5. SEQUENCE

5.1. Real Sequence.

A mapping $f : \mathbb{N} \to \mathbb{R}$ is said to be a *sequence in* $\mathbb{R}$, or a *real sequence.* The f-images $f(1), f(2), f(3), \ldots \ldots$ are real numbers. The image of the nth element, $f(n)$, is said to be the nth *element* (or the nth term) of the real sequence.

We shall be mainly concerned with real sequences and we shall use the term 'sequence' to mean a 'real sequence'.

A sequence f is generally denoted by the symbol $\{f(n)\}$. Also the symbol $\{f(1), f(2), f(3), \ldots \ldots\}$ is used to denote the sequence f.

The *range* of the real sequence $\{f(n)\}$ is a subset of $\mathbb{R}$, denoted by the symbol $\{f(n) : n \in \mathbb{N}\}$.

The symbols like $\{u_n\}, \{v_n\}, \{x_n\}$, etc. shall also be used to denote a sequence.

Examples.

1. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n, n \in \mathbb{N}$. Then $f(1) = 1, f(2) = 2, \ldots \ldots$ The sequence is denoted by $\{n\}$. It is also denoted by $\{1, 2, 3, \ldots \ldots\}$.

2. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n^2, n \in \mathbb{N}$. The sequence is $\{n^2\}$. It is also denoted by $\{1^2, 2^2, 3^2, \ldots \ldots\}$.

3. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{n}{n+1}, n \in \mathbb{N}$. The sequence is $\{\frac{n}{n+1}\}$. It is also denoted by $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \ldots\}$.

4. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n, n \in \mathbb{N}$. The sequence is $\{(-1)^n\}$. It is also denoted by $\{-1, 1, -1, \ldots \ldots\}$. The range of the sequence is $\{-1, 1\}$.

5. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \sin \frac{n\pi}{2}, n \in \mathbb{N}$. The sequence is $\{1, 0, -1, 0, 1, 0, \ldots \ldots\}$. The range of the sequence is $\{-1, 0, 1\}$.

6. Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $\{2, 2, 2, \ldots \ldots\}$. It is called a *constant sequence* .

Sometimes it is convenient to specify $f(1)$ and describe $f(n+1)$ in terms of $f(n)$ for all $n \geq 1$.

For example, $f(1) = \sqrt{2}$ and $f(n+1) = \sqrt{2f(n)}$ for $n \geq 1$ defines the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2}\sqrt{2}}, \ldots \ldots\}$.

5.2. Bounded Sequence.

A real sequence $\{f(n)\}$ is said to be *bounded above* if there exists a real number G such that $f(n) \leq G$ for all $n \in \mathbb{N}$. G is said to be *an upper bound* of the sequence.

A real sequence $\{f(n)\}$ is said to be *bounded below* if there exists a real number g such that $f(n) \geq g$ for all $n \in \mathbb{N}$. g is said to be *a lower bound* of the sequence.

A real sequence $\{f(n)\}$ is said to be a *bounded sequence* if there exist real numbers G, g such that $g \leq f(n) \leq G$ for all $n \in \mathbb{N}$.

Therefore a real sequence is bounded if and only if it is bounded above as well as bounded below. In this case, the range of the sequence is a bounded set.

For a real sequence $\{f(n)\}$ bounded above, the range of the sequence is a set bounded above and by the supremum property of $\mathbb{R}$, the range set has the least upper bound, which is also called the *least upper bound* of the sequence $\{f(n)\}$ and is denoted by $\sup\{f(n)\}$.

The least upper bound of a real sequence $\{f(n)\}$ is a real number M satisfying the following conditions :

(i) $f(n) \leq M$ for all $n \in \mathbb{N}$,

(ii) for each pre-assiged positive ϵ, there exists a *natural number* k such that $f(k) > M - \epsilon$.

By similar arguments, for a real sequence $\{f(n)\}$ bounded below, there exists a *greatest lower bound* and it is denoted by $\inf\{f(n)\}$.

The greatest lower bound of a real sequence $\{f(n)\}$ is a real number m satisfying the following conditions :

(i) $f(n) \geq m$ for all $n \in \mathbb{N}$,

(ii) for each pre-assigned positive ϵ, there exists a *natural number* k such that $f(k) < m + \epsilon$.

For a real sequence $\{f(n)\}$ unbounded above, we define $\sup\{f(n)\} = \infty$

For a real sequence $\{f(n)\}$ unbounded below, we define $\inf\{f(n)\} = -\infty$.

Examples.

1. The sequence $\{\frac{1}{n}\}$ is a bounded sequence. 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

2. The sequence $\{n^2\}$ is bounded below and unbounded above. Here

$\sup\{f(n)\} = \infty, \inf\{f(n)\} = 1.$

3. The sequence $\{-2n\}$ is bounded above and unbounded below. Here $\sup\{f(n)\} = -2, \inf\{f(n)\} = -\infty.$

4. Let $f(n) = (-1)^n n, n \in \mathbb{N}$. The sequence $\{f(n)\}$ is unbounded above and unbounded below. The sequence is $\{-1, 2, -3, 4, \dots \dots\}$.

Here $\sup\{f(n)\} = \infty, \inf\{f(n)\} = -\infty.$

5.3. Limit of a sequence.

Let $\{f(n)\}$ be a real sequence. A *real number* l is said to be a *limit* of the sequence $\{f(n)\}$ if corresponding to a pre-assigned positive ϵ there exists a *natural number* k (depending on ϵ) such that

$$| f(n) - l | < \epsilon \text{ for all } n \geq k$$

$$\text{i.e., } l - \epsilon < f(n) < l + \epsilon \text{ for all } n \geq k.$$

To be explicit, a real number l is said to be a limit of the sequence $\{f(n)\}$, if for a pre-assigned positive ϵ there exists a natural number k such that *all elements* of the sequence, excepting the first $k-1$ at most, lie in the ϵ- neighbourhood of l.

Theorem 5.3.1. A sequence can have at most one limit.

Proof. If possible, let a sequence $\{f(n)\}$ have two distinct limits l_1 and l_2 where $l_1 < l_2$.

Let $\epsilon = \frac{1}{2}(l_2 - l_1)$. Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 - \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint.

Since l_1 is a limit of the sequence, for the chosen ϵ, there exists a natural number k_1 such that

$$l_1 - \epsilon < f(n) < l_1 + \epsilon \text{ for all } n \geq k_1.$$

Since l_2 is a limit of the sequence, for the same chosen ϵ, there exists a natural number k_2 such that

$$l_2 - \epsilon < f(n) < l_2 + \epsilon \text{ for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.

Then $l_1 - \epsilon < f(n) < l_1 + \epsilon$ and $l_2 - \epsilon < f(n) < l_2 + \epsilon$ for all $n \geq k$.

This cannot happen since the neighbourhoods $N(l_1, \epsilon)$ and $N(l_2, \epsilon)$ are disjoint. Therefore our assumption that $l_1 \neq l_2$ is wrong.

Hence $l_1 = l_2$ and this proves the theorem.

5.4. Convergent sequence.

A real sequence $\{f(n)\}$ is said to be a *convergent sequence* if it has a limit $l \in \mathbb{R}$. In this case the sequence is said to converge to l.

We write $\lim_{n \to \infty} f(n) = l$, or $\lim f(n) = l$.

A sequence is said to be a *divergent sequence* if it is not convergent.

Examples.

1. The sequence $\{\frac{1}{n}\}$ converges to 0.

Let us choose a positive ϵ.

By Archimedean property of $\mathbb{R}$, there exists a natural number k such that $0 < \frac{1}{k} < \epsilon$. This implies $0 < \frac{1}{n} < \epsilon$ for all $n \geq k$.

It follows that $| \frac{1}{n} - 0 | < \epsilon$ for all $n \geq k$.

This proves $\lim \frac{1}{n} = 0$.

2. The sequence $\{\frac{n^2+1}{n^2}\}$ converges to 1.

Let us choose a positive ϵ.

Now $| \frac{n^2+1}{n^2} - 1 | < \epsilon$ will hold if $\frac{1}{n^2} < \epsilon$, i.e., if $n > \frac{1}{\sqrt{\epsilon}}$.

Let $k = [\frac{1}{\sqrt{\epsilon}}] + 1$. [For example, if $\epsilon = .01$ then $k = 11$; if $\epsilon = .001$ then $k = 32$.] Then k is a natural number and $| \frac{n^2+1}{n^2} - 1 | < \epsilon$ for all $n \geq k$.

This proves $\lim \frac{n^2+1}{n^2} = 1$.

3. Let $f(n) = 2$ for all $n \in \mathbb{N}$. The sequence is $\{2, 2, 2, \ldots \ldots\}$. We prove that the sequence converges to 2.

Let us choose a positive ϵ.

Now $| f(n) - 2 | < \epsilon$ holds for all $n \geq 1$.

Therefore $\lim f(n) = 2$.

Note. A constant sequence is a convergent sequence.

Theorem 5.4.1. A convergent sequence is bounded.

Proof. Let $\{f(n)\}$ be a convergent sequence and let l be its limit. Let us choose $\epsilon = 1$. For this chosen ϵ there exists a natural number k such that $\quad l - 1 < f(n) < l + 1$ for all $n \geq k$.

Let $B = \max\{f(1), f(2), \ldots, f(k-1), l+1\}$;
$\quad b = \min\{f(1), f(2), \ldots, f(k-1), l-1\}$.

Then $b \leq f(n) \leq B$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{f(n)\}$ is a bounded sequence.

Corollary. An unbounded sequence is not convergent.

Note. A bounded sequence may not be a convergent sequence.

For example, the sequence $\{(-1)^n\}$ is a bounded sequence but the sequence does not converge to a limit.

5.5. Limit theorems.

Theorem 5.5.1. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to u and v respectively.

Then (i) $\lim(u_n + v_n) = u + v$;

(ii) if $c \in \mathbb{R}, \lim(cu_n) = cu$;

(iii) $\lim u_n v_n = uv$;

(iv) $\lim \frac{u_n}{v_n} = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of non zero real numbers and $v \neq 0$.

Proof. (i) To show that $\lim(u_n + v_n) = u + v$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that $|(u_n + v_n) - (u + v)| < \epsilon$ for all $n \geq k$.

Using triangle inequality, we have

$|(u_n + v_n) - (u + v)| = |(u_n - u) + (v_n - v)| < |u_n - u| + |v_n - v|$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k_1 such that $|u_n - u| < \frac{\epsilon}{2}$ for all $n \geq k_1$.

Since $\lim v_n = v$, there exists a natural number k_2 such that $|v_n - v| < \frac{\epsilon}{2}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $|u_n - u| < \frac{\epsilon}{2}$ and $|v_n - v| < \frac{\epsilon}{2}$ for all $n \geq k$. It follows that $|(u_n + v_n) - (u + v)| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim(u_n + v_n) = u + v$.

(ii) Let us assume $c \neq 0$. When $c = 0$ the theorem is obvious.

To show that $\lim cu_n = cu$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$|cu_n - cu| < \epsilon$ for all $n \geq k$.

We have $|cu_n - cu| = |c||u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $|u_n - u| < \frac{\epsilon}{|c|}$ for all $n \geq k$.

It follows that $|cu_n - cu| < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim cu_n = cu$.

(iii) To show that $\lim u_n v_n = uv$, we need to establish that for a pre-assigned positive ϵ there exists a natural number k such that

$|u_n v_n - uv| < \epsilon$ for all $n \geq k$.

We have $|u_n v_n - uv| = |u_n(v_n - v) + v(u_n - u)|$

$\leq |u_n||v_n - v| + |v||u_n - u|$.

Since $\{u_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number B_1 such that $|u_n| < B_1$ for all $n \in \mathbb{N}$.

Let $B = \max\{B_1, |v|\}$.

Then $|u_n v_n - uv| < B|v_n - v| + B|u_n - u|$.

Let $\epsilon > 0$. Since $\lim u_n = u$ and $\lim v_n = v$, there exist natural numbers k_1 and k_2 such that

$| u_n - u |< \frac{\epsilon}{2B}$ for all $n \geq k_1$ and $| v_n - v |< \frac{\epsilon}{2B}$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| u_n - u |< \frac{\epsilon}{2B}$ and $| v_n - v |< \frac{\epsilon}{2B}$ for all $n \geq k$.

It follows that $| u_n v_n - uv |< B.\frac{\epsilon}{2B} + B.\frac{\epsilon}{2B}$ for all $n \geq k$

or, $| u_n v_n - uv |< \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim u_n v_n = uv$.

(iv) First we prove that if $\lim v_n = v$ where $\{v_n\}$ is a sequence of non-zero real numbers and $v \neq 0, \lim 1/v_n = 1/v$.

Let $\alpha = \frac{1}{2} | v |$. Then $\alpha > 0$. Since $\lim v_n = v$, there exists a natural number k_1 such that

$| v_n - v |< \alpha$ for all $n \geq k_1$.

We have $| | v_n | - | v | |\leq v_n - v |< \alpha$ for all $n \geq k_1$

or, $| v | -\alpha <| v_n |<| v | +\alpha$ for all $n \geq k_1$.

Therefore $| v_n |> \frac{1}{2} | v |$ for all $n \geq k_1$.

Now $| \frac{1}{v_n} - \frac{1}{v} |= \frac{|v_n - v|}{|v||v_n|} < \frac{2}{|v|^2} | v_n - v |$ for all $n \geq k_1$.

Let $\epsilon > 0$. Since $\lim v_n = v$, there exists a natural number k_2 such that $| v_n - v |< \frac{|v|^2}{2}\epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $| \frac{1}{v_n} - \frac{1}{v} |< \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim \frac{1}{v_n} = \frac{1}{v}$.

The proof of the theorem is now completed by considering the convergence of the product of two sequences $\{u_n\}$ and $\{\frac{1}{v_n}\}$.

Therefore $\lim \frac{u_n}{v_n} = \lim(u_n.\frac{1}{v_n}) = u.\frac{1}{v} = \frac{u}{v}$.

Note. If $\{u_n\}, \{v_n\}, \{w_n\}$ be three convergent sequences of real numbers that converge to u, v, w respectively, then

(i) $\lim(u_n + v_n + w_n) = u + v + w$ and
(ii) $\lim(u_n v_n w_n) = uvw$.

The theorem can be generalised to the sum and the product of a *finite* number of convergent sequences.

Theorem 5.5.2. Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{| u_n |\}$ converges to $| u |$.

Proof. We have $| | u_n | - | u | |\leq| u_n - u |$.

Let $\epsilon > 0$. Since $\lim u_n = u$, there exists a natural number k such that $| u_n - u |< \epsilon$ for all $n \geq k$.

It follows that $| | u_n | - | u | |< \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim | u_n |=| u |$.

Note 1. The converse of the theorem is not true. That is, if $\{|u_n|\}$ is a convergent sequence it does not necessarily imply that $\{u_n\}$ is a convergent sequence.

For example, let $u_n = (-1)^n$. Then the sequence $\{|u_n|\}$ converges to 1 but the sequence $\{u_n\}$ is a divergent sequence.

Note 2. The theorem states that $\lim |u_n| = |\lim u_n|$, provided the limit in R.H. S. exists.

Theorem 5.5.3. Let $\{u_n\}$ be a convergent sequence of real numbers and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$.

Proof. Let $\lim u_n = u$ and if possible let $u < 0$.

Let us choose a positive ϵ such that $u + \epsilon < 0$.

Since $\lim u_n = u$, there exists a natural number k_1 such that

$$u - \epsilon < u_n < u + \epsilon \text{ for all } n \geq k_1.$$

Let $k = \max\{k_1, m\}$.

Then by hypothesis, $u_n > 0$ for all $n \geq k$ and we have from above $u_n < u + \epsilon < 0$ for all $n \geq k$.

This is a contradiction. Therefore $\lim u_n \geq 0$.

Note 1. The theorem also says that a convergent sequence of positive numbers may converge to 0. For example, for the sequence $\{u_n\}$ where $u_n = \frac{1}{n}$; $u_n > 0$ for all $n \in \mathbb{N}$ but $\lim u_n = 0$.

Note 2. If $\{u_n\}$ be a convergent sequence and $u_n \geq 0$ for all $n \geq m$ (m being a natural number) then $\lim u_n \geq 0$.

Theorem 5.5.4. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences and there exists a natural number m such that $u_n > v_n$ for all $n \geq m$.

Then $\lim u_n \geq \lim v_n$.

Proof. Let $\lim u_n = u, \lim v_n = v$ and $w_n = u_n - v_n$.

Then $\{w_n\}$ is a convergent sequence such that $w_n > 0$ for all $n \geq m$ and $\lim w_n = u - v$.

By the previous theorem, $u - v \geq 0$.

Consequently, $\lim u_n \geq \lim v_n$.

Note. If $\{u_n\}$ and $\{v_n\}$ be two convergent sequences and $u_n \geq v_n$ for all $n \geq m$ then $\lim u_n \geq \lim v_n$.

If $w_n = u_n - v_n$ then $\{w_n\}$ is a convergent sequence such that $w_n \geq 0$ for all $n \geq m$ and $\lim w_n = u - v$.

So $u - v \geq 0$ and therefore $\lim u_n \geq \lim v_n$.

Corollary 1. If $\{x_n\}$ is a convergent sequence of points in $[a,b]$ and $\lim x_n = c$, then $c \in [a,b]$.

Corollary 2. If $\{x_n\}$ is a convergent sequence of points in (a,b) and $\lim x_n = c$, then $c \in [a,b]$. [Here c may not be in (a,b)].

Theorem 5.5.5. (Sandwich theorem)

Let $\{u_n\}, \{v_n\}, \{w_n\}$ be three sequences of real numbers and there is a natural number m such that

$$u_n < v_n < w_n \text{ for all } n \geq m.$$

If $\lim u_n = \lim w_n = l$ then $\{v_n\}$ is convergent and $\lim v_n = l$.

Proof. Let $\epsilon > 0$. It follows from the convergence of the sequences $\{u_n\}$ and $\{w_n\}$ that there exist natural numbers k_1 and k_2 such that

$| u_n - l |< \epsilon$ for all $n \geq k_1$ and $| w_n - l |< \epsilon$ for all $n \geq k_2$.

Let $k_3 = \max\{k_1, k_2\}$.

Then $l - \epsilon < u_n < l + \epsilon$ and $l - \epsilon < w_n < l + \epsilon$ for all $n \geq k_3$.

Let $k = \max\{k_3, m\}$.

Then $l - \epsilon < u_n < v_n < w_n < l + \epsilon$ for all $n \geq k$.

Consequently, $| v_n - l |< \epsilon$ for all $n \geq k$.

This shows that the sequence $\{v_n\}$ is convergent and $\lim v_n = l$.

Note. If $u_n \leq v_n \leq w_n$ for all $n \geq m$ and $\lim u_n = \lim w_n = l$ then $\lim v_n = l$.

Worked Examples.

1. Prove that $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = 3$.

$\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = \lim\limits_{n\to\infty} \frac{u_n}{v_n}$ where $u_n = 3 + \frac{2}{n} + \frac{1}{n^2}$ and $v_n = 1 + \frac{1}{n^2}$.

But $\lim u_n = 3$ and $\lim v_n = 1$.

Therefore $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = \lim \frac{u_n}{v_n} = 3$.

2. Prove that $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

$$\begin{aligned}
\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&= \lim_{n\to\infty} u_n v_n \text{ where } u_n = \tfrac{1}{\sqrt{n}}, v_n = \tfrac{1}{\sqrt{1+\frac{1}{n}}+1} \\
&= 0, \text{ since } \lim u_n = 0 \text{ and } \lim v_n = \tfrac{1}{2}.
\end{aligned}$$

3. Prove that

$\lim\limits_{n\to\infty} (\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}) = 1$.

Let $u_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$.

$$\text{We have } \frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$$
$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$$
$$\cdots \qquad \cdots$$
$$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}.$$

Therefore $u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

$$\text{Again, } \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$$
$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

Therefore $u_n > \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$.

Thus $\frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}}$ for all $n \geq 2$.

But $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$ and $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$.

By Sandwich theorem, $\lim u_n = 1$.

5.6. Null sequence.

A sequence $\{u_n\}$ is said to be a *null sequence* if $\lim u_n = 0$.

Theorem 5.6.1. If $\{u_n\}$ be null sequence then $\{|u_n|\}$ is a null sequence and conversely.

Proof. Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k such that $|u_n| < \epsilon$ for all $n \geq k$.

As $||u_n| - 0| = |u_n|$, it follows that $||u_n| - 0| < \epsilon$ for all $n \geq k$.

This proves $\lim |u_n| = 0$.

Conversely, let $\lim |u_n| = 0$.

Let $\epsilon > 0$. There exists a natural number k such that

$||u_n| - 0| < \epsilon$ for all $n \geq k$. That is, $|u_n| < \epsilon$ for all $n \geq k$.

This proves $\lim u_n = 0$.

5.7. Divergent sequence.

A real sequence $\{f(n)\}$ is said to *diverge* to ∞ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$$f(n) > G \text{ for all } n \geq k.$$

In this case we write $\lim f(n) = \infty$ and also say that the sequence $\{f(n)\}$ tends to ∞.

A real sequence $\{f(n)\}$ is said to diverge to $-\infty$ if corresponding to a pre-assigned positive number G, however large, there exists a natural number k such that

$$f(n) < -G \text{ for all } n \geq k.$$

In this case we write $\lim f(n) = -\infty$ and also say that the sequence $\{f(n)\}$ tends to $-\infty$.

A real sequence $\{f(n)\}$ is said to be a *properly divergent sequence* if it either diverges to ∞, or diverges to $-\infty$.

Theorem 5.7.1. A sequence diverging to ∞ is unbounded above but bounded below.

Proof. Let a sequence $\{f(n)\}$ diverge to ∞. Then for each pre-assigned positive number G there exists a natural number k such that $f(k) > G$.

Therefore there does not exist a real number B such that $f(n) \leq B$ holds for all $n \in \mathbb{N}$. In other words, $\{f(n)\}$ is unbounded above.

Let $G > 0$. Then there exists a natural number k such that

$$f(n) > G \text{ for all } n \geq k.$$

Let $b = \min\{f(1), f(2), \dots, f(k-1), G\}$. Then $f(n) \geq b$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{f(n)\}$ is bounded below.

Note. A sequence unbounded above but bounded below may not diverge to ∞.

For example, let us consider the sequence $\{f(n)\}$ where $f(n) = n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \dots \dots\}$.

The sequence is unbounded above and bounded below, 0 being a lower bound. The sequence does not diverge to ∞, because for a pre-assigned positive number G there does not exist a natural number k such that $f(n) > G$ holds for all $n \geq k$.

Theorem 5.7.2. A sequence diverging to $-\infty$ is unbounded below but bounded above.

Proof left to the reader.

Note. A sequence unbounded below but bounded above may not diverge to $-\infty$.

Definitions. A bounded sequence that is not convergent is said to be an *oscillatory sequence of finite oscillation.*

An unbounded sequence that is not properly divergent is said to be an *oscillatory sequence of infinite oscillation.*

An oscillatory sequence is therefore neither convergent nor properly divergent. It is called an *improperly divergent sequence.*

Examples.

1. The sequence $\{2^n\}$ diverges to ∞.

2. The sequence $\{-n^2\}$ diverges to $-\infty$.

3. The sequence $\{(-1)^n\}$ is a bounded sequence, but not convergent. It is an oscillatory sequence of finite oscillation.

4. The sequence $\{(-1)^n n\}$ is an unbounded sequence, and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

5.8. Some important limits.

1. $\lim r^n = 0$ if $|r| < 1$.

Case 1. $r = 0$. In this case the sequence is $\{0, 0, 0, \ldots \ldots\}$.
The sequence converges to 0.
That is, $\lim r^n = 0$ when $r = 0$.

Case 2. $r \neq 0$ and $|r| < 1$.
$\frac{1}{|r|} > 1$, since $|r| < 1$. Let $\frac{1}{|r|} = a + 1$ where $a > 0$.
$|r^n - 0| = |r^n| = |r|^n = \frac{1}{(a+1)^n}$.
We have $(1+a)^n > na$ for all $n \in \mathbb{N}$.
So $|r^n - 0| < \frac{1}{na}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Then $|r^n - 0| < \epsilon$ holds if $n > \frac{1}{a\epsilon}$.
Let $k = [\frac{1}{a\epsilon}] + 1$. Then k is a natural number and $|r^n - 0| < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim r^n = 0$.

Combining the cases, $\lim r^n = 0$ if $|r| < 1$.

2. $\lim a^{1/n} = 1$ if $a > 0$.

Case 1. $a = 1$. In this case the sequence converges to 1.

Case 2. $a > 1$. Then $a^{1/n} > 1$. Let $a^{1/n} = 1 + x_n$ where $x_n > 0$.
Then $a = (1 + x_n)^n$
$> 1 + nx_n$ for $n > 1$.
Let $\epsilon > 0$. Then $|a^{1/n} - 1| < \epsilon$ holds if $\frac{a-1}{n} < \epsilon$ i.e., if $n > \frac{a-1}{\epsilon}$.
Let $k = [\frac{a-1}{\epsilon}] + 1$. Then k is a natural number and $|a^{1/n} - 1| < \epsilon$ for all $n \geq k$.
Since ϵ is arbitrary, $\lim a^{1/n} = 1$.

Case 3. $0 < a < 1$. Let $b = 1/a$. Then $b > 1$ and
$\lim a^{1/n} = \lim \frac{1}{b^{1/n}} = 1$, by case 2.
Combining the cases, $\lim a^{1/n} = 1$ if $a > 0$.

3. If $\lim x_n = 0$ and $a > 0$, then $\lim a^{x_n} = 1$.

We have $\lim a^{1/n} = 1$ and $\lim a^{-1/n} = 1$.
Let us choose $\epsilon > 0$. There exist natural numbers k_1, k_2 such that

$1-\epsilon < a^{1/n} < 1+\epsilon$ for all $n \geq k_1$ and $1-\epsilon < a^{-1/n} < 1+\epsilon$ for all $n \geq k_2$

Let $k = \max\{k_1, k_2\}$.

Then $1-\epsilon < a^{1/k} < 1+\epsilon$ and $1-\epsilon < a^{-1/k} < 1+\epsilon$.

Since $\lim x_n = 0$, there exists a natural number p such that $-\frac{1}{k} < x_n < \frac{1}{k}$ for all $n \geq p$.

Let $a > 1$. Then $a^{-1/k} < a^{x_n} < a^{1/k}$ for all $n \geq p$
or, $1-\epsilon < a^{-1/k} < a^{x_n} < a^{1/k} < 1+\epsilon$ for all $n \geq p$.

Let $0 < a < 1$. Then $a^{1/k} < a^{x_n} < a^{-1/k}$ for all $n \geq p$
or, $1-\epsilon < a^{1/k} < a^{x_n} < a^{-1/k} < 1+\epsilon$ for all $n \geq p$.
Therefore if $a > 0, 1-\epsilon < a^{x_n} < 1+\epsilon$ for all $n \geq p$.
This implies $\lim a^{x_n} = 1$.

Corollary 1. If $\lim x_n = l$ and $a > 0$, then $\lim a^{x_n} = a^l$.

Corollary 2. If $\lim x_n = l$, then $\lim e^{x_n} = e^l$.

4. If $\lim x_n = 0$, then $\lim \log(1+x_n) = 0$.

Let $\epsilon > 0$. $-\epsilon < \log(1+x_n) < \epsilon$ will hold if $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1$
Since $\epsilon > 0, e^{\epsilon} - 1 > 0$ and $e^{-\epsilon} - 1 < 0$.

By Archimedean property of $\mathbb{R}$, there exists a natural number m_1 such that $0 < \frac{1}{m_1} < e^{\epsilon} - 1$ and also there exists a natural number m_2 such that $0 < \frac{1}{m_2} < 1 - e^{-\epsilon}$.

Let $m = \max\{m_1, m_2\}$.
Then $0 < \frac{1}{m} < e^{\epsilon} - 1$ and $0 < \frac{1}{m} < 1 - e^{-\epsilon}$.
Combining, $e^{-\epsilon} - 1 < -\frac{1}{m} < \frac{1}{m} < e^{\epsilon} - 1$.
Since $\lim x_n = 0$, there exists a natural number k such that
$| x_n - 0 | < \frac{1}{m}$ for all $n \geq k$
or, $-\frac{1}{m} < x_n < \frac{1}{m}$ for all $n \geq k$.
Consequently, $e^{-\epsilon} - 1 < x_n < e^{\epsilon} - 1$ for all $n \geq k$
or, $-\epsilon < \log(1+x_n) < \epsilon$ for all $n \geq k$.
This proves $\lim \log(1+x_n) = 0$.

Corollary. Let $\{x_n\}$ be a sequence such that $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim x_n = c > 0$. Then $\lim(\log x_n - \log c) = \lim \log \frac{x_n}{c} = \lim \log(1 + \frac{x_n - c}{c}) = 0$, since $\lim \frac{x_n - c}{c} = 0$.

Therefore $\lim \log x_n = \log c$.

5. If $u_n > 0$ and $\lim u_n = u > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = v$, then $\lim(u_n)^{v_n} = u^v$.

By definition, $(u_n)^{v_n} = e^{v_n \log u_n}$.
As $\lim u_n = u, \lim \log u_n = \log u$. So $\lim(v_n \log u_n) = v \log u$.

By corollary 2 of Ex. 3, $\lim e^{v_n \log u_n} = e^{v \log u} = u^v$
or, $\lim (u_n)^{v_n} = u^v$.

6. $\lim n^{1/n} = 1$.

$n^{1/n} > 1$ for all $n > 1$.
Let $n^{1/n} = 1 + x_n$ where $x_n > 0$.
Then $n = (1 + x_n)^n$

$$= 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots + x_n^n$$
$$> \tfrac{1}{2}n(n-1)x_n^2.$$

Clearly, $x_n^2 < \frac{2}{n-1}$ for all $n > 1$
or, $| x_n | < \frac{\sqrt{2}}{\sqrt{n-1}}$.
Let $\epsilon > 0$. Then $| n^{1/n} - 1 | = | x_n | < \epsilon$ holds if $n > 1 + \frac{2}{\epsilon^2}$.

Let $k = [1 + \frac{2}{\epsilon^2}] + 1$. Then k is a natural number and $| n^{1/n} - 1 | < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, $\lim n^{1/n} = 1$.

7. Behaviour of the sequence $\{r^n\}$ for different real values of r.

Case 1. $r > 1$. Let $r = 1 + a$ where $a > 0$.
Then $r^n = (1 + a)^n > 1 + na$ for $n > 1$.
Let $G > 0$. Then $1 + na > G$ holds if $n > \frac{G-1}{a}$.

Let $k = [\frac{G-1}{a}] + 1$. Then k is a natural number and $r^n > G$ for all $n \geq k$.

Since G is an arbitrary positive number, $\lim r^n = \infty$.
Therefore in this case the sequence diverges to ∞.

Case 2. $r = 1$. In this case the sequence is $\{1, 1, 1, \ldots \ldots\}$ and the sequence converges to 1.

Case 3. $| r | < 1$. In this case the sequence converges to 0, by Example 1.

Case 4. $r = -1$. In this case the sequence is $\{-1, 1, -1, \ldots \ldots\}$. The sequence is bounded but not convergent. The sequence is an oscillatory sequence of finite oscillation.

Case 5. $r < -1$. let $r = -s$. Then $s > 1$.

The sequence is $\{(-1)^n s^n\}$. It is an unbounded sequence. It neither diverges to ∞ nor diverges to $-\infty$. It is an oscillatory sequence of infinite oscillation.

Theorem 5.8.1. Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim \frac{u_{n+1}}{u_n} = l$.

(i) If $0 \leq l < 1$ then $\lim u_n = 0$,

(ii) if $l > 1$ then $\lim u_n = \infty$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \geq k.$$

Let $l + \epsilon = r$. Then $0 < r < 1$.

Therefore $\frac{u_{n+1}}{u_n} < r$ for all $n \geq k$.

Hence we have $\frac{u_{k+1}}{u_k} < r, \frac{u_{k+2}}{u_{k+1}} < r, \cdots, \frac{u_n}{u_{n-1}} < r$ for $n \geq k+1$.

Multiplying, $\frac{u_n}{u_k} < r^{n-k}$ for $n \geq k+1$

or, $u_n < \frac{u_k}{r^k}.r^n$ for $n \geq k+1$.

Now $\lim r^n = 0$ since $0 < r < 1$; and $\frac{u_k}{r^k}$ is a fixed positive number.

Therefore $\lim u_n = 0$.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number m such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = s$. Then $s > 1$.

Therefore $\frac{u_{n+1}}{u_n} > s$ for all $n \geq m$.

Hence we have $\frac{u_{m+1}}{u_m} > s, \frac{u_{m+2}}{u_{m+1}} > s, \cdots, \frac{u_n}{u_{n-1}} > s$ for $n \geq m+1$.

Multiplying, $\frac{u_n}{u_m} > s^{n-m}$ for $n \geq m+1$

or, $u_n > \frac{u_m}{s^m}.s^n$ for $n \geq m+1$.

Now $\lim s^n = \infty$ since $s > 1$; and $\frac{u_m}{s^m}$ is a fixed positive number.

Therefore $\lim u_n = \infty$.

Note. If $\lim \frac{u_{n+1}}{u_n} = 1$, no definite conclusion can be made about the nature of the sequence. For example, (i) if $u_n = \frac{n+1}{n}$ then $\lim \frac{u_{n+1}}{u_n} = 1$ and $\lim u_n = 1$; (ii) if $u_n = \frac{1}{n}$ then $\lim \frac{u_{n+1}}{u_n} = 1$ and $\lim u_n = 0$.

Theorem 5.8.2. Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim \sqrt[n]{u_n} = l$.

(i) If $0 \leq l < 1$ then $\lim u_n = 0$.

(ii) If $l > 1$ then $\lim u_n = \infty$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim \sqrt[n]{u_n} = l$, there exists a natural number k such that

$$l - \epsilon < \sqrt[n]{u_n} < l + \epsilon \text{ for all } n \geq k.$$

Let $l + \epsilon = r$. Then $0 < r < 1$ and $\sqrt[n]{u_n} < r$ for all $n \geq k$.

So we have $0 < u_n < r^n$ for all $n \geq k$.

Since $\lim r^n = 0$, $\lim u_n = 0$, by Sandwich theorem.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim \sqrt[n]{u_n} = l$, there exists a natural number m such that

$$l - \epsilon < \sqrt[n]{u_n} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = s$. Then $s > 1$ and $\sqrt[n]{u_n} > s$ for all $n \geq m$.
So we have $u_n > s^n$ for all $n \geq m$.
Since $s > 1, \lim s^n = \infty$ and therefore $\lim u_n = \infty$.

Note. If $\lim \sqrt[n]{u_n} = 1$, no definite conclusion can be made about the nature of the sequence $\{u_n\}$.

For example, (i) if $u_n = \frac{n+1}{n}$ then $\lim \sqrt[n]{u_n} = 1$ and $\lim u_n = 1$; (ii) if $u_n = \frac{n+1}{2n}$ then $\lim \sqrt[n]{u_n} = 1$ and $\lim u_n = \frac{1}{2}$.

Worked Examples.

1. A sequence $\{u_n\}$ is defined by $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for $n \geq 1$ and $0 < u_1 < u_2$. Prove that the sequence $\{u_n\}$ converges to $\frac{u_1+2u_2}{3}$.

$$u_2 - u_1 > 0$$
$$u_3 - u_2 = \tfrac{1}{2}(u_2 + u_1) - u_2 = -\tfrac{1}{2}(u_2 - u_1)$$
$$u_4 - u_3 = \tfrac{1}{2}(u_3 + u_2) - u_3 = \tfrac{1}{2}(u_2 - u_3) = (-\tfrac{1}{2})^2(u_2 - u_1)$$
$$\cdots \quad \cdots \quad \cdots$$
$$u_n - u_{n-1} = (-\tfrac{1}{2})^{n-2}(u_2 - u_1).$$

Therefore
$$u_n - u_1 = (u_2 - u_1)[1 + (-\tfrac{1}{2}) + (-\tfrac{1}{2})^2 + \cdots + (-\tfrac{1}{2})^{n-2}]$$
$$= \tfrac{2(u_2-u_1)}{3}[1 - (-\tfrac{1}{2})^{n-1}].$$

Now $\lim(u_n - u_1) = \frac{2}{3}(u_2 - u_1)$ since $\lim(-\frac{1}{2})^{n-1} = 0$.

Therefore $\lim u_n = u_1 + \frac{2}{3}(u_2 - u_1) = \frac{u_1+2u_2}{3}$.

2. If $x_n = (a^n + b^n)^{1/n}$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim x_n = b$.

$x_n = b[(\frac{a}{b})^n + 1]^{1/n}$
$> b$ for all $n \in \mathbb{N}$, since $(\frac{a}{b})^n + 1 > 1$ for all $n \in \mathbb{N}$.
Again, $0 < a < b \Rightarrow a^n < b^n$ for all $n \in \mathbb{N}$.
Therefore $a^n + b^n < 2b^n$
or, $x_n < 2^{1/n}.b$ for all $n \in \mathbb{N}$.

Let $u_n = b$ for all $n \in \mathbb{N}, v_n = 2^{1/n}b$ for all $n \in \mathbb{N}$.
Then $\lim u_n = b$ and $\lim v_n = b$ since $\lim 2^{1/n} = 1$.
Now $u_n < x_n < v_n$ for all $n \in \mathbb{N}$.
Since $\lim u_n = \lim v_n = b, \lim x_n = b$ by Sandwich theorem.

3. A sequence $\{u_n\}$ is defined by $u = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for $n \geq 1$. Prove that $\lim u_n = 2$.

$$u_1 = 2^{1/2}, u_2 = \sqrt{2\sqrt{2}} = 2^{1/2+1/2^2} = 2^{1-\frac{1}{2^2}},$$
$$u_3 = 2^{1/2+1/2^2+1/2^3} = 2^{1-\frac{1}{2^3}},$$
$$\cdots \quad \cdots$$
$$u_n = 2^{1/2+1/2^2+\cdots+1/2^n} = 2^{1-\frac{1}{2^n}}.$$

$\lim u_n = \lim 2^{1-1/2^n} = \lim 2^{x_n}$ where $x_n = 1 - \frac{1}{2^n}$.

As $\lim x_n = 1$, we have $\lim u_n = \lim 2^{x_n} = 2$, since $\lim x_n = l$ and $a > 0 \Rightarrow \lim a^{x_n} = a^l$.

4. If $u_n > 0$ for all n and $\lim \sqrt[n]{u_n} = \mu > 0$ prove that $\lim \sqrt[n]{(n+1)u_{n+1}} = \mu$.

$\lim \sqrt[n]{n+1} = \lim\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

Since $\lim n^{\frac{1}{n}} = 1$, it follows that $\lim(n+1)^{\frac{1}{n+1}} = 1$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim(n+1)^{\frac{1}{n+1}} = 1$, we have $\lim \sqrt[n]{n+1} = 1$, by Ex 5 of 5.8.

$\lim \sqrt[n]{u_{n+1}} = \lim\{(u_{n+1})^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$.

Since $\lim \sqrt[n]{u_n} = \mu$, it follows that $\lim(u_{n+1})^{\frac{1}{n+1}} = \mu$.

Since $\lim \frac{n+1}{n} = 1$ and $\lim(u_{n+1})^{\frac{1}{n+1}} = \mu > 0$, we have $\lim \sqrt[n]{u_{n+1}} = \mu$.

Therefore $\lim \sqrt[n]{(n+1)u_{n+1}} = \lim(\sqrt[n]{n+1}.\sqrt[n]{u_{n+1}}) = \mu$.

5.9. Monotone sequence.

A real sequence $\{f(n)\}$ is said to be a *monotone increasing sequence* if $f(n+1) \geq f(n)$ for all $n \in \mathbb{N}$.

A real sequence $\{f(n)\}$ is said to be a *monotone decreasing sequence* if $f(n+1) \leq f(n)$ for all $n \in \mathbb{N}$.

A real sequence $\{f(n)\}$ is said to be a *monotone sequence* if it is either a monotone increasing sequence or a monotone decreasing sequence.

Note. If $f(n+1) > f(n)$ for all $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a *strictly monotone increasing sequence.*

If $f(n+1) < f(n)$ for all $n \in \mathbb{N}$, the sequence $\{f(n)\}$ is said to be a *strictly monotone decreasing sequence.*

If for some natural number $m, f(n+1) \geq f(n)$ for all $n \geq m$ the sequence $\{f(n)\}$ is said to be an 'ultimately' monotone increasing sequence.

If for some natural number $m, f(n+1) \leq f(n)$ for all $n \geq m$ the sequence $\{f(n)\}$ is said to be an 'ultimately' monotone decreasing sequence.

Examples.

1. Let $f(n) = 2^n, n \geq 1$.
Then $f(n+1) > f(n)$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{f(n)\}$ is a monotone increasing sequence. It is also strictly monotone.

2. Let $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, n \geq 1$.

Then $f(n+1) - f(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$= \frac{1}{(2n+1)(2n+2)} > 0$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{f(n)\}$ is a monotone increasing sequence. It is also strictly monotone.

3. Let $f(n) = \frac{1}{n}, n \geq 1$.

Then $f(n+1) - f(n) < 0$ for all $n \in \mathbb{N}$.

Therefore the sequence $\{f(n)\}$ is a monotone decreasing sequence. It is also strictly monotone.

4. The sequence $\{(-2)^n\}$ is neither a monotone increasing sequence, nor a monotone decreasing sequence. Therefore it is not a monotone sequence.

Theorem 5.9.1. A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound.

Proof. Let $\{f(n)\}$ be a monotone increasing sequence bounded above and let M be its its least upper bound.

Then (i) $f(n) \leq M$ for all $n \in \mathbb{N}$ and

(ii) for a pre-assigned positive ϵ, there exists a natural number k such that $f(k) > M - \epsilon$.

Since $\{f(n)\}$ is a monotone increasing sequence,

$M - \epsilon < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots\cdots \leq M$.

That is, $M - \epsilon < f(n) < M + \epsilon$ for all $n \geq k$.

This shows that the sequence $\{f(n)\}$ is convergent and $\lim f(n) = M$.

Theorem 5.9.2. A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

Similar proof.

Theorem 5.9.3. A monotone increasing sequence that is unbounded above diverges to ∞.

Proof. Let $\{f(n)\}$ be a monotone increasing sequence, not bounded above. Since the sequence is unbounded above, for a pre-assigned positive number G, however large, there exists a natural number k such that $f(k) > G$.

Since the sequence $\{f(n)\}$ is monotone increasing,

$$G < f(k) \leq f(k+1) \leq f(k+2) \leq \cdots\cdots$$

That is, $f(n) > G$ for all $n \geq k$.

This proves that the sequence $\{f(n)\}$ diverges to ∞.

Theorem 5.9.4. A monotone decreasing sequence that is unbounded below diverges to $-\infty$.

Similar proof.

Note. A monotone sequence has a definite behaviour. It is either convergent, or properly divergent.

The theorems on monotone sequences are important and useful in the sense that the convergence of the sequence can be established without prior knowledge of the limit. The limit of the sequence, however, can be determined if the l.u.b. of the increasing sequence (or the g.l.b. of the decreasing sequence) be evaluated.

Theorem 5.9.5. (Cantor's theorem on nested intervals)

Let $\{[a_n, b_n]\}$ be a sequence of *closed and bounded* intevals such that
(i) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$, and
(ii) $\lim \delta_n = 0$ where $\delta_n = b_n - a_n =$ length of $[a_n, b_n]$.

Then $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$ contains precisely one point.

Proof. $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$.

So we have $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$.

Also $a_n \leq b_n$ for $n \in \mathbb{N}$.

Therefore $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$.

This shows that the sequence $\{a_n\}$ is a monotone increasing sequence, bounded above and the sequence $\{b_n\}$ is a monotone decreasing sequence, bounded below.

Hence both the sequences are convergent. Let $\lim a_n = l, \lim b_n = m$.

Since $\lim(b_n - a_n) = 0$, $l = m = \alpha$, say,

Therefore α is the least upper bound of the sequence $\{a_n\}$ and the greatest lower bound of the sequence $\{b_n\}$.

Hence $a_n \leq \alpha$ and $\alpha \leq b_n$ for all $n \in \mathbb{N}$.'

This implies $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$. That is, $\alpha \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

We now prove that α is the only point in $\bigcap\limits_{n=1}^{\infty} [a_n, b_n]$.

If possible, let $\beta \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$. Then $a_n \leq \beta \leq b_n$ for all $n \in \mathbb{N}$.

Let us define a sequence $\{u_n\}$ by $u_n = \beta$ for all $n \in \mathbb{N}$. Then $\lim u_n = \beta$. Now $a_n \leq u_n \leq b_n$ for $n \geq 1$ and $\lim a_n = \lim b_n = \alpha$.

By Sandwich theorem, $\lim u_n = \alpha$ and therefore $\beta = \alpha$.
This proves that α is unique.

Note. The theorem says that a nested sequence of closed and bounded intervals has a non-empty intersecion.

A nested sequence of open and bounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of open bounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

A nested sequence of closed and unbounded intervals $\{I_n\}$ may not have a non-empty intersection.

For example, let $I_n = \{x \in \mathbb{R} : x \geq n\}$ for all $n \in \mathbb{N}$. Then $\{I_n\}$ is a nested sequence of closed and unbounded intervals since $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Here $\bigcap\limits_{n=1}^{\infty} I_n = \phi$.

5.10. Some important sequences.

1. The sequence $\{(1+\frac{1}{n})^n\}$ is a monotone increasing sequence, bounded above.

Let $u_n = (1+\frac{1}{n})^n$. Then $u_{n+1} = (1+\frac{1}{n+1})^{n+1}$.

Let us consider $n+1$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, \cdots, 1+\frac{1}{n}$ (n times) and 1.

Applying A.M.> G.M., we have $\frac{n(1+\frac{1}{n})+1}{n+1} > (1+\frac{1}{n})^{\frac{n}{n+1}}$

$$\text{or, } (1+\tfrac{1}{n+1})^{n+1} > (1+\tfrac{1}{n})^n$$

$$\text{i.e., } u_{n+1} > u_n \text{ for all } n \in \mathbb{N}.$$

This shows that the sequence $\{u_n\}$ is a monotone increasing sequence.

$$\begin{aligned} \text{Now } u_n &= 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2.1}{n!}\frac{1}{n^n} \\ &= 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots\frac{2}{n}.\frac{1}{n} \\ &< 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} \text{ for all } n \geq 2. \end{aligned}$$

We have $n! > 2^{n-1}$ for all $n > 2$. Utilising this
$1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$, for $n > 2$.

Also $1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+2[1-(\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$.
It follows that $u_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence bounded above, is convergent. The limit of the sequence is denoted by e.

Since $u_1 = 2$, it follows that $2 < u_n < 3$ for all $n \geq 2$.

2. The sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ is a monotone increasing sequence, bounded above. And $\lim x_n = e$.

$x_{n+1} - x_n = [1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n+1)!}]-[1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}] = \frac{1}{(n+1)!} > 0$ for all $n \geq 1$.

So $x_{n+1} > x_n$ for all $n \geq 1$.

This shows that the sequence $\{x_n\}$ is a monotone increasing sequence.

$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$, for $n \geq 3$, since $n! > 2^{n-1}$ for all $n \geq 3$.

Again $1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2[1 - (\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$

It follows that $x_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{x_n\}$ is bounded above.

Thus the sequence $\{x_n\}$ being a monotone increasing sequence bounded above, is convergent.

Let $u_n = (1 + \frac{1}{n})^n$.

$$\begin{aligned} \text{Then } u_n &= 1 + 1 + \frac{n(n-1)}{2!}.\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2.1}{n!}.\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots\frac{2}{n}.\frac{1}{n}. \\ &< 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \text{ for all } n \geq 2. \end{aligned}$$

Therefore $\lim u_n \leq \lim x_n$ (since both the limits exist).

or, $e \leq \lim x_n$ ... ... (A)

Let us choose a natural number m. Then for each $n > m$,

$u_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{m-1}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots\frac{2}{n}.\frac{1}{n}$

$> 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n})(1 - \frac{2}{n})\cdots(1 - \frac{m-1}{n})$.

Keeping m fixed, let $n \to \infty$. Then

$$\lim_{n\to\infty} u_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$$

or, $e \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$

or, $x_m \leq e$.

The inequality holds for all natural numbers m.

Proceeding to limit as $m \to \infty$, $\lim_{m\to\infty} x_m \leq e$ ... ... (B)

From (A) and (B), $\lim x_n = e$.

3. The sequence $\{(1 + \frac{1}{n})^{n+1}\}$ is a monotone decreasing sequence with limit e.

Let $v_n = (1 + \frac{1}{n})^{n+1}$.

Let us consider $n + 2$ positive numbers $1 - \frac{1}{n+1}, 1 - \frac{1}{n+1}, \cdots, 1 - \frac{1}{n+1}$ [$(n + 1)$ times] and 1.

Applying A.M.> G.M., we have $\frac{(n+1)(1-\frac{1}{n+1})+1}{n+2} > (1 - \frac{1}{n+1})^{\frac{n+1}{n+2}}$

or, $(\frac{n+1}{n+2})^{n+2} > (\frac{n}{n+1})^{n+1}$
or, $(\frac{n+1}{n})^{n+1} > (\frac{n+2}{n+1})^{n+2}$
or, $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$
i.e., $v_n > v_{n+1}$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{v_n\}$ is a monotone decreasing sequence.
Again $v_n = 1 + \frac{n+1}{n} + \frac{(n+1)n}{2!}.\frac{1}{n^2} + \cdots + \frac{1}{n^{n+1}} > 1$ for all $n \in \mathbb{N}$.
This shows that the sequence $\{v_n\}$ is bounded below.
Hence the sequence $\{v_n\}$ is convergent.

Let $u_n = (1+\frac{1}{n})^n$. Then $v_n - u_n = (1+\frac{1}{n})^n.\frac{1}{n}$
and $\lim(v_n - u_n) = \lim(1+\frac{1}{n})^n.\frac{1}{n} = 0$.
This implies $\lim v_n = \lim u_n$, since both the limits exist.
As $\lim u_n = e$, it follows that $\lim v_n = e$.

Note. $v_n - u_n = (1+\frac{1}{n})^n.\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Since $u_n < u_{n+1}$ and $v_{n+1} < v_n$ for all $n \in \mathbb{N}$, we have $u_n < u_{n+1} < v_{n+1} < v_n$.

Let $I_n = [u_n, v_n]$ be an interval. Then $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.
The sequence of intervals $\{I_n\}$ is such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$; and (ii) $\lim | I_n | = 0$.

By Cantor's theorem on nested intervals, $\bigcap_{n=1}^{\infty} I_n$ is a singleton set and the set is $\{e\}$.

Worked Examples.

1. Prove that the sequence $\{u_n\}$ defined by
$u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for all $n \geq 1$ converges to 2.

The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots \cdots\}$
$u_{n+1}^2 - u_n^2 = 2(u_n - u_{n-1})$
or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = 2(u_n - u_{n-1})$.
Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.
But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots \cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $2u_n = u_{n+1}^2 > u_n^2$ for all $n \in \mathbb{N}$.
That is, $u_n^2 - 2u_n < 0$ for all $n \in \mathbb{N}$
or, $u_n(u_n - 2) < 0$ for all $n \in \mathbb{N}$.
But $u_n > 0$. Therefore $u_n < 2$ for all $n \in \mathbb{N}$.
This shows that the sequence $\{u_n\}$ is bounded above and therefore it is convergent.

Let $\lim u_n = l$.

By definition, $u_{n+1}^2 = 2u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 2l$. Therefore l is either 0 or 2. But l cannot be 0 since the sequence $\{u_n\}$ is monotone increasing and $u_1 = \sqrt{2} > 1$.

Therefore $l = 2$. That is, the sequence converges to 2.

2. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

The sequence is $\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \cdots \cdots\}$

$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$.

or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots \cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $u_n^2 < u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$

or, $u_n^2 - u_n - 7 < 0$

or, $(u_n - \alpha)(u_n - \beta) < 0$ where α, β are the roots of the equation $x^2 - x - 7 = 0$. One of the roots is negative and the other is positive. Let $\alpha < 0$. & $\beta < 0$.

Since $u_n > 0$ for all $n \in \mathbb{N}, u_n - \alpha > 0$. Consequently, $u_n < \beta$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{u_n\}$ is bounded above and therefore the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.

By definition, $u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 7 + l$.

Therefore $(l - \alpha)(l - \beta) = 0$.

But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$. Therefore $l = \beta$. That is, the sequence converges to the positive root of the equation $x^2 - x - 7 = 0$.

3. Let $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n; v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n, n \geq 2$.

Show that $\{u_n\}$ is a monotone decreasing sequence and $\{v_n\}_{n=2}^{\infty}$ is a monotone increasing one and they converge to the same limit.

$u_{n+1} - u_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log(1 + \frac{1}{n})$.

$v_{n+1} - v_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \log(1 + \frac{1}{n})$.

As the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is a strictly monotone decreasing sequence converging to e, $(1+\frac{1}{n})^{n+1} > e$ for all $n \in \mathbb{N}$.

Therefore $\log(1+\frac{1}{n}) > \frac{1}{n+1}$ for all $n \in \mathbb{N}$

or, $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a strictly monotone decreasing sequence.

As the sequence $\{(1+\frac{1}{n})^n\}$ is a strictly monotone increasing sequence converging to e, $(1+\frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

Therefore $\log(1+\frac{1}{n}) < \frac{1}{n}$ for all $n \in \mathbb{N}$

or, $v_{n+1} > v_n$ for all $n \geq 2$.

Therefore the sequence $\{v_n\}$ is a strictly monotone increasing sequence.

Again $\frac{1}{n} > \log\frac{n+1}{n} = \log(n+1) - \log n$.

Therefore $1 > \log 2 - \log 1$, $\frac{1}{2} > \log 3 - \log 2, \cdots, \frac{1}{n} > \log(n+1) - \log n$.

So we have $1 + \frac{1}{2} + \cdots + \frac{1}{n} > \log(n+1) > \log n$.

Hence $u_n > 0$ for all $n \in \mathbb{N}$.

Therefore $\{u_n\}$ is a monotone decreasing sequence bounded below.

Hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = \gamma$.

Now $u_n - v_n = \frac{1}{n}$ for $n \geq 2$. Therefore $\lim v_n = \gamma$.

Thus the sequences $\{u_n\}$ and $\{v_n\}$ converge to the same limit γ.

Note 1. This limit γ is called **Euler's constant.**

Since $u_1 = 1$ and $\{u_n\}$ is a strictly monotone decreasing sequence, $\gamma < 1$. Since $v_2 = 1 - \log 2 = 1 - .69315 > .3$ and $\{v_n\}$ is a monotone increasing sequence, $\gamma > .3$. Therefore $.3 < \gamma < 1$.

The approximation of γ upto six places of decimal is given by $\gamma = 0.577215$.

Note 2. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ is denoted by γ_n. Then the sequence $\{\gamma_n\}$ converges to γ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \gamma_n + \log n$.

Evaluation of the limit of some sequences can be done by the help of Euler's constant.

For example, if $s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, then

$$\begin{aligned}
\lim s_n &= \lim[(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2n}) - (1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n})] \\
&= \lim[(\gamma_{2n} + \log 2n) - (\gamma_n + \log n)] \\
&= \lim[\gamma_{2n} - \gamma_n + \log 2] \\
&= \log 2, \text{ since } \lim \gamma_{2n} = \lim \gamma_n = \gamma.
\end{aligned}$$

4. Two sequences $\{x_n\}, \{y_n\}$ are defined by
$x_{n+1} = \frac{1}{2}(x_n + y_n); y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0, y_1 > 0$.
Prove that both the sequences converge to a common limit.

Case 1. Let $x_1 \neq y_1$.
$x_2 = \frac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} = y_2$.
Let us assume that $x_k > y_k$.
Then $x_{k+1} = \frac{1}{2}(x_k + y_k) > \sqrt{x_k y_k} = y_{k+1}$.
$x_k > y_k$ implies $x_{k+1} > y_{k+1}$ and $x_2 > y_2$.
By the principle of induction, $x_n > y_n$ for all $n \geq 2$.

$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n$ for all $n \geq 2$
$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n . y_n} = y_n$ for all $n \geq 2$.
So we have $y_2 < y_3 < y_4 < \cdots \cdots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim x_n = l$, $\lim y_n = m$.
$x_{n+1} = \frac{1}{2}(x_n + y_n)$ for all $n \in \mathbb{N}$.
Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l + m)$. i.e., $l = m$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

Case 2. Let $x_1 = y_1$.
In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$.
Therefore $\{x_n\}$ and $\{y_n\}$ both converge to the same limit x_1.

5. If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$, prove that the sequence $\{u_n\}$ converges to 3.

$u_n^2 - 2u_n u_{n+1} + 9 = 0$. This is a quadratic equation in u_n having real roots. Therefore $4u_{n+1}^2 - 36 \geq 0$.
This implies $u_{n+1} \geq 3$ for all $n \geq 1$, since $u_{n+1} > 0$ for all $n \geq 1$.

$$u_n - u_{n+1} = u_n - \tfrac{1}{2}(u_n + \tfrac{9}{u_n})$$
$$= \tfrac{1}{2}(u_n - \tfrac{9}{u_n}) = \tfrac{1}{2}(\tfrac{u_n^2 - 9}{u_n}) \geq 0 \text{ for all } n \geq 2.$$

Therefore $u_{n+1} \leq u_n$ for all $n \geq 2$.
This shows that the sequence $\{u_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.
$u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$. Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l + \frac{9}{l})$. This gives $l = 3$, since $l > 0$.

$(1+x/n)^{n}$ e^{x}

Exercises 7

1. (i) Give an example of a sequence of rational numbers that converges to an irrational number.

(ii) Give an example of a sequence of irrational numbers that converges to a rational number.

(iii) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n + v_n\}$ is convergent.

(iv) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent.

2. Find $\sup\{u_n\}$ and $\inf\{u_n\}$ where

(i) $u_n = (-1)^n + \cos\frac{n\pi}{4}$, (ii) $u_n = \frac{(-1)^n}{n} + \sin\frac{n\pi}{2}$.

3. Let $\{u_n\}$ be a bounded sequence and $\lim v_n = 0$. Prove that $\lim u_n v_n = 0$. Utilise this to prove that

(i) $\lim \frac{\sin n}{n} = 0$, (ii) $\lim \frac{(-1)^n n}{n^2+1} = 0$, (iii) $\lim(-1)^n a_n = 0$ if $\lim a_n = 0$.

4. Let $\{u_n\}, \{v_n\}$ be two real sequences with $\lim u_n = l, \lim v_n = m$.
If $x_n = \max\{u_n, v_n\}, y_n = \min\{u_n, v_n\}$ prove that the sequence $\{x_n\}$ converges to $\max\{l, m\}$ and the sequence $\{y_n\}$ converges to $\min\{l, m\}$.

[**Hint.** $\max\{a, b\} = \frac{1}{2}\{a + b + \mid a - b \mid\}, \min\{a, b\} = \frac{1}{2}\{a + b - \mid a - b \mid\}$ for all $a, b \in \mathbb{R}$.]

5. If $\{u_n\}$ be a bounded sequence and $x_r = \min\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, $y_r = \max\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, for $r \geq 1$, prove that $\{x_n\}$ and $\{y_n\}$ are both monotone convergent sequences.
If $\lim x_n = \lim y_n = l$ prove that the sequence $\{u_n\}$ converges to l.

[**Hint.** $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all n. $x_n \leq u_n \leq y_n$ for all n.]

6. Prove that the sequence $\{u_n\}$ is a null sequence.

(i) $u_n = \frac{n!}{n^n}$, (ii) $u_n = \frac{4^{3n}}{3^{4n}}$, (iii) $u_n = \frac{b^n}{n!}, b > 1$.

7. Use Sandwich theorem to prove that

(i) $\lim(\sqrt{n+1} - \sqrt{n}) = 0$, (ii) $\lim(2^n + 3^n)^{1/n} = 3$,

(iii) $\lim[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}] = 0$, (iv) $\lim \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n} = 0$.

[**Hint.** (iv) Let $u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$. Then $u_n < \frac{2.4\ldots2n}{3.5\ldots(2n+1)}$, since $\frac{n}{n+1} < \frac{n+1}{n+2}$ for all $n \geq 1$. Therefore $u_n^2 = u_n.u_n < \frac{1}{2n+1}$ for all $n \geq 1$.]

8. If $0 < u_1 < 1$ and $u_{n+1} = 1 - \sqrt{1 - u_n}$ for $n \geq 1$, prove that

(i) the sequence $\{u_n\}$ converges to 0 and (ii) $\lim \frac{u_{n+1}}{u_n} = \frac{1}{2}$.

[**Hint.** $1 - u_{n+1} = (1 - u_n)^{1/2} = (1 - u_{n-1})^{1/2^2} = \cdots = (1 - u_1)^{1/2^n}$.]

9. Prove that (i) $\lim \sqrt[n]{n+1} = 1$, (ii) $\lim \sqrt[n+1]{n} = 1$,

(iii) $\lim \frac{(n+1)^{2n}}{(n^2+1)^n} = e^2$, (iv) $\lim\{(1+\frac{1}{n^2})(1+\frac{2}{n^2})(1+\frac{3}{n^2})\}^{n^2} = e^6$.

[**Hint.** $\frac{(n+1)^{2n}}{(n^2+1)^n} = \{(1+\frac{1}{n})^n\}^2/\{(1+\frac{1}{n^2})^{n^2}\}^{\frac{n}{n^2}}$.]

10. Prove that the sequence $\{u_n\}$ defined by

(i) $u_1 = \sqrt{3}$ and $u_{n+1} = \sqrt{3u_n}$ for $n \geq 1$, converges to 3;

(ii) $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6+u_n}$ for $n \geq 1$, converges to 3.

11. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6+u_n}$ for $n \geq 1$. Show that

(i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$;

(ii) the sequence $\{u_n\}$ is monotone decreasing if $u_1 > 3$.

Find $\lim u_n$.

12. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \frac{3(1+u_n)}{5+u_n}$ for $n \geq 1$. Prove that

(i) the sequence $\{u_n\}$ is a decreasing sequence if $u_1 > 1$;

(ii) the sequence $\{u_n\}$ is an increasing sequence if $0 < u_1 < 1$.

(iii) $\lim u_n = 1$ in both cases.

13. Prove that the sequence $\{u_n\}$ defined by

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1}+u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1+3u_2}{4}$,

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \frac{u_{n+1}+2u_n}{3}$ for $n \geq 1$, converges to $\frac{2u_1+3u_2}{5}$,

(iii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to the limit $\sqrt[3]{u_1u_2^2}$,

(iv) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to the limit $3/(\frac{1}{u_1} + \frac{2}{u_2})$.

14. If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}(s_n + \frac{4}{s_n})$ for $n \geq 1$, prove that the sequence $\{s_n\}$ is a monotone decreasing sequence bounded below and $\lim s_n = 2$.

15. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$x_{n+1} = \sqrt{x_ny_n}$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$ converge to a common limit.

16. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$x_{n+1} = \frac{1}{2}(x_n + y_n)$, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$ converge to a common limit l where $l^2 = x_1y_1$.

17. Prove that the sequence $\{\gamma_n\}$ where $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$ is convergent. Hence find

(i) $\lim[1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}]$, (ii) $\lim[\frac{1}{1.3} + \frac{1}{2.5} + \cdots + \frac{1}{n(2n+1)}]$.

18. Let S be a non-empty subset of $\mathbb{R}$ having a limit point l. Show that there exists a sequence $\{u_n\}$ of distinct elements of S such that $\lim u_n = l$.

19. Let S be an infinite subset of $\mathbb{R}$ that is bounded above and let $\sup S \notin S$. Show that there exists a monotone increasing sequence $\{u_n\}$ with $u_n \in S$, such that $\lim u_n = \sup S$. $S = \{\{1 - \frac{1}{n}\} \mid n \in N\}$.

5.11. Subsequence.

Let $\{u_n\}$ be a real sequence and $\{r_n\}$ be a strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$ Then the sequence $\{u_{r_n}\}$ is said to be a *subsequence* of the sequence $\{u_n\}$. The elements of the subsequence $\{u_{r_n}\}$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Let $r : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers such that $r_1 < r_2 < \cdots < r_n < \cdots \cdots$ and $u : \mathbb{N} \to \mathbb{R}$ be a real sequence. Then the composite mapping $u o r : \mathbb{N} \to \mathbb{R}$ is said to be a *subsequence* of the real sequence u. The elements of the subsequence $u o r$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots \ldots$

Examples.

1. Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots \ldots\}$
   $= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots \cdots\}$ is a subsequence of $\{\frac{1}{n}\}$.

2. Let $u_n = \frac{1}{n}$ and $r_n = 2n - 1$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_1, u_3, u_5, \ldots \ldots\}$
   $= \{1, \frac{1}{3}, \frac{1}{5}, \cdots \cdots\}$ is a subsequence of $\{\frac{1}{n}\}$.

3. Let $u_n = (-1)^n$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots \ldots\}$
   $= \{1, 1, 1, \ldots \ldots\}$ is a subsequence of $\{(-1)^n\}$.

4. Let $u_n = 1 + 1/n$ and $r_n = n^2$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{1+1, 1+\frac{1}{2^2}, 1+\frac{1}{3^2}, \cdots \cdots\}$ is a subsequence of $\{1+\frac{1}{n}\}$.

Theorem 5.11.1. If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converges to l.

Proof. Let $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{u_{r_n}\}$ is subsequence of the sequence $\{u_n\}$.

Let $\epsilon > 0$. Since $\lim u_n = l$, there exists a natural number k such that $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$.

Therefore $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k_0$.

Since ϵ is arbitrary, $\lim\limits_{n\to\infty} u_{r_n} = l$.

Note. If there exist two different subsequences $\{u_{r_n}\}$ and $\{u_{k_n}\}$ of a sequence $\{u_n\}$ such that $\{u_{r_n}\}$ and $\{u_{k_n}\}$ converge to two different limits, then the sequence $\{u_n\}$ is not convergent.

If a sequence $\{u_n\}$ has a divergent subsequence then $\{u_n\}$ is divergent.

Worked Examples.

1. Prove that $\lim(1+\frac{1}{2n})^n = \sqrt{e}$.

Let $u_n = (1+\frac{1}{n})^n, v_n = (1+\frac{1}{2n})^{2n}$ and $w_n = (1+\frac{1}{2n})^n$ for all $n \in \mathbb{N}$.

$\{u_n\}$ is a convergent sequence and $\lim u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ and therefore $\lim v_n = e$.

Now $w_n = \sqrt{v}_n$ for all $n \in \mathbb{N}$. Therefore $\lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2. Prove that the sequence $\{(-1)^n\}$ is divergent.

Let $u_n = (-1)^n, v_n = u_{2n}, w_n = u_{2n-1}$.

Then $\{v_n\}$ is the subsequence $\{1, 1, 1, \ldots \ldots\}$ and $\lim v_n = 1$,

$\{w_n\}$ is the subsequence $\{-1, -1, -1, \ldots \ldots\}$ and $\lim w_n = -1$.

Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is divergent.

Theorem 5.11.2. If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$.

Proof. Let us choose $\epsilon > 0$. Since $\lim u_{2n} = l$, there exists a natural number k_1 such that $| u_{2n} - l | < \epsilon$ for all $n \geq k_1$.

Since $\lim u_{2n-1} = l$, there exists a natural number k_2 such that $| u_{2n-1} - l | < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then k is a natural number and for all $n \geq k$, $l - \epsilon < u_{2n} < l + \epsilon$ and $l - \epsilon < u_{2n-1} < l + \epsilon$.

That is, $l - \epsilon < u_n < l + \epsilon$ for all $n \geq 2k - 1$.

As $2k - 1$ is a natural number, it follows that $\lim u_n = l$.

Note 1. If two subsequences of a sequence converge to the same limit l, the sequence $\{u_n\}$ may not be convergent.

For example, let $u_n = \sin \frac{n\pi}{4}$.

Then the subsequence $\{u_{8n-7}\}$ is $\{\sin \frac{\pi}{4}, \sin \frac{9\pi}{4}, \sin \frac{17\pi}{4}, \cdots\}$ and this converges to $\frac{1}{\sqrt{2}}$.

The subsequence $\{u_{8n-5}\}$ is $\{\sin \frac{3\pi}{4}, \sin \frac{11\pi}{4}, \sin \frac{19\pi}{5}, \cdots\}$ and this converges to $\frac{1}{\sqrt{2}}$.

But the sequence $\{u_n\}$ is not convergent.

2. If $k \in \mathbb{N}$ and k subsequences $\{u_{kn}\}$, $\{u_{kn-1}\}, \{u_{kn-2}\}, \cdots \{u_{kn-k+1}\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$.

Worked Examples (continued.)

4. Prove that the sequence $\{u_n\}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$, is convergent.

$u_3 - u_1 = \frac{u_1+u_2}{2} - u_1 = \frac{(u_2-u_1)}{2} > 0$, i.e., $u_1 < u_3$.

$u_3 - u_2 = \frac{u_1+u_2}{2} - u_2 = \frac{(u_1-u_2)}{2} < 0$, i.e., $u_3 < u_2$.

$\frac{u_1 + 2u_2}{3}$.

So $u_1 < u_3 < u_2$.

Similarly, $u_3 < u_4 < u_2, u_3 < u_5 < u_4, u_5 < u_6 < u_4, \cdots \cdots$

The inequalities give

$u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence bounded above, u_2 being an upper bound; and the sequence $\{u_{2n}\}$ is a monotone decreasing sequence bounded below, u_1 being a lower bound .

So both the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ are convergent.

Let $\lim u_{2n} = l, \lim u_{2n-1} = m$.

Now $2u_{2n+2} = u_{2n} + u_{2n+1}$ for all $n \in \mathbb{N}$.

Proceeding to limit as $n \to \infty$, we have $2l = l + m$, i.e., $l = m$.

Thus the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit l and therefore the sequence $\{u_n\}$ is convergent.

5. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

(i) Prove that the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.

(ii) Find $\lim u_n$.

$u_{n+1} - u_n = \frac{6}{1+u_n} - u_n = \frac{6-u_n-u_n^2}{1+u_n} = \frac{(2-u_n)(3+u_n)}{1+u_n}$.

Therefore $u_n < 2 \Rightarrow u_n < u_{n+1}$; $u_n > 2 \Rightarrow u_n > u_{n+1}$.

Again $u_n < 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} > 2; u_n > 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} < 2$.

It follows that

$u_n < 2 \Rightarrow u_n < 2 < u_{n+1}$; $u_n > 2 \Rightarrow u_{n+1} < 2 < u_n$ ... (i)

$u_{n+2} - u_n = \frac{6(1+u_n)}{7+u_n} - u_n = \frac{6-u_n-u_n^2}{7+u_n} = \frac{(2-u_n)(3+u_n)}{7+u_n}$.

$u_n < 2 \Rightarrow u_n < u_{n+2}$; $u_n > 2 \Rightarrow u_n > u_{n+2}$ ... (ii)

Case 1. Let $u_1 < 2$. Then $u_2 > 2$.

From (i) $u_1 < 2 \Rightarrow u_1 < 2 < u_2$; $u_2 > 2 \Rightarrow u_3 < 2 < u_2$; $u_3 < 2 \Rightarrow u_3 < 2 < u_4$; $u_4 > 2 \Rightarrow u_5 < 2 < u_4; \cdots$

From (ii) $u_1 < 2 \Rightarrow u_1 < u_3; u_3 < 2 \Rightarrow u_3 < u_5; \cdots$
$u_2 > 2 \Rightarrow u_2 > u_4;\ u_4 > 2 \Rightarrow u_4 > u_6;\ \cdots$
Therefore $u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the subsequence $\{u_{2n-1}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n}\}$ is a monotone decreasing sequence, bounded below. Hence both the subsequences are convegent.

Let $\lim u_{2n-1} = l$, $\lim u_{2n} = m$.

We have $u_{2n} = \frac{6}{1+u_{2n-1}}$, $u_{2n+1} = \frac{6}{1+u_{2n}}$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $m = \frac{6}{1+l}, l = \frac{6}{1+m}$.

Therefore $l = m$ and the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.

Case 2. $u_1 > 2$.

From (i) and (ii) it follows that $u_2 < u_4 < u_6 < \cdots < u_5 < u_3 < u_1$.

The subsequence $\{u_{2n}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n-1}\}$ is a monotone decreasing sequence, bounded below.

Hence both the sequences are convergent.

Proceeding as in case 1, it can be shown that they converge to a common limit.

(ii) Let the limit be l. We have $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l = \frac{6}{1+l}$. This gives $l = 2$, or $l = -3$.

As $u_n > 0$ for all $n \in \mathbb{N}, l \neq -3$. Therefore $\lim u_n = 2$.

Theorem 5.11.3. Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).

Proof. (i) Let $\{u_n\}$ be a monotone increasing sequence. Then for any two natural numbers p, q with $p > q, u_p \geq u_q$.

Let $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers. This implies $r_{n+1} > r_n$ for all $n \in \mathbb{N}$.

$r_{n+1} > r_n \Rightarrow u_{r_{n+1}} \geq u_{r_n}$ for all n.

This proves that $\{u_{r_n}\}$ is a monotone increasing subseqeunce.

(ii) Similar proof for a monotone decreasing sequence $\{u_n\}$.

Theorem 5.11.4. A monotone sequence of real numbers having a convergent subsequence with limit l, is convergent with limit l.

Proof. Let $\{u_n\}$ be a monotone increasing sequence and $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$ such that $\lim u_{r_n} = l$.

Since $\{u_n\}$ is a monotone increasing sequence, the subsequence $\{u_{r_n}\}$ is also monotone increasing, by Theorem 5.11.3.

Since $\{u_{r_n}\}$ is a convergent sequence, it is bounded above.

We assert that the sequence $\{u_n\}$ is bounded above. If not, let $\{u_n\}$ be unbounded above. Then being a monotone increasing sequence it must diverge to ∞ and therefore for a pre-assigned positive number G, however large, there must exist a natural number k such that $u_n > G$ for all $n \geq k$. Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$. Consequently, $u_{r_n} > G$ holds for all $n \geq k_0$.

Since G is arbitrary, the sequence $\{u_{r_n}\}$ must diverge to ∞, a contradiction. So our assertion is established and the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence, bounded above, is convergent.

Let $\lim u_n = m$. Then $\{u_{r_n}\}$ being subsequence of $\{u_n\}$ converges to m, by Theorem 5.11.1. Therefore $l = m$. This completes the proof.

Theorem 5.11.5. A monotone sequence of real numbers having a divergent subsequence is properly divergent.

Proof. Let $\{u_n\}$ be a montone increasing sequence having a divergent subsequence $\{u_{r_n}\}$. Since the sequence $\{u_n\}$ is monotone increasing, the subsequence $\{u_{r_n}\}$ is also monotone increasing and therefore it is a properly divergent subsequence. Consequently, the subsequence $\{u_{r_n}\}$ is unbounded above. Hence the sequence $\{u_n\}$ must be unbounded above and therefore it is properly divergent.

Similar proof if $\{u_n\}$ be a monotone decreasing sequence.

Worked Example.

1. Prove that the sequence $\{u_n\}$ is divergent where $u_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

$u_{n+1} - u_n = \frac{1}{n+1} > 0$ for all n. Therefore the sequence $\{u_n\}$ is a monotone increasing sequence.

Let $r_n = 2^n$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers and so the sequence $\{u_{r_n}\}$ is a subsequence of $\{u_n\}$.

Now $u_{r_n} = u_{2^n}$

$$
\begin{aligned}
&= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n}\\
&= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \cdots + \tfrac{1}{8}) + \cdots + (\tfrac{1}{2^{n-1}+1} + \cdots + \tfrac{1}{2^n})\\
&> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \cdots + \tfrac{1}{8}) + \cdots + (\tfrac{1}{2^n} + \cdots + \tfrac{1}{2^n})\\
&= 1 + \tfrac{1}{2} + 2\tfrac{1}{2^2} + 2^2.\tfrac{1}{2^3} + \cdots + 2^{n-1}.\tfrac{1}{2^n}\\
&= 1 + \tfrac{n}{2}.
\end{aligned}
$$

Let $v_n = 1 + \frac{n}{2}$. Then $u_{r_n} > v_n$ for all $n > 2$ and $\lim v_n = \infty$. Therefore $\lim u_{r_n} = \infty$.

Thus the sequence $\{u_n\}$ is a monotone increasing sequence having a properly divergent subsequence $\{u_{r_n}\}$ and therefore the sequence $\{u_n\}$ is properly divergent.

Theorem 5.11.6. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{u_n\}$ be a sequence of real numbers. An elerment u_k is said to be a *peak* of the sequence $\{u_n\}$ if $u_k \geq u_n$ for all $n > k$, i.e., u_k is greater than or equal to all subsequent elements beyond u_k. A sequence may or may not have a peak or else it may have a finite or an infinite number of peaks.

We consider the following cases.

Case 1. Let the sequence $\{u_n\}$ have infinitely many peaks.

Let the peaks be $u_{r_1}, u_{r_2}, u_{r_3}, \ldots$ (u_{r_1} being the first peak, u_{r_2} being the second,...). Then $u_{r_1} \geq u_{r_2} \geq u_{r_3} \geq \cdots$

The subsequence $\{u_{r_1}, u_{r_2}, u_{r_3}, \ldots\}$ is a monotone decreasing subsequence.

Case 2. Let the sequence have either no peak or a finite number of peaks.

Let the peaks be arranged in ascending order of the subscripts as $u_{r_1}, u_{r_2}, \ldots, u_{r_m}$. Let $s_1 = r_m + 1$. Then u_{s_1} is not a peak and there is no peak beyond the element u_{s_1}.

Since u_{s_1} is not a peak, there is an $s_2 \in \mathbb{N}$ with $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$.

Since u_{s_2} is not a peak, there is an $s_3 \in \mathbb{N}$ with $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$.

Proceeding thus we obtain natural numbers s_i such that $s_1 < s_2 < s_3 < \cdots$ and $u_{s_1} < u_{s_2} < u_{s_3} < \cdots$

Clearly, the subsequence $\{u_{s_n}\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

This completes the proof.

Examples

1. Let $u_n = \sin \frac{n\pi}{2}, n \in \mathbb{N}$. The sequence is $\{1, 0, -1, 0, 1, 0, -1, 0, \cdots\}$
   Here $u_1 \geq u_n$ for all $n > 1$. Therefore u_1 is a peak.
   $u_5 \geq u_n$ for all $n > 5$. Therefore u_5 is a peak.
   Here $u_9 \geq u_n$ for all $n > 9$. Therefore u_9 is a peak.
   ... ...

The infinitely many peaks are $u_1, u_5, u_9, \ldots$

The subsequence $\{u_1, u_5, u_9, u_{13}, \ldots\}$ is a monotone subsequence of the sequence $\{u_n\}$.

2. Let $u_n = n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \cdots\}$.

Here the sequence $\{u_n\}$ has no peak.

u_1 is not a peak. Let $s_1 = 1$. Since u_{s_1} is not a peak, there is a natural number $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$. Here $s_2 = 2$.

Since u_{s_2} is not a peak, there is a natural number $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$. Here $s_3 = 4$.

By similar arguments, $s_4 = 6, s_5 = 8, \ldots$

Thus $\{u_1, u_2, u_4, u_6, u_8, \ldots\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

5.12. Subsequential limit.

Let $\{u_n\}$ be a real sequence. A real number l is said to be a *subsequential limit* of the sequence $\{u_n\}$ if there exists a subsequence of $\{u_n\}$ that converges to l.

Theorem 5.12.1. A real number l is a subsequential limit of a sequence $\{u_n\}$ if and only if every neighbourhood of l contains infinitely many elements of the sequence $\{u_n\}$.

Proof. Let l be a subsequential limit of the sequence $\{u_n\}$. Then there exists a subsequence $\{u_{r_n}\}$ such that $\lim_{n\to\infty} u_{r_n} = l$.

Let us choose a positive ϵ. Then there exists a natural number k such that $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k$.

Therefore $l - \epsilon < u_n < l + \epsilon$ for infinitely many values of n.

Since ϵ is arbitrary, every neighbourhood of l contains infinite number of elements of the sequence $\{u_n\}$.

Conversely, let the sequence $\{u_n\}$ be such that for each pre-assigned positive ϵ the ϵ-neighbourhood of l contains infinitely many elements of the sequence.

Let $\epsilon = 1$. Then $l - 1 < u_n < l + 1$ for infinitely many values of n. Therefore the set $S_1 = \{n : l - 1 < u_n < l + 1\}$ is an infinite subset of the set $\mathbb{N}$. By the well ordering property of the set $\mathbb{N}$, S_1 has a least element, say r_1.

Therefore $l - 1 < u_{r_1} < l + 1$.

Let $\epsilon = \frac{1}{2}$. Then $l - \frac{1}{2} < u_n < l + \frac{1}{2}$ for infinitely many values of n. Therefore the set $S_2 = \{n : l - \frac{1}{2} < u_n < l + \frac{1}{2}\}$ is an infinite subset of $\mathbb{N}$ and hence there exists a natural number $r_2 (> r_1)$ in S_2 such that $l - \frac{1}{2} < u_{r_2} < l + \frac{1}{2}$.

Continuing thus, we obtain a strictly increasing sequence of natural numbers $\{r_1, r_2, r_3, \ldots \ldots\}$ such that $l - \frac{1}{n} < u_{r_n} < l + \frac{1}{n}$ for all $n \in \mathbb{N}$.

By Sandwich theorem, $\lim u_{r_n} = l$.

In other words the subsequence $\{u_{r_n}\}$ converges to l.

That is, l is a subsequential limit of the sequence $\{u_n\}$.

Note. The limit of a sequence, if it exists, is also a subsequential limit of the sequence.

Theorem 5.12.2. (Bolzano-Weierstrass theorem)

Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let $\{u_n\}$ be a bounded sequence. Then there is a closed and bounded interval, say $I = [a, b]$, such that $u_n \in I$ for every $n \in \mathbb{N}$.

Let $c = \frac{a+b}{2}$ and $I' = [a, c], I'' = [c, b]$. Then at least one of the intervals I' and I'' contains infinitely many elements of $\{u_n\}$.

Let $I_1 = [a_1, b_1]$ be such an interval. Then $I_1 \subset I$ and $| I_1 |$ = the length of the interval $= \frac{1}{2}(b - a)$.

Let $c_1 = \frac{a_1+b_1}{2}$ and $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. Then at least one of the intervals I_1' and I_1'' contains infinitely many elements of $\{u_n\}$. Let $I_2 = [a_2, b_2]$ be such an interval.

Then $I_2 \subset I_1$ and $| I_2 | = \frac{1}{2} | I_1 |$.

Continuing thus, we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(ii) $| I_n | = \frac{1}{2^n}(b - a)$ and therefore $\lim\limits_{n \to \infty} | I_n | = 0$; and

(iii) each I_n contains infinitely many elements of $\{u_n\}$.

By Cantor's theorem on nested intervals, there exists a unique point α such that $\alpha \in \bigcap\limits_{n=1}^{\infty} I_n$.

We prove that α is a subsequential limit of the sequence $\{u_n\}$.

Let us choose $\epsilon > 0$. There exists a natural number k such that

$0 < \frac{b-a}{2^k} < \epsilon$. That is, $| I_k | < \epsilon$.

Since $\alpha \in I_k$ and $| I_k | < \epsilon, I_k$ is entirely contained in the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon)$ and consequently, the ϵ-neighbourhood of α contains infinitely many elements of $\{u_n\}$.

Since ϵ is arbitrary, each neighbourhood of α contains infinitely many elements of $\{u_n\}$. Therefore α is a subsequential limit of $\{u_n\}$.

Therefore there exists a subsequence of $\{u_n\}$ that converges to α. In other words, $\{u_n\}$ has a convergent subsequence.

This completes the proof.

Note. Another version of the theorem is -- Every bounded sequence of real numbers has a subsequential limit.

Examples.

1. The sequence $\{u_n\}$ where $u_n = \sin\frac{n\pi}{2}, n \geq 1$ is a bounded sequence since $|u_n| \leq 1$ for all $n \geq 1$.

(i) The subsequence $\{u_1, u_5, u_9, \ldots \ldots\}$, i.e., $\{u_{4n-3}\}$ is a convergent subsequence that converges to 1.

(ii) The subsequence $\{u_2, u_4, u_6, \ldots \ldots\}$, i.e., $\{u_{2n}\}$ is a convergent subsequence that converges to 0.

(iii) The subsequence $\{u_1, u_3, u_5, \ldots \ldots\}$, i.e., $\{u_{2n-1}\}$ is a $\boxed{\text{divergent}}$ subsequence.

$S_n = \sin\frac{n\pi}{2}$, inwhich S.S is C

Note. The example 1(iii) shows that a bounded sequence may have a ~~divergent~~ Not Congent subsequence.

2. The sequence $\{u_n\}$ where $u_n = n^{(-1)^n}$ is an unbounded sequence.

The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots \ldots\}$.

(i) The sequence $\{u_2, u_4, u_6, \ldots \ldots\}$, i.e., $\{u_{2n}\}$ is a properly divergent subsequence.

(ii) The sequence $\{u_1, u_3, u_5, \ldots \ldots\}$, i.e., $\{u_{2n-1}\}$ is a convergent subsequence.

Note. The example 2(ii) shows that an unbounded sequence may have a convergent subsequence.

5.13. Characterisation of a compact set.

Theorem 5.13.1. Let K be a non-empty subset of $\mathbb{R}$. Then K is compact if and only if every sequence in K has a subsequence convergent to a point in K.

Proof. Let K be a compact set. Let $\{x_n\}$ be a sequence in K.

Since K is compact, K is a closed and bounded set. Since $\{x_n\}$ is a sequence in K, it is a bounded sequence and by Bolzano-Weierstrass theorem it has a convergent subsequence, say $\{x_{r_n}\}$. Let $\lim\limits_{n\to\infty} x_{r_n} = l$.

We prove that $l \in K$.

Let $l \notin K$. Then $l \in \mathbb{R} - K$. Since K is a closed set, it follows that $\mathbb{R} - K$ is an open set and l is an interior point of $\mathbb{R} - K$. So there exists a neighbourhood $N(l)$ of l such that $N(l) \subset \mathbb{R} - K$.

Hence $N(l)$ contains no element of the sequence $\{x_{r_n}\}$ and therefore l cannot be the limit of the sequence $\{x_{r_n}\}$, a contradiction.

Therefore $l \in K$.

Thus every sequence in K has a subsequence convergent to a point in K.

Conversely, suppose that K is a non-empty subset of $\mathbb{R}$ with the property that every sequence in K has a subsequence convergent to a point in K. Let T be an infinite subset of K.

Let $x_1 \in T, x_2 \in T - \{x_1\}, x_3 \in T - \{x_1, x_2\}, \ldots \ldots$

Continuing thus we obtain a sequence $\{x_n\}$ of distinct elements in K. By hypothesis there is a subsequence $\{x_{r_n}\}$ which converges to some point x in K. Therefore x is a limit point of the set T.

Thus K is such that every infinite subset of K has a limit point in K and therefore K is compact. [Theorem 3.16.4]

This completes the proof.

Worked Examples.

1. If S and T are disjoint compact subsets of $\mathbb{R}$ prove that $d(S,T) > 0$, where $d(S,T) = \inf\{| x - y |: x \in S, y \in T\}$.

Let $P = \{| x - y |: x \in S, y \in T\}$.

Since S and T are disjoint subsets, none is empty. Let $s \in S, t \in T$.

Then $| s - t | \in P$ and therefore P is a non-empty subset of $\mathbb{R}$.

Also P is a set of non-negative elements and therefore P is bounded below, 0 being a lower bound.

By the infimum property of $\mathbb{R}$, $\inf P$ exists and $\inf P \geq 0$. We prove that $\inf P > 0$.

If $\inf P = 0$, then for a pre-assigned positive ϵ there exist points $x' \in S, y' \in T$ such that $0 \leq | x' - y' | < 0 + \epsilon$.

Since S and T are disjoint $x' \neq y'$. Therefore $0 < | x' - y' | < \epsilon$.

For $\epsilon = \frac{1}{n}$, there exist points $x_n \in S, y_n \in T$ such that

$$0 < | x_n - y_n | < \tfrac{1}{n}.$$

This holds for all $n \in \mathbb{N}$.

$\{x_n\}$ is a sequence of points in S. Since S is compact, there is a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ converging to a point, say x, in S.

Now $| x_{r_n} - y_{r_n} | < \frac{1}{r_n}$ for all $n \in \mathbb{N}$.

Therefore $\lim(x_{r_n} - y_{r_n}) = 0$ and since $\lim x_{r_n} = x$, we have $\lim y_{r_n} = x$.

But $\{y_{r_n}\}$ is a convergent sequence in T and since T is compact, $\lim y_{r_n} \in T$. That is, $x \in T$.

Therefore $x \in S$ and $x \in T$. This contradicts that S and T are disjoint.

Hence $d(S,T) = \inf P > 0$.

2. Let K be a non-empty compact set in $\mathbb{R}$ and $p \in \mathbb{R}$. Prove that there exists a point c in K such that $\sup\{| p - x |: x \in K\} = | p - c |$.

Since K is compact, K is bounded. Therefore the set $H = \{| p - x |: x \in K\}$ is a bounded set. Since K is non-empty, H is non-empty.

By the supremum property of $\mathbb{R}$, $\sup H$ exists. Let $\sup H = M$.

Let $\epsilon > 0$. Then there exists an element in the set, say $| p - x_0 |$, such that $M - \epsilon < | p - x_0 | \leq M$..

Let $\epsilon = 1$. Then there is an element $x_1 \in K$ such that
$M - 1 < | p - x_1 | \leq M$.
Let $\epsilon = \frac{1}{2}$. Then there exists an element $x_2 \in K$ such that
$M - \frac{1}{2} < | p - x_2 | \leq M$.
... ...

Proceeding in this way we obtain a sequence $\{x_n\}$ in K. Since K is compact, $\{x_n\}$ has a convergent subsequence $\{x_{r_n}\}$ that converges to a point c in K.

Now $M - \frac{1}{r_n} < | p - x_{r_n} | \leq M$ for all $n \in \mathbb{N}$.
$\lim x_{r_n} = c \Rightarrow \lim(p - x_{r_n}) = p - c \Rightarrow \lim | p - x_{r_n} | = | p - c |$.
Also $\lim M - \frac{1}{r_n} = M$.
By Sandwich theorem, $| p - c | = M$.

5.14. The upper limit and the lower limit.

Let $\{u_n\}$ be a bounded sequence of real numbers. Then by Bolzano-Weierstrass theorem there is a convergent subsequence of $\{u_n\}$. In other words, there is a subsequential limit of $\{u_n\}$. Since $\{u_n\}$ is bounded, the set S of all subsequential limits of $\{u_n\}$ is a bounded set.

Case 1. S is a finite set. Then S has a greatest element. → Maximum element

Case 2. S is an infinite set. Being a bounded set, S has a least upper bound. sup S

Let u^* be the lub of S. Then there is an element of S greater than $u^* - 1$. That is, $\{u_n\}$ has a subsequential limit $l_1 > u^* - 1$. Since every neighbourhood of l_1 contains an infinite number of elements of $\{u_n\}$, there is a natural number r_1 such that $u_{r_1} > u^* - 1$.

There is an element of S greater than $u^* - \frac{1}{2}$. That is, $\{u_n\}$ has a subsequential limit $l_2 > u^* - \frac{1}{2}$. Therefore there is a natural number $r_2 > r_1$ such that $u_{r_2} > u^* - \frac{1}{2}$.

Proceeding similarly, we obtain a strictly increasing sequence of natural numbers $\{r_1, r_2, r_3, \ldots \ldots\}$ such that $u_{r_n} > u^* - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. There is a natural number k such that $0 < \frac{1}{n} < \epsilon$ for all $n \geq k$.

Therefore $| u_{r_n} - u^* | < \frac{1}{n} < \epsilon$ for all $n \geq k$.

Since ϵ is arbitrary, the subsequence $\{u_{r_n}\}$ converges to u^*.

Therefore u^* is a subsequential limit of the sequence $\{u_n\}$.

Since u^* is the lub of the set S, u^* is the *greatest subsequential limit* of the sequence $\{u_n\}$.

In a similar manner it can be established that a bounded sequence $\{u_n\}$ has a *least subsequential limit* u_*.

Definition. Let $\{u_n\}$ be a bounded sequence of real numbers.

The greatest subsequential limit of $\{u_n\}$ is said to be *the upper limit* or *the limit superior* of $\{u_n\}$ and this is denoted by $\overline{\lim}\, u_n$ or $\lim\sup u_n$.

The least subsequential limit of $\{u_n\}$ is said to be *the lower limit* or *the limit inferior* of $\{u_n\}$ and this is denoted by $\underline{\lim}\, u_n$ or $\lim\inf u_n$.

If $\{u_n\}$ is unbounded above then we define $\overline{\lim}\, u_n = \infty$

If $\{u_n\}$ is unbounded below then we define $\underline{\lim}\, u_n = -\infty$

If $\{u_n\}$ be unbounded above but bounded below, then $\underline{\lim}\, u_n$ is defined to be the *least subsequential limit.* If there is no subsequential limit, we define $\underline{\lim}\, u_n = \infty$.

If $\{u_n\}$ be unbounded below but bounded above, then $\overline{\lim}\, u_n$ is defined to be the *greatest subsequential limit.* If there is no subsequential limit, we define $\overline{\lim}\, u_n = -\infty$.

Examples.

1. Let $u_n = (-1)^n(1 + \frac{1}{n}), n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = 1, \underline{\lim}\, u_n = -1$.

2. Let $u_n = \frac{1}{n}, n \geq 1$. Then the sequence $\{u_n\}$ is a bounded sequence. $\overline{\lim}\, u_n = \underline{\lim}\, u_n = 0$.

3. Let $u_n = (-1)^n n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and unbounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = -\infty$.

4. Let $u_n = n^{(-1)^n \cdot n}, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and bounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = 0$.

5. Let $u_n = n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and bounded below. $\overline{\lim}\, u_n = \infty, \underline{\lim}\, u_n = \infty$.

6. Let $u_n = -n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded below and bounded above. $\overline{\lim}\, u_n = -\infty, \underline{\lim}\, u_n = -\infty$.

Let $\{u_n\}$ be a bounded sequence and $u^* = \overline{\lim}\, u_n, u_* = \lim u$

Let B be the least upper bound and b be the greatest lower bound of the sequence $\{u_n\}$. Then $b \le u^* \le B$ and also $b \le u_* \le B$.

Since u^* is a subsequential limit of $\{u_n\}$, each ϵ-neighbourhood of u^* contains infinite number of elements of the sequence $\{u_n\}$.

Therefore for a positive $\epsilon, u^* - \epsilon < u_n < u^* + \epsilon$ for infinitely many values of n.

Also $u_n > u^* + \epsilon$ for at most a finite number of elements of$\{u_n\}$.

Because, if $u_n > u^* + \epsilon$ for infinitely many values of n, then there is a subsequence $\{u_{r_n}\}$ whose elements lie in the closed interval $[u^* + \epsilon, B]$, and $\{u_{r_n}\}$ being itself a bounded sequence, must have a subsequential limit l lying in $[u^* + \epsilon, B]$. l being also a subsequential limit of $\{u_n\}$, is greater than u^* and thereby u^* fails to be the upper limit of $\{u_n\}$.

Thus the upper limit u^* satisfies the following conditions.

For each positive ϵ,

(i) $u_n > u^* - \epsilon$ for infinitely many values of n, and

(ii) there exists a natural number k such that $u_n < u^* + \epsilon$ for all $n \ge k$.

By similar arguments, the lower limit u_* satisfies the following conditions.

For each positive ϵ,

(i) $u_n < u_* + \epsilon$ for infinitely many values of n, and

(ii) there exists a natural number k such that $u_n > u_* - \epsilon$ for all $n \ge k$.

Theorem 5.14.1. A bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Proof. Let $\{u_n\}$ be a convergent sequence and $\lim u_n = l$.

Since $\{u_n\}$ is convergent, every subsequence of $\{u_n\}$ converges to l. Therefore l is the greatest as well as the least subsequential limit.

That is, $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Conversely, let $\{u_n\}$ be a bounded sequence such that $\overline{\lim}\, u_n = \underline{\lim}\, u_n$.

Let $\overline{\lim}\, u_n = \underline{\lim}\, u_n = l$.

Since $\overline{\lim}\, u_n = l$, for a pre-assigned positive ϵ there exists a natural number k_1 such that $u_n < l + \epsilon$ for all $n \ge k_1$.

Since $\underline{\lim}\, u_n = l$, corresponding to the same ϵ there exists a natural number k_2 such that $u_n > l - \epsilon$ for all $n \ge k_2$.

Let $k = \max\{k_1, k_2\}$. Then $l - \epsilon < u_n < l + \epsilon$ for all $n \ge k$.

This proves that $\lim u_n = l$.

In other words, the sequence $\{u_n\}$ is convergent.

Note. The theorem can be restated as –

A bounded sequence is convergent if and only if it has only one subsequential limit.

Theorem 5.14.2. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences. Then

(i) $\overline{\lim}\, u_n + \overline{\lim}\, v_n \geq \overline{\lim}\, (u_n + v_n)$

(ii) $\underline{\lim}\, u_n + \underline{\lim}\, v_n \leq \underline{\lim}\, (u_n + v_n)$.

Proof. (i) Since $\{u_n\}$ and $\{v_n\}$ are both bounded sequences, the sequence $\{u_n + v_n\}$ is a bounded sequence.

Let $\overline{\lim}\, u_n = l_1, \overline{\lim}\, v_n = l_2, \overline{\lim}\, (u_n + v_n) = p$.

Let us choose $\epsilon > 0$.

Since $\overline{\lim}\, u_n = l_1$, there exists a natural number k_1 such that

$$u_n < l_1 + \tfrac{\epsilon}{2} \text{ for all } n \geq k_1.$$

Since $\overline{\lim}\, v_n = l_2$, there exists a natural number k_2 such that

$$v_n < l_2 + \tfrac{\epsilon}{2} \text{ for all } n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.

Then $u_n < l_1 + \frac{\epsilon}{2}$ and $v_n < l_2 + \frac{\epsilon}{2}$ for all $n \geq k$.

So $u_n + v_n < l_1 + l_2 + \epsilon$ for all $n \geq k$.

It follows that no subsequential limit of $\{u_n + v_n\}$ can be greater than $l_1 + l_2 + \epsilon$. Since $\epsilon (> 0)$ is arbitrary, every subsequential limit $\leq l_1 + l_2$. Hence $p \leq l_1 + l_2$.

(ii) proof left to the reader.

Note. Strict inequality in both (i) and (ii) may occur.

For example; if $u_n = \sin \frac{n\pi}{2}, n \in \mathbb{N}; v_n = \cos \frac{n\pi}{2}, n \in \mathbb{N}$ then

$\underline{\lim}\, (u_n + v_n) = -1, \underline{\lim}\, u_n = -1, \underline{\lim}\, v_n = -1.$

$\overline{\lim}\, u_n = 1, \overline{\lim}\, v_n = 1, \overline{\lim}\, (u_n + v_n) = 1.$

So in this case $\underline{\lim}\, u_n + \underline{\lim}\, v_n < \underline{\lim}\, (u_n + v_n)$ and $\overline{\lim}\, u_n + \overline{\lim}\, v_n > \overline{\lim}\, (u_n + v_n)$.

5.15. Cauchy criterion.

We discussed several methods of establishing convergence of a real sequence. In most of the methods, a prior knowledge of the limit is necessary. If however a sequence is monotone, the convergence can be established without any pre-conceived limit.

Cauchy's method of establishing convergence of a sequence does not require any knowledge of its limit, nor does it require the sequence to be monotone.

The method is very powerful as it is concerned only with the elements of the sequence.

Theorem 5.15.1. (Cauchy's general principle of convergence)

A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+p} - u_n |< \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \ldots \ldots$

Proof. Let $\{u_n\}$ be convergent and $\lim u_n = l$. Then for a pre-assigned positive ϵ there exists a natural number m such that

$$| u_n - l |< \tfrac{\epsilon}{2} \text{ for all } n \geq m.$$

Therefore $| u_{n+p} - l |< \frac{\epsilon}{2}$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

$$\begin{aligned} \text{Now } | u_{n+p} - u_n | &\leq | u_{n+p} - l | + | u_n - l | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq m \text{ and } p = 1, 2, 3, \ldots \end{aligned}$$

That is, $| u_{n+p} - u_n |< \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the condition is necessary.

We now prove that the sequence $\{u_n\}$ is convergent under the stated condition. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$| u_{n+p} - u_n |< 1 \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Therefore $| u_{k+p} - u_k |< 1$ for $p = 1, 2, 3, \ldots \ldots$

or, $u_k - 1 < u_{k+p} < u_k + 1$ for $p = 1, 2, 3, \ldots \ldots$

Let $B = \max\{u_1, u_2, \ldots, u_k, u_k + 1\}, b = \min\{u_1, u_2, \ldots, u_k, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$.

This proves that $\{u_n\}$ is a bounded sequence.

By Bolzano-Weierstrass theorem, the sequence $\{u_n\}$ has a convergent subsequence. Let l be the limit of that subsequence. Then l is a subsequential limit of $\{u_n\}$.

Let $\epsilon > 0$. Then by the given condition, there exists a natural number m such that

$$| u_{n+p} - u_n |< \tfrac{\epsilon}{3} \text{ for all } n \geq m \text{ and } p = 1, 2, 3, \ldots$$

Taking $m = n$, it follows that

$$| u_{m+p} - u_m |< \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \ldots \quad \text{(i)}$$

Since l is a subsequential limit of $\{u_n\}$, each ϵ-neighbourhood of l contains infinite number of elements of $\{u_n\}$. Therefore there exists a natural number $q > m$ such that $| u_q - l |< \frac{\epsilon}{3}$.

As $q > m$, it follows from (i) that $| u_q - u_m |< \frac{\epsilon}{3}$.

$$\begin{aligned} \text{Now } | u_{m+p} - l | &\leq | u_{m+p} - u_m | + | u_m - u_q | + | u_q - l | \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \text{ for } p = 1, 2, 3, \ldots \ldots \end{aligned}$$

Therefore $| u_n - l |< \epsilon$ for all $n \geq m + 1$.

Since ϵ is arbitrary, the sequence $\{u_n\}$ converges to l.

In other words, $\{u_n\}$ is a convergent sequence. This completes the proof.

Note. The condition stated in the theorem is called the "*Cauchy condition*" for convergence of a sequence.

Therefore a sequence $\{u_n\}$ is convergent if and only if the *Cauchy condition* is satisfied.

Worked Examples.

1. Use Cauchy's general principle of convergence to prove that the sequence $\{\frac{n}{n+1}\}$ is convergent.

Let $u_n = \frac{n}{n+1}$. Let p be a natural number.

Then $u_{n+p} = \frac{n+p}{n+p+1}$.

$$\begin{aligned} |u_{n+p} - u_n| &= \left| \frac{n+p}{n+p+1} - \frac{n}{n+1} \right| \\ &= \frac{p}{(n+p+1)(n+1)} \\ &< \frac{1}{n+1} < \frac{1}{n} \text{ for all } p, \text{ since } \frac{p}{n+p+1} < 1 \text{ for all } p. \end{aligned}$$

Let $\epsilon > 0$. Then $\frac{1}{n} < \epsilon$ holds for $n > \frac{1}{\epsilon}$.

Let $m = [\frac{1}{\epsilon}] + 1$. Then m is a natural number and $|u_{n+p} - u_n| < \epsilon$ for all $n \geq m$ and $p = 1, 2, 3, \ldots \ldots$

This proves that the sequence $\{u_n\}$ is convergent.

2. Use Cauchy's general principle of convergence to prove that the sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is not convergent.

Let p be a natural number.

$|u_{n+p} - u_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}$.

Let us choose $n = m$ and $p = m$.

$$\begin{aligned} \text{Then } |u_{2m} - u_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} \\ &> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} \\ &= \frac{1}{2}. \end{aligned}$$

If we choose $\epsilon = \frac{1}{2}$ then no natural number k can be found such that $|u_{n+p} - u_n| < \epsilon$ will hold for all $n \geq k$ and for every natural number p.

This shows that Cauchy condition is not satisfied by the sequence and the sequence $\{u_n\}$ is not convergent.

Cauchy sequence.

Definition. A sequence $\{u_n\}$ is said to be a *Cauchy sequence* if for a pre-assigned positive ϵ there exists a natural number k such that

$$|u_m - u_n| < \epsilon \text{ for all } m, n \geq k.$$

Replacing m by $n+p$ where $p = 1, 2, 3, \ldots \ldots$ the above condition can be equivalently stated as

$$| u_{n+p} - u_n | < \epsilon \text{ for all } n \geq k \text{ and } p = 1, 2, 3, \ldots \ldots$$

Theorem 5.15.2. A Cauchy sequence of real numbers is convergent.

Proof. Let $\{u_n\}$ be a Cauchy sequence. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\epsilon = 1$. Then there exists a natural number k such that

$$| u_m - u_n | < 1 \text{ for all } m, n \geq k.$$

Therefore $| u_k - u_n | < 1$ for all $n \geq k$.

or, $u_k - 1 < u_n < u_k + 1$ for all $n \geq k$.

Let $B = \max\{u_1, u_2, \ldots, u_{k-1}, u_k + 1\}$,

$b = \min\{u_1, u_2, \ldots, u_{k-1}, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$ and this proves that the sequence $\{u_n\}$ is bounded.

By Bolzano-Weierstrass theorem, $\{u_n\}$ has a convergent subsequence.

Let l be the limit of that convergent subsequence. Then l is a subsequential limit of $\{u_n\}$.

We now prove that the sequence $\{u_n\}$ converges to l.

Let us choose $\epsilon > 0$. There exists a natural number k such that

$$| u_m - u_n | < \tfrac{\epsilon}{2} \text{ for all } m, n \geq k \ \ldots \ldots \ \text{(i)}$$

Since l is a subsequential limit of $\{u_n\}$, there exists a natural number $q > k$ such that $| u_q - l | < \frac{\epsilon}{2}$.

Since $q > k$, from (i) $| u_q - u_n | < \frac{\epsilon}{2}$ for all $n \geq k$.

$$\begin{aligned} \text{Now } \ | u_n - l | &= | u_n - u_q | + | u_q - l | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } n \geq k. \end{aligned}$$

That is, $| u_n - l | < \epsilon$ for all $n \geq k$.

This implies $\lim u_n = l$. In other words, the sequence $\{u_n\}$ is convergent and the theorem is done.

Theorem 5.15.3. A convergent sequence is a Cauchy sequence.

Proof. Let $\{u_n\}$ be a convergent sequence and let $\lim u_n = l$.

For a pre-assigned positive ϵ there exists a natural number k such that $| u_n - l | < \frac{\epsilon}{2}$ for all $n \geq k$.

If m, n be natural numbers $\geq k$, then

$$| u_m - l | < \tfrac{\epsilon}{2} \text{ and } | u_n - l | < \tfrac{\epsilon}{2}.$$

$$\begin{aligned} \text{Now } \ | u_m - u_n | &\leq | u_m - l | + | l - u_n | \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } m, n \geq k. \end{aligned}$$

That is, $| u_m - u_n | < \epsilon$ for all $m, n \geq k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Worked Examples (continued).

3. Prove that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Let $u_n = \frac{1}{n}$. Let us choose a positive ϵ. There is a natural number k such that $\frac{2}{k} < \epsilon$.

$$\text{Then } \mid u_m - u_n \mid = \mid \frac{1}{m} - \frac{1}{n} \mid \leq \frac{1}{m} + \frac{1}{n} < \epsilon \text{ if } m, n \geq k.$$

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

4. Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence.

Let $u_n = (-1)^n$. Then

$$\begin{aligned} \mid u_m - u_n \mid &= \mid (-1)^m - (-1)^n \mid. \\ \mid u_m - u_n \mid &= 0 \text{ if } m \text{ and } n \text{ are both odd or both even,} \\ \mid u_m - u_n \mid &= 2 \text{ if one of } m, n \text{ is odd and the other is even.} \end{aligned}$$

Let us choose $\epsilon = \frac{1}{2}$. Then it is not possible to find a natural number k such that $\mid u_m - u_n \mid < \epsilon$ for all $m, n \geq k$.

Hence $\{u_n\}$ is not a Cauchy sequence.

5. Prove that the sequence $\{u_n\}$ where $u_1 = 0, u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \geq 1$ is a Cauchy sequence.

$u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1} = -\frac{1}{2}(u_{n+1} - u_n)$
or, $\mid u_{n+2} - u_{n+1} \mid = \frac{1}{2} \mid u_{n+1} - u_n \mid$ for all $n \in \mathbb{N}$.

Therefore $\mid u_{n+2} - u_{n+1} \mid = \frac{1}{2} \mid u_{n+1} - u_n \mid = \frac{1}{2^2} \mid u_n - u_{n-1} \mid = \cdots = \frac{1}{2^n} \mid u_2 - u_1 \mid = \frac{1}{2^n}$.

Let $m > n$. Then $\mid u_m - u_n \mid$
$\leq \mid u_m - u_{m-1} \mid + \mid u_{m-1} - u_{m-2} \mid + \cdots + \mid u_{n+1} - u_n \mid$
$= (\frac{1}{2})^{m-2} + (\frac{1}{2})^{m-3} + \cdots + (\frac{1}{2})^{n-1}$
$= \frac{4}{2^n}[1 - (\frac{1}{2})^{m-n}] < \frac{4}{2^n}$.

Let $\epsilon > 0$. Then there exists a natural number k such that $\frac{4}{2^n} < \epsilon$ for all $n \geq k$.

Hence $\mid u_m - u_n \mid < \epsilon$ for all $m, n \geq k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

6. Prove that the sequence $\{u_n\}$ satisfying the condition $\mid u_{n+2} - u_{n+1} \mid \leq c \mid u_{n+1} - u_n \mid$ for all $n \in \mathbb{N}$, where $0 < c < 1$, is a Cauchy sequence.

$$\begin{aligned} |u_{n+2} - u_{n+1}| &\le c\,|u_{n+1} - u_n| \\ &\le c^2\,|u_n - u_{n-1}| \\ &\le \cdots \\ &\le c^n\,|u_2 - u_1|. \end{aligned}$$

Let $m > n$.

$$\begin{aligned} \text{Then } |u_m - u_n| &\le |u_m - u_{m-1}| + \cdots + |u_{n+1} - u_n| \\ &\le |u_2 - u_1|\,\{c^{m-2} + c^{m-3} + \cdots + c^{n-1}\} \\ &= |u_2 - u_1|\,c^{n-1}.\tfrac{1-c^{m-n}}{1-c} \\ &< \tfrac{c^{n-1}}{1-c}\,|u_2 - u_1|. \end{aligned}$$

Let $\epsilon > 0$. Since $0 < c < 1$, the sequence $\{c^{n-1}\}$ is a convergent sequence. Therefore there exists a natural number k such that

$$\tfrac{c^{n-1}}{1-c}\,|u_2 - u_1| < \epsilon \text{ for all } n \ge k.$$

It follows that $|u_m - u_n| < \epsilon$ for all $m, n \ge k$ and this proves that the sequence $\{u_n\}$ is a Cauchy sequence.

5.16. Cauchy's theorems on limits.

Theorem 5.16.1. If $\lim u_n = l$ then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

Proof. **Case 1.** $l = 0$.

Since $\{u_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number B such that $|u_n| < B$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim u_n = 0$, there exists a natural number k_1 such that $|u_n| < \frac{\epsilon}{2}$ for all $n \ge k_1$.

$$\begin{aligned} \text{Now } \left|\tfrac{u_1+u_2+\cdots+u_n}{n}\right| &\le \left|\tfrac{u_1+u_2+\cdots+u_{k_1-1}}{n}\right| + \left|\tfrac{u_{k_1}+u_{k_1+1}+\cdots+u_n}{n}\right| \\ &\le \left|\tfrac{|u_1|+|u_2|+\cdots+|u_{k_1-1}|}{n}\right| + \left|\tfrac{|u_{k_1}|+|u_{k_1+1}|+\cdots+|u_n|}{n}\right| \\ &< \tfrac{B(k_1-1)}{n} + \tfrac{n-k_1+1}{n}.\tfrac{\epsilon}{2} \text{ for all } n \ge k_1 \\ &< \tfrac{Bk_1}{n} + \tfrac{\epsilon}{2} \text{ for all } n \ge k_1. \end{aligned}$$

Since $\lim \frac{1}{n} = 0$, there exists a natural number k_2 such that $\frac{Bk_1}{n} < \frac{\epsilon}{2}$ for all $n \ge k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\left|\frac{u_1+u_2+\cdots+u_n}{n}\right| < \epsilon$ for all $n \ge k$.

This proves that $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$.

Case 2. $l \ne 0$.

Let $v_n = u_n - l$. Then $\lim v_n = 0$.

Now $\frac{u_1+u_2+\cdots+u_n}{n} - l = \frac{v_1+v_2+\cdots+v_n}{n}$.

By case 1, $\lim \frac{v_1+v_2+\cdots+v_n}{n} = 0$. Therefore $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

This completes the proof.

Note. The converse of the theorem is not true.

Let us consider the sequence $\{u_n\}$ where $u_n = (-1)^n$.

Then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$ but the sequence $\{u_n\}$ is not convergent.

Corollary. If $\lim u_n = l$ where $u_n > 0$ for all n and $l \neq 0$, then $\lim \sqrt[n]{u_1 u_2 \ldots u_n} = l$.

Since each u_n is positive and $\lim u_n = l > 0$, the sequence $\{\log u_n\}$ converges to $\log l$, by the Corollary of 4.(Art 5.8).

Therefore $\lim \frac{\log u_1 + \log u_2 + \cdots + \log u_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(u_1 u_2 \ldots u_n)} = \log l$.

It follows that, $\lim \sqrt[n]{(u_1 u_2 \ldots u_n)} = l$.

Worked Examples.

1. Prove that $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

Let $u_n = \frac{1}{n}$. Then $\lim u_n = 0$.

By Cauchy's theorem, $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$.

2. Prove that $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Let $u_n = \sqrt[n]{n}$. Then $\lim u_n = 1$.

By Cauchy's theorem, $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$.

Theorem 5.16.2. If $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l (\neq 0)$ then $\lim \sqrt[n]{u_n} = l$.

Proof. Let $v_1 = u_1, v_2 = \frac{u_2}{u_1}, v_3 = \frac{u_3}{u_2}, \cdots, v_n = \frac{u_n}{u_{n-1}}, \cdots$

Then $v_n > 0$ for all $n \in \mathbb{N}$ and $\lim v_n = l > 0$

This implies $\lim \log v_n = \log l$.

By the first theorem, $\lim \frac{\log v_1 + \log v_2 + \cdots + \log v_n}{n} = \log l$.

or, $\lim \log \sqrt[n]{(v_1 v_2 \ldots v_n)} = \log l$.

It follows that $\lim \sqrt[n]{(v_1 v_2 \ldots v_n)} = l$. That is, $\lim \sqrt[n]{u_n} = l$.

Theorem 5.16.3. Let $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l$ (finite or infinite). Then $\lim \sqrt[n]{u_n} = l$.

Proof. **Case 1.** $0 < l < \infty$.

Let us choose $\epsilon > 0$ such that $l - \epsilon > 0$. Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that $l - \frac{\epsilon}{2} < \frac{u_{n+1}}{u_n} < l + \frac{\epsilon}{2}$ for all $n \geq k$.

Then $l - \frac{\epsilon}{2} < \frac{u_{k+1}}{u_k} < l + \frac{\epsilon}{2}$.

$$l - \frac{\epsilon}{2} < \frac{u_{k+2}}{u_k} < l + \frac{\epsilon}{2}$$

$$\cdots \qquad \cdots$$

$$l - \frac{\epsilon}{2} < \frac{u_n}{u_{n+1}} < l + \frac{\epsilon}{2}.$$

We have $(l-\frac{\epsilon}{2})^{n-k} < \frac{u_n}{u_k} < (l+\frac{\epsilon}{2})^{n-k}$ for all $n > k$
or, $(l-\frac{\epsilon}{2})^n.B < u_n < A.(l+\frac{\epsilon}{2})^n$, where $A = \frac{u_k}{(l+\frac{\epsilon}{2})^k} > 0, B = \frac{u_k}{(l-\frac{\epsilon}{2})^k} > 0$.
or, $(l-\frac{\epsilon}{2})B^{\frac{1}{n}} < u_n^{\frac{1}{n}} < A^{\frac{1}{n}}(l+\frac{\epsilon}{2})$.

Since $A > 0, \lim A^{\frac{1}{n}} = 1$. Since $B > 0, \lim B^{\frac{1}{n}} = 1$.

Since $\lim A^{\frac{1}{n}}(l+\frac{\epsilon}{2}) = l+\frac{\epsilon}{2}$, there exists a natural number k_2 such that $A^{\frac{1}{n}}(l+\frac{\epsilon}{2}) < l+\epsilon$ for all $n \geq k_2$.

Since $\lim B^{\frac{1}{n}}(l-\frac{\epsilon}{2}) = l-\frac{\epsilon}{2}$, there exists a natural number k_3 such that $B^{\frac{1}{n}}(l-\frac{\epsilon}{2}) > l-\epsilon$ for all $n \geq k_3$.

Let $k_0 = \max\{k_1, k_2, k_3\}$. Then $l-\epsilon < u_n^{\frac{1}{n}} < l+\epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = l$.

Case 2. $l = 0$.

Let $\epsilon > 0$. There exists a natural number k such that

$$0 < \frac{u_{n+1}}{u_n} < \frac{\epsilon}{2} \text{ for all } n \geq k.$$

Therefore $0 < \frac{u_{k+1}}{u_k} < \frac{\epsilon}{2}, 0 < \frac{u_{k+2}}{u_{k+1}} < \frac{\epsilon}{2}, \cdots\cdots, 0 < \frac{u_n}{u_{n-1}} < \frac{\epsilon}{2}$.

We have $0 < \frac{u_n}{u_k} < (\frac{\epsilon}{2})^{n-k}$ for all $n > k$
or, $0 < u_n < \frac{u_k}{(\frac{\epsilon}{2})^k}.(\frac{\epsilon}{2})^n$
or, $0 < u_n < \Lambda(\frac{\epsilon}{2})^n$ where $\Lambda = u_k(\frac{2}{\epsilon})^k > 0$
or, $0 < u_n^{\frac{1}{n}} < \Lambda^{\frac{1}{n}}\frac{\epsilon}{2}$.

Since $\Lambda > 0, \lim \Lambda^{\frac{1}{n}} = 1$.

Since $\lim \Lambda^{\frac{1}{n}}\frac{\epsilon}{2} = \frac{\epsilon}{2}$, there exists a natural number k_1 such that $\Lambda^{\frac{1}{n}}\frac{\epsilon}{2} < \epsilon$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $0 < u_n^{\frac{1}{n}} < \epsilon$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = 0$.

Case 3. $\lim \frac{u_{n+1}}{u_n} = \infty$.

Let us choose $G > 0$. There exists a natural number k such that $\frac{u_{n+1}}{u_n} > G+1$ for all $n \geq k$.

Therefore $\frac{u_{k+1}}{u_k} > G+1, \frac{u_{k+2}}{u_{k+1}} > G+1, \cdots, \frac{u_n}{u_{n-1}} > G+1$.

We have $\frac{u_n}{u_k} > (G+1)^{n-k}$ for all $n > k$
or, $u_n > \mu(G+1)^n$ where $\mu = \frac{u_k}{(G+1)^k} > 0$
or, $u_n^{\frac{1}{n}} > \mu^{\frac{1}{n}}(G+1)$.

Since $\mu > 0, \lim \mu^{\frac{1}{n}} = 1$.

Since $\lim \mu^{\frac{1}{n}}(G+1) = G+1$, there exists a natural number k_1 such that $\mu^{\frac{1}{n}}(G+1) > G$ for all $n \geq k_1$.

Let $k_0 = \max\{k, k_1\}$. Then $u_n^{\frac{1}{n}} > G$ for all $n > k_0$.

Therefore $\lim u_n^{\frac{1}{n}} = \infty$. This completes the proof.

Note. The converse of the theorem is not true.

For example, let us consider the sequence $\{u_n\}$ where $u_n = \frac{3+(-1)^n}{2}$. The sequence is $\{1, 2, 1, 2, 1, 2, \ldots \ldots\}$.

Here $\lim \sqrt[n]{u_n} = 1$, since $\lim \sqrt[2n]{u_{2n}} = \lim 2^{\frac{1}{2n}} = 1$ and $\lim (u_{2n-1})^{\frac{1}{2(n-1)}} = 1$. But $\lim \frac{u_{n+1}}{u_n}$ does not exist.

Theorem 5.16.4. If $u_n > 0$ for all $n \in \mathbb{N}$ then
$\underline{\lim} \frac{u_{n+1}}{u_n} \le \underline{\lim} \sqrt[n]{u_n} \le \overline{\lim} \sqrt[n]{u_n} \le \overline{\lim} \frac{u_{n+1}}{u_n}$.

Proof. Let $\underline{\lim} \frac{u_{n+1}}{u_n} = \lambda_*, \overline{\lim} \frac{u_{n+1}}{u_n} = \lambda^*, \underline{\lim} \sqrt[n]{u_n} = \mu_*, \overline{\lim} \sqrt[n]{u_n} = \mu^*$.
We first prove $\mu^* \le \lambda^*$

Case 1. Let $\lambda^* = \infty$. Then $\mu^* \le \lambda^*$ trivially.

Case 2. Let λ^* be finite.

Let us choose $\epsilon > 0$. Since $\overline{\lim} \frac{u_{n+1}}{u_n} = \lambda^*$, there exists a natural number k such that $\frac{u_{n+1}}{u_n} < \lambda^* + \epsilon$ for all $n \ge k$.

Then $\frac{u_{k+1}}{u_k} < \lambda^* + \epsilon, \frac{u_{k+2}}{u_{k+1}} < \lambda^* + \epsilon, \cdots, \frac{u_n}{u_{n-1}} < \lambda^* + \epsilon$

So $u_n < u_k(\lambda^* + \epsilon)^{n-k}$ for all $n > k$.

Hence for all $n > k, u_n < A(\lambda^* + \epsilon)^n$ where $A = \frac{u_k}{(\lambda^*+\epsilon)^k} > 0$.

Therefore $\sqrt[n]{u_n} < A^{1/n}(\lambda^* + \epsilon)$ for all $n > k$.

Consequently, $\overline{\lim} \sqrt[n]{u_n} \le \overline{\lim} A^{1/n}(\lambda^* + \epsilon)$
$= \lambda^* + \epsilon$ since $\lim A^{1/n} = 1$.

Since ϵ is arbitrary, $\overline{\lim} \sqrt[n]{u_n} \le \lambda^*$, i.e., $\mu^* \le \lambda^*$.

In a similar manner we can prove $\lambda_* \le \mu_*$.

Also the inequality $\mu_* \le \mu^*$ follows from the property of the limit inferior and the limit superior of a sequence.

This completes the proof.

Note. If $u_n > 0$ for all n and $\lim \frac{u_{n+1}}{u_n}$ exists, then it follows from the theorem that $\lim \sqrt[n]{u_n}$ also exists.

Worked Examples (continued).

3. Prove that $\lim \sqrt[n]{n} = 1$.

Let $u_n = n$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = 1 > 0$.
It follows that from the theorem that $\lim \sqrt[n]{n} = 1$.

4. Prove that $\lim \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Let $u_n = \frac{n!}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \frac{1}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{1}{e}$, i.e., $\lim \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

5. Prove that $\lim \frac{\{(n+1)(n+2)\cdots(2n)\}^{1/n}}{n} = \frac{4}{e}$.

Let $u_n = \frac{(n+1)(n+2)\cdots 2n}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = \lim \frac{2(2n+1)}{n+1} \cdot \frac{1}{(1+\frac{1}{n})^n} = \frac{4}{e} > 0$.

It follows from the theorem that $\lim \sqrt[n]{u_n} = \frac{4}{e}$.

Exercises 8

1. Establish the convergence and find the limit of the sequence $\{u_n\}$, where u_n is

(i) $(1+\frac{1}{3n})^n$, (ii) $(1+\frac{1}{n^2})^{n^2}$, (iii) $(1+\frac{1}{3n+1})^n$, (iv) $(1+\frac{1}{n^2+2})^{n^2}$.

2. Prove that the sequence $\{u_n\}$ is convergent by showing that the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit.

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{3}(u_{n+1} + 2u_n)$ for $n \geq 1$;

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$;

(iii) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$.

3. Prove that the sequence

(i) $\{2, \frac{2}{1+2}, \frac{2}{1+\frac{2}{1+2}}, \cdots\cdots\}$ converges to 1;

(ii) $\{6, \frac{6}{1+6}, \frac{6}{1+\frac{6}{1+6}}, \cdots\cdots\}$ converges to 2.

4. $\{x_n\}$ and $\{y_n\}$ are bounded sequences and a sequence $\{z_n\}$ is defined by $z_1 = x_1, z_2 = y_1; z_3 = x_2, z_4 = y_2; z_5 = x_3, z_6 = y_3; \ldots \;\ldots$. Prove that the sequence $\{z_n\}$ is convergent if and only if both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit.

5. (a) If $\overline{\lim}\, u_n = \infty$ prove that there exists a properly divergent subsequence $\{u_{r_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{r_n} = \infty$.

(b) If $\underline{\lim}\, u_n = -\infty$ prove that there exists a properly divergent subsequence $\{u_{r_n}\}$ of the sequence $\{u_n\}$ such that $\lim u_{r_n} = -\infty$.

6. A sequence $\{u_n\}$ is such that every subsequence of $\{u_n\}$ has a subsequence that converges to 0. Prove that $\lim u_n = 0$.

[**Hint.** Prove that $\overline{\lim}\, u_n$ and $\underline{\lim}\, u_n$ are both finite. Assume $\overline{\lim}\, u_n = l, \underline{\lim}\, u_n = m$. Prove that $l = 0, m = 0$.]

7. Find $\overline{\lim}\, u_n$ and $\underline{\lim}\, u_n$ where $u_n =$

(i) $(-1)^n(1+\frac{1}{n})$, (ii) $n + \frac{(-1)^n}{n}$, (iii) $n^{(-1)^n}$, (iv) $(\cos\frac{n\pi}{4})^{(-1)^n}$.

8. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences and $u_n > 0, v_n > 0$ for all $n \in \mathbb{N}$. Prove that

(i) $\overline{\lim}\, u_n . \overline{\lim}\, v_n \geq \overline{\lim}\, u_n v_n$; (ii) $\underline{\lim}\, u_n . \underline{\lim}\, v_n \leq \underline{\lim}\, u_n v_n$.

E

9. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences such that the sequence $\{v_n\}$ is convergent. Prove that

(i) $\overline{\lim}\ (u_n + v_n) = \overline{\lim}\ u_n + \lim v_n$; (ii) $\underline{\lim}\ (u_n + v_n) = \underline{\lim}\ u_n + \lim v_n$.

10. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences with $u_n > 0, v_n > 0$ for all $n \in \mathbb{N}$ such that the sequence $\{v_n\}$ is convergent. Prove that

(i) $\overline{\lim}\ (u_n v_n) = \overline{\lim}\ u_n . \lim v_n$; (ii) $\underline{\lim}\ (u_n v_n) = \underline{\lim}\ u_n . \lim v_n$.

11. Let $\{u_n\}$ be a bounded sequence of real numbers and E be the set of all subsequential limits of $\{u_n\}$. Prove that E is a non-empty closed and bounded set and $\sup E \in E$, $\inf E \in E$.

12. $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Prove directly that

(i) $\{u_n + v_n\}$ is a Cauchy sequence, (ii) $\{u_n v_n\}$ is a Cauchy sequence.

13. Establish from definition that $\{u_n\}$ is a Cauchy sequence, where

(i) $u_n = \frac{n}{n+1}$, (ii) $u_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$,

(iii) $| u_{n+2} - u_{n+1} | \leq \frac{1}{2} | u_{n+1} - u_n |$ for all $n \in \mathbb{N}$.

[**Hint.** (ii) $(n+1)! \geq 2^n$ for $n \geq 2$. (iii) $|u_{n+2} - u_{n+1}| \leq (\frac{1}{2})^n |u_2 - u_1|$.]

14. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$ and $\{y_n\}$ is a sequence in $\mathbb{R}$ such that $| x_n - y_n | < \frac{1}{n}$ for all $n \geq 1$. Prove that $\{y_n\}$ is a Cauchy sequence and $\lim x_n = \lim y_n$.

15. If $\{u_n\}$ be a Cauchy sequence in $\mathbb{R}$ having a subsequence converging to a real number l, prove that $\lim u_n = l$.

16.(i) Let $u_1 = 2$ and $u_{n+1} = 2 + \frac{1}{u_n}$ for $n \geq 1$. Prove that the sequence $\{u_n\}$ converges to the limit $\sqrt{2} + 1$.

(ii) Let $u_1 > 0$ and $u_{n+1} = \frac{1}{2+u_n}$ for $n \geq 1$. Prove that the sequence $\{u_n\}$ converges to the limit $\sqrt{2} - 1$.

[**Hint.** (i) $|u_{n+2} - u_{n+1}| < \frac{1}{4}|u_{n+1} - u_n|$. (ii) $|u_{n+2} - u_{n+1}| < \frac{1}{4}|u_{n+1} - u_n|$.]

17. If $u_n = \frac{1}{1.n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots + \frac{1}{n.1}$, prove that $\lim u_n = 0$.

[**Hint.** $(n+1)u_n = (1 + \frac{1}{n}) + (\frac{1}{2} + \frac{1}{n-1}) + \cdots + (\frac{1}{n} + 1)$.]

18. Prove that $\lim\limits_{n\to\infty} \dfrac{1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}}{2n+1} = 0$.

19. Prove that (i) $\lim\limits_{n\to\infty} \frac{1}{n}\{(2n+1)(2n+2)\cdots(2n+n)\}^{\frac{1}{n}} = \frac{27}{4e}$,

(ii) $\lim\limits_{n\to\infty} \frac{1}{n}\{(a+1)(a+2)\cdots(a+n)\}^{\frac{1}{n}} = \frac{1}{e}$ if $a > 0$.

[**Hint.** (i) Let $u_n = \frac{(2n+1)(2n+2)\cdots(2n+n)}{n^n}$. Then $\lim \frac{u_{n+1}}{u_n} = \frac{27}{4e}$.]

6.1. Infinite Series.

Let $\{u_n\}$ be a sequence. Then the sequence $\{s_n\}$ defined by

$$s_1 = u_1, s_2 = u_1 + u_2, s_3 = u_1 + u_2 + u_3, \cdots \cdots$$

is represented by the symbol $u_1 + u_2 + u_3 + \cdots \cdots$, which is said to be an *infinite series* (or a *series*) generated by the sequence $\{u_n\}$. The series is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or by Σu_n. u_n is said to be the *nth term* of the series. The elements s_k of the sequence $\{s_n\}$ are called the partial sums of the series Σu_n.

The sequence $\{s_n\}$ is called the *sequence of partial sums* of the series Σu_n.

If $\{u_n\}$ be a real sequence then Σu_n is a series of real numbers.

We shall be mainly concerned with the series of real numbers.

The infinite series Σu_n is said to be *convergent* or *divergent* according as the sequence $\{s_n\}$ is convergent or divergent.

In case of convergence, if $\lim s_n = s$ then s is said to be the *sum* of the series Σu_n.

If, however, $\lim s_n = \infty$ (or $-\infty$) the series Σu_n is said to *diverge* to ∞ (or $-\infty$).

Examples.

1. Let us consider the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \cdots$

Let the series be $\sum\limits_{n=1}^{\infty} u_n$. Then $u_n = \frac{1}{n(n+1)}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

$$\begin{aligned} \text{Then } s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) \\ &= 1 - \frac{1}{n+1}, \end{aligned}$$

and $\lim s_n = 1$. Hence the series Σu_n is convergent and the sum of the series is 1.

2. Let us consider the series $1 + 2 + 3 + \cdots \cdots$

Let $s_n = 1+2+3+\cdots+n$. Then $s_n = \frac{n(n+1)}{2}$ and $\lim s_n = \infty$.
Hence the series is divergent.

3. Let us consider the series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots$

Let $s_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$.
Then $s_n = 2(1-\frac{1}{2^n}) = 2-\frac{1}{2^{n-1}}$ and $\lim s_n = 2$ since $\lim(\frac{1}{2})^{n-1} = 0$.
Therefore the series is convergent and the sum of the series is 2.

4. Let us consider the series $1-1+1-1+\cdots\cdots$

Let $s_n = 1-1+1-1+\cdots+(-1)^{n-1}$.

$$\begin{aligned} \text{Then } s_n &= 0 \text{ if } n \text{ be even,} \\ &= 1 \text{ if } n \text{ be odd.} \end{aligned}$$

The sequence $\{s_n\}$ is divergent. Therefore the series is divergent.

5. Geometric series.

A. Let us consider the series $1+a+a^2+\cdots$ where $|a|<1$.

Let $s_n = 1+a+a^2+\cdots+a^{n-1}$. Then $s_n = \frac{1-a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a}$.
$\lim s_n = \frac{1}{1-a}$ since $\lim a^n = 0$.
Therefore the series is convergent and the sum of the series is $\frac{1}{1-a}$.

B. Let us consider the series $1+a+a^2+\cdots$ where $|a|\geq 1$.
Let $s_n = 1+a+a^2+\cdots+a^{n-1}$.

Case 1. $a=1$. In this case $s_n = n$ and $\lim s_n = \infty$.
Therefore the series is divergent.

Case 2. $a>1$. In this case $s_n = \frac{a^n-1}{a-1}$ and $\lim s_n = \infty$ since $\lim a^n = \infty$ in this case.
Therefore the series is divergent.

Case 3. $a=-1$. In this case $s_n = 1$ if n be odd,
$= 0$ if n be even.

The sequence $\{s_n\}$ is divergent. Therefore the series is divergent.

Case 4. $a<-1$. In this case the sequence $\{s_n\}$ is divergent and therefore the series is divergent.

From (A) and (B), the geometric series $1+a+a^2+\cdots\cdots$ is convergent if $|a|<1$, and divergent if $|a|\geq 1$.

6. Harmonic series.

$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{1}{n}$.

$$\begin{aligned}
\text{Let } s_n &= u_1 + u_2 + \cdots + u_n. \\
\text{Then } s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) = 1 + 2.\tfrac{1}{2} \\
s_8 &= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) \\
&> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) = 1 + 3.\tfrac{1}{2} \\
s_{16} &> 1 + 4.\tfrac{1}{2} \\
&\cdots \quad \cdots \\
s_{2^n} &> 1 + n.\tfrac{1}{2}.
\end{aligned}$$

Therefore $\lim\limits_{n \to \infty} s_{2^n} = \infty$.

The sequence $\{s_n\}$ is a monotone increasing sequence, since $s_{n+1} - s_n = a_{n+1} > 0$ for all $n \in \mathbb{N}$. Since the subsequence $\{s_{2^n}\}$ diverges to ∞, the sequence $\{s_n\}$ is unbounded above and therefore the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Theorem 6.1.1. Let m be a natural number. Then the two series $u_1 + u_2 + u_3 + \cdots \cdots$ and $u_{m+1} + u_{m+2} + u_{m+3} + \cdots \cdots$ converge or diverge together.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n, t_n = u_{m+1} + u_{m+2} + \cdots + u_{m+n}$.
Then $t_n = s_{m+n} - s_m$, where s_m is a fixed number.
If $\{s_n\}$ converges then $\{t_n\}$ converges and conversely.
If $\{s_n\}$ diverges then $\{t_n\}$ diverges and conversely.
Therefore both the sequences $\{s_n\}$ and $\{t_n\}$ and consequently the series Σu_n and Σu_{m+n} converge or diverge together.

Note. The theorem states that we can remove from the beginning a finite number of terms from a given series, or add to the beginning a finite number of terms to a given series without changing its behaviour regarding convergence or divergence.

Theorem 6.1.2. If Σu_n and Σv_n be two convergent series having the sums s and t respectively then

(i) the series $\Sigma(u_n + v_n)$ converges to the sum $s + t$;

(ii) the series $\Sigma k u_n$, where k is a real number, converges to the sum ks.

The proof is immediate.

Theorem 6.1.3. (Cauchy's principle of convergence)

A necessary and sufficient condition for the convergence of a series Σu_n is that corresponding to a pre-assigned positive ϵ there exists a natural number m such that

$|\ u_{n+1}+u_{n+2}+\cdots+u_{n+p}\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

Proof. Let $s_n=u_1+u_2+\cdots+u_n$.

Let Σu_n be convergent. Then the sequence $\{s_n\}$ is convergent. Therefore by Cauchy's principle of convergence for the sequence, corresponding to a pre-assigned positive ϵ there exists a natural number m such that

$|\ s_{n+p}-s_n\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

or, $|\ u_{n+1}+u_{n+2}+\cdots+u_{n+p}\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

Conversely, let us assume that for a pre-assigned positive ϵ there exists a natural number m such that

$|\ u_{n+1}+u_{n+2}+\cdots+u_{n+p}\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

Then $|\ s_{n+p}-s_n\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

This implies that the sequence $\{s_n\}$ is convergent by Cauchy's principle of convergence. Therefore Σu_n is convergent.

This completes the proof.

Theorem 6.1.4. A necessary condition for the convergence of a series Σu_n is $\lim u_n=0$.

Proof. Let Σu_n be convergent. Then for a pre-assigned positive ϵ there exists a natural number m such that

$|\ u_{n+1}+u_{n+2}+\cdots+u_{n+p}\ |<\epsilon$ for all $n\geq m$ and for every natural number p.

Taking $p=1$, $|\ u_{n+1}\ |<\epsilon$ for all $n\geq m$.

This implies $\lim u_n=0$.

Note. The converse of the theorem is not true.

That is, $\lim u_n=0$ does not necessarily imply convergence of the series Σu_n. Because the sufficient condition for the convergence of Σu_n states that for a chosen positive ϵ there must exist a natural number m such that

$|\ u_{n+1}+u_{n+2}+\cdots+u_{n+p}\ |<\epsilon$ for all $n\geq m$ and for $p=1,2,3,\cdots$.

Therfore the sum of p consecutive terms of the series must be less than ϵ whatever natural number p may be. The condition must be satisfied for all p and not for only a particular p.

Let us consider the series Σu_n where $u_n=\frac{1}{n}$.

Here $\lim u_n=0$. But Σu_n is a divergent series.

Here $|\ s_{n+p}-s_n\ |=|\ \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\ |$.

If we take $p = n$, $\left| s_{n+p} - s_n \right| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$
$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$.

Therefore $| s_{n+p} - s_n |$ cannot be made less than a chosen positive $\epsilon < \frac{1}{2}$ for every natural number p.

Worked Examples.

1. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is convergent.

Let the series be $\sum_{n=1}^{\infty} u_n$. Then $u_n = (-1)^{n+1}\frac{1}{n}$.

Let $s_n = u_1 + u_2 + \cdots + u_n$. Then

$$\begin{aligned} | s_{n+p} - s_n | &= \left| \tfrac{1}{n+1} - \tfrac{1}{n+2} + \cdots + (-1)^{p-1}\tfrac{1}{n+p} \right| \\ &= \left| \tfrac{1}{n+1} - (\tfrac{1}{n+2} - \tfrac{1}{n+3}) - \cdots \right| \\ &< \tfrac{1}{n+1}. \end{aligned}$$

Let $\epsilon > 0$. Then $| s_{n+p} - s_n | < \epsilon$ holds if $n > \frac{1}{\epsilon} - 1$.

Let $m = [\frac{1}{\epsilon} - 1] + 2$. Then m is a natural number and $| s_{n+p} - s_n | < \epsilon$ for all $n \geq m$ and for $p = 1, 2, 3, \cdots$

This proves that the sequence $\{s_n\}$ is convergent and consequently the series Σu_n is convergent.

2. Prove that the series $\sum_{n=1}^{\infty} u_n$ where $u_n = \frac{n}{n+1}$, is divergent.

Here $\lim u_n = 1$. Since $\lim u_n$ is not $0, \Sigma u_n$ is divergent because a necessary condition for convergence of the series Σu_n is $\lim u_n = 0$.

6.2. Series of positive terms.

A series Σu_n is said to be a *series of positive terms* if u_n is a positive real number for all $n \in \mathbb{N}$.

Theorem 6.2.1. A series of positive real numbers Σu_n is convergent if and only if the sequence $\{s_n\}$ of partial sums is bounded above.

Proof. $s_n = u_1 + u_2 + \cdots + u_n$. Then $s_{n+1} - s_n = u_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence the sequence $\{s_n\}$ is a monotone increasing sqeuence. Therefore $\{s_n\}$ is convergent if and only if it is bounded above.

Consequently, the series Σu_n is convergent if and only if the sequence $\{s_n\}$ is bounded above.

Note. If not bounded above, the sequence $\{s_n\}$ being a monotone increasing sequence, diverges to ∞. In this case the series diverges to ∞.

Therefore a series of positive real numbers either converges to a real number, or diverges to ∞.

Introduction and removal of brackets.

Let Σu_n be a series of positive real numbers. Let the terms of the series be arranged in groups without changing the order of the terms. Let us denote the nth group by v_n. Then a new series Σv_n is obtained.

Example.

Let $\Sigma u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots$

Let us introduce brackets and the series takes the form
$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots$

If the new series be Σv_n then
$v_1 = 1, v_2 = \frac{1}{2}, v_3 = \frac{1}{3} + \frac{1}{4}, v_4 = \frac{1}{5} + \cdots + \frac{1}{8}, v_5 = \frac{1}{9} + \cdots + \frac{1}{16}, \cdots$

Σv_n is obtained from Σu_n by introduction of brackets and Σu_n is obtained from Σv_n by removal of brackets.

Theorem 6.2.2. Let Σu_n be a series of positive real numbers and Σv_n is obtained from Σu_n by grouping its terms. Then

(i) if Σu_n converges to the sum s, so does Σv_n ;

(ii) if Σv_n converges to the sum t, so does Σu_n.

Proof. Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + \cdots + u_{r_2}, v_3 = u_{r_2+1} + \cdots + u_{r_3}, \cdots \cdots$

Then $\{r_n\}$ is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.

Let Σu_n be convergent and the sum of the series be s. Then $\lim s_n = s$. The sequence $\{s_{r_n}\}$ being a subsequence of the convergent sequence $\{s_n\}$, is convergent and $\lim s_{r_n} = s$. That is, $\lim t_n = s$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

Let Σv_n be convergent and the sum of the series be t. Then $\lim t_n = t$. That is, $\lim s_{r_n} = t$.

$\{s_{r_n}\}$ is a convergent subsequence of the monotone increasing sequence $\{s_n\}$. By Theorem 5.11.4, the sequence $\{s_n\}$ is convergent and $\lim s_n = t$.

This proves that the series Σu_n is convergent and the sum of the series is also t. This completes the proof.

Note. The theorem does not hold if Σu_n be a series of arbitrary terms.

Let us consider the series

$1 - 1 + 1 - 1 + \cdots \cdots \ldots \quad \ldots$(A)

Introducing brackets we get the series

$(1-1)+(1-1)+\cdots\cdots\cdots$ ...(B)

Introducing brackets in another manner we get the series

$1-(1-1)-(1-1)-\cdots\cdots\cdots$ ...(C)

The series (A) is divergent but the series (B) converges to 0 and the series (C) converges to 1.

Re-arrangement of terms.

Let Σu_n be a given series. If a new series Σv_n is obtained by using each term of Σu_n exactly once, the order of the terms being disturbed, then Σv_n is called a re-arrangement of Σu_n.

If $f:\mathbb{N}\to\mathbb{N}$ be a bijective mapping, $\Sigma u_{f(n)}$ is a re-arrangement of Σu_n and conversely if Σv_n be a re-arrangement of the series Σu_n then $v_n = u_{f(n)}$ for some bijection $f:\mathbb{N}\to\mathbb{N}$.

For example, let $f(n) = n+1$ if n be odd ,
$\qquad = n-1$ if n be even.

$f(1)=2, f(2)=1, f(3)=4, f(4)=3, \cdots\cdots$

$\Sigma u_{f(n)} = u_2+u_1+u_4+u_3+\cdots\cdots$ is a re-arrangement of Σu_n.

Theorem 6.2.3. Let Σu_n be a convergent series of positive real numbers.Then any re-arrangement of Σu_n is convergent and the sum remains unaltered.

Proof. Let Σu_n converge to s and Σv_n be a re-arrangement of Σu_n. Then $v_n = u_{f(n)}$ for some bijection $f:\mathbb{N}\to\mathbb{N}$.

Let $s_n = u_1+u_2+\cdots+u_n,\ t_n = v_1+v_2+\cdots+v_n$.

Since $u_n > 0$, the sequence $\{s_n\}$ is a monotone increasing sequence. As Σu_n converges to s, $\lim s_n = s$. Therefore the sequence $\{s_n\}$ is bounded above and $s_n \le s$ for all $n\in\mathbb{N}$.

$$\begin{aligned} t_n &= v_1+v_2+\cdots+v_n \\ &= u_{f(1)}+u_{f(2)}+\cdots+u_{f(n)} \\ &\le u_1+u_2+\cdots+u_{m(n)}, \text{ where } m(n)=\max\{f(1),\ldots,f(n)\}. \end{aligned}$$

But $u_1+u_2+\cdots+u_{m(n)} = s_{m(n)} \le s$.

Thus the sequence $\{t_n\}$ is bounded above and being a monotone increasing sequence, it is convergent. Let $\lim t_n = t$. Then $t\le s$.

$$\begin{aligned} s_n &= u_1+u_2+\cdots+u_n \\ &= v_{f^{-1}(1)}+v_{f^{-1}(2)}+\cdots+v_{f^{-1}(n)} \\ &\le v_1+v_2+\cdots+v_{k(n)}, \text{ where } k(n)=\max\{f^{-1}(1),\ldots,f^{-1}(n)\}. \end{aligned}$$

But $v_1+v_2+\cdots+v_{k(n)} = t_{k(n)} \le t$.

$s_n\le t \Rightarrow \lim s_n \le t$, i.e., $s\le t$

It follows that $s=t$.

This proves that the series Σv_n is convergent and the sum of the series is also s.

6.3. Tests for convergence of a series of positive terms.

The convergence or divergence of a particular series is decided by examining the sequence of partial sums of the series. In most cases the expression for s_n (the nth partial sum) becomes not so nice as can be easily handled to determine its nature in a straightforward manner. Some other elegent methods will be applied to the series that will decide the convergence of the series without prior knowledge of the nature of the sequence $\{s_n\}$. These methods, called 'tests for convergence', will be discussed here.

Theorem 6.3.1.(Comparison test [First type]).

A. Let Σu_n and Σv_n be two series of positive real numbers and there is a natural number m such that $u_n \leq kv_n$ for all $n \geq m, k$ being a fixed positive number.

Then (i) Σu_n is convergent if Σv_n is convergent

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.

$$\begin{aligned} \text{Then } s_n - s_m &= u_{m+1} + u_{m+2} + \cdots + u_n \\ &\leq k(v_{m+1} + v_{m+2} + \cdots + v_n) \\ &= k(t_n - t_m) \end{aligned}$$

or, $s_n \leq kt_n + h$ where $h = s_m - kt_m$, a finite number.

(i) Let Σv_n be convergent. Then the sequence $\{t_n\}$ is bounded.

Let B be an upper bound. Then $t_n < B$ for all $n \in \mathbb{N}$.

Therefore $s_n < kB + h$ for all $n \geq m$.

This shows that the sequence $\{s_n\}$ is bounded above. $\{s_n\}$ being a monotone increasing sequence bounded above, is convergent.

Therefore Σu_n is convergent.

(ii) Let Σu_n is divergent. Then the sequence $\{s_n\}$ is not bounded above.

Since $s_n \leq kt_n + h$, the sequence $\{t_n\}$ is not bounded above. Therefore the series Σv_n is divergent.

B. Limit form.

Let Σu_n and Σv_n be two series of positive real numbers and $\lim \frac{u_n}{v_n} = l$ where l is a *non-zero* finite number.

Then the two series Σu_n and Σv_n converge or diverge together.

Proof. $l > 0$. Let us choose a positive ϵ such that $l - \epsilon > 0$. There a natural number m such that $l - \epsilon < \frac{u_n}{v_n} < l + \epsilon$ for all $n \geq m$.

Therefore $u_n < kv_n$ for all $n \geq m$ where $k = l + \epsilon > 0$ ... ... (i) and $v_n < k'u_n$ for all $n \geq m$ where $k' = \frac{1}{l-\epsilon} > 0$ ... ... (ii)

By comparison test A, it follows from (i) that Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

By comparison test A, it follows from (ii) that Σv_n is convergent if Σu_n is convergent and Σu_n is divergent if Σv_n is divergent.

Therefore the two series Σu_n and Σv_n converge or diverge together.

Note. If $\lim \frac{u_n}{v_n} = 0$, then for a pre-asigned positive number ϵ there exists a natural number m such that $0 < \frac{u_n}{v_n} < \epsilon$, for all $n \geq m$.

Therefore Σu_n is convergent if Σv_n is convergent.

If $\lim \frac{u_n}{v_n} = \infty$, then for a pre-asigned positive number G there exists a natural number m such that $\frac{u_n}{v_n} > G$, for all $n \geq m$.

Therefore Σu_n is divergent if Σv_n is divergent.

For example, if Σu_n be a convergent series of positive real numbers, then the series $\Sigma \frac{1}{n^p} u_n$ is convergent for all $p > 0$ and if Σu_n be a divergent series of positive real numbers, then the series $\Sigma n^p u_n$ is divergent for all $p > 0$.

In order to make use of the Comparison test we need to have a collection of series whose nature are known. The series $\Sigma \frac{1}{n^p}$ discussed in the following theorem will be an addition to the collection.

Theorem 6.3.2. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots$ converges for $p > 1$ and diverges for $p \leq 1$.

Proof. **Case 1.** $p > 1$. Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n^p}$.

Let Σv_n be obtained from Σu_n by grouping the terms as

$1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}) + (\frac{1}{8^p} + \cdots + \frac{1}{15^p}) + \cdots$

Then $v_1 = 1, v_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$,

$v_3 = \frac{1}{4^p} + \cdots + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{2^{2(p-1)}}$,

$v_4 = \frac{1}{8^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{2^{3(p-1)}}$,

$\cdots \quad \cdots \quad \cdots$

Let $w_n = \{\frac{1}{2^{p-1}}\}^{n-1}$. Then $v_n < w_n$ for all $n \geq 2$.

But Σw_n is a geometric series of common ratio $\frac{1}{2^{p-1}}$.

Since $p > 1, 0 < \frac{1}{2^{p-1}} < 1$ and hence Σw_n is convergent.

Therefore Σv_n is convergent by Comparison test.

Since Σv_n is obtained from Σu_n by introduction of brackets, Σu_n is convergent.

Case 2. $p = 1$. In this case the series is $1 + \frac{1}{2} + \frac{1}{3} + \cdots\ \cdots$

Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

$$\begin{aligned} \text{Then } s_{2n} - s_n &= \tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{2n} \\ &> \tfrac{1}{2n} + \tfrac{1}{2n} + \cdots + \tfrac{1}{2n} = \tfrac{1}{2}. \end{aligned}$$

This shows that the sequence $\{s_n\}$ is not a Cauchy sequence and therefore is not convergent. Hence the series $\Sigma\frac{1}{n}$ is not convergent.

Case 3. $0 < p < 1$. Then $\frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}, \cdots \cdots$

Therefore $\frac{1}{n^p} > \frac{1}{n}$ for all $n \geq 2$.

But $\Sigma\frac{1}{n}$ is divergent. Therefore $\Sigma\frac{1}{n^p}$ is divergent by Comparison test.

Case 4. $p \leq 0$. Then $\lim \frac{1}{n^p} \neq 0$ and therefore $\Sigma\frac{1}{n^p}$ is not convergent.

This completes the proof.

Worked Examples.

1. Test the convergence of the series $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}$.

Let $v_n = \frac{1}{n}$. Then $\lim \frac{u_n}{v_n} = \lim \frac{n(n+2)}{2(n+1)^2} = \frac{1}{2}$.

Since Σv_n is divergent, Σu_n is divergent by Comparison test.

2. Test the convergence of the series $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{1}{n(n+1)^2}$.

Let $v_n = \frac{1}{n^3}$. Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But Σv_n is convergent. Therefore Σu_n is convergent by Comparison test.

3. Test the convergence of the series Σu_n where $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$. Let $v_n = \frac{1}{n^2}$.

Then $\lim \frac{u_n}{v_n} = \lim \dfrac{2n^2}{n^2\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\}} = 1$.

Since Σv_n is convergent, Σu_n is convergent by Comparison test.

Theorem 6.3.3. (Comparison test [Second type])

Let Σu_n and Σv_n be two series of positive real numbers and there is natural number m such that

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n} \text{ for all } n \geq m.$$

Then (i) Σu_n is convergent if Σv_n is convergent,

(ii) Σv_n is divergent if Σu_n is divergent.

Proof. $\frac{u_{m+1}}{u_m} \leq \frac{v_{m+1}}{v_m}, \frac{u_{m+2}}{u_{m+1}} \leq \frac{v_{m+2}}{v_{m+1}}, \cdots, \frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}}$, where $n > m$.

Therefore $\frac{u_n}{u_m} \leq \frac{v_n}{v_m}$ for all $n > m$

or, $u_n \le \frac{u_m}{v_m} v_n$ for all $n > m$.
or, $u_n \le k v_n$ for all $n > m$ and $k(= \frac{u_m}{v_m})$ is a positive number.

By Comparison test (first type), Σu_n is convergent if Σv_n is convergent and Σv_n is divergent if Σu_n is divergent.

Theorem 6.3.4. (D'Alembert's ratio test)

Let Σu_n be a series of positive real numbers and let $\lim \frac{u_{n+1}}{u_n} = l$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.
Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number m such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \ge m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$.
We have $\frac{u_{m+1}}{u_m} < r, \frac{u_{m+2}}{u_{m+1}} < r, \cdots\cdots, \frac{u_n}{u_{n-1}} < r$ where $n > m$.
Consequently, $\frac{u_n}{u_m} < r^{n-m}$ for all $n > m$
or, $u_n < \frac{u_m}{r^m} . r^n$ for all $n > m$.

$\frac{u_m}{r^m}$ is a positive number and Σr^n is a geometric series of common ratio r where $0 < r < 1$ and therefore Σr^n is convergent.

Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.
Since $\lim \frac{u_{n+1}}{u_n} = l$, there exists a natural number k such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \text{ for all } n \ge k.$$

Let $l - \epsilon = p$. Then $p > 1$.
We have $\frac{u_{k+1}}{u_k} > p, \frac{u_{k+2}}{u_{k+1}} > p, \cdots, \frac{u_n}{u_{n-1}} > p$ where $n > k$.
Consequently, $\frac{u_n}{u_k} > p^{n-k}$ for all $n > k$ or, $u_n > \frac{u_k}{p^k} . p^n$ for all $n > k$.

$\frac{u_k}{p^k}$ is a positive number and Σp^n is a geometric series of common ratio $p > 1$ and therefore Σp^n is divergent.

Therefore Σu_n is divergent by Comparison test.

Note. When $l = 1$, the test fails to give a decision.
Let $u_n = \frac{1}{n}$. Then Σu_n is a divergent series and $\lim \frac{u_{n+1}}{u_n} = 1$
Let $u_n = \frac{1}{n^2}$. Then Σu_n is a convergent series and $\lim \frac{u_{n+1}}{u_n} = 1$.

Although for both the series $\lim \frac{u_{n+1}}{u_n} = 1$, one is a convergent series and the other is a divergent series.

Therefore if $\lim \frac{u_{n+1}}{u_n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Theorem 6.3.5. (Cauchy's root test)

Let Σu_n be a series of positive real numbers and let $\lim u_n^{1/n} = 1$.

Then Σu_n is convergent if $l < 1$, Σu_n is divergent if $l > 1$.

Proof. **Case 1.** $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number m such that

$$l - \epsilon < u_n^{1/n} < l + \epsilon \text{ for all } n \geq m$$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq m$.

Let $l + \epsilon = r$. Then $o < r < 1$ and $u_n < r^n$ for all $n \geq m$.

But Σr^n is a geometric series of common ratio r where $0 < r < 1$. So Σr^n is convergent.

Therefore Σu_n is convergent by Comparison test.

Case 2. $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim u_n^{1/n} = l$, there exists a natural number k such that

$$l - \epsilon < u_n^{1/n} < l + \epsilon \text{ for all } n \geq k$$

or, $(l - \epsilon)^n < u_n < (l + \epsilon)^n$ for all $n \geq k$.

Let $l - \epsilon = p$. Then $p > 1$ and $u_n > p^n$ for all $n \geq k$.

But Σp^n is a geometric series of common ratio $p > 1$. So Σp^n is divergent.

Therefore Σu_n is divergent by Comparison test.

Note. When $l = 1$, the test fails to give a decision.

Let $u_n = 1/n$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a divergent series.

Let $u_n = 1/n^2$. Then $\lim u_n^{1/n} = 1$ and Σu_n is a convergent series.

Although for both the series $\lim u_n^{1/n} = 1$, one is a convergent series and the other is a divergent series.

Thus if $\lim u_n^{1/n} = 1$, nothing can be said about the convergence or divergence of the series Σu_n.

Worked Examples (continued).

4. Test the convergence of the series $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \frac{2n-1}{n!}$.

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)(2n-1)}$ and $\lim \frac{u_{n+1}}{u_n} = 0 < 1$.

By D'Alembert's ratio test, Σu_n is convergent.

5. Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \cdots, x > 0$.

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Since $x > 0, \Sigma u_n$ is a series of positive terms. $\frac{u_{n+1}}{u_n} = \frac{nx}{n+1}$ and $\lim \frac{u_{n+1}}{u_n} = x$.

By D'Alembert's ratio test,

Σu_n is convergent if $x < 1$, Σu_n is divergent if $x > 1$.

When $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots$ and this is divergent.

6. Test the convergence of the series $1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots \cdots$

Ignoring the first term, let $\sum\limits_{n=1}^{\infty} u_n$ be the series. Then $u_n = \frac{n^n}{n!}$.

$\frac{u_{n+1}}{u_n} = (\frac{n+1}{n})^n$ and $\lim \frac{u_{n+1}}{u_n} = e > 1$.

Σu_n is divergent by D'Alembert's ratio test.

Therefore the given series is divergent.

7. Test the convergence of the series

$1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \cdots \cdots$

Here $u_n = \{2^{n+(-1)^n}\}^{-1}$ and $\lim u_n^{1/n} = \lim\{2^{1+\frac{(-1)^n}{n}}\}^{-1} = \frac{1}{2}$.

Therefore the series is convergent by Cauchy's root test.

Note. Here $\frac{u_{n+1}}{u_n} = \frac{1}{8}$ if n be odd,

$= 2$ if n be even.

$\lim \frac{u_{n+1}}{u_n}$ does not exist and therefore the convergence of the series cannot be decided by D'Alembert's ratio test.

It follows that the root test is more powerful than the ratio test in deciding convergence of a series of positive real numbers.

The fact is explained by the theorem 5.16.4 which states that if $u_n > 0$ then

$\underline{\lim} \frac{u_{n+1}}{u_n} \leq \underline{\lim}\, u_n^{1/n} \leq \overline{\lim}\, u_n^{1/n} \leq \overline{\lim} \frac{u_{n+1}}{u_n}$.

If for some series Σu_n of positive terms $\lim \frac{u_{n+1}}{u_n}$ exists and equals l, then $\lim u_n^{1/n}$ also exists and equals l. Therefore if the ratio test decides the convergence of the series Σu_n then the root test also does.

But if for some series Σu_n of positive terms $\lim u_n^{1/n}$ exists and equals l, then $\lim \frac{u_{n+1}}{u_n}$ does not necessarily exist. Therefore if the root test decides the convergence of the series Σu_n, the ratio test may not do so.

Theorem 6.3.6. (General form of ratio test)

Let Σu_n be a series of positive real numbers and let

$$\overline{\lim} \frac{u_{n+1}}{u_n} = R, \quad \underline{\lim} \frac{u_{n+1}}{u_n} = r.$$

Then Σu_n is convergent if $R < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $R < 1$.

Let us choose a positive ϵ such that $R + \epsilon < 1$.

Since $\overline{\lim} \frac{u_{n+1}}{u_n} = R$, there exists a natural number m such that

$$\frac{u_{n+1}}{u_n} < R + \epsilon \text{ for all } n \geq m.$$

Let $R + \epsilon = p$. Then $0 < p < 1$.

We have $\frac{u_{m+1}}{u_m} < p, \frac{u_{m+2}}{u_{m+1}} < p, \cdots, \frac{u_n}{u_{n-1}} < p$ where $n > m$.

Consequently, $\frac{u_n}{u_m} < p^{n-m}$ for all $n > m$

or, $u_n < \frac{u_m}{p^m} p^n$ for all $n > m$.

$\frac{u_m}{p^m}$ is a positive number and Σp^n is convergent since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim} \frac{u_{n+1}}{u_n} = r$, there exists a natural number k such that

$$\frac{u_{n+1}}{u_n} > r - \epsilon \text{ for all } n \geq k.$$

Let $r - \epsilon = q$. Then $q > 1$.

We have $\frac{u_{k+1}}{u_k} > q, \frac{u_{k+2}}{u_{k+1}} > q, \cdots, \frac{u_n}{u_{n-1}} > q$.

Consequently, $\frac{u_n}{u_k} > q^{n-k}$ for all $n > k$

or, $u_n > \frac{u_k}{q^k} q^n$ for all $n > k$.

$\frac{u_k}{q^k}$ is a positive number and Σq^n is divergent since $q > 1$.

Therefore Σu_n is divergent by Comparison test.

Theorem 6.3.7. (General form of root test)

Let Σu_n be a series of positive real numbers and let $\overline{\lim}\, u_n^{1/n} = r$.

Then Σu_n is convergent if $r < 1$, Σu_n is divergent if $r > 1$.

Proof. **Case 1.** $r < 1$.

Let us choose a positive ϵ such that $r + \epsilon < 1$.

Since $\overline{\lim}\, u_n^{1/n} = r$, there exists a natural number m such that

$$u_n^{1/n} < r + \epsilon \text{ for all } n \geq m.$$

Let $r + \epsilon = p$. Then $0 < p < 1$ and $u_n < p^n$ for all $n \geq m$.

But Σp^n is a convergent series since $0 < p < 1$.

Therefore Σu_n is convergent by Comparison test.

Case 2. $r > 1$.

Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\overline{\lim}\, u_n^{1/n} = r, u_n^{1/n} > r - \epsilon$ for infinite number of values of n.

That is, infinite number of elements of the sequence $\{u_n\}$ are greater than 1 and therefore $\lim u_n$ cannot be 0.

Therefore Σu_n is divergent, since a necessary condition for convergence of the series Σu_n is $\lim u_n = 0$.

Worked Examples (continued).

8. Test the convergence of the series.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \cdots$

Let $\sum\limits_1^\infty u_n$ be the given series.

Then $u_{2n} = \frac{1}{3^n}, u_{2n+1} = \frac{1}{2^{n+1}}, u_{2n-1} = \frac{1}{2^n}$.

$\lim \frac{u_{2n}}{u_{2n-1}} = \lim(\frac{2}{3})^n = 0, \lim \frac{u_{2n+1}}{u_{2n}} = \lim \frac{1}{3}(\frac{3}{2})^{n+1} = \infty$.

It follows that $\limsup \frac{u_{n+1}}{u_n} = \infty, \liminf \frac{u_{n+1}}{u_n} = 0$.

Clearly, the ratio test gives no decision.

$\lim(u_{2n})^{1/2n} = \frac{1}{\sqrt{3}}, \lim(u_{2n+1})^{1/(2n+1)} = \frac{1}{\sqrt{2}}$.

It follows that $\limsup(u_n)^{1/n} = \frac{1}{\sqrt{2}} < 1$.

Therefore Σu_n is convergent by the root test.

9. Test the series

$a + b + a^2 + b^2 + a^3 + b^3 + \cdots \cdots$ where $0 < a < b < 1$.

Let $\sum\limits_1^\infty u_n$ be the given series.

Here $\frac{u_{2n}}{u_{2n-1}} = (\frac{b}{a})^n, \frac{u_{2n+1}}{u_{2n}} = a(\frac{a}{b})^n$. $\lim \frac{u_{2n}}{u_{2n-1}} = \infty, \lim \frac{u_{2n+1}}{u_{2n}} = 0$.

It follows that $\overline{\lim}\, \frac{u_{n+1}}{u_n} = \infty, \underline{\lim}\, \frac{u_{n+1}}{u_n} = 0$.

The ratio test gives no decision.

$\lim u_{2n}^{1/2n} = \lim(b^n)^{1/2n} = \sqrt{b}$,

$\lim(u_{2n+1})^{1/(2n+1)} = \lim(a^{n+1})^{\frac{1}{2n+1}} = \sqrt{a}$.

It follows that $\limsup(u_n)^{1/n} = \sqrt{b} < 1$.

Therefore Σu_n is convergent by the root test.

Note. Here the ratio test does not decide convergence of the series but the root test does. The root test is more powerful than the ratio test for deciding convergence of a series positive real numbers.

Theorem 6.3.8. (Cauchy's condensation test)

Let $\{f(n)\}$ be a monotone decreasing sequence of positive real numbers and a be a positive integer > 1.

Then the series $\sum_1^\infty f(n)$ and $\sum_1^\infty a^n f(a^n)$ converge or diverge together.

Proof. Grouping the terms of $\Sigma f(n)$ as

$\{f(1)\}+\{f(2)+\cdots+f(a)\}+\{f(a+1)+\cdots+f(a^2)\}+\cdots$ and ignoring the first term, let Σv_n be the new series.

Then $v_n = f(a^{n-1}+1) + f(a^{n-1}+2) + \cdots + f(a^n)$ for all $n \geq 1$.

The number of terms in v_n is $a^n - a^{n-1}$. Since $\{f(n)\}$ is a monotone decreasing sequence, each term of $v_n \leq f(a^{n-1}+1)$ and $\geq f(a^n)$.

Therefore $(a^n - a^{n-1})f(a^n) \leq v_n$ for all $n \geq 1$

or, $\frac{a-1}{a} a^n f(a^n) \leq v_n$ for all $n \geq 1$.

Let $w_n = a^n f(a^n)$. Then $w_n \leq \frac{a}{a-1} v_n$ for all $n \geq 1$.

$\frac{a}{a-1}$ is positive. By Comparison test,

Σw_n is convergent if Σv_n is convergent

and Σv_n is divergent if Σw_n is divergent. ... ... (A)

$$\begin{aligned} \text{Again, } v_n &\leq (a^n - a^{n-1})f(a^{n-1}+1) \\ &\leq (a^n - a^{n-1})f(a^{n-1}) \text{ for all } n \geq 2. \end{aligned}$$

That is, $v_n \leq (a-1)w_{n-1}$ for all $n \geq 2$.

$a - 1$ is positive. By Comparison test,

Σv_n is convergent if Σw_n is convergent

and Σw_n is divergent if Σv_n is divergent. ... ... (B)

From (A) and (B), Σv_n and Σw_n converge or diverge together.

But Σv_n and $\Sigma f(n)$ converge or diverge together.

Therefore $\Sigma f(n)$ and Σw_n, i.e., $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ converge or diverge together.

Worked Examples (continued).

10. Test the convergence of the series $\sum_1^\infty \frac{1}{n}$.

Let $f(n) = \frac{1}{n}$. Then $\{f(n)\}$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 1$ and therefore $\Sigma 2^n f(2^n)$ is divergent.

It follows that $\Sigma f(n)$ is divergent, i.e., $\Sigma 1/n$ is divergent.

11. Discuss the convergence of the series $\sum_1^\infty 1/n^p$, $p > 0$.

Let $f(n) = 1/n^p$. As $p > 0$, the sequence $\{f(n)\}$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$2^n f(2^n) = 2^n . \frac{1}{2^{np}} = \frac{1}{2^{n(p-1)}}$.

But $\Sigma(\frac{1}{2^{p-1}})^n$ is a geometric series and it converges if $p > 1$ and diverges if $p \leq 1$.

Therefore $\Sigma 1/n^p$ is convergent when $p > 1$ and is divergent when $0 < p \leq 1$.

12. Discuss the convergence of the series $\sum\limits_2^\infty \dfrac{1}{n(\log n)^p}$, $p > 0$.

Let $f(n) = \frac{1}{n(\log n)^p}, n \geq 2$. As $\{\log n\}$ is an increasing sequence and $p > 0$, $\{\log(n+1)\}^p > \{\log n\}^p$ and therefore $(n+1)\{\log(n+1)\}^p > n(\log n)^p$.

Therefore $\{f(n)\}_2^\infty$ is a monotone decreasing sequence of positive real numbers.

By Cauchy's condensation test, the two series $\Sigma f(n)$ and $\Sigma 2^n f(2^n)$ converge or diverge together.

$\Sigma 2^n f(2^n) = \Sigma \frac{1}{(n \log 2)^p}$ and this converges when $p > 1$ and diverges when $p \leq 1$.

Therefore $\sum\limits_{n=2}^\infty f(n)$ is convergent when $p > 1$ and divergent when $0 < p \leq 1$.

If the limits $\lim \frac{u_{n+1}}{u_n}$ or $\lim \sqrt[n]{u_n}$ be equal to 1, D'Alembert's ratio test and Cauchy's root test fail to decide convergence of the series Σu_n. In such cases it is often helpful to use a more delicate test due to Raabe.

Theorem 6.3.9. (Raabe's test)

Let Σu_n be a series of positive real numbers and let $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$.

Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$, there exists a natural number m such that
$$l - \epsilon < n(\frac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $nu_n - nu_{n+1} > ru_{n+1}$ for all $n \geq m$

or, $nu_n - (n+1)u_{n+1} > (r-1)u_{n+1}$ for all $n \geq m$.

We have $mu_m - (m+1)u_{m+1} > (r-1)u_{m+1}$
$(m+1)u_{m+1} - (m+2)u_{m+2} > (r-1)u_{m+2}$
... ...
$(n-1)u_{n-1} - nu_n > (r-1)u_n$ where $n > m$.

Consequently, $mu_m - nu_n > (r-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

$$\text{or, } u_{m+1} + u_{m+2} + \cdots + u_n < \tfrac{1}{r-1}(mu_m - nu_n) < \tfrac{1}{r-1}mu_m$$

or, $s_n - s_m < \frac{1}{r-1}mu_m$ where $s_n = u_1 + u_2 + \cdots + u_n$
or, $s_n < \frac{1}{r-1}mu_m + s_m$ for all $n > m$.

This shows that the sequence $\{s_n\}$ is bounded above and therefore the series Σu_n is convergent.

Case 2. $l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$. There exists a natural number k such that

$$l - \epsilon < n(\tfrac{u_n}{u_{n+1}} - 1) < l + \epsilon \text{ for all } n \geq k.$$

Let $l + \epsilon = p$. Then $p < 1$.
We have $n(\frac{u_n}{u_{n+1}} - 1) < p < 1$ for all $n \geq k$.
Therefore $n(u_n - u_{n+1}) < pu_{n+1} < u_{n+1}$ for all $n \geq k$
or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.
We have $ku_k < (k+1)u_{k+1}$
$(k+1)u_{k+1} < (k+2)u_{k+2}$
... ...
$(n-1)u_{n-1} < nu_n$ where $n > k$.
Consequently, $nu_n > ku_k$ for all $n > k$
or, $u_n > ku_k.\frac{1}{n}$.
ku_k is a positive number and $\Sigma\frac{1}{n}$ is a divergent series.
Therefore Σu_n is divergent by Comparison test.

Note. If $l = 1$, the test is inconclusive. This can be established by taking the series Σu_n where $u_n > 0$ for all $n \in \mathbb{N}$ and $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{2}{n \log n}, n \geq 2$ and the series Σv_n where $v_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.

Theorem 6.3.10. (General form of Raabe's test)

Let Σu_n be a series of positive real numbers and let $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$ and $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$.
Then Σu_n is convergent if $r > 1$, Σu_n is divergent if $R < 1$.

Proof. **Case 1.** $r > 1$.
Let us choose a positive ϵ such that $r - \epsilon > 1$.

Since $\underline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = r$, there exists a natural number m such that $n(\frac{u_n}{u_{n+1}} - 1) > r - \epsilon$ for all $n \geq m$.

Let $r - \epsilon = k$. Then $k > 1$.

We have $nu_n - nu_{n+1} > ku_{n+1}$ for all $n \geq m$

or, $nu_n - (n+1)u_{n+1} > (k-1)u_{n+1}$ for all $n \geq m$

We have $mu_m - (m+1)u_{m+1} > (k-1)u_{m+1}$

$$(m+1)u_{m+1} - (m+2)u_{m+2} > (k-1)u_{m+2}$$
$$\cdots \quad \cdots$$
$$(n-1)u_{n-1} - nu_n > (k-1)u_n \quad \text{where } n > m.$$

Consequently, $mu_m - nu_n > (k-1)(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

$$\begin{aligned} \text{or, } u_{m+1} + u_{m+2} + \cdots + u_n &< \tfrac{1}{k-1}(mu_m - nu_n) \\ &< \tfrac{1}{k-1} mu_m. \end{aligned}$$

Let $s_n = u_1 + u_2 + \cdots + u_n$.

Then $s_n < s_m + \frac{1}{k-1} mu_m$ for all $n > m$.

This shows that the sequence $\{s_n\}$ is bounded above and therefore the series Σu_n is convergent.

Case 2. $R < 1$.

Let us choose a positive ϵ such that $R + \epsilon < 1$.

Since $\overline{\lim}\, n(\frac{u_n}{u_{n+1}} - 1) = R$, there exists a natural number k such that $n(\frac{u_n}{u_{n+1}} - 1) < R + \epsilon$ for all $n \geq k$.

Let $R + \epsilon = p$. Then $p < 1$.

We have $n(u_n - u_{n+1}) < pu_{n+1}$ for all $n \geq k$

i.e., $n(u_n - u_{n+1}) < u_{n+1}$ for all $n > k$

or, $nu_n < (n+1)u_{n+1}$ for all $n \geq k$.

We have $ku_k < (k+1)u_{k+1}$

$$(k+1)u_{k+1} < (k+2)u_{k+2}$$
$$\cdots \quad \cdots$$
$$(n-1)u_{n-1} < nu_n \quad \text{where } n > k.$$

Therefore $ku_k < nu_n$ for all $n > k$

or, $u_n > ku_k . \frac{1}{n}$ for all $n > k$.

ku_k is positive and $\Sigma 1/n$ is a divergent series.

Therefore Σu_n is divergent by Comparison test.

Worked Example (continued).

13. Test the convergence of the series

$$1 + \frac{1}{2} . \frac{1}{3} + \frac{1.3}{2.4} . \frac{1}{5} + \frac{1.3.5}{2.4.6} . \frac{1}{7} + \cdots \cdots$$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)} \cdot \frac{1}{2n-1}$ for all $n \geq 2$.

Therefore $\frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.

D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim\limits_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = \lim\limits_{n\to\infty} \frac{6n^2-n}{(2n-1)^2} = \frac{3}{2} > 1$.

Therefore Σu_n is convergent by Raabe's test.

Theorem 6.3.11. (Logarithmic test)

Let Σu_n be a series of positive real numbers and $\lim n \log \frac{u_n}{u_{n+1}} = l$. Then Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

Proof. **Case 1.** $l > 1$.

Let us choose a positive ϵ such that $l - \epsilon > 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number m such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq m.$$

Let $l - \epsilon = r$. Then $r > 1$.

We have $n \log \frac{u_n}{u_{n+1}} > r > 1$ for all $n \geq m$

or, $\frac{u_n}{u_{n+1}} > e^{r/n}$ for all $n \geq m$.

Since $\{(1 + \frac{1}{n})^n\}$ is a monotonic increasing sequence converging to e and e is irrational, $(1 + \frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

It follows that $\frac{u_n}{u_{n+1}} > (1 + \frac{1}{n})^r$ for all $n \geq m$

$$= \frac{(n+1)^r}{n^r}.$$

Let $v_n = \frac{1}{n^r}$. Then $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ for all $n \geq m$.

That is, $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ for all $n \geq m$.

By Comparison test, Σu_n is convergent since Σv_n is convergent.

Case 2. $0 \leq l < 1$.

Let us choose a positive ϵ such that $l + \epsilon < 1$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number p such that

$$l - \epsilon < n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq p.$$

Let $l + \epsilon = k$. Then $0 < k < 1$.

$n \log \frac{u_n}{u_{n+1}} < k$ for all $n \geq p$

or, $\frac{u_n}{u_{n+1}} < e^{k/n}$ for all $n \geq p$.

Since $\{(1 + \frac{1}{n-1})^n\}_{n=2}^{\infty}$ is a monotone decreasing sequence converging to e and e is irrational, $(1 + \frac{1}{n-1})^n > e$ for all $n \geq 2$.

Therefore $\frac{u_n}{u_{n+1}} < (\frac{n}{n-1})^k$ for all $n \geq p > 1$.

Let $w_n = \frac{1}{(n-1)^k}$ for $n \geq 2$. Then $\sum\limits_{2}^{\infty} w_n$ is divergent and $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq p > 1$.

By Comparison test, Σu_n is divergent.

Case 3. $l < 0$.

Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim n \log \frac{u_n}{u_{n+1}} = l$, there exists a natural number q such that

$$n \log \frac{u_n}{u_{n+1}} < l + \epsilon \text{ for all } n \geq q.$$

Let $l + \epsilon = s$. Then $s < 0$ and $n \log \frac{u_n}{u_{n+1}} < s < 0$ for all $n \geq q$

or, $n \log \frac{u_{n+1}}{u_n} > -s > 0$

or, $n \log \frac{u_{n+1}}{u_n} > p' > 0$ (where $p' = -s$) for all $n \geq q$

or, $\frac{u_{n+1}}{u_n} > e^{p/n}$ for all $n \geq q$.

Since $e > (1 + \frac{1}{n})^n$ for all $n \in \mathbb{N}$, it follows that $\frac{u_{n+1}}{u_n} > (1 + \frac{1}{n})^{p'}$ for all $n \geq q$.

Let $w_n = n^{p'}$. Then $\frac{u_{n+1}}{u_n} > \frac{w_{n+1}}{w_n}$ for all $n \geq q$.

As Σw_n is a divergent series, Σu_n is divergent by Compasiron test.

Worked Example (continued).

14. Test the convergence of the series

$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots \quad \cdots, x > 0$

Ignoring the first term, let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{n^n x^n}{n!}$. $\frac{u_{n+1}}{u_n} = (1 + \frac{1}{n})^n x$ and $\lim \frac{u_{n+1}}{u_n} = ex$.

By D'Alembert's ratio test,

Σu_n is convergent if $0 < x < 1/e$, Σu_n is divergent if $x > 1/e$.

When $x = 1/e$, let us apply Logarithmic test.

$\lim n \log \frac{u_n}{u_{n+1}} = \lim n[1 + n \log \frac{n}{n+1}] = \lim[n + n^2 \log \frac{n}{n+1}] = \frac{1}{2}$.

By Logarithmic test, Σu_n is divergent when $x = 1/e$.

So the given series is convergent if $0 < x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

Theorem 6.3.12. (Kummer's test)

Let Σu_n and $\Sigma 1/d_n$ be two series of positive real numbers and let $w_n = \frac{u_n}{u_{n+1}} d_n - d_{n+1}$.

If $\lim w_n = k > 0$ then Σu_n is convergent.

If $\lim w_n = k < 0$ and $\Sigma 1/d_n$ is divergent then Σu_n is divergent.

Proof. Case 1. $k > 0$.

Let us choose a positive ϵ such that $k - \epsilon > 0$.

Since $\lim w_n = k$, there exists a natural number m such that
$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq m.$$

Let $k - \epsilon = r$. Then $r > 0$ and $\frac{u_n d_n}{u_{n+1}} - d_{n+1} > r$ for all $n \geq m$

or, $u_n d_n - u_{n+1} d_{n+1} > r u_{n+1}$ for all $n \geq m$.

Then we have $u_m d_m - u_{m+1} d_{m+1} > r u_{m+1}$

$u_{m+1} d_{m+1} - u_{m+2} d_{m+2} > r u_{m+2}$

$\ldots \quad \ldots \quad \ldots$

$u_{n-1} d_{n-1} - u_n d_n > r u_n$, where $n > m$.

So $u_m d_m - u_n d_n > r(u_{m+1} + u_{m+2} + \cdots + u_n)$ for all $n > m$

or, $u_{m+1} + u_{m+2} + \cdots + u_n < \frac{1}{r}(u_m d_m - u_n d_n)$
$$< \tfrac{1}{r} u_m d_m.$$

or, $s_n - s_m < \frac{1}{r} u_m d_m$, where $s_n = u_1 + u_2 + \cdots + u_n$

or, $s_n < s_m + \frac{1}{r} u_m d_m$ for all $n > m$.

The sequence $\{s_n\}$ is bounded above and therefore Σu_n is convergent.

Case 2. $k < 0$.

Let us choose a positive ϵ such that $k + \epsilon < 0$.

Then there exists a natural number p such that
$$k - \epsilon < w_n < k + \epsilon \text{ for all } n \geq p.$$

So $\frac{u_n d_n}{u_{n+1}} - d_{n+1} < 0$ for all $n \geq p$.

or, $u_n d_n < u_{n+1} d_{n+1}$ for all $n \geq p$.

We have $u_p d_p < u_{p+1} d_{p+1}$

$u_{p+1} d_{p+1} < u_{p+2} d_{p+2}$

$\ldots \quad \ldots \quad \ldots$

$u_{n-1} d_{n-1} < u_n d_n$ for all $n > p$.

So $u_p d_p < u_n d_n$ for all $n > p$.

or, $u_n > \frac{u_p d_p}{d_n}$ for all $n > p$.

$u_p d_p$ is positive and $\Sigma \frac{1}{d_n}$ is a divergent series.

Therefore Σu_n is divergent by Comparison test.

Corollary 1. If we take $d_n = n$ then

$w_n = n \frac{u_n}{u_{n+1}} - (n+1) = n(\frac{u_n}{u_{n+1}} - 1) - 1.$

$\lim w_n = \lim n(\frac{u_n}{u_{n+1}} - 1) - 1.$

Let $\lim n(\frac{u_n}{u_{n+1}} - 1) = l$. Then Kummer's test gives

Σu_n is convergent if $l > 1$, Σu_n is divergent if $l < 1$.

This is Rabbe's test.

Corollary 2. If we take $d_n = 1$ then $w_n = \frac{u_n}{u_{n+1}} - 1$ and $\lim w_n = \lim(\frac{u_n}{u_{n+1}} - 1)$.

Let $\lim \frac{u_{n+1}}{u_n} = l$. Then Kummer's test gives
Σu_n is convergent if $\frac{1}{l} > 1$, i.e., if $l < 1$;
Σu_n is divergent if $\frac{1}{l} < 1$, i.e., if $l > 1$.
This is D'Alembert's ratio test.

Theorem 6.3.13. (Gauss's test).

Let Σu_n be a series of positive real numbers and let $\frac{u_n}{u_{n+1}} = 1+\frac{a}{n}+\frac{b_n}{n^p}$, where $p > 1$ and the sequence $\{b_n\}$ is bounded.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

Proof. **Case 1.** $a \neq 1$.
$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{a}{n} + \frac{b_n}{n^p}) = a$, since $\lim \frac{b_n}{n^{p-1}} = 0$.
By Raabe's test,
Σu_n is convergent if $a > 1$ and Σu_n is divergent if $a < 1$.

Case 2. $a = 1$.
Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}$.
Let us apply Kummer's test by taking $d_n = n \log n$.

$$\begin{aligned} \text{Then } w_n &= \frac{d_n u_n}{u_{n+1}} - d_{n+1} \\ &= n \log n(1 + \frac{1}{n} + \frac{b_n}{n^p}) - (n+1)\log(n+1) \\ &= (n+1)\log n + \frac{b_n \log n}{n^{p-1}} - (n+1)\log(n+1) \\ &= (n+1)\log\frac{n}{n+1} + \frac{\log n}{n^{p-1}} b_n. \end{aligned}$$

$$\begin{aligned} \lim w_n &= \lim(n+1)\log(1 - \tfrac{1}{n+1}) + \lim \tfrac{\log n}{n^{p-1}}.b_n \\ &= -1, \text{ since } \lim \log(1 - \tfrac{1}{n+1})^{n+1} = \log e^{-1} = -1 \text{ and} \end{aligned}$$

$\lim \frac{\log n}{n^{p-1}} = 0$ and $\{b_n\}$ is a bounded sequence.
By Kummer's test, Σu_n is divergent.

Worked Examples (continued).

15. Examine the convergence of the series $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \cdots\cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \frac{2^2.4^2\cdots(2n-2)^2}{3^2.5^2\cdots(2n-1)^2}$ for all $n \geq 2$.

$\frac{u_{n+1}}{u_n} = \frac{4n^2}{4n^2+4n+1}$ and $\lim \frac{u_{n+1}}{u_n} = 1$.
D'Alembert's ratio test gives no decision.

Let us apply Raabe's test.

$\lim n(\frac{u_n}{u_{n+1}} - 1) = \lim n(\frac{4n+1}{4n^2+4n+1}) = 1$.
Raabe's test gives no decision.

Let us apply Gauss's test.

$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{1}{4n^2}$.

$\frac{u_n}{u_{n+1}}$ is of the form $1 + \frac{a}{n} + \frac{b_n}{n^2}$, where $a = 1$ and $b_n = \frac{1}{4}$ and so $\{b_n\}$ is a bounded sequence.

By Gauss's test, Σu_n is divergent.

16. Hypergeometric series.

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

where $\alpha, \beta, \gamma, x > 0$.

Ignoring the first term, let $\sum\limits_1^{\infty} u_n$ be the series.

Then $u_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1.2\cdots.n\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$ for $n \geq 1$.

$\frac{u_{n+1}}{u_n} = \frac{(\alpha+n)(\beta+n)}{(1+n)(\gamma+n)}x$ and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = x$.

By D'Alembert's ratio test, Σu_n is convergent if $0 < x < 1$ and Σu_n is divergent if $x > 1$.

When $x = 1$,

$\frac{u_n}{u_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)}$

$= 1 + (\frac{(\gamma+1-\alpha-\beta)n+(\gamma-\alpha\beta)}{n^2+(\alpha+\beta)n+\alpha\beta})$

$= 1 + (\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2})[1 + \frac{\alpha+\beta}{n} + \frac{\alpha\beta}{n^2}]^{-1}$

$= 1 + (\frac{\gamma+1-\alpha-\beta}{n} + \frac{\gamma-\alpha\beta}{n^2})[1 - \frac{\alpha+\beta}{n} - \frac{\alpha\beta}{n^2} + \cdots]$

$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + \frac{1}{n^2}[(\gamma-\alpha\beta) - (\alpha+\beta)(\gamma+1-\alpha-\beta) +$ terms containing $\frac{1}{n}$ and higher powers of $\frac{1}{n}]$

$= 1 + \frac{\gamma+1-\alpha-\beta}{n} + \frac{\phi(n)}{n^2}$, where $\lim\limits_{n\to\infty} \phi(n)$ is finite and therefore $\{\phi(n)\}$ is bounded.

By Gauss's test, when $x = 1$,
Σu_n is convergent if $\gamma + 1 - \alpha - \beta > 1$ and
Σu_n is divergent if $\gamma + 1 - \alpha - \beta \leq 1$.

Therefore the series is convergent if $0 < x < 1$ and divergent if $x > 1$.

When $x = 1$, the series is convergent if $\gamma > \alpha + \beta$ and divergent if $\gamma \leq \alpha + \beta$.

The order symbol O.

Let f and ϕ be two functions of n defined for all $n \geq m$, m being a natural number; and ϕ be ultimately monotone with $\phi(n) > 0$ for sufficiently large n.

If there exists a natural number $m_o \geq m$ such that $|f(n)| \leq k\phi(n)$ for all $n \geq m_o, k$ being a positive constant, we write $f = O(\phi)$.

Thus $O(\phi)$ denotes a function f such that $f(n) = h(n)\phi(n)$ where h is a bounded function of n.

In particular, $f = O(1)$ means that f is a bounded function of n.

Examples.

1. Let $f(n) = 5n^2 + 3n + 1$. Then $f(n) = O(n^2)$, since $f(n) \leq 5n^2$ for all $n \geq 1$.

2. Let $f(n) = \frac{(-1)^n}{n}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{|f(n)|}{\frac{1}{n}} \to 1$ as $n \to \infty$

3. Let $f(n) = \frac{2n^2-3n+1}{5n^3-3}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{2}{5}$ as $n \to \infty$.

4. Let $f(n) = \frac{1}{\sqrt{n^2-1}+\sqrt{n^2+1}}$. Then $f(n) = O(\frac{1}{n})$, since $\frac{f(n)}{\frac{1}{n}} \to \frac{1}{2}$ as $n \to \infty$.

5. Let $f(n) = 2\sin\frac{n\pi}{4}$. Then $f(n) = O(1)$, since $| \ f(n) \ | \leq 2$ for all $n \geq 1$.

Alternative form of Gauss's test.

Let Σu_n be a series of positive real numbers and let
$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + O(\frac{1}{n^p})$ where $p > 1$.

Then Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \leq 1$.

$O(\frac{1}{n^p})$ denotes a sequence f such that $f(n) = h(n).\frac{1}{n^p}$, where h is a bounded sequence.

Therefore $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{h(n)}{n^p}$, where $h(n)$ is a bounded sequence and $p > 1$.

By Gauss's test, Σu_n is convergent if $a > 1$, Σu_n is divergent if $a \geq 1$.

Worked Example (continued).

17. Test the convergence of the series
$(\frac{1}{2})^2 + (\frac{1.3}{2.4})^2 + (\frac{1.3.5}{2.4.6.})^2 + \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $u_n = \{\frac{1.3.5...(2n-1)}{2.4.6...2n}\}^2$ for all $n \geq 1$.

$$\begin{aligned}\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} &= (1 + \tfrac{1}{n})^2(1 + \tfrac{1}{2n})^{-2} \\ &= (1 + \tfrac{2}{n} + \tfrac{1}{n^2})[1 - \tfrac{1}{n} + O(\tfrac{1}{n^2})] \\ &= 1 + \tfrac{1}{n} + O(\tfrac{1}{n^2}).\end{aligned}$$

By Gauss's test, Σu_n is divergent.

Theorem 6.3.14. **(De Morgan and Bertrand's test)**

Let Σu_n be a series of positive real numbers and $\lim[n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n = l$.

Then Σu_n is convergent if $l > 1$; and divergent if $l < 1$.

Proof. Let $b_n = [n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n$.

Then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n \log n}$.

Let $w_n = \frac{u_n}{u_{n+1}} d_n - d_{n+1}$ where $d_n = n \log n$.

Then $\sum\limits_{n=2}^{\infty} \frac{1}{d_n}$ is a divergent series and

$$\begin{aligned} w_n &= \frac{u_n}{u_{n+1}} n \log n - (n+1)\log(n+1) \\ &= (1 + \tfrac{1}{n} + \tfrac{b_n}{n \log n}) n \log n - (n+1)\log(n+1) \\ &= (n+1)\log \tfrac{n}{n+1} + b_n. \end{aligned}$$

$\lim w_n = \lim \log \frac{1}{(1+\frac{1}{n})^{n+1}} + l = -1 + l$.

By Kummer's test, Σu_n is convergent if $l - 1 > 0$, i.e., if $l > 1$ and Σu_n is divergent if $l - 1 < 0$, i.e., if $l < 1$.

Worked Example (continued).

18. Test the convergence of the series

$(\frac{1}{2})^3 + (\frac{1.4}{2.5})^3 + (\frac{1.4.7}{2.5.8})^3 + \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = \{\frac{1.4.7\ldots(3n-2)}{2.5.8\ldots(3n-1)}\}^3$.

$\frac{u_{n+1}}{u_n} = (\frac{3n+1}{3n+2})^3$.

$$\begin{aligned} \lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) &= \lim n[\frac{(3n+2)^3 - (3n+1)^3}{(3n+1)^3}] \\ &= \lim \frac{27n^3 + 27n^2 + 7n}{27n^3 + 27n^2 + 9n + 1} = 1. \end{aligned}$$

Raabe's test gives no decision.

Let us apply De Morgan and Bertrand's test.

$$\begin{aligned} \lim_{n\to\infty} [n(\frac{u_n}{u_{n+1}} - 1) - 1]\log n &= \lim \frac{(-2n-1)\log n}{27n^3 + 27n^2 + 9n + 1} \\ &= \lim \frac{-2n^3 - n^2}{27n^3 + 27n^2 + 9n + 1} \cdot \frac{\log n}{n^2} \\ &= \frac{-2}{27}.0, \text{ since } \lim \frac{\log n}{n^2} = 0 \\ &= 0 < 1. \end{aligned}$$

By De Morgan and Bertrand's test, Σu_n is divergent.

Theorem 6.3.15. **(Abel's theorem or Pringsheim's theorem)**

If Σu_n be a convergent series positive real numbers and $\{u_n\}$ is a monotone decreasing sequence then $\lim n u_n = 0$.

Proof. Since Σu_n is convergent, for a pre-assigned positive ϵ there exists a natural number m such that

$| u_{n+1} + u_{n+2} + \cdots + u_{n+p} | < \frac{\epsilon}{2}$ for all $n \geq m$ and for every natural number p.

Let $n = m$.

Then $u_{m+1} + u_{m+2} + \cdots + u_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

But $u_{m+1} + u_{m+2} + \cdots + u_{m+p} \geq pu_{m+p}$, since $\{u_n\}$ is a monotone decreasing sequence.

Consequently, $pu_{m+p} < \frac{\epsilon}{2}$ for every natural number p.

Let $p = m$. Then $2mu_{2m} < \epsilon$ ... ... (i)

Let $p = m + 1$. Then $(m + 1)u_{2m+1} < \frac{\epsilon}{2}$.

Therefore $(2m + 1)u_{2m+1} < (2m + 2)u_{2m+1} < \epsilon$ ... ... (ii)

From (i) and (ii) $nu_n < \epsilon$ for all $n \geq 2m$.

This shows that $\lim nu_n = 0$.

Note. If $\{u_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim nu_n = 0$, then Σu_n is not necessarily convergent.

For example, let $u_n = \frac{1}{n \log n}, n > 1$. Then $u_{n+1} < u_n$ for all $n > 1$ and $\lim nu_n = 0$. But $\sum\limits_{2}^{\infty} u_n$ is a divergent series.

Worked Examples (continued).

19. Prove that the series $(\frac{1}{2})^p + (\frac{1.3}{2.4})^p + (\frac{1.3.5}{2.4.6})^p + \cdots$ is convergent for $p > 2$ and divergent for $p \leq 2$.

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

$$\begin{aligned} \text{Then } \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p &= (1 + \tfrac{1}{n})^p(1 + \tfrac{1}{2n})^{-p} \\ &= \{1 + \tfrac{p}{n} + O(\tfrac{1}{n^2})\}\{1 - \tfrac{p}{2n} + O(\tfrac{1}{n^2})\} \\ &= 1 + \tfrac{p}{2n} + O(\tfrac{1}{n^2}). \end{aligned}$$

By Gauss's test,

the series Σu_n is convergent if $\frac{p}{2} > 1$, i.e., if $p > 2$ and divergent if $\frac{p}{2} \leq 1$, i.e., if $p \leq 2$.

20. If Σu_n be a divergent series of positive real numbers prove that the series $\Sigma \frac{u_n}{1+u_n}$ is divergent.

Let $s_n = u_1 + u_2 + \cdots + u_n$.

Since the series Σu_n is a divergent series of positive real numbers, the sequence $\{s_n\}$ is a monotone increasing sequence and $\lim s_n = \infty$.

Therefore for every natural number n we can choose a natural number p such that $s_{n+p} > 1 + 2s_n$.

Now $\frac{u_{n+1}}{1+u_{n+1}} + \frac{u_{n+2}}{1+u_{n+2}} + \cdots + \frac{u_{n+p}}{1+u_{n+p}}$

$> \frac{u_{n+1}}{1+s_{n+p}} + \frac{u_{n+2}}{1+s_{n+p}} + \cdots + \frac{u_{n+p}}{1+s_{n+p}}$, since $s_{n+p} \geq s_{n+1} > u_{n+1}$, $s_{n+p} \geq s_{n+2} > u_{n+2}, \cdots, s_{n+p} > u_{n+p}$

$= \frac{s_{n+p}-s_n}{1+s_{n+p}}$

$> \frac{\frac{1}{2}(1+s_{n+p})}{1+s_{n+p}} = \frac{1}{2}$.

This shows that Cauchy's principle of convergence is not satisfied by the series $\Sigma\frac{u_n}{1+u_n}$. Hence the series is divergent.

Exercises 9

1. If Σu_n be a convergent series of positive real numbers prove that Σu_n^2 is convergent.

[**Hint.** There exists an $m \in \mathbb{N}$ such that $u_n < 1$ for all $n \geq m$. $\therefore$ $u_n^2 < u_n$ for all $n \geq m$.]

2. If Σu_n be a convergent series of positive real numbers prove that $\Sigma\frac{u_n}{n}$ is convergent.

[**Hint.** $u_n \cdot \frac{1}{n} < \frac{u_n^2+1/n^2}{2}$.]

3. If Σu_n be a convergent series of positive real numbers prove that the series Σu_{2n} is convergent.

Hint. Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = u_2 + u_4 + \cdots + u_{2n}$. Then $t_n < s_{2n}$ for all $n \in \mathbb{N}$. The sequence $\{t_n\}$ is a monotone increasing sequence bounded above.

4. If Σu_n be a convergent series of positive real numbers prove that $\Sigma\frac{u_n}{1+u_n}$ is convergent.

[**Hint.** Let $v_n = \frac{u_n}{1+u_n}$. Then $\lim \frac{u_n}{v_n} = 1$.]

5. If Σu_n be a series of positive real numbers and $v_n = \frac{u_1+u_2+\cdots+u_n}{n}$, prove that Σv_n is divergent.

[**Hint.** $v_1 + v_2 + \cdots + v_n > u_1(1 + \frac{1}{2} + \cdots + \frac{1}{n})$.]

6. If Σu_n be a divergent series of positive real numbers and $s_n = u_1 + u_2 + \cdots + u_n$, prove that the series $\Sigma\frac{u_n}{s_n}$ is divergent.

[**Hint.** Since $\{s_n\}$ is a monotone increasing sequence diverging to ∞, for every natural number n, we can choose a natural number p such that $s_{n+p} > 2s_n$. Then $\frac{u_{n+1}}{s_{n+1}} + \frac{u_{n+2}}{s_{n+2}} + \cdots + \frac{u_{n+p}}{s_{n+p}} > \frac{1}{2}$.]

7. If $\{a_1, a_2, a_3, \cdots\}$ be the collection of those natural numbers that end with 1, prove that the series $\Sigma\frac{1}{a_n}$ is divergent.

Hint. Considering the collection as an increasing sequence of natural numbers, $a_n = 10n - 9$ for all $n \in \mathbb{N}$. $\frac{1}{a_n} > \frac{1}{10n}$ for all $n \in \mathbb{N}$.

8. Test the convergence of the following series

(i) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots\cdots$ (ii) $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \cdots\cdots$

(iii) $\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \cdots\cdots$ (iv) $\sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \sin\frac{\pi}{6} + \cdots\cdots$

(v) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots\cdots$ (vi) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \cdots\cdots$

[**Hint.** (iv) $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.]

9. Test the series Σu_n for convergence where $u_n =$

(i) $\frac{2^n+1}{3^n+2}$, (ii) $\sqrt{n^4+1} - n^2$, (iii) $\sqrt[3]{n^3+1} - n$, (iv) $\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$,

(v) $\frac{1}{\sqrt{n}} \tan\frac{1}{n}$, (vi) $\frac{1}{n}\sin\frac{1}{n}$, (vii) $\frac{3^n}{2^n+3^n}$, (viii) $\frac{1}{n\log n}, n \geq 2$,

(ix) $\frac{1}{n \log n (\log\log n)}, n \geq 3$.

10. Test the convergence of the following series

(i) $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots\cdots$

(ii) $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots\cdots$

(iii) $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \cdots\cdots$

(iv) $(\frac{2^2}{1^2} - \frac{2}{1})^{-1} + (\frac{3^3}{2^3} - \frac{3}{2})^{-2} + (\frac{4^4}{3^4} - \frac{4}{3})^{-3} + \cdots\cdots$

(v) $1 + \frac{1}{2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^4} + \frac{1}{2^5} + \frac{1}{4^6} + \cdots\cdots$

(vi) $1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2^2.3} + \frac{1}{2^2.3^2} + \frac{1}{2^3.3^2} + \cdots$

(vii) $\frac{1}{3} + \frac{1}{5} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\cdots$

(viii) $\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3^5} + \frac{1}{3^4} + \cdots\cdots$

(ix) $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots\cdots$

[**Hint.** $\log(1+x) < x$ for $x > 0$.]

(x) $\tan\frac{\pi}{4} + \tan\frac{\pi}{8} + \tan\frac{\pi}{12} + \cdots\cdots$

[**Hint.** $x < \tan x$ for $0 < x < \frac{\pi}{2}$.]

(xi) $(\frac{1}{2})^{\log 1} + (\frac{1}{2})^{\log 2} + (\frac{1}{2})^{\log 3} + \cdots\cdots$

(xii) $\frac{1}{3} + (\frac{1}{3})^{1+\frac{1}{2}} + (\frac{1}{3})^{1+\frac{1}{2}+\frac{1}{3}} + \cdots\cdots$

(xiii) $\frac{1}{4} + (\frac{1}{4})^{1+\frac{1}{3}} + (\frac{1}{4})^{1+\frac{1}{3}+\frac{1}{5}} + \cdots\cdots$

[**Hint.** (xi), (xii), (xiii). Use Logarithmic test.]

(xiv) $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots\cdots, x > 0$

(xv) $\frac{2}{3} + \frac{2.4}{3.5}x + \frac{2.4.6}{3.5.7}x^2 + \cdots\cdots, x > 0$

(xvi) $1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots\cdots, x > 0$

(xvii) $\frac{3}{7} + \frac{3.6}{7.10}x + \frac{3.6.9}{7.10.13}x^2 + \cdots\cdots, x > 0$

(xviii) $\frac{1+x}{1!} + \frac{(1+2x)^2}{2!} + \frac{(1+3x)^3}{3!} + \cdots\cdots, x > 0$

[**Hint.** Compare with the series $\Sigma \frac{n^n x^n}{n!}, x > 0.$]

(xix) $1 + \frac{2^2}{3^2}x + \frac{2^2.4^2}{3^2.5^2}x^2 + \cdots\cdots, x > 0$

(xx) $1 + \frac{1!}{2!}x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots\cdots, x > 0$

11. Prove that the series $\frac{a}{b} + \frac{a(a+c)}{b(b+c)} + \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + \cdots\cdots, a, b, c > 0$ is convergent if $b > a + c$ and divergent if $b \leq a + c$.

12. Prove that the series $1 + \frac{\alpha^2}{1.\beta} + \frac{\alpha^2(\alpha+1)^2}{1.2.\beta(\beta+1)} + \frac{\alpha^2(\alpha+1)^2(\alpha+2)^2}{1.2.3.\beta(\beta+1)(\beta+2)} + \cdots\cdots, \alpha, \beta > 0$ is convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$.

6.4. Series of arbitrary terms.

Let Σu_n be a series of positive and negative real numbers.

Let $u'_n = |u_n|$. Then $\Sigma u'_n$ is a series of positive real numbers.

If $\Sigma u'_n$ is convergent then Σu_n is said to be an *absolutely convergent series.*

Theorem 6.4.1. An absolutely convergent series is convergent.

Proof. Let Σu_n be a series of positive and negative real numbers and be absolutely convergent. Then $\Sigma |u_n|$ is a convergent series of positive terms.

Let us choose a positive ϵ. Then there exists a natural number m such that

$|\ |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|\ | < \epsilon$ for all $n \geq m$ and for every natural number p.

That is, $|u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

But $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}|$.

Therefore $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$ for all $n \geq m$ and for every natural number p.

By Cauchy's principle of convergence, Σu_n is convergent.

Examples.

1. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots$ is convergent since it is absolutely convergent.

2. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$ is convergent since it is absolutely convergent.

3. For a fixed value of x the series $\Sigma \frac{\sin nx}{n^2}$ is absolutely convergent.

$\Sigma \frac{\sin nx}{n^2}$ is a series of arbitrary terms.

$\left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2}$. Since $\Sigma \frac{1}{n^2}$ is a convergent series, $\Sigma \left| \frac{\sin nx}{n^2} \right|$ is a convergent series, by Comparison test.

Consequently, $\Sigma \frac{\sin nx}{n^2}$ is an absolutely convergent series.

Theorem 6.4.2. If the series Σu_n be absolutely convergent and $\{v_n\}$ be a bounded sequence, then the series $\Sigma u_n v_n$ is absolutely convergent.

Proof. There exists a positive real number k such that $|v_n| < k$ for all $n \in \mathbb{N}$.

Now $|u_{n+1}v_{n+1}| + |u_{n+2}v_{n+2}| + \cdots + |u_{n+p}v_{n+p}|$
$< k\{| u_{n+1} | + | u_{n+2} | + \cdots + | u_{n+p} |\}$.

Since Σu_n is absolutely convergent, the series $\Sigma|u_n|$ is convergent.

Therefore for a chosen positive ϵ there exists a natural number m such that $| u_{n+1} | + | u_{n+2} | + \cdots + | u_{n+p} | < \epsilon/k$ for all $n \geq m$ and for every natural number p.

Therefore $| u_{n+1}v_{n+1} | + | u_{n+2}v_{n+2} | + \cdots + | u_{n+p}v_{n+p} | < \epsilon$ for all $n \geq m$ and for every natural number p.

By Cauchy's principle of convergence, the series $\Sigma | u_n v_n |$ is convergent and consequently, the series $\Sigma u_n v_n$ is absolutely convergent.

Worked Example.

1. Test the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots\cdots$

Let $\sum_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1} \frac{n+1}{n^3} = \frac{(-1)^{n+1}}{n^2} . (1 + \frac{1}{n}) = a_n b_n$, say, where $a_n = \frac{(-1)^{n+1}}{n^2}, b_n = 1 + \frac{1}{n}$.

The series Σa_n is absolutely convergent and $\{b_n\}$ is a bounded sequence and therefore the series $\Sigma a_n b_n$, i.e., the given series is absolutely convergent.

Theorem 6.4.3. Ratio test.

Let Σu_n be a series of arbitrary terms and let $\lim \frac{|u_{n+1}|}{|u_n|} = l$.

Then (i) Σu_n is absolutely convergent if $l < 1$,

(ii) Σu_n is divergent if $l > 1$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

There exists a natural number m such that

$$l - \epsilon < \frac{|u_{n+1}|}{|u_n|} < l + \epsilon \text{ for all } n \geq m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$ and $\frac{|u_{n+1}|}{|u_n|} < r$ for all $n \geq m$.

Then $\frac{|u_{m+1}|}{|u_m|} < r, \frac{|u_{m+2}|}{|u_{m+1}|} < r, \cdots, \frac{|u_n|}{|u_{n-1}|} < r$.

Consequently, $\frac{|u_n|}{|u_m|} < r^{n-m}$ for all $n > m$

or, $|u_n| < \frac{|u_m|}{r^m} r^n$ for all $n > m$.

But Σr^n is a convergent series, since $0 < r < 1$.

By Comparison test, the series $\Sigma |u_n|$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

There exists a natural number k such that

$$l - \epsilon < \frac{|u_{n+1}|}{|u_n|} < l + \epsilon \text{ for all } n \geq k.$$

Therefore $\frac{|u_{n+1}|}{|u_n|} > l - \epsilon > 1$ for all $n \geq k$.

Hence the sequence $\{|u_n|\}$ is ultimately a monotone increasing sequence of positive real numbers.

So $\lim |u_n| \neq 0$ and this implies $\lim u_n \neq 0$. Consequently, the series Σu_n is divergent.

Theorem 6.4.4. Root test.

Let Σu_n be a series of arbitrary terms and let $\lim |u_n|^{1/n} = l$.

Then (i) Σu_n is absolutely convergent if $l < 1$,

(ii) Σu_n is divergent if $l > 1$.

Proof. (i) Let us choose a positive ϵ such that $l + \epsilon < 1$.

There exists a natural number m such that

$$l - \epsilon < |u_n|^{1/n} < l + \epsilon \text{ for all } n \geq m.$$

Let $l + \epsilon = r$. Then $0 < r < 1$.

We have $|u_n|^{1/n} < r$ for all $n \geq m$.

or, $|u_n| < r^n$ for all $n \geq m$.

But Σr^n is a convergent series, since $0 < r < 1$.

By Comparison test, the series $\Sigma \mid u_n \mid$ is convergent. Therefore the series Σu_n is absolutely convergent.

(ii) Let us choose a positive ϵ such that $l - \epsilon > 1$.

There exists a natural number k such that

$$l - \epsilon < |u_n|^{1/n} < l + \epsilon \text{ for all } n \geq k.$$

Therefore $|u_n| > 1$ for all $n \geq k$.

So $\lim |u_n| \neq 0$ and this implies $\lim u_n \neq 0$. Consequently, the series Σu_n is divergent.

Worked Examples (continued).

2. Examine the convergence of the series

$1 - \frac{2^2}{2!} + \frac{3^3}{3!} - \frac{4^4}{4!} + \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1}\frac{n^n}{n!}$.

$\frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)^{n+1}}{n+1}\cdot\frac{1}{n^n} = (1+\frac{1}{n})^n$ and $\lim \frac{|u_{n+1}|}{|u_n|} = e > 1$.

By Ratio test, the series Σu_n is divergent.

3. Examine the convergence of the series $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots \cdots$

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series. Then $u_n = (-1)^{n+1}\frac{(n!)^2}{(2n)!}$ for $n \geq 2$.

$\lim \frac{|u_{n+1}|}{|u_n|} = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$.

By Ratio test, the series Σu_n is absolutely convergent.

Alternating series.

Definition. If $u_n > 0$ for all n, the series $\sum\limits_{1}^{\infty}(-1)^{n+1}u_n$ called an *alternating series.*

Theorem 6.4.5. (Leibnitz's test)

If $\{u_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim u_n = 0$ then the alternating series

$$u_1 - u_2 + u_3 - u_4 + \cdots \text{ is convergent.}$$

Proof. Let $s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$.

Then $s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0$ for all $n \in \mathbb{N}$.

The sequence $\{s_{2n}\}$ is a monotone increasing sequence.

$s_{2n+1} - s_{2n-1} = -u_{2n} + u_{2n+1} \leq 0$ for all $n \in \mathbb{N}$.

The sequence $\{s_{2n+1}\}$ is a monotone decreasing sequence.

Again $s_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n}$

$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - u_{2n} < u_1$

The sequence $\{s_{2n}\}$ is bounded above.

$s_{2n+1} = u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n+1}$

$= (u_1 - u_2) + (u_3 - u_4) + \cdots + u_{2n+1} > u_1 - u_2$

The sequence $\{s_{2n+1}\}$ is bounded below.

Therefore both the sequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ are convergent.

Now $\lim(s_{2n+1} - s_{2n}) = \lim u_{2n+1} = 0$.

This shows that both the sequences $\{s_{2n+1}\}$ and $\{s_{2n}\}$ converge to the same limit.

Hence the sequence $\{s_n\}$ is convergent and consequently the series $\Sigma(-1)^{n+1}u_n$ is convergent.

Note. If s be the sum of the series and s_n be the nth partial sum then $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

$$s - s_n = (-1)^{n+2}\{u_{n+1} - u_{n+2} + u_{n+3} - \cdots\}$$
or, $$(-1)^n(s - s_n) = u_{n+1} - u_{n+2} + u_{n+3} - \cdots$$
$$= u_{n+1} - (u_{n+2} - u_{n+3}) - \cdots < u_{n+1}.$$

Also $(-1)^n(s - s_n) = (u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \cdots > 0$.

Combining, we have $0 < (-1)^n(s - s_n) < u_{n+1}$ for all $n \in \mathbb{N}$.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent by Leibnitz's test.

2. The series $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \cdots$ is convergent by Leibnitz's test.

3. The series $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots$ is convergent by Leibnitz's test, since $\lim \frac{1.3.....(2n-1)}{2.4.....2n} = 0$ and $\frac{1.3.....(2n-1)}{2.4.....2n} > \frac{1.3.....(2n+1)}{2.4.....(2n+2}$.

Theorem 6.4.6.(Abel's test)

If the sequence $\{b_n\}$ is a monotone bounded sequence and Σa_n is a convergent series then the series $\Sigma a_n b_n$ is convergent.

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$, $t_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Then $$t_n = s_1 b_1 + (s_2 - s_1)b_2 + (s_3 - s_2)b_3 + \cdots + (s_n - s_{n-1})b_n$$
$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_n b_{n+1}.$$

Since Σa_n is convergent, the sequence $\{s_n\}$ is convergent. Since the sequence $\{b_n\}$ is monotonic and bounded, $\{b_n\}$ is convergent.

Therefore $s_n b_{n+1}$ tends to a limit ... ... (i)

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n = b_1 - b_{n+1}$ tends to a definite limit since $\{b_n\}$ is convergent. Therefore Σd_n is absolutely convergent.

Since the sequence $\{s_n\}$ is bounded and the series Σd_n is absolutely convergent, the series $\Sigma s_n d_n$ is absolutely convergent, by Theorem 6.4.2.

Therefore the sequence $\{\Sigma s_n d_n\}$ is convergent ... ... (ii)

From (i) and (ii) it follows that the sequence $\{t_n\}$ is convergent and this proves that the series $\Sigma a_n b_n$ is convergent.

Examples.

1. The series $\sum_{2}^{\infty} \frac{(-1)^{n+1}}{n \log n}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n}$ is a convergent series and the sequence $\{\frac{1}{\log n}\}_2^{\infty}$ is a monotone decreasing sequence bounded below.

2. The series $\sum\limits_{1}^{\infty} \frac{(-1)^{n+1} \cdot n^n}{(n+1)^{n+1}}$ is convergent by Abel's test, since $\Sigma \frac{(-1)^{n+1}}{n+1}$ is a convergent series and the sequence $\{(1+\frac{1}{n})^{-n}\}$ is a monotone decreasing sequence bounded below.

Theorem 6.4.7.(Dirichlet's test)

If the sequence $\{b_n\}$ is a monotone sequence converging to 0 and the sequence of partial sums $\{s_n\}$ of the series Σa_n is bounded, then the series $\Sigma a_n b_n$ is convergent.

Proof. $s_n = a_1 + a_2 + \cdots + a_n$.

Let $t_n = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Then
$$\begin{aligned} t_n &= s_1 b_1 + (s_2 - s_1) b_2 + (s_3 - s_2) b_3 + \cdots + (s_n - s_{n-1}) b_n \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_n(b_n - b_{n+1}) + s_n b_{n+1}. \end{aligned}$$

Since the sequence $\{s_n\}$ is bounded and $\lim b_n = 0, \lim s_n b_{n+1} = 0$.

Let $d_n = b_n - b_{n+1}$. Then either $d_n \geq 0$ for all n, or ≤ 0 for all n; and $d_1 + d_2 + \cdots + d_n (= b_1 - b_{n+1})$ tends to a definite limit since $\lim b_n = 0$. Therefore Σd_n is absolutely convergent.

Since the sequence $\{s_n\}$ is bounded and the series Σd_n is absolutely convergent, the series $\Sigma s_n d_n$ is absolutely convergent by Theorem 6.4.2.

Hence the series $\Sigma s_n d_n$ is convergent and therefore the sequence $\{t_n\}$ is convergent.

This proves that the series $\Sigma a_n b_n$ is convergent.

Note. Leibnitz's test is a particular case of Dirichlet's test. If $\{b_n\}$ is a monotone decreasing sequence converging to 0, then the series $\sum\limits_{1}^{\infty}(-1)^{n+1} b_n$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded.

This is Leibnitz's test for an alternating series.

Examples.

1. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $\{\frac{1}{\sqrt{n}}\}$ is a monotone decreasing sequence converging to 0.

2. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ is convergent by Dirichlet's test, since the sequence of partial sums $\{s_n\}$ of the series $\Sigma(-1)^{n+1}$ is bounded and the sequence $\{\frac{1}{\log(n+1)}\}$ is a monotone decreasing sequence converging to 0.

6.5. Conditionally convergent series.

Definition. A series Σu_n is called *conditionally convergent* if Σu_n is convergent but $\Sigma \mid u_n \mid$ is not convergent.

A conditionally convergent series is also called a *semi convergent* series or a *non-absolutely convergent* series.

Examples.

1. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is convergent, but the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ is divergent.

Therefore the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$ is conditionally convergent.

2. Let $u_n = \frac{1}{2n-1}$. Then $\{u_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim u_n = 0$.

By Leibnitz's test, $\Sigma(-1)^{n+1} u_n$ is convergent. But Σu_n is a divergent series. Therefore $\Sigma(-1)^{n+1} u_n$ is a conditionally convergent series.

3. Show that the series

$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots \cdots, a > 0$

is (i) absolutely convergent if $p > 1$,

(ii) conditionally convergent if $0 < p \leq 1$.

Let Σu_n be the given series and $v_n = \mid u_n \mid$.

Then Σv_n is a series of positive real numbers and $v_n = \frac{1}{(n+a)^p}$.

Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$.

By Comparison test, Σv_n is convergent if $p > 1$,

Σv_n is divergent if $0 < p \leq 1$.

Case 1. $p > 1$.

In this case Σu_n is an alternating series and $\Sigma \mid u_n \mid$ is convergent. Therefore Σu_n is absolutely convergent.

Case 2. $0 < p \leq 1$.

In this case $\{v_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim v_n = 0$.

By Leibnitz's test, $\Sigma(-1)^{n+1} v_n$, i.e., Σu_n is convergent.

Since $\Sigma \mid u_n \mid$ is divergent, Σu_n is conditionally convergent.

Let Σu_n be a series of positive real numbers and let

$p_n = u_n$ if $u_n > 0$ $\quad q_n = 0$ if $u_n \geq 0$

$\quad = 0$ if $u_n \leq 0$; $\quad\quad = u_n$ if $u_n < 0$.

Then Σp_n is a series of positive real numbers along with some 0's and Σq_n is a series of negative real numbers along with some 0's.

For example, for the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \cdots$
$\Sigma p_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$ and $\Sigma q_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \cdots$

$p_n = \frac{u_n + |u_n|}{2}, q_n = \frac{u_n - |u_n|}{2}$ and $u_n = p_n + q_n$.

Theorem 6.5.1. Let Σu_n be a series of arbitrary terms and $p_n = \frac{u_n+|u_n|}{2}, q_n = \frac{u_n-|u_n|}{2}$.

(i) If Σu_n is absolutely convergent then both Σp_n and Σq_n are convergent.

(ii) If Σu_n is conditionally convergent then both Σp_n and Σq_n are divergent.

Proof. (i) Since Σu_n is absolutely convergent, both Σu_n and $\Sigma |u_n|$ are convergent.

But $p_n = \frac{u_n+|u_n|}{2}$, $q_n = \frac{u_n-|u_n|}{2}$. Hence Σp_n and Σq_n are both convergent.

(ii) Since Σu_n is conditionally convergent, Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Now $|u_n| = 2p_n - u_n$ ... ... (A)

If we assume that Σp_n is convergent then if follows from (A) that $\Sigma |u_n|$ is convergent, a contradiction. Therefore Σp_n is divergent.

Again $|u_n| = u_n - 2q_n$ ... ... (B)

If we assume that Σq_n is convergent then if follows from (B) that $\Sigma |u_n|$ is convergent, a contradiction. Therefore Σq_n is divergent.

Note. For all n, $p_n \geq 0$ and $q_n \leq 0$. From (i) it follows that in an absolutely convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both convergent.

From (ii) it follows that in a conditionally convergent series, the series formed by the positive terms alone and the series formed by the negative terms alone are both divergent.

Introduction and removal of brackets.

Theorem 6.5.2. Let Σu_n be a series of positive and negative real numbers and Σv_n is obtained from Σu_n by grouping it terms. Then

(i) if Σu_n converges to the sum s, then Σv_n also converges to s,

(ii) if Σv_n converges, then Σu_n may not be convergent.

Proof. (i) Let $v_1 = u_1 + u_2 + \cdots + u_{r_1}, v_2 = u_{r_1+1} + u_{r_1+2} + \cdots + u_{r_2}, \cdots, v_n = u_{r_{n-1}+1} + u_{r_{n-1}+2} + \cdots + u_{r_n}, \cdots \cdots$

Then $\{r_n\}$ is a strictly increasing sequence of natural numbers.

Let $s_n = u_1 + u_2 + \cdots + u_n$, $t_n = v_1 + v_2 + \cdots + v_n$.
Then $t_n = u_1 + u_2 + \cdots + u_{r_n} = s_{r_n}$.
Since Σu_n converges to the sum s, $\lim s_n = s$.

The sequence $\{t_n\}$ is a subsequence of the sequence $\{s_n\}$ and therefore the sequence $\{t_n\}$ also converges to the sum s.
In other words, the series Σv_n converges to the sum s.

(ii) That the converse is not true can be established by the following example.
Let $u_n = (-1)^{n+1}$. Then the series Σu_n is $1 - 1 + 1 - 1 + \cdots$
This is not a convergent series.

Let Σv_n be obtained from Σu_n by grouping the terms as
$(1-1) + (1-1) + (1-1) + \cdots \cdots$
Then Σv_n is clearly a convergent series.

Examples.

1. Prove that $\log 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent, by Leibnitz's test.
We have $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ when $-1 < x \leq 1$.
So $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$.

Grouping the terms of the series as $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6}) + \cdots$ we have the series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

By Theorem 6.5.2, the sum of the series is $\log 2$.

2. Prove that $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots$

The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ is convergent by Leibnitz's test.
We have $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ when $-1 \leq x \leq 1$ (Gregory's series)
So $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

Grouping the terms of the series as $(1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + \cdots$ we have the series $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \cdots$

By Theorem 6.5.2, the sum of the series is $\frac{\pi}{4}$.
Hence $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \cdots = \frac{\pi}{8}$.

Re-arrangement of terms.

Theorem 6.5.3. If the terms of an absolutely convergent series be rearranged the series remains convergent and its sum remains unaltered.

Proof. Let Σu_n be an absolutely convergent series and let the terms be re-arranged in any manner.

Let the new series be Σv_n. Then every u is a v and every v is a u.

Let $\Sigma|u_n| = s$. Then $\Sigma|u_n| + u_n$ is a series of positive real numbers and $u_n + |u_n| leq 2|u_n|$.

By Comparison test, $\Sigma(|u_n| + u_n)$ is convergent.

Let $\Sigma(|u_n| + u_n) = s'$. Then $\Sigma u_n = s' - s$.

Since $\Sigma|u_n|$ and $\Sigma|u_n| + u_n)$ are convergent series of positive real numbers their sums are not altered by re-arrangement of terms.

Therefore $\Sigma|v_n| = s$ and $\Sigma(|v_n| + v_n) = s'$.

Consequently, $\Sigma v_n = s' - s$. This shows that Σv_n is convergent and $\Sigma v_n = \Sigma u_n$. This proves the theorem.

We state here without proof an important theorem of Riemann about the behaviour of a conditionally convergent series.

Theorem 6.5.4. (Riemann's theorem)

By appropriate re-arrangement of terms, a conditionally convergent series Σu_n can be made

(i) to converge to any number l, or (ii) to diverge to $+\infty$, or

(iii) to diverge to $-\infty$, or (iv) to oscillate finitely, or

(v) to oscillate infinitely.

Worked Example.

Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\log 2$, but the re-arranged series

$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \cdots$ converges to $\frac{1}{2}\log 2$.

We have $\lim\limits_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) = \gamma$.

Let $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n = \gamma_n$. Then $\lim\limits_{n\to\infty}\gamma_n = \gamma$.

Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}$.

Then $s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}) - 2(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n})$

$= \log 2n + \gamma_{2n} - (1 + \frac{1}{2} + \cdots + \frac{1}{n})$

$= \log 2n + \gamma_{2n} - (\log n + \gamma_n) = \log 2 + \gamma_{2n} - \gamma_n$.

Therefore $\lim s_{2n} = \log 2$.

$s_{2n+1} = s_{2n} + \frac{1}{2n+1}$. Therefore $\lim s_{2n+1} = \lim s_{2n} = \log 2$.

This proves $\lim s_n = \log 2$. That is, the series converges to $\log 2$.

Let $t_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots\cdots$ to n terms.

Then $t_{3n} = (1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + \cdots + (\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n})$

$= (1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}) - \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n})$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n}) - (\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}) - \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{2n})$

$= \log 2n + \gamma_{2n} - \frac{1}{2}(\log n + \gamma_n) - \frac{1}{2}(\log 2n + \gamma_{2n})$

$= \frac{1}{2}(\log 2n + \gamma_{2n}) - \frac{1}{2}(\log n + \gamma_n)$

$= \frac{1}{2}\log 2 + \frac{1}{2}\gamma_{2n} - \frac{1}{2}\gamma_n.$

Therefore $\lim t_{3n} = \frac{1}{2}\log 2$.

Again $t_{3n+1} = t_{3n} + \frac{1}{2n+1}$ and $t_{3n+2} = t_{3n+1} - \frac{1}{4n+2}$. Therefore $\lim t_{3n+1} = \lim t_{3n} = \frac{1}{2}\log 2$ and $\lim t_{3n+2} = \lim t_{3n+1} = \frac{1}{2}\log 2$.

This proves that $\lim t_n = \frac{1}{2}\log 2$ and hence the series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots\cdots$ converges to $\frac{1}{2}\log 2$.

6.6. Multiplication of series.

Let $a_0 + a_1 + a_2 + a_3 + \cdots$, $b_0 + b_1 + b_2 + b_3 + \cdots$ be two series. Then the product $(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$ contains doubly infinite number of terms of the type $a_i b_j$ and they can be arranged in the form of a doubly infinite array

$$\begin{array}{ccccc}
a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\
a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\
a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\
a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

This array extends to the right and also downwards. We can arrange the terms of the array in the form of an infinite series in many ways.

Two particular arrangements are described below.

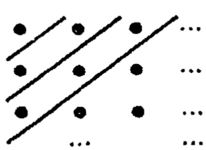

Figure 1

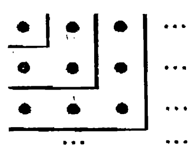

Figure 2

1. We take the first term as a_0b_0 in which the sum of the suffixes in the product a_0b_0 is 0; the second term as $a_0b_1 + a_1b_0$ in which the sum of the suffixes in each product a_ib_j is 1; the third term as $a_0b_2 + a_1b_1 + a_2b_0$ in which the sum of the suffixes in each product a_ib_j is 2; and so on.

The series takes the form

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots \quad \text{(i)}$$

The nth term of the series is the sum of all products lying between the $(n-1)$th and the nth lines as shown in the figure 1 where dots represent the products (the dot appearing in the ijth entry represents the product a_ib_j).

If $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$ then the series (i) is $c_0 + c_1 + c_2 + \cdots$. The series $c_0 + c_1 + c_2 + \cdots$ is said to be the *Cauchy product* of the series $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$

[Note that c_n is the co-efficient of x^n in the product of the two series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$]

2. We take the first term as a_0b_0 in which both the suffixes are 0; the second term as $a_1b_0 + a_1b_1 + a_0b_1$ where each term contains the suffix 1 but no higher suffix; the third term as $a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2$ where each term contains the suffix 2 but no higher suffix; and so on.

The series takes the form $a_0b_0 + (a_1b_0 + a_1b_1 + a_0b_1) + (a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2) + \cdots$

The nth term of the series is the sum of all products lying between the $(n-1)$th and the nth squares as shown in the figure 2 where dots represent the products (the dot appearing in the ijth entry represents the product a_ib_j).

The sum of the first $(n+1)$ terms of the series is $(a_0 + a_1 + a_2 + a_3 + \cdots + a_n)(b_0 + b_1 + b_2 + b_3 + \cdots + b_n)$.

Theorem 6.6.1. If $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$ be two convergent series of positive terms with s and t as their sums, then the series $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (i.e., the Cauchy product) is convergent and has the sum st.

Proof. Let us arrange the terms of the product $(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$ in the form of a doubly infinite array

$$\begin{array}{ccccc}
a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \ldots \\
a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \ldots \\
a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \ldots \\
a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

This array extends to the right and also downwards. Let us arrange the elements of the array in the form of an infinite series in two ways –

(i) $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (A)

(ii) $a_0b_0 + (a_1b_0 + a_1b_1 + a_0b_1) + (a_2b_0 + a_2b_1 + a_2b_2 + a_1b_2 + a_0b_2) + \cdots$ ... (B)

Let $s_n = a_0 + a_1 + a_2 + \cdots + a_n$, $t_n = b_0 + b_1 + b_2 + \cdots + b_n$. Then $\lim s_n = s$, $\lim t_n = t$.

Let σ_n be the sum of the first $(n+1)$ terms of the series (B).

Then $\sigma_0 = s_0t_0$, $\sigma_1 = s_1t_1$, $\sigma_2 = s_2t_2, \ldots$ $\sigma_n = s_nt_n, \ldots$

$\lim \sigma_n = \lim s_nt_n = st$. Therefore the series (B) is convergent and has the sum st.

Since the series (B) is a convergent series of positive terms, the series remains convergent with the same sum st after removal of brackets. The series (A) is obtained from the resulting series by rearrangement of terms and then by introduction of brackets. Hence the series (A) remains convergent with the same sum st.

This completes the proof.

Theorem 6.6.2. If $a_0 + a_1 + a_2 + a_3 + \cdots$ and $b_0 + b_1 + b_2 + b_3 + \cdots$ be two absolutely convergent series with s and t as their sums, then the series $a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$ (i.e., the Cauchy product) is absolutely convergent and has the sum st.

Proof. Since an absolutely convergent series remains convergent either by rearrangement of terms or by introduction of brackets and in either case the sum remains unaltered, the theorem can be established by following the same lines of proof as discussed in the previous theorem.

The following theorem due to Mertens is a further extension of the previous one and it is stated below without proof.

Theorem 6.6.3. (Mertens)

If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ be convergent series with sums s and t respectively and one of the series, say $\sum\limits_{n=0}^{\infty} a_n$ be absolutely convergent, then the series $\sum\limits_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$, is convergent and its sum is st.

Note. If both the series $\Sigma\, a_n$ and $\Sigma\, b_n$ be non-absolutely convergent then their Cauchy product may not be convergent.

For example, let us consider the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$.

The series is non-absolutely convergent. Let the series be $\sum\limits_{n=1}^{\infty} a_n$.

Let the Cauchy product of tne series $\sum\limits_{n=1}^{\infty} a_n . \sum\limits_{n=1}^{\infty} a_n$ be $\sum\limits_{n=1}^{\infty} c_n$.

Then $c_n = (-1)^{n-1}[\frac{1}{\sqrt{1.n}} + \frac{1}{\sqrt{2(n-1}} + \frac{1}{\sqrt{3(n-2}} + \cdots + \frac{1}{\sqrt{n.1}}]$.

$r(n-r+1) = (\frac{n+1}{2})^2 - (\frac{n+1}{2} - r)^2 \leq (\frac{n+1}{2})^2$ for all r satisfying $1 \leq r \leq n$.

$|c_n| \geq \frac{2n}{n+1}$ and this implies $\lim c_n \neq 0$. The necessary condition for convergence of the series $\sum\limits_{n=1}^{\infty} c_n$ is not satisfied.

This establishes that the Cauchy product of two non-absolutely convergent series may not be convergent.

If, however, the Cauchy product of two non-absolutely convergent series be convergent, then the following theorem due to Abel establishes that the sum of the Cauchy product is the product of the sums of the series.

Theorem 6.6.4. (Abel)

If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be convergent series with sums A and B respectively and if their Cauchy product $\sum\limits_{n=1}^{\infty} c_n$ be convergent with the sum C, then $C = AB$.

First we prove the following lemma.

Lemma. If $\lim u_n = u, \lim v_n = v$, then $\lim \frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n} = uv$.

Proof. Let $u_n = u + \alpha_n$, $v_n = v + \beta_n$. Then $\lim \alpha_n = 0, \lim \beta_n = 0$.

$\frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n}$

$= \frac{(u+\alpha_n)(v+\beta_1)+(v+\alpha_{n-1})(v+\beta_2)+\cdots+(u+\alpha_1)(v+\beta_n)}{n} = uv + \frac{u}{n}[\beta_1 + \beta_2 + \cdots + \beta_n] + \frac{v}{n}[\alpha_1 + \alpha_2 + \cdots + \alpha_n] + \frac{\alpha_n\beta_1 + \alpha_{n-1}\beta_2 + \cdots + \alpha_1\beta_n}{n}$ ... (i)

By Cauchy's theorem, $\lim \frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{n} = 0, \lim \frac{\beta_1+\beta_2+\cdots+\beta_n}{n} = 0$.

Since $\lim \alpha_n = 0$, the sequence $\{\alpha_n\}$ is bounded. So there exists a positive real number k such that $\alpha_n < k$ for all n.

Therefore $\lim \frac{\alpha_n\beta_1+\alpha_{n-1}\beta_2+\cdots+\alpha_1\beta_n}{n} \leq k. \lim \frac{\beta_1+\beta_2+\cdots+\beta_n}{n} = 0$.

From (i) it follows that $\lim \frac{u_n v_1 + u_{n-1} v_2 + \cdots + u_1 v_n}{n} = uv$.

Proof of the theorem.

Let $s_n = a_1+a_2+\cdots+a_n$, $t_n = b_1+b_2+\cdots+b_n$, $p_n = c_1+c_2+\cdots+c_n$.
Then $c_1 = a_1b_1$, $c_2 = a_1b_2 + a_2b_1$, $c_3 = a_1b_3 + a_2b_2 + a_3b_1, \cdots$

$$p_n = (a_1b_1) + (a_1b_2 + a_2b_1) + \cdots + (a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)$$
$$= a_1t_n + a_2t_{n-1} + \cdots + a_nt_1.$$

$$p_1 + p_2 + \cdots + p_n = s_nt_1 + s_{n-1}t_2 + \cdots + s_1t_n.$$

Since $\lim s_n = A, \lim t_n = B$; by the lemma we have
$\lim \frac{s_nt_1+s_{n-1}t_2+\cdots+s_1t_n}{n} = AB$ ... (ii)

Again since $\lim p_n = C$, by Cauchy's theorem we have
$\lim \frac{p_1+p_2+\cdots+p_n}{n} = C$ ... (iii)

From (ii) and (iii) it follows that $AB = C$.

Examples.

1. The series $1 + x + x^2 + x^3 + \cdots$ is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{1-x}$, $|x| < 1$.

(i) Let us consider the product $(1 + x + x^2 + \cdots + x^n + \cdots)(1 + x + x^2 + \cdots + x^n + \cdots)$.

Let the Cauchy product be the series $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. This series is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{(1-x)^2}$, $|x| < 1$.

Here $c_n = n + 1$.

Therefore $1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots = \frac{1}{(1-x)^2}$, $|x| < 1$.

(ii) Let us consider the product $(1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots)(1 + x + x^2 + \cdots + x^n + \cdots)$.

Let the Cauchy product be the series $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. This series is absolutely convergent for $|x| < 1$ and the sum of the series is $\frac{1}{(1-x)^3}$, $|x| < 1$.

Here $c_n = 1 + 2 + 3 + \cdots + (n+1) = \frac{1}{2}(n+1)(n+2)$.

Therefore $1+3x+6x^2+\cdots+\frac{1}{2}(n+1)(n+2)x^n+\cdots = \frac{1}{(1-x)^3}$, $|x| < 1$.

2. The series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ is a convergent series of positive terms and the sum of the series is e.

Let us consider the product $(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots)(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots)$.

Let the Cauchy product be the series $c_0 + c_1 + c_2 + c_3 + \cdots$. This series is convergent and the sum of the series is e^2.

$$c_n = \frac{1}{n!} + \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!} + \cdots + \frac{1}{(n-1)!1!} + \frac{1}{n!}$$
$$= \frac{1}{n!}[1 + {}^n c_1 + {}^n c_2 + \cdots + 1] = \frac{2^n}{n!}.$$

Therefore $1 + \frac{2}{1!} + \frac{2^2}{2!} + \cdots + \frac{2^n}{n!} + \cdots = e^2$.

Exercises 10

1. If $\Sigma a_n{}^2$ and $\Sigma b_n{}^2$ be both convergent prove that the series $\Sigma a_n b_n$ is absolutely convergent.

2. If $\{u_n\}$ be a sequence of real numbers and Σu_n^2 is convergent, prove that $\Sigma(u_n/n)$ is absolutely convergent.

3. If $\{a_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim a_n = 0$, prove that the following series are convergent.

(i) $a_1 - \frac{1}{2}(a_1 + a_2) + \frac{1}{3}(a_1 + a_2 + a_3) - \cdots\cdots$

(ii) $a_1 - \frac{1}{2}(a_1 + a_3) + \frac{1}{3}(a_1 + a_3 + a_5) - \cdots\cdots$

(iii) $a_1 - \frac{1}{3}(a_1 + a_3) + \frac{1}{5}(a_1 + a_3 + a_5) - \cdots\cdots$

Hint. (i) Let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Then $b_n > 0, b_{n+1} \le b_n$ for all $n \ge 1$ and $\lim b_n = 0$; (ii) The sequence $\{a_{2n-1}\}$ is a monotone decreasing null sequence.

4. Prove that the following series are convergent.

(i) $1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots\cdots$

(ii) $1 - \frac{1}{3.4} + \frac{1}{5.4^2} - \frac{1}{7.4^3} + \cdots\cdots$

(iii) $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\cdots$

(iv) $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \cdots\cdots$

(v) $1 - \frac{1}{2}(1 + \frac{1}{3}) + \frac{1}{3}(1 + \frac{1}{3} + \frac{1}{5}) - \cdots\cdots$

(vi) $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots\cdots$

(vii) $\frac{1}{2^2 \log 2} - \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} - \cdots\cdots$

(viii) $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots, 0 < x < 1$.

5. Prove that the following series are conditionally convergent.

(i) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots\cdots$

(ii) $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots\cdots$

(iii) $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots\cdots$

(iv) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \cdots\cdots$

(v) $\sin\frac{\pi}{2} - \sin\frac{\pi}{4} + \sin\frac{\pi}{6} - \cdots\cdots$

(vi) $(\frac{1}{2})^2 - (\frac{1.3}{2.4})^2 + (\frac{1.3.5}{2.4.6})^2 - \cdots\cdots$

(vii) $(\frac{1}{2})^2 - (\frac{1.4}{2.5})^2 + (\frac{1.4.7}{2.5.8})^2 - \cdots\cdots$

6. Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ converges to $\log 2$, but the rearranged series

(i) $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \cdots\cdots$ converges to 0,

(ii) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots\cdots$ converges to $\frac{3}{2}\log 2$,

(iii) $1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} + \cdots\cdots$ converges to $\log 2$,

(iv) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots\cdots$ converges to $\frac{1}{2}\log 12$.

7. Prove that the series

$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\cdots$ is divergent.

[**Hint.** Let s_n be the nth partial sum. Then $s_{3n} > (\frac{1}{3} + \frac{1}{3} - \frac{1}{3}) + (\frac{1}{6} + \frac{1}{6} - \frac{1}{6}) + \cdots + (\frac{1}{3n} + \frac{1}{3n} - \frac{1}{3n})$.]

8. If Σu_n be a convergent series, show that the following series are convergent.

(i) $\Sigma \frac{u_n}{n}$, (ii) $\Sigma n^{1/n} u_n$, (iii) $\Sigma \frac{u_n}{\log(n+1)}$.

9. Test the series for convergence. If the series be convergent, determine whether it is absolutely or conditionally convergent.

(i) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n\sqrt{n}}$, (ii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}n^2}{2^n}$, (iii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}n}{n^2+1}$,

(iv) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 3^n}{n!}$, (v) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}(n!)^2}{(2n)!}.5^n$, (vi) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}(n!)^2}{(2n)!}.3^n$,

(vii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}\log(n+1)}{n+1}$, (viii) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n+1)\log(n+1)}$.

7. LIMITS

7.1. Limit of a function.

Let f be a real function defined on a domain $D \subset \mathbb{R}$. In order that f may have a limit $l(\in \mathbb{R})$ at a point c, for x sufficiently close to c, $f(x)$ should be arbitrarily close to l. For this to be meaningful, it is necessary that c be a limit point of the domain D. Keeping this requirement in view, we give the formal definition.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to any neighbourhood V of l there exists a neighbourhood W of c such that $f(x) \in V$ for all $x \in [W - \{c\}] \cap D$.

This is expressed by the symbol $\lim\limits_{x \to c} f(x) = l$.

Equivalent definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number l is said to be a *limit of f at c* if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$,

where $N'(c, \delta) = \{x \in \mathbb{R} : 0 <\mid x - c \mid< \delta\} = (c - \delta, c + \delta) - \{c\}$.

Note 1. In order that we may enquire if $\lim\limits_{x \to c} f(x)$ exists, c must be a *limit point* of the domain D of the function f.

Note 2. The definition demands that all values of f in some deleted δ-neighbourhood $N'(c, \delta)$ contained in D must lie in the chosen ϵ-neighbourhood of l. It does not matter whether c belongs to D or not. Even if $c \in D$, $f(c)$ need not lie in the ϵ-neighbourhood of l.

Theorem 7.1.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. Then f can have at most one limit at c.

Proof. Suppose, on the contrary, there exist two different limits, l, m of the function f at c.

Since $l \neq m$, we assume $m > l$, without loss of generality. Let $\epsilon = \frac{m-l}{2} > 0$. Then the neighbourhoods $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Since l is a limit of f at c, there exists a positive δ_1 such that

$$l - \epsilon < f(x) < l + \epsilon \text{ for all } x \in N'(c, \delta_1) \cap D.$$

Since m is a limit of f at c, there exists a positive δ_2 such that

$$m - \epsilon < f(x) < m + \epsilon \text{ for all } x \in N'(c, \delta_2) \cap D.$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $l - \epsilon < f(x) < l + \epsilon$ and $m - \epsilon < f(x) < m + \epsilon$ for all $x \in N'(c, \delta) \cap D$. This is a contradiction, since the neighbourhoods $(l - \epsilon, l + \epsilon)$ and $(m - \epsilon, m + \epsilon)$ are disjoint.

Therefore $l = m$ and the theorem is done.

Worked Examples.

1. Show that $\lim\limits_{x \to 2} f(x) = 4$, where $f(x) = \frac{x^2-4}{x-2}, x \neq 2$.

Here the domain D of f is $\mathbb{R} - \{2\}$. 2 is a limit point of D.

When $x \in D, \mid f(x) - 4 \mid = \mid \frac{x^2-4}{x-2} - 4 \mid = \mid x - 2 \mid$.

Let us choose $\epsilon > 0$.

$\mid f(x) - 4 \mid < \epsilon$ whenever $\mid x - 2 \mid < \epsilon$ and $x \in D$, i.e., for all $x \in D$ satisfying $0 < \mid x - 2 \mid < \epsilon$.

Therefore $\mid f(x) - 4 \mid < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].

So we have $\lim\limits_{x \to 2} f(x) = 4$.

2. Show that $\lim\limits_{x \to 2} f(x) = 4$,

$$\begin{aligned} \text{where } f(x) &= \frac{x^2-4}{x-2}, x \neq 2 \\ &= 10, \ x = 2. \end{aligned}$$

Here the domain D of f is $\mathbb{R}$. 2 is a limit point of D.

When $x \in D - \{2\}, \mid f(x) - 4 \mid = \mid \frac{x^2-4}{x-2} - 4 \mid = \mid x - 2 \mid$.

Let us choose $\epsilon > 0$.

$\mid f(x) - 4 \mid < \epsilon$ whenever $\mid x - 2 \mid < \epsilon$ and $x \neq 2$.

Therefore $\mid f(x) - 4 \mid < \epsilon$ for all $x \in N'(2, \delta) \cap D$ [taking $\delta = \epsilon$].

So we have $\lim\limits_{x \to 2} f(x) = 4$.

3. Show that $\lim\limits_{x \to 0} f(x) = 0$ where $f(x) = \sqrt{x}, x \geq 0$.

Here the domain D of f is $\{x \in \mathbb{R} : x \geq 0\}$. 0 is a limit point of D.

Let us choose $\epsilon > 0$.

When $x \geq 0, \mid f(x) - 0 \mid = \sqrt{x}$.

Therefore $\mid f(x) - 0 \mid < \epsilon$ for all x satisfying $0 < x < \epsilon^2$, i.e., for all $x \in N'(0, \delta) \cap D$ [taking $\delta = \epsilon^2$].

So we have $\lim\limits_{x \to 0} f(x) = 0$.

Note. Here $N'(0, \delta) \cap D = (0, \delta)$, since $D = \{x \in \mathbb{R} : x \geq 0\}$.

Theorem 7.1.2. (Sequential criterion)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Then $\lim\limits_{x \to c} f(x) = l$ if and only if for *every* sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ, there exists a positive δ such that

$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$ ... ... (i)

Let $\{x_n\}$ be a sequence in $D - \{c\}$ converging to c.

Since $\lim x_n = c$, there exists a natural number k such that

$c - \delta < x_n < c + \delta$ for all $n \geq k$.

Therefore from (i) $l - \epsilon < f(x_n) < l + \epsilon$ for all $n \geq k$.

This proves that $\lim f(x_n) = l$.

Conversely, let for every sequence $\{x_n\}$ in $D - \{c\}$ converging to $c, \lim f(x_n) = l$. We prove that $\lim\limits_{x \to c} = l$.

If not, there exists a neighbourhood V of l such that for every neighbourhood W of c there exists at least one element $x_w \in [W - \{c\}] \cap D$ for which $f(x_w)$ does not belong to V.

Let $W_1 = N(c, 1)$. Then there exists an element $x_1 \in N'(c, 1) \cap D$ such that $f(x_1) \notin V$.

Let $W_2 = N(c, \frac{1}{2})$. Then there exists an element $x_2 \in N'(c, \frac{1}{2}) \cap D$ such that $f(x_2) \notin V$.

Proceeding in this manner, we obtain a sequence $\{x_1, x_2, x_3, \ldots\}$ in D such that $\lim x_n = c$, since $x_n \in W_n = N(c, \frac{1}{n})$ for all $n \in \mathbb{N}$; but the sequence $f(x_n)$ does not converge to l, since $f(x_n)$ does not belong to the neighbourhood V of l for all $n \in \mathbb{N}$. This is a contradiction to the hypothesis and therefore $\lim\limits_{x \to c} f(x) = l$.

This completes the proof.

Worked Examples (continued).

4. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist where $f(x) = \sin \frac{1}{x}, x \neq 0$.

Here the domain D of f is $\mathbb{R} - \{0\}$. 0 is a limit point of D.

Let us consider the sequence $\{x_n\}$ in D defined by $x_n = \frac{2}{(4n-3)\pi}, n \in \mathbb{N}$. The sequence is $\{\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \cdots\}$ and this converges to 0.

The sequence $\{f(x_n)\}$ is $\{\sin \frac{\pi}{2}, \sin \frac{5\pi}{2}, \sin \frac{9\pi}{2}, \cdots\}$, i.e., $\{1, 1, 1, \ldots\}$ and this converges to 1.

Let us consider the sequence $\{y_n\}$ in D defined by $y_n = \frac{1}{n\pi}, n \in \mathbb{N}$.

The sequence is $\{\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \cdots\}$ and this converges to 0.

The sequence $\{f(y_n)\}$ is $\{\sin\pi, \sin 2\pi, \sin 3\pi, \cdots\}$, i.e., $\{0, 0, 0, \ldots\}$ and this converges to 0.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in D both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits.

Therefore $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.

5. Show that $\lim\limits_{x\to 0} [x]$ does not exist.

Let $f(x) = [x]$. The domain of f is $\mathbb{R}$. In order to examine whether $\lim\limits_{x\to 0} f(x)$ exists or not, it is sufficient to consider the function f in an arbitrary nieghbourhood of 0, say $N(0,1)$.

$$\begin{aligned} f(x) &= -1, \text{ if } -1 < x < 0 \\ &= 0, \quad \text{if } 0 \le x < 1. \end{aligned}$$

Let us consider the sequence $\{x_n\}$ in $N(0,1)$ defined by $x_n = \frac{1}{n+1}, n \in \mathbb{N}$. The sequence $\{x_n\}$ converges to 0.

The sequence $\{f(x_n)\}$ is $\{0, 0, 0, \ldots\}$. This converges to 0.

Let us consider the sequence $\{y_n\}$ in $N(0,1)$ defined by $y_n = -\frac{1}{n+1}, n \in \mathbb{N}$. The sequence $\{y_n\}$ converges to 0.

The sequence $\{f(y_n)\}$ is $\{-1, -1, -1, \cdots\}$. This converges to -1.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in $N(0,1)$ both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits. Therefore $\lim\limits_{x\to 0} f(x)$, i.e., $\lim\limits_{x\to 0} [x]$ does not exist.

6. Show that $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist.

Let $f(x) = \operatorname{sgn} x$. Then
$$\begin{aligned} f(x) &= 1 \text{ for } x > 0 \\ &= 0 \text{ for } x = 0 \\ &= -1 \text{ for } x < 0. \end{aligned}$$

Here the domain of f is $\mathbb{R}$. 0 is a limit point of the domain of f.

Let us consider the sequence $\{x_n\}$ in $\mathbb{R}$ defined by $x_n = \frac{1}{n}, n \in \mathbb{N}$. Then $\lim x_n = 0$. $f(x_n) = 1$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = 1$.

Let us consider the sequence $\{y_n\}$ in $\mathbb{R}$ defined by $y_n = \frac{-1}{n}, n \in \mathbb{N}$. Then $\lim y_n = 0$. $f(y_n) = -1$ for all $n \in \mathbb{N}$ and $\lim f(y_n) = -1$.

Thus we have two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ both converging to 0 but $\lim f(x_n) \neq \lim f(y_n)$. Therefore $\lim\limits_{x\to 0} f(x)$ does not exist.

Theorem 7.1.3. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If f has a limit $l \in \mathbb{R}$ at c then f is bounded on $N(c) \cap D$ for some neighbourhood $N(c)$ of c.

Proof. Let $\lim_{x \to c} f(x) = l$.

Let us choose $\epsilon = 1$. Then there exists a positive δ such that $| f(x) - l | < 1$ for all $x \in N'(c, \delta) \cap D$.

But $| f(x) - l | \geq | \, | f(x) | - | l | \, |$.

It follows that $| f(x) | < | l | + 1$ for all $x \in N'(c, \delta) \cap D$.

Therefore if $c \notin D$, $| f(x) | < | l | + 1$ for all $x \in N(c, \delta) \cap D$, showing that f is bounded on $N(c, \delta) \cap D$.

If, however, $c \in D$, let $B = \max\{| f(c) | + 1, | l | + 1\}$.

Then $| f(x) | < B$ for all $x \in N(c, \delta) \cap D$, showing that f is bounded on $N(c, \delta) \cap D$.

This completes the proof.

Corollary. If f be not bounded on $N(c, \delta) \cap D$ for some δ-neighbourhood $N(c, \delta)$ of c then $\lim_{x \to c} f(x)$ does not exist in $\mathbb{R}$.

For example, $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Let $f(x) = \frac{1}{x}, x \in D$. Here $D = \mathbb{R} - \{0\}$ and $0 \in D'$. f is unbounded on every neighbourhood of 0. Therefore $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$.

Theorem 7.1.4. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$ and $\lim_{x \to c} f(x) = l$.

(i) If $l > 0$ then there exists a positive δ such that $f(x) > 0$ for all $x \in N'(c, \delta) \cap D$.

(ii) If $l < 0$ then there exists a positive δ such that $f(x) < 0$ for all $x \in N'(c, \delta) \cap D$.

Proof. (i) Let us choose a positive ϵ such that $l - \epsilon > 0$.

Since $\lim_{x \to c} f(x) = l$, there exists a $\delta > 0$ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Since $l - \epsilon > 0, f(x) > 0$ for all $x \in N'(c, \delta) \cap D$.

(ii) Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim_{x \to c} f(x) = l$, there exists a $\delta > 0$ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Since $l + \epsilon < 0, f(x) < 0$ for all $x \in N'(c, \delta) \cap D$.

This completes the proof.

Theorem 7.1.5. Let $D \subset \mathbb{R}$ and f and g are functions on D to $\mathbb{R}$.

Let $c \in D'$ and $\lim_{x \to c} f(x) = l, \lim_{x \to c} g(x) = m$. Then

(i) $\lim_{x \to c} (f + g)(x) = l + m$, where $f + g : D \to \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x), x \in D$.

(ii) if $k \in \mathbb{R}$, $\lim\limits_{x \to c}(k.f)(x) = kl$, where $kf : D \to \mathbb{R}$ is defined by $(kf)(x) = k.f(x), x \in D$.

(iii) $\lim\limits_{x \to c}(f.g)(x) = lm$, where $f.g : D \to \mathbb{R}$ is defined by $(f.g)(x) = f(x).g(x), x \in D$.

Proof. (i)
$$(f+g)(x) - (l+m) = \mid \overline{f(x)+g(x)} \mid - \overline{l+m} \mid \\ \leq \mid f(x) - l \mid + \mid g(x) - m \mid.$$

Let us choose $\epsilon > 0$.

Since $\lim\limits_{x \to c} f(x) = l$, there exists a positive δ_1 such that

$\mid f(x) - l \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta_1) \cap D$.

Since $\lim\limits_{x \to c} g(x) = m$, there exists a positive δ_2 such that

$\mid g(x) - m \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $\mid f(x) - l \mid < \frac{\epsilon}{2}$ and $\mid g(x) - m \mid < \frac{\epsilon}{2}$ for all $x \in N'(c, \delta) \cap D$.

Hence
$$\mid (f+g)(x) - \overline{l+m} \mid \leq \mid f(x) - l \mid + \mid g(x) - m \mid \\ < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } x \in N'(c, \delta) \cap D.$$

That is, $\mid (f+g)(x) - \overline{l+m} \mid < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves that $\lim\limits_{x \to c}(f+g)(x) = l + m$.

Another Proof. Let $\{x_n\}$ be a sequence in $D - \{c\}$ converging to c. Since $\lim\limits_{x \to c} f(x) = l$ and $\lim\limits_{x \to c} g(x) = m$, we have

$\lim\limits_{n \to \infty} f(x_n) = l$ and $\lim\limits_{n \to \infty} g(x_n) = m$, by sequential criterion for limits.

Therefore $\lim\limits_{n \to \infty} [f(x_n) + g(x_n)] = l + m$.

That is, $\lim\limits_{n \to \infty} (f+g)(x_n) = l + m$.

Since $\{x_n\}$ is an arbitrary sequence in $D - \{c\}$ converging to c, it follows from the sequential criterion for limits that $\lim\limits_{x \to c}(f+g)(x) = l+m$.

(ii) Proof left to the reader.

(iii)
$$\mid (f.g)(x) - lm \mid = \mid f(x)g(x) - lm \mid \\ = \mid (f(x) - l)g(x) + l(g(x) - m) \mid \\ \leq \mid f(x) - l \mid \mid g(x) \mid + \mid l \mid \mid g(x) - m \mid.$$

Since $\lim\limits_{x \to c} g(x)$ exists, there exists a positive number B and a positive δ_1 such that $\mid g(x) \mid < B$ for all $x \in N(c, \delta_1) \cap D$.

Let $k = \max\{B, \mid l \mid\}$. Then $k > 0$ and

$\mid (f.g)(x) - lm \mid < k\{\mid f(x) - l \mid + \mid g(x) - m \mid\}$ for all $x \in N(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim\limits_{x \to c} f(x) = l$, there exists a positive δ_2 such that $\mid f(x) - l \mid < \frac{\epsilon}{2k}$ for all $x \in N'(c, \delta_2) \cap D$.

Since $\lim_{x \to c} g(x) = m$, there exists a positive δ_3 such that
$| g(x) - m | < \frac{\epsilon}{2k}$ for all $x \in N'(c, \delta_3) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
Then $| (f.g)(x) - lm | < k.\frac{\epsilon}{2k} + k.\frac{\epsilon}{2k}$ for all $x \in N'(c, \delta) \cap D$.

That is, $| (f.g)(x) - lm | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.
This proves that $\lim_{x \to c} (f.g)(x) = lm$.

Note 1. Using sequential criterion for limits as in the alternative proof of part (i), the proof of the parts (ii) and (iii) can also be done.

Note 2. Let $f_1, f_2, \ldots, f_n$ be n functions each defined on some domain $D \subset \mathbb{R}$ and let $c \in D'$. If $\lim_{x \to c} f_1(x) = l_1, \lim_{x \to c} f_2(x) = l_2, \cdots, \lim_{x \to c} f_n(x) = l_n$ then

(i) $\lim_{x \to c} (f_1 + f_2 + \cdots + f_n)(x) = l_1 + l_2 + \cdots + l_n$, where $(f_1 + f_2 + \cdots + f_n)(x) = f_1(x) + f_2(x) + \cdots + f_n(x), x \in D$;

(ii) $\lim_{x \to c} (f_1 f_2 \ldots f_n)(x) = l_1 l_2 \ldots l_n$, where $(f_1 f_2 \ldots f_n)(x) = f_1(x) f_2(x) \ldots f_n(x)$, $x \in D$.

In particular, if $f_1 = f_2 = \cdots = f_n = f$ and $\lim_{x \to c} f(x) = l$, then $\lim_{x \to c} [f(x)]^n = l^n$. Therefore if n be a positive integer and $\lim_{x \to c} f(x) = l$, then $\lim_{x \to c} [f(x)]^n = l^n$.

Theorem 7.1.6. Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$ and $g(x) \neq 0$ for all $x \in D$. Let $c \in D'$ and $\lim_{x \to c} f(x) = l, \lim_{x \to c} g(x) = m \neq 0$.

Then $\lim_{x \to c} \frac{f}{g}(x) = \frac{l}{m}$ where the function $\frac{f}{g} : D \to \mathbb{R}$ is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

Proof. First we prove that $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \frac{|g(x) - m|}{|g(x)||m|}$.

Let $p = \frac{1}{2} | m |$. Then $p > 0$. Since $\lim_{x \to c} g(x) = m$, there exists a positive δ_1 such that $| g(x) - m | < p$ for all $x \in N'(c, \delta_1) \cap D$.

Since $|| g(x) | - | m || \leq | g(x) - m |$, it follows that $|| g(x) | - | m || < p$ for all $x \in N'(c, \delta_1) \cap D$.

Therefore $| m | - p < | g(x) | < | m | + p$ for all $x \in N'(c, \delta_1) \cap D$
or, $| g(x) | > \frac{1}{2} | m |$ for all $x \in N'(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim_{x \to c} g(x) = m$, there exists a positive δ_2

such that

$| g(x) - m | < \frac{1}{2} | m |^2 \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\left| \frac{1}{g(x)} - \frac{1}{m} \right| < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves that $\lim\limits_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

The proof of the theorem is completed by considering the limit of the product of the functions f and $1/g$.

Hence $\lim\limits_{x \to c} \frac{f}{g}(x) = \lim\limits_{x \to c} \left\{ f(x) . \frac{1}{g(x)} \right\} = l . \frac{1}{m} = \frac{l}{m}$.

Theorem 7.1.7. Let $D \subset \mathbb{R}$ and f and g be functions on D to $\mathbb{R}$. Let $c \in D'$. If f is bounded on $N'(c) \cap D$ for some deleted neighbourhood $N'(c)$ of c and $\lim\limits_{x \to c} g(x) = 0$ then $\lim\limits_{x \to c} (f.g)(x) = 0$.

Proof. Since f is bounded on $N'(c) \cap D$ for some neighbourhood $N(c)$ of c, there exists a positive number B and a positive δ_1 such that $| f(x) | < B$ for all $x \in N'(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since $\lim\limits_{x \to c} g(x) = 0$, there exists a positive δ_2 such that $| g(x) - 0 | < \frac{\epsilon}{B}$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $| f(x) | < B$ for all $x \in N'(c, \delta) \cap D$ and $| g(x) | < \frac{\epsilon}{B}$ for all $x \in N'(c, \delta) \cap D$.

Therefore $| f.g(x) - 0 | = | f(x) || g(x) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves $\lim\limits_{x \to c} (f.g)(x) = 0$.

Worked Examples (continued).

7. Prove that $\lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0$.

Here $\lim\limits_{x \to 0} x = 0$ and $\sin \frac{1}{x^2}$ is bounded in some deleted neighbourhood of 0. Therefore $\lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0$.

8. Prove that $\lim\limits_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0$

Let $f(x) = \sqrt{x} \sin \frac{1}{x}, x \in D$. Then $D = \{x \in \mathbb{R} : x > 0\}$.
$0 \in D'$. Let $g(x) = \sqrt{x}, x \in D, h(x) = \sin \frac{1}{x}, x \in D$.

Here $\lim\limits_{x \to 0} g(x) = 0$ and h is bounded on $N'(0) \cap D$ for some neighbourhood $N(0)$ of 0.

Therefore $\lim\limits_{x \to 0} g(x)h(x) = 0$, i.e., $\lim\limits_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0$.

Theorem 7.1.8.(a) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If $f(x) \le b$ for all $x \in D - \{c\}$ and $\lim_{x \to c} f(x) = l$, then $l \le b$.

Proof. Let $\{x_n\}$ be any sequence in $D - \{c\}$ converging to c.
Since $\lim_{x \to c} f(x) = l$, $\lim_{n \to \infty} f(x_n) = l$.

Let us define a sequence $\{u_n\}$ by $u_n = b$ for all $n \in \mathbb{N}$.
Then $f(x_n) \le u_n$ for all $n \in \mathbb{N}$.
Since $\lim f(x_n) = l$ and $\lim u_n = b, l \le b$, by Theorem 5.5.4.

Note. If there exists a positive δ and a real number b such that $f(x) \le b$ for all $x \in N'(c, \delta) \cap D$ and $\lim_{x \to c} f(x) = l$, then $l \le b$.

Theorem 7.1.8.(b) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$. If $f(x) \ge a$ for all $x \in D - \{c\}$ and $\lim_{x \to c} f(x) = l$, then $l \ge a$.

Proof. Let $\{x_n\}$ be any sequence in $D - \{c\}$ convergeing to c.
Since $\lim_{x \to c} f(x) = l$, $\lim_{n \to \infty} f(x_n) = l$.

Let us define a sequence $\{v_n\}$ by $v_n = a$ for all $n \in \mathbb{N}$.
Then $f(x_n) \ge v_n$ for all $n \in \mathbb{N}$.
Since $\lim_{x \to c} f(x) = l$ and $\lim v_n = a, l \ge a$, by Theorem 5.5.4.

Note. If there exists a positive δ and a real number a such that $f(x) \ge a$ for all $x \in N'(c, \delta) \cap D$ and $\lim_{x \to c} f(x) = l$, then $l \ge a$.

Theorem 7.1.9. (Sandwich theorem)

Let $D \subset \mathbb{R}$ and f, g, h be functions on D to $\mathbb{R}$. Let $c \in D'$.
If $f(x) \le g(x) \le h(x)$ for all $x \in D - \{c\}$ and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$ then $\lim_{x \to c} g(x) = l$.

Proof. Let us choose $\epsilon > 0$.
Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that
$l - \epsilon < f(x) < l + \epsilon$ for all $x \in N'(c, \delta_1) \cap D$.

Since $\lim_{x \to c} h(x) = l$, there exists a positive δ_2 such that
$l - \epsilon < h(x) < l + \epsilon$ for all $x \in N'(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $l - \epsilon < f(x) < l + \epsilon$ and $l - \epsilon < h(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.
Therefore $l - \epsilon < f(x) \le g(x) \le h(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$
or, $l - \epsilon < g(x) < l + \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This proves $\lim_{x \to c} g(x) = l$.

Note 1. If $f(x) < g(x) < h(x)$ for all $x \in D - \{c\}$ and if $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = l$, then $\lim\limits_{x \to c} g(x) = l$.

Note 2. If there exists a positive δ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in N'(c, \delta) \cap D$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = l$, then $\lim\limits_{x \to c} g(x) = l$.

Worked Example (continued).

9. Show that $\lim\limits_{x \to 0} x \cos \frac{1}{x} = 0$.

Let $f(x) = \cos \frac{1}{x}, x \in D$. The domain of f is $D = \{x \in \mathbb{R} : x \neq 0\}$.
$-1 \leq f(x) \leq 1$ for all $x \in D$.

Hence $-x \leq xf(x) \leq x$ for all $x > 0$ and $x \leq xf(x) \leq -x$ for all $x < 0$.

Therefore $-\mid x \mid \leq xf(x) \leq \mid x \mid$ for all $x \neq 0$.
$\lim\limits_{x \to 0} \mid x \mid = 0$ and $\lim\limits_{x \to 0} - \mid x \mid = 0$.
By Sandwich theorem, $\lim\limits_{x \to 0} xf(x) = 0$, i.e., $\lim\limits_{x \to 0} x \cos \frac{1}{x} = 0$.

Theorem 7.1.10. (Cauchy's principle)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D'$.

A necessary and sufficient condition for the existence of $\lim\limits_{x \to c} f(x)$ is that for a pre-assigned positive ϵ there exists a positive δ such that
$\mid f(x') - f(x'') \mid < \epsilon$ for every pair of points $x', x'' \in N'(c, \delta) \cap D$.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ there exists a $\delta > 0$ such that $l - \frac{\epsilon}{2} < f(x) < l + \frac{\epsilon}{2}$ for all $x \in N'(c, \delta) \cap D$.

So for every pair of points $x', x'' \in N'(c, \delta) \cap D$, we have
$l - \frac{\epsilon}{2} < f(x') < l + \frac{\epsilon}{2}$ and $l - \frac{\epsilon}{2} < f(x'') < l + \frac{\epsilon}{2}$.

$$\begin{aligned} \text{But } \mid f(x') - f(x'') \mid &= \mid f(x') - l + l - f(x'') \mid \\ &\leq \mid f(x') - l \mid + \mid f(x'') - l \mid < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2}. \end{aligned}$$

It follows that $\mid f(x') - f(x'') \mid < \epsilon$ for all $x', x'' \in N'(c, \delta) \cap D$, proving that the condition is necessary.

Let us assume that for a given $\epsilon > 0$ we can find a $\delta > 0$ such that the given condition holds.

Let us take a sequence $\{x_n\}$ such that $x_n \in D - \{c\}$ for all $n \in \mathbb{N}$ and $\lim x_n = c$. Then there exists a natural number k such that $c - \delta < x_n < c + \delta$ for all $n \geq k$.

In other words, $x_n \in N'(c, \delta) \cap D$ for all $n \geq k$.

Hence for every natural number p and every $n \geq k$,
$x_n \in N'(c, \delta) \cap D$ and $x_{n+p} \in N'(c, \delta) \cap D$.

By the condition, $| f(x_n) - f(x_{n+p}) | < \epsilon$ for all $n \geq k$ and $p = 1, 2, 3, \cdots$

This proves that the sequence $\{f(x_n)\}$ is a Cauchy sequence and is therefore convergent.

Thus for every sequence $\{x_n\}$ in $D - \{c\}$ such that $\lim x_n = c$, the sequence $\{f(x_n)\}$ is convergent.

We now prove that all such sequences $\{f(x_n)\}$ converge to a common limit. Let $\{p_n\}$ and $\{q_n\}$ be two sequences in $D - \{c\}$ such that $\lim p_n = c, \lim q_n = c$ and $\lim f(p_n) = p, \lim f(q_n) = q$.

Let us consider the sequence $\{x_n\}$ where $x_{2n-1} = p_n, x_{2n} = q_n$ i.e., $\{x_n\} = \{p_1, q_1, p_2, q_2, \cdots\}$

Then $\lim x_n = c$ and therfore $\{f(x_n)\}$ is a convergent sequence. $\{f(x_{2n-1})\}$ and $\{f(x_{2n})\}$ are subsequences of $\{f(x_n)\}$. Since $\{f(x_n)\}$ is a convergent sequence, $\lim f(x_{2n-1}) = \lim f(x_{2n})$.

Therefore $p = q = l$, say.

Thus for every sequence $\{x_n\}$ in $D - \{c\}$ such that $\lim x_n = c$, $\{f(x_n)\}$ converges to l.

By the sequential criterion for limits, $\lim\limits_{x \to c} f(x)$ exists and this proves that the condition is sufficient. This completes the proof.

Worked Examples (continued).

10. A function $f : (0, 1) \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x) &= 1, \text{ if } x \text{ is rational, } x \in (0, 1) \\ &= -1, \text{ if } x \text{ is irrational, } x \in (0, 1). \end{aligned}$$

Using Cauchy's principle prove that $\lim\limits_{x \to a} f(x)$ does not exist, where $a \in [0, 1]$.

Here the domain of f is $D = \{x \in \mathbb{R} : 0 < x < 1\}$. $a \in D'$.

Let us choose $\epsilon = 1$. In order that $\lim\limits_{x \to a} f(x)$ should exist, it is necessary that there exists a $\delta > 0$ such that $| f(x') - f(x'') | < 1$ for every pair of points $x', x'' \in N'(a, \delta) \cap (0, 1)$.

Whatever $\delta (> 0)$ may be, the set $N'(a, \delta) \cap (0, 1)$ contains rational and irrational points. Let x' be rational and x'' be irrational in $N'(a, \delta) \cap (0, 1)$.

Then $f(x') = 1, f(x'') = -1$ and $| f(x') - f(x'') | = 2$.

Therefore $| f(x') - f(x'') | \not< \epsilon$ for some pair of points $x', x'' \in N'(a, \delta) \cap (0, 1)$ for every $\delta > 0$.

Therefore Cauchy's condition for the existence of $\lim\limits_{x \to a} f(x)$ is not satisfied and $\lim\limits_{x \to a} f(x)$ does not exist.

11. Using Cauchy's principle prove that $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist.

Let $f(x) = \cos \frac{1}{x}, x \neq 0$. Here the domain D of f is $\{x \in \mathbb{R} : x \neq 0\}$.

Let us choose $\epsilon = \frac{1}{2}$. In order that $\lim\limits_{x \to 0} f(x)$ should exist, it is necessary that there exists a $\delta > 0$ such that $| f(x') - f(x'') | < \frac{1}{2}$ for every pair of points $x', x'' \in N'(0, \delta) \cap D$.

For a given positive δ we can find a natural number n such that $\left| \frac{1}{2n\pi} - \frac{2}{(4n+1)\pi} \right| < \delta$, because $\left| \frac{1}{2n\pi} - \frac{2}{(4n+1)\pi} \right| = \frac{1}{\pi} \frac{1}{2n(4n+1)}$ and this can be made less than δ for a suitable natural number n.

Let $x' = \frac{1}{2n\pi}$ and $x'' = \frac{2}{(4n+1)\pi}$. Then $x', x'' \in N'(0, \delta) \cap D$ and $f(x') = 1, f(x'') = 0$.

Therefore $| f(x') - f(x'') | \not< \epsilon$ for some pair of points $x', x'' \in N'(0, \delta) \cap D$ for every $\delta > 0$.

Therefore Cauchy's condition for the existence of $\lim\limits_{x \to 0} \cos \frac{1}{x}$ is not satisfied and $\lim\limits_{x \to 0} \cos \frac{1}{x}$ does not exist.

7.2. One sided limits.

There are cases where a function f does not have a limit at a limit point c of its domain D, but the restriction of the function f to an interval at one side of c (either right or left) may have a limit.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) =$ sgn x does not possess a limit at 0 but the restriction of f to $(0, \infty)$ does have a limit at 0 and also the restriction of f to $(-\infty, 0)$ does have a limit at 0.

In the former case we say that f has a *right hand limit* at 0 and in the latter case we say that f has a *left hand limit* at 0.

Definitions.

Right hand limit. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Let c be a limit point of $D_1 = D \cap (c, \infty) = \{x \in D : x > c\}$.

f is said to have a *right hand limit* $l (\in \mathbb{R})$ at c if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$$| f(x) - l | < \epsilon \text{ for all } x \in N'(c, \delta) \cap D_1$$

i.e., $l - \epsilon < f(x) < l + \epsilon$ for all x in D satisfying $c < x < c + \delta$.

In this case we write $\lim\limits_{x \to c+} f(x) = l$.

Left hand limit. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Let c be a limit point of $D_2 = D \cap (-\infty, c) = \{x \in D : x < c\}$.

f is said to have a *left hand limit* $l(\in \mathbb{R})$ at c if corresponding to a pre-assigned positive ϵ there exists a positive δ such that

$$| f(x) - l | < \epsilon \text{ for all } x \in N'(c,\delta) \cap D_2$$

i.e., $l - \epsilon < f(x) < l + \epsilon$ for all x in D satisfying $c - \delta < x < c$.

In this case we write $\lim\limits_{x \to c-} f(x) = l$.

Note. In order that we may enquire if $\lim\limits_{x \to c+} f(x)$ exists, the domain D of the function f must be such that c is a limit point of $D \cap (c, \infty)$.

Similarly, in order that we may enquire if $\lim\limits_{x \to c-} f(x)$ exists, D must be such that c is a limit point of $D \cap (-\infty, c)$.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of $D_1 = D \cap (c, \infty)$. Then $\lim\limits_{x \to c+} f(x) = l$ if and only if for every sequence $\{x_n\}$ in D_1 converging to c, the sequence $\{f(x_n\}$ converges to l.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of $D_2 = D \cap (-\infty, c)$. Then $\lim\limits_{x \to c-} f(x) = l$ if and only if for every sequence $\{x_n\}$ in D_2 converging to c, the sequence $\{f(x_n\}$ converges to l.

Note. It is possible that both the right hand limit and the left hand limit may exist, or both may not exist, or one of them exists while the other does not.

Worked Examples.

1. Let $f(x) = \operatorname{sgn} x$. Examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1. $f(x) = 1$ for all $x \in D_1$. Therefore $\lim\limits_{x \to 0+} f(x) = 1$.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2. $f(x) = -1$ for all $x \in D_2$. Therefore $\lim\limits_{x \to 0-} f(x) = -1$.

Note. Here both the right hand limit and the left hand limit of f at 0 exist. f is defined at 0 but $f(0) \neq \lim\limits_{x \to 0+} f(x)$ and also $f(0) \neq \lim\limits_{x \to 0-} f(x)$.

2. Let $f(x) = e^{1/x}$. Examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R} - \{0\}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1.

f is unbounded on $N(0) \cap D_1$ for any neighbourhood $N(0)$ of 0.

Therefore $\lim\limits_{x \to 0+} f(x)$ does not exist in $\mathbb{R}$.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2.

We have $e^t > t > 0$ for all $t > 0$. Taking $t = -\frac{1}{x}$, we have $e^{-\frac{1}{x}} > -\frac{1}{x} > 0$ for all $x < 0$ and this implies $0 < e^{\frac{1}{x}} < -x$ for all $x < 0$.

By Sandwich theorem, $\lim\limits_{x \to 0-} f(x) = 0$.

3. Let $f(x) = \sin \frac{1}{x}$. Using sequential criterion for limits examine if $\lim\limits_{x \to 0+} f(x)$ and $\lim\limits_{x \to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R} - \{0\}$.

Let $D_1 = D \cap (0, \infty)$. Then $D_1 = \{x \in \mathbb{R} : x > 0\}$. 0 is a limit point of D_1.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n\pi}, n \in \mathbb{N}$.

Then $x_n \in D_1$ for all $n \in \mathbb{N}$ and $\lim x_n = 0$.

$f(x_n) = 0$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = 0$.

Let us consider the sequence $\{y_n\}$ where $y_n = \frac{2}{(4n+1)\pi}, n \in \mathbb{N}$.

$y_n \in D_1$ for all $n \in \mathbb{N}$ and $\lim y_n = 0$.

$f(y_n) = 1$ for all $n \in \mathbb{N}$ and therefore $\lim f(y_n) = 1$.

Therefore by sequential criterion $\lim\limits_{x \to 0+} f(x)$ does not exist.

Let $D_2 = D \cap (-\infty, 0)$. Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2.

Considering two sequences $\{u_n\}$ and $\{v_n\}$ where $u_n = -\frac{1}{n\pi}$ and $v_n = -\frac{2}{(4n+1)\pi}$ we can establish that $\lim\limits_{x \to 0-} f(x)$ does not exist.

Theorem 7.2.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of both the sets $D_1 (= D \cap (c, \infty))$ and $D_2 (= D \cap (-\infty, c))$. Then $\lim\limits_{x \to c} f(x) = l \in \mathbb{R}$ if and only if $\lim\limits_{x \to c+} f(x) = l = \lim\limits_{x \to c-} f(x)$.

Proof. Let $\lim\limits_{x \to c} f(x) = l$. Then for a pre-assigned positive ϵ, there exists a positive δ such that

$$| f(x) - l | < \epsilon \text{ for all } x \in N'(c, \delta) \cap D.$$

That is, for all x in D satisfying $0 < | x - c | < \delta, | f(x) - l | < \epsilon$.

Therefore for all x in D satisfying $c < x < c + \delta$, $|f(x) - l| < \epsilon$; and also for all x in D satisfying $c - \delta < x < c$, $|f(x) - l| < \epsilon$.

It follows that $\lim\limits_{x \to c+} f(x) = l$ and $\lim\limits_{x \to c-} f(x) = l$.

Conversely, let $\lim\limits_{x\to c+} f(x) = l$ and $\lim\limits_{x\to c-} f(x) = l$.

Let $\epsilon > 0$. Then there exists a $\delta_1 > 0$ such that

$|f(x) - l| < \epsilon$ for all x in D satisfying $c < x < c + \delta_1$.

Also there exists a $\delta_2 > 0$ such that

$|f(x) - l| < \epsilon$ for all x in D satisfying $c - \delta_2 < x < c$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x) - l| < \epsilon$ for all x in D satisfying $c - \delta < x < c$ and $c < x < c + \delta$, i.e., for all $x \in N'(c, \delta) \cap D$.

Therefore $\lim\limits_{x\to c} f(x) = l$. This proves the theorem.

This completes the proof.

Worked Examples (continued).

4. Find $\lim\limits_{x\to 0+} \sin x$, $\lim\limits_{x\to 0-} \sin x$, $\lim\limits_{x\to 0} \sin x$.

In $0 < x < \frac{\pi}{2}$, $0 < \sin x < x$.
Let $\phi(x) = 0$ in $0 < x < \frac{\pi}{2}$. Then $\lim\limits_{x\to 0+} \phi(x) = 0$. Also $\lim\limits_{x\to 0+} x = 0$.
By Sandwich theorem, $\lim\limits_{x\to 0+} \sin x = 0$.

In $-\frac{\pi}{2} < x < 0$, $x < \sin x < 0$.
Let $\psi(x) = 0$ in $-\frac{\pi}{2} < x < 0$. Then $\lim\limits_{x\to 0-} \psi(x) = 0$. Also $\lim\limits_{x\to 0-} x = 0$.
By Sandwich theorem, $\lim\limits_{x\to 0-} \sin x = 0$.

Since $\lim\limits_{x\to 0+} \sin x = \lim\limits_{x\to 0-} \sin x = 0$, we have $\lim\limits_{x\to 0} \sin x = 0$.

5. Prove that $\lim\limits_{x\to 0} \cos x = 1$. Deduce that $\lim\limits_{x\to 0+} \cos x = \lim\limits_{x\to 0-} \cos x = 1$.

$|\cos x - 1| = |2\sin^2 \frac{x}{2}| < 2\frac{x^2}{4}$, since $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.
Let us choose $\epsilon > 0$.
Then $|\cos x - 1| < \epsilon$ for all x satisfying $\frac{x^2}{2} < \epsilon$,
i.e., for all x satisfying $-\sqrt{2\epsilon} < x < \sqrt{2\epsilon}$.
Therefore $\lim\limits_{x\to 0} \cos x = 1$.

Let $f(x) = \cos x$. The domain D of f is $\mathbb{R}$.
Let $D_1 = D \cap (0, \infty)$ and $D_2 = D \cap (-\infty, 0)$.

Since $\lim\limits_{x\to 0} \cos x = 1$ and 0 is a limit point of both D_1 and D_2, each c
$\lim\limits_{x\to 0-} \cos x$ and $\lim\limits_{x\to 0-} \cos x$ exists and equals 1.

6. Prove that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

In $0 < x < \frac{\pi}{2}$, $\quad \sin x < x < \tan x$.
Therefore $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ in $0 < x < \frac{\pi}{2}$.

As $\lim\limits_{x\to 0+} \frac{1}{\cos x} = 1$, we have $\lim\limits_{x\to 0+} \frac{x}{\sin x} = 1$, by Sandwich theorem.

Therefore $\lim\limits_{x\to 0+} \frac{\sin x}{x} = 1$.

In $-\frac{\pi}{2} < x < 0$, $\tan x < x < \sin x$.

Therefore $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ in $-\frac{\pi}{2} < x < 0$.

As $\lim\limits_{x\to 0-} \frac{1}{\cos x} = 1$, we have $\lim\limits_{x\to 0-} \frac{x}{\sin x} = 1$, by Sandwich theorem.

Therefore $\lim\limits_{x\to 0-} \frac{\sin x}{x} = 1$.

Since $\lim\limits_{x\to 0+} \frac{\sin x}{x} = \lim\limits_{x\to 0-} \frac{\sin x}{x} = 1$, it follows that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

7. Evaluate the one sided limits – (i) $\lim\limits_{x\to 0-} [x]$, (ii) $\lim\limits_{x\to 0+} [x]$.

Show that $\lim\limits_{x\to 0} [x]$ does not exist.

Let $f(x) = [x]$. The domain of f is $\mathbb{R}$.

Since we like to evaluate $\lim\limits_{x\to 0-} f(x)$ and $\lim\limits_{x\to 0+} f(x)$, we are interested in the nature of the function f in the neighbourhood of 0.

$$f(x) = -1, \text{ if } -1 < x < 0$$
$$= 0, \text{ if } 0 \le x < 1.$$

$\lim\limits_{x\to 0-} f(x) = -1$, $\lim\limits_{x\to 0+} f(x) = 0$.

Since $\lim\limits_{x\to 0-} f(x) \neq \lim\limits_{x\to 0+} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist.

8. Evaluate $\lim\limits_{x\to 3}([x]-[\frac{x}{3}])$.

Let $f(x) = [x] - [\frac{x}{3}]$. The domain of f is $\mathbb{R}$.

Since we like to evaluate $\lim\limits_{x\to 3} f(x)$, we are interested in the nature of the function f in the neighbourhood of 3.

$[x] = 2$, if $2 < x < 3$ $\qquad$ $[\frac{x}{3}] = 0$, if $0 < x < 3$
$\quad = 3$, if $3 \le x < 4$. $\qquad$ $\quad = 1$, if $3 \le x < 6$.

Therefore $f(x) = 2$, if $2 < x < 3$
$\qquad = 2$, if $3 \le x < 4$.

$\lim\limits_{x\to 3-} f(x) = 2$ and $\lim\limits_{x\to 3+} f(x) = 2$. Therefore $\lim\limits_{x\to 3} f(x) = 2$.

7.3. Infinite limits.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. We have seen that if f is not bounded on $N(c) \cap D$ for some neighbourhood $N(c)$ of c, f does not approach a finite limit l.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. If corresponding to a pre-assigned positive number G, there exists a positive δ such that

$f(x) > G$ for all $x \in N'(c, \delta) \cap D$,

then we say that f tends to ∞ as $x \to c$ and we write $\lim_{x \to c} f(x) = \infty$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. If corresponding to a pre-assigned positive number G, there exists a positive δ such that

$f(x) < -G$ for all $x \in N'(c, \delta) \cap D$,

then we say that f tends to $-\infty$ as $x \to c$ and we write $\lim_{x \to c} f(x) = -\infty$.

Note. In both these cases we say that limit of the function f at c *exists in* $\mathbb{R}^*$. When $\lim_{x \to c} f(x) = l, (l \in \mathbb{R})$ we say that limit of the function f at c *exists* and it is expressed by saying that the limit of the function f at c *exists in* $\mathbb{R}$.

As in the case of finite limits, the sequential criteria can be formulated in the case of infinite limits.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. Then $\lim_{x \to c} f(x) = \infty$ if and only if for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ diverges to ∞.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of D. Then $\lim_{x \to c} f(x) = -\infty$ if and only if for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ diverges to $-\infty$.

Worked Examples.

1. Show that $\lim_{x \to 0} f(x) = \infty$, where $f(x) = \frac{1}{x^2}$.

In every neighbourhood of 0, f is unbounded above.

Let us choose $G > 0$. Then $f(x) > G$ for all x satisfying $-\frac{1}{\sqrt{G}} < x < \frac{1}{\sqrt{G}}, x \neq 0$.

That is, $f(x) > G$ for all $x \in N'(0, \delta)$ where $\delta = \frac{1}{\sqrt{G}}$.

Therefore $\lim_{x \to 0} f(x) = \infty$.

2. Show that $\lim_{x \to 0} f(x)$ does not exist in $\mathbb{R}^*$, where $f(x) = \frac{1}{x}$.

Here the domain f is $D = \mathbb{R} - \{0\}$. 0 is a limit point of D.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}, n \in \mathbb{N}$.

Then $x_n \in D$ and $\lim x_n = 0, f(x_n) = n$, $\lim f(x_n) = \infty$.

Let us consider the sequence $\{y_n\}$ where $y_n = -\frac{1}{n}, n \in \mathbb{N}$.

Then $y_n \in D$ and $\lim y_n = 0, f(y_n) = -n$, $\lim f(y_n) = -\infty$.

We have two sequences $\{x_n\}$ and $\{y_n\}$ in D both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ approach to two different limits in $\mathbb{R}^*$. Therefore $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}^*$.

Theorem 7.3.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of both the sets $D_1 = D \cap (c, \infty)$ and $D_2 = D \cap (-\infty, c)$. Then $\lim\limits_{x \to c} f(x) = \infty(\infty)$ if and only if $\lim\limits_{x \to c+} f(x) = \lim\limits_{x \to c-} f(x) = \infty(-\infty)$.

Proof left to the reader.

Worked Example (continued).

3. Examine if $\lim\limits_{x \to \frac{\pi}{2}} \tan x$ exists.

Let $f(x) = \tan x$. The domain of f is $D = \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n$ is an integer $\}$. $D_1 = D \cap (\frac{\pi}{2}, \infty) \neq \phi, D_2 = D \cap (-\infty, \frac{\pi}{2}) \neq \phi$. Also $\frac{\pi}{2}$ is a limit point of both D_1 and D_2.

In $\frac{\pi}{2} < x < \pi$, f is a monotone increasing function unbounded below. Therefore $\lim\limits_{x \to \frac{\pi}{2}+} f(x) = -\infty$.

In $0 < x < \frac{\pi}{2}$, f is a monotone increasing function unbounded above. Therefore $\lim\limits_{x \to \frac{\pi}{2}-} f(x) = \infty$. [Theorem 7.6.1]

Since $\lim\limits_{x \to \frac{\pi}{2}+} f(x)$ and $\lim\limits_{x \to \frac{\pi}{2}-} f(x)$ both exist in $\mathbb{R}^*$ and $\lim\limits_{x \to \frac{\pi}{2}+} f(x) \neq \lim\limits_{x \to \frac{\pi}{2}-} f(x)$, $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exist.

7.4. Limits at infinity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. We say that f tends to $l (\in \mathbb{R})$ as $x \to \infty$ if corresponding to a pre-assigned positive ϵ there exists a real number $G > c$ such that $| f(x) - l | < \epsilon$ for all $x > G$.

In this case we write $\lim\limits_{x \to \infty} f(x) = l$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. We say that f tends to $l (\in \mathbb{R})$ as $x \to -\infty$ if corresponding to a pre-assigned positive ϵ there exists a real number $G < c$ such that $| f(x) - l | < \epsilon$ for all $x < G$.

In this case we write $\lim\limits_{x \to -\infty} f(x) = l$.

Note. In order that we may enquire if $\lim\limits_{x \to \infty} f(x)$ exists, the domain D of f must be such that $(c, \infty) \subset D$ for some $c \in \mathbb{R}$.

In order that we may enquire if $\lim\limits_{x \to -\infty} f(x)$ exists, the domain D of f must be such that $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$.

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to \infty} f(x) = l(\in \mathbb{R})$ if and only if for every sequence $\{x_n\}$ in (c, ∞) diverging to ∞, the sequence $\{f(x_n)\}$ converges to l.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to -\infty} f(x) = l(\in \mathbb{R})$ if and only if for every sequence $\{x_n\}$ in $(-\infty, c)$ diverging to $-\infty$, the sequence $\{f(x_n)\}$ converges to l.

Worked Examples.

1. Prove that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

Let $f(x) = \frac{1}{x}$. The domain of f is $D = \mathbb{R} - \{0\}.(0, \infty) \subset D$.
Let us choose $\epsilon > 0$.
When $x \neq 0, \mid f(x) - 0 \mid = \frac{1}{x}$
Therefore $\mid f(x) - 0 \mid < \epsilon$ for all $x > G$ where $G = 1/\epsilon > 0$.

Hence $\lim\limits_{x \to \infty} f(x) = 0$.

2. Show that $\lim\limits_{x \to \infty} x \sin x$ does not exist in $\mathbb{R}^*$.

Let $f(x) = x \sin x$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{\pi}{2} + 2n\pi, n \in \mathbb{N}$. Then $\lim x_n = \infty$.

$f(x_n) = f(\frac{\pi}{2} + 2n\pi) = \frac{\pi}{2} + 2n\pi$ for all $n \in \mathbb{N}$ and therefore $\lim f(x_n) = \infty$.

Let us consider the sequence $\{y_n\}$ where $y_n = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{N}$. Then $\lim y_n = \infty$.

$f(y_n) = f(-\frac{\pi}{2} + 2n\pi) = \frac{\pi}{2} - 2n\pi$ for all $n \in \mathbb{N}$ and therefore $\lim f(y_n) = -\infty$.

We have two sequences $\{x_n\}$ and $\{y_n\}$ both diverging to ∞ but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ approach to two different limits in $\mathbb{R}^*$.

By sequential criterion, $\lim\limits_{x \to \infty} f(x)$ does not exist in $\mathbb{R}^*$.

Theorem 7.4.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function.

Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. If $f(x) > 0$ for all $x \in (c, \infty)$ and $\lim\limits_{x \to \infty} f(x) = l$, then $l \geq 0$.

Proof. If possible, let $l < 0$.

Let us choose a positive ϵ such that $l + \epsilon < 0$.

Since $\lim\limits_{x \to \infty} f(x) = l$, there exists a real number $G > c$ such that $| f(x) - l | < \epsilon$ for all $x > G$

or, $l - \epsilon < f(x) < l + \epsilon$ for all $x > G$.

$l + \epsilon < 0 \Rightarrow f(x) < 0$ for all $x > G$.

But by hypothesis $f(x) > 0$ for all $x > c$. Thus we arrive at a contradiction. Therefore $l \geq 0$.

Theorem 7.4.2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be functions.

Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Let $f(x) < g(x)$ for all $x \in (c, \infty)$.

If $\lim\limits_{x \to \infty} f(x) = l$ and $\lim\limits_{x \to \infty} g(x) = m$ then $l \leq m$.

Proof. Let $h(x) = g(x) - f(x), x \in (c, \infty)$.

Let us choose $\epsilon > 0$. Since $\lim\limits_{x \to \infty} f(x) = l$, there exists a real number $G_1 > c$ such that $|f(x) - l| < \frac{\epsilon}{2}$ for all $x > G_1$.

Since $\lim\limits_{x \to \infty} g(x) = m$ there exists a real number $G_2 > c$ such that $| g(x) - m | < \frac{\epsilon}{2}$ for all $x > G_2$

Let $G = \max\{G_1, G_2\}$. Then $| f(x) - l | < \frac{\epsilon}{2}$ and $| g(x) - m | < \frac{\epsilon}{2}$ for all $x > G$.

$$\text{We have } \quad | h(x) - (m - l) | \;\; = \;\; | g(x) - m - f(x) + l |$$
$$\leq \;\; | f(x) - l | + | g(x) - m |$$
$$< \;\; \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \text{ for all } x > G.$$

Therefore $| h(x) - (m - l) | < \epsilon$ for all $x > G$.

That is, $\lim\limits_{x \to \infty} h(x) = m - l$.

Since $h(x) > 0$ for all $x \in (c, \infty)$ and $\lim\limits_{x \to \infty} h(x) = m - l$, $m - l \geq 0$, by Theorem 7.4.1. That is, $l \leq m$.

Theorem 7.4.3. Let $D \subset \mathbb{R}$ and $(c, \infty) \subset D$. Let f, g, h be functions on D to $\mathbb{R}$ such that $f(x) < g(x) < h(x)$ for all $x \in (c, \infty)$ and $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} h(x) = l$ then $\lim\limits_{x \to \infty} g(x) = l$.

Proof left to the reader.

Theorem 7.4.4. Cauchy criterion

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to \infty} f(x) = l (\in \mathbb{R})$ if and only if for a pre-assigned positive ϵ there exists a positive number $G > c$ such that $| f(x') - f(x'') | < \epsilon$ for every pair of points $x', x'' > G$.

Worked Example.

1. Using Cauchy's principle prove that $\lim_{x\to\infty} \cos x$ does not exist.

Let $f(x) = \cos x, x \in \mathbb{R}$. Here the domain of f is $\mathbb{R}$.

Let us choose $\epsilon = \frac{1}{2}$. In order that $\lim_{x\to\infty} f(x)$ should exist, it is necessary that there exists a positive G such that

$| f(x') - f(x'') | < \frac{1}{2}$ for every pair of points $x', x'' > G$.

For a given positive G we can find a natural number n such that $2n\pi > G$, say $2k\pi > G$.

Let $x' = (2k+1)\pi$ and $x'' = 2k\pi$. Then $x', x'' > G$ and $f(x') = -1, f(x'') = 1$. Therefore $|f(x') - f(x'')| \not< \epsilon$ for some pair of points $x', x'' > G$ for every $G > 0$.

This shows that Cauchy's condition for the existence of $\lim_{x\to\infty} \cos x$ is not satisfied. Therefore $\lim_{x\to\infty} \cos x$ does not exist.

7.5. Infinite limits at infinity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. If corresponding to a pre-assigned positive number G there exists a real number $k > c$ such that

$f(x) > G$ (or $< -G$) for all $x > k$

then we say that f tends to ∞ (or, $-\infty$) as $x \to \infty$.

In this case we write $\lim_{x\to\infty} f(x) = \infty$ (or, $-\infty$).

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. If corresponding to a pre-assigned positive number G there exists a real number $k < c$ such that

$f(x) > G$ (or $< -G$) for all $x < k$

then we say that f tends to ∞ (or, $-\infty$) as $x \to -\infty$.

In this case we write $\lim_{x\to-\infty} f(x) = \infty$ (or, $-\infty$).

Sequential criterion.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = \infty$ (or, $-\infty$) if and only if for every sequence $\{x_n\}$ in (c, ∞) diverging to ∞, the sequence $\{f(x_n)\}$ diverges to ∞ (or, $-\infty$).

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$. Then $\lim_{x\to-\infty} f(x) = \infty$ (or, $-\infty$) if and only if for every sequence

$\{x_n\}$ in $(-\infty, c)$ diverging to $-\infty$, the sequence $\{f(x_n)\}$ diverges to ∞ (or, $-\infty$).

Theorem 7.5.1. Let $g : D \to \mathbb{R}$ be a function on D and $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then $\lim\limits_{x \to \infty} g(x) = l$ $(l \in \mathbb{R})$ if and only if $\lim\limits_{x \to 0+} g(\frac{1}{x}) = l$.

Proof. Let $f(x) = \frac{1}{x}$. Then $\lim\limits_{x \to 0+} f(x) = \infty$.

Let $\lim\limits_{x \to \infty} g(x) = l$. Then for a pre-assigned positive ϵ there exists a positive number $d > c$ such that $| g(x) - l | < \epsilon$ for all $x > d$.

Since $\lim\limits_{x \to 0+} f(x) = \infty$, for the chosen positive number d, there exists a positive number δ such that $f(x) > d$ for all $x \in (0, \delta)$.

Therefore $| gf(x) - l | < \epsilon$ for all $x \in (0, \delta)$.

This implies that $\lim\limits_{x \to 0+} gf(x) = l$, i.e., $\lim\limits_{x \to 0+} g(\frac{1}{x}) = l$.

Conversely, let $\lim\limits_{x \to 0+} gf(x) = l$.

Then for a pre-assigned positive ϵ there exists a positive δ such that $l - \epsilon < gf(x) < l + \epsilon$ for all $x \in (0, \delta) \cap D$, D being the domain of gf.
$x \in (0, \delta) \Rightarrow f(x) > \frac{1}{\delta}$. Hence $l - \epsilon < g(x) < l + \epsilon$ for all $x > \frac{1}{\delta}$.
This implies that $\lim\limits_{x \to \infty} g(x) = l$.

This completes the proof.

Note 1. The thoerem can be generalised to include the cases $l = \infty$ and $l = -\infty$.

Note 2. The theorem in the generalised form says that in order to evaluate $\lim\limits_{x \to \infty} f(x)$ it is sufficient to evaluate $\lim\limits_{y \to 0+} f(\frac{1}{y})$.

Theorem 7.5.2. Let $g : D \to \mathbb{R}$ be a function on D and $(-\infty, c) \subset D$ for some $c \in \mathbb{R}$.

Then $\lim\limits_{x \to -\infty} g(x) = l$ if and only if $\lim\limits_{x \to 0-} g(\frac{1}{x}) = l$.

Proof left to the reader.

Worked Examples.

1. Find $\lim\limits_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}$.

$$\lim_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} = \lim_{y \to 0+} \frac{\frac{1}{\sqrt{y}} - \frac{1}{y}}{\frac{1}{\sqrt{y}} + \frac{1}{y}} = \lim_{y \to 0+} \frac{\sqrt{y} - 1}{\sqrt{y} + 1} = -1.$$

2. Find $\lim\limits_{x \to \infty} \frac{\sin x}{x}$.

$$\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{y \to 0+} \frac{\sin \frac{1}{y}}{\frac{1}{y}} = \lim_{y \to 0+} y \sin \frac{1}{y} = 0.$$

7.6. Limits of monotone functions.

Theorem 7.6.1. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to b-} f(x) = \sup\limits_{x \in (a,b)} f(x)$

(ii) If f is bounded below on I then $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,b)} f(x)$

(iii) If f is unbounded above on I then $\lim\limits_{x \to b-} f(x) = \infty$

(iv) If f is unbounded below on I then $\lim\limits_{x \to a+} f(x) = -\infty$.

Proof. (i) Let $M = \sup\limits_{x \in (a,b)} f(x)$.

Let $\epsilon > 0$. Then $f(x) \leq M$ for all $x \in (a, b)$ and there exists a point $x_o \in (a, b)$ such that $M - \epsilon < f(x_o) \leq M$.

Let $\delta = b - x_o$. Then $f(b - \delta) = f(x_o) > M - \epsilon$.
Since f is monotone increasing, $f(x) > M - \epsilon$ for all $x \in (b - \delta, b)$.
Hence $M - \epsilon < f(x) \leq M < M + \epsilon$ for all $x \in (b - \delta, b)$
or, $| f(x) - M | < \epsilon$ for all $x \in (b - \delta, b)$.

This implies $\lim\limits_{x \to b-} f(x) = M$. That is, $\lim\limits_{x \to b-} f(x) = \sup\limits_{x \in (a,b)} f(x)$.

(ii) Let $m = \inf\limits_{x \in (a,b)} f(x)$.

Let $\epsilon > 0$. Then $f(x) \geq m$ for all $x \in (a, b)$ and there exists a point $x_o \in (a, b)$ such that $m \leq f(x_o) < m + \epsilon$.

Let $\delta = x_o - a$. Then $f(a + \delta) = f(x_o) < m + \epsilon$.
Since f is monotone increasing, $f(x) < m + \epsilon$ for all $x \in (a, a + \delta)$.
Hence $m - \epsilon < m \leq f(x) < m + \epsilon$ for all $x \in (a, a + \delta)$
or, $| f(x) - m | < \epsilon$ for all $x \in (a, a + \delta)$.

This implies $\lim\limits_{x \to a+} f(x) = m$. That is, $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,b)} f(x)$.

(iii) Let $G > 0$. Since f is unbounded above on (a, b), there exists a point x_o in (a, b) such that $f(x_o) > G$.

Let $\delta = b - x_o$. Then $f(b - \delta) = f(x_o) > G$.
Since f is monotone increasing, $f(x) > G$ for all $x \in (b - \delta, b)$.

This implies $\lim\limits_{x \to b-} f(x) = \infty$.

(iv) Proof similar to (iii).

Theorem 7.6.2. Let $I = (a, b)$ be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone decreasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to a+} f(x) = \sup\limits_{x \in (a,b)} f(x)$

(ii) If f is bounded below on I then $\lim\limits_{x \to b-} f(x) = \inf\limits_{x \in (a,b)} f(x)$

(iii) If f is unbounded above on I then $\lim\limits_{x \to a+} f(x) = \infty$

(iv) If f is unbounded below on I then $\lim\limits_{x \to b-} f(x) = -\infty$.

Proof left to the reader.

Theorem 7.6.3. Let $I = (a, b)$ be a bounded open interval and $c \in (a, b)$. If $f : I \to \mathbb{R}$ be a monotone function on I then $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist.

Proof. **Case 1.** Let f be monotone increasing on I. Then f is monotone increasing on (a, c) and f is bounded above on (a, c), $f(c)$ being an upper bound. Let M be the supremum of f on (a, c). Then $M \leq f(c)$.

By Theorem 7.6.1, $\lim\limits_{x \to c-} f(x) = M \leq f(c)$.

Also f is monotone increasing on (c, b) and f is bounded below on (c, b), $f(c)$ being a lower bound. Let m be the infimum of f on (c, b). Then $f(c) \leq m$.

By Theorem 7.6.1, $\lim\limits_{x \to c+} f(x) = m \geq f(c)$.

Therefore $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist and $\lim\limits_{x \to c-} f(x) \leq \lim\limits_{x \to c+} f(x)$.

Case 2. Let f be monotone decreasing on I. Then f is monotone decreasing on (a, c) and f is bounded below on (a, c), $f(c)$ being a lower bound. Let m be the infimum of f on (a, c). Then $f(c) \leq m$.

By Theorem 7.6.2, $\lim\limits_{x \to c-} f(x) = m \geq f(c)$.

Also f is monotone decreasing on (c, b) and f is bounded above on (c, b), $f(c)$ being an upper bound. Let M be the supremum of f on (c, b). Then $M \leq f(c)$.

By Theorem 7.6.2, $\lim\limits_{x \to c+} f(x) = M \leq f(c)$.

Therefore $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist and $\lim\limits_{x \to c-} f(x) \leq \lim\limits_{x \to c+} f(x)$.

Theorem 7.6.4. Let $a \in \mathbb{R}$ and $I = (a, \infty)$.

Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to \infty} f(x) = \sup\limits_{x \in (a,\infty)} f(x)$.

(ii) If f is unbounded above on I then $\lim\limits_{x \to \infty} f(x) = \infty$.

(iii) If f is bounded below on I then $\lim\limits_{x \to a+} f(x) = \inf\limits_{x \in (a,\infty)} f(x)$.

(iv) If f is unbounded below on I then $\lim\limits_{x \to a+} f(x) = -\infty$.

Proof left to the reader.

Theorem 7.6.5. Let $a \in \mathbb{R}$ and $I = (-\infty, a)$.

Let $f : I \to \mathbb{R}$ be a monotone increasing function on I.

(i) If f is bounded above on I then $\lim\limits_{x \to a-} f(x) = \sup\limits_{x \in (-\infty,a)} f(x)$.

(ii) If f is unbounded above on I then $\lim\limits_{x \to a-} f(x) = \infty$.

(iii) If f is bounded below on I then $\lim\limits_{x \to -\infty} f(x) = \inf\limits_{x \in (-\infty,a)} f(x)$.

(iv) If f is unbounded below on I then $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Proof left to the reader.

Similar theorems can be formulated for a monotone decreasing function f.

Examples.

1. Let $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

f is a monotone increasing function on $(-\frac{\pi}{2}, \frac{\pi}{2})$. f is unbounded above on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore $\lim\limits_{x \to \frac{\pi}{2}-} \tan x = \infty$.

f is unbounded below on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore $\lim\limits_{x \to -\frac{\pi}{2}+} \tan x = -\infty$.

2. Let $f(x) = \log x, x > 0$.

f is a monotone increasing function on $(0, \infty)$. f is unbounded above on $(0, \infty)$. Therefore $\lim\limits_{x \to \infty} f(x) = \infty$.

f is unbounded below on $(0, \infty)$. Therefore $\lim\limits_{x \to 0+} \log x = -\infty$.

3. Let $f(x) = \frac{1}{x}, x > 0$.

f is a monotone decreasing function on $(0, \infty)$. f is bounded below on $(0, \infty)$. Therefore $\lim\limits_{x \to \infty} f(x) = \inf\limits_{x \in (0,\infty)} f(x) = 0$.

f is unbounded above on $(0, \infty)$. Therefore $\lim\limits_{x \to 0+} f(x) = \infty$.

4. Let $f(x) = \frac{1}{x}, x < 0$.

f is a monotone decreasing function on $(-\infty, 0)$. f is bounded above on $(-\infty, 0)$. $\sup\limits_{x \in (-\infty, 0)} f(x) = 0$.

Therefore $\lim\limits_{x \to -\infty} f(x) = \sup\limits_{x \in (-\infty, 0)} f(x) = 0$.

f is unbounded below on $(-\infty, 0)$. Therefore $\lim\limits_{x \to 0-} f(x) = -\infty$.

5. Let $f(x) = \frac{1}{x^2}, x \in \mathbb{R} - \{0\}$.

f is a monotone decreasing function on $(0, \infty)$. f is bounded below and unbounded above on $(0, \infty)$.

$\sup\limits_{x \in (0, \infty)} f(x) = \infty$, $\inf\limits_{x \in (0, \infty)} f(x) = 0$.

$\lim\limits_{x \to 0+} f(x) = \infty$, $\lim\limits_{x \to \infty} f(x) = 0$.

f is a monotone increasing function on $(-\infty, 0)$.f is unbounded above and bounded below on $(-\infty, 0)$.

$\sup\limits_{x \in (-\infty, 0)} f(x) = \infty$, $\inf\limits_{x \in (-\infty, 0)} f(x) = 0$.

$\lim\limits_{x \to 0-} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = 0$.

7.7. Some important limits.

1. Prove that $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x = e$.

We have $\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n = e$.

Since $x \to \infty$, we assume $x > 1$.

Let $[x] = k$. Then $k \leq x < k + 1$ and $1 + \frac{1}{k+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{k}$.

It follows that $(1 + \frac{1}{k+1})^k < (1 + \frac{1}{x})^x < (1 + \frac{1}{k})^{k+1}$.

Taking limit as $x \to \infty$ and noting that as $x \to \infty, k \to \infty$

we have $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x \geq \lim\limits_{k \to \infty} (1 + \frac{1}{k+1})^k$

$$= \lim_{k \to \infty} \frac{1}{(1+\frac{1}{k+1})^{k+1}} (1 + \frac{1}{k+1}) = e.$$

Also $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x \leq \lim\limits_{k \to \infty} (1 + \frac{1}{k})^{k+1}$

$$= \lim_{k \to \infty} (1 + \tfrac{1}{k})^k . (1 + \tfrac{1}{k}) = e.$$

It follows that $\lim\limits_{x \to \infty} (1 + \frac{1}{x})^x = e$.

2. Prove that $\lim_{x \to -\infty} (1 + \frac{1}{x})^x = e$.

Let $x = -y$. As $x \to \infty, y \to -\infty$.

Then $\lim_{x \to -\infty} (1 + \frac{1}{x})^x = \lim_{y \to \infty} (1 - \frac{1}{y})^{-y}$

$$= \lim_{y \to \infty} (\tfrac{y}{y-1})^y$$

$$= \lim_{t \to \infty} (1 + \tfrac{1}{t})^{t+1} \quad \text{where } t = y - 1$$

$$= \lim_{t \to \infty} [(1 + \tfrac{1}{t})^t . (1 + \tfrac{1}{t})]$$

$$= e.$$

3. Prove that $\lim_{x \to 0} (1 + x)^{1/x} = e$.

We have $\lim_{x \to \infty} (1 + \frac{1}{x})^x = e$.

Let $y = \frac{1}{x}$. As $x \to \infty, y \to 0+$.

Then $e = \lim_{x \to \infty} (1 + \frac{1}{x})^x = \lim_{y \to 0+} (1 + y)^{1/y}$ ... ... (i)

Also we have $e = \lim_{x \to -\infty} (1 + \frac{1}{x})^x$.

Let $y = \frac{1}{x}$. As $x \to -\infty, y \to 0-$.

Then $e = \lim_{x \to -\infty} (1 + \frac{1}{x})^x = \lim_{y \to 0-} (1 + y)^{1/y}$ ... ... (ii)

From (i) and (ii), $\lim_{y \to 0} (1 + y)^{1/y} = e$.

That is, $\lim_{x \to 0} (1 + x)^{1/x} = e$.

Exercises 11

1. Use sequential criterion for limits to show that the following limits do not exist.

(i) $\lim_{x \to 0} \cos \frac{1}{x^2}$, (ii) $\lim_{x \to 0} \frac{1}{x} \sin \frac{1}{x}$, (iii) $\lim_{x \to \infty} x^{1+\sin x}$ (iv) $\lim_{x \to \infty} x^2 \text{sgn} \cos x$.

Hint. (iii) Take $x_n = \frac{3\pi}{2} + 2n\pi; y_n = \frac{\pi}{2} + 2n\pi$.

(iv) Take $x_n = 2n\pi; y_n = \pi + 2n\pi$.

2. Let $f(x) = x, \ x \in \mathbb{Q}$

$= 2 - x, \ x \in \mathbb{R} - \mathbb{Q}$.

Show that (i) $\lim_{x \to 1} f(x) = 1$; (ii) $\lim_{x \to c} f(x)$ does not exist, if $c \neq 1$.

3. Show that the following limits do not exist.

(i) $\lim_{x \to 0} \frac{|\sin x|}{x}$, (ii) $\lim_{x \to 0} \frac{1}{e^{1/x} + 1}$, (iii) $\lim_{x \to 0} \frac{2x+|x|}{2x-|x|}$.

4. Evaluate the limits

(i) $\lim\limits_{x\to 0+} \sqrt{x-[x]}$, $\lim\limits_{x\to 0-} \sqrt{x-[x]}$;

(ii) $\lim\limits_{x\to 0+} \dfrac{1}{e^{\frac{1}{x}}+1}$, $\lim\limits_{x\to 0-} \dfrac{1}{e^{\frac{1}{x}}+1}$;

(iii) $\lim\limits_{x\to 0+} \{[x]+[1-x]\}$, $\lim\limits_{x\to 0-} \{[x]+[1-x]\}$.

(iv) $\lim\limits_{x\to 0+} x[\frac{1}{x}]$, $\lim\limits_{x\to 0-} x[\frac{1}{x}]$.

(v) $\lim\limits_{x\to 0+} [\frac{\sin x}{x}]$, $\lim\limits_{x\to 0-} [\frac{\sin x}{x}]$.

(vi) $\lim\limits_{x\to 0+} \{\frac{\sin x}{x}\}$, $\lim\limits_{x\to 0-} \{\frac{\sin x}{x}\}$, where $\{x\} = x-[x]$ = fractional part of x for all $x \in \mathbb{R}$.

Hint. (iv) For all non-zero real x, $[\frac{1}{x}] = \frac{1}{x} - \theta(x)$, where $0 \le \theta(x) < 1$.

(v) For all $x \in (0, \frac{\pi}{2})$, $0 < \sin x < x \Rightarrow 0 < \frac{\sin x}{x} < 1 \Rightarrow [\frac{\sin x}{x}] = 0$.

(vi) For all $x \in \mathbb{R}$, $[x] + \{x\} = x$. For all $x \in (0, \frac{\pi}{2})$, $[\frac{\sin x}{x}] = 0$ and this implies $\{\frac{\sin x}{x}\} = \frac{\sin x}{x}$.

5. Evaluate the limits

(i) $\lim\limits_{x\to\infty} \frac{x^2+3x}{x^2+x+1}$, (ii) $\lim\limits_{x\to\infty} \frac{\sin x}{x+\cos x}$, (iii) $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+3}$,

(iv) $\lim\limits_{x\to\infty} (1+\frac{2}{x})^x$, (v) $\lim\limits_{x\to\infty} (x-\sqrt{x^2-1})$, (vi) $\lim\limits_{x\to\infty} (\sqrt[3]{x+1} - \sqrt[3]{x})$.

8. CONTINUITY

8.1. Continuity.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D. f$ is said to be *continuous* at c if given any neighbourhood V of $f(c)$ there exists a neighbourhood W of c such that for all $x \in W \cap D, f(x) \in V$.

Equivalent definitions.

1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D$. f is said to be *continuous* at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D$. f is said to be *continuous* at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(c+h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$ and $c + h \in D$.

Note. In order that we may enquire if a function f is continuous at a point c, c must belong to the domain of f.

Theorem 8.1.1. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If c be an isolated point of D then f is continuous at c.

Proof. Since c is an isolated point of D, there exists a neighbourhood $N(c, \delta)$ of c such that $N(c, \delta) \cap D = \{c\}$.

Let $\epsilon > 0$. Then $| f(x) - f(c) | < \epsilon$ holds for $x = c$.
Therefore there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that f is continuous at c.

Theorem 8.1.2. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. If $c \in D \cap D'$ then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

Proof. Let f be continuous at c. Then for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

We also have $| f(x) - f(c) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

This implies that $\lim_{x \to c} f(x) = f(c)$.

Conversely, let $\lim_{x \to c} f(x) = f(c)$. Then for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N'(c, \delta) \cap D$.

Also for $x = c$, $| f(x) - f(c) | = | f(c) - f(c) | = 0 < \epsilon$.

Combining, we have $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that f is continuous at c.

Continuity on a set.

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $A \subset D$. f is said to be *continuous on* A if f be continuous at every point of A.

Continuity on an interval.

Let I be an interval and $f : I \to \mathbb{R}$ be a function.

(i) Let c be an interior point of I.

f is said to be continuous at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap I$.

Equivalently, f is said to be continuous at c if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(c + h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$ and $c + h \in I$.

(ii) Let a be the left end point of I and $a \in I$.

f is said to be continuous at a if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(a) | < \epsilon$ for all $x \in N(a, \delta) \cap I$

i.e., $| f(x) - f(a) | < \epsilon$ for all $x \in [a, a + \delta) \cap I$.

Equivalently, f is said to be continuous at a if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(a + h) - f(a) | < \epsilon$ for all h satisfying $| h | < \delta$ and $a + h \in I$.

(iii) Let b be the right end point of I and $b \in I$.

f is said to be continuous at b if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(x) - f(b) | < \epsilon$ for all $x \in N(b, \delta) \cap I$

i.e., $| f(x) - f(b) | < \epsilon$ for all $x \in (b - \delta, b] \cap I$.

Equivalently, f is said to be continuous at b if for a pre-assigned positive ϵ there exists a positive δ such that

$| f(b + h) - f(b) | < \epsilon$ for all h satisfying $| h | < \delta$ and $b + h \in I$.

Theorem 8.1.3. (Sequential criterion for continuity)

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D \cap D'$. f is continuous at c if and only if for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Proof. Let f be continuous at c. Let $\{x_n\}$ be a sequence in D such that $\lim x_n = c$.

Since f is continuous at c, for a pre-assigned positive ϵ, there exists a positive δ such that $| f(x) - f(c) |< \epsilon$ for all $x \in N(c, \delta) \cap D$.

Since $\lim x_n = c$, there exists a natural number m such that $| x_n - c |< \delta$ for all $n \geq m$.

Therefore for all $n \geq m$, $x_n \in N(c, \delta)$ and this implies $x_n \in N(c, \delta) \cap D$ for all $n \geq m$, since $x_n \in D$ for all $n \in \mathbb{N}$.

We have $| f(x_n) - f(c) |< \epsilon$ for all $n \geq m$.

This shows that $\lim f(x_n) = f(c)$.

Conversely, let $\lim f(x_n) = f(c)$ for every sequence $\{x_n\}$ in D converging to c. We prove that f is continuous at c.

Let f be not continuous at c. Then there exists a neighbourhood $N(f(c), \epsilon_o)$ of $f(c)$ such that no matter what neighbourhood $N(c, \delta)$ of c we consider, there will exist at least one point $p(\delta) \in N(c, \delta) \cap D$ such that $f(p) \notin N(f(c), \epsilon_o)$.

Let $\delta = 1$. Then there is a point, say x_1 in $N(c, 1) \cap D$ such that $f(x_1) \notin N(f(c), \epsilon_o)$.

Let $\delta = \frac{1}{2}$. Then there is a point x_2 in $N(c, \frac{1}{2}) \cap D$ such that $f(x_2) \notin N(f(c), \epsilon_o)$.

... ...

Thus we obtain a sequence of points $\{x_n\}$ in D such that $|x_n - c| < \frac{1}{n}$ for all $n \in \mathbb{N}$ and therefore $\lim x_n = c$. But the sequence $\{f(x_n)\}$ does not converge to $f(c)$ since $f(x_n) \notin N(f(c), \epsilon_o)$ for all $n \in \mathbb{N}$.

This is a contradiction to the hypothesis that for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Therefore our assumption is not tenable and f is continuous at c.

Worked Examples.

1. Let $k \in \mathbb{R}$. Prove that the function f defined by $f(x) = k, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. $| f(x) - f(c) |=| k - k |= 0$, for all $x \in \mathbb{R}$.

Let us choose $\epsilon > 0$.

Then $| f(x) - f(c) |< \epsilon$ for all $x \in N(c, \delta)$ for every positive δ.

This implies f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

2. Prove that the function f defined by $f(x) = x, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. $\mid f(x) - f(c) \mid = \mid x - c \mid$.
Let us choose $\epsilon > 0$.
Then $\mid f(x) - f(c) \mid < \epsilon$ whenever $\mid x - c \mid < \epsilon$.
That is, $\mid f(x) - f(c) \mid < \epsilon$ for all $x \in N(c, \delta)$ where $\delta = \epsilon$.
So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

3. A function f is defined on $\mathbb{R}$ by $f(x) = \cos\frac{1}{x}, x \neq 0$
$= 0, \ x = 0.$

Prove that f is not continuous at 0.

Let us consider a sequence $\{x_n\}$ where $x_n = \frac{1}{2\pi n}, n \in \mathbb{N}$. Then $\lim x_n = 0$; $f(x_n) = 1$ for all $n \in \mathbb{N}$. Therefore $\lim f(x_n) = 1$.

We have a sequence $\{x_n\}$ in $\mathbb{R}$ that converges to 0 but $\lim f(x_n) \neq f(0)$, proving that f is not continuous at 0.

4. A function f is defined on $\mathbb{R}$ by $f(x) = 1, x \in \mathbb{Q}$
$= 0, x \in \mathbb{R} - \mathbb{Q}.$

Prove that f is continuous at no point $c \in \mathbb{R}$.

Case 1. Let c be a rational point. Let $\{x_n\}$ be a sequence of irrational points such that $\lim x_n = c$.

Then $f(x_n) = 0$ for all $n \in \mathbb{N}$. Therefore $\lim f(x_n) = 0$.

But $f(c) = 1$. Thus there exists a sequence $\{x_n\} \in \mathbb{R}$ that converges to c but the sequence $\{f(x_n)\}$ does not converge to $f(c)$. By the sequential criterion for continuity, f is not continuous at c.

Case 2. Let c be an irrational point. Let $\{y_n\}$ be a sequence of rational points such that $\lim y_n = c$.

Then $f(y_n) = 1$ for all $n \in \mathbb{N}$. Therefore $\lim f(y_n) = 1$.

But $f(c) = 0$. Thus there exists a sequence $\{y_n\} \in \mathbb{R}$ that converges to c but the sequence $\{f(y_n)\}$ does not converge to $f(c)$. By the sequential criterion for continuity, f is not continuous at c.

Since every real number is either a rational number or an irrational number, it follows that f is not continuous at any point $c \in \mathbb{R}$.

Note. This function f is called *Dirichlet's function.* Dirichlet's function is everywhere discontinuous on $\mathbb{R}$.

5. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

Taking $x = y = 0$, we have $f(0) = 2f(0)$.

This implies $f(0) = 0$ ... ... (i)

Taking $y = -x$, we have $f(x) + f(-x) = 0$.
This implies $f(-x) = -f(x)$ ... ... (ii)

Let x be a positive integer, say n.

$$\begin{aligned} \text{Then } f(x) &= f(1+1+\cdots+1) \\ &= f(1)+f(1)+\cdots+f(1) \ \ (n \text{ times }) \\ &= nf(1) = kn = kx. \end{aligned}$$

So $f(x) = kx$ if x be a positive integer ... ...(iii)

Let x be a negative integer, say $-n$.

$$\begin{aligned} f(x) = f(-n) &= -f(n) \text{ by (ii)} \\ &= -kn = kx. \end{aligned}$$

So $f(x) = kx$ if x be a negative integer ... ...(iv)
From (i), (iii) and (iv) it follows that $f(x) = kx$ if x be an integer ...(v)

Let x be a rational number, say $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.
$f(qx) = f(p) = kp$ by (v).

$$\begin{aligned} \text{Also } f(qx) &= f(x+x+\cdots+x) \\ &= f(x)+f(x)+\cdots+f(x)[q \text{ times }] \\ &= qf(x). \end{aligned}$$

Therefore $qf(x) = kp$
or, $f(x) = \frac{kp}{q} = kx$.
So $f(x) = kx$ if x be a rational number ... ... (vi)

Let x be an irrational number α.
Let us consider a sequence of rational points $\{x_n\}$ converging to α.
Since f is continuous at $\alpha, \lim f(x_n) = f(\alpha)$.
But $\lim f(x_n) = \lim kx_n$, since x_n is rational.
As $\lim kx_n = k\alpha$, it follows that $f(\alpha) = k\alpha$.
So $f(x) = kx$ if x be an irrational number ... ...(vii)

From (v), (vi) and (vii) it follows that $f(x) = kx$ for all $x \in \mathbb{R}$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that either $f(x) = 0$ for all $x \in \mathbb{R}$, or $f(x) = a^x$ for all $x \in \mathbb{R}$, where a is some positive real number.

Taking $x = y = 0$, we have $f(0) = f(0)f(0)$.
This implies either $f(0) = 0$, or $f(0) = 1$.

Case 1. $f(0) = 0$.
For any real $c, f(c) = f(c+0) = f(c)f(0) = 0$.
Thus in this case $f(x) = 0$ for all $x \in \mathbb{R}$.

Case 2. $f(0) = 1$.

For any real $c, f(c) = f(\frac{c}{2} + \frac{c}{2}) = [f(\frac{c}{2})]^2 \geq 0 \ldots \quad \ldots$(i)

Also $1 = f(0) = f(c - c) = f(c)f(-c) \ldots \quad \ldots$(ii)

From (i) and (ii) it follows that $f(c) > 0$ for all real c in this case.

Let x be a positive integer, say n.

$$\begin{aligned} \text{Then } f(x) &= f(1 + 1 + \cdots + 1) \\ &= f(1)f(1)\cdots f(1) \ (n \text{ factors }) \\ &= [f(1)]^n = a^n \text{ where } a = f(1) > 0. \end{aligned}$$

So $f(x) = a^x$ if x be a positive integer ... ...(iii)

Let x be a negative integer, say $-n$.

$f(x) = f(-n) = \frac{1}{f(n)}$ by (ii) $= \frac{1}{a^n} = a^{-n} = a^x$.

So $f(x) = a^x$ if x be a negative integer ... ...(iv)

From (iii) and (iv) it follows that $f(x) = a^x$ if x be an integer ... ...(v)

Let x be a rational number, say $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

$f(p) = f(\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q})$ (q times) $= [f(\frac{p}{q})]^q$.

or, $a^p = [f(x)]^q$

or, $f(x) = a^{\frac{p}{q}} = a^x$.

So $f(x) = a^x$ if x be a rational number ... ...(vi)

Let x be an irrational number α.

Let us consider a sequence of rational points $\{x_n\}$ converging to α. Since f is continuous at $\alpha, \lim f(x_n) = f(\alpha)$.

But $\lim f(x_n) = \lim a^{x_n}$, since x_n is rational.

As $\lim a^{x_n} = a^\alpha$ by Corollary 1 of Example 3, 5.8, it follows that $f(\alpha) = a^\alpha$. So $f(x) = a^x$ if x be an irrational number ... ...(vii)

From (v), (vi) and (vii) it follows that $f(x) = a^x$ for all $x \in \mathbb{R}$.

Theorem 8.1.4. Let $D \subset \mathbb{R}$ and f and g are functions on D to $\mathbb{R}$. Let $c \in D$ and f and g are continuous at c. Then

(i) $f + g$ is continuous at c

(ii) if $k \in \mathbb{R}, kf$ is continuous of c

(iii) fg is continuous at c

(iv) if $g(x) \neq 0$ for all $x \in D, f/g$ is continuous at c.

Proof. (i) $|(f + g)(x) - (f + g)(c)|$

$= |f(x) + g(x) - \overline{f(c) + g(c)}|$

$\leq |f(x) - f(c)| + |g(x) - g(c)|$.

Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ_1 such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_1) \cap D$.

Since g is continuous at c, there exists a positive δ_2 such that $|g(x) - g(c)| < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_2) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $| f(x) - f(c) | < \frac{\epsilon}{2}$ and $| g(x) - g(c) | < \frac{\epsilon}{2}$ for all $x \in N(c, \delta) \cap D$.

Therefore $| (f + g)(x) - (f + g)(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that $f + g$ is continuous at c.

(ii) Proof left to the reader.

(iii) $| fg(x) - fg(c) | = | f(x).g(x) - f(c).g(c) |$
$= | f(x)\{g(x) - g(c)\} + g(c)\{f(x) - f(c)\} |$
$\leq | f(x) || g(x) - g(c) | + | g(c) || f(x) - f(c) |$.

Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ_1, such that $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta_1) \cap D$.

Since $| \; |f(x)| - |f(c)| \; | \leq | f(x) - f(c) |$, it follows that $| \; |f(x)| - |f(c)| \; | < \epsilon$ for all $x \in N(c, \delta_1) \cap D$
or, $| f(c) | - \epsilon < | f(x) | < | f(c) | + \epsilon$ for all $x \in N(c, \delta_1) \cap D$.

Let $| f(c) | + \epsilon = B_1$. Then $| f(x) | < B_1$ for all $x \in N(c, \delta_1) \cap D$.
Let $B = \max\{B_1, | g(c) |\}$. Then $B > 0$ and
$| fg(x) - fg(c) | < B | g(x) - g(c) | + B | f(x) - f(c) |$.

Since f is continuous at c, there exists a positive δ_2 such that $| f(x) - f(c) | < \frac{\epsilon}{2B}$ for all $x \in N(c, \delta_2) \cap D$.

Since g is continuous at c, there exists a positive δ_3 such that $| g(x) - g(c) | < \frac{\epsilon}{2B}$ for all $x \in N(c, \delta_3) \cap D$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
Then $| fg(x) - fg(c) | < (\frac{\epsilon}{2B} + \frac{\epsilon}{2B})B$ for all $x \in N(c, \delta) \cap D$
i.e., $| fg(x) - fg(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that fg is continuous at c.

(iv) First we prove that $\frac{1}{g}$ is continuous at c if $g(x) \neq 0$ for all $x \in D$.
$| \frac{1}{g}(x) - \frac{1}{g}(c) | = | \frac{1}{g(x)} - \frac{1}{g(c)} | = \frac{|g(x) - g(c)|}{|g(x)||g(c)|}$.

Let us choose $\epsilon = \frac{1}{2} | g(c) |$. Since g is continuous at c, there exists a positive δ_1 such that
$| g(x) - g(c) | < \frac{1}{2} | g(c) |$ for all $x \in N(c, \delta_1) \cap D$.

Now $| \; |g(x)| - |g(c)| \; | \leq |g(x) - g(c)| < \frac{1}{2}|g(c)|$.
Therefore $| g(x) | > \frac{1}{2} | g(c) |$ for all $x \in N(c, \delta_1) \cap D$.

Let us choose $\epsilon > 0$. Since g is continuous at c, there exists a positive δ_2 such that

$| g(x) - g(c) | < \frac{|g(c)|^2 \epsilon}{2}$ for all $x \in N(c, \delta_2) \cap D$.
Let $\delta = \min\{\delta_1, \delta_2\}$.
Then $| \frac{1}{g}(x) - \frac{1}{g}(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
This shows that $\frac{1}{g}$ is continuous at c.

The proof of the theorem is completed by considering the product of two functions f and $\frac{1}{g}$.

Note 1. If $f_1, f_2, \ldots, f_n$ be n functions on D and each of them be continuous at $c \in D$, then $f_1 + f_2 + \cdots + f_n$ is continuous at c.

Note 2. (iv) holds if $g(x) \neq 0$ in some neighbourhood $N(c, \delta) \subset D$. The existence of such a neighbourhood is guaranteed if $g(c) \neq 0$, by the neighbourhood property of continuity to be discussed later. Therefore (iv) holds under the single condition $g(c) \neq 0$.

Theorem 8.1.5. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be both continuous on D. Then

(i) $f + g$ is continuous on D

(ii) if $k \in \mathbb{R}, kf$ is continuous on D

(iii) fg is continuous on D

(iv) if $g(x) \neq 0$ on $D, f/g$ is continuous on D.

Immediate consequence of the Theorem 8.1.4.

Remark. The set of all real functions continuous on the closed and bounded interval $[a, b]$ is denoted by $C[a, b]$.

Theorem 8.1.6. Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be continuous at $c \in D$. Then $|f|$ is continuous at c.

Proof. $|f| : D \to \mathbb{R}$ is defined by $|f|(x) = |f(x)|$, $x \in D$.

$| \; |f|(x) - |f|(c) \; | = | \; |f(x)| - |f(c)| \; | \leq |f(x) - f(c)|$.

Let us choose $\epsilon > 0$.

Since f is continuous at c, there exists a positive δ such that

$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

Therefore $| \; |f|(x) - |f|(c) \; | < \epsilon$ for all $x \in N(c, \delta) \cap D$.

This shows that $|f|$ is continuous at c.

An immediate consequence of this theorem is the following theorem.

Theorem 8.1.7. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous on D. Then $|f|$ is continuous on D.

Note. If $| f |$ be continuous on D then f may not be continuous on D. For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= 1, x \in \mathbb{Q}, \\ &= -1, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$

Then $|f| : \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = 1, x \in \mathbb{R}$.

Here $|f|$ is continuous on $\mathbb{R}$ but f is not continuous on $\mathbb{R}$.

Definition. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ be functions. We define the function $\sup(f, g) : D \to \mathbb{R}$ by
$\sup(f, g)(x) = \sup\{f(x), g(x)\}, x \in D$.

We define the function $\inf(f, g) : D \to \mathbb{R}$ by
$\inf(f, g)(x) = \inf\{f(x), g(x)\}, x \in D$.

Theorem 8.1.8. Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be continuous at $c \in D$. Then $\sup(f, g)$ and $\inf(f, g)$ are continuous at c.

Proof.
$$\begin{aligned} \sup(f, g)(x) &= \sup\{f(x), g(x)\} \\ &= \tfrac{1}{2}(f(x) + g(x)) + \tfrac{1}{2} \mid f(x) - g(x) \mid \\ &= \tfrac{1}{2}(f + g)(x) + \tfrac{1}{2} \mid f - g \mid (x), x \in D. \end{aligned}$$

$$\begin{aligned} \inf(f, g)(x) &= \inf\{f(x), g(x)\} \\ &= \tfrac{1}{2}(f(x) + g(x)) - \tfrac{1}{2} \mid f(x) - g(x) \mid \\ &= \tfrac{1}{2}(f + g)(x) - \tfrac{1}{2} \mid f - g \mid (x), x \in D. \end{aligned}$$

Since f and g are continuous at c, $f + g, f - g, \mid f - g \mid$ are continuous at c. It follows that $\sup(f, g)$ and $\inf(f, g)$ are continuous at c.

Theorem 8.1.9. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D. Then $\sup(f, g)$ and $\inf(f, g)$ are continuous on D.

Immediate consequence of the Theorem 8.1.8.

Theorem 8.1.10. Let $A, B \subset \mathbb{R}$ and $f : A \to \mathbb{R}, g : B \to \mathbb{R}$ be functions such that $f(A) \subset B$.

Let $c \in A$ and f is continuous at c and g is continuous at $f(c) \in B$. Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous at c.

Proof. Let W be a neighbourhood of $g \circ f(c)$.

Since g is continuous at $f(c)$, there exists a neighbourhood V of $f(c)$ such that $y \in V \cap B \Rightarrow g(y) \in W$.

Since f is continuous at c, corresponding to the neighbourhood V of $f(c)$ there exists a neighbourhood U of c such that $x \in U \cap A \Rightarrow f(x) \in V$.

Since $f(A) \subset B, f(x) \in V \Rightarrow f(x) \in V \cap B$.

$$\begin{aligned} \text{Therefore } x \in U \cap A \quad &\Rightarrow \quad f(x) \in V \\ &\Rightarrow \quad f(x) \in V \cap B \\ &\Rightarrow \quad gof(x) \in W. \end{aligned}$$

Thus corresponding to a chosen neighbourhood W of $g \circ f(c)$ there exists a neighbourhood U of c such that for all $x \in U \cap A$, $g \circ f(x) \in W$.

This proves that $g \circ f$ is continuous at c.

Theorem 8.1.11. Let $A, B \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous on A and let $g : B \to \mathbb{R}$ be continuous on B and $f(A) \subset B$.

Then the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.

8.2. Continuity of some important functions.

1. Polynomial function.

Let $f(x) = a_o x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ for all $x \in \mathbb{R}$, where $a_o, a_1, \ldots, a_n$ are real numbers. Then f is a polynomial function.

f is the sum of $n+1$ functions $f_0, f_1, f_2, \ldots, f_n$ where $f_i = a_i x^{n-i}, i = 0, 1, 2, \ldots, n$. Each f_i is continuous on $\mathbb{R}$. Therefore by Theorem 8.1.5, f is continuous on $\mathbb{R}$.

2. Rational function.

Let $p(x)$ and $q(x)$ be polynomial functions on $\mathbb{R}$.

There are at most a finite number of real roots, say $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $q(x)$. If $x \neq \alpha_1, \alpha_2, \ldots, \alpha_m$ then we can define a function f by

$f(x) = \frac{p(x)}{q(x)}, x \neq \alpha_1, \alpha_2, \ldots, \alpha_m$.

By Theorem 8.1.4, if $q(c) \neq 0$ then f is continuous at c.

That is, if c be not a root of $q(x)$ then f is continuous at c.

So a rational function is continuous for all $x \in \mathbb{R}$ for which the function is defined.

3. Trigonometric functions.

(a) Let $f(x) = \sin x, x \in \mathbb{R}$. Let $c \in \mathbb{R}$.

$$\begin{aligned} |\sin x - \sin c| &= 2 \left| \cos \tfrac{x+c}{2} \sin \tfrac{x-c}{2} \right| \\ &\leq 2 \left| \sin \tfrac{x-c}{2} \right|, \text{ since } |\cos x| \leq 1 \\ &\leq 2 \left| \tfrac{x-c}{2} \right|, \text{ since } |\sin x| \leq |x| \\ &= |x - c|. \end{aligned}$$

Let us choose $\epsilon > 0$.

Then $|\sin x - \sin c| < \epsilon$ for all x satisfying $|x - c| < \epsilon$.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

(b) Let $f(x) = \cos x, x \in \mathbb{R}$. Let $c \in \mathbb{R}$.

$$\begin{aligned}
|\cos x - \cos c| &= 2 \left| \sin \tfrac{x+c}{2} \sin \tfrac{x-c}{2} \right| \\
&\leq 2 \left| \sin \tfrac{x-c}{2} \right|, \text{ since } \left| \sin \tfrac{x+c}{2} \right| \leq 1 \\
&\leq 2 \left| \tfrac{x-c}{2} \right|, \text{ since } |\sin x| \leq |x| \\
&= |x - c|
\end{aligned}$$

Let us choose $\epsilon > 0$.
Then $|\cos x - \cos c| < \epsilon$ for all x satisfying $|x - c| < \epsilon$.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

(c) Let $f(x) = \tan x$.

f is not defined at the points $(2n+1)\frac{\pi}{2}$ (n being an integer) where the denominator $\cos x = 0$.
Let $c \in \mathbb{R}$ and $c \neq (2n+1)\frac{\pi}{2}$. Then $\lim\limits_{x \to c} \tan x = \tan c$.

So f is continuous at c when $c \neq (2n+1)\frac{\pi}{2}$.

Thus f is continuous on its domain.

(d) The functions $\cot x$, cosec $x, \sec x$ are continuous on their respective domains.

4. Exponential function.
Let $a > 0$ and $f(x) = a^x, x \in \mathbb{R}$.
Let $c \in \mathbb{R}$. Let $\{x_n\}$ be any sequence in $\mathbb{R}$ converging to c. Then $\lim a^{x_n} = a^c$, by the corollary of the worked out limit 3 in art. 5.8.

So f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

Corollary. The function f defined by $f(x) = e^x, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

5. Logarithmic function.

Let $f(x) = \log x, x > 0$.
Let $c > 0$. Let $\{x_n\}$ be any sequence such that $x_n > 0$ for all $n \in \mathbb{N}$ and $\lim x_n = c$, then $\lim \log x_n = \log c$, by the corollary of the limit 4 in art.5.8.

So f is continuous at c. Since c is arbitrary, f is continuous on $(0, \infty)$.

6. Square root function.

Let $f(x) = \sqrt{x}, x \geq 0$. The domain of f is $D = \{x \in \mathbb{R} : x \geq 0\}$
Let $c > 0, f(c) = \sqrt{c}$.

$$\begin{aligned}
\text{When } x \geq 0, |f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\
&= \left| \tfrac{x-c}{\sqrt{x}+\sqrt{c}} \right| \leq \tfrac{1}{\sqrt{c}} |x - c|.
\end{aligned}$$

Let us choose $\epsilon > 0$.
Then $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \sqrt{c}\epsilon$ and $x \geq 0$.

That is, $| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$. $[\delta = \sqrt{c}\epsilon$
So f is continuous at c.
Also $\lim_{x \to 0} f(x) = f(0)$, showing that f is continuous at 0.

Thus f is continuous for all $x \geq 0$.

7. Some composite functions.

(a) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be such that $f(x) \geq 0$ for all $x \in D$ and f is continuous on D. Then $\sqrt{f}$ is continuous on D.

To prove this, let $g(x) = \sqrt{x}$.

Then the composite function $gf : D \to \mathbb{R}$ is defined by $gf(x) = \sqrt{f(x)}, x \in D$.

Since f is continuous on D and g is continuous on $f(D)$, the composite function gf, i.e., $\sqrt{f}$ is continuous on D.

Worked Examples.

(i) Prove that the function $h(x) = \sqrt{x^2 + 3}, x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = x^2 + 3, x \in \mathbb{R}$ and $g(x) = \sqrt{x}, x \geq 0$. $f(x) > 0$ for $x \in \mathbb{R}$. f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

(ii) Prove that the function $h(x) = \sqrt{\sin x}, x \in [0, \pi]$ is continuous on $[0, \pi]$.

h is the composite function gf where $f(x) = \sin x, x \in [0, \pi]$ and $g(x) = \sqrt{x}, x \geq 0$.

$f(x) \geq 0$ for $x \in [0, \pi]$. f is continuous on $[0, \pi] = D$, say. g is continuous on $f(D)$.

So gf is continuous on $[0, \pi]$. That is, h is continuous on $[0, \pi]$.

(iii) Prove that the function $h(x) = \sqrt{x + \sqrt{x}}, x \geq 0$ is continuous on $[0, \infty)$.

h is the composite function gf where $f(x) = x + \sqrt{x}, x \geq 0$ and $g(x) = \sqrt{x}, x \geq 0$.

$f(x) \geq 0$ for $x \geq 0$. f is continuous on $[0, \infty) = D$, say. g is continuous on $f(D)$.

So gf is continuous on $[0, \infty)$. That is, h is continuous on $[0, \infty)$.

(b) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be such that $f(x) > 0$ for all $x \in D$ and f is continuous on D. Then $\log f$ is continuous on D.

To prove this, let $g(x) = \log x, x > 0$

The composite function $gf : D \to \mathbb{R}$ is defined by $g(x) = \log f(x), x \in D$.

Since f is continuous on D and g is continuous on $f(D)$, the composite function gf, i.e., $\log f$ is continuous on D.

Worked Examples (continued).

(iv) Prove that the function $h(x) = \log(x^2 + 3)$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = x^2 + 3, x \in \mathbb{R}$ and $g(x) = \log x, x > 0$.

$f(x) > 0$ for $x \in \mathbb{R}$. f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

(v) Prove that the function $h(x) = \log \sin x$ is continuous on $(0, \pi)$.

h is the composite function gf where $f(x) = \sin x, x \in (0, \pi)$ and $g(x) = \log x, x > 0$.

gf is continuous on $(0, \pi)$. That is, h is continuous on $(0, \pi)$.

(c) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is continuous on D. Then e^f is continuous on D.

To prove this let $g(x) = e^x$.

The composite function $gf : D \to \mathbb{R}$ is defined on D and $gf(x) = e^{f(x)}, x \in D$.

Since f is continuous on D and g is continuous on $f(D)$ the composite function gf, i.e., e^f is continuous on D.

Worked Example (continued).

(vi) Prove the function $h(x) = e^{\sin x}$ is continuous on $\mathbb{R}$.

h is the composite function gf where $f(x) = \sin x, x \in \mathbb{R}$ and $g(x) = e^x, x \in \mathbb{R}$. $f(x) \in [-1, 1]$ for $x \in \mathbb{R}$.

f is continuous on $\mathbb{R}$ and g is continuous on $f(\mathbb{R})$, i.e., on $[-1, 1]$.

So gf is continuous on $\mathbb{R}$. That is, h is continuous on $\mathbb{R}$.

8.3. Limits of composite functions.

Theorem 8.3.1. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $g : D \to \mathbb{R}$ where $f(A) \subset D$.

Let c be a limit point of A and $\lim\limits_{x \to c} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim\limits_{x \to c} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to c} gf(x) = m$.

Proof. (i) Since g is continuous at l, for a pre-assigned positive ϵ there exists a positive δ such that

$| g(y) - g(l) | < \epsilon$ for all $y \in N(l, \delta) \cap D$.

Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that

$f(x) \in N(l, \delta)$ for all $x \in N'(c, \delta_1) \cap A$.

Since $f(A) \subset D, x \in N'(c, \delta_1) \cap A \Rightarrow f(x) \in N(l, \delta) \cap D$.

Therefore $x \in N'(c, \delta_1) \cap A \Rightarrow | g(y) - g(l) | < \epsilon$

i.e., $x \in N'(c, \delta_1) \cap A \Rightarrow | gf(x) - g(l) | < \epsilon$.

This proves $\lim_{x \to c} gf(x) = g(l)$.

(ii) Since $\lim_{y \to l} g(y) = m$, for a pre-assigned positive ϵ there exists a positive δ such that $| g(y) - m | < \epsilon$ for all $y \in N'(l, \delta) \cap D$.

Since $\lim_{x \to c} f(x) = l$, there exists a positive δ_1 such that $f(x) \in N(l, \delta)$ for all $x \in N'(c, \delta_1) \cap A$.

Since $f(A) \subset D, x \in N'(c, \delta_1) \cap A \Rightarrow f(x) \in N(l, \delta) \cap D$
$\Rightarrow f(x) \in N'(l, \delta) \cap D$, since $l \notin D$.

Therefore $x \in N'(c, \delta_1) \cap A \Rightarrow | g(y) - m | < \epsilon$

i.e., $| gf(x) - m | < \epsilon$. This proves $\lim_{x \to c} gf(x) = m$.

Note. In case (ii) if $\lim_{y \to l} g(y) = \infty$ (or $-\infty$) then $\lim_{x \to c} gf(x) = \infty$ (or $-\infty$).

As an immediate corollary it follows that

(i) if $\lim_{x \to c} f(x) = l > 0$ then $\lim_{x \to c} \log f(x) = \log \lim_{x \to c} f(x)$, since $\log x$ is continuous on its domain $(0, \infty)$;

(ii) if $\lim_{x \to c} f(x) = l \geq 0$ then $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$, since $\sqrt{x}$ is continuous on its domain $[0, \infty)$;

(iii) if $\lim_{x \to c} f(x) = l (\in \mathbb{R})$ then $\lim_{x \to c} e^{f(x)} = e^{\lim_{x \to c} f(x)}$, since e^x is continuous on its domain $(-\infty, \infty)$.

Extension of the theorem.

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $g : D \to \mathbb{R}$ where $f(A) \subset D$.

(a) Let c be a limit point of A and $\lim_{x \to c} f(x) = \infty$.

If for some $b \in \mathbb{R}, (b, \infty) \subset D$ and $\lim_{y \to \infty} g(y) = m$ then $\lim_{x \to c} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(b) Let c be a limit point of A and $\lim_{x \to c} f(x) = -\infty$.

If for some $b \in \mathbb{R}$, $(-\infty, b) \subset D$ and $\lim_{y \to -\infty} g(y) = m$ then $\lim_{x \to c} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(c) For some $a \in \mathbb{R}$, let $(a, \infty) \subset A$ and $\lim_{x \to \infty} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim_{x \to \infty} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to \infty} gf(x) = m$ where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

(d) For some $a \in \mathbb{R}$, let $(-\infty, a) \subset A$ and $\lim_{x \to -\infty} f(x) = l$.

(i) If $l \in D$ and g is continuous at l then $\lim_{x \to -\infty} gf(x) = g(l)$.

(ii) If $l \notin D$ but $l \in D'$ and $\lim_{y \to l} g(y) = m$ then $\lim_{x \to -\infty} gf(x) = m$, where $m \in \mathbb{R}$, or $m = \infty$, or $m = -\infty$.

Some other similar extensions of the theorem can be formulated.

A *word of caution* : $m = \infty(-\infty)$ stands for the phrase "$\lim gf(x) = \infty(-\infty)$".

Examples.

1. $\lim_{x \to 0} \sin\sqrt{x} = 0$.

Let $f(x) = \sqrt{x}, x \geq 0; g(x) = \sin x, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \geq 0\}, D = \mathbb{R}$ and $f(A) \subset D$.
$gf(x) = \sin\sqrt{x}, x \geq 0$.
$0 \in A'$ and $\lim_{x \to 0} f(x) = 0.0 \in \mathbb{R}$ and g is continuous at 0.
Therefore $\lim_{x \to 0} \sin\sqrt{x} = \lim_{x \to 0} gf(x) = g(0) = 0$.

2. $\lim_{x \to 0} \sqrt{1 + \sqrt{x}} = 1$.

Let $f(x) = 1 + \sqrt{x}, x \geq 0; g(x) = \sqrt{x}, x \geq 0$.
Here $A = \{x \in \mathbb{R} : x \geq 0\}, D = \{x \in \mathbb{R}/x\}, f(A) \subset D$.
$gf(x) = \sqrt{1 + \sqrt{x}}, x \geq 0$.
$0 \in A'$ and $\lim_{x \to 0} f(x) = 1.1 \in D$ and g is continuous at 1.
Therefore $\lim_{x \to 0} \sqrt{1 + \sqrt{x}} = \lim_{x \to 1} gf(x) = g(1) = 1$.

3. $\lim_{x \to 0+} e^{1/x} = \infty, \lim_{x \to 0-} e^{1/x} = 0, \lim_{x \to \infty} e^{1/x} = 1, \lim_{x \to -\infty} e^{1/x} = 1$.

Let $f(x) = \frac{1}{x}, x \neq 0; g(x) = e^x, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \neq 0\}, D = \mathbb{R}, f(A) \subset D$. $gf(x) = e^{1/x}, x \neq 0$.

$0 \in A'$, $\lim_{x \to 0} f(x)$ does not exist.
But $\lim_{x \to 0+} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = \infty$.
Therefore $\lim_{x \to 0+} gf(x) = \infty$, i.e., $\lim_{x \to 0+} e^{1/x} = \infty$.

$\lim_{x \to 0-} f(x) = -\infty$ and $\lim_{x \to -\infty} g(x) = 0$.
Therefore $\lim_{x \to 0-} gf(x) = 0$, i.e., $\lim_{x \to 0-} e^{1/x} = 0$.

$\lim_{x \to \infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x \to \infty} gf(x) = g(0) = 1$, i.e., $\lim_{x \to \infty} e^{1/x} = 1$.

$\lim_{x \to -\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x \to -\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x \to -\infty} e^{1/x} = 1$.

4. $\lim_{x \to 0+} e^{-1/x} = 0$, $\lim_{x \to 0-} e^{-1/x} = \infty$, $\lim_{x \to \infty} e^{-1/x} = 1$, $\lim_{x \to -\infty} e^{-1/x} = 1$.

Let $f(x) = \frac{1}{x}, x \neq 0$; $g(x) = e^{-x}, x \in \mathbb{R}$.
Here $A = \{x \in \mathbb{R} : x \neq 0\}, D = \mathbb{R}, f(A) \subset D$.
$gf(x) = e^{-1/x}, x \neq 0$.

$0 \in A'$, $\lim_{x \to 0} f(x)$ does not exist.
But $\lim_{x \to 0+} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = 0$.
Therefore $\lim_{x \to 0+} gf(x) = 0$, i.e., $\lim_{x \to 0+} e^{-1/x} = 0$.

$\lim_{x \to 0-} f(x) = -\infty$ and $\lim_{x \to -\infty} g(x) = \infty$.
Therefore $\lim_{x \to 0-} gf(x) = 0$, i.e., $\lim_{x \to 0-} e^{-1/x} = 0$.

$\lim_{x \to \infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x \to \infty} gf(x) = g(0) = 1$, i.e., $\lim_{x \to \infty} e^{-1/x} = 1$.

$\lim_{x \to -\infty} f(x) = 0, 0 \in D$ and g is continuous at 0.
Therefore $\lim_{x \to -\infty} gf(x) = g(0) = 1$, i.e., $\lim_{x \to -\infty} e^{-1/x} = 1$.

5. Some important limits.

(i) $\lim_{x \to 0} \dfrac{\log(1+x)}{x} = 1$, (ii) $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$,

(iii) $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a, a > 0$.

(i) Let $f(x) = (1+x)^{1/x}, x > -1$ but $x \neq 0; g(x) = \log x, x > 0$.

Here $A = (-1, 0) \cup (0, \infty)$ $D = \{x \in \mathbb{R} : x > 0\}$.$f(A) \subset D$.
$gf(x) = \frac{\log(1+x)}{x}, x \in A$.

$0 \in A'$ and $\lim\limits_{x \to 0}(1+x)^{1/x} = e$. $e \in D$ and g is continuous at e.
Therefore $\lim\limits_{x \to e} g(x) = g(e) = 1$.

Hence $\lim\limits_{x \to 0} gf(x) = 1$, i.e., $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$.

(ii) Let $\log(1+x) = y$. Then $1 + x = e^y$, i.e., $x = e^y - 1$.
As $x \to 0, y \to 0$.

From (i) $1 = \lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{y}{e^y - 1}$

Therefore $\lim\limits_{y \to 0} \dfrac{e^y - 1}{y} = 1$, i.e., $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$.

(iii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^{x \log_e a} - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^{x \log_e a} - 1}{x \log_e a} . \log_e a$

$= \lim\limits_{y \to 0} \dfrac{e^y - 1}{y} . \log_e a$ [let $x \log_e a = y$; as $x \to 0, y \to 0$]

$= \log_e a$.

8.4. Discontinuity.

We have seen that a function defined on a domain D may be continuous at all points of D, or may be continuous at some points of D and discontinuous at the other points of D, or may be discontinuous at every point of D.

If c does not belong to the domain of f, it is certain that f is discontinuous at c but the nature of discontinuity at c depends on the behaviour of the function f in the immediate neighbourhood of c. Even if c belongs to the domain of f, a discontinuity of f at c can occur in a variety of ways.

We now discuss different types of discontinuity of a finction f at a point c irrespective of the cases whether c belongs to or does not belong to the domain of f.

Discontinuity at an end point of an interval.

I. Let c be the left end point of the interval I and let f be continuous on (c, d) but discontinuous at c, $(c, d) \subset I$.

Three cases may arise.

(a) $\lim\limits_{x \to c} f(x)$ exists but $f(c) \neq \lim\limits_{x \to c} f(x)$.

In this case f is discontinuous at c. This type of discontinuity is called a *removable discontinuity*. c is said to be a *point of removable discontinuity.*

Note. The discontinuity at c can be removed by suitably defining f at c. That is why the discontinuity is called a removable discontinuity.

Example.

1. Let $f(x) = \frac{x^2-4}{x-2}, x > 2$
$\qquad\qquad = 10, x = 2.$

Here 2 is the left end point of the interval $[2, \infty)$, the domain of f. $f(2) = 10, \lim_{x \to 2} f(x) = 4 \neq f(2)$.

f is not continuous at 2. 2 is a point of removable discontinuity.

Note. If we define f by $f(x) = \frac{x^2-4}{x-2}, x > 2$
$\qquad\qquad\qquad\qquad = 4, x = 2$

then f becomes continuous at 2.

(b) $\lim_{x \to c} f(x)$ does not exist but there exists a neighbourhood $N(c, \delta)$ of c such that f is bounded on $N'(c, \delta) \cap I$.

In this case f is discontinuous at c whether f is defined at c or not. This type of discontinuity is called an *oscillatory discontinuity.* c is said to be a *point of oscillatory discontinuity.*

Example (continued).

2. Let $f(x) = \sin \frac{1}{x}, x > 0$
$\qquad\qquad = 0, x = 0.$

Here 0 is the left end point of the interval $[0, \infty)$, the domain of f. $f(0) = 0$. $\lim_{x \to 0} f(x)$ does not exist. f is discontinuous at 0. f is bounded on $(0, \delta)$ for $\delta > 0$. 0 is a point of oscillatory discontinuity.

(c) f is unbounded on every neighbourhood of c. In this case f is discontinuous at c whether f is defined at c or not.

This type of discontinuity is called an *infinite discontinuity.* c is said to be a *point of infinite discontinuity.*

If $\lim_{x \to c} f(x) = \infty$ (or $-\infty$), c is said to be a point of infinite discontinuity.

If $\lim_{x \to c} f(x)$ does not exist in $\mathbb{R}^*$, c is said to be a point of infinite oscillatory discontinuity.

Examples (continued).

3. Let $f(x) = \log x, x > 0$.

Here $\lim_{x\to 0} f(x) = -\infty$. f is discontinuous at 0. 0 is a point of infinite discontinuity.

4. Let $f(x) = \frac{1}{\sqrt{x-3}}, x > 3$.

Here $\lim_{x\to 3} f(x) = \infty$. f is discontinuous at 3. 3 is a point of infinite discontinuity.

5. Let $f(x) = \frac{1}{x}\sin\frac{1}{x}, x > 0$
$= 0, x = 0$.

Here $\lim_{x\to 0} f(x)$ does not exist. f is unbounded in $(0, \delta)$ for each $\delta > 0$. $\overline{\lim}_{x\to 0} f(x) = \infty, \underline{\lim}_{x\to 0} f(x) = -\infty$.

f is discontinuous at 0. 0 is a point of infinite oscillatory discontinuity.

6. Let $f(x) = \frac{1}{x} \mid \sin\frac{1}{x} \mid, x > 0$
$= 0, x = 0$.

f is unbounded on $(0, \delta)$ for each $\delta > 0$.

Here $\overline{\lim}_{x\to 0} f(x) = \infty$, $\underline{\lim}_{x\to 0} f(x) = 0$.

f is discontinuous at 0. 0 is a point of infinite oscillatory discontinuity.

II. Let c be the right end point of the interval I and let f be continuous on (a, c) but discontinuous at $c, (a, c) \subset I$.

Three cases as in **I** may arise and we have three types of discontinuity at c.

Discontinuity at an interior point of an interval.

Let c be an interior point of the interval I and let f be continuous on (a, c) and (c, b), but discontinuous at $c, (a, b) \subset I$.

Three cases may arise.

(a) (i) $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+}$ both exist and $\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x)$.

Subcase (i'). If f is not defined at c then f is discontinuous at c.

Subcase (i''). If f is defined at c but $f(c) \neq \lim_{x\to c} f(x)$, then f is discontinuous at c.

This type of discontinuity [either in (i') or in (i'')] is called a *removable discontinuity.* c is said to be a *point of removable discontinuity.*

Note. The discontinuity at c can be removed by suitably defining f at c.

(ii) $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist and $\lim_{x\to c-} f(x) \neq \lim_{x\to c+} f(x)$.

In this case f is discontinuous at c whether f is defined at c or not.

This type of discontinuity is called a *jump discontinuity.* c is said to be a *point of jump discontinuity.*

$\lim\limits_{x \to c+} f(x) - \lim\limits_{x \to c-} f(x)$ [i.e., $f(c+0) - f(c-0)$] is defined to be the *total jump* of f at c and it is denoted by $J_f(c)$.

If f is defined at c, $f(c+0) - f(c)$ is defined to be the *right hand jump* of f at c; and $f(c) - f(c-0)$ is defined to be the *left hand jump* of f at c.

The discontinuities discussed in (a) are called *simple discontinuities* or *discontinuities of the first kind.*

Examples (continued).

7. Let $f(x) = \frac{x^2-4}{x-2}, x \neq 2$
$\qquad = 10, x = 2.$

Here $\lim\limits_{x \to 2} f(x) = 4, \quad f(2) = 10.$

f is discontinuous at 2. 2 is a point of removable discontinuity.

Note. If we define f by $f(x) = \frac{x^2-4}{x-2}, x \neq 2$
$\qquad = 4, x = 2$

then f becomes continuous at 2.

8. Let $f(x) = [x], 0 < x < 2.$

$$\begin{aligned} f(x) &= 0, 0 < x < 1 \\ &= 1, 1 \leq x < 2. \end{aligned}$$

Here $\lim\limits_{x \to 1-} f(x) = 0$, $\lim\limits_{x \to 1+} f(x) = 1, f(1) = 1.$

f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 1 - 0 = 1.$

Right hand jump at $1 = f(1+0) - f(1) = 1 - 1 = 0.$

Left hand jump at $1 = f(1) - f(1-0) = 1 - 0 = 1.$

9. Let $f(x) = x - [x], 0 < x < 2$

$$\begin{aligned} f(x) &= x, 0 < x < 1 \\ &= x - 1, 1 \leq x < 2 \end{aligned}$$

Here $\lim\limits_{x \to 1-} f(x) = 1$, $\lim\limits_{x \to 1+} f(x) = 0, \quad f(1) = 0.$

f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 0 - 1 = -1.$

Right hand jump at $1 = f(1+0) - f(1) = 0.$

Left hand jump at $1 = f(1) - f(1-0) = 0 - 1 = -1.$

(b) At least one of $\lim\limits_{x\to c-} f(x)$ and $\lim\limits_{x\to c+} f(x)$ does not exist. But f is bounded on some deleted neighbourhood $N'(c,\delta)$ of c.

In this case f is discontinuous at c whether f is defined at c or not. This type of discontinuity is called an *oscillatory discontinuity.*

Since f is bounded on *some* deleted neighbourhood $N'(c,\delta)$ of c,each of $\overline{\lim}_{x\to c+} f(x)$ $(\overline{f(c+0)})$, $\underline{\lim}_{x\to c+} f(x)$ $(\underline{f(c+0)})$, $\overline{\lim}_{x\to c-} f(x)$ $(\overline{f(c-0)})$, $\underline{\lim}_{x\to c-} f(x)$ $(\underline{f(c-0)})$ exist finitely.

Let $M_f(c) = \max\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$;

$m_f(c) = \min\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$.

Then both $M_f(c)$ and $m_f(c)$ are finite. $M_f(c) - mf(c)$ is defined to be the *saltus* of f at c. It is denoted by $s_f(c)$.

Note. In particular, if both $f(c+0)$ and $f(c-0)$ exist,
$S_f(c) = | f(c+0) - f(c-0) |$.

The discontinuities discussed in **(b)** are called *discontinuities of the second kind.*

Examples (continued).

10. Let $f(x) = \sin\frac{1}{x}, x \neq 0$
$\qquad = 0, x = 0.$

Here $\lim\limits_{x\to 0} f(x)$ does not exist. f is bounded on $N(0,\delta)$ for $\delta > 0$.
$\overline{f(0+0)} = 1, \underline{f(0+0)} = -1, \overline{f(0-0)} = 1, \underline{f(0-0)} = -1.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - (-1) = 2$.

11. Let $f(x) = |\sin\frac{1}{x}|, x \neq 0$
$\qquad = 0, x = 0.$

Here $\overline{f(0+0)} = 1, \underline{f(0+0)} = 0, \overline{f(0-0)} = 1, \underline{f(0-0)} = 0.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - 0 = 1$.

12. Let $f(x) = \operatorname{sgn} x\, |\sin\frac{1}{x}|, x \neq 0$
$\qquad = 0, x = 0.$

Here $\overline{f(0+0)} = 1, \underline{f(0+0)} = 0, \overline{f(0-0)} = 0, \underline{f(0-0)} = -1.$

f is discontinuous 0. 0 is a point of oscillatory discontinuity. The saltus at $0 = 1 - (-1) = 2$.

(c) f is unbounded on every neighbourhood of c.

Since f unbounded on every neighbourhood of c, at least one of $\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}$ is ∞ or $-\infty$.

If each of $\overline{f(c+0)}$ and $\underline{f(c+0)}$ is ∞ (or $-\infty$) f is said to have an *infinite discontinuity* at the right of c.

If each of $\overline{f(c-0)}$ and $\underline{f(c-0)}$ is ∞ (or $-\infty$) f is said to have an *infinite discontinuity* at the left of c.

In either case c is said to be a point of *infinite discontinuity.*

If one or both of $\overline{f(c+0)}$ and $\underline{f(c+0)}$ be infinite and $\overline{f(c+0)} \neq \underline{f(c+0)}$, f is said to have an *infinite oscillatory discontinuity* at the right of c.

Similar definition for an *infinite oscillatory discontinuity* at the left of c.

Let $M_f(c) = \max\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}$;

$$m_f(c) = \min\{\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}\}.$$

If f be unbounded above on every neighbourhood of $c, M_f(c) = \infty$.
If f be unbounded below on every neighbourhood of $c, m_f(c) = -\infty$.

$M_f(c) - m_f(c)$ is said to be the *oscillation* of f at c. It is denoted by $w_f(c)$. $w_f(c)$ is infinite if at least one of $M_f(c)$ and $m_f(c)$ be ∞ (or $-\infty$) and $M_f(c) \neq m_f(c)$.

Examples (continued).

13. Let $f(x) = \frac{1}{x}$.

$\overline{f(0+0)} = \underline{f(0+0)} = \infty, \quad \overline{f(0-0)} = \underline{f(0-0)} = -\infty.$

f has an infinite discontinuity at the right of 0.
f has an infinite discontinuity at the left of 0.
0 is a point of infinite discontinuity.

14. Let $f(x) = \frac{1}{x^2}$.

$\overline{f(0+0)} = \underline{f(0+0)} = \overline{f(0-0)} = \underline{(0-0)} = \infty.$
0 is a point of infinite discontinuity.

15. Let $f(x) = \tan x$.

$\overline{f(\frac{\pi}{2}+0)} = \underline{f(\frac{\pi}{2}+0)} = -\infty, \overline{f(\frac{\pi}{2}-0)} = \underline{f(\frac{\pi}{2}-0)} = \infty.$
$\frac{\pi}{2}$ is a point of infinite discontinuity.

16. Let $f(x) = \frac{1}{x}\sin\frac{1}{x}$.

$\overline{f(0+0)} = \infty, \underline{f(0+0)} = -\infty, \overline{f(0-0)} = \infty, \underline{f(0-0)} = -\infty.$

f has an oscillatory infinite discontinuity at the right of 0
f has an oscillatory infinite discontinuity at the left of 0.

17. Let $f(x) = \frac{1}{x} \mid \sin \frac{1}{x} \mid$.

$\overline{f(0+0)} = \infty, \underline{f(0+0)} = 0, \quad \overline{f(0-0)} = 0, \underline{f(0-0)} = -\infty.$
f has an oscillatory infinite discontinuity at the right of 0
f has an oscillatory infinite discontinuity at the left of 0.

Worked Examples.

1. A function $f : [0,1] \to \mathbb{R}$ is defined by
$f(x) = x$, x is rational in $[0,1]$
$\quad = 1 - x$, x is irrational in $[0,1]$.

Show that (i) f is injective on $[0,1]$, (ii) f assumes every real number in $[0,1]$, (iii) f is continuous at $\frac{1}{2}$ and discontinuous at every other point in $[0,1]$.

(i) Let $x_1, x_2 \in [0,1]$ and $x_1 \neq x_2$.

Case 1. x_1, x_2 are both rational. Then $f(x_1) = x_1, f(x_2) = x_2$. As $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

Case 2. x_1 is rational, x_2 is irrational. Then $f(x_1) = x_1, f(x_2) = 1 - x_2$. As $f(x_1)$ is rational and $f(x_2)$ is irrational, $f(x_1) \neq f(x_2)$.

Case 3. x_1 is irrational, x_2 is rational. Then $f(x_1) = 1 - x_1, f(x_2) = x_2$. As $f(x_1)$ is irrational and $f(x_2)$ is rational, $f(x_1) \neq f(x_2)$.

Case 4. x_1, x_2 are both irrational. Then $f(x_1) = 1 - x_1, f(x_2) = 1 - x_2$. As $x \neq x_2$, $f(x_1) \neq f(x_2)$.

Therefore $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Hence f is injective on $[0,1]$.

(ii) Let $p \in [0,1]$.

If p be rational then the pre-image of p is p, since $p \in [0,1]$ and $f(p) = p$. If p be irrational then the pre-image of p is $1 - p$, since $1 - p \in [0,1]$ and $f(1-p) = p$.

Thus every element in $[0,1]$ has a pre-image. In other words, f assumes every real number in $[0,1]$.

(iii) Let $c = \frac{1}{2}$. Then $f(c) = \frac{1}{2}$.

$|f(x) - f(c)| = |x - \frac{1}{2}|$, if x be rational in $[0,1]$
$\qquad = |(1-x) - \frac{1}{2}| = |x - \frac{1}{2}|$, if x be irrational in $[0,1]$.

Therefore $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}|$ for all $x \in [0,1]$.

Let us choose $\epsilon > 0$. Then $|f(x) - f(\frac{1}{2})| < \epsilon$ holds for all x satisfying $|x - \frac{1}{2}| < \epsilon$. This proves that f is continuous at $\frac{1}{2}$.

Let $c \in [0,1], c \neq \frac{1}{2}$.

Let us consider a sequence of rational points $\{x_n\}$ in $[0,1]$ such that $\lim x_n = c$. Then $\lim f(x_n) = \lim x_n = c$.

Let us consider a sequence of irrational points $\{y_n\}$ in $[0,1]$ such that $\lim y_n = c$. Then $\lim f(y_n) = \lim(1-y_n) = 1-c$.

Since $c \neq 1-c$, f is not continuous at c by sequential criterion for continuity.

Thus f is discontinuous at every point other than $\frac{1}{2}$ in $[0,1]$.

2. A function $f : \mathbb{R}$ is defined by
$$f(x) = x, x \in \mathbb{Q}$$
$$= 0, x \in \mathbb{R} - \mathbb{Q}.$$

Show that f is continuous at 0 and f has a discontinuity of the second kind at every other point in $\mathbb{R}$.

$$\begin{aligned} | f(x) - f(0) | = | f(x) | &= | x | \text{ if } x \in \mathbb{Q} \\ &= 0 \text{ if } x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$

Let $\epsilon > 0$. Then $| f(x) - f(0) | < \epsilon$ for all x in $(0-\epsilon, 0+\epsilon)$.

Therefore f is continuous at 0.

Let $c \in \mathbb{R}$ and $c \neq 0$.

Let us take a sequence of rational numbers $\{x_n\}$ such that $x_n > c$ for all $n \in \mathbb{N}$ and $\lim x_n = c$. Then $\lim f(x_n) = \lim x_n = c$.

Let us take a sequence of irrational numbers $\{y_n\}$ such that $y_n > c$ for all $n \in \mathbb{N}$ and $\lim y_n = c$. Then $\lim f(y_n) = 0$.

$\lim\limits_{x \to c+} f(x)$ does not exist since for two different sequences $\{x_n\}$ and $\{y_n\}$ in (c, ∞) both converging to c, the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits.

By similar arguments, $\lim\limits_{x \to c-} f(x)$ does not exist.

It follows that f is discontinuous at c and it is a discontinuity of the second kind.

3. Find the points of discontinuities of the function f defined by
$$f(x) = \lim_{n \to \infty} \frac{(1+\sin \pi x)^n - 1}{(1+\sin \pi x)^n + 1}, x \in \mathbb{R}.$$

Case I. Let x be an integer.

Then $\sin \pi x = 0$ and therefore $f(x) = 0$.

Case II. Let $2m < x < 2m+1, m$ being an integer.

Then $2m\pi < \pi x < (2m+1)\pi$.

Therefore $0 < \sin \pi x < 1$ and $1 < 1 + \sin \pi x < 2$.

$$f(x) = \lim_{n \to \infty} \frac{1 - \frac{1}{(1+\sin \pi x)^n}}{1 + \frac{1}{(1+\sin \pi x)^n}} = 1, \text{ since } \lim_{n \to \infty} (1+\sin \pi x)^n = \infty.$$

Case III. Let $2m-1 < x < 2m, m$ being an integer.

Then $(2m-1)\pi < \pi x < 2m\pi$.
Therefore $-1 < \sin \pi x < 0$ and $0 < 1 + \sin \pi x < 1$.

$$f(x) = \lim_{n\to\infty} \frac{(1+\sin \pi x)^n - 1}{(1+\sin \pi x)^n + 1} = -1, \text{ since } \lim_{n\to\infty} (1+\sin \pi x)^n = 0.$$

$$\begin{aligned} \text{Thus } f(x) &= 0 \text{ if } x = 0, \pm 1, \pm 2, \cdots \\ &= 1 \text{ if } 2m < x < 2m+1, m \text{ being an integer} \\ &= -1 \text{ if } 2m-1 < x < 2m, m \text{ being an integer.} \end{aligned}$$

Let us examine continuity of f at $x = 2m, m$ being an integer.

$\lim_{x\to 2m-} f(x) = -1$, $\lim_{x\to 2m+} f(x) = 1$ and $f(2m) = 0$.

Therefore f is discontinuous at $2m$.

Let us examine continuity of f at $x = 2m-1, m$ being an integer.

$\lim_{x\to (2m-1)-} f(x) = 1$, $\lim_{x\to (2m-1)+} f(x) = 1$ and $f(2m-1) = 0$.

Therefore f is discontinuous at $2m-1$.

Clearly, f is continuous at c if c be not an integer.

Thus f is discontinuous at $c \in \mathbb{R}$ when c is an integer.

Note. Each point of discontinuity is a point of jump discontinuity.

4. Find the points of discontinuity of the function f defined by

$$f(x) = \lim_{n\to\infty} [\lim_{t\to 0} \frac{\sin^2(n!\pi x)}{\sin^2(n!\pi x) + t^2}], x \in \mathbb{R}.$$

Case I. Let x be rational.

Then by taking n sufficiently large, $n!x$ can be made an integer, so that $\sin(n!\pi x) = 0$.

Therefore $f(x) = \lim_{t\to 0} \frac{0}{0+t^2} = 0$.

Case II. Let x be irrational. Then $0 < \sin^2(n!\pi x) < 1$.

Therefore $f(x) = \lim_{n\to\infty} [\lim_{t\to 0} \frac{1}{1 + \frac{t^2}{\sin^2(n!\pi x)}}] = 1$.

Thus f is defined by $f(x) = 0$ if x be rational
$= 1$ if x be irrational.

f is discontinuous at all points in $\mathbb{R}$.

5. A function is defined on $[0,1]$ by $f(0) = 1$ and

$$\begin{aligned} f(x) &= 0, \text{ if } x \text{ be irrational} \\ &= \tfrac{1}{n}, \text{ if } x = \tfrac{m}{n} \text{ where } m, n \text{ are positive integers prime to each other.} \end{aligned}$$

Prove that f is continuous at every irrational point in $[0,1]$ and discontinuous at every rational point in $[0,1]$.

Let $a \in [0,1]$ be rational. Let $\{x_n\}$ be a sequence of irrational points such that $x_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\lim x_n = a$. Then $\lim f(x_n) = 0$.

But $f(a) > 0$. So f is discontinuous at a.

Let $\alpha \in (0,1)$ be irrational. Let us choose a positive ϵ. There is a natural number k such that $0 < \frac{1}{k} < \epsilon$ (by Archimedean property).

In $(0,1)$ there are only a finite number of rational points $\frac{m}{n}$ with n less than k.

Hence there exists a positive δ such that the neighbourhood $(\alpha - \delta, \alpha + \delta) \subset (0,1)$ contains no rational point $\frac{m}{n}$ with n less than k.

Therefore for all $x \in N(\alpha, \delta) \subset (0,1)$,
$| f(x) - f(\alpha) | = 0$, if x be irrational
$| f(x) - f(\alpha) | = \frac{1}{n} \leq \frac{1}{k} < \epsilon$, if $x = \frac{m}{n}$.

Thus $| f(x) - f(\alpha) | < \epsilon$ for all $x \in N(\alpha, \delta) \subset (0,1)$.

This proves that f is continuous at α.

Thus f is continuous at every irrational point in $[0,1]$ and discontinuous at every rational point in $[0,1]$.

Exercises 12

1. Give an example of functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the sum $f + g$ is continuous at c.

2. Give an example of functions f and g which are not continuous at a point $c \in \mathbb{R}$ but the product fg is continuous at c.

3. Let $f(x) = \text{sgn } x$, $g(x) = x(1 - x^2)$.
Show that the composite function $g \circ f$ is continuous at 0.

Note. Here f is discontinuous at 0 and g is continuous at $f(0)$, but still the composite gf is continuous at a 0. The converse implication of the theorem 8.1.10 is not true.

4. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at one point $c \in \mathbb{R}$, prove that f is continuous at every point in $\mathbb{R}$.

[**Hint.** Let f be continuous at c. Then $\lim\limits_{h \to 0} f(c+h) = f(c)$, i.e., $\lim\limits_{h \to 0} f(h) = 0$.]

5. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x = 0$, prove that f is continuous on $\mathbb{R}$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

[**Hint.** Let $c \in \mathbb{R}$. Consider a sequence of rational points $\{c_n\}$ converging to c. Use sequential criterion for continuity.]

7. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ for all $x, y \in \mathbb{R}$. Prove that $f(x) = ax + b$, $(a, b \in \mathbb{R})$ for all $x \in \mathbb{R}$.

[**Hint.** $f(x) = \frac{1}{2}[f(2x)+f(0)]$, $f(x)+f(y) = \frac{1}{2}[f(2x)+f(2y)]+f(0) = f(x+y)+f(0)$. Let $\phi(x) = f(x) - f(0)$. Then ϕ is continuous on $\mathbb{R}$ and $\phi(x+y) = \phi(x) + \phi(y)$. Worked Ex.5, page 248.]

8. Let $f : (-1, 1) \to \mathbb{R}$ be continuous at 0. If $f(x) = f(x^2)$ for all $x \in (-1, 1)$, prove that $f(x) = f(0)$ for all $x \in (-1, 1)$.

[**Hint.** Let $c \in (-1, 1)$. Consider the sequence $\{c^{2^n}\}$ in $(-1, 1)$ converging to 0. Use sequential criterion for continuity.]

9. Prove that the function f is continuous on the indicated interval.

(i) $f(x) = e^{\sqrt{x}}$, $x \in [0, \infty)$; (ii) $f(x) = \log \sin x$, $x \in (0, \pi)$.

10. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= 2x, x \in \mathbb{Q} \\ &= 1 - x, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$
Prove that f is continuous at $\frac{1}{3}$ and discontinuous at every other point.

11. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= x^2 + 1, x \in \mathbb{Q} \\ &= x, x \in \mathbb{R} - \mathbb{Q}. \end{aligned}$$
Prove that f has a discontinuity of the second kind at every point c in $\mathbb{R}$.

12. Find the points of discontinuity of the functions.

(i) $f(x) = [\sin x], x \in \mathbb{R}$; (ii) $f(x) = (-1)^{[x]}, x \in \mathbb{R}$;

(iii) $f(x) = [x] + [-x], x \in \mathbb{R}$; (iv) $f(x) = \lim\limits_{n\to\infty} \dfrac{(1 + \sin\frac{\pi}{x})^n - 1}{(1 + \sin\frac{\pi}{x})^n + 1}, x \in (0, 1)$;

(v) $f(x) = \lim\limits_{n\to\infty} \dfrac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}, x \in \mathbb{R}$.

13. Examine the nature of discontinuity of f at 0.

$$\begin{aligned} \text{(i) } f(x) &= \tfrac{1}{\sqrt{x}},\ x > 0, & \text{(ii) } f(x) &= \tfrac{\sin x}{x}, x \neq 0 \\ &= 0,\ x = 0. & &= 0,\ x = 0. \\ \text{(iii) } f(x) &= \log \sin x, 0 < x < \pi & \text{(iv) } f(x) &= [x] + [1 - x]. \\ &= 0, x = 0. & & \\ \text{(v) } f(x) &= \cos \tfrac{1}{x}, x \neq 0 & \text{(vi) } f(x) &= \tfrac{1}{x} \sin \tfrac{1}{x}, x \neq 0 \\ &= 0,\ x = 0. & &= 0,\ x = 0. \\ \text{(vii) } f(x) &= \tfrac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0 & \text{(viii) } f(x) &= 1 + 2^{1/x}, x \neq 0 \\ &= 0, x = 0. & &= 0, x = 0. \end{aligned}$$

14. A function f is said to be *piecewise continuous* on an interval I if f be continuous on I except at a finite number of points of jump discontinuity.

Show that f is piecewise continuous on the indicated interval I.

(i) $f(x) = [x]$, $I = [0, 3]$; (ii) $f(x) = x - [x]$, $I = [0, 3]$;

(iii) $f(x) = [2x]$, $I = [0, 3]$; (iv) $f(x) = sgn\ x$, $I = [-2, 2]$.

8.5. Properties of continuous functions.

Theorem 8.5.1. (Neighbourhood property)

Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be continuous on D. Let $c \in D$. If $f(c) \neq 0$ then there exists a suitable $\delta > 0$ such that for all $x \in N(c, \delta) \cap D$, $f(x)$ keeps the same sign as $f(c)$.

Proof. **Case 1.** $f(c) > 0$. Let us choose a positive ϵ such that $f(c) - \epsilon > 0$.
Since f is continuous at c, there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
or, $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in N(c, \delta) \cap D$.
Therefore $f(x) > f(c) - \epsilon > 0$ for all $x \in N(c, \delta) \cap D$.

Case 2. $f(c) < 0$. Let us choose a positive ϵ such that $f(c) + \epsilon < 0$.
Since f is continuous at c, there exists a positive δ such that
$| f(x) - f(c) | < \epsilon$ for all $x \in N(c, \delta) \cap D$.
or, $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in N(c, \delta) \cap D$.
Therefore $f(x) < f(c) + \epsilon < 0$ for all $x \in N(c, \delta) \cap D$.

So in any case $f(x)$ keeps the same sign as $f(c)$ for all $x \in N(c, \delta) \cap D$ for some $\delta > 0$.

Note. This property is a *local property* of a continuous function. It is also called the *sign preserving property* of a continuous function.

Corollary. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and $c \in \mathbb{R}$. If $f(c) \neq 0$ then there exists a positive δ such that $f(x)$ keeps the same sign as $f(c)$ for all $x \in N(c, \delta)$.

Worked Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) > 0\}$ is an open set in $\mathbb{R}$.

Case I. Let $f(x) \leq 0$ for all $x \in \mathbb{R}$. Then $S = \phi$ and S is an open set.

Case II. Let $f(x) > 0$ for all $x \in \mathbb{R}$. Then $S = \mathbb{R}$ and S is an open set.

Case III. Let S be a proper subset of $\mathbb{R}$.

Let $c \in S$. Then $f(c) > 0$. Since f is continuous on $\mathbb{R}$ and $f(c) > 0$, by the neighbourhood property there exists a positive δ such that for all $x \in N(c, \delta)$, $f(x) > 0$. Therefore $N(c, \delta) \subset S$.

Thus $c \in S \Rightarrow N(c, \delta) \subset S$.

This shows that c is an interior point of S.

Since c is arbitrary, every point of S is an interior point of S and therefore S is an open set.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) < 0\}$ is an open set in $\mathbb{R}$.

Similar proof as in Example 1.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) \neq 0\}$ is an open set in $\mathbb{R}$.

Similar proof as in Example 1.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

Let $P = \{x \in \mathbb{R} : f(x) > 0\}$, $T = \{x \in \mathbb{R} : f(x) < 0\}$. Then $P \cup T \cup S = \mathbb{R}$.

But $P \cup T$ is an open set, since P and T are open sets in $\mathbb{R}$ by Examples 1 and 2.

Therefore S being the complement of an open set in $\mathbb{R}$, is a closed set.

Theorem 8.5.2. Let $I = [a, b]$ be a closed and bounded inteval and a function $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof. If possible, let f be not bounded on I. Then for each natural number n, there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. Thus we obtain a sequence $\{x_n\}$ such that $x_n \in [a, b]$ for all $n \in \mathbb{N}$ and $| f(x_n) |> n$ for all $n \in \mathbb{N}$.

Since $[a, b]$ is a bounded interval, the sequence $\{x_n\}$ is bounded.

By Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{r_n}\}$ converging to l, say.

Since $[a, b]$ is a closed set and each element of the convergent sequence $\{x_{r_n}\}$ belongs to $[a, b]$, the limit l belongs to $[a, b]$.

Since $l \in [a, b], f$ is continuous at l. Since the sequence $\{x_{r_n}\}$ converges to l and f is continuous at l, the sequence $\{f(x_{r_n}\}$ must converge to $f(l)$, by the sequential criterion for continuity.

Therefore the sequence $\{f(x_{r_n})\}$ must be bounded. But by construction, $| f(x_{r_n}) |> r_n$ and since $\{r_n\}$ is a strictly increasing sequence of natural numbers, $r_n \geq n$.

So $| f(x_{r_n}) |> n$ and this implies that the sequence $\{f(x_{r_n})\}$ is not bounded. Thus we arrive at a contradiction.

Therefore f is bounded on I and the theorem is proved.

Another Proof.

Let $c \in [a, b]$. Then f is continuous at c.

Let $\epsilon > 0$. Then there exists a positive δ such that $|f(x) - f(c)| < \epsilon$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$.

But $|f(x)| - |f(c)| \leq |f(x) - f(c)|$ and this gives $|f(x)| < |f(c)| + \epsilon$

for all $x \in (c-\delta, c+\delta) \cap [a,b]$. This shows that f is bounded on $(c-\delta, c+\delta) \cap [a,b]$.

Thus for every $x \in [a,b]$ there exists an open interval $I_x = (x-\delta_x, x+\delta_x)$ such that f is bounded on $I_x \cap [a,b]$.

The set of all open intervals $\{I_x : x \in [a,b]\}$ forms an open cover of the closed and bounded interval $[a,b]$.

By Heine-Borel theorem, there exists a finite number of these open intervals, say $I_{x_1}, I_{x_2}, \ldots, I_{x_m}$, such that $[a,b] \subset I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_m}$.

For each $i = 1, 2, \ldots, m$, f is bounded on $I_i \cap [a,b]$ and therefore there exists a positive real number M_i such that $|f(x)| \leq M_i$ for all $x \in I_i \cap [a,b]$.

Let $M = \max\{M_1, M_2, \ldots, M_m\}$. Then $|f(x)| \leq M$ for all $x \in (I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_m}) \cap [a,b]$, i.e., for all $x \in [a,b]$.

Therefore f is bounded on $[a,b]$ and the proof is complete.

Note. Since f is bounded on $[a,b]$, the set $\{f(x) : x \in [a,b]\}$ is a non-empty bounded subset of $\mathbb{R}$. Therefore there exist real numbers M, m such that $M = \sup\limits_{x \in [a,b]} f(x), m = \inf\limits_{x \in [a,b]} f(x)$.

Theorem 8.5.3. Let $I = [a,b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on I. Then there is a point c in I such that $f(c) = \sup\limits_{x \in I} f(x)$ and also there is a point d in I such that $f(d) = \inf\limits_{x \in I} f(x)$.

Proof. The set $f(I) = \{f(x) : x \in I\}$ is a bounded set. Since this is a non-empty bounded subset of $\mathbb{R}$, $\sup f(I)$ and $\inf f(I)$ exist.

Let $M = \sup f(I)$. Then
there exists a point $x_1 \in I$ such that $M - 1 < f(x_1) \leq M$,
there exists a point $x_2 \in I$ such that $M - \frac{1}{2} < f(x_2) \leq M$,
there exists a point $x_3 \in I$ such that $M - \frac{1}{3} < f(x_3) \leq M$,
... ...

We obtain a sequence of points $\{x_n\}$ in I such that
$M - \frac{1}{n} < f(x_n) \leq M$ for all $n \in \mathbb{N}$.

Since I is bounded, the sequence $\{x_n\}$ is bounded and therefore there exists a convergent subsequence $\{x_{r_n}\}$ that converges to a limit c, say.

Since I is a closed set and the elements of the convergent sequence $\{x_{r_n}\}$ belong to I, the limit $c \in I$. Therefore f is continuous at c.

Since $\{x_{r_n}\}$ is a sequence in I converging to c and f is continuous at c, the sequence $\{f(x_{r_n})\}$ converges to $f(c)$.

Now $M - \frac{1}{r_n} < f(x_{r_n}) \leq M$ for all $n \in \mathbb{N}$ and $\lim(M - \frac{1}{r_n}) = M$.
By Sandwich theorem, $\lim f(x_{r_n}) = M$. That is, $f(c) = M$.

Taking $m = \inf f(I)$ it can be proved in a similar manner that there exists a point d in I such that $f(d) = m$.

Another Proof.

The set $f(I) = \{f(x) : x \in I\}$ is a bounded set. Since this is a non-empty bounded subset of $\mathbb{R}$, $\sup f(I)$ and $\inf f(I)$ exist.

Let $M = \sup f(I), m = \inf f(I)$. Then for all $x \in [a,b], m \leq f(x) \leq M$.

We shall prove that $f(x) = M$ for some $c \in [a,b]$.

If not, let $f(x) < M$ for all $x \in [a,b]$. Then $M - f(x) > 0$ for all $x \in [a,b]$.

Let $\phi(x) = \frac{1}{M-f(x)}, x \in [a,b]$. Then ϕ is continuous on $[a,b]$ and therefore ϕ is bounded on $[a,b]$.

Let B be an upper bound of ϕ on $[a,b]$. Then $B > 0$ and $0 < \frac{1}{M-f(x)} < B$ for all $x \in [a,b]$.

Therefore $f(x) < M - \frac{1}{B}$ for all $x \in [a,b]$. This contradicts that $M = \sup f(I)$.

Hence there exists a point c in $[a,b]$ such that $f(c) = M$.

In a similar manner it can be proved that there exists a point d in $[a,b]$ such that $f(d) = m$.

This completes the proof.

Note 1. $M(= \sup\limits_{x\in[a,b]} f(x))$ is called the *global maximum* of the function f on $[a,b]$. $m(= \inf\limits_{x\in[a,b]} f(x))$ is called the *global minimum* of the function f on $[a,b]$.

Note 2. A function f continuous on a bounded open interval I may not be bounded on I. Even if it is bounded on I, it may not attain the supremum or the infimum of f at some point of I.

For example, let $I = (0,1)$ and $f : I \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}, x \in (0,1)$. Then f is continuous on I and f is not bounded on I.

Let $I = (2,3)$ and $f : I \to \mathbb{R}$ be defined by $f(x) = x^2, x \in (2,3)$. Then f is continuous on I and f is bounded on I. $\sup\limits_{x\in I} f(x) = 9$, $\inf\limits_{x\in I} f(x) = 4$. But there is no point $c \in I$ such that $f(c) = 9$ and there is no point $d \in I$ such that $f(d) = 4$.

Note 3. A function f continuous on a closed interval I may not be bounded on I.

For example, let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}, x \geq 0$. f is continuous on $[0,\infty)$ but f is not bounded on $[0,\infty)$.

Theorem 8.5.4. (Bolzano)

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a)$ and $f(b)$ are of opposite signs then there exists at least a point c in the open interval (a, b) such that $f(c) = 0$.

Proof. Let $I_1 = [a, b] = [a_1, b_1]$, say. Without loss of generality, let us assume that $f(a_1) < 0$ and $f(b_1) > 0$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then $f(c_1)$ is either 0 or $\neq 0$. If $f(c_1) = 0$ the theorem is proved. If $f(c_1) \neq 0$ then either $f(c_1) < 0$, or $f(c_1) > 0$.

If $f(c_1) < 0$ we consider the closed interval $[c_1, b_1]$ and call it $I_2 = [a_2, b_2]$. If $f(c_1) > 0$ we consider the closed interval $[a_1, c_1]$ and call it $I_2 = [a_2, b_2]$.

Thus if $f(c_1) \neq 0$, the closed interval $I_2 = [a_2, b_2]$ is such that

(i) f is continuous on I_2 and $f(a_2) < 0, f(b_2) > 0$;

(ii) $I_2 \subset I_1$;

(iii) $| I_2 |= \frac{1}{2}(b - a)$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then $f(c_2)$ is either 0 or $\neq 0$.

If $f(c_2) = 0$ the theorem is proved. If $f(c_2) \neq 0$, then either $f(c_2) < 0$, or $f(c_2) > 0$.

If $f(c_2) < 0$ we consider the closed interval $[c_2, b_2]$ and call it $I_3 = [a_3, b_3]$. If $f(c_2) > 0$ we consider the closed interval $[a_2, c_2]$ and call it $I_3 = [a_3, b_3]$.

Thus if $f(c_2) \neq 0$ the closed interval $I_3 = [a_3, b_3]$ is such that

(i) f is continuous on I_3 and $f(a_3) < 0, f(b_3) > 0$;

(ii) $I_3 \subset I_2 \subset I_1$;

(iii) $| I_3 |= \frac{b-a}{2^2}$.

Let $c_3 = \frac{a_3+b_3}{2}$. Then $f(c_2)$ is either 0 or $\neq 0$.

Continuing in this manner, either we obtain a point c_k in (a, b) such that $f(c_k) = 0$, in which case the theorem is proved, or we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) f is continuous on I_n and $f(a_n) < 0, f(b_n) > 0$ for all $n \in \mathbb{N}$;

(ii) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(iii) $| I_n |= \frac{b-a}{2^{n-1}}$ and therefore $\lim | I_n |= 0$.

Thus $\{I_n\}$ is a sequence of nested closed and bounded intervals with $\lim | I_n |= 0$.

By *nested intervals theorem* there exists one and only one point α such that (i) $\alpha \in [a_n, b_n]$ for all $n \in \mathbb{N}$, and (ii) $\lim a_n = \alpha = \lim b_n$.

From (i) it follows that $\alpha \in [a, b]$. Since f is continuous on $[a, b]$, f is continuous at α.

From (ii) it follows that $\{a_n\}$ is a sequence of points in $[a, b]$ converging to α. Since f is continuous at $\alpha, \lim f(a_n) = f(\alpha)$.

But $f(a_n) < 0$ for all $n \in \mathbb{N}$ and this implies $\lim f(a_n) \leq 0$.

That is, $f(\alpha) \leq 0$ ... ... (A)

Also from (ii) it follows that $\{b_n\}$ is a sequence of points in $[a, b]$ converging to α. Since f is continuous at $\alpha, \lim f(b_n) = f(\alpha)$.

But $f(b_n) > 0$ for all $n \in \mathbb{N}$ and this implies $\lim f(b_n) \geq 0$.

That is, $f(\alpha) \geq 0$ ... ... (B)

From (A) and (B) it follows that $f(\alpha) = 0$.

Since $\alpha \in [a, b]$ and $f(a) < 0, f(b) > 0$ it follows that $\alpha \in (a, b)$.

Thus $\alpha = c$ and the theorem is proved.

Theorem 8.5.5. (Intermediate value theorem)

Let $[a, b]$ be a closed an bounded interval and a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval (a, b).

Proof. Without loss of generality, we assume that $f(a) < f(b)$.

Let μ be a real number such that $f(a) < \mu < f(b)$.

Let us consider the function $\phi : [a, b] \to \mathbb{R}$ defined by $\phi(x) = f(x) - \mu, x \in [a, b]$.

ϕ is continuous on $[a, b]$, since f is continuous on $[a, b]$.

$\phi(a) = f(a) - \mu < 0, \phi(b) = f(b) - \mu > 0$.

As $\phi(a)$ and $\phi(b)$ are of opposite signs, by Bolzano's theorem there exists at least one point c in (a, b) such that $\phi(c) = 0$.

Therefore $f(c) - \mu = 0$, i.e., $f(c) = \mu$.

Thus f attains μ at a point c in (a, b) and the theorem is done.

Note. Let $I = [a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be such that $f(a) \neq f(b)$ and f attains every value between $f(a)$ and $f(b)$ at least once in (a, b). Still f may not be continuous on $[a, b]$.

For example, let $f : [0, 2] \to \mathbb{R}$ be defined by $f(0) = 0, f(2) = 2$ and

$f(x) = x, 0 < x \leq 1$

$\quad = 3 - x, 1 < x < 2.$

f assumes every value between 0 and 2 on $[0, 2]$. But f is not continuous on $[0, 2]$ since f is not continuous at 1 and 2.

Theorem 8.5.6. Let $[a, b]$ be a closed and bounded interval and a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $\sup\limits_{x\in[a,b]} f(x) \neq \inf\limits_{x\in[a,b]} f(x)$ and μ be a real number lying between $\sup\limits_{x\in[a,b]} f(x)$ and $\inf\limits_{x\in[a,b]} f(x)$ then there is a point p in (a, b) such that $f(p) = \mu$.

Proof. Since f is continuous on the closed and bounded interval $[a, b]$, there is a point c in $[a, b]$ such that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and there is a point d in $[a, b]$ such that $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Without loss of generality, let us assume that $c < d$.

Therefore $[c, d] \subset [a, b]$ and f is continuous on $[c, d]$.

Since μ lies between $f(c)$ and $f(d)$, by the intermediate value theorem there is a point p in (c, d) such that $f(p) = \mu$.

Therefore $p \in (a, b)$ and the theorem is proved.

Theorem 8.5.7. Let $I = [a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be continuous on I. Then $f(I) = \{f(x) : x \in I\}$ is a closed and bounded interval.

Proof. Since f is continuous on I, f is bounded on I.

Let $M = \sup\limits_{x\in I} f(x), m = \inf\limits_{x\in I} f(x)$. Then there is a point c in I such that $f(c) = M$ and a point d in I such that $f(d) = m$.

Therefore $M \in f(I), m \in f(I)$ and $m \leq M$.

Case 1. $m = M$. In this case f is a constant and $f(I)$ reduces to the point m and $f(I)$ is the closed interval $[m, m]$.

Case 2. $m < M$. Let $J = [m, M]$. We prove that $J = f(I)$.

Let $p \in f(I)$. Then there is a point x_0 in I such that $f(x_0) = p$.

Since $M = \sup\limits_{x\in I} f(x)$, and $m = \inf\limits_{x\in I} f(x), m \leq f(x_0) \leq M$.

Thus $p \in f(I) \Rightarrow p \in [m, M]$ and therefore $f(I) \subset J$ ... ... (i)

Let $q \in J$ and $q \neq m, q \neq M$.

Since f is continuous on (c, d) (or (d, c)) and $f(d) < q < f(c)$, there is a point x_1 in (c, d) (or (d, c)) such that $f(x_1) = q$.

Therefore $q \in f(I)$. Also $m \in f(I)$ and $M \in f(I)$.

Thus $x \in J$ implies $x \in f(I)$ and therefore $J \subset f(I)$ ... ... (ii)

From (i) and (ii) it follows that $J = f(I)$. This completes the proof.

Note 1. The continuous image of a closed and bounded interval $[a, b]$ is the closed and bounded interval $[m, M]$. In particular, if f be a constant (and hence continuous) on $[a, b]$, the image reduces to a point.

Note 2. The continuous image of an open bounded interval may not be an open bounded interval. For example, let $I = (-1, 1)$ and $f(x) = x^2, x \in I$. Then f is continuous on I but $f(I) = [0, 1)$, which is not an open interval. Let $I = (0, 1)$ and $f(x) = x^2, x \in I$. Then $f(I) = (0, 1)$, an open interval.

Theorem 8.5.8. Let I be an interval and $f : I \to \mathbb{R}$ be continuous on I. Then $f(I)$ is an interval.

[A subset S of $\mathbb{R}$ is an interval if for any two points $c, d \in S$ with $c < d$, the closed interval $[c, d] \subset S$]

Proof. Let $p, q \in f(I)$ and $p < q$. There exist points c, d in I such that $f(c) = p, f(d) = q$.

Let $r \in (p, q)$. Then $p < r < q$.

By the intermediate value theorem, there exists a point x_0 in (c, d) [or (d, c)] such that $f(x_0) = r$.

Thus $r \in (p, q) \Rightarrow r \in f(I)$ and therefore $(p, q) \subset f(I)$.

Also $p \in f(I)$ and $q \in f(I)$. So $[p, q] \subset f(I)$.

This proves that $f(I)$ is an interval.

Worked Examples (continued).

5. A function $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$.

If $f(0) = 0$ or $f(1) = 1$, the existence is proved.

We assume $f(0) \neq 0$ and $f(1) \neq 1$.

Let us consider the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x, x \in [0, 1]$.

g is continuous on $[0, 1]$ and $g(0) = f(0) > 0$, since $f(0) \in [0, 1]$ and $f(0) \neq 0$. Also $g(1) = f(1) - 1 < 0$, since $f(1) \in [0, 1]$ and $f(1) \neq 1$.

By the Intermediate value theorem there exists a point c in $(0, 1)$ such that $g(c) = 0$. Therefore $f(c) = c$.

Note. c is said to be a *fixed point* of the continuous map f.

6. A function $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and assumes only rational values. If $f(\frac{1}{2}) = \frac{1}{2}$, prove that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$.

Let us take a point x_1 such that $0 \leq x_1 < \frac{1}{2}$ and consider the closed interval $[x_1, \frac{1}{2}]$. f is continuous on $[x_1, \frac{1}{2}]$.

Let $f(x_1) = p$. Then p is rational.

We prove that $p = \frac{1}{2}$. If not, let $p \neq \frac{1}{2}$.

Then f is continuous on $[x_1, \frac{1}{2}]$ and $f(x_1) \neq f(\frac{1}{2})$.

Let q be an irrational number lying between p and $\frac{1}{2}$.

By the Intermediate value theorem, $f(x) = q$ at some point c in $(x_1, \frac{1}{2})$ and this is a contradiction to the hypothesis that f assumes only rational values.

Hence $p = \frac{1}{2}$ and therefore $f(x) = \frac{1}{2}$ in $[0, \frac{1}{2})$.

Let us take a point x_2 such that $\frac{1}{2} < x_2 \leq 1$ and consider the closed interval $[\frac{1}{2}, x_2]$.

Proceeding with similar arguments we can prove $f(x) = \frac{1}{2}$ in $(\frac{1}{2}, 1]$.

Also $f(\frac{1}{2}) = \frac{1}{2}$. It follows that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$.

Theorem 8.5.9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then for every open subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in $\mathbb{R}$.

Proof. If $f^{-1}(G)$ be empty, then it is open in $\mathbb{R}$.

Let $f^{-1}(G)$ be non-empty and let $c \in f^{-1}(G)$. Then $f(c) \in G$.

Since G is an open set and $f(c) \in G$, $f(c)$ is an interior point of G and so there exists a positive ϵ such that $N(f(c), \epsilon) \subset G$.

Since f is continuous at c, there exists a $\delta > 0$ such that for all $x \in N(c, \delta)$, $f(x) \in N(f(c), \epsilon) \subset G$.

This implies $N(c, \delta) \subset f^{-1}(G)$. Hence c is an interior point of $f^{-1}(G)$.

Thus every point of $f^{-1}(G)$ is an interior point of $f^{-1}(G)$ and therefore $f^{-1}(G)$ is an open set.

Note. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ implies that the inverse image of any open subset is open in $\mathbb{R}$. The converse implication is also true.

We have the following theorem in this respect.

Theorem 8.5.10. A function $f : \mathbb{R} \to \mathbb{R}$ is such that $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

Proof. Let $c \in \mathbb{R}$. Then $f(c) \in \mathbb{R}$.

Let $\epsilon > 0$. Then the neighbourhood $G = (f(c) - \epsilon, f(c) + \epsilon)$ is an open set in $\mathbb{R}$ and by hypothesis, $f^{-1}(G)$ is an open set in $\mathbb{R}$.

$c \in f^{-1}(G)$, since $f(c) \in G$. Since $f^{-1}(G)$ is an open set, c is an interior point of $f^{-1}(G)$ and therefore there exists a positive δ such that the neighbourhood $N(c, \delta) \subset f^{-1}(G)$, i.e., for all $x \in N(c, \delta)$, $f(x) \in G$, i.e., $|f(x) - f(c)| < \epsilon$ for all $x \in N(c, \delta)$.

This proves that f is continuous at c. Since c is arbitrary, f is continuous on $\mathbb{R}$.

Note. We observe that a continuous function $f : \mathbb{R} \to \mathbb{R}$ pulls back open sets of $\mathbb{R}$ into open sets of $\mathbb{R}$. But f may not map an open set of $\mathbb{R}$ into an open set of $\mathbb{R}$. For example, let f be a constant function on $\mathbb{R}$, say

$f(x) = 2$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$, but the image of an open set, say $\{x \in \mathbb{R} : 0 < x < 1\}$ is the singleton set $\{2\}$ which is not an open set in $\mathbb{R}$.

Theorem 8.5.11. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if $f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$.

Proof. Let f be continuous on $\mathbb{R}$ and let F be a closed set in $\mathbb{R}$. Then $\mathbb{R} - F$ is open in $\mathbb{R}$ and $f^{-1}(\mathbb{R} - F)$ is open in $\mathbb{R}$, by Theorem 8.5.9.

Since $\mathbb{R} - f^{-1}(F) = f^{-1}(\mathbb{R} - F)$, it follows that $f^{-1}(F)$ is closed in $\mathbb{R}$.

Conversely, let $f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$. We shall prove that f is continuous on $\mathbb{R}$.

Let G be an open set in $\mathbb{R}$. Then $\mathbb{R} - G$ is closed in $\mathbb{R}$ and by hypothesis, $f^{-1}(\mathbb{R} - G)$ is closed in $\mathbb{R}$.

Since $\mathbb{R} - f^{-1}(G) = f^{-1}(\mathbb{R} - G)$, it follows that $\mathbb{R} - f^{-1}(G)$ is closed in $\mathbb{R}$ and therefore $f^{-1}(G)$ is open in $\mathbb{R}$.

f is continuous on $\mathbb{R}$, by Theorem 8.5.10.

Theorem 8.5.12. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$. Then the set $S = \{x \in \mathbb{R} : f(x) < g(x)\}$ is an open set in $\mathbb{R}$.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$, $x \in \mathbb{R}$. Then h is continuous on $\mathbb{R}$ and $S = \{x \in \mathbb{R} : h(x) < 0\}$.

Using the neighbourhood property of the function h and proceeding as in the worked Example 1, Art.8.5, the theorem can be established.

Another proof.

If $S = \phi$ then S is an open set.

Let $S \neq \phi$ and let $a \in S$. Then $f(a) < g(a)$. let us choose a real number b such that $f(a) < b < g(a)$.

Let us consider the open set $I = (b, \infty)$. Since g is continuous on $\mathbb{R}$, $g^{-1}(I)$ is an open set in $\mathbb{R}$. $a \in g^{-1}(I)$ and $g(x) > b$ for all $x \in g^{-1}(I)$.

Let us consider the open set $J = (-\infty, b)$. Since f is continuous on $\mathbb{R}$, $f^{-1}(J)$ is an open set in $\mathbb{R}$. $a \in f^{-1}(J)$ and $f(x) < b$ for all $x \in f^{-1}(J)$.

$f^{-1}(J) \cap g^{-1}(I)$ is an open set in $\mathbb{R}$ containing a and $f(x) < b < g(x)$ for all $x \in f^{-1}(J) \cap g^{-1}(I)$.

Therefore $f^{-1}(J) \cap g^{-1}(I) \subset S$. Thus $a \in S \Rightarrow f^{-1}(J) \cap g^{-1}(I) \subset S$. So a is an interior point of S. Therefore S is an open set.

This completes the proof.

Note. The set $S = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ is an open set in $\mathbb{R}$.

Theorem 8.5.13. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$. Then the set $S = \{x \in \mathbb{R} : f(x) = g(x)\}$ is a closed set in $\mathbb{R}$.

Proof. Let $A = \{x \in \mathbb{R} : f(x) \neq g(x)\}$. Then A is the complement of S in $\mathbb{R}$.

If $A = \phi$ then A is an open set.

Let $A \neq \phi$ and let $a \in A$. Then $f(a) \neq g(a)$. Let $|f(a) - g(a)| = k > 0$. Let $\epsilon = \frac{k}{3}$. Since f and g are continuous at a, there exists a positive δ such that $|f(x) - f(a)| < \epsilon$ and $|g(x) - g(a)| < \epsilon$ for all $x \in N(a, \delta)$.

$|f(a) - g(a)| \leq |f(x) - f(a)| + |f(x) - g(x)| + |g(x) - g(a)|$
or, $|f(x) - g(x)| \geq \epsilon$ for all $x \in N(a, \delta)$.

Therefore $N(a, \delta) \subset A$. Thus $a \in A \Rightarrow N(a, \delta) \subset A$. Therefore a is an interior point of A. So A is an open set in $\mathbb{R}$.

Consequently, S is a closed set in $\mathbb{R}$.

Corollary 1. If f and g are continuous on $\mathbb{R}$ and $f(x) = g(x)$ at all rational points, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Let $S = \{x \in \mathbb{R} : f(x) = g(x)\}$. Then by hypothesis, $\mathbb{Q} \subset S \subset \mathbb{R}$.
S is a closed subset in $\mathbb{R}$, by the theorem.
$\mathbb{Q} \subset S \Rightarrow \bar{\mathbb{Q}} \subset \bar{S} \Rightarrow \mathbb{R} \subset S$, since $\bar{\mathbb{Q}} = \mathbb{R}$ and $\bar{S} = S$.
Therefore $S = \mathbb{R}$, i.e., $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Corollary 2. If f is continuous on $\mathbb{R}$ and $f(x) = k$, a constant, at all rational points, then $f(x) = k$ for all $x \in \mathbb{R}$.

Considering the continuous function g defined by $g(x) = k$ for all $x \in \mathbb{R}$, this can be established.

Alternative proof of Intermediate value theorem.

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. If $f(a) \neq f(b)$ then for every real number r lying between $f(a)$ and $f(b)$ there is a point c in (a, b) such that $f(c) = r$.

Proof. Suppose on the contrary, there does not exist a point c in (a, b) such that $f(c) = r$.

Let us define a function $g : \mathbb{R} \to \mathbb{R}$ by
$$\begin{aligned} g(x) &= f(a), x \in (-\infty, a) \\ &= f(x), x \in [a, b] \\ &= f(b), x \in (b, \infty). \end{aligned}$$

Then g is continuous on $\mathbb{R}$ and $g = f$ on $[a, b]$.

Let $G_1 = (-\infty, r)$, $G_2 = (r, \infty)$. Then $\mathbb{R} = G_1 \cup \{r\} \cup G_2$.

G_1 and G_2 are open sets in $\mathbb{R}$. Since g is continuous on $\mathbb{R}$, $g^{-1}(G_1)$ and $g^{-1}(G_2)$ are both open sets in $\mathbb{R}$.

Since $g^{-1}(r) = \phi$, $g^{-1}(G_1) = \mathbb{R} - g^{-1}(G_2)$. Since $g^{-1}(G_2)$ is open, $g^{-1}(G_1)$ is closed. Thus $g^{-1}(G_1)$ is both open and closed in $\mathbb{R}$.

But $g^{-1}(G_1)$ is non-empty, since $a \in g^{-1}(G_1)$ and $g^{-1}(G_1) \neq \mathbb{R}$, since $b \notin g^{-1}(G_1)$.

So $g^{-1}(G_1)$ is neither $\mathbb{R}$ nor ϕ and at the same time $g^{-1}(G_1)$ is both open and closed. This is a contradiction, since the only subsets in $\mathbb{R}$ which are both open and closed are ϕ and $\mathbb{R}$.

Hence our assumption is wrong and the theorem is proved.

Definition. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy the *intermediate-value property* on $[a, b]$ if for every x_1, x_2 satisfying $a \le x_1 < x_2 \le b$ and for every k between $f(x_1)$ and $f(x_2)$ there exists a $c \in (x_1, x_2)$ such that $f(c) = k$.

Note. A function $f : [a, b] \to \mathbb{R}$ which satisfies the intermediate-value property on $[a, b]$ need not be continuous on $[a, b]$.

For example, the function f defined on $[-1, 1]$ by
$$f(x) = \sin \tfrac{1}{x}, x \neq 0$$
$$= 0, x = 0$$
is discontinuous on $[-1, 1]$ but it satisfies the intermediate-value property on $[-1, 1]$.

Theorem 8.5.14. If a function $f : [a, b] \to \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ then f has no simple discontinuity on $[a, b]$. I st kind,

Proof. Let f has a simple discontinuity at $c \in [a, b]$.

If c be an interior point of $[a, b]$ then $\lim\limits_{x \to c+} f(x)$ and $\lim\limits_{x \to c-} f(x)$ both exist finitely.

If $c = a$ then $\lim\limits_{x \to c+} f(x)$ exists finitely.

If $c = b$ then $\lim\limits_{x \to c-} f(x)$ exists finitely.

It is sufficient to prove that
(i) $\lim\limits_{x \to c+} f(x) = f(c)$ for all $c \in [a, b)$; (ii) $\lim\limits_{x \to c-} f(x) = f(c)$ for all $c \in (a, b]$.

Let $c \in [a, b)$ and let $\lim\limits_{x \to c+} f(x) = l$.

Case 1. Let $l < f(c)$. Let us choose a positive ϵ such that $l + \epsilon < f(c)$.

Since $\lim\limits_{x \to c+} f(x) = l$, there exists a positive δ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c + \delta) \cap [a, b]$.

Therefore $l - \epsilon < f(x) < l + \epsilon < f(c)$ for all $x \in (c, c + \delta) \cap [a, b]$... ...(i)

Let $x_1 \in (c, c + \delta) \cap [a, b]$. Then $l - \epsilon < f(x_1) < l + \epsilon < f(c)$, by (i).

Since f satisfies intermediate value property on $[c, x_1] \subset [a, b]$, there exists a point p in (c, x_1) such that $f(p) = l + \epsilon$.

But $p \in (c, x_1) \Rightarrow p \in (c, c+\delta) \cap [a, b] \Rightarrow f(p) < l + \epsilon$, a contradiction. Therefore $l \geq f(c)$.

Case 2. Let $l > f(c)$. Let us choose a positive ϵ such that $l - \epsilon > f(c)$.

Since $\lim\limits_{x \to c+} f(x) = l$, there exists a positive δ such that $l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c+\delta) \cap [a, b]$.

Therefore $f(c) < l - \epsilon < f(x) < l + \epsilon$ for all $x \in (c, c+\delta) \cap [a, b]$... ...(ii)

Let $x_2 \in (c, c+\delta) \cap [a, b]$. Then $f(c) < l - \epsilon < f(x_2) < l + \epsilon$, by (ii).

Since f satisfies intermediate value property on $[c, x_2] \subset [a, b]$, there exists a point q in (c, x_2) such that $f(q) = l - \epsilon$.

But $q \in (c, x_2) \Rightarrow q \in (c, c+\delta) \cap [a, b] \Rightarrow l - \epsilon < f(q)$, a contradiction. Therefore $l \leq f(c)$.

Combining the cases, we have $\lim\limits_{x \to c+} f(x) = f(c)$ for all $c \in [a, b)$.

Similarly, it can be proved that $\lim\limits_{x \to c-} f(x) = f(c)$ for all $c \in (a, b]$.

This completes the proof.

8.6. Monotone functions and continuity.

Theorem 8.6.1. Let $I = (a, b)$ be an interval. Let $f : I \to \mathbb{R}$ be monotone increasing on I. Then at any point $c \in I$,

(i) $f(c-0) = \sup\limits_{x \in (a,c)} f(x)$, **(ii)** $f(c+0) = \inf\limits_{x \in (c,b)} f(x)$,

(iii) $f(c-0) \leq f(c) \leq f(c+0)$.

Proof. (i) If $x \in I$ and $x < c$ then $f(x) \leq f(c)$.

Hence the set $\{f(x) : a < x < c\}$ is bounded above, $f(c)$ being an upper bound. The set, being non-empty, has a least upper bound, say u.

Then $u \leq f(c)$, and for a pre-assigned positive ϵ, there exists a point x_0 in (a, c) such that $u - \epsilon < f(x_0) \leq u$.

Let $x_0 = c - \delta, 0 < \delta < c - a$.

Since f is monotonic increasing on (a, c),

$u - \epsilon < f(x_0) \leq f(x) \leq u < u + \epsilon$ for all x in $x_0 < x < c$.

Consequently, $| f(x) - u | < \epsilon$ for all x in $x_0 < x < c$.

This implies that $\lim\limits_{x \to c-} f(x) = u$, i.e., $f(c-0) = u = \sup\limits_{x \in (a,c)} f(x)$

(ii) If $x \in I$ and $x > c$, then $f(x) \geq f(c)$.

Hence the set $\{f(x) : c < x < b\}$ is bounded below, $f(c)$ being a lower bound. The set being non-empty, has a greatest lower bound, say l.

Then $l \geq f(c)$, and for a pre-assigned positive ϵ, there exists a point x_1 in (c, b) such that $l \leq f(x_1) < l + \epsilon$.

Let $x_1 = c + \delta, 0 < \delta < b - c$.

Since f is monotone increasing on (c, b), for all x in $c < x < x_1$

$l - \epsilon < l \leq f(x) \leq f(x_1) < l + \epsilon$ for all x in $c < x < x_1$

Consequently, $| f(x) - l |< \epsilon$ for all x in $c < x < x_1 + \delta$

This implies that $\lim\limits_{x \to c+} f(x) = l$, i.e., $f(c + 0) = l = \inf\limits_{x \in (c,b)} f(x)$

(iii) We have $f(c - 0) = u \leq f(c)$ and $f(c + 0) = l \geq f(c)$.

Therefore $f(c - 0) \leq f(c) \leq f(c + 0)$.

Note. If $f : I \to \mathbb{R}$ be monotone decreasing on $I = (a, b)$ then at any point $c \in I$, **(i)** $f(c - 0) = \inf\limits_{x \in (a,c)} f(x)$, **(ii)** $f(c + 0) = \sup\limits_{x \in (c,b)} f(x)$,

(iii) $f(c - 0) \geq f(c) \geq f(c + 0)$.

Corollary 1. If $f : (a, b) \to \mathbb{R}$ be monotone on (a, b) then at every point $c \in (a, b), f(c-0)$ and $f(c+0)$ both exist. Therefore a monotone function f cannot have a discontinuity of the second kind in its domain.

Corollary 2. If f be monotone increasing on $I = [a, b]$ then for any two points $c, d \in I$ with $c < d, f(c + 0) \leq f(d - 0)$.

Theorem 8.6.2. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be monotone increasing on I. Then **(i)** $f(a + 0) = \inf\limits_{x \in (a,b)} f(x)$, **(ii)** $f(b - 0) = \sup\limits_{x \in (a,b)} f(x)$,

(iii) $f(a) \leq f(a + 0); f(b - 0) \leq f(b)$.

Proof left to the reader.

Theorem 8.6.3. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be monotone decreasing on I. Then **(i)** $f(a + 0) = \sup\limits_{x \in (a,b)} f(x)$, **(ii)** $f(b - 0) = \inf\limits_{x \in (a,b)} f(x)$,

(iii) $f(a) \geq f(a + 0)$; $f(b - 0) \geq f(b)$.

Proof left to the reader.

Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$ and $c \in (a, b)$.
Since $f(c - 0)$ and $f(c + 0)$ both exist,
the *jump* of f at c is defined by $J(c) = f(c + 0) - f(c - 0)$.

The jump at a is defined by $J(a) = f(a + 0) - f(a)$.

The jump at b is defined by $J(b) = f(b) - f(b - 0)$.

(i) If f be monotone increasing on I and $c \in I, J(c) \geq 0$.

(ii) If f be monotone decreasing on I and $c \in I, J(c) \leq 0$.

Theorem 8.6.4. If a function $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$ then the set of points of discontinuities of f in $[a, b]$ is a countable set.

Proof. First let us consider the case when f is monotone increasing on $[a, b]$. Let S be the set of all points of discontinuity of f. Since f is increasing on $[a, b]$, $J(x) \geq 0$ for all $x \in [a, b]$ and $S = \{x \in [a, b] : J(x) > 0\}$.

Let T be the set of points of discontinuity of f on (a, b).

For any two points c, d in T with $c < d$,

$f(c - 0) < f(c + 0) \leq f(d - 0) < f(d + 0)$.

Thus the open intervals $(f(c - 0), f(c + 0))$ and $(f(d - 0), f(d + 0))$ are disjoint.

Let us choose a rational point r_x in $(f(x - 0), f(x + 0))$ for every $x \in T$ and define a mapping $\phi : T \to \mathbb{Q}$ by $\phi(x) = r_x$ for all $x \in T$.

Then ϕ is injective because, for any two distinct elements x_1, x_2 in T the intervals $(f(x_1 - 0), f(x_1 + 0))$ and $(f(x_2 - 0), f(x_2 + 0))$ are disjoint and therefore $r_{x_1} \neq r_{x_2}$.

Since $\mathbb{Q}$ is an enumerable set, $\phi(T)$ being a proper subset of $\mathbb{Q}$ is countable. Since ϕ is injective and $\phi(T)$ is countable, T is countable. Therefore $T \cup \{a, b\}$ is also a countable set.

Thus S, being a subset of $T \cup \{a, b\}$, is a countable set.

The case when f is monotone decreasing on $[a, b]$ can be similarly dealt with and the theorem is done.

Corollary. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone on $\mathbb{R}$. Then the set of points of discontinuity of f is a countable set.

$\mathbb{R}$ can be considered as the union of an enumerable number of closed intervals $\{[0, 1] \cup [1, 2] \cup \cdots\} \cup \{[-1, 0] \cup [-2, -1] \cup \cdots\}$.

Theorem 8.6.5. If a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

Proof. $f(a) \neq f(b)$ since f is injective on $[a, b]$.

Case 1. $f(a) < f(b)$. Let $x_1 \in (a, b)$. We prove that $f(a) < f(x_1) < f(b)$.

If not, then either (i) $f(x_1) < f(a) < f(b)$, or (ii) $f(a) < f(b) < f(x_1)$.

If $f(x_1) < f(a) < f(b)$ then by the intermediate value theorem on the interval $[x_1, b]$, there is a point $x' \in (x_1, b)$ such that $f(x') = f(a)$ and this contradicts that f is injective on $[a, b]$.

If $f(a) < f(b) < f(x_1)$ then by the intermediate value theorem on the interval $[a, x_1]$, there is a point $x'' \in (a, x_1)$ such that $f(x'') = f(b)$ and this again contradicts that f is injective on $[a, b]$.

Therefore $f(a) < f(x_1) < f(b)$ when $a < x_1 < b$.

Let $x_2 \in (a, b)$ such that $a < x_1 < x_2 < b$.
Since $a < x_1 < x_2, f(a) < f(x_1) < f(x_2)$, by what we have proved.
Similarly, since $x_1 < x_2 < b, f(x_1) < f(x_2) < f(b)$.
Therefore $a < x_1 < x_2 < b \Rightarrow f(a) < f(x_1) < f(x_2) < f(b)$.

Hence f is strictly increasing on $[a, b]$.

Case 2. $f(a) > f(b)$. By similar arguments we can prove that f is strictly decreasing on $[a, b]$.

The theorem can be extended to hold on any kind of interval.

Theorem 8.6.5(a). Let I be an interval and $f : I \to \mathbb{R}$ is continuous and injective on I. Then f is strictly monotone on I.

Proof. Let $r, s \in I$ and $r < s$. Since f is injective on I, $f(r) \neq f(s)$.

Case 1. $f(r) < f(s)$.

Let c, d be arbitrary points in I with $c < d$. Let $a = \min\{c, r\}, b = \max\{d, s\}$. Then $[a, b]$ is a closed and bounded interval contained in I and containing the points r, s, c, d.

Since $[a, b] \subset I, f$ is continuous and injective on $[a, b]$ and by the Theorem 8.6.5, f is strictly monotone on $[a, b]$. Since $r, s \in [a, b]$ and $f(r) < f(s)$, f must be strictly increasing on $[a, b]$ and therefore $f(c) < f(d)$. As c, d are arbitrary points in I, f is strictly increasing on I.

Case 2. $f(r) > f(s)$.

In this case we can prove f is strictly decreasing on I.

Theorem 8.6.6. Let $I = [a, b]$ be closed and bounded interval and $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then there exists an inverse function $g : J \to \mathbb{R}$ where $J = f(I)$, such that

(i) g is strictly monotone on J and **(ii)** g is continuous on J.

We prove the theorem for the case when f is strictly increasing on I. The proof for the other case (when f is strictly decreasing) is similar.

Proof. f is strictly increasing on I. Since f is continuous on I, f is bounded on I. Since f is strictly increasing on $I, \sup f = f(b)$ and $\inf f = f(a)$. Therefore $J = [f(a), f(b)]$.

Since f is strictly increasing on I, $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) < f(x_2)$. So $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

This proves that f is injective on I. Also since $f(I) = J, f$ is surjective.

Therefore f is bijective and hence there exists an inverse function $g : J \to I$ such that $x \in I$ and $f(x) = y \Leftrightarrow y \in J$ and $g(y) = x$.

First we prove that g is strictly increasing on J.

Let $y_1, y_2 \in J$ with $y_1 < y_2$. Then there exist x_1, x_2 in I such that $y_1 = f(x_1), y_2 = f(x_2)$ and since f is strictly increasing on $I, y_1 < y_2 \Rightarrow x_1 < x_2$. That is, $y_1 < y_2$ in $J \Rightarrow g(y_1) < g(y_2)$ in I.

This proves that g is strictly increasing on J.

To prove that g is continuous on J, let $d \in J$ and $d = f(c)$. Then $g(d) = c$.

Let $\{y_n\}$ be a sequence in J converging to d.

Let $f(x_n) = y_n$, $n = 1, 2, 3, ...$

Then $g(y_n) = x_n$ and $\{x_n\}$ is a sequence in I. I being a bounded set, the sequence $\{x_n\}$ is a bounded sequence and therefore it must have a convergent subsequence.

Let $\{x_{r_n}\}$ be a convergent subsequence of the sequence $\{x_n\}$. I being a closed set, $\lim x_{r_n} \in I$.

The continuity of f at $\lim x_{r_n}$ gives $f(\lim x_{r_n}) = \lim f(x_{r_n}) = \lim y_{r_n}$.

But $\lim y_{r_n} = d$, since $\lim y_n = d$. Therefore $f(\lim x_{r_n}) = d = f(c)$.

Since f is injective on I, it follows that $\lim x_{r_n} = c$.

Thus every convergent subsequence of $\{x_n\}$ converges to c.

It follows that $\underline{\lim}\ x_n = \overline{\lim}\ x_n = c$ and therefore $\lim\ x_n = c$, i.e., $\lim\ g(y_n) = g(d)$.

Thus every sequence $\{y_n\}$ in J converging to d, the sequence $\{g(y_n)\}$ converges to $g(d)$. This proves that g is continuous at d.

Since $d \in J$, it follows that g is continuous on J.

This completes the proof.

Note. The theorem can be extended to any kind of interval. If I be an interval and $f : I \to \mathbb{R}$ be strictly monotone and continuous on I then the inverse function f^{-1} is continuous and strictly monotone on $J[= f(I)]$.

Theorem 8.6.7. If a function $f : [a, b] \to \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

Proof. In the proof of the Theorem 8.6.5, the injectivity of f on $[a, b]$ and the intermediate-value property of f on $[a, b]$ were only utilised. Therefore it follows from the proof of the Theorem 8.6.5 that f is strictly monotone on $[a, b]$.

Theorem 8.6.8. If $f : [a, b] \longrightarrow \mathbb{R}$ satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is continuous on $[a, b]$.

Proof. Since f satisfies the intermediate-value property on $[a, b]$ and f is injective on $[a, b]$ then f is strictly monotone on $[a, b]$, by the Theorem 8.6.7. Since f is strictly monotone on $[a, b]$, f cannot have a discontinuity of the second second kind in $[a, b]$, by the Corollary 1 of the Theorem 8.6.1 and since f satisfies the intermediate-value property on $[a, b]$, f has no discontinuity of the first kind on $[a, b]$ by the Theorem 8.5.10.

Therefore f is continuous on $[a, b]$.

Examples.

1. The exponential function and its inverse.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = e^x, x \in \mathbb{R}$. The range of f is $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. f is continuous and strictly increasing on $\mathbb{R}$.

Hence there exists an inverse function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that g is continuous and strictly increasing on $\mathbb{R}^+$.

g is defined by $g(y) = \log y, y \in \mathbb{R}^+$. The range of g is $\mathbb{R}$.

$gf(x) = x$ for all $x \in \mathbb{R}$, i.e., $\log(e^x) = x$ for all $x \in \mathbb{R}$, and

$fg(y) = y$ for all $y \in \mathbb{R}^+$, i.e., $e^{\log y} = y$ for all $y \in \mathbb{R}^+$.

g is called the *logarithm function.*

2. The nth power function and its inverse.

Case 1. Let n be an even positive integer and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. The range of f is $[0, \infty)$.

f is not injective on $\mathbb{R}$, but f is continuous on $\mathbb{R}$ and strictly increasing on $[0, \infty)$

Let $I = [0, \infty)$ and let $f : I \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n, x \in I$.

Then f is continuous and strictly increasing on I and $f(I) = [0, \infty)$.

Hence there exists an inverse function $g : [0, \infty) \longrightarrow [0, \infty)$ such that g is continuous and strictly increasing on $[0, \infty)$.

g is defined by $g(y) = \sqrt[n]{y}, y \in [0, \infty)$.
$gf(x) = x$ for all $x \in [0, \infty)$, i.e., $\sqrt[n]{x^n} = x$ for all $x \in [0, \infty)$, and
$fg(y) = y$ for all $y \in [0, \infty)$, i.e., $(\sqrt[n]{y})^n = y$ for all $y \in [0, \infty)$.

g is called the *nth root function* (n even) and it is defined on $[0, \infty)$.

Case 2. Let n be an odd positive integer.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n, x \in \mathbb{R}$. Then f is continuous and srtictly increasing on $\mathbb{R}$. The range of f is $\mathbb{R}$.

Hence there exists an inverse function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and strictly increasing on $\mathbb{R}$.

g is defined by $g(y) = \sqrt[n]{y}$, $y \in \mathbb{R}$.

g is called the *nth root function* (n odd) and it is defined on $\mathbb{R}$.

3. Sine function and its inverse.

Let $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ be defined by $f(x) = \sin x, x = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then f is strictly increasing and continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of f is $[-1, 1]$.

Hence there exists an inverse function $g : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that g is continuous and strictly increasing on $[-1, 1]$.
g is defined by $g(y) = \sin^{-1} y, y \in [-1, 1]$.

g is called the *principal inverse sine function.*

4. Cosine function and its inverse.

Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \cos x, x \in [0, \pi]$. Then f is continuous and strictly decreasing on $[0, \pi]$. The range of f is $[-1, 1]$.

Hence there exists an inverse function $g : [-1, 1] \to [0, \pi]$ such that g is continuous and strictly decreasing on $[-1, 1]$.
g is defined by $g(y) = \cos^{-1} y, y \in [-1, 1]$. The range of g is $[0, \pi]$.

g is called the *principal inverse cosine function.*

5. Tangent function and its inverse.

Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then f is continuous and strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. The range of f is $\mathbb{R}$.
Hence there exists an inverse function $g : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ such that g is continuous and strictly increasing on $\mathbb{R}$.
g is defined by $g(y) = \tan^{-1} y, y \in \mathbb{R}$. The range of g is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

g is called the *principal inverse tangent function.*

6. Cotangent function and its inverse.

Let $f : (0, \pi) \to \mathbb{R}$ be defined by $f(x) = \cot x, x \in (0, \pi)$. Then f is continuous and strictly increasing on $(0, \pi)$. The range of f is $\mathbb{R}$.
Hence there exists an inverse function $g : \mathbb{R} \to (0, \pi)$ such that g is continuous and strictly decreasing on $\mathbb{R}$.
g is defined by $g(y) = \cot^{-1} y, y \in \mathbb{R}$. The range of g is $(0, \pi)$.

g is called the *principal inverse cotangent function.*

8.7. Uniform continuity.

Let I be an interval and a function $f : I \to \mathbb{R}$ be continuous on I. Let $c \in I$. Then for a pre-assigned positive ϵ there exists a positive δ such that for all $x \in N(c, \delta) \cap I$, $|f(x) - f(c)| < \epsilon$.

If we move to another point $c' \in I$ and keep the same ϵ fixed then it may happen that the same δ does not work but a smaller δ may be necessary for c' to fulfil the requirement of the condition for continuity.

Thus δ depends not only on ϵ but also on the point c and therefore δ can be expressed as $\delta(\epsilon, c)$. Let $\delta_0 = \inf\{\delta(\epsilon, c) : c \in I\}$. $\delta_0 \geq 0$ since $\delta(\epsilon, c) > 0$ for all $c \in I$.

If $\delta_0 > 0$, then for all $c \in I$ and $x \in N(c, \delta_0) \cap I$, $|\ f(x) - f(c)\ | < \epsilon$. That is, δ_0 works uniformly over the entire interval I in the sense that for any two points $x_1, x_2 \in I$ satisfying $|x_1 - x_2| < \delta_0$, $|f(x_1) - f(x_2)| < \epsilon$ holds. In this case f is said to be *uniformly continuous* on I.

Every function continuous on an interval I may not be uniformly continuous on I, because a positive δ_0 as $\inf\{\delta(\epsilon, c) : c \in I\}$ may not be available.

Definition. A function $f : I \to \mathbb{R}$ is said to be *uniformly continuous* on I if corresponding to a pre-assigned positive ϵ there exists a positive δ such that for any two points x_1, x_2 in I

$$|\ x_2 - x_1\ | < \delta \Rightarrow |\ f(x_2) - f(x_1)\ | < \epsilon.$$

Note 1. The definition of uniform continuity shows that uniform continuity is a property of the function on an interval (or on a set) but continuity is a property of the function at a point. This is expressed by saying that continuity of a function is a *local property* while uniform continuity of a function is a *global property.*

2. It follows from the definition of uniform continuity that if a function f be uniformly continuous on an interval I, then it is also uniformly continuous on any subinterval $I_1 \subset I$.

Worked Examples.

1. Show that the function f defined by $f(x) = \frac{1}{x}$, $x \in [1, \infty)$ is uniformly continuous on $[1, \infty)$.

Let $c \geq 1$. Then for all $x \geq 1$,
$|\ f(x) - f(c)\ | = |\ \frac{1}{x} - \frac{1}{c}\ | = |\ \frac{x-c}{cx}\ | \leq |\ x - c\ |$, since $|\ cx\ | \geq 1$.

Let us choose $\epsilon > 0$. Then for all $x \geq 1$, satisfying $|\ x - c\ | < \epsilon$, $|\ f(x) - f(c)\ | < \epsilon$, whatever $c (\geq 1)$ may be.

This shows that f is uniformly continuous on $[1, \infty)$.

2. Show that the function f defined by $f(x) = \sin x, x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Let $c \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} | f(x) - f(c) | = | \sin x - \sin c | &= 2 \left| \sin \tfrac{x-c}{2} \right| \left| \cos \tfrac{x+c}{2} \right| \\ &\leq 2 \left| \sin \tfrac{x-c}{2} \right| \\ &\leq 2 . \tfrac{|x-c|}{2}, \text{ since } | \sin x | \leq | x | \\ &\qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Let us choose $\epsilon > 0$. Then for all $x \in \mathbb{R}$, satisfying $| x - c | < \epsilon$, $| f(x) - f(c) | < \epsilon$, whatever $c (\in \mathbb{R})$ may be.
This shows that f is uniformly continuous on $\mathbb{R}$.

3. Let $f(x) = x^2, x \in \mathbb{R}$. Show that f is uniformly continuous on any closed interval $[a, b], a \geq 0$; but f is not uniformly continuous on $[a, \infty), a \geq 0$.

First part. Let us choose $\epsilon > 0$. f will be uniformly continuous on $[a, b]$ if we can find a $\delta > 0$ such that for any two points x_1, x_2 in $[a, b]$, $| x_2 - x_1 | < \delta \Rightarrow | f(x_2) - f(x_1) | < \epsilon$.

$| f(x_2) - f(x_1) | = | x_2^2 - x_1^2 | = | x_2 - x_1 | | x_2 + x_1 | < 2b | x_2 - x_1 |$, since $0 \leq x_1 \leq b, 0 \leq x_2 \leq b$.

If we choose $\delta = \frac{\epsilon}{2b}$, then for any two points x_1, x_2 in $[a, b]$ satisfying $| x_2 - x_1 | < \delta$, the inequality $| f(x_2) - f(x_1) | < \epsilon$ holds.

This shows that f is uniformly continuous on $[a, b], a \geq 0$.

Second part. Let us choose $\epsilon > 0$. Then for any two points x_1, x_2 in $[a, b]$ satisfying $| x_2 - x_1 | < \delta$, the inequality $| f(x_2) - f(x_1) | < \epsilon$ will hold if we choose $\delta = \frac{\epsilon}{2b}$.

But as b takes larger and larger values, δ gets smaller and smaller. So it is not possible to find a *single* positive δ which will work for *all* $b > a$. It follows that f is not uniformly continuous on $[a, \infty), a \geq 0$.

Theorem 8.7.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is continuous on I.

Proof. Since f is uniformly continuous on I, for a pre-asigned positive ϵ there exists a positive δ such that for any two points x_1, x_2 in I,

$$| x_2 - x_1 | < \delta \Rightarrow | f(x_2) - f(x_1) | < \epsilon.$$

Let $c \in I$. Taking $x_1 = c$, the condition yields
$| f(x_2) - f(c) | < \epsilon$ for all $x_2 \in I$ satisfying $| x_2 - c | < \delta$,
i.e., $| f(x) - f(c) | < \epsilon$ for all $x \in I$ satisfying $| x - c | < \delta$.
This proves that f is continuous at c.
Since c is arbitrary, f is continuous on I. This completes the proof.

Theorem 8.7.2. Let $I = [a, b]$ be a closed and bounded interval and and a function $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Proof. If possible, let f be not uniformly continuous on I.

Then there exists a positive ϵ_0 for which no positive δ will work, i.e., for each positive δ there exist points x, y in $[a, b]$ such that $| x - y | < \delta$ but $| f(x) - f(y) | \not< \epsilon_0$.

Let $\delta = 1$. Then there exist points x_1, y_1 in $[a, b]$ such that
$| x_1 - y_1 | < 1$ but $| f(x_1) - f(y_1) | \geq \epsilon_0$.
Let $\delta = \frac{1}{2}$. Then there exist points x_2, y_2 in $[a, b]$ such that
$| x_2 - y_2 | < \frac{1}{2}$ but $| f(x_2) - f(y_2) | \geq \epsilon_0$.
... ...

Thus we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that
$| x_n - y_n | < \frac{1}{n}$ but $| f(x_n) - f(y_n) | \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Since $a \leq x_n \leq b, a \leq y_n \leq b$, both the sequences $\{x_n\}$ and $\{y_n\}$ are bounded sequences.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\{x_n\}$, say $\{x_{r_n}\}$ and let l be the limit of the subsequence $\{x_{r_n}\}$. Since $[a, b]$ is a closed interval, $l \in [a, b]$.

Let us consider the subsequence $\{y_{r_n}\}$ of the sequence $\{y_n\}$.

Since $| x_{r_n} - y_{r_n} | < \frac{1}{r_n}$ for all $n \in \mathbb{N}$ and since $\lim \frac{1}{r_n} = 0$ and $\lim x_{r_n} = l$, it follows that the subsequence $\{y_{r_n}\}$ converges to l.

Since $l \in [a, b], f$ is continuous at l.

Since $\{x_{r_n}\}$ converges to l and f is continuous at $l, \lim f(x_{r_n}) = f(l)$.
Since $\{y_{r_n}\}$ converges to l and f is continuous at $l, \lim f(y_{r_n}) = f(l)$.

Thus both the sequences $\{f(x_{r_n})\}$ and $\{f(y_{r_n})\}$ converge to a common limit. But this is contradicted by the condition $| f(x_{r_n}) - f(y_{r_n}) | \geq \epsilon_0$ for all $n \in \mathbb{N}$.

So our assumption that f is not uniformly continuous on $[a, b]$ is not tenable. Therefore f is uniformly continuous on $[a, b]$ and this completes the proof.

Theorem 8.7.3. Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Proof. Since f is uniformly continuous on D, for a pre-assigned positive ϵ there exists a positive δ such that for every pair of points x', x'' in D satisfying $| x' - x'' | < \delta$, $| f(x') - f(x'') | < \epsilon$ in $\mathbb{R}$.

Since $\{x_n\}$ is a Cauchy sequence, there exists a natural number k such that $| x_m - x_n | < \delta$ for all $m, n > k$.

It follows that for all $m, n > k$, $| f(x_m) - f(x_n) | < \epsilon$ in $\mathbb{R}$. This shows that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

This completes the proof.

Note. If $f : D \to \mathbb{R}$ be continuous on D but not uniformly continuous on D and $\{x_n\}$ be a Cauchy sequence in D then $\{f(x_n)\}$ may not be a Cauchy sequence in $\mathbb{R}$.

For example, let $f(x) = \frac{1}{x}$, $x \in (0, 1]$.

Then f is continuous on $(0, 1]$. Let us consider the sequence $\{x_n\}$ in $(0, 1]$ where $x_n = \frac{1}{n}, n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in $(0, 1]$. But $\{f(x_n)\} = \{1, 2, 3, 4, \ldots\}$. This is not a Cauchy sequence in $\mathbb{R}$.

Theorem 8.7.4. Let I be a bounded interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is bounded on I.

Proof. Let us assume that f is not bounded on I. Then there is a sequence $\{x_n\}$ in I such that $| f(x_n) | > n$ for $n = 1, 2, 3, \ldots$

Since $\{x_n\}$ is a sequence in a bounded interval I, it is a bounded sequence and therefore it has a convergent subsequence, say $\{x_{r_n}\}$ in I. Since $\{x_{r_n}\}$ is a convergent sequence in I, it is a Cauchy sequence in I.

Since f is uniformly continuous on $I, \{f(x_{r_n})\}$ must be a Cauchy sequence in $\mathbb{R}$. But by construction, $| f(x_{r_n}) | > r_n > n$, for $n = 1, 2, 3, \ldots$ and this shows that $\{f(x_{r_n})\}$ can not be a Cauchy sequence and we arrive at a contradiction. This proves that f is bounded on I.

Note. If I be a bounded interval and $f : I \to \mathbb{R}$ be continuous on I, then f may not be bounded on I. For example, let $f(x) = \frac{1}{x}, x \in (0, 1)$. Then f is continuous on the bounded interval $(0, 1)$ but f is not bounded on $(0, 1)$.

If I be a bounded interval and f is continuous and bounded on I, f may not be uniformly continuous on I. For example, let $f(x) = \sin \frac{1}{x}, x \in (0, 1)$. f is continuous on $(0, 1)$ and bounded on $(0, 1)$. But f is not uniformly continuous on $(0, 1)$.

Worked Examples (continued).

4. Prove that the function $f(x) = \sin \frac{1}{x}$, $x \in (0, 1)$ is not uniformly continuous on $(0, 1)$.

Let us assume that f is uniformly continuous on $(0, 1)$. Then for every Cauchy sequence $\{x_n\}$ in $(0, 1)$, the sequence $\{f(x_n)\}$ must be a Cauchy sequence in $\mathbb{R}$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{2}{n\pi}, n \in \mathbb{N}$. This is a Cauchy sequence in $(0, 1)$. The sequence $\{f(x_n)\}$ is $\{1, 0, -1, 0, \ldots \ldots\}$.

This is a divergent sequence and therefore this is not a Cauchy sequence in $\mathbb{R}$. Therefore f is not uniformly continuous on $(0, 1)$.

5. Prove that the function $f(x) = \frac{1}{x}$, $x \in (0, 1)$ is not uniformly continuous on $(0, 1)$.

Let us assume that f is uniformly continuous on $(0, 1)$. Then for every Cauchy sequence $\{x_n\}$ in $(0, 1)$, the sequence $\{f(x_n)\}$ must be a Cauchy sequence in $\mathbb{R}$.

Let us consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n+1}, n \in \mathbb{N}$. This is a Cauchy sequence in $(0, 1)$. The sequence $\{f(x_n)\}$ is $\{2, 3, 4, \ldots\}$. This is not a Cauchy sequence in $\mathbb{R}$. Therefore f is not uniformly continuous on $(0, 1)$.

Note. This example shows that a function continuous on an open bounded interval may not be uniformly continuous on that interval.

The following theorem gives a necessary and sufficient condition under which a function continuous on an open bounded interval will be uniformly continuous on that interval.

Theorem 8.7.5. Let a function f be continuous on an open bounded interval (a, b). Then f is uniformly continuous on (a, b) if and only if $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Proof. Let f be continuous on an open bounded interval (a, b) and let $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Let us define a function g on $[a, b]$ by $g(x) = f(x)$, for all $x \in (a, b)$ and $g(a) = \lim\limits_{x \to a+} f(x)$, $g(b) = \lim\limits_{x \to b-} f(x)$.

g is continuous on (a, b), since f is continuous on (a, b).

$g(a) = \lim\limits_{x \to a+} f(x)$ (by definition) $= \lim\limits_{x \to a+} g(x)$ and

$g(b) = \lim\limits_{x \to b-} f(x)$ (by definition) $= \lim\limits_{x \to b-} g(x)$.

Therefore g is right continuous at a and left continuous at b and consequently, g is continuous on $[a, b]$.

By Theorem 8.7.2, g is uniformly continuous on $[a, b]$.

By Theorem 8.7.1, g is uniformly continuous on (a, b). Since $g = f$ on (a, b), it follows that f is uniformly continuous on (a, b).

Conversely, let f be uniformly continuous on (a, b). We prove that both the limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ exist finitely.

Let $\{x_n\}$ be a sequence in (a, b) converging to a. Then $\{x_n\}$ is a Cauchy sequence in (a, b). Since f is uniformly continuous on (a, b), the sequence $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Let $\lim_{n\to\infty} f(x_n) = l$.

Let $\{y_n\}$ be another sequence in (a, b) converging to a. Then the sequence $\{x_n - y_n\}$ is a sequence in (a, b) converging to 0.

Let $\epsilon > 0$. Since f is uniformly continuous on (a, b), there exists a positive δ such that for any two points $x_1, x_2 \in (a, b)$

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \tfrac{\epsilon}{2}.$$

Since $\{x_n - y_n\}$ is a sequence in (a, b) converging to 0, there exists a natural number k such that $|x_n - y_n| < \delta$ for all $n \geq k$.

Therefore $|f(x_n) - f(y_n)| < \frac{\epsilon}{2}$ for all $n \geq k$.

$|f(y_n) - l| \leq |f(y_n) - f(x_n)| + |f(x_n) - l| < \epsilon$ for all $n \geq k$. This proves that $\lim_{n\to\infty} f(y_n) = l$.

Thus for every sequence $\{x_n\}$ in (a, b) converging to a, the sequence $\{f(x_n)\}$ converges to the limit l. This implies $\lim_{x\to a+} f(x) = l$.

In a similar manner it can be proved that $\lim_{x\to b-} f(x)$ exists finitely.

This completes the proof.

Definition. Let a function f be continuous on an interval I. A function g is said to be a *continuous extension* of f to $\mathbb{R}$ if g be continuous on $\mathbb{R}$ and $g(x) = f(x)$ for all $x \in I$.

If a function f be continuous on a closed and bounded interval $[a, b]$, then the function g defined on $\mathbb{R}$ by

$$\begin{aligned} g(x) &= f(a), \text{ for } x < a \\ &= f(x), \text{ for } x \in [a, b] \\ &= f(b), \text{ for } x > b \end{aligned}$$

is clearly a continuous extension of f to $\mathbb{R}$.

If a function f be continuous on an open interval (a, b), then f may not have a continuous extension to $\mathbb{R}$.

The following theorem specifies the conditions under which a function f continuous on an open bounded interval (a, b) may have a continuous extension to $\mathbb{R}$.

Theorem 8.7.6. Let a function f be continuous on an open bounded interval (a, b). Then f admits of a continuous extension to $\mathbb{R}$ if and only if f be uniformly continuous on (a, b).

Proof. Let f be uniformly continuous on (a, b). Then both the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist finitely.

Let $\lim_{x\to a+} f(x) = l$ and $\lim_{x\to b-} f(x) = m$.

Let us define a function g on $\mathbb{R}$ by $g(x) = l$, for $x \le a$

$= f(x)$, for $x \in (a, b)$

$= m$, for $x \ge b$.

Since $g(x) = f(x)$ for all $x \in (a, b)$, $l = \lim_{x\to a+} f(x) = \lim_{x\to a+} g(x)$ and $m = \lim_{x\to b-} f(x) = \lim_{x\to b-} g(x)$.

But by definition, $l = g(a)$ and $m = g(b)$. Therefore g is right continuous at a and left continuous at b and consequently, g is continuous at a and continuous at b.

Since f is uniformly continuous on (a, b), f is continuous on (a, b) and since $g = f$ on (a, b), g is continuous on (a, b).

Also by definition, g is continuous on $(-\infty, a)$ and on (b, ∞).

Consequently, g is continuous on $\mathbb{R}$.

Conversely, let f be continuous on an open bounded interval (a, b) and g be a continuous extension of f to $\mathbb{R}$. Then g is continuous on $\mathbb{R}$ and $g(x) = f(x)$ for all $x \in (a, b)$.

Since g is continuous on $\mathbb{R}$, g is continuous at a.

Therefore $\lim_{x\to a+} g(x) = \lim_{x\to a-} g(x) = g(a)$.

Since $f(x) = g(x)$ for all $x \in (a, b)$, $g(a) = \lim_{x\to a+} g(x) = \lim_{x\to a+} f(x)$.

This shows that the limit $\lim_{x\to a+} f(x)$ exists finitely.

Since g is continuous at b, it can be shown that the limit $\lim_{x\to b-} f(x)$ exists finitely. Consequently, f is uniformly continuous on (a, b). This completes the proof.

Worked Examples (continued).

6. If $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, prove that f is uniformly continuous on $[0, \infty)$.

Let $\epsilon > 0$. Then there exists a real number $p > 0$ such that $|f(x)| < \frac{\epsilon}{2}$ for all $x \ge p$ ... ... (i)

Since f is continuous at p, there exists a positive δ_1 such that $|f(x) - f(p)| < \frac{\epsilon}{2}$ for all x satisfying $|x - p| < \delta_1$ ... ... (ii)

Since f is continuous on $[0, p]$, it is uniformly continuous on $[0, p]$. Hence there exists a positive δ_2 such that for all $x_1, x_2 \in [0, p]$ with $|x_1 - x_2| < \delta_2$, $|f(x_1) - f(x_2)| < \epsilon$ ... ... (iii)

Let $\delta = \min\{\delta_1, \delta_2\}$ and let $a, b \in [0, \infty)$ with $|a - b| < \delta$.

Case (i). Let $a, b \in [p, \infty)$. Then by (i), $|f(a)| < \frac{\epsilon}{2}$ and $|f(b)| < \frac{\epsilon}{2}$; and therefore $|f(a) - f(b)| < |f(a)| + |f(b)| < \epsilon$.

Case (ii). Let $a, b \in [0, p]$. Then $|a - b| < \delta \Rightarrow |a - b| < \delta_2$ and by (iii), $|f(a) - f(b)| < \epsilon$.

Case (iii). Let $a \in [0, p], b \in [p, \infty)$ with $|a - b| < \delta$.
$|a - b| < \delta \Rightarrow |a - p| < \delta_1, |b - p| < \delta_1$. Then $|f(a) - f(p)| < \frac{\epsilon}{2}, |f(b) - f(p)| < \frac{\epsilon}{2}$ by (ii) and therefore $|f(a) - f(b)| < |f(a) - f(p)| + |f(b) - f(p)| < \epsilon$.

Therefore we have $|f(a) - f(b)| < \epsilon$, whenever $a, b \in [0, \infty)$ with $|a - b| < \delta$. This proves that f is uniformly continuous on $[0, \infty)$.

Q 7. If $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$, prove that f is uniformly continuous on $\mathbb{R}$.

Let f be continuous at a point $c \in \mathbb{R}$.
Let us choose $\epsilon > 0$. There exists a positive δ such that
$| f(c + h) - f(c) | < \epsilon$ for all h satisfying $| h | < \delta$.
But $| f(c + h) - f(c) | = | f(c) + f(h) - f(c) | = | f(h) |$.
Continuity of f at c implies $| f(h) | < \epsilon$ for all h satisfying $| h | < \delta$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $| x_1 - x_2 | < \delta$.
Then $| f(x_1 - x_2) | < \epsilon$.

$f(x + y) = f(x) + f(y)$ gives $f(0 + 0) = f(0) + f(0)$
or, $f(0) = 2f(0)$ or, $f(0) = 0$.
Also $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$.
Therefore $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
$| f(x_1 - x_2) | = | f(x_1) + f(-x_2) | = | f(x_1) - f(x_2) |$.

Thus $| f(x_1) - f(x_2) | < \epsilon$ for any two points x_1, x_2 in $\mathbb{R}$ satisfying $| x_1 - x_2 | < \delta$. δ depends on ϵ only and not on the points x_1, x_2 in $\mathbb{R}$. This proves that f is uniformly continuous on $\mathbb{R}$.

Q 8. Let A be a non-empty subset of $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f_A(x) = \inf\{| x - a | : a \in A\}$. Prove that f_A is uniformly continuous on $\mathbb{R}$.

Let $x_1, x_2 \in \mathbb{R}$. $f_A(x_1) = \inf\{| x_1 - a | : a \in A\}$ and $f_A(x_2) = \inf\{| x_2 - a | : a \in A\}$.
$| x_1 - a | \leq | x_1 - x_2 | + | x_2 - a |$ for all $a \in A$.
This implies $\inf\{| x_1 - a | : a \in A\} \leq | x_1 - x_2 | + \inf\{| x_2 - a | : a \in A\}$
or, $f_A(x_1) \leq | x_1 - x_2 | + f_A(x_2)$
or, $f_A(x_1) - f_A(x_2) \leq | x_1 - x_2 |$.

Similarly, $f_A(x_2) - f_A(x_1) \leq | x_1 - x_2 |$.

We have, $-\mid x_1 - x_2 \mid \leq f_A(x_2) - f_A(x_1) \leq \mid x_1 - x_2 \mid$
or, $\mid f_A(x_1) - f_A(x_2) \mid \leq \mid x_1 - x_2 \mid$.

Let $\epsilon > 0$. Then $\mid f_A(x_1) - f_A(x_2) \mid < \epsilon$ for all $x_1, x_2 \in \mathbb{R}$ satisfying $\mid x_1 - x_2 \mid < \epsilon$. This proves that f_A is uniformly continuous on $\mathbb{R}$.

~~Note.~~ Since f_A is uniformly continuous on $\mathbb{R}$, f_A is continuous on $\mathbb{R}$. The set $\{x \in \mathbb{R} : f_A(x) = 0\}$ is a closed set. (worked Ex.4, Page 277).

The set $\{x \in \mathbb{R} : f_A(x) = 0\} = \bar{A}$, by worked Ex.1, Page 65. Therefore if A be a non-empty closed set in $\mathbb{R}$, then the continuous function f_A defined by $f_A(x) = \inf\{\mid x - a \mid : a \in A\}$ is such that the set $\{x \in \mathbb{R} : f_A(x) = 0\} = A$.

Thus for a given closed set $A \subset \mathbb{R}$ there always exists a continuous function f on $\mathbb{R}$ such that $f(x) = 0$ on A.

Lipschitz function.

Definition. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on I if there exists a positive real number M such that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$ for any two points $x_1, x_2 \in I$.

In this case f is also said to be a *Lipschitz function* on I.

For example, let $f(x) = x^2$, $x \in [0, 2]$. Then
$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| \leq 4|x_1 - x_2|$ for all $x_1, x_2 \in [0, 2]$.
Therefore f satisfies Lipschitz condition with $M = 4$ on $[0, 2]$.
$M \in [4, \infty)$

Theorem 8.7.7. Let $f : I \to \mathbb{R}$ be a Lipschitz function on I. Then f is uniformly continuous on I.

Proof. Since f is a Lipschitz function on I, there exists a positive real number k such that $\mid f(x_1) - f(x_2) \mid \leq k \mid x_1 - x_2 \mid$ for all $x_1, x_2 \in I$.

Let $\epsilon > 0$. Then for all points x_1, x_2 in I satisfying $\mid x_1 - x_2 \mid < \frac{\epsilon}{k}$, $\mid f(x_1) - f(x_2) \mid < k.\frac{\epsilon}{k} = \epsilon$.

This proves that f is uniformly continuous on I.

Worked Example (continued).

9. Let $f(x) = \log x$, $x \in (0, \infty)$. Show that f is uniformly continuous on $[a, \infty)$, where $a > 0$.

Let $x_1, x_2 \in [a, \infty)$, where $a > 0$.
If $x_1 < x_2$ then $\frac{x_2}{x_1} > 1$ and therefore

$$\begin{aligned} 0 < \log \tfrac{x_2}{x_1} &< \tfrac{x_2}{x_1} - 1, \text{ since } \log(1+x) < x \text{ if } x > 0 \\ &\leq \tfrac{x_2 - x_1}{a}. \end{aligned}$$

If $x_2 < x_1$ then $\frac{x_1}{x_2} > 1$ and therefore

$$0 < \log \frac{x_1}{x_2} < \frac{x_1}{x_2} - 1, \text{ since } \log(1+x) < x \text{ if } x > 0$$
$$\leq \frac{x_1 - x_2}{a}.$$

In either case, $|\log x_2 - \log x_1| \leq \frac{1}{a}|x_2 - x_1|$.

This shows that f is a Lipschitz function on $[a, \infty)$ with $M = \frac{1}{a}$ and therefore f is uniformly continuous on $[a, \infty)$.

8.8. Continuity on a compact set.

Theorem 8.8.1. Let $D \subset \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be continuous on D. Then $f(D)$ is a compact set in $\mathbb{R}$.

The theorem says that continuous image of a compact set in $\mathbb{R}$ is a compact set.

Proof. Let $\mathcal{G}$ be a family of open intervals $\{I_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that $f(D) \subset \bigcup_{\alpha \in \Lambda} I_\alpha$. Then $\mathcal{G}$ is an open cover of $f(D)$.

Let $c \in D$. Then $f(c) \in f(D)$ and there exists an open interval of the family $\mathcal{G}$, say I_c, such that $f(c) \in I_c$.

Since I_c is an open interval, it is open set.

So $f(c)$ is an interior point of I_c and there exists a neighborhood of $f(c)$, say $N(f(c), \epsilon_c)$ such that $N(f(c), \epsilon_c) \subset I_c$.

Since f is continuous at c, there exists a $\delta_c > 0$ such that $f(x) \in N(f(c), \epsilon_c)$ for all $x \in N(c, \delta_c) \cap D$.

Clearly, the set of neighbourhoods $\{N(c, \delta_c) : c \in D\}$ covers D.

Since D is compact, there exists a finite subcollection of the family of the neighbourhoods $\{N(c, \delta_c) : c \in D\}$ which also covers D. Therefore there exists a finite number of points $c_1, c_2, \ldots, c_m$ in D such that $D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \cdots \cup N(c_m, \delta_{c_m})$.

Let $p \in f(D)$. Then there exists a point $q \in D$ such that $f(q) = p$.

Since $D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \cdots \cup N(c_m, \delta_{c_m})$, $q \in N(c_k, \delta_{c_k})$ for some natural number $k \leq m$.

But $x \in N(c_i, \delta_{c_i}) \Rightarrow f(x) \in N(f(c_i), \epsilon_{c_i}) \subset I_{c_i}$ for each $i = 1, 2, \ldots, m$.

As $q \in N(c_k, \delta_{c_k}), p \in N(f(c_k), \epsilon_{c_k}) \subset I_{c_k}$.

Since p is arbitrary, $f(D) \subset I_{c_1} \cup I_{c_2} \cup \cdots \cup I_{c_m}$.

Thus a finite subcollection of $\mathcal{G}$ covers $f(D)$ and therefore $f(D)$ is compact.

This completes the proof.

Another proof.

Let $\{y_n\}$ be a sequence in $f(D)$. For each $n \in \mathbb{N}$, let us choose $x_n \in D$ such that $f(x_n) = y_n$.

Since D is compact and $\{x_n\}$ is a sequence in D, there is a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $\{x_{r_n}\}$ converges to a point, say c, of D.

Since f is continuous at c, the sequence $\{f(x_{r_n})\}$ converges to $f(c)$. That is, the subsequence $\{y_{r_n}\}$ of $\{y_n\}$ converges to a point $f(c)$ of $f(D)$.

Therefore every sequence in $f(D)$ has a subsequence that converges to a point of $f(D)$. Consequently, $f(D)$ is compact.

Corollary. Since $f(D)$ is a compact set in $\mathbb{R}$, it is closed and bounded.

Since $f(D)$ is a closed and bounded set, $\sup f(D)$ and $\inf f(D)$ both exist and both belong to $f(D)$. (worked Example 1, Page 63)

Therefore there exists a point x^* in D such that $f(x^*) = \sup f(D)$ and there exists a point x_* in D such that $f(x_*) = \inf f(D)$.

Theorem 8.8.2. Let $D \subset \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be one-to-one and continuous on D. Then $f^{-1} : E \to D$ is continuous on E where $E = f(D) \subset \mathbb{R}$.

Proof. Since D is a compact set and f is continuous on D, E is compact. Let $b \in E$. Since f is one-to-one on D and $f(D) = E$, the inverse function $f^{-1} : E \to \mathbb{R}$ exists. Let $f^{-1}(b) = a$.

Let us choose a positive ϵ. Let $A = N(a, \epsilon) \cap D$.

Then $D - A = D - N(a, \epsilon)$.

Since D is a closed set and $N(a, \epsilon)$ is an open set, $D - A$ is a closed subset of D. Since D is compact and $D - A$ is a closed subset of D, $D - A$ is compact. (worked Ex.10, Page 97)

Let $D_1 = D - A$. Since $f : D \to E$ is continuous on D, the restriction function $\frac{f}{D_1} : D_1 \to E$ is also continuous on D_1.

Therefore $\frac{f}{D_1}(D_1)(= f(D_1))$ is compact.

Let $f(D_1) = E_1$. Now $a \notin D_1$ and f is one-to-one on D implies $f(a) \notin E_1$.

Since E_1 is compact and $b \notin E_1$ it follows that b is not a limit point of E_1. So there exists a neighbourhood $N(b, \delta)$ of b such that $[N(b, \delta) \cap E] \cap E_1 = \phi$.

Thus for every point $y \in N(b, \delta) \cap E$, $f^{-1}(y) \in N(a, \epsilon) \cap D$. Therefore f^{-1} is continuous at b.

Since b is arbitrary, f^{-1} is continuous on E.
This completes the proof.

Another proof. Since f is one-to-one, the inverse mapping $f^{-1}: f(D) \to D$ exists. Since D is compact and f is continuous on D, $f(D)$ is compact.

Case 1. Let $y_0 \in f(D)$ and y_0 be a limit point point of $f(D)$. Then there exists a sequence of distinct points $\{y_n\}$ in $f(D)$ such that $\lim y_n = y_0$.

Since f is injective, there exists a sequence of distinct points $\{x_n\}$ in D such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $x_n = f^{-1}(y_n)$ for all $n \in \mathbb{N}$. f^{-1} will be continuous at y_0 if we can prove that $\lim f^{-1}(y_n) = f^{-1}(y_0)$. i.e., if we can prove that $\lim x_n = x_0$ where $x_0 = f^{-1}(y_0)$.

Let us assume on the contrary, that the sequence $\{x_n\}$ does not converge to x_0. Then there exists a positive ϵ and a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $| x_{r_n} - x_0 | \geq \epsilon$ for all $n \in \mathbb{N}$.

Since D is compact and $\{x_{r_n}\}$ is a sequence in D, there exists a convergent subsequence $\{x'_n\}$ of the sequence $\{x_{r_n}\}$ such that $\{x'_n\}$ converges to a point, say x' in D.

As $| x'_n - x_0 | \geq \epsilon$ holds for all $n \in \mathbb{N}$, it follows that $x' \neq x_0$.

Since f is continuous on D, f is continuous at x' and $\lim f(x'_n) = f(x')$. Since f is injective, $x' \neq x_0 \Rightarrow f(x') \neq y_0$.

Let $f(x'_n) = y'_n$. Then $\{y'_n\}$ is a subsequence of the sequence $\{y_n\}$ and as $\lim y_n = y_0, \lim y'_n$ must be y_0.

Thus we arrive at a contradiction. Therefore our assumption that the sequence $\{x_n\}$ does not converge to x_0 is not tenable. We conclude $\lim x_n = x_0$ and thereby f^{-1} is continuous at y_0.

Case 2. Let $y_0 \in f(D)$ and y_0 be an isolated point of $f(D)$. Then f^{-1} is continuous at y_0.

Since y_0 is an arbitrary point of $f(D)$, f^{-1} is continuous on $f(D)$.
This completes the proof.

Theorem 8.8.3. Let $D \subset \mathbb{R}$ be a compact set and a function $f: D \to \mathbb{R}$ is continuous on D. Then f is uniformly continuous on D.

Proof. Let $c \in D$. Then f is continuous at c. Therefore for a pre-assigned positive ϵ there exists a positive δ_c such that

$| f(x) - f(c) | < \frac{\epsilon}{2}$ for all $x \in N(c, \delta_c) \cap D$.

Let $\mathcal{G}$ be the family of neighbourhoods $\{N(c, \frac{1}{2}\delta_c) : c \in D\}$. Clearly, $\mathcal{G}$ is an open cover of D. Since D is compact, there is a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers D.

Let $\mathcal{G}' = \{N(c_1, \frac{1}{2}\delta_{c_1}), N(c_2, \frac{1}{2}\delta_{c_2}), \cdots, N(c_m, \frac{1}{2}\delta_{c_m})\}$.

Let $\delta = \min\{\frac{1}{2}\delta_{c_1}, \frac{1}{2}\delta_{c_2}, \cdots, \frac{1}{2}\delta_{c_m}\}$ and let $x_1, x_2 \in D$ such that $|x_1 - x_2| < \delta$.

Since $x_1 \in D, x_1 \in N(c_k, \frac{1}{2}\delta_{c_k})$ for some natural number $k \leq m$.

Therefore $|x_1 - c_k| < \frac{1}{2}\delta_{c_k}$.

$$\begin{aligned} |x_2 - c_k| &\leq |x_2 - x_1| + |x_1 - c_k| \\ &< \delta + \tfrac{1}{2}\delta_{c_k} \\ &\leq \tfrac{1}{2}\delta_{c_k} + \tfrac{1}{2}\delta_{c_k} = \delta_{c_k}. \text{ This shows that } x_2 \in N(c_k, \delta_{c_k}). \end{aligned}$$

Since $x_1 \in N(c_k, \frac{1}{2}\delta_{c_k}) \cap D, |f(x_1) - f(c_k)| < \frac{\epsilon}{2}$.

Since $x_2 \in N(c_k, \delta_{c_k}) \cap D, |f(x_2) - f(c_k)| < \frac{\epsilon}{2}$.

Hence $|f(x_2) - f(x_1)| < \epsilon$.

Thus for all $x_1, x_2 \in D$ satisfying $|x_2 - x_1| < \delta, |f(x_2) - f(x_1)| < \epsilon$.
This shows that f is uniformly continuous on D.
This completes the proof.

Corollary. If $D \subset \mathbb{R}$ be compact and a map $f : D \to \mathbb{R}$ be continuous on D, then f maps a Cauchy sequence in D to a Cauchy sequence in $\mathbb{R}$.

Exercises 13

1. (i) Give an example of a function f which satisfies the intermediate-value property on a closed and bounded interval $[a, b]$ but is not continuous on $[a, b]$.

(ii) Give an example of a function f which is monotone increasing on a closed and bounded interval $[a, b]$ but does not satisfy the intermediate-value property on $[a, b]$.

2. Let $c \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at c. If for every positive δ there is a point y in $(c - \delta, c + \delta)$ such that $f(y) = 0$, prove that $f(c) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $c \in \mathbb{R}$ such that $f(c) > \mu$. Prove that there exists a neighbourhood U of c such that $f(x) > \mu$ for all $x \in U$.

4. Let a function $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that the set $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

Give an example of a function f continuous on $\mathbb{R}$ such that
(i) $Z(f)$ is a bounded enumerable set; (ii) $Z(f)$ is an unbounded enumerable set.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a *fixed point* of f if $f(c) = c$ holds. Prove that the set of all fixed points of f is a closed set.

6. Let $I = [a, b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on I and $f(x) > 0$ for all $x \in I$. Prove that there exists a positive number α such that $f(x) \geq \alpha$ for all $x \in I$.

7. A function $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and f assumes only rational values on $[0, 1]$. Prove that f is a constant.

8. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $f(a) < g(a)$, $f(b) > g(b)$. Show that there exists a point c in (a, b) such that $f(c) = g(c)$.

Deduce that $\cos x = x^2$ for some $x \in (0, \frac{\pi}{2})$.

9. A function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $x_1, x_2, x_3 \in [a, b]$. Prove that there is a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+f(x_3)}{3}$.

[**Hint.** There exist p, q in $[a, b]$ such that $f(p) \le f(x) \le f(q)$ for all $x \in [a, b]$. Then $f(p) \le \frac{f(x_1)+f(x_2)+f(x_3)}{3} \le f(q)$. Apply intermediate value theorem to the function f on $[p, q]$ or $[q, p]$.]

10. A function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $x_1, x_2, \ldots, x_n \in [a, b]$. Prove that there is a point $c \in [a, b]$ such that $f(c) = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$.

11. Give an example of a function f which is continuous on a closed interval I but (i) f is not bounded on I; (ii) $f(I)$ is not a closed interval.

12. A function $f : [0, 1] \to [0, 1]$ is defined by
$$f(x) = x, \ x \in \mathbb{Q}$$
$$= 1 - x, \ x \in \mathbb{R} - \mathbb{Q}.$$

Prove that (i) f is injective as well as surjective and $f^{-1} = f$;
(ii) f is not continuous on $[0, 1]$.

[**Note.** This example shows that continuity of f is not necessary for the existence of f^{-1}.]

13. If $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ be both continuous functions on $[a, b]$ having the same range $[0, 1]$, prove that $f(c) = g(c)$ for some $c \in [a, b]$.

[**Hint.** $f(p) = 0, f(q) = 1$ for some $p, q \in [a, b]$. If $g(p) \ne 0$ and $g(q) \ne 1$, consider $f - g$. Then $(f - g)(p) < 0, (f - g)(q) > 0$. If $g(p) = 0, c = p$. If $g(q) = 1, c = q$.]

14. A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists at least a point c in $[0, 1]$ such that $f(c) = f(c + 1)$.

[**Hint.** If $f(0) = f(1)$ then $c = 0, 1$. If $f(0) \ne f(1)$, consider g on [0,1] defined by $g(x) = f(x) - f(x + 1)$.]

15. If $f : (-\infty, 0] \to \mathbb{R}$ be continuous on $(-\infty, 0]$ and $\lim\limits_{x \to -\infty} f(x) = 0$, prove that f is uniformly continuous on $(-\infty, 0]$.

16. Prove that the following functions are uniformly continuous on the indicated interval.

(i) $f(x) = \sqrt{x}$, on $[1, \infty)$;

(ii) $f(x) = \frac{1}{1+x^2}$, on $\mathbb{R}$;

(iii) $f(x) = x \sin \frac{1}{x}, \ x \ne 0$
$= 0, \ x = 0$, on $[-1, 1]$;

(iv) $f(x) = \tan x$, on $[a, b]$ where $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$.

9. DIFFERENTIATION

9.1. Differentiability. Derivative.

Let $I = [a, b]$ be an interval and a function $f : I \longrightarrow \mathbb{R}$.

(i) Let c be an interior point of I.

f is said to be *differentiable* at c if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

If the limit be l, l is said to be the *derivative* of f at c and is denoted by $f'(c)$.

Since c is an interior point of the domain of f, in order that $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ may exist, both the limits $\lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$ and $\lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$ should exist and should be equal.

(ii) Let c be the left end point a.

f is said to be differentiable at a if $\lim\limits_{x \to a+} \dfrac{f(x) - f(a)}{x - a}$ exists. If l be the limit, l is called the derivative of f at a and is denoted by $f'(a)$.

(iii) Let c be the right end point b.

f is said to be differentiable at b if $\lim\limits_{x \to b-} \dfrac{f(x) - f(b)}{x - b}$ exists. If l be the limit, l is called the derivative of f at b and is denoted by $f'(b)$.

Note. If $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \infty$ (or $-\infty$) then f is said to have the derivative ∞ (or $-\infty$) at c and we write $f'(c) = \infty$ (or $-\infty$). However, f is said to be differentiable at c if $f'(c)$ is *finite*. This is also expressed by saying that "$f'(c)$ exists".

Right hand derivative, Left hand derivative.

Let I be an interval and $f : I \longrightarrow \mathbb{R}$. Let $c \in I$.

If $\lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$ exists and equals l, l is called the *right hand derivative* of f at c and it is denoted by $Rf'(c)$ (or by $f'_+(c)$).

If $\lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$ exists and equals l, l is called the *left hand derivative* of f at c and it is denoted by $Lf'(c)$ (or by $f'_-(c)$).

Therefore if c be an interior point of the domain of f, the derivative of f at c exists if and only if $Rf'(c)$ and $Lf'(c)$ both exist and be equal.

If c be the left end point of the interval I, the derivative of f at c, if it exists, is the right hand derivative of f at c.

If c be the right end point of the interval I, the derivative of f at c, if it exists, is the left hand derivative of f at c.

Theorem 9.1.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then f is continuous at c.

Proof. For all $x \in I$, but $x \neq c$, $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}.(x-c)$.

Since f is differentiable at c, $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists and the limit is finite. Also, since $\lim\limits_{x \to c}(x - c) = 0$,

$$\begin{aligned}\lim_{x \to c}[f(x) - f(c)] &= \lim \tfrac{f(x)-f(c)}{x-c}.(x-c) \\ &= f'(c).0 \\ &= 0, \text{ since } f'(c) \text{ is finite.}\end{aligned}$$

Therefore $\lim\limits_{x \to c} f(x) = f(c)$ and this shows that f is continuous at c.

Note. The continuity of f at a point $c \in I$ does not ensure differentiability of f at c.

For example, let $f(x) = |x|, x \in \mathbb{R}$.
At $x = 0, f(x) = 0$. Also f is continuous at 0.

$$\lim_{x \to 0+} \tfrac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \tfrac{|x|}{x} = \lim_{x \to 0+} \tfrac{x}{x} = 1$$

$$\lim_{x \to 0-} \tfrac{f(x)-f(0)}{x-0} = \lim_{x \to 0-} \tfrac{|x|}{x} = \lim_{x \to 0-} \tfrac{-x}{x} = -1.$$

As $Rf'(0) \neq Lf'(0)$, f is not differentiable at 0.

Hence continuity at a point c does not imply differentiability at c.

Remark. If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$, it is possible to define differentiability of f at a point $c \in D$, provided $c \in D'$ also.

If $c \in D \cap D', f$ is said to be differentiable at c if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists. If l be the limit, then l is called the derivative of f at c and is denoted by $f'(c)$.

If $D_1 = D \cap [c, \infty)$ and c be a limit point of D_1, then $\lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c}$, if it exists, is said to be the right hand derivative of f at c and it is denoted by $Rf'(c)$.

If $D_2 = D \cap (-\infty, c]$ and c be a limit point of D_2, then

$\lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c}$, if it exists, is said to be the left hand derivative of f at c and it is denoted by $Lf'(c)$.

If c be a limit point of both D_1 and D_2, then f is differentiable at c if and only if $Rf'(c)$ and $Lf'(c)$ both exist and are real.

Let I be an interval and $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$. Then $f'(c)$ exists. Let A be a subset of I such that at every point of A, f is differentiable.

Then $f'(x)$ exists for each $x \in A$. f' can be considered as a function on A. f' is said to be the *derived function* of f on A. If f' be a function of x, then $f'(x)$ is expressed by $f'(x) = \frac{d}{dx} f(x), x \in A$ or by $f'(x) = Df(x), x \in A$.

Examples.

1. Let $k \in \mathbb{R}$ and $f(x) = k, x \in \mathbb{R}$. Find the derived function f' and its domain.

Let $c \in \mathbb{R}$. When $x \neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{k-k}{x-c} = 0$.

Therefore $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = 0$. That is, $f'(c) = 0$ for all $c \in \mathbb{R}$.

The derived function f' is defined by $f'(x) = 0, x \in \mathbb{R}$. The domain of f' is $\mathbb{R}$.

2. Let $f(x) = x^2, x \in \mathbb{R}$. Find the derived function f' and its domain.

Let $c \in \mathbb{R}$. When $x \neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{x^2-c^2}{x-c} = x + c$.

Therefore $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim\limits_{x \to c}(x + c) = 2c$.

That is, $f'(c) = 2c$ for all $c \in \mathbb{R}$.

The derived function f' is defined by $f'(x) = 2x, x \in \mathbb{R}$. The domain of f' is $\mathbb{R}$.

3. Let $f(x) = \sqrt{x}, x \in [0, \infty)$. Find the derived function f' and its domain.

Let $c \in [0, \infty)$. When $x \geq 0$ but $\neq c$, $\frac{f(x)-f(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{1}{\sqrt{x}+\sqrt{c}}$.

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}} \text{ provided } c \neq 0.$$

That is, $f'(c) = \frac{1}{2\sqrt{c}}$ if $c \in (0, \infty)$.

The domain of f' is $\{x \in \mathbb{R} : x > 0\}$.

The derived function f' is defined by $f'(x) = \frac{1}{2\sqrt{x}}, x \in (0, \infty)$.

Note. Here the domain of f' is a proper subset of the domain of f.

f' $D = (0, \infty)$ $= [0, \infty)$

f D //

$$
\begin{aligned}
f(x) &= x, \quad 0 \le x \le \\
&= 2 - x^2, \quad 1 < x < 2. \\
&= x - x^2, \quad 2 \le x \le 3.
\end{aligned}
$$

Find the derived function f' and its domain.

$$
\begin{aligned}
f'(x) &= 1 \text{ for } x \in (0,1) \\
&= -2x \text{ for } x \in (1,2) \\
&= 1 - 2x \text{ for } x \in (2,3).
\end{aligned}
$$

$\lim\limits_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0+} \frac{x}{x} = 1$. Therefore $Rf'(0) = 1$.

Hence f is differentiable at 0 and $f'(0) = 1$.

$\lim\limits_{x \to 1-} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1-} \frac{x-1}{x-1} = 1$. Therefore $Lf'(1) = 1$.

$\lim\limits_{x \to 1+} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1+} \frac{(2-x^2)-1}{x-1} = -2$. Therefore $Rf'(1) = -2$.

Hence f is not differentiable at 1.

$\lim\limits_{x \to 2-} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x \to 2-} \frac{(2-x^2)-(-2)}{x-2} = \lim\limits_{x \to 2-} -(x+2) = -4$. Therefore $Lf'(2) = -4$.

$\lim\limits_{x \to 2+} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x \to 2+} \frac{(x-x^2)-(-2)}{x-2} = \lim\limits_{x \to 2+} -(x+1) = -3$. Therefore $Rf'(2) = -3$.

Hence f is not differentiable at 2.

$\lim\limits_{x \to 3-} \frac{f(x)-f(3)}{x-3} = \lim\limits_{x \to 3-} \frac{x-x^2-(-6)}{x-3} = -5$. Therefore $Lf'(3) = -5$.

Hence f is differentiable at 3 and $f'(3) = -5$.

The derived function f' is defined by
$$
\begin{aligned}
f'(x) &= 1, \quad 0 \le x < 1 \\
&= -2x, \quad 1 < x < 2 \\
&= 1 - 2x, \quad 2 < x \le 3.
\end{aligned}
$$

The domain of f' is $[0,1) \cup (1,2) \cup (2,3]$.

Note. The domain of f' is a proper subset of the domain of f.

Theorem 9.1.2. Let I be an interval and $c \in I$. Let the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at c. Then

(i) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$

(ii) if $k \in \mathbb{R}$, kf is differentiable at c and $(kf)'(c) = kf'(c)$

(iii) $f.g$ is differentiable at c and $(f.g)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) if $g(c) \neq 0, f/g$ is differentiable at c and $(f/g)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$.

Proof. Proofs of **(i)** and **(ii)** are left to the reader.

(iii) Let $h = f.g$. Then for $x \in I, x \neq c$,

$\frac{h(x)-h(c)}{x-c} = \frac{f(x)g(x)-f(c)g(c)}{x-c}$

$= \frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c}$

$= \frac{f(x)-f(c)}{x-c}.g(x) + f(c).\frac{g(x)-g(c)}{x-c}$.

Since g is continuous at c by Theorem 9.1.1, $\lim\limits_{x\to c} g(x) = g(c)$.

Since f is differentiable at c, $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$.

Since g is differentiable at c, $\lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c} = g'(c)$.

Therefore $\lim\limits_{x\to c} \dfrac{h(x)-h(c)}{x-c}$

$= \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}. \lim\limits_{x\to c} g(x) + f(c). \lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c}$

$= f'(c)g(c) + f(c)g'(c)$.

Therefore h is differentiable at c and $g'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) Let $h = f/g$. Since g is differentiable at c, g is continuous at c. Since $g(c) \neq 0$, there exists a neighbourhood $N(c)$ of c such that $g(x) \neq 0$ for all $x \in N(c) \cap I$. Therefore for $x \in N(c) \cap I, x \neq c$,

$\frac{h(x)-h(c)}{x-c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \frac{f(x)g(c)-f(c)g(x)}{g(x)g(c).(x-c)}$

$= \frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}$

$= \frac{1}{g(x)g(c)}[\frac{f(x)-f(c)}{x-c}.g(c) - f(c).\frac{g(x)-g(c)}{x-c}]$.

Since g is continuous at c by Theorem 9.1.1, $\lim\limits_{x\to c} g(x) = g(c)$.

Since f and g are differentiable at c,

$\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$ and $\lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c} = g'(c)$.

Therefore $\lim\limits_{x\to c} \dfrac{h(x)-h(c)}{x-c} = \dfrac{1}{g(c).g(c)}[f'(c)g(c) - f(c)g'(c)]$.

Therefore $h(x)$ is differentiable at c and $h'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$.

Theorem 9.1.3. Let I be an interval and the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be both differentiable on a subset $D \subset I$. Then

(i) $f+g$ is differentiable on D and $(f+g)'(x) = f'(x) + g'(x), x \in D$

(ii) if $k \in \mathbb{R}, kf$ is differentiable on D and $(kf)'(x) = kf'(x), x \in D$

(iii) $f.g$ is differentiable on D and $(f.g)'(x) = f'(x)g(x) + f(x)g'(x), x \in D$

(iv) if $g'(x) \neq 0$ on $D, f/g$ is differentiable on D and $(f/g)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}, x \in D.$

This is an immediate consequence of the Theorem 9.1.2.

gof

Theorem 9.1.4. Let I and J be intervals. Let $f : I \to \mathbb{R}, g : J \to \mathbb{R}$ and $f(I) \subset J$. Let $c \in I$ and f is differentiable at c and g is differentiable at $f(c)$. Then the composite function gf is differentiable at c and $(gf)'(c) = g'(f(c)).f'(c)$.

Proof. Let $f(c) = d$. Since g is differentiable at d, $\lim\limits_{y \to d} \dfrac{g(y)-g(d)}{y-d} = g'(d)$. Since f is differentiable at c, $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$.

Let us define a function $G : J \to \mathbb{R}$ by

$$\begin{aligned} G(y) &= \tfrac{g(y)-g(d)}{y-d} \text{ if } y \in J \text{ and } y \neq d \\ &= g'(d) \text{ if } y = d. \end{aligned}$$

$$\begin{aligned} \text{Then } \lim_{y \to d} G(y) &= \lim_{y \to d} \frac{g(y)-g(d)}{y-d} \\ &= g'(d), \text{ since } g \text{ is differentiable at } d \\ &= G(d), \text{ by definition .} \end{aligned}$$

This shows that G is continuous at d.

Since f is continuous at c and G is continuous at $d(= f(c))$, the composite function Gf is continuous at c. Hence $\lim\limits_{x \to c} Gf(x) = Gf(c)$.

$$\begin{aligned} \text{But } \lim_{x \to c} Gf(x) &= \lim_{x \to c} \frac{g(f(x))-g(d)}{f(x)-d}, \text{ by definition of } G \\ &= \lim_{x \to c} \frac{g(f(x))-g(f(c))}{f(x)-f(c)}. \end{aligned}$$

Therefore $\lim\limits_{x \to c} \dfrac{g(f(x))-g(f(c))}{f(x)-f(c)} = g'(d)$, since $Gf(c) = g'(d)$.

We also have $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$.

Hence $\lim\limits_{x \to c} \dfrac{g(f(x))-g(f(c))}{x-c} = g'(d).f'(c)$.

Therefore the function gf is differentiable at c and $(gf)'(c) = g'(d)f'(c) = g'(f(c))f'(c)$.

This completes the proof.

As an immediate consequence of the theorem we have the following theorem.

Theorem 9.1.5. Let I and J be intervals and $f : I \to \mathbb{R}, g : J \to \mathbb{R}$ be functions such that $f(I) \subset J$. If f is differentiable on I and g is differentiable on $f(I)$ then the composite function gf is differentiable on I and $(gf)'(x) = g'(f(x)).f'(x), x \in I$.

Example (continued).

5. Find the derived function of $f(x) = x^\alpha, x > 0$ and $\alpha \in \mathbb{R}$.

$f(x) = x^\alpha = e^{\alpha \log x}$.

Let $g(x) = \alpha \log x, x > 0$ and $h(x) = e^x; x \in \mathbb{R}$.

Then $f(x) = hg(x)$, $x > 0$ and $f'(x) = h'(g(x)).g'(x)$, $x > 0$.

But $h'(x) = e^x$ and hence $h'(g(x)) = e^{\alpha \log x} = x^\alpha$; $g'(x) = \frac{\alpha}{x}$.

Therefore $f'(x) = x^\alpha . \frac{\alpha}{x} = \alpha x^{\alpha - 1}, x > 0$.

Theorem 9.1.6. Let $I \subset \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ and let $g : J \to \mathbb{R}$ be the inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$ then g is differentiable at $d(= f(c))$ and $g'(d) = \frac{1}{f'(c)}$.

Proof. Let $y \in J, y \neq d$. Let $g(y) = x \in I$. Then $f(x) = y$ and since f is strictly monotone on $I, x \neq c$.

$\frac{g(y)-g(d)}{y-d} = \frac{g(f(x))-g(f(c))}{f(x)-f(c)}$.

Since f is strictly monotone and continuous on I, g is continuous on J. As $y \to d, g(y) \to g(d)$.

Since $gf(x) = x$ for all $x \in I$, it follows that $x \to c$ as $y \to d$.

Therefore $\lim\limits_{y \to d} \frac{g(y)-g(d)}{y-d} = \lim\limits_{x \to c} \frac{g(f(x))-g(f(c))}{f(x)-f(c)} = \lim\limits_{x \to c} \frac{x-c}{f(x)-f(c)}$.

Since f is differentiable at c, $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ and since $f'(c) \neq 0$, $\lim\limits_{x \to c} \frac{x-c}{f(x)-f(c)} = \frac{1}{f'(c)}$.

Therefore $\lim\limits_{y \to d} \frac{g(y)-g(d)}{y-d} = \frac{1}{f'(c)}$. That is, $g'(d) = \frac{1}{f'(c)}$.

Examples (continued).

6. Let $f(x) = x^2, x \in [0, \infty)$. f is strictly increasing and continuous on $[0, \infty)$. Let $I = [0, \infty)$. Then $f(I) = [0, \infty)$.

The inverse function g defined by $g(y) = \sqrt{y}, y \in [0, \infty)$ is continuous on $[0, \infty)$.

f is differentiable on $[0, \infty)$ and $f'(x) = 2x, x \in [0, \infty)$.

$f'(x) \neq 0$ on $(0, \infty)$. Let $I_1 = (0, \infty)$. Then $f(I_1) = (0, \infty)$.

Hence $g'(y)$ exists for all $y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}$, $y \in (0, \infty)$.

7. Let $f(x) = e^x, x \in \mathbb{R}$. f is strictly increasing and continuous on $\mathbb{R}$. $f(\mathbb{R}) = (0, \infty)$.

The inverse function g defined by $g(y) = \log y, y \in (0, \infty)$ is continuous on $(0, \infty)$.

f is differentiable on $\mathbb{R}$ and $f'(x) = e^x \neq 0$ on $\mathbb{R}$.

Hence $g'(y)$ exists for all $y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\log y}} = \frac{1}{y}$, $y \in (0, \infty)$.

8. Let $f(x) = \sin x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.$f$ is strictly increasing and continuous on $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $f(I) = [-1, 1]$. The inverse function g defined by $g(y) = \sin^{-1} y, y \in [-1, 1]$ is continuous on $[-1, 1]$.

f is differentiable on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $f'(x) = \cos x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$f'(x) \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $I_1 = (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(I_1) = (-1, 1)$.

Hence $g'(y)$ exists for all $y \in (-1, 1)$ and
$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

That is, $D \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

9. Let $f(x) = \cos x, x \in [0, \pi]$. f is strictly decreasing and continuous on $I = [0, \pi]$. $f(I) = [-1, 1]$. The inverse function g defined by $g(y) = \cos^{-1} y, y \in [-1, 1]$ is continuous on $[-1, 1]$.

f is differentiable on $[0, \pi]$ and $f'(x) = -\sin x, x \in [0, \pi]$.

$f'(x) \neq 0$ on $(0, \pi)$. Let $I_1 = (0, \pi)$. Then $f(I_1) = (-1, 1)$.

Hence $g'(y)$ exists for all $y \in (-1, 1)$ and

$$\begin{aligned} g'(y) = \frac{1}{f'(x)} = \frac{1}{-\sin x} &= \frac{1}{-\sqrt{1-\cos^2 x}} \text{ since } \sin x > 0 \text{ in } (0, \pi) \\ &= \frac{1}{-\sqrt{1-y^2}}, y \in (-1, 1). \end{aligned}$$

That is, $D \cos^{-1} y = \frac{-1}{\sqrt{1-y^2}}$, $y \in (-1, 1)$.

10. Let $f(x) = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. f is strictly increasing and continuous on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. $f(I) = \mathbb{R}$. The inverse function g defined by $g(y) = \tan^{-1} y, y \in \mathbb{R}$, is continuous on $\mathbb{R}$.

f is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f'(x) = \sec^2 x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$f'(x) \neq 0$ on $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence $g'(y)$ exists for all $y \in \mathbb{R}$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}, y \in \mathbb{R}$.

That is, $D \tan^{-1} y = \frac{1}{1+y^2}$, $y \in \mathbb{R}$.

11. Let $f(x) = \sec x$, $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Let $I_1 = [0, \frac{\pi}{2}), I_2 = (\frac{\pi}{2}, \pi]$.

f is strictly increasing and continuous on I_1. $f(I_1) = [1, \infty)$.

The inverse function $g : [1, \infty) \to \mathbb{R}$ defined by $g(y) = \sec^{-1} y$ is continuous on $[1, \infty)$.

f is differentiable on I_1 and $f'(x) = \sec x \tan x, x \in I_1$.

$f'(x) \neq 0$ on $(0, \frac{\pi}{2})$.

Hence $g'(y)$ exists for all $y \in (1, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec x \tan x}$.

f is strictly increasing and continuous on I_2. $f(I_2) = (-\infty, -1]$.

The inverse function $g : (-\infty, -1] \to \mathbb{R}$ defined by $g(y) = \sec^{-1} y$ is continuous on $(-\infty, -1]$.

f is differentiable on I_2 and $f'(x) = \sec x \tan x, x \in I_2$.

$f'(x) \neq 0$ on $(\frac{\pi}{2}, \pi)$. Hence $g'(y)$ exists for all $y \in (-\infty, -1)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec x \tan x}$.

But $\sec x \tan x = y\sqrt{y^2 - 1}, x \in (0, \frac{\pi}{2})$ and $\sec x \tan x = -y\sqrt{y^2 - 1}$, $x \in (\frac{\pi}{2}, \pi)$.

That is, $D \sec^{-1} y = \frac{1}{|y|\sqrt{y^2-1}}, y \in (-\infty, -1) \cup (1, \infty)$.

12. Let $f(x) = \operatorname{cosec} x, x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.

Let $I_1 = [-\frac{\pi}{2}, 0)$, $I_2 = (0, \frac{\pi}{2}]$.

f is strictly decreasing and continuous on I_1. $f(I_1) = (-\infty, -1]$.

The inverse function $g : (-\infty, -1] \to \mathbb{R}$ defined by $g(y) = \operatorname{cosec}^{-1} y$ is continuous on $(-\infty, -1]$.

f is differentiable on I_1 and $f'(x) = -\operatorname{cosec} x \cot x, x \in [-\frac{\pi}{2}, 0)$

$f'(x) \neq 0$ on $(-\frac{\pi}{2}, 0)$. Hence $g'(y)$ exists for all $y \in (-\infty, -1)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{-\operatorname{cosec} x \cot x}$.

f is strictly decreasing and continuous on I_2. $f(I_2) = [1, \infty)$.

The inverse function $g : [1, \infty) \to \mathbb{R}$ defined by $g(y) = \operatorname{cosec}^{-1} y$ is continuous on $[1, \infty)$.

f is differentiable on I_2 and $f'(x) = -\operatorname{cosec} x \cot x, x \in I_2$.

$f'(x) \neq 0$ on $(0, \frac{\pi}{2})$. Hence $g'(y)$ exists for all $y \in (1, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{-\operatorname{cosec} x \cot x}$.

But $\operatorname{cosec} x \cot x = -y\sqrt{y^2 - 1}, x \in [-\frac{\pi}{2}, 0)$ and $\operatorname{cosec} x \cot x = y\sqrt{y^2 - 1}$, $x \in (0, \frac{\pi}{2}]$.

That is, $D \operatorname{cosec}^{-1} y = \frac{-1}{|y|\sqrt{y^2-1}}, y \in (-\infty, -1) \cup (1, \infty)$.

13. Let $f(x) = \cot x, x \in (0, \pi)$. f is strictly decreasing and continuous on $I = (0, \pi)$. $f(I) = \mathbb{R}$.

The inverse function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = \cot^{-1} y, y \in \mathbb{R}$ is continuous on $\mathbb{R}$.

f is differentiable on $(0, \pi)$ and $f'(x) = -\operatorname{cosec}^2 x, x \in (0, \pi)$.

$f'(x) \neq 0$ on $(0, \pi)$. Hence $g'(y)$ exists for all $y \in \mathbb{R}$ and $g'(y) = \frac{1}{f'(x)} = \frac{-1}{\operatorname{cosec}^2 x} = -\frac{1}{1+y^2}$.

That is, $D \cot^{-1} y = -\frac{1}{1+y^2}, y \in \mathbb{R}$.

Worked Examples.

1. A function f is defined on some neighbourhood of c and f is differentiable at c. Prove that $\lim\limits_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Show by an example that the limit may exist even if $f'(c)$ does not exist.

$$\begin{aligned}\lim_{h \to 0+} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0+} \left[\frac{f(c+h)-f(c)}{2h} + \frac{f(c-h)-f(c)}{-2h}\right] \\ &= \lim_{h \to 0+} \frac{f(c+h)-f(c)}{2h} + \lim_{k \to 0-} \frac{f(c+k)-f(c)}{2k} \\ &= \tfrac{1}{2} Rf'(c) + \tfrac{1}{2} Lf'(c), \text{ since } f'(c) \text{ exists} \\ &= f'(c).\end{aligned}$$

$$\begin{aligned}\lim_{h \to 0-} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0-} \left[\frac{f(c+h)-f(c)}{2h} + \frac{f(c-h)-f(c)}{-2h}\right] \\ &= \lim_{h \to 0-} \frac{f(c+h)-f(c)}{2h} + \lim_{k \to 0+} \frac{f(c+k)-f(c)}{2k} \\ &= \tfrac{1}{2} Lf'(c) + \tfrac{1}{2} Rf'(c) \\ &= f'(c).\end{aligned}$$

Therefore $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c-h)}{2h} = f'(c)$.

Second part. Let $f(x) = |x|, c = 0$.

$$\begin{aligned}\text{Then } \lim_{h \to 0} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h \to 0} \frac{f(h)-f(-h)}{2h} \\ &= \lim_{h \to 0} \frac{|h| - |-h|}{2h} = 0.\end{aligned}$$

But $f'(0)$ does not exist.

2. A function f is defined by $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0$.

Show that f is differentiable at 0 but f' is not continuous at 0.

$\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$, since $\lim\limits_{x \to 0} x = 0$ and $\sin \frac{1}{x}$ is bounded on some deleted neighbourhood of 0.

Hence $f'(0) = 0$. When $x \neq 0, f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

Thus the derived function f' is defined by

$$f'(x) = 2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}, x \neq 0$$
$$= 0, \ x = 0.$$

$\lim\limits_{x\to 0} f'(x)$ does not exist, since $\lim\limits_{x\to 0} \cos \frac{1}{x}$ does not exist.

Therefore f' is not continuous at 0.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 0$ and

$f(x) = 0$, if x is irrational

$= \frac{1}{q}$, if $x = \frac{p}{q}$, where $p \in \mathbb{Z}, q \in \mathbb{N}$ and $gcd(p,q) = 1$.

Show that f is not differentiable at 0.

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{f(x)}{x}$.

Let $\phi(x) = \frac{f(x)}{x}$. Let $\{x_n\}$ be a sequence of rational points converging to 0 where $x_n = \frac{1}{n}, n \in \mathbb{N}$.

Then $\lim\limits_{n\to\infty} \phi(x_n) = \lim\limits_{n\to\infty} n.\frac{1}{n} = 1$.

Let $\{y_n\}$ be a sequence of irrational points converging to 0.

$$\lim_{n\to\infty} \phi(y_n) = \lim_{n\to\infty} \frac{f(y_n)}{y_n} = 0, \text{ since } f(y_n) = 0 \text{ for all } n \in \mathbb{N}.$$

Therefore $\lim\limits_{x\to 0} \phi(x)$ does not exist, since for two sequences $\{x_n\}$ and $\{y_n\}$ both converging to 0, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ converge to two different limits.

Hence f is not differentiable at 0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$ and $f(c) \neq 0$. Let $g(x) = |f(x)|, x \in \mathbb{R}$. Show that g is differentiable at c and

$$g'(c) = f'(c), \text{ if } f(c) > 0$$
$$= -f'(c), \text{ if } f(c) < 0.$$

Let $h(x) = |x|, x \in \mathbb{R}$. Then $g(x) = h(f(x)), x \in \mathbb{R}$.

$g'(c) = h'(f(c)).f'(c)$, provided h is differentiable at $f(c)$.

If $f(c) > 0$, then $f(x) > 0$ for all x in some neighbourhood of c, since f is continuous at c. Therefore $\lim\limits_{x\to c} \frac{|f(x)|-|f(c)|}{f(x)-f(c)} = \lim\limits_{x\to c} \frac{f(x)-f(c)}{f(x)-f(c)} = 1$.

That is, $h'(f(c)) = 1$ if $f(c) > 0$. Similarly, $h'(f(c)) = -1$ if $f(c) < 0$.

Therefore $g'(c) = f'(c)$, if $f(c) > 0$

$= -f'(c)$, if $f(c) < 0$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$ and $f(c) = 0$. Let $g(x) = |f(x)|, x \in \mathbb{R}$. Show that g is differentiable at c if and only if $f'(c) = 0$.

Let $h(x) = |x|, x \in \mathbb{R}$. Then $g(x) = h(f(x)), x \in \mathbb{R}$.

$$\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = \lim_{x\to c} \frac{|f(x)|-|f(c)|}{f(x)-f(c)} . \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{|f(x)|}{f(x)} . \frac{f(x)-f(c)}{x-c}.$$

$\lim\limits_{x \to c} \frac{|f(x)|}{f(x)}$ does not exist, in general. But $\frac{|f(x)|}{f(x)}$ is bounded in some deleted neighbourhood $N'(c)$ of c, since $|\frac{|f(x)|}{f(x)}| = 1$ for all $x \in N'(c)$.

Therefore $\lim\limits_{x \to c} \frac{|f(x)|}{f(x)} \cdot \frac{f(x)-f(c)}{x-c}$ exists if and only if $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$, i.e., if and only if $f'(c) = 0$ and in this case $\lim\limits_{x \to c} \frac{g(x)-g(c)}{x-c} = 0$, i.e., $g'(c) = 0$.

6. Let $f(x) = \sin^{-1} \frac{2x}{1+x^2}$, $x \in \mathbb{R}$. Find the derived function f'.

Let $g(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$, $h(x) = \sin^{-1} x$, $|x| \leq 1$.

Since $|g(x)| \leq 1$ for all $x \in \mathbb{R}$, the composite function hg is defined for all $x \in \mathbb{R}$.

Let $c \in \mathbb{R}$. Then $|g(c)| \leq 1$ and $|g(c)| = 1$ if $c = \pm 1$. Since h is differentiable for all $x \in (-1, 1)$, h is not differentiable at $g(c)$ if $c = \pm 1$.

Let $c \in \mathbb{R}$ and $c \neq \pm 1$.

Then $f'(c) = h'(g(c)).g'(c) = \frac{1}{\sqrt{1-(\frac{2c}{1+c^2})^2}} \cdot \frac{2(1-c^2)}{(1+c^2)^2} = \frac{2(1-c^2)}{\sqrt{(1-c^2)^2}.(1+c^2)}$.

Therefore $f'(c) = \frac{2}{1+c^2}$, if $c^2 < 1$, i.e., if $|c| < 1$

$= \frac{-2}{1+c^2}$, if $c^2 > 1$, i.e., $|c| > 1$.

Hence $f'(x) = \frac{2}{1+x^2}$, if $|x| < 1$; $f'(x) = \frac{-2}{1+x^2}$, if $|x| > 1$; and f is not differentiable at ± 1.

7. Let $f(x) = x^3 + 2x + 3$, $x \in \mathbb{R}$. Show that f has an inverse function g on $\mathbb{R}$. Find the derivative of g at the points corresponding to $x = 0$, $x = -1$.

$f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$. Therefore f is a continuous and strictly increasing function on $\mathbb{R}$.

Let $y \in \mathbb{R}$ has a pre-image x in $\mathbb{R}$. Then $x^3 + 2x + (3 - y) = 0$. This is a cubic equation in x and it has a real root. This means that each y has a pre-image and therefore $f(\mathbb{R}) = \mathbb{R}$.

f admits of an inverse function g on $\mathbb{R}$.

$f'(x) \neq 0$ on $\mathbb{R}$. Therefore g is differentiable at every point in $\mathbb{R}$ and $g'(y) = \frac{1}{f'(x)}$, where $f(x) = y$.

$f(0) = 3$. $g'(3) = \frac{1}{f'(0)} = \frac{1}{2}$. $f(-1) = 0$. $g'(0) = \frac{1}{f'(-1)} = \frac{1}{5}$.

9.2. Higher order derivatives.

Let I be an interval and a function $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. If f be differentiable at every point of some subinterval $I_1(c)$ such that $c \in I_1(c) \subset I$, then $f' : I_1(c) \to \mathbb{R}$ is a function on $I_1(c)$.

If f' be differentiable at c then the derivative of f' at c is called the *second order derivative* of f at c and is denoted by $f''(c)$ or by $f^{(2)}(c)$.

This is to note that c may also be an end point of the sub-interval $I_1(c)$.

If f' be differentiable at every point of some sub-interval $I_2(c)$ such that $c \in I_2(c) \subset I_1(c)$, then $f'' : I_2(c) \longrightarrow \mathbb{R}$ is a function on $I_2(c)$.

If f'' be differentiable at c then the derivative of f'' at c is called the *third order derivative* of f at c and is denoted by $f'''(c)$ or by $f^{(3)}(c)$.

In a similar manner we define the nth order derivative $f^{(n)}(c)$ whenever the derivative exists.

This is to emphasize that in order that the nth derivative of f may exist at c, $f^{(n-1)}$ must be defined on some sub-interval containing c, allowing the possibility of c to be an end point also of such subinterval.

Exercises 14

1. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= x, x < 1 \\ &= 2 - x, 1 \le x \le 2 \\ &= x^2 - 3x + 2, x > 2. \end{aligned}$$
Show that $f'(x)$ does not exist at 1 and 2.

2. A function f is defined on some neighbourhood $N(0)$ of 0 by
$$\begin{aligned} f(x) &= \tfrac{x}{1+e^{1/x}}, x \ne 0 \\ &= 0,\ x = 0. \end{aligned}$$
Find $Lf'(0)$ and $Rf'(0)$. Show that f is not differentiable at 0.

3. A function f is defined by
$$\begin{aligned} f(x) &= x\left(\tfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), x \ne 0 \\ &= 0,\ x = 0. \end{aligned}$$
Show that f is continuous at 0 but not differentiable at 0.

4. $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x| + |x - 1| + |x - 2|$, $x \in \mathbb{R}$.
Find the derived function f' and specify the domain of f'.

5. Find $f'(x)$ if (i) $f(x) = \sin^{-1} 2x\sqrt{1 - x^2}$, $|x| \le 1$,
(ii) $f(x) = \sin^{-1}(3x - 4x^3)$, $|x| \le 1$,
(iii) $f(x) = \cos^{-1}(8x^4 - 8x^2 + 1)$, $|x| \le 1$.

6. $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$\begin{aligned} f(x) &= e^{-1/x^2} \sin \tfrac{1}{x}, x \ne 0 \\ &= 0,\ x = 0. \end{aligned}$$
Show that f' is continuous at 0.

7. A function f is defined on $(-1, 1)$ by
$$\begin{aligned} f(x) &= x^\alpha \sin \tfrac{1}{x^\beta}, x \ne 0 \\ &= 0, x = 0. \end{aligned}$$
Prove that (i) if $0 < \beta < \alpha - 1$, f' is continuous at 0;
(ii) if $0 < \alpha - 1 \le \beta$, f' is discontinuous at 0.

8. $f(x) = x^2 \sin \frac{1}{x}, x \neq 0$ and $g(x) = x,\ x \in \mathbb{R}$.
$\quad = 0,\ x = 0;$

Show that $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist, but $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$.

9. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} f(x) &= 0, \text{ if } x = 0 \text{ or } x \text{ is irrational} \\ &= \tfrac{1}{q^3}, \text{ if } x = \tfrac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } gcd(p,q) = 1. \end{aligned}$$

Show that f is differentiable at 0 and $f'(0) = 0$.

[**Hint.** For $x \neq 0, 0 \leq \left| \frac{f(x)}{x} \right| \leq x^2$.]

10. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, if x is rational
$= 0$, if x is irrational.

Show that f is differentiable at 0 and $f'(0) = 0$.

[**Hint.** $0 \leq \frac{f(x)}{x} \leq x$ for $x > 0$ and $x \leq \frac{f(x)}{x} \leq 0$ for $x < 0$.]

11. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$, if x is rational
$= \sin x$, if x is irrational.

Show that f is differentiable at 0 and $f'(0) = 1$.

[**Hint.** $\cos x \leq \frac{f(x)}{x} \leq 1$ for all $x \in N'(0, \frac{\pi}{2})$.]

12. Let $f(x) = x^5 + 4x + 1, x \in \mathbb{R}$.
(i) Show that f has an inverse function g differentiable on $\mathbb{R}$.
(ii) Find $g'(1)$, $g'(6)$.

9.3. Sign of the derivative.

Let $I \subset \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$.

Let c be an *interior point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$ and $f(x) > f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$ and $f(x) < f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

Let c be the *left end point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$;

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c < x < c + \delta$.

Let c be the *right end point* of I.

f is said to be *increasing* at c if there exists a positive δ such that $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$;

f is said to be *decreasing* at c if there exists a positive δ such that $f(x) > f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$.

Theorem 9.3.1. Let $I \subset \mathbb{R}$ be an interval and a function $f : I \to \mathbb{R}$ be differentiable at $c \in I$.

(i) If $f'(c) > 0$ then f is increasing at c

(ii) if $f'(c) < 0$ then f is decreasing at c.

Proof. (i) $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$. Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c, \delta) \cap I$.

Let c be an interior point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in (c-\delta, c) \cap I$ and for all $x \in (c, c+\delta) \cap I$.

Therefore $f(x) < f(c)$ for all $x \in (c - \delta, c) \cap I$ and $f(x) > f(c)$ for all $x \in (c, c + \delta) \cap I$. This proves that f is increasing at c.

Let c be the left end point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in I$ such that $c < x < c + \delta$.

Therefore $f(x) > f(c)$ for all $x \in I$ such that $c < x < c + \delta$.

This proves that f is increasing at c.

Let c be the right end point of I.

Then $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in I$ such that $c - \delta < x < c$.

Therefore $f(x) < f(c)$ for all $x \in I$ satisfying $c - \delta < x < c$.

This proves that f is increasing at c.

(ii) Similar proof.

Note 1. A function f may be increasing (or decreasing) at a point c in its domain without being differentiable at c. For example, the function f defined by
$$\begin{aligned} f(x) &= x, x < 1 \\ &= 2x - 1, x \geq 1 \end{aligned}$$
is increasing at 1 but f is not differentiable at 1.

The function f defined by
$$\begin{aligned} f(x) &= 1 - x, x < 0 \\ &= 1 - 2x, x \geq 0 \end{aligned}$$
is decreasing at 0 but f is not differentiable at 0.

Note 2. If f is increasing at a point c then $f'(c)$ may not be positive. For example, let $f(x) = x^3, x \in \mathbb{R}$. f is increasing at 0, but $f'(0) = 0$.

If f is decreasing at a point c then $f'(c)$ may not be negative. For example, let $f(x) = -x^3, x \in \mathbb{R}$. f is decreasing at 0, but $f'(0) = 0$.

Note 3. If $f'(x) > 0$ at c, it does not follow that f increases monotonically in some neighbourhood of c. For if x_1, x_2 be any two points in a small neighbourhood of c such that $c < x_1 < x_2$ then it has only been proved that $f(c) < f(x_1)$ and $f(c) < f(x_2)$ and we cannot conclude that $f(x_1) < f(x_2)$.

For example, let $f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}, x \neq 0$
$= 0, x = 0.$

Then f is increasing at 0. But in every neighbourhood of $0, f'(x)$ assumes both positive and negative values. Therefore f is not monotonic in any neighbourhood of 0.

9.4. Properties of the derivative.

We have seen that if a function f be continuous on a closed and bounded interval $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$. We have a similar theorem for a derived function.

Theorem 9.4.1. (Darboux)

Let $I = [a, b]$ and a function $f : I \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$. If k be a real number lying between $f'(a)$ and $f'(b)$ then there exists a point c in (a, b) such that $f'(c) = k$.

Proof. Without loss of generality, let $f'(a) < k < f'(b)$.

Let us define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - kx, x \in [a, b]$.

g is differentiable on $[a, b]$ and therefore g is continuous on $[a, b]$. Consequently, g will attain the minimum value (the greatest lower bound) at some point c in $[a, b]$.

$g'(a) = f'(a) - k < 0$. This implies g is decreasing at a.

Therefore there exists a positive δ such that $g(x) < g(a)$ for all $x \in [a, b]$ satisfying $a < x < a + \delta$.

This shows that $g(a)$ is not the minimum value of g on $[a, b]$.

$g'(b) = f'(b) - k > 0$. This implies g is increasing at b.

Therefore there exists a positive δ such that $g(x) < g(b)$ for all $x \in [a, b]$ satisfying $b - \delta < x < b$.

This shows that $g(b)$ is not the minimum value of g on $[a, b]$.

Thus $c \neq a, c \neq b$ and therefore $a < c < b$.

Since $c \in (a, b)$, $g'(c)$ exists. We prove that $g'(c) = 0$.

Let $g'(c) > 0$. Then there exists a positive δ such that $g(x) < g(c)$ for all $x \in [a, b]$ satisfying $c - \delta < x < c$. This contradicts that $g(c)$ is the minimum value of g on $[a, b]$. Therefore $g'(c) \not> 0$.