcal{G} = \{G_a : a \in K\}$.

Let $b \in K$. Since K has no least element, there is an element c in K such that $c < b$ and therefore $b \in G_c$.

Thus $b \in K \Rightarrow b \in G_c$ for some $c \in K$. This shows that the family $\mathcal{G}$ is an open cover of K.

Since K is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers K.

Let $\mathcal{G}' = \{G_{a_1}, G_{a_2}, \ldots, G_{a_m}\}$. Then each $a_i \in K$ and $K \subset G_{a_1} \cup G_{a_2} \cup \cdots \cup G_{a_m}$.

Let $a_0 = \min\{a_1, a_2, \ldots, a_m\}$. Then $a_0 \in K$. But $a_0 \notin \bigcup\limits_{i=1}^{m} G_{a_i}$.

We arrive at a contradiction and therefore K has a least element.

Note. In a similar manner we can prove that K has a greatest element.

Theorem 3.16.5. Lindelof's theorem.

If S be a subset of $\mathbb{R}$ every open cover of S has a countable subcover. That is, if $\mathcal{G}$ be a collection of open sets in $\mathbb{R}$ that covers S then there is a countable subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

Proof. Let $\mathcal{G}$ be a collection of open sets $\{G_\alpha : \alpha \in \Lambda$, Λ being the index set$\}$ in $\mathbb{R}$ such that $\mathcal{G}$ covers S.

Let $x \in S$. Then x belongs to at least one open set, say G_λ, of the collection. Therefore there exists an open interval I_x such that $x \in I_x \subset G_\lambda$.

Let us take an open interval $J(x)$ with rational end points such that $x \in J(x) \subset I_x$.

Let $\mathcal{G}'$ be the collection of all distinct open intervals $\{J(x) : x \in S\}$. Obviously $\mathcal{G}'$ covers S.

The set of all open intervals in $\mathbb{R}$ with rational end points is enumerable. The collection $\{J(x) : x \in S\}$ being a subset of this is countable. Therefore the collection $\mathcal{G}'$ can be enumerated as $\mathcal{G}' = \{J_1, J_2, J_3, \ldots\}$.

Now corresponding to each $J_m \in \mathcal{G}'$ let us choose a point $x_m \in S$ such that $x_m \in J_m \subset I_{x_m} \subset G_{\lambda_m}$, say. Then $J_m \subset G_{\lambda_m} \in \mathcal{G}$.

Thus for each $J_m \in \mathcal{G}'$ there corresponds one $G_{\lambda_m} \in \mathcal{G}$.

Let $\mathcal{G}''$ be the family of open sets $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots\}$.

Then $\mathcal{G}''$ is a countable subcollection of $\mathcal{G}$ and $\mathcal{G}''$ covers S.

This completes the proof.

Another proof of Heine-Borel theorem.

If S be closed and bounded subset of $\mathbb{R}$ then every open cover of S has a finite subcover.

Proof. Let $\mathcal{G}$ be a collection of open sets in $\mathbb{R}$ that covers S. By Lindelof's theorem, there exists a countable subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S. Let $\mathcal{G}' = \{J_1, J_2, \ldots, J_n, \ldots\}$. Then $S \subset \bigcup_{i=1}^{\infty} J_i$.

Let $C_1 = S - J_1$,
$C_2 = S - (J_1 \cup J_2)$,
$C_3 = S - (J_1 \cup J_2 \cup J_3)$,
$\ldots \quad \ldots \quad \ldots$

$C_k \subset S$ for all $k \in \mathbb{N}$. Also $C_1 \supset C_2 \supset C_3 \supset \cdots$
Since S is a bounded set each C_k is a bounded set.

Since each J_i is open, $C_k = S - (J_1 \cup J_2 \cup \cdots \cup J_k)$ is a closed set.

Therefore the collection $\{C_1, C_2, C_3, \ldots\}$ is a countable collection of closed and bounded sets in $\mathbb{R}$ and $C_1 \supset C_2 \supset C_3 \supset \cdots$

We shall prove that $c_m = \phi$ for some $m \in \mathbb{N}$.

If none of the sets of the collection $\{C_1, C_2, C_3, \ldots \quad \ldots\}$ be empty then the collection is an enumerable collection of non-empty closed and bounded sets with $C_1 \supset C_2 \supset C_3 \supset \cdots$

By Cantor's intersection theorem, there exists a point x in $\mathbb{R}$ such that $x \in \bigcap_{i=1}^{\infty} C_i$. [Theorem 3.11.2]

But $C_k \subset S$ for all $k \in \mathbb{N}$. Therefore $x \in S$ ... ... (A)

Again $x \in \bigcap_{i=1}^{\infty} C_i \Rightarrow x \notin \bigcup_{i=1}^{\infty} J_i$ ... ... (B)

(A) and (B) together imply that $\{J_1, J_2, J_3, \ldots\}$ is not a cover of S, a contradiction.

So our assumption that none of the sets of the collection $\{C_1, C_2, C_3, \ldots\}$ is empty is wrong.

Therefore at least one of the sets, say C_m, is empty. Consequently, $S \subset J_1 \cup J_2 \cup \cdots \cup J_m$. That is, a finite subcollection of $\mathcal{G}$ also covers S.

This proves the theorem.

Exercises 5

1. Define a compact set. Use your definition to prove that

(i) the set $\mathbb{R}$ is not compact; (ii) the set $\mathbb{Z}$ is not compact;

(iii) the set $\mathbb{N}$ is not compact.

[**Hint.** Let $I_n = \{x \in \mathbb{R} : -n < x < n\}$. Then the family $\mathcal{F}$ of open intervals $\{I_n : n \in \mathbb{N}\}$ is an open cover of the set.]

2. Let $\mathcal{F}$ be the family of open intervals $\{I_n : n \in \mathbb{N}\}$, where $I_n = \{x \in \mathbb{R} : \frac{1}{n+2} < x < 1 - \frac{1}{n+2}\}$. Show that the family $\mathcal{F}$ is an open cover of the interval $I = \{x \in \mathbb{R} : 0 < x < 1\}$. Does there exist a finite subfamily of $\mathcal{F}$ that can cover I? Justify your answer.

3. *For each* $x \in (0,2)$, *let* $I_x = (\frac{x}{2}, \frac{x+2}{2})$. Show that the family $\mathcal{G} = \{I_x : x \in (0,2)\}$ *is an open cover of the set* $S = \{x \in \mathbb{R} : 0 < x < 2\}$. Show that no *finite subcollection of* $\mathcal{G}$ *can cover* S.

Hint. Let $c \in S$. Then $0 < c < 2$. This implies $0 < \frac{c}{2} < c < \frac{c+2}{2} < 2 \Rightarrow c \in I_c$.

4. *Give an example of an open cover of the set* $(0,1]$ *which does not have a finite sub cover.*

5. *Give an example of an open cover of the set* $[0,\infty)$ *which does not have a finite sub cover.*

6. *Use the definition of a compact set to prove that the union of two compact sets in* $\mathbb{R}$ *is a compact set.*

Give an example to show that the union of an infinite number of compact sets in $\mathbb{R}$ *is not necessarily a compact set.*

7. *Use the definition of a compact set to prove that*

(i) the intersection of two compact sets in $\mathbb{R}$ is compact,

(ii) the intersection of an infinite collection of compact sets in $\mathbb{R}$ is compact.

8. Let A and B be subsets of $\mathbb{R}$ of which A is closed and B is compact. Prove that $A \cap B$ is compact.

9. Give an alternative proof of the theorem- A compact set K in $\mathbb{R}$ is closed.

[**Hint.** Let $y \in \mathbb{R} - K$. For every $x \in K$, there exists $\epsilon_x > 0$ such that $N(x,\epsilon_x) \cap N(y,\epsilon_x) = \phi$. The family of neighbourhoods $\{N(x,\epsilon_x) : x \in K\}$ is an open cover of K. Extract a finite sub cover. Show that $y \notin K'$.]

10. *If* p be a limit point of a set $S \subset \mathbb{R}$ prove that there exists a countably *infinite subset of* S having p as its only limit point.

11. Let S be a subset of $\mathbb{R}$ such that every infinite subset of S has at least one limit point in S. Prove that S is a closed set.

4. REAL FUNCTIONS

4.1. Real function.

Let X be a non-empty set. A function $f : X \longrightarrow \mathbb{R}$ is called a *real valued function* on X. For each $x \in X$, the f-image, which is also called the value of f at x, denoted by $f(x)$, is a real number.

For example, the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ defined by $f(z) = |z|, z \in \mathbb{C}$ is a real valued function of complex numbers.

Let D be a non-empty subset of $\mathbb{R}$. A function $f : D \longrightarrow \mathbb{R}$ is said to be a *real valued function of real numbers*. Such a function is also called a *real function*.

D is said to be the *domain* of f. The set $f(D) = \{f(x) : x \in D\}$ is a subset of $\mathbb{R}$ and it is called the *range* of f.

Examples.

1. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = c, x \in \mathbb{R}$. The range of the function f is the singleton set $\{c\}$. f is called a **constant function**. f is also expressed as $f(x) = c, x \in \mathbb{R}$.

2. Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and $f : D \longrightarrow \mathbb{R}$ is defined by $f(x) = \frac{1}{x}, x \neq 0$. The range of f is $\{x \in \mathbb{R} : x \neq 0\}$.

3. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \longrightarrow \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 0\}$. f is also expressed as $f(x) = \sqrt{x}, x \geq 0$. f is called the **square root function**.

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined $f(x) = \sin x, x \in \mathbb{R}$. The range of f is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$. f is also expressed as $f(x) = \sin x, x \in \mathbb{R}$. f is called the real **sine function**.

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = |x|, x \in \mathbb{R}$. The range of the function is $\{x \in \mathbb{R} : x \geq 0\}$. f is equivalently expressed as
$$\begin{aligned} f(x) &= x, x > 0 \\ &= 0, x = 0 \\ &= -x, x < 0. \end{aligned}$$
f is called the **absolute value function.**

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \text{sgn } x, x \in \mathbb{R}$.

$$\begin{aligned} \operatorname{sgn} x &= \tfrac{|x|}{x}, x \neq 0 \\ &= 0, x = 0. \end{aligned}$$

The range of f is the finite set $\{-1, 0, 1\}$. f is equivalently expressed as

$$\begin{aligned} f(x) &= 1, x > 0 \\ &= 0, x = 0 \\ &= -1, x < 0. \end{aligned}$$

f is called the **signum function.**

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$. $[x]$ is the greatest integer not greater than x. The range of the function is $\mathbb{Z}$. f is equivalently expressed as

$$\begin{aligned} f(x) &= 0, 0 \leq x < 1 \\ &= 1, 1 \leq x < 2 \\ &= 2, 2 \leq x < 3 \\ &\cdots \quad \cdots \\ &= -1, -1 \leq x < 0 \\ &= -2, -2 \leq x < -1 \\ &\cdots \quad \cdots \end{aligned}$$

f is called the **greatest integer function.**

For every $x \in \mathbb{R}$, $x \geq [x]$. The difference between x and its integral part $[x]$ is called the *fractional part* of x and is denoted by $\{x\}$.

Therefore $\{x\} = x - [x]$ for all real x. It also follows that $0 \leq \{x\} < 1$ for all real x.

For example, $\{.3\} = .3$, $\{2.3\} = .3$, $\{2\} = 0$, $\{-.3\} = .7$.

Definition.

A function f defined on $I = [a, b]$ is said to be a **piecewise constant function** on I (or a **step function** on I) if there exist finite number of points $x_0, x_1, \ldots, x_n (a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b)$ such that f is a constant on each open subinterval (x_{k-1}, x_k) of $[a, b]$. That is, for each $k = 1, 2, \ldots, n$ there is a real number s_k such that $f(x) = s_k$ for all $x \in (x_{k-1}, x_k)$. $f(x_{k-1}), f(x_k)$ need not be same as s_k, $k = 1, 2, \ldots, n$.

4.2. Injective function, Surjective function.

Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *injective* (or one-one) if for two distinct elements x_1, x_2 in D the functional values $f(x_1)$ and $f(x_2)$ are distinct.

Let $D \subset \mathbb{R}, E \subset \mathbb{R}$. A function $f : D \to E$ is said to be surjective (or onto) if $f(D) = E$.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin\ x, x \in \mathbb{R}$ is not injective, because two distinct points π and 2π in the domain $\mathbb{R}$ have the same functional value. f is not surjective, because the range of $f = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, a proper subset of the codomain set $\mathbb{R}$.

4.3. Equal functions.

Let $D \subset \mathbb{R}$. The functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ having the same domain D are said to be *equal* if $f(x) = g(x)$ for all $x \in D$.

Examples.

1. Let $f(x) = |x|, x > 0; g(x) = x, x > 0$

Then f and g have the same domain $\{x \in \mathbb{R} : x > 0$ and $f(x) = g(x)$ for all x in the domain. Therefore $f = g$.

2. Let $f(x) = \sqrt{\frac{2x}{x-1}}, x \in A \subset \mathbb{R}; g(x) = \frac{\sqrt{2x}}{\sqrt{x-1}}, x \in B \subset \mathbb{R}$.

Here $A = \{x \in \mathbb{R} : x > 1\} \cup \{x \in \mathbb{R} : x \leq 0\}$, $B = \{x \in \mathbb{R} : x > 1\}$.

f and g have different domains. Therefore $f \neq g$.

4.4. Restriction function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let D_o be a non-empty subset of D. The function $g : D_o \to \mathbb{R}$ defined by $g(x) = f(x), x \in D_o$ is said to be the *restriction* of f to D_o and g is denoted by f/D_o.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \text{ sgn } x, x \in \mathbb{R}$.

Let $D_o = \{x \in \mathbb{R} : x > 0\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 1, x > 0$.

Let $D_1 = \{x \in \mathbb{R} : x < 0\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = -1, x < 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x], x \in \mathbb{R}$.

Let $D_o = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Then the restriction function f/D_o is defined by $f/D_o\ (x) = 0, 0 \leq x < 1$.

Let $D_1 = \{x \in \mathbb{R} : 1 \leq x < 2\}$. Then the restriction function f/D_1 is defined by $f/D_1\ (x) = 1, 1 \leq x < 2$.

3. Let $D = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \sqrt{1 - \sin 2x}, x \in D$.

Let $D_o = \{x \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{4}\}$. Then the restriction function f/D_o is defined by $f/D_o(x) = \cos x - \sin x, 0 \leq x \leq \pi/4$.

Let $D_1 = \{x \in \mathbb{R} : \frac{\pi}{4} \leq x \leq \frac{\pi}{2}\}$. Then the restriction function f/D_1 is defined by $f/D_1(x) = \sin x - \cos x, \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

4.5. Composite function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $g : E \to \mathbb{R}$ be a function on E where $f(D) \subset E \subset \mathbb{R}$. Then for each $x \in D, f(x) \in E$ and therefore $g(f(x)) \in \mathbb{R}$. We can conceive of a real function $h : D \to \mathbb{R}$ such that $h(x) = g(f(x)), x \in D$. Then h is said to be the *composite function* of f and g and the function h is expressed as gf or as $g \circ f$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f : D \to \mathbb{R}$ is defined by $f(x) = \sqrt{x}, x \in D$, and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = e^x, x \in \mathbb{R}$.

$f(D) = \{x \in \mathbb{R} : x \geq 0\}$. $f(D)$ is a subset of the domain of g. The composite function $g \circ f : D \to \mathbb{R}$ is defined by $g \circ f(x) = e^{\sqrt{x}}, x \in D$, i.e., $g \circ f(x) = e^{\sqrt{x}}, x \geq 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1, x \in \mathbb{R}$. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $g : D \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}, x \in D$. The range of f is $\{x \in \mathbb{R} : x \geq 1\}$ and this is a subset of the domain of g.

The composite function $gf : \mathbb{R} \to \mathbb{R}$ is defined by $gf(x) = \sqrt{x^2 + 1}, x \in \mathbb{R}$.

4.6. Inverse function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be an injective function. Let $f(D) = E \subset \mathbb{R}$. Then $f : D \to E$ is injective as well as surjective.

Let $x \in D$. Then $f(x) = y \in E$. Each y in E has exactly one pre-image x in D. We can define a function $g : E \to D$ by $g(y) = x, y \in E$ where $f(x) = y$.

Therefore $gf(x) = x$ for all $x \in D$ and $fg(y) = y$ for all $y \in E$.

g is said to be the *inverse* of f and is denoted by f^{-1}.

The domain of the inverse function f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$.

Examples.

1. Let $D = \{x \in \mathbb{R} : x \geq 0\}$ and $f(x) = x^2, x \in D$. $f(D) = \{x \in \mathbb{R} : x \geq 0\} = E$, say. Then $f : D \to E$ is injective as well as surjective.
The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sqrt{y}, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \geq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;

i.e., $\sqrt{x^2} = x$ for all $x \geq 0$ and $(\sqrt{y})^2 = y$ for all $y \geq 0$.

This inverse function is called **the square root function.**

2. Let $D = \{x \in \mathbb{R} : x \leq 0\}$ and $f(x) = x^2, x \leq 0$. The range of is $\{x \in \mathbb{R} : x \geq 0\} = E$,say. Then $f : D \to E$ is injective as well a surjective.
The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = -\sqrt{y}, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \leq 0$ and $ff^{-1}(y) = y$ for all $y \geq 0$;

i.e., $-\sqrt{x^2} = x$ for all $x \leq 0$ and $(-\sqrt{y})^2 = y$ for all $y \geq 0$.

This inverse function is called the **negative square root** function

Note. The function $f(x) = x^2, x \in \mathbb{R}$ admits of two inverse function the *principal* inverse function is the function described in Example 1.

3. The real sine function defined on $\mathbb{R}$ is not injective on $\mathbb{R}$. The rang of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$. The $f : D \to E$ defined by $f(x) = \sin x, x \in D$ is injective as well as surjectiv The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \sin^{-1} y, y \in$

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$;

i.e., $\sin^{-1}(\sin x) = x$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $\sin(\sin^{-1} y) = y$, f $-1 \leq y \leq 1$.

This inverse function is called the **principal inverse sine functio** The domain of the inverse function is $\{y \in \mathbb{R} : -1 \leq y \leq 1\}$ and t range is $\{x \in \mathbb{R} : \frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$

Therefore $-\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$ for $-1 \leq y \leq 1$.

Note. If instead of D, we choose $D_1 = \{x \in \mathbb{R} : 3\pi/2 \leq x \leq 5\pi/2\}$ the domain then the function $f(x) = \sin x, x \in D_1$, is injective as well surjective and therefore it admits of an inverse function $f^{-1} : E \to$ satisfying the conditions

$f^{-1}f(x) = x$ for all $x \in D_1$ and $ff^{-1}(y) = y$ for all $y \in E$.

But this inverse function differs from the principal inverse sine fun tion as they have different ranges.

Equivalently, we can define *many* inverse sine functions on the san domain E with their respective ranges different. This is expressed saying that inverse of real sine function is a *many-valued function* ar this is denoted by Sin^{-1} (or Arc sin). The *principal* inverse function denoted by $\sin^{-1}$ (or arc sin).

Thus sin (Arc $\sin y$) $= y$, for $-1 \leq y \leq 1$ but Arc sin $(\sin x) \neq x$, general.

4. The real cosine function $f(x) = \cos x, x \in \mathbb{R}$ is not injective. Th range of the function is $E = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$.

Let us consider the subset $D = \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$. Then t function $f : D \to E$ defined by $f(x) = \cos x, x \in D$ is injective as well surjective.

The inverse function $f^{-1} : E \to D$ is defined by $f^{-1}(y) = \cos^{-1} y, y \in E$.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in E$, i.e., $\cos^{-1}(\cos x) = x$, for $0 \leq x \leq \pi$ and $\cos(\cos^{-1} y) = y$, for $-1 \leq y \leq 1$.

This inverse function is called the *principal inverse cosine function.* The domain of this inverse function is $\{y \in \mathbb{R} : -1 \leq y \leq 1\}$ and the range is $\{x \in \mathbb{R} : 0 \leq x \leq \pi\}$.

Therefore $0 \leq \cos^{-1} y \leq \pi$ for $-1 \leq y \leq 1$.

Note. If instead of D we choose $D_1 = \{x \in \mathbb{R} : 2\pi \leq x \leq 3\pi\}$ as the domain, then the function $f : D_1 \to E$ defined by $f(x) = \cos x, x \in D_1$ is injective as well as surjective and therefore it admits of an inverse function $f^{-1} : E \to D_1$.

Thus as in the case of sine function, we can define *many* inverse cosine functions on the same domain E with their respective ranges different. Therefore inverse cosine function is also a many valued function and this is denoted by Cos^{-1} or (Arc cos). The *principal* inverse function is denoted by $\cos^{-1}$ (or arc cos).

5. Let n be an integer and $I_n = \{x \in \mathbb{R} : (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}\}$. Let $I = \bigcup_{n \in z} I_n$. The real tangent function is defined on the domain I and the range of the function is $\mathbb{R}$. Let us consider the subset $D = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$. Then $f : D \to \mathbb{R}$ defined by $f(x) = \tan x, x \in D$ is injective as well as surjective.

The inverse function $f^{-1} : \mathbb{R} \to D$ is defined by $f^{-1}(y) = \tan^{-1} y, y \in \mathbb{R}$.

Also $f^{-1}f(x) = x$ for all $x \in D$ and $ff^{-1}(y) = y$ for all $y \in \mathbb{R}$, i.e., $\tan^{-1}(\tan x) = x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan(\tan^{-1} y) = y$, for $y \in \mathbb{R}$.

This inverse function is called the **principal inverse tangent function.** The domain of the inverse function is $\mathbb{R}$ and the range is $\{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$

Therefore $-\frac{\pi}{2} < \tan^{-1} y < \frac{\pi}{2}$ for all $y \in \mathbb{R}$.

Note. If instead of D, we choose $D_1 = \{x \in \mathbb{R} : \frac{3\pi}{2} < x < 5\frac{\pi}{2}\}$ as the domain then the function $f : D_1 \to \mathbb{R}$ defined by $f(x) = \tan x, x \in D_1$, is injective as well as surjective and therefore it admits of an inverse function $f^{-1} : \mathbb{R} \to D_1$.

Thus we can define many inverse tangent functions on $\mathbb{R}$ with their respective ranges different. Therefore inverse tangent function is also a many valued function and this is denoted by Tan^{-1} (or Arc tan).

The principal inverse function is denoted by $\tan^{-1}$ (or arc tan).

6. Let n be an integer and $I_n = \{x \in \mathbb{R} : n\pi < x < (n+1)\pi\}$. Let $I = \bigcup\limits_{n \in z} I_n$. The real cotangent function is defined on the domain I and the range of the function is $\mathbb{R}$. Let $D = \{x \in \mathbb{R} : 0 < x < \pi\}$ then $f(x) = \cot x, x \in D$ is injective as well as surjective.
The inverse function $f^{-1} : \mathbb{R} \to D$ is defined by $f^{-1}(y) = \cot^{-1} y, y \in \mathbb{R}$.

This inverse function is called the **principal inverse cotangent function**. The domain of the inverse function is $\mathbb{R}$ and the range is $\{x \in \mathbb{R} : 0 < x < \pi\}$.

Therefore $0 < \cot^{-1} y < \pi$ for all $y \in \mathbb{R}$.

4.7. Algebraic operations on functions.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be functions on D.

(i) The sum function $f + g$ is defined on D by
$$(f+g)(x) = f(x) + g(x), x \in D.$$

(ii) The product function $f.g$ is defined on D by
$$f.g(x) = f(x)g(x), x \in D.$$

(iii) Let $k \in \mathbb{R}$. The function kf is defined on D by
$$kf(x) = k.f(x), x \in D.$$

(iv) If $g(x) \neq 0, x \in D$, the quotient f/g is defined on D by
$$f/g(x) = f(x)/g(x), x \in D.$$

Note. If f and g be two functions on $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ respectively then the function $f + g$ and $f.g$ are defined on $A \cap B$. If $g(x) \neq 0, x \in D \subset A \cap B$, then f/g is defined on D.

Examples.

1. Let $f(x) = \sqrt{x}, x \geq 0; g(x) = x, x \in \mathbb{R}$.
Then $(f+g)(x) = \sqrt{x} + x, x \geq 0. fg(x) = x\sqrt{x}, x \geq 0$.
$g(x) \neq 0$ on $\{x \in \mathbb{R} : x \neq 0\}$. $\therefore$ $f/g(x) = \frac{\sqrt{x}}{x}, x > 0$.

2. Let $f(x) = \sqrt{x}, x \geq 0; g(x) = \sqrt{x-1}, x \geq 1$.
Then $(f+g)(x) = \sqrt{x} + \sqrt{x-1}, x \geq 1$
$f.g(x) = \sqrt{x}.\sqrt{x-1}, x \geq 1$
$f/g(x) = \frac{\sqrt{x}}{\sqrt{x-1}}, x > 1$.

Note. If $h(x) = \sqrt{x(x-1)}$, then the domain of h is $\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > 1\}$. Therefore $f.g \neq h$.

If $h(x) = \sqrt{\frac{x}{x-1}}$, then the domain of h is $\{x \in \mathbb{R} : x < 0\} \cup \{x \in \mathbb{R} : x > 1\}$. Therefore $f/g \neq h$.

4.8. Monotone functions.

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **monotone increasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

$f : I \to \mathbb{R}$ is said to be **monotone decreasing** on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *monotone* on I if f is either monotone increasing or monotone decreasing on I.

A function $f : I \to \mathbb{R}$ is said to be *strictly increasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

$f : I \to \mathbb{R}$ is said to be *strictly decreasing* on I if $x_1, x_2 \in I$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

A function $f : I \to \mathbb{R}$ is said to be *strictly monotone* on I if f is either strictly increasing or strictly decreasing on I.

Let $I = [a, b]$ be a closed and bounded interval.
A function $f : I \to \mathbb{R}$ is said to be monotone increasing on I if $x_1, x_2 \in I$ and $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) \leq f(x_2)$.

Similar definitons for a monotone decreasing function.

Let I be an interval and $f : I \to \mathbb{R}, g : I \to \mathbb{R}$ are both monotone increasing (decreasing) on I. Then

(i) $f + g$ is monotone increasing (decreasing) on I.

(ii) if $k \in \mathbb{R}$ and $k > 0, kf$ is monotone increasing (decreasing) on I.

(iii) if $k \in \mathbb{R}$ and $k < 0, kf$ is monotone decreasing (increasing) on I.

Examples.

1. Let $f(x) = 1 - x, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly decreasing on $\mathbb{R}$.

2. Let $f(x) = x^2, x \in \mathbb{R}$.
$x_1, x_2 \in \mathbb{R}$ and $0 \leq x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2 \leq 0 \Rightarrow f(x_1) > f(x_2)$.
Therefore f is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$.

3. Let $f(x) = \operatorname{sgn} x, x \in [-1, 1]$.
$x_1 < 0, x_2 < 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
$x_1 < 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
$x_1 > 0, x_2 > 0$ and $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$.
Therefore f is monotone increasing on $[-1, 1]$.

4.9. Even function, odd function.

For $a \in \mathbb{R}^*$, let D be the symmetric interval $(-a, a)$.

A function $f : D \to \mathbb{R}$ is said to be an *even* function if $f(-x) = f(x)$ for all $x \in D$.

A function $f : D \to \mathbb{R}$ is said to be an *odd* function if $f(-x) = -f(x)$ for all $x \in D$.

For example,the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2, f(x) = \cos x$ are even functions on $\mathbb{R}$ and defined by $f(x) = x, f(x) = \operatorname{sgn} x, f(x) = \sin x$ are odd functions on $\mathbb{R}$.

If f be an odd function on $(-a, a)$ then $f(0) = 0$.

Let f be an odd function on $(-a, a)$, for some $a \in \mathbb{R}^*$. If $(x, f(x))$ be a point on the graph of f then $(-x, -f(x))$ is also a point on the graph. It follows that the graph of f is symmetrical about the origin.

Let f be an even function on $(-a, a)$, for some $a \in \mathbb{R}^*$. If (x, y) be a point on the graph of f then $(-x, y)$ is also a point on the graph. It follows that the graph of f is symmetrical about the y axis.

4.10. Power functions.

A. Positive Integral powers.

Case 1. Let n be an even positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $[0, \infty)$.

f is not injective on $\mathbb{R}$ since $f(c) = f(-c)$ for all $c \in \mathbb{R}$.

Let $x_1, x_2 \in [0, \infty)$ and $0 \le x_1 < x_2$. Then $f(x_1) < f(x_2)$. f is a strictly increasing function on $[0, \infty)$.

Let $x_1, x_2 \in (-\infty, 0]$ and $x_1 < x_2 \le 0$. Then $f(x_1) > f(x_2)$. f is a strictly decreasing function on $(-\infty, 0]$.

If we restrict the domain of f to $[0, \infty)$, then the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n, x \in [0, \infty)$ is a strictly increasing function on $[0, \infty)$ and therefore f is injective on $[0, \infty)$.

For each $y \in (0, \infty)$ there exists a unique $x \in (0, \infty)$ such that $x^n = y$ [2.4.23, worked Ex 8]. This together with $f(0) = 0$ shows that f is surjective.

Therefore f is a bijective function and the inverse function f^{-1} is defined by $f^{-1}(x) = x^{\frac{1}{n}}, x \in [0, \infty)$.

This inverse function is called the *nth rootfunction* (n even positive integer) and the domain of this function is $[0, \infty)$.

Case 2. Let n be an odd positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $\mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$.

Therefore f is a strictly increasing function on $\mathbb{R}$. Consequently f is injective.

For each $y \in (0, \infty)$ there exists a unique $x > 0$ such that $x^n = y$ [2.4.23, worked Ex 8].

f is an odd function. Hence for each $y \in (-\infty, 0)$ there exists a unique $x < 0$ such that $x^n = y$. Also $f(0) = 0$.

Thus for each $y \in \mathbb{R}$ there is a unique $x \in \mathbb{R}$ such that $x^n = y$. Consequently, f is surjective.

Since f is a bijective function, the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined by $f^{-1}(x) = x^{\frac{1}{n}}, x \in \mathbb{R}$.

This inverse function is called the *nth rootfunction* (n odd positive integer) and the domain of this function is $\mathbb{R}$.

B. Negative integral powers.

Let n be a positive integer. We define $x^{-n} = (\frac{1}{x})^n$ for all $x \neq 0$.

C. Rational powers.

In A, we have defined the nth root function $x^{\frac{1}{n}}$ (n even positive integer) for all $x \geq 0$ and the nth root function $x^{\frac{1}{n}}$ (n odd positive integer) for all $x \in \mathbb{R}$.

Therefore for all positive integers n, the nth root function $x^{\frac{1}{n}}$ is defined for all real $x \geq 0$ and $x^{\frac{1}{n}} \geq 0$ for all real $x \geq 0$.

Let r be a positive rational number, say $\frac{p}{q}$, where $p \in \mathbb{N}, q \in \mathbb{N}$.

We define $x^r = x^{\frac{p}{q}} = (x^{\frac{1}{q}})^p$ for all $x \geq 0$; and $x^{-r} = (\frac{1}{x})^r$ for all $x > 0$. Also we define $x^0 = 1$ for all $x > 0$.

Thus for *all rational* n, the power function $f(x) = x^n$ is defined for all real $x > 0$.

D. Irrational powers.

The power function for irrational powers is defined in terms of exponential functions discussed in 4.11.

Theorem 4.10.1. If $x \in \mathbb{R}, x > 0$ and r be a positive rational number p/q then $x^r = (x^p)^{1/q}$.

By definition, $x^r = (x^{1/q})^p$.

$x^{1/q} > 0$, since $x > 0$. Let $y = x^r$. Then $y = (x^{1/q})^p$.

$$\begin{aligned} y^q = \{(x^{1/q})^p\}^q &= (x^{1/q})^{pq}, \text{ since } p, q \text{ are positive integers} \\ &= (x^{1/q})^{qp} \\ &= \{(x^{1/q})^q\}^p = x^p. \end{aligned}$$

Since $x^p > 0$, $y = (x^p)^{1/q}$. That is, $(x^{1/q})^p = (x^p)^{1/q}$.

Note. The theorem says that it is immeterial whether we define $x^{p/q}$ by $(x^{1/q})^p$ or by $(x^p)^{1/q}$.

Theorem 4.10.2. If $x \in \mathbb{R}, x > 0$ and r be a positive rational number and $r = p/q = m/n$ where p, q, m, n are natural numbers then
$x^r = (x^{1/q})^p = (x^{1/n})^m$.

Here $np = qm$. Let $y = (x^{1/q})^p$.

Then $y^n = \{(x^{1/q})^p\}^n = (x^{1/q})^{pn} = (x^{1/q})^{qm} = \{(x^{1/q})^q\}^m = x^m$.
Therefore $y = (x^m)^{1/n} = (x^{1/n})^m$, by the previous theorem.
That is, $(x^{1/q})^p = (x^{1/n})^m$.

Note. The theorem says that although r can be written in many ways, the definition of x^r is unambiguous.

Theorem 4.10.3. If $x \in \mathbb{R}, x > 0$ and r, s are rational numbers then
(i) $x^r.x^s = x^{r+s}$; (ii) $(x^r)^s = x^{rs}$.

Proof left to the reader.

Theorem 4.10.4. If $x, y \in \mathbb{R}$ and $x > 0, y > 0$ and r is a rational number then
(i) $(xy)^r = x^r y^r$; (ii) $(\frac{x}{y})^r = \frac{x^r}{y^r}$.

Proof left to the reader.

Theorem 4.10.5. If $x \in \mathbb{R}, x > 0$ and r, s are rational numbers, then
(i) $x^r > 0$;
(ii) for $x > 1$, $x^r < x^s$ if $r < s$; and for $0 < x < 1$, $x^r > x^s$ if $r < s$.

Proof. (i) Let $r = \frac{p}{q}$ where $q \in \mathbb{N}, p \in \mathbb{Z}$. Then $x^r = (x^{\frac{1}{q}})^p = y^p$, say.
$x > 0 \Rightarrow x^{\frac{1}{q}} > 0$, i.e., $y > 0$ and $y > 0 \Rightarrow y^p > 0$, i.e., $x^r > 0$.
Therefore $x^r > 0$.

(ii) $x > 1 \Rightarrow \cdots < x^{-2} < x^{-1} < 1 < x < x^2 < \cdots$
It follows that $x^m < x^n$ if m, n are integers and $m < n$.

Let $r = \frac{u}{q}, s = \frac{v}{q}$ where u, v, q are integers and $q > 1$. $r < s \Rightarrow u < v$.

$x > 1 \Rightarrow x^{\frac{1}{q}} > 1 \Rightarrow (x^{\frac{1}{q}})^u < (x^{\frac{1}{q}})^v$, since $u < v$.
That is, $x^r < x^s$ if $r < s$.

Second part. $0 < x < 1 \Rightarrow \frac{1}{x} > 1$.

$$\begin{aligned} \tfrac{1}{x} > 1 \text{ and } r < s \quad &\Rightarrow \quad (\tfrac{1}{x})^r < (\tfrac{1}{x})^s, \text{ by the first part} \\ &\Rightarrow \quad x^r > x^s. \end{aligned}$$

This completes the proof.

4.11. Exponential function.

We define a^x where a and x are real numbers and $a > 0$.

Case 1. Let $a > 1, x \in \mathbb{R}$.

If x be a rational number, a^x is already defined. We define a^x when x is irrational.

There exist rational numbers r and s such that $r < x < s$.

Since $a > 1$ and $r < s$, we have $a^r < a^s$.

Let $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

S is a non-empty subset of $\mathbb{R}$ having a^s as an upper bound. By the supremum property of $\mathbb{R}$, $\sup S$ exists.

We define $a^x = \sup S$.

Case 2. Let $0 < a < 1$, $x \in \mathbb{R}$.

We define $a^x = b^{-x}$ where $b = \frac{1}{a} > 1$.

Case 3. Let $a = 1, x \in \mathbb{R}$.

In this case $a^x = 1$.

Theorem 4.11.1. If $a \in \mathbb{R}, a > 0$ and $x \in \mathbb{R}$ then $a^x > 0$.

Proof. When $x \in \mathbb{Q}$, this reduces to the Theorem 4.10.5 (i).

We prove the theorem when x is irrational.

Case 1. $a > 1$.

$a^x = \sup S$ where $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

Each element of S is positive by Theorem 4.10.5 (i) and a^x being the supremum of S must be positive.

Thus $a^x > 0$ for all irrational x.

Case 2. $0 < a < 1$.

$a^x = b^{-x}$ where $b = \frac{1}{a} > 1$.

Since $b > 1$ and $-x$ is irrational, $b^{-x} > 0$ by case 1.

Thus $a^x > 0$ for all irrational x.

Case 3. $a = 1$.

Then $a^x = 1 > 0$.

Combining all cases, the theorem is done.

Theorem 4.11.2. If $a \in \mathbb{R}, a > 0, x \in \mathbb{R}$ and r, s are rational numbers such that $r < x < s$, then

(i) for $a > 1, a^r < a^x < a^s$; and (ii) for $0 < a < 1, a^r > a^x > a^s$.

Proof. (i) **Case 1.** $x \in \mathbb{Q}$.

$r < x < s \Rightarrow a^r < a^x < a^s$, by Theorem 4.10.5 (ii).

Case 2. x is irrational.

We have $a^x = \sup S$ where $S = \{a^r : r \in \mathbb{Q} \text{ and } r < x\}$.

First we prove that $a^x \notin S$. If not, let $a^x \in S$. Then $a^x = a^{r_1}$ for some rational number $r_1 < x$. By Density property of $\mathbb{R}$, there exists $r_2 \in \mathbb{Q}$ such that $r_1 < r_2 < x$.

Then $a^{r_2} \in S$, by the definition of S and $a^{r_1} < a^{r_2}$, by Theorem 4.10.5 (ii). That is, $a^x < a^{r_2}$ and this contradicts that $a^x = \sup S$.

Therefore $a^x \notin S$.

$$\begin{aligned} r < x \quad &\Rightarrow \quad a^r \in S \\ &\Rightarrow \quad a^r < a^x \text{ since } a^x = \sup S \text{ and } a^x \notin S \ldots \qquad \ldots\text{(i)} \end{aligned}$$

$$\begin{aligned} x < s \quad &\Rightarrow \quad s > r \text{ for all rational } r < x \\ &\Rightarrow \quad a^s > a^r \text{ for all rational } r < x \\ &\Rightarrow \quad a^s \text{ is an upper bound of } S \ldots \qquad \ldots\text{(ii)} \\ &\Rightarrow \quad a^s \geq a^x, \text{ since } a^x = \sup S. \end{aligned}$$

If possible, let $a^s = a^x$. By Density property of $\mathbb{R}$, there exists $s_1 \in \mathbb{Q}$ such that $x < s_1 < s$.

$$\begin{aligned} s_1 < s \quad &\Rightarrow \quad a^{s_1} < a^s \text{ by Theorem 4.10.5 (ii)} \\ &\Rightarrow \quad a^{s_1} < \sup S, \text{ since } a^s = a^x = \sup S. \end{aligned}$$

Again $x < s_1 \quad \Rightarrow \quad a^{s_1}$ is an upper bound of S, by (ii).

We arrive at a contradiction. Therefore $a^x < a^s \quad \ldots \qquad \ldots$(iii)

From (i) and (iii) $\quad a^r < a^x < a^s$.

(ii) $0 < a < 1$.

$0 < a < 1 \Rightarrow \frac{1}{a} > 1$. By (i) $(\frac{1}{a})^r < (\frac{1}{a})^x < (\frac{1}{a})^s$.

Therefore $a^s < a^x < a^r$.

This completes the proof.

Theorem 4.11.3. If $a \in \mathbb{R}, a > 0$ and $x_1, x_2 \in \mathbb{R}$, then

(i) for $a > 1, a^{x_1} < a^{x_2}$ if $x_1 < x_2$;

(ii) for $0 < a < 1, a^{x_1} > a^{x_2}$ if $x_1 < x_2$.

Proof. (i) Let r_1, r_2 be rational numbers such that $r_1 < x_1 < r_2 < x_2$. Also $a^{r_2} < a^{x_2}$, by Theorem 4.11.2.

Then $a^{r_1} < a^{x_1} < a^{r_2}$, by Theorem 4.11.2. Therefore $a^{x_1} < a^{x_2}$.

(ii) Similar proof.

Theorem 4.11.4. If $a \in \mathbb{R}, a > 0$ and $x, y \in \mathbb{R}$ then $a^{x+y} = a^x \cdot a^y$.

Proof. **Case 1.** $x, y \in \mathbb{Q}$.

In this case the theorem reduces to Theorem 4.10.3.

Case 2. x, y are both irrational.

Subcase (i) $a > 1$.

$a^x = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x\}$, $a^y = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < y\}$.
Since $x, y \in \mathbb{R}$, $a^x > 0$ and $a^y > 0$.

Let us choose ϵ such that $0 < \epsilon < \min\{a^x(a^x + a^y), a^y(a^x + a^y)\}$.

There exist rational numbers p, q such that $p < x, q < y$ and $0 < a^x - \frac{\epsilon}{A} < a^p < a^x$ and $0 < a^y - \frac{\epsilon}{A} < a^q < a^y$, where $A = a^x + a^y > 0$.

Therefore $(a^x - \frac{\epsilon}{A})(a^y - \frac{\epsilon}{A}) < a^p.a^q < a^x.a^y$
or, $a^x.a^y - \epsilon < a^{p+q} < a^x.a^y$.
$p + q < x + y \Rightarrow a^{p+q} < a^{x+y}$, by Theorem 4.11.3.
Therefore $a^x.a^y - \epsilon < a^{x+y}$ ... ... (i)

$a^{x+y} = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x + y\}$.
Let $\epsilon > 0$. There exists a rational number s such that $s < x + y$ and $a^{x+y} - \epsilon < a^s < a^{x+y}$.

Let $x+y-s = 2k$. Let $u, v \in \mathbb{Q}$ such that $x-k < u < x, y-k < v < y$. Then $s < u + v < x + y$.

$u < x, v < y \Rightarrow a^u < a^x, a^v < a^y \Rightarrow a^{u+v} < a^x.a^y$
$s < u + v \Rightarrow a^{x+y} - \epsilon < a^s < a^{u+v}$.
Therefore $a^{x+y} - \epsilon < a^x.a^y$ ... ... (ii)

From (i) and (ii) $a^{x+y} - \epsilon < a^x.a^y < a^{x+y} + \epsilon$.

As ϵ is arbitrary, it follows that $a^{x+y} = a^x.a^y$.

Subcase (ii) $0 < a < 1$.

In this case $\frac{1}{a} > 1$ and $(\frac{1}{a})^{x+y} = (\frac{1}{a})^x.(\frac{1}{a})^y$, by subcase (i)
or, $a^{x+y} = a^x.a^y$.

Subcase (iii) $a = 1$. In this case $a^{x+y} = 1$, $a^x = 1, a^y = 1$ and hence $a^{x+y} = a^x.a^y$.

Case 3. One of x, y is rational and the other is irrational. Let $x \in \mathbb{Q}, y \in \mathbb{R} - \mathbb{Q}$. Then $x + y$ is irrational.

Subcase (i) $a > 1$.

$a^{x+y} = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < x + y\}$.

Let $\epsilon > 0$. There exists a rational number s such that $s < x + y$ and $a^{x+y} - \epsilon < a^s < a^{x+y}$.

Let $x + y - s = k$. Let $u \in \mathbb{Q}$ such that $y - k < u < y$.
Then $s < x + u < x + y$.
$u < y \Rightarrow a^u < a^y$.

$$\begin{aligned} s < x+u \quad &\Rightarrow \quad a^s < a^{x+u} \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^{x+u} \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^x.a^u \\ &\Rightarrow \quad a^{x+y} - \epsilon < a^x.a^y \ldots \quad \ldots \text{(i)} \end{aligned}$$

$a^y = \sup\{a^r : r \in \mathbb{Q} \text{ and } r < y\}$.

There exists a rational number p such that $p < y$ and $a^y - \frac{\epsilon}{a^x} < a^p < a^y$.

Therefore $a^x(a^y - \frac{\epsilon}{a^x}) < a^x.a^p < a^x.a^y$
or, $a^x.a^y - \epsilon < a^{x+p} < a^x.a^y$

$x + p < x + y \Rightarrow a^{x+p} < a^{x+y}$.
Therefore $a^x.a^y - \epsilon < a^{x+y}$.
or, $a^x.a^y < a^{x+y} + \epsilon \ldots \quad \ldots$ (ii)

From (i) and (ii) $a^{x+y} - \epsilon < a^x.a^y < a^{x+y} + \epsilon$.
Since ϵ is arbitrary, $a^{x+y} = a^x.a^y$.

Subcase (ii) $0 < a < 1$.
Similar proof.

Subcase (iii) $a = 1$.
Similar proof.

Combining all cases, the proof is complete.

Definition. If $a \in \mathbb{R}$ and $a > 0$ the function $f(x) = a^x, x \in \mathbb{R}$ is called the **exponential function**. The domain of the exponential function is $\mathbb{R}$.

When $a = 1$, $f(x) = 1$ for all $x \in \mathbb{R}$.

When $a > 1$, the exponential function is a strictly increasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

When $0 < a < 1$, the exponential function is a strictly decreasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

In particular, the exponential function $f(x) = e^x, x \in \mathbb{R}$ is a strictly increasing function on $\mathbb{R}$, since $e > 1$. The range of the function is $(0, \infty)$.

4.12. Logarithmic function.

Let $a \in \mathbb{R}, a > 1$.
In this case the function $f(x) = a^x, x \in \mathbb{R}$ is strictly increasing on $\mathbb{R}$. The range of the function is $(0, \infty)$.
Therefore the function is bijective and the inverse function $f^{-1} : (0, \infty) \to \mathbb{R}$ exists.

Let $a \in \mathbb{R}, 0 < a < 1$.

In this case, the function $f(x) = a^x, x \in \mathbb{R}$ is strictly decreasing on $\mathbb{R}$. The range of the function is $(0, \infty)$.

Therefore the function is bijective and the inverse function f^{-1} : $(0, \infty) \to \mathbb{R}$ exists.

In both the cases the inverse function is called the *logarithmic function* and it is denoted by $\log_a x$. a is called the *base* of the logarithmic function. The logarithmic function is a monotone function on $(0, \infty)$ (monotone increasing if $a > 1$, monotone decreasing if $0 < a < 1$) and the range of this function is $\mathbb{R}$.

Also we have $\log_a(a^x) = x$ for all $x \in \mathbb{R}$ and $a^{\log_a x} = x$ for $x > 0$.

In particular, the inverse function $\log_e x$ is called the **natural logarithmic** function. The domain of the function is $(0, \infty)$ and the range of the function is $\mathbb{R}$.

Also we have $\log_e(e^x) = x$ for all $x \in \mathbb{R}$ and $e^{\log_e x} = x$ for all $x > 0$.

[The base e in the natural logarithmic function is often dropped and it is expressed as $\log x$.]

Remark. The exponential function a^x is defined on $\mathbb{R}$ for all real $a > 0$.

For $a = 1$, the function a^x is a constant function.

For $a > 1$, the function a^x is a strictly increasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

For $0 < a < 1$, the function a^x is a strictly decreasing function on $\mathbb{R}$. The range of the function is $(0, \infty)$.

For $a > 0 (\neq 1)$ the exponential function admits of an inverse function (called the logarithmic function) $\log_a x$ on $(0, \infty)$. The range of the logarithmic function is $\mathbb{R}$.

For $a > 1$, the logarithmic function $\log_a x$ is a strictly increasing function on $(0, \infty)$.

For $0 < a < 1$, the logarithmic function $\log_a x$ is a strictly decreasing function on $(0, \infty)$.

In particular, when $a = e$, the logarithmic function $\log_e x$ is called the natural logarithmic function.

Theorem 4.12.1. If $a \in \mathbb{R}, a > 0 (\neq 1)$ and $x, y \in \mathbb{R}, x > 0, y > 0$, then $\log_a x + \log_a y = \log_a xy$.

Proof. Since $x > 0, y > 0$ and $a > 0 (\neq 1), \log_a x \in \mathbb{R}, \log_a y \in \mathbb{R}$.

$$\begin{aligned} a^{\log_a x + \log_a y} &= a^{\log_a x}.a^{\log_a y}, \text{ by Theorem 4.11.4} \\ &= xy. \end{aligned}$$

Since $xy > 0$ and $a^{\log_a x + \log_a y} = xy$ it follows from the property of the inverse function that

$\log_a xy = \log_a x + \log_a y$.

In particular, $\log_e xy = \log_e x + \log_e y$.

Corollary. $\log_a \frac{1}{x} = -\log_a x$.

Theorem 4.12.2. If $a \in \mathbb{R}, a > 0 (\neq 1)$ and $x \in \mathbb{R}, x > 0$ then $\log_a x^n = n \log_a x$ if n be a rational number.

Proof left to the reader.

In particular, $\log_e x^n = n \log_e x$.

Definition. If $x \in \mathbb{R}, x > 0$ and $\alpha \in \mathbb{R}$ we define the **power function** x^α by $x^\alpha = e^{\alpha \log x}, x > 0$.

This definition is consistent with the definition of the power function for rational α. Because, if $\alpha = \frac{m}{n}$ where $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $x > 0$ then $\log_e x^{\frac{m}{n}} = \frac{m}{n} \log_e x$.

By the property of the inverse function, $x^\alpha = x^{\frac{m}{n}} = e^{\log_e x^{\frac{m}{n}}} = e^{\frac{m}{n} \log_e x} = e^{\alpha \log_e x}$.

Theorem 4.12.3. If $x > 0$ and $\alpha, \beta \in \mathbb{R}$ then

(i) $x^{\alpha+\beta} = x^\alpha . x^\beta$;

(ii) $(x^\alpha)^\beta = x^{\alpha\beta}$;

(iii) $x^\alpha > 0$;

(iv) for $x > 1, x^\alpha < x^\beta$ if $\alpha < \beta$; and for $0 < x < 1, x^\alpha > x^\beta$ if $\alpha < \beta$

(v) $\log_e x^\alpha = \alpha \log_e x$.

Proof. (v) $\log_e x^\alpha = \log_e(e^{\alpha \log_e x})$, by definition
$= \alpha \log_e x$, since $\log_e e^x = x$ for all $x \in \mathbb{R}$.

Proofs for other parts left to the reader.

The general exponential function a^x can be expressed in terms of the exponential function e^x by $a^x = e^{x \log_e a}$.

For $a \in \mathbb{R}, x \in \mathbb{R}$ and $a > 0$, we have $a^x > 0$.

Therefore for all $x \in \mathbb{R}$, $a^x = e^{\log_e (a^x)}$, since $e^{\log_e x} = x$ for all $x > 0$.

Since $a \in \mathbb{R}, a > 0$ and $x \in \mathbb{R}, \log_e(a^x) = x \log_e a$, by Theorem 4.12.3 (v).

Consequently, $a^x = e^{x \log_e a}$.

4.13. Hyperbolic functions.

The hyperbolic functions $\sinh x$ and $\cosh x$ are defined by $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

The other hyperbolic functions $\tanh x$, $\coth x$, $\operatorname{cosech} x$ and $\operatorname{sech} x$ are defined by $\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, $\operatorname{sech} x = \frac{1}{\cosh x}$.

Domain and Range:

sinh x **:** Since the domains of e^x and e^{-x} are both $\mathbb{R}$, the domain of $\sinh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $e^x - e^{-x} = 2y$
or, $e^{2x} - 2ye^x - 1 = 0$.
Therefore $e^x = y \pm \sqrt{y^2 + 1}$.
Since $e^x > 0$ for all real $x, x = \log(y + \sqrt{y^2+1}) \in \mathbb{R}$.
So the range of the function $\sinh x$ is $\mathbb{R}$.

cosh x **:** Since the domains of e^x and e^{-x} are both $\mathbb{R}$, the domain of $\cosh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $e^x + e^{-x} = 2y$
or, $e^{2x} - 2ye^x + 1 = 0$.
Therefore $e^x = y \pm \sqrt{y^2 - 1}$.
Since $e^x > 0$ for all real x, $y \geq 1$ and $x = \log(y \pm \sqrt{y^2 - 1})$ shows that, there are two pre-images of y.
So the range of the function $\cosh x$ is $\{y \in \mathbb{R} : y \geq 1\}$.

tanh x **:** Since $\cosh x \geq 1$, the domain of $\tanh x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$
or, $e^{2x} - 1 = (e^{2x} + 1)y$.
Therefore $e^{2x} = \frac{y+1}{1-y}, y \neq 1$.
But $e^{2x} > 0$ for all real x. $-1 < y < 1$.
So the range of the function $\tanh x$ is $\{y \in \mathbb{R} : -1 < y < 1\}$.

coth x **:** $\sinh x = 0$ gives $x = 0$.

So the domain of $\coth x$ is $\{x \in \mathbb{R} : x \neq 0\}$.
Since $-1 < \tanh x < 1$, therefore $|\coth x| > 1$.
So the range of the function $\coth x$ is $\{y \in \mathbb{R} : |y| > 1\}$.

sech x **:** Since $\cosh x \geq 1$ for all real x, the domain of $\operatorname{sech} x$ is $\mathbb{R}$.

Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.

Then $\frac{2}{e^x+e^{-x}} = y$
or, $ye^{2x} - 2e^x + y = 0$.
Therefore $e^x = \frac{1\pm\sqrt{1-y^2}}{y}, y \neq 0$.
Since $e^{2x} > 0$ for all real $x, 0 < y \leq 1$.
So the range of the function sech x is $\{y \in \mathbb{R} : 0 < y \leq 1\}$.

cosech x : sinh $x = 0$ gives $x = 0$.
So the domain of cosech x is $\{x \in \mathbb{R} : x \neq 0\}$.
Let $y \in \mathbb{R}$, the co-domain set and let x be a pre-image of y.
Then $\frac{2}{e^x-e^{-x}} = y$
or, $ye^{2x} - 2e^x - y = 0$.
Therefore $e^x = \frac{1\pm\sqrt{y^2+1}}{y}, y \neq 0$.

$$\text{Since } e^x > 0 \text{ for all real } x, x = \log \frac{1+\sqrt{y^2+1}}{y}, \text{ for } y > 0$$
$$= \log \frac{1-\sqrt{y^2+1}}{y}, \text{ for } y < 0.$$

So the range of the function cosech x is $\{y \in \mathbb{R} : y \neq 0\}$.

Properties.

1. $\cosh^2 x - \sinh^2 x = 1, x \in \mathbb{R}$
$\text{sech}^2 x = 1 - \tanh^2 x, x \in \mathbb{R}$
$\text{cosech}^2 x = \coth^2 x - 1, x \neq 0$.

2. $\sinh(x + y) = \sinh x \cosh x + \cosh x \sinh y$
$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
$\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$.

4.14. Bounded function.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. f is said to be *bounded above* on D if there exists a real number B such that $f(x) \leq B$ for all $x \in D$. B is said to be *an upper bound* of f on D. f is said to be *bounded below* on D if there exists a real number b such that $f(x) \geq b$ for all $x \in D$. b is said to be a *lower bound* of f on D.

f is said to be **bounded** on D if f is bounded above as well as bounded below on D.

In other words, f is bounded on D if the range set $f(D)$ be a bounded set in $\mathbb{R}$.

f is said to be *unbounded* on D if f is either *unbounded* above or *unbounded* below or both.

Let f be bounded above on D. Then the range set $f(D) = \{f(x) : x \in D\}$ is a non-empty subset of $\mathbb{R}$ bounded above. Therefore by the supremum property of $\mathbb{R}$, the subset $f(D)$ has a least upper bound M. M is called the *supremum* of f on D and is expressed as $M = \sup\limits_{x \in D} f(x)$.

Similarly, if f be bounded below on D there exists a real number m which is called the *infimum* of f on D and is expressed as $m = \inf\limits_{x \in D} f(x)$.

Let f be a bounded function on D. Then M, the supremum of f on D, satisfies the following conditions :

(i) $f(x) \leq M$ for all $x \in D$,

(ii) for each pre-assigned positive ϵ there exists an element y in D such that $M - \epsilon < f(y) \leq M$.

Also m, the infimum of f on D, satisfies the following conditions:

(i) $f(x) \geq m$ for all $x \in D$,

(ii) for each pre-assigned positive ϵ there exists an element y in D such that $m \leq f(y) < m + \epsilon$.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}, x \in \mathbb{R}$. Then f is bounded above as well as bounded below and $\sup\limits_{x \in \mathbb{R}} f(x) = 1$, $\inf\limits_{x \in \mathbb{R}} f(x) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x, x \in \mathbb{R}$. Then f is unbounded above but bounded below and $\inf\limits_{x \in \mathbb{R}} f(x) = 0$.

Definition. Let $f : D \to \mathbb{R}$ be bounded on D. Then $\sup\limits_{x \in D} f(x) - \inf\limits_{x \in D} f(x)$ is said to be the *oscillation* of f on D.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let $c \in D$. f is said to be *bounded at c* if there exists a neighbourhood $N(c)$ of c such that f is bounded on $N(c) \cap D$.

If f be bounded on D then it follows from the definition that f is bounded at each point of D. But f can be bounded at each point of D without being bounded on D.

For example, the function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is bounded at each point of the interval $(0,1)$, but is not bounded on the interval $(0,1)$.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is bounded at each point of $\mathbb{R}$, but is not bounded on $\mathbb{R}$.

The following theorem gives a condition for which boundedness of a function f at each point of a set implies boudedness on the whole set.

Theorem 4.14.1. If a function $f : D \to \mathbb{R}$ be bounded at each point of D and D is a closed and bounded set in $\mathbb{R}$, then f is bounded on D.

Proof. Let $x \in D$. Since f is bounded at each point of D, there is a neighbourhood $N(x)$ of x such that f is bounded on $N(x) \cap D$.

Let us consider the collection of neighbourhoods $\mathcal{G} = \{N(x) : x \in D\}$ such that f is bounded on $N(x) \cap D$. Clearly, $\mathcal{G}$ is an open cover of D. Since D is a closed and bounded set in $\mathbb{R}$, by Heine-Borel theorem, there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers D.

Let $\mathcal{G}' = \{N(x_1), N(x_2), \ldots, N(x_m)\}$. Then $D \subset N(x_1) \cup N(x_2) \cup \ldots \cup N(x_m)$ and f is bounded on $D \cap N(x_i)$ for $i = 1, 2, \ldots, m$.

So there exists a positive M_i such that $|f(x)| \le M_i$ for all $x \in D \cap N(x_i)$ and this holds for $i = 1, 2, \ldots, m$.

Let $M = \max\{M_1, M_2, M_3, \ldots, M_m\}$.

Let $x \in D$. Then $x \in D \cap N(x_k)$ for some $k \in \{1, 2, \ldots, m\}$ and therefore $|f(x)| \le M_k \le M$.

This proves that f is bounded on D and this completes the proof.

Note. In particular, if f be bounded at each point of a closed and bounded interval $[a, b]$, then f is bounded on $[a, b]$.

Exercises 6

1. Determine the domain of the real function f.

(i) $f(x) = \log \frac{2+x}{2-x}$, (ii) $f(x) = \cos^{-1} \frac{2x}{1+x}$, (iii) $f(x) = \sqrt{2 + x - x^2}$,

(iv) $f(x) = \sqrt{-x} + \frac{1}{\sqrt{2+x}}$, (v) $f(x) = \log \sin x$, (vi) $f(x) = \frac{x}{[x+1]}$.

2. Show that the pair of functions f and g are not equal by specifying their domains.

(i) $f(x) = \sqrt{x(x-3)}$; $g(x) = \sqrt{x}.\sqrt{x-3}$

(ii) $f(x) = \log x^2$; $g(x) = 2 \log x$

(iii) $f(x) = \sqrt{\frac{x}{x-3}}$; $g(x) = \frac{\sqrt{x}}{\sqrt{x-3}}$

(iv) $f(x) = \log \frac{x+3}{x}$; $g(x) = \log(x+3) - \log x$.

Show that in each of the cases, g is a restriction of f.

3. Determine which of the following functions are even and which are odd.

(i) $f(x) = \log \frac{1+x}{1-x}$, $x \in (-1, 1)$ (ii) $f(x) = \log(x + \sqrt{1 + x^2})$, $x \in \mathbb{R}$

(iii) $f(x) = \sqrt[3]{(x+1)^2} + \sqrt[3]{(x-1)^2}$, $x \in \mathbb{R}$

(iv) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$, $x \in \mathbb{R}$.

4. Prove that every function $f : D \to \mathbb{R}$, where D is a symmetric interval (i.e., $x \in D \Rightarrow -x \in D$) can be expressed as the sum of an even and an odd function.

Express f as the sum of an even and an odd function, where

(i) $f(x) = \sqrt{1+x}$, $-1 \le x \le 1$, (ii) $f(x) = x + \sqrt{1+x^2}$, $x \in \mathbb{R}$.

5. A function $f : D \to \mathbb{R}$ is said to be a *periodic function* if there exists a positive real number p such that for all $n \in \mathbb{Z}$, $f(x+np) = f(x)$ holds in D.

The least positive p is said to be the *period* of f.

For example , (i) let $f(x) = \sin x, x \in \mathbb{R}$. Then $p = 2\pi$ since $\sin(x+2n\pi) = \sin x$ for all $n \in \mathbb{Z}$;

(ii) let $f(x) = x - [x], x \in \mathbb{R}$. Then $f(x+n) = f(x)$ for all $n \in \mathbb{Z}$, since for all real x, $[x+n] = [x] + n$, if n be an integer. Therefore $p = 1$.

Find the period of the periodic function f, where

(i) $f(x) = a\sin 3x + b\cos 3x, a, b \in \mathbb{R}$, (ii) $f(x) = \sqrt{\tan x}$,

(iii) $f(x) = \sin^2 x$.

6. Verify the following.

(i) The domain of the inverse function $\sinh^{-1} x$ is $\mathbb{R}$ and $\sinh^{-1} x = \log(x + \sqrt{x^2+1}), x \in \mathbb{R}$.

(ii) The domain of the inverse function $\cosh^{-1} x$ is $\{x \in \mathbb{R} : x \ge 1\}$ and $\cosh^{-1} x$ has two values $\pm\log(x + \sqrt{x^2-1}), x \ge 1$.

Note : $\cosh^{-1} x$ has two branches, the principle branch is given by $\cosh^{-1} x = \log(x + \sqrt{x^2-1}), x \ge 1$.

(iii) The domain of the inverse function $\tanh^{-1} x$ is $\{x \in \mathbb{R} : |x| < 1\}$ and $\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}, -1 < x < 1$.

(iv) The domain of the inverse function $\coth^{-1} x$ is $\{x \in \mathbb{R} : |x| > 1\}$ and $\coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1}, |x| > 1$.

(v) The domain of the inverse function $\operatorname{sech}^{-1}x$ is $\{x \in \mathbb{R} : 0 < x \le 1\}$ and $\operatorname{sech}^{-1}x = \log\frac{1+\sqrt{1-x^2}}{x}, 0 < x \le 1$.

(vi) The domain of the inverse function $\operatorname{cosech}^{-1}x$ is $\{x \in \mathbb{R} : x \ne 0\}$ and

$$\begin{aligned} \operatorname{cosech}^{-1}x &= \log\frac{1+\sqrt{x^2+1}}{x}, x > 0 \\ &= \log\frac{1-\sqrt{x^2+1}}{x}, x < 0. \end{aligned}$$