Real Analysis(H2) (MA4.101a)

IIIT-H, Semester Monsoon 22, Quiz 2

Date: 13th January 2023, Duration: 45 minutes

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1. Calculate the line integral of a vector field $\vec{V} = -y\hat{i} + x\hat{j}$ counterclockwise around a triangle with vertices $(0,0), (L,0), (0,L)$. [6]

2. You are given two vector fields

$$\vec{F}(x,y,z) = \hat{i}yz + \hat{j}xz + \hat{k}xy\,, \qquad \vec{G}(x,y,z) = -\hat{i}y + \hat{j}x + 0\hat{k}\,.$$

Determine in which of these vector fields a line integral around a closed curve will vanish. [2]

3. Given a vector field

$$\vec{F} = -\hat{i}\frac{y}{x^2+y^2} + \hat{j}\frac{x}{x^2+y^2}\,,$$

calculate the value of $\int (\vec{\nabla} \times \vec{F}).d\vec{S}$ over a circle of radius R centered at the origin in the $x-y$ plane with normal vector $\hat{k}$. *Hint: The circle includes the origin.* [5]

4. Given $r = \hat{i}x + \hat{j}y + \hat{k}z$, evaluate the surface integral $\oint \vec{r}.d\vec{S}$ over a right circular cylinder of radius R and length L. [3]

5. Prove that

$$\vec{\nabla} \times (\phi\vec{a}) = \vec{\nabla}\phi \times \vec{a} + \phi\vec{\nabla} \times \vec{a}\,,$$

where ϕ is a scalar and $\vec{a}$ is a vector. [5]

6. Evaluate the line integral [3]

$$I = \oint \Big[(e^x y + \cos x \sin y)dx + (e^x + \sin x \cos y)dy\Big]\,,$$

around an ellipse $x^2/a^2 + y^2/b^2 = 1$.

7. Determine the limits if they exist

   (a) $\lim_{z\to 0} \frac{\mathrm{Im}(z)}{z}$. [3]

   (b) $\lim_{z\to 0} \frac{\bar{z}^2}{z}$, where $\bar{z}$ indicates the complex conjugate of z. [3]

Useful formulas:

In cylindrical polar coordinate (ρ, ϕ, z), we have $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $z = z$, where $\rho \geq 0$, $0 \leq \varphi \leq 2\pi$, and $-\infty \leq z \leq \infty$. The unit vectors are $\hat{e}_\rho = \hat{i}\cos\phi + \hat{j}\sin\phi$, $\hat{e}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi$, and $\hat{e}_z = \hat{k}$.