

# Real Analysis (UG1, Monsoon 2022)

Midsem Exam [15 marks]

A Wednesday!

## 1 Instructions

- You are allowed to bring at most one A4 sheet with only “handwritten” notes (no xerox, e-print, etc.).
- Give satisfactory reasoning. State clearly which theorem or axioms you are using.
- Please read questions carefully before you begin to answer. Turn both sides of the question paper.

## Question A [ $3 \times 2.5 = 7.5$ marks]

Let  $C[0, 1]$  be the set of functions  $f : [0, 1] \rightarrow \mathbb{R}$  such that functions  $f$  are continuous over  $[0, 1]$ . In other words,  $C[0, 1]$  is the set of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ . Consider the set  $C[0, 1]$  to be equipped with metric  $d_p$  defined for  $p \geq 1$  as

$$d_p(f, g) = \left( \int_0^1 |f(x) - g(x)|^p dx \right)^{\frac{1}{p}}.$$

Answer any 3 of the following questions.

- ✓ 1. Show that  $(C[0, 1], d_p)$  is a metric space (for  $p \geq 1$ ).
2. Consider  $p, q \geq 1$  and  $p \neq q$ . Are metrics  $d_p$  and  $d_q$  equivalent over  $C[0, 1]$ ? Provide satisfactory justification.
3. State a necessary and sufficient condition for a subset  $S \subset C[0, 1]$  to be compact.
- ✓ 4. Consider a mapping  $G : C[0, 1] \rightarrow \mathbb{R}$  defined by

$$G(f) = \int_0^1 |f(x)| dx.$$

Is  $G$  continuous? Provide satisfactory explanation.

## Question B [ $2.5 \times 3 = 7.5$ marks]

Answer any 3 of the following.

- ✓ 1. Prove that no order can be defined in the complex field  $(\mathbb{C}, +, \cdot)$  (the set of complex numbers with conventional addition and multiplication rules) that turns it into an ordered field.
- ✓ 2. Let  $A$  be a nonempty set of real numbers which is bounded from below. Let  $-A$  be the set of all numbers  $-x$ , where  $x \in A$ . Prove that

$$\inf A = -\sup(-A).$$

3. Construct a bounded set of real numbers with exactly three limit points.
- ✓ 4. Prove that the convergence of a series  $\sum_{n=1}^{\infty} a_n$ , where  $a_n \in \mathbb{R}$  for all  $n \in \mathbb{N}$ , implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \quad \text{if } a_n \geq 0.$$

5. If a series  $\sum_{n=1}^{\infty} a_n$  of real numbers converges, and if a sequence  $\{b_n\}$  of real numbers is monotonic and bounded, prove that the series  $\sum_{n=1}^{\infty} a_n b_n$  converges.