EC5.203 Communication Theory (3-1-0-4):

Introductory Class

Instructor: Dr. Sachin Chaudhari

Email: sachin.chaudhari@iiit.ac.in

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and
Jan. 09, 2025

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

My Background: *Academics and Industry*

BE (Electronics)
1998-2002

ME (Telecommunications)
2002-2004

ESQUBE

Sr. Wireless Engg.
2004-2007

IISc Startup

PhD (Signal Processing for Communications) (2007-2012)
Post-Doc (2013-2014)

Aalto University
School of Engineering

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

Assistant Professor since Dec. 2014
Associate Professor since July 2021

I am also a Senior IEEE member

IEEE
HYDERABAD SECTION

Aalto University

- Formerly famous as Helsinki University of Technology (TKK)
- Founded in 1849, got the university status in 1908
- Best in Finland and a top ranking university in Europe, especially in ICT
- Famous Alumni:
  - **Alvar Aalto (World Famous Designer),**
  - Founders of **Linux, AngryBirds, Supercell, SSH, F-Secure**
  - CEOs of **Nokia, Kone, MySQL**
  - **Noble Laurete** in Chemistry

- Merged with Helsinki School of Business and University of Art and Design in Helsinki to form Aalto University in 2010
- Motivation behind Aalto: A unique interdisciplinary university to create new innovative thought

My Background: *Research Areas*

- **Signal Processing and Machine Learning for Wireless Communication**
  - Internet of Things (IoT)
  - 5G and Beyond
  - Satellite Communications
  - Cognitive Radios
- During PhD and post-doc
  - Spectrum Sensing for Cognitive Radios
  - **Encor2** (2013-2014): TEKES funded project titled *Enabling methods for dyNamic spectrum access and COgnitive Radios*
  - **SENDORA** (2008-2010): An EU FP7 project titled *SEnsor Networks for Dynamic and cOgnitive Radio Access*
- At IIITH
  - Cognitive Radios for 5G and Beyond (2015-2019)
  - IoT for Smart Cities (2018-ongoing)
    - IoT-enabled Smart Cities: Pollution, Health and Governance *(DST and PRIF)*
    - CoE on IoT for Smart Cities *(India-EU Collaboration Project for Standardization)*
    - Smart City Living Lab (*MEITY and Smart City Mission*)
    - 5G Use Case Lab at IIITH (*DoT*)
  - Satellite Communication for IoT (2020-ongoing)

Teaching Assistants

- Baly Naganjani (MS, ECE)
- Nikhil Singh Parihar (MS, ECE)
- M. Manasa (MS, ECE)
- Vishnupriya (UG4, ECE)

Outline

- Course Intro
  - What is a communication system?
    - Channel Issues
    - Analog Communication
    - Digital Communication
  - Motivation and Importance

- Course Administration
  - Syllabus
  - Resources
  - Evaluation
  - Assignments
  - Tutorials

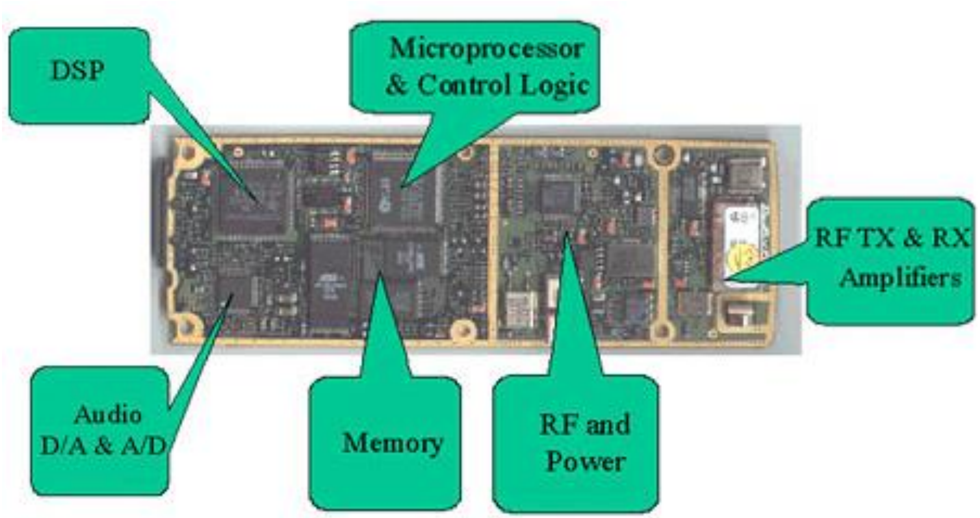
Introduction and Motivation

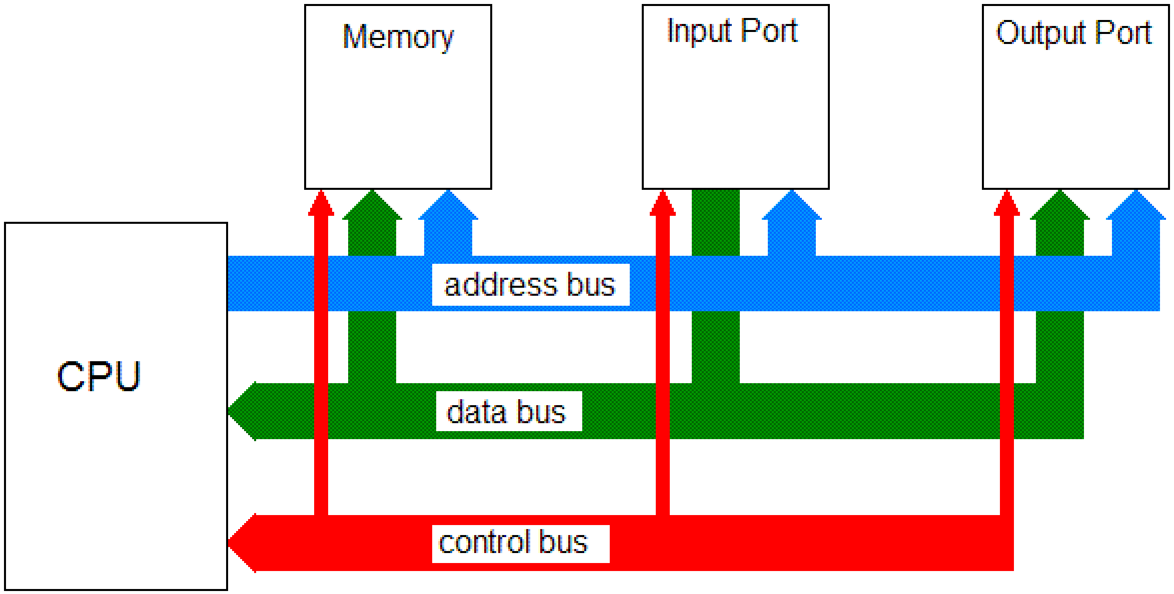
What is Communication?

- Communication is defined as transmission of information from source to destination via a transmission medium

- Examples:
  - AM and FM Radios
  - TV
  - Phone call/Whatsapp Message/Internet
  - Accessing Intranet over Ethernet
  - Microprocessor and peripherals

Examples: Microprocessor and Peripherals





Communication: A slightly different perspective

- Information transfer can be across space or time
  - Separated in space (telephony, web browsing,…)
  - Separated in time ??
    - *Storage of information - CDs, DVDs, hard drives, cloud*

Popular Communication Systems

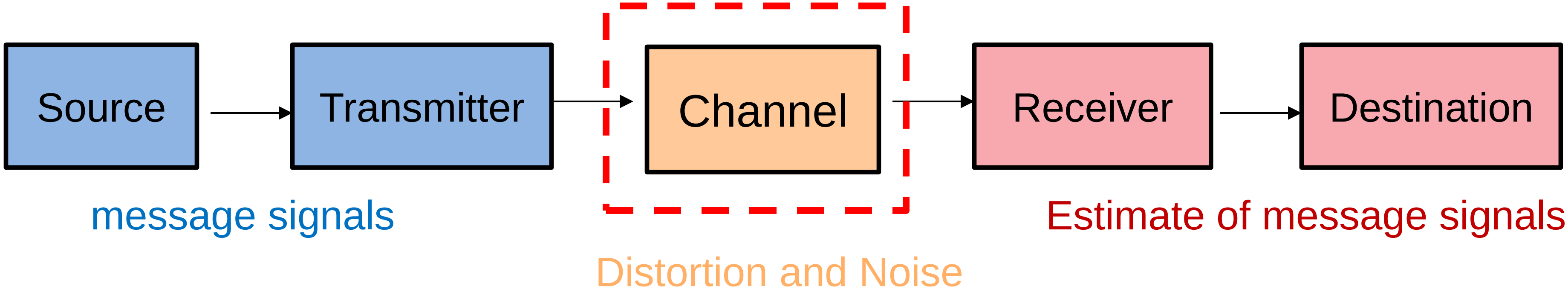
- Cellular Communication (2G/3G/4G/5G)
- LAN
  - Ethernet
  - WLAN
- Broadcasting: radio and TV
- Bluetooth
- Sensor networks
- Satellite communication
- Many more……..

Why Wireless Communication?

- Wired communication will always be better than wireless
- Wireless provides “Anytime Anywhere communication”
- A necessity in today’s world
- Mobile phone has become an addiction
  - Teenagers and technology: 'I'd rather give up my kidney than my phone’
  - Digital communication is not just prevalent in teenagers' lives. It IS teenagers' lives
  - https://www.theguardian.com/lifeandstyle/2010/jul/16/teenagers-mobiles-facebook-social-networking

Key Steps in Communication Link

- Insertion of information into a signal, termed the transmitted signal, compatible with the physical medium of interest.
- Propagation of the signal through the physical medium (termed the channel) in space or time
- Extraction of information from the signal (termed the received signal) obtained after propagation through the medium.



Questions?

CHANNEL

The Main Resource and
the Main Challenge!

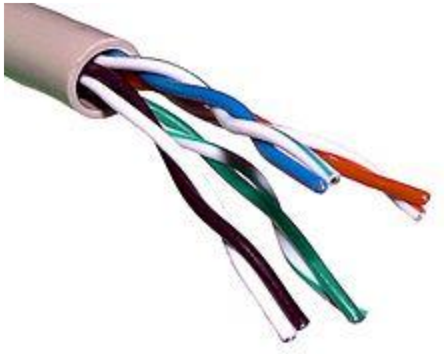
Types of Channel

Classification based on Medium

- Guided channels
  - Copper wire (twisted pair)
  - Coaxial cable
  - Optical fibre
  - microwave guides

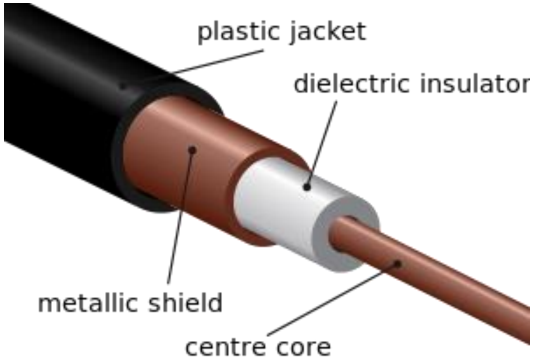
- Unguided channels
  - wireless channel
  - underwater acoustic channel

Wired Medium



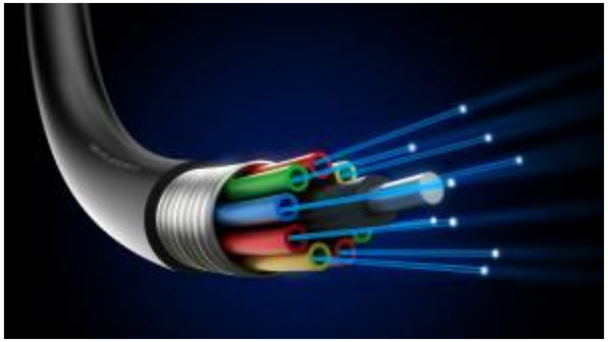
Unshielded Twisted Pair

https://en.wikipedia.org/wiki/Twisted_pair



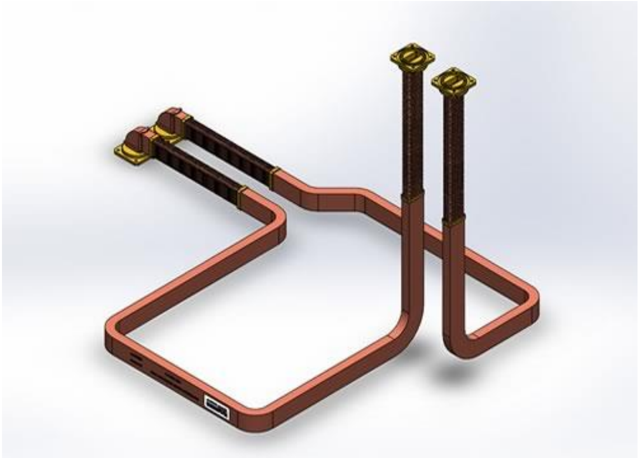
Coaxial Cable

https://en.wikipedia.org/wiki/Coaxial_cable



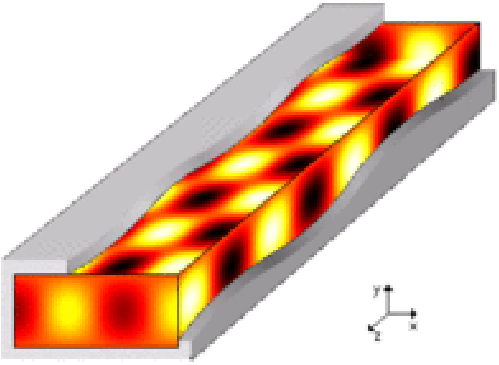
Fiber Optics

http://www.slideshare.net/subrata11/optical-fiber-55382806



Waveguide

http://www.megaind.com/



Waveguide

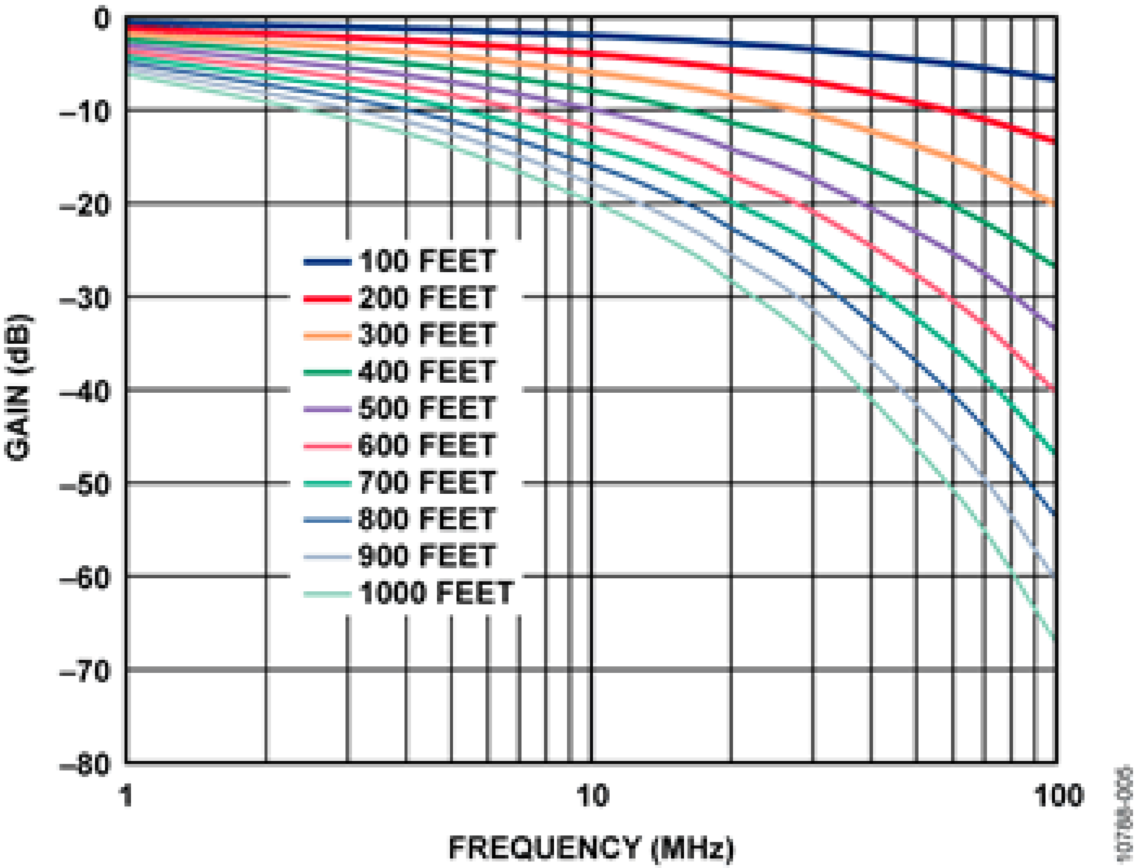
https://en.wikipedia.org/wiki/Waveguide

Channel: the challenges!

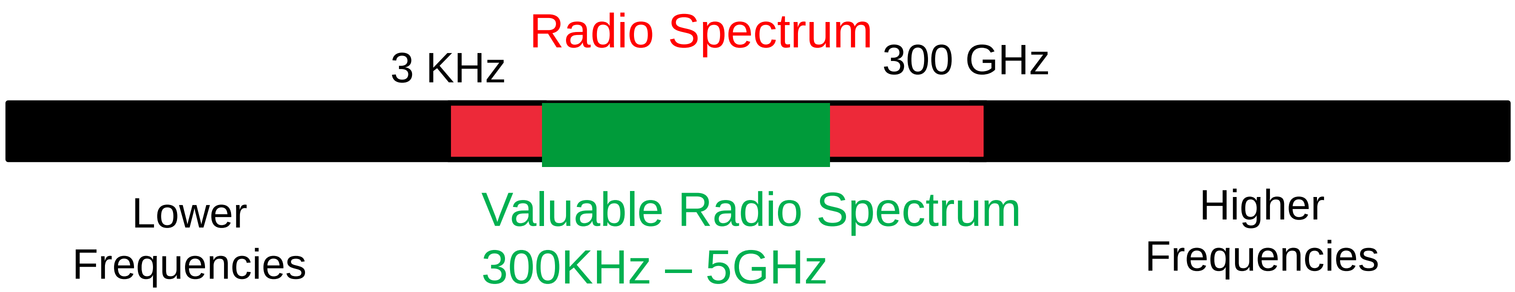
- Limited Spectrum
  - Limited wireless spectrum: High frequencies high attenuation, at low frequencies: large antenna sizes and low bandwidth
  - Wired spectrum: the medium has inherent spectrum of bands it will allow.
  - Cannot transmit in all bands as it may not be allowed.
- Needs to be shared by many users
  - Congestion, Collision, Call Drops
- Distorts the signal
  - Attenuation at different frequencies
  - Noise/Interference
  - Multipath
- Main resource or infrastructure (such as road or pipe)
  - Limits the amount of information that can be transmitted

Wired: *Limited Spectrum*

- Channel is the most challenging part in communication
  - Wired spectrum: the medium has inherent spectrum of bands it will allow



Wireless: *Limited Spectrum*



- Low radio frequencies: Large antenna heights
- High radio frequencies
  - High attenuation
  - Above 5 GHz only line of sight

Channel: the challenges!

- Limited Spectrum
  - Limited wireless spectrum: High frequencies high attenuation, at low frequencies: large antenna sizes and low bandwidth
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- Distorts the signal
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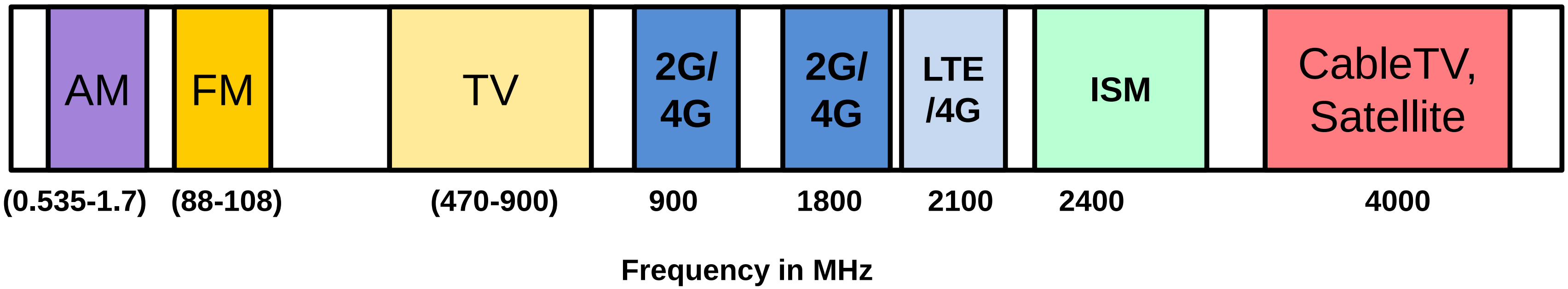
  - Multipath
- Main resource or infrastructure (such as road or pipe)
  - Limits the amount of information that can be transmitted

Wireless: *Need for sharing spectrum*

- Shared spectrum across different technologies



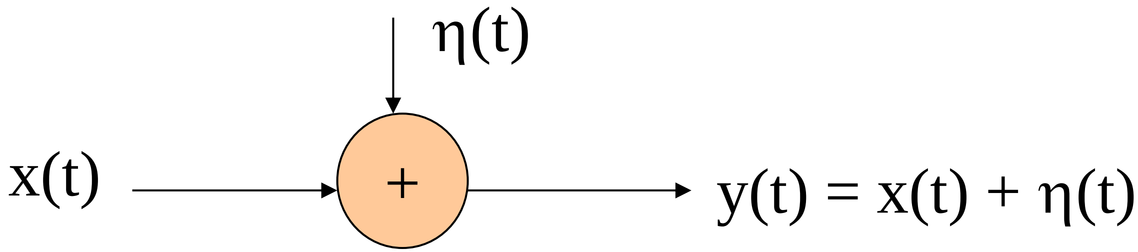
- Shared spectrum across same technology
  - By the end of 2012, the number of **mobile-connected devices** exceeded the number of people on earth (approx. 7 billion).
  - Leads to congestion, collisions, call drops

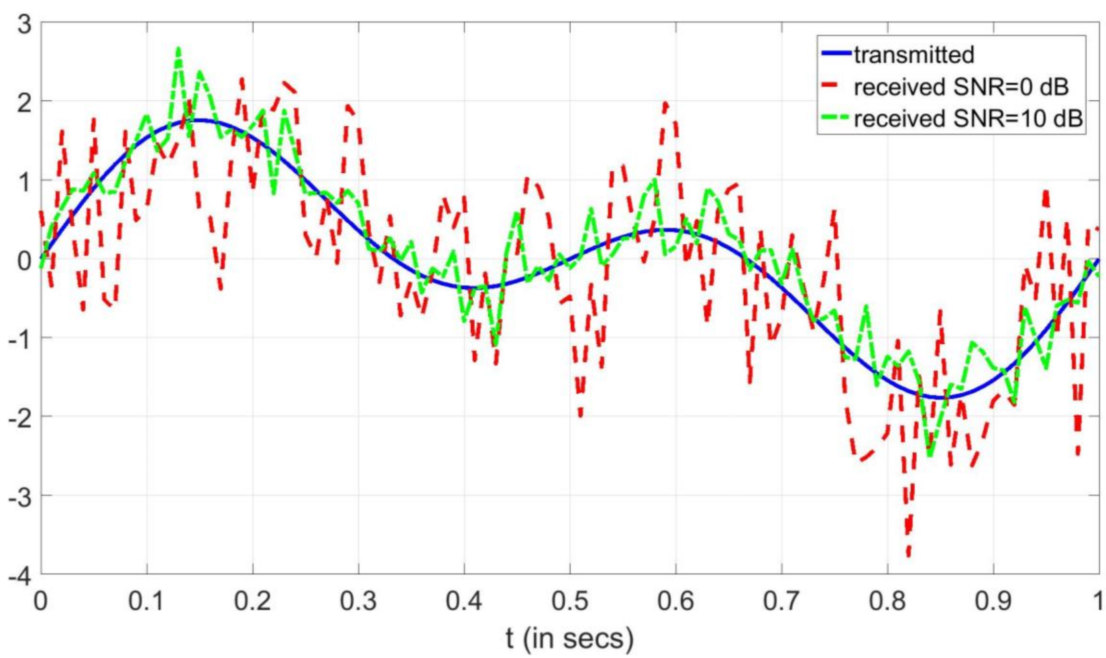
Channel: the challenges!

- Limited Spectrum
  - Limited wireless spectrum: High frequencies high attenuation, at low frequencies: large antenna sizes and low bandwidth
  - Wired spectrum: the medium has inherent spectrum of bands it will allow.
  - Cannot transmit in all bands as it may not be allowed.
- Needs to be shared by many users
  - Congestion, Collision, Call Drops
- Distorts the signal
  - Attenuation at different frequencies
  - Noise/Interference
  - Multipath
- Main resource or infrastructure (such as road or pipe)
  - Limits the amount of information that can be transmitted

Channel Distortion: Noise

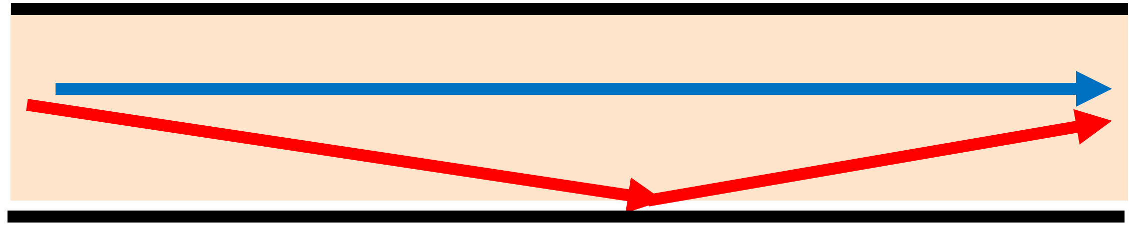
- Sources: Thermal noise at the receiver, Interference from other sources, man-made EMI, powerline interference
- Modelled as AWGN (additive White Gaussian noise)

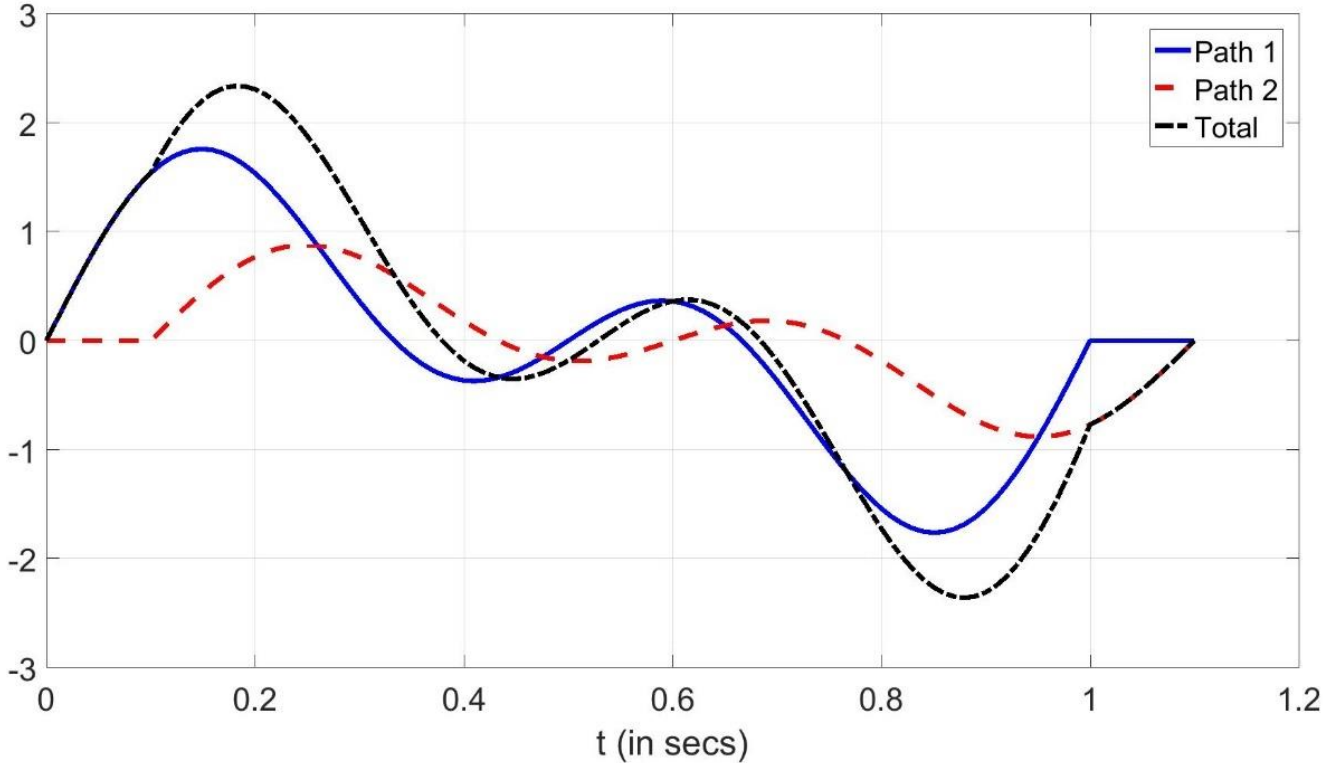




Channel Distortion: *Multipath*

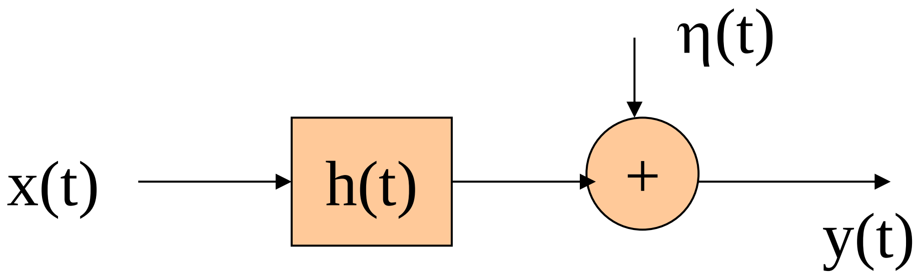
- Guided Medium: Time Invariant





Channel: Linear Time-Invariant filter

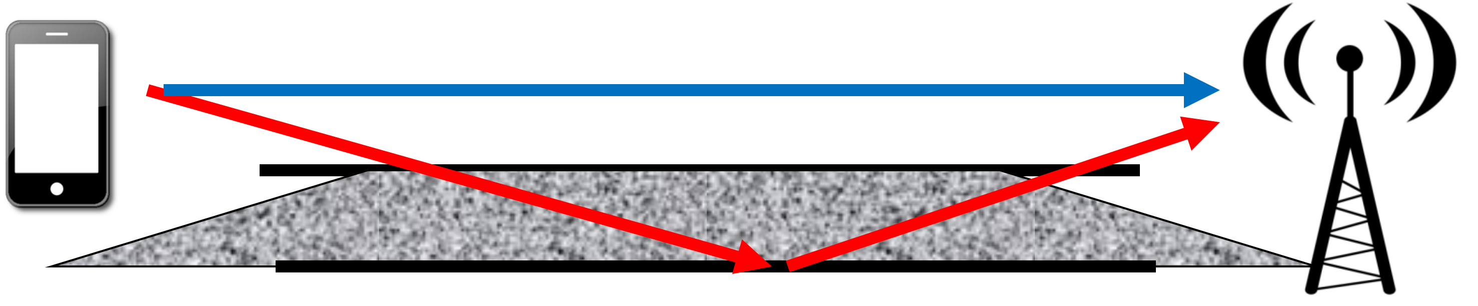
- Channels whose time/ frequency characteristics does not change in time are modeled in this fashion.
- E.g: Wired Channels

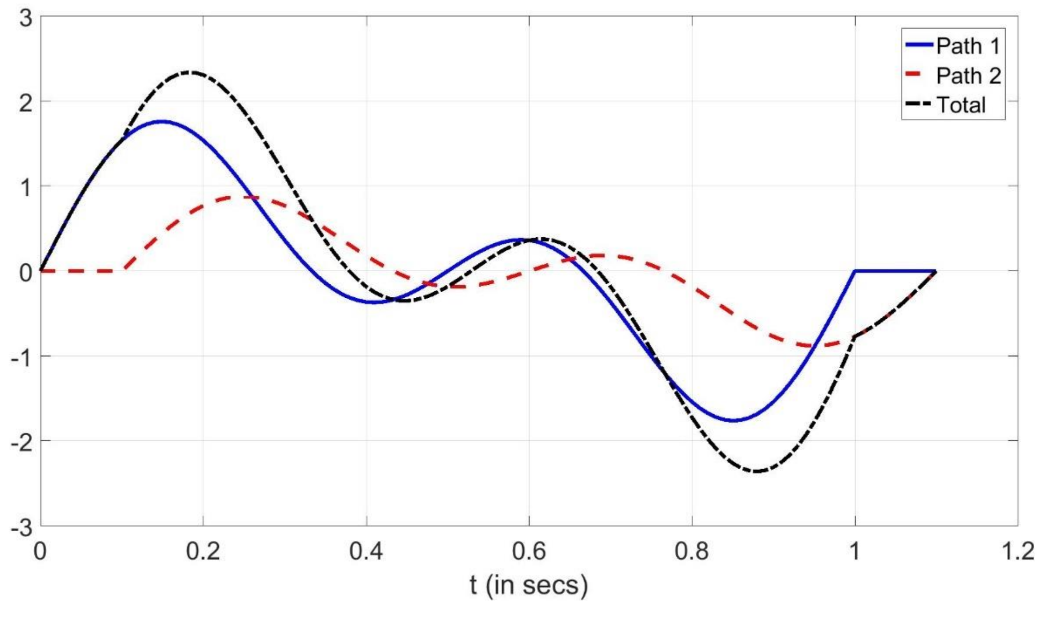


$y(t) = h(t)*x(t) + \eta(t)$

Channel Distortion: *Multipath*

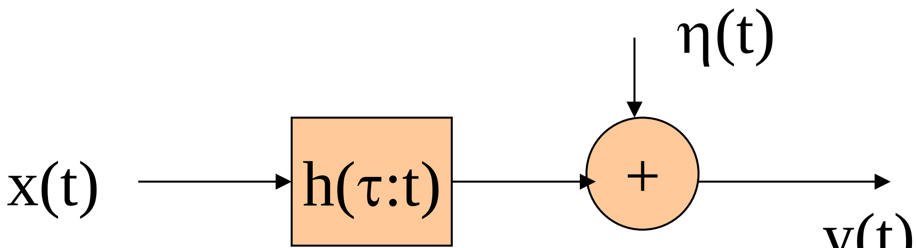
- Wireless Channel: Delay and Time Variance





Channel: Linear Time-Variant filter

- Channels whose time/ frequency characteristics change in time are modeled in this fashion.
- h(τ : t) is the time varying impulse response of the filter. E.g: Wireless Channels with multi-path.



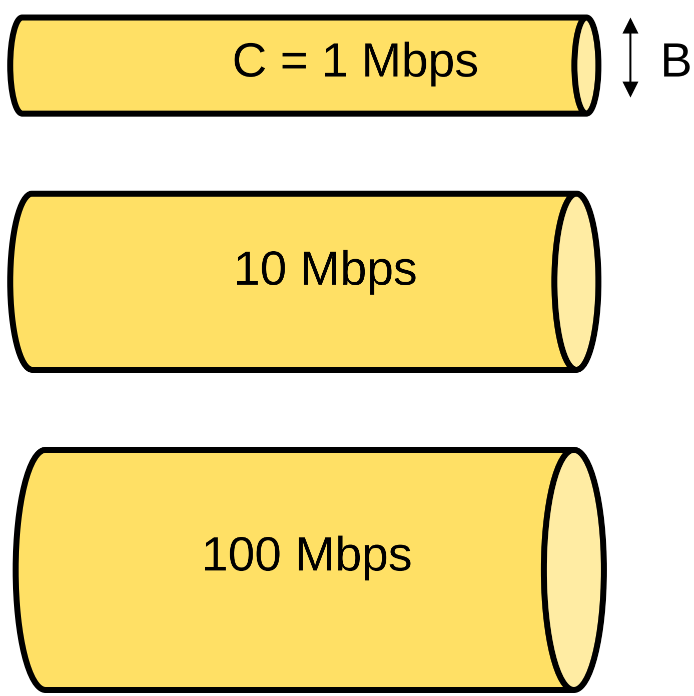
$$y(t) = h(\tau{:}t) * x(t) + \eta(t)$$

Limited Capacity: *Shannon's Theorem*

- Capacity of AWGN channel is given by

$$C = B \log_2(1 + \mathrm{SNR})$$

where B is the channel bandwidth, $\mathrm{SNR} = P/N$ is the signal to noise ratio, P is the signal power, N is the noise power.



Typical Wired Media Bandwidth

Typical Media	Max. Bandwidth	Max. Speed	Max. Physical Distance
50-Ohm coaxial cable (Thinnet)	Few MHz	10 Mbps	185m
75-Ohm coaxial cable (thicknet)	Few MHz	10 Mbps	500m
CAT 5 100 Base-TX Ethernet	100 MHz	100 Mbps	100m
CAT 5e 1000 Base-TX Ethernet	100 MHz	1000 Mbps	100m
Fiber 100 Base-FX Ethernet	Few GHz	100 Mbps	2000m
Fiber 1000 base LX Ethernet	Few GHz	1000 Mbps (1 Gbps)	5000m

Typical Wireless Bandwidth

Typical Media	Max. Theoretical Bandwidth	Max. Speed
WLAN 802.11a	20 MHz	54 Mbps
LTE 3G	20 MHz	100 Mbps
LTE-Advanced 4G	20-100 MHz	1 Gbps
WLAN 802.11n	20-40 MHz	600 Mbps

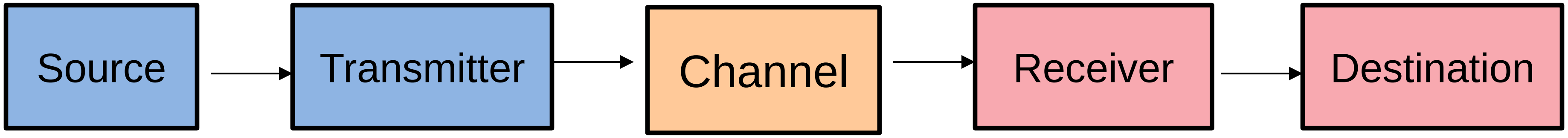
Channel: the bright side!

- Without the issues and challenges in the channel, there would not had been any ECE branch and several billions dollar companies such as Qualcomm, AT&T, Bell Labs, Nokia,

Types of Communication Systems: Analog and Digital

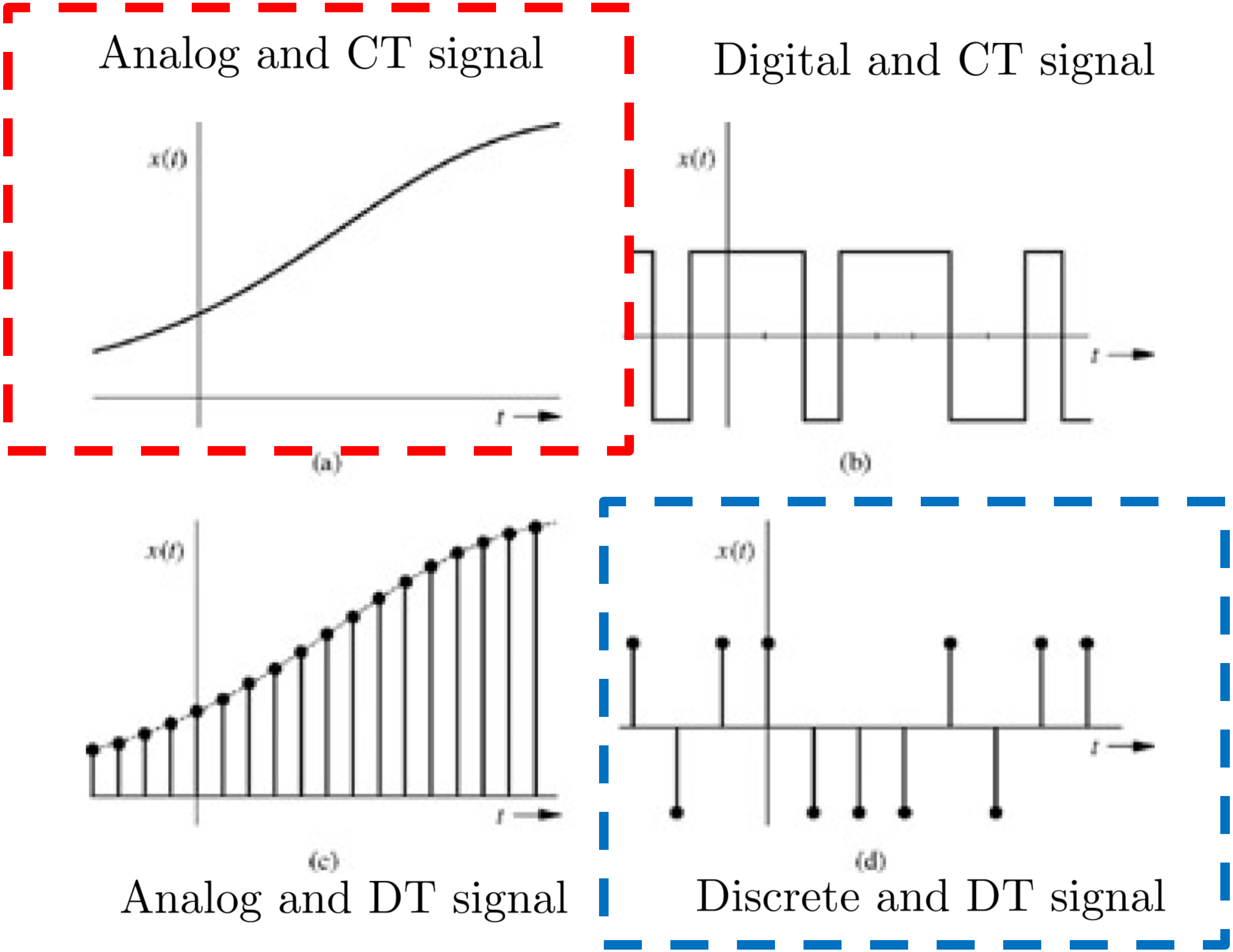
Type of Communication Systems

- Depends on the message signals



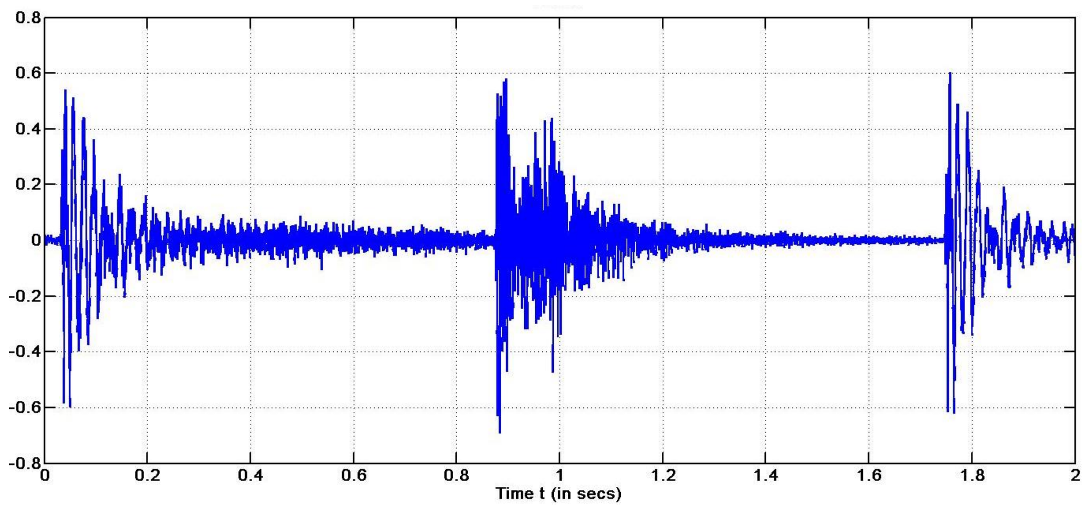
message signals

Analog and Digital: S&S



Analog Communications

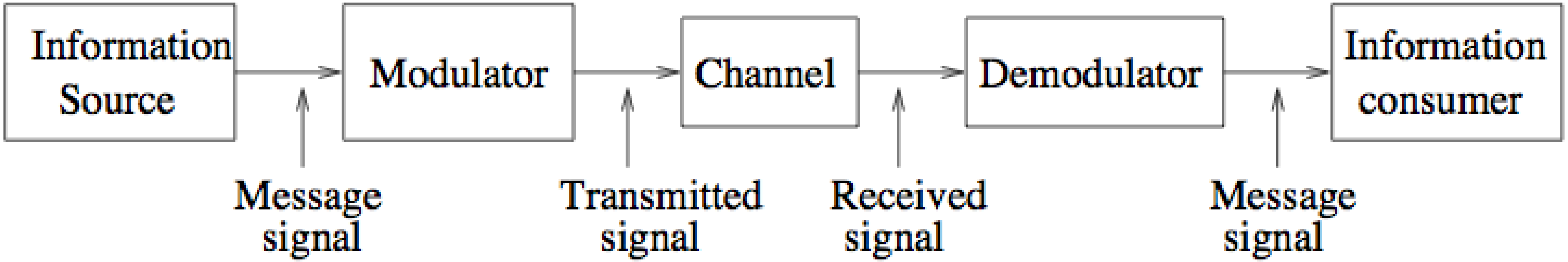
- Message signal is analog
  - Continuous time signal which takes continuum of values.
- Example: Audio signals, Speech,
- Transmitted signals over physical medium are also analog



Analog Communication Systems

- Examples
  - AM (amplitude modulation) and FM (frequency modulation) radios
  - Analog television
  - first generation cellular technology (AMPS),
  - vinyl records, audio cassettes, and VHS

Analog Communication Systems



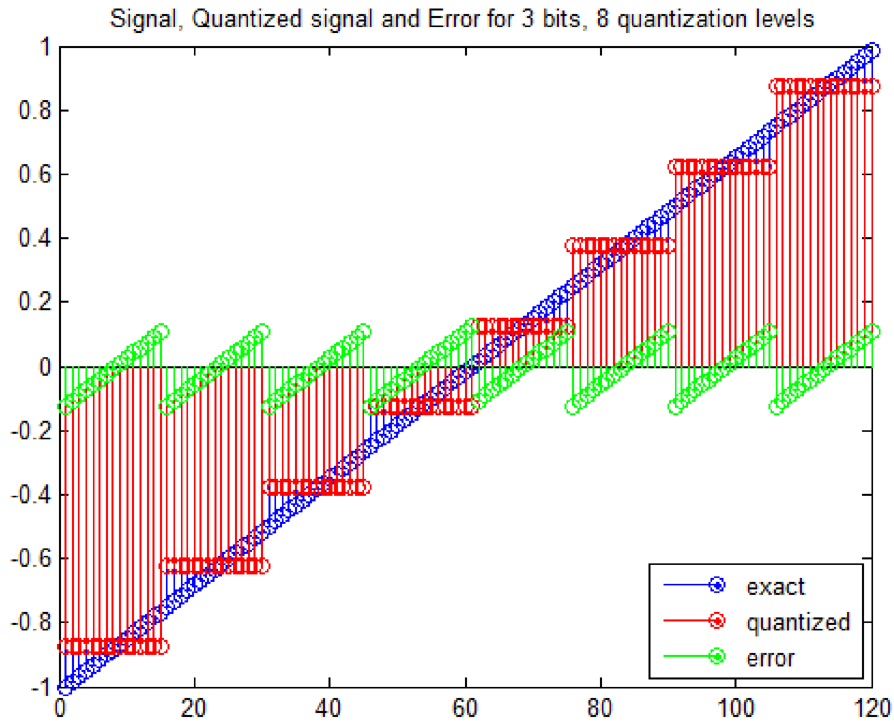
- Modulator: Moves the signal to higher frequency for transmission and multiplexing
- The "obvious" thing to do
  - Message waveforms are analog
  - Waveforms sent over the channel must be analog
- But not the right thing to do
  - Not efficient
  - Analog communication is rendered obsolete by digital communication

Digital Communications

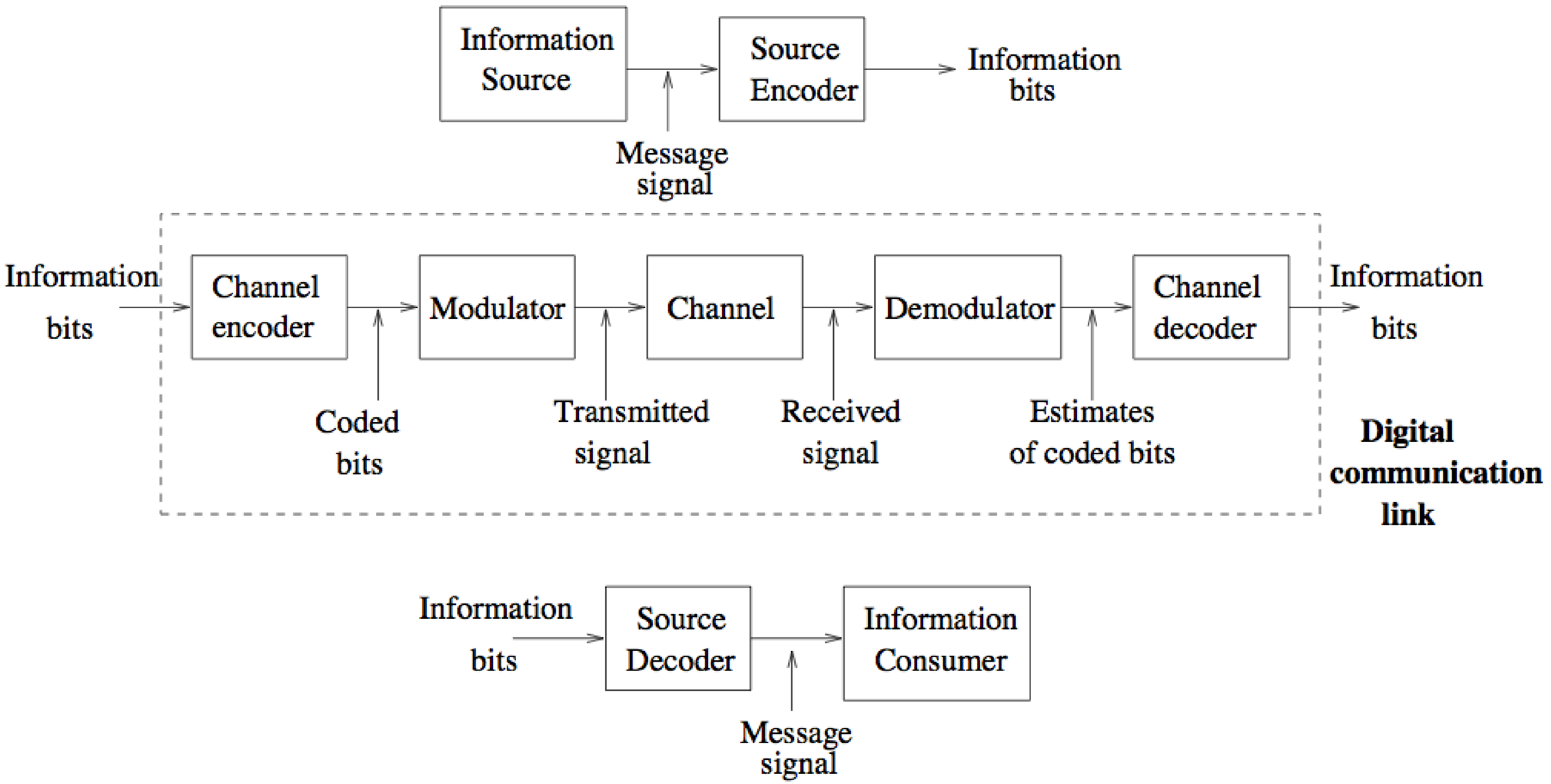
- **Message signal is digital**
  - Discrete time signal which takes only discrete level of values
  - Any signal which can be converted to 0’s and 1’s
- Example
  - Demographic data: Census data (Number of people in family, Gender, Age, Income (rounded to some decimal),
  - Text in any language (ASCII code: A 0041H, B0042H,…)
  - Data generated or stored by computer and mobiles

Digital Communications

- Analog signal can be converted to digital signal or sequence by sampling and quantization
  - Songs/Movies stored in CD/DVD, Data in hard-drive



Digital Communication Systems



Digital Communications: *Transmitter*

- **Source Encoder**
  - Finite number of messages to bits
  - Source compression
    - Obtains a digital representation of source signals using minimum binary digits. This leads to source compression. Removes Redundancy
    - Compression based on statistics: Example Huffman coding

Message	Binary Mapping
m_1	00
m_2	01
m_3	10
m_4	11

Message	Probability	Huffman Coding
m_1	0.8	0
m_2	0.1	11
m_3	0.08	100
m_4	0.02	101

      - Avg. number of bits/sample = 2
      - Avg. number of bits/sample = 0.8(1) + 0.1(2)+0.08(3) +0.02(3) = 1.3
    - Compression based on redundancy For e.g.: ZIP, PNG, MPEG, JPEG
      - 2000000000 (2 billion) can be represented by 2E9 (3 characters for 10)
  - Information bit rate depends on the message and nature of application

Digital Communications: *Transmitter*

- **Channel Encoder**
  – Introduces controlled redundancy to combat channel errors.

Example of repetition code (3,1)

- Let P_b be bit error probability that a bit will be flipped.
- Using repetition code, we send 111 for 1 and 000 for 0.
- Decide 1 was sent if majority of bits are 1
- Probability of error for this repetition code is

$$\begin{aligned} P_e &= P(2 \text{ bits in error}) + P(3 \text{ bits in error}) \\ &= \binom{3}{2} P_b^2 (1 - P_b) + \binom{3}{3} P_b^3 = 3P_b^2 - 2P_b^3 \\ &\approx 3P_b^2 \end{aligned}$$

- For $P_b = 0.1$, $P_e \approx 0.03$; For $P_b = 0.01$, $P_e \approx 0.0003$.

Digital Communications: *Transmitter*

- **Digital Modulator**
  - Converts the bit stream into a waveform suitable for transmission over the channel
    - For e.g: If the channel has band-pass response, then the modulator up-converts the frequency band of the information source to the necessary band.
  - Power and bandwidth constraints

Digital Communication: *Receiver*

- **Digital Demodulator**
  - It obtains the corrupt waveform and converts it to a bit-stream. For e.g. It down-coverts the frequency of the waveform and converts it in bits

- **Channel Decoder**
  - Obtains an estimate of the information bits. Uses the "Redundancy" to combat channel variations

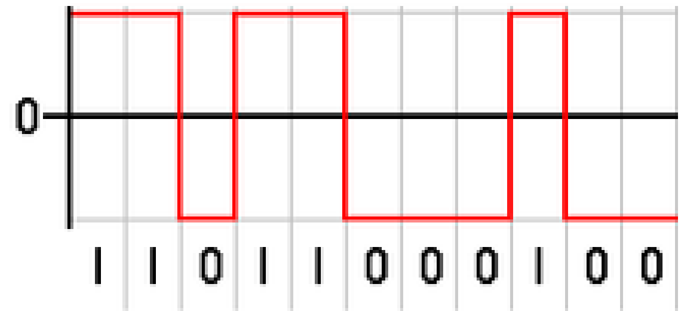
- **Source Decoder**
  - It obtains an estimate of the actual information transmitted
  - Decompress or unzip the signal
  - Bits to message mapping

Digital Communication Systems

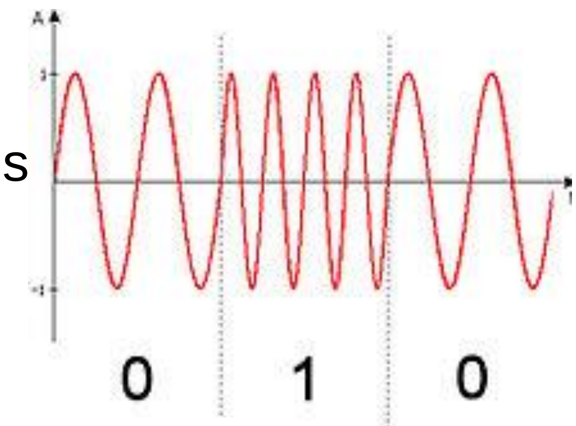
- Examples
  - Cellular 2G/3G/4G/5G
  - TV and Radio broadcasting (DVB-T and DRM)
  - CD/DVD
  - Hard drive

Digital Communications

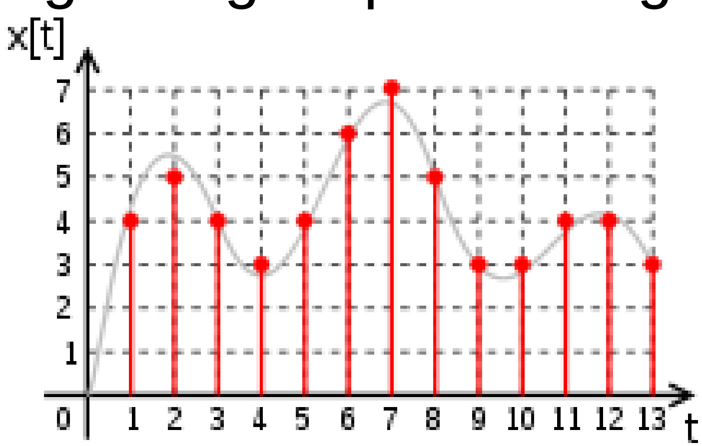
- In digital electronics, digital signal transmitted as a pulse train in baseband



- Digital communication in passband
  - Distinct finite number of analog waveforms



- Digital signal processing and storage



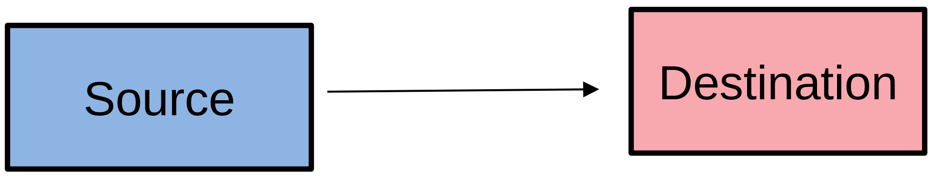
Quantization Levels	Binary mapping
l_1	000
l_2	001
l_3	010
l_4	011
l_5	100
l_6	101
l_7	110
l_8	111

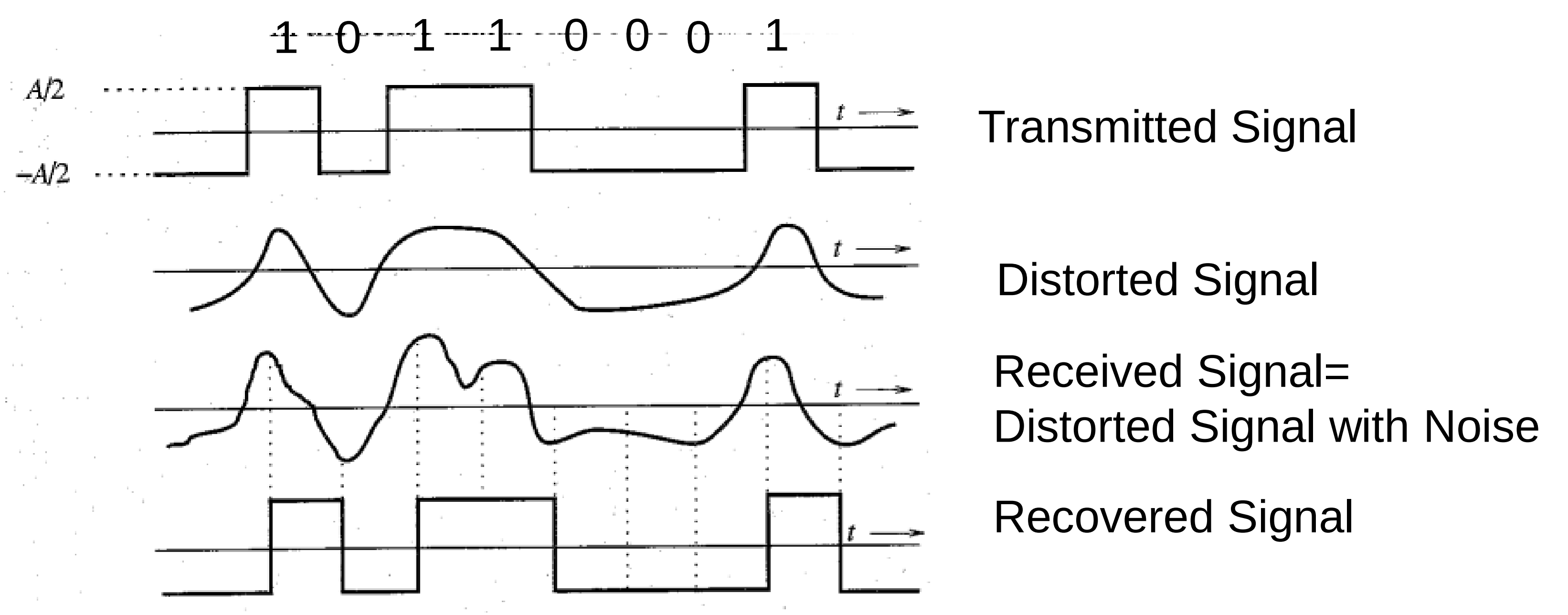
Transition from Analog to Digital

- Content is often analog (speech, image, video)
- Signals sent over physical channels are analog
  - Currents, voltages, EM waves are continuous-valued, continuous-time functions
- Many communication systems have shifted
  - Analog cellular to digital cellular (CDMA, GSM, OFDM)
  - Analog TV/radio to Digital TV/radio
  - VR and Audio casettes to CDs, VHS to DVD

Why Digital?

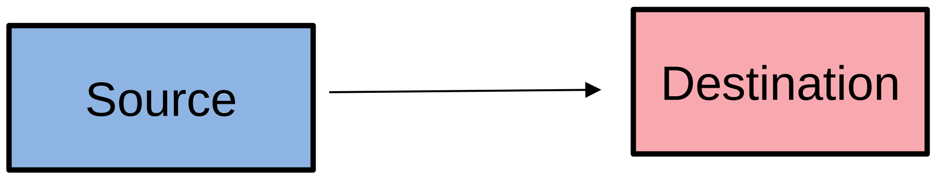
- Robust against distortion and noise

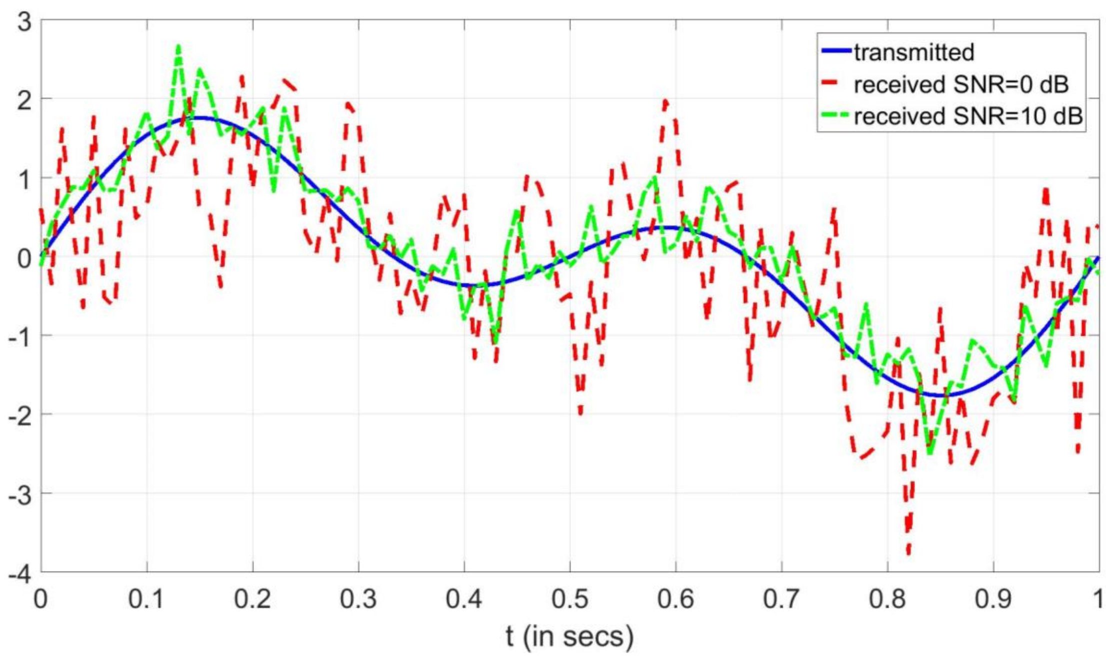




Why Digital?

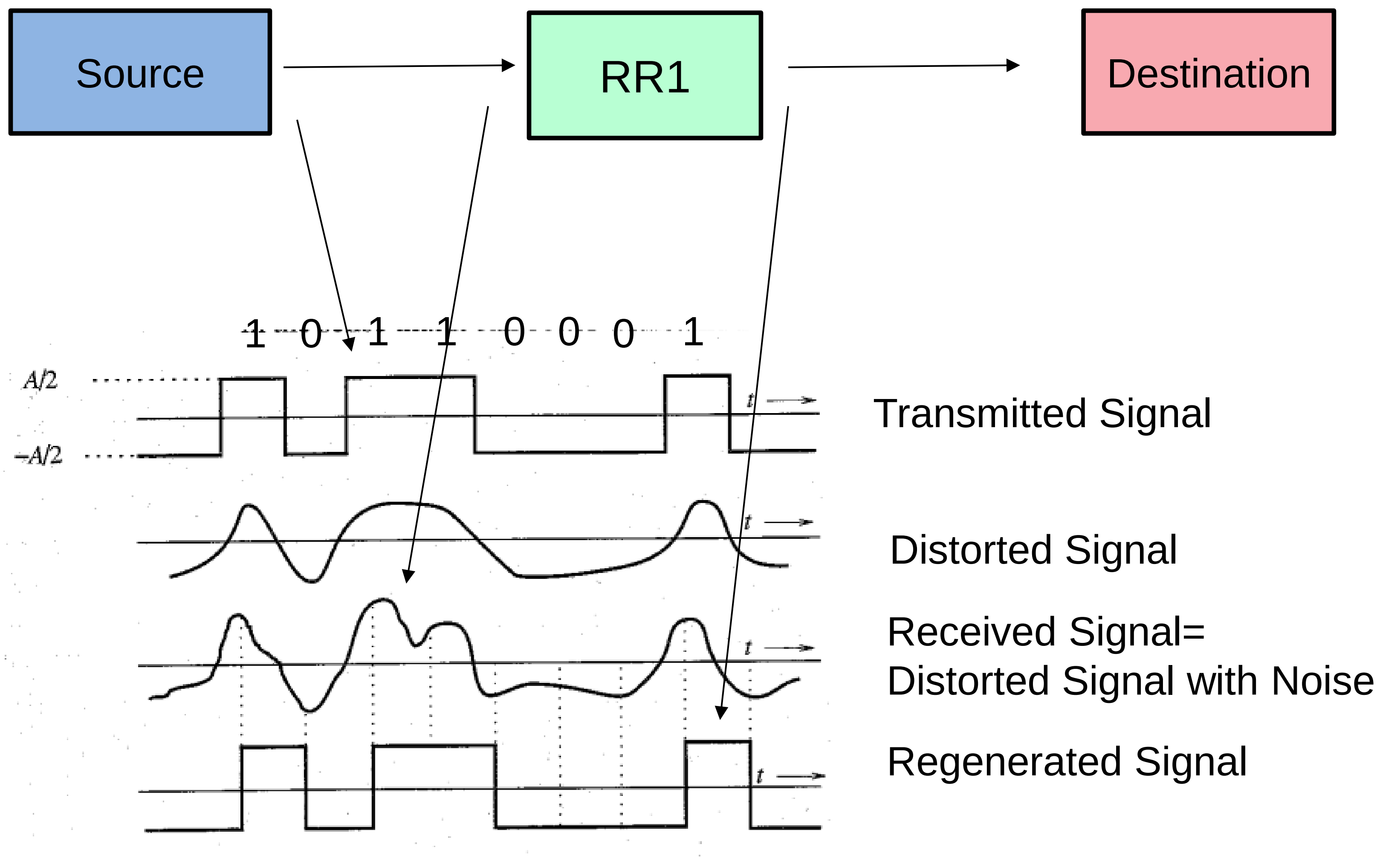
- Robust against distortion and noise
- Case of Analog





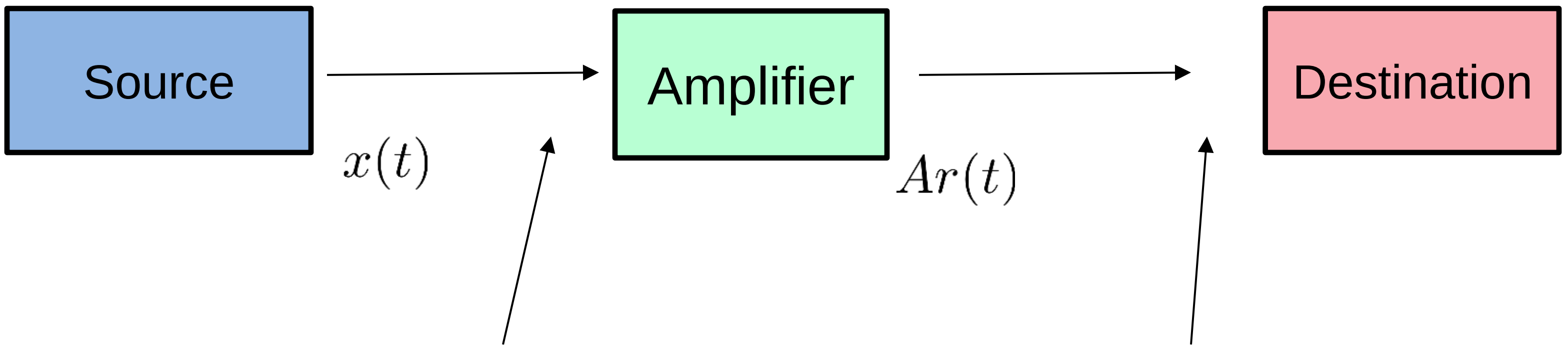
Why Digital?

- Viability of regenerative repeaters for digital comm.



Why Digital?

- Case of Analog and AWGN channels



$$r(t) = \alpha_1 x(t) + w_1(t)$$

$$\begin{aligned} y(t) &= \alpha_2 A r(t) + w_2(t) \\ &= \alpha_2 A(\alpha_1 x(t) + w_1(t)) + w_2(t) \\ &= \alpha_2 A \alpha_1 x(t) + \alpha_2 A w_1(t) + w_2(t) \end{aligned}$$

- Signal strength increases after each amplifier. SNR keeps decreasing after each stage of amplification.

Why Digital?

- Robust as compared to analog communication
  - Viability of regenerative repeaters for digital comm.
- Design of digital communication systems based on source-channel separation principle
  - Source-independent and channel-optimized: Huge Gains
  - Not possible in Analog
- Digital hardware is flexible and allows the use of microprocessors and VLSI circuits
  - Scalability
- Digital hardware is much cheaper and more robust
- Digital signals can be coded for arbitrary low error rates
  - Shannon's theorem
- Possibility of encryption
- Digital signal storage is easy and inexpensive to store
- And Many more…

Is there still a role for analog?

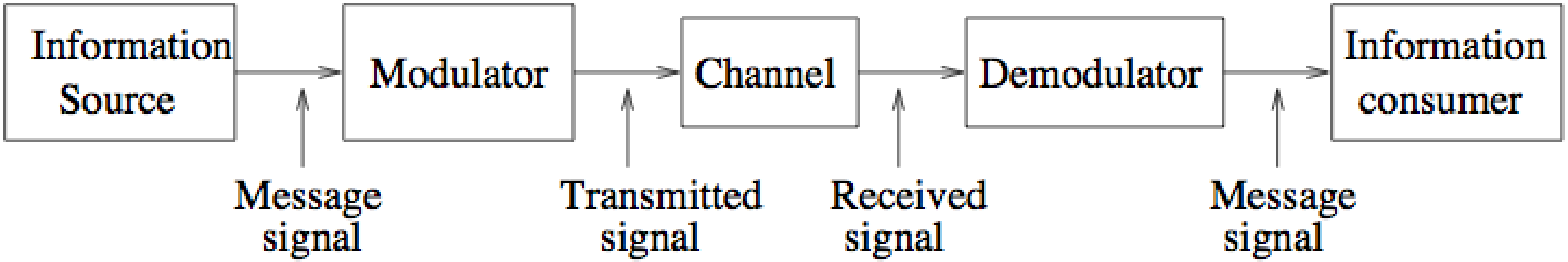
- **Of course! The physical world is analog**
- Signal transmission: Need to convert digital data to analog signals that can be sent over the physical channel
- Signal reception: Need to convert analog received signals back into digital data
- RF signal processing still a challenge

Today's Class

Importance of this subject

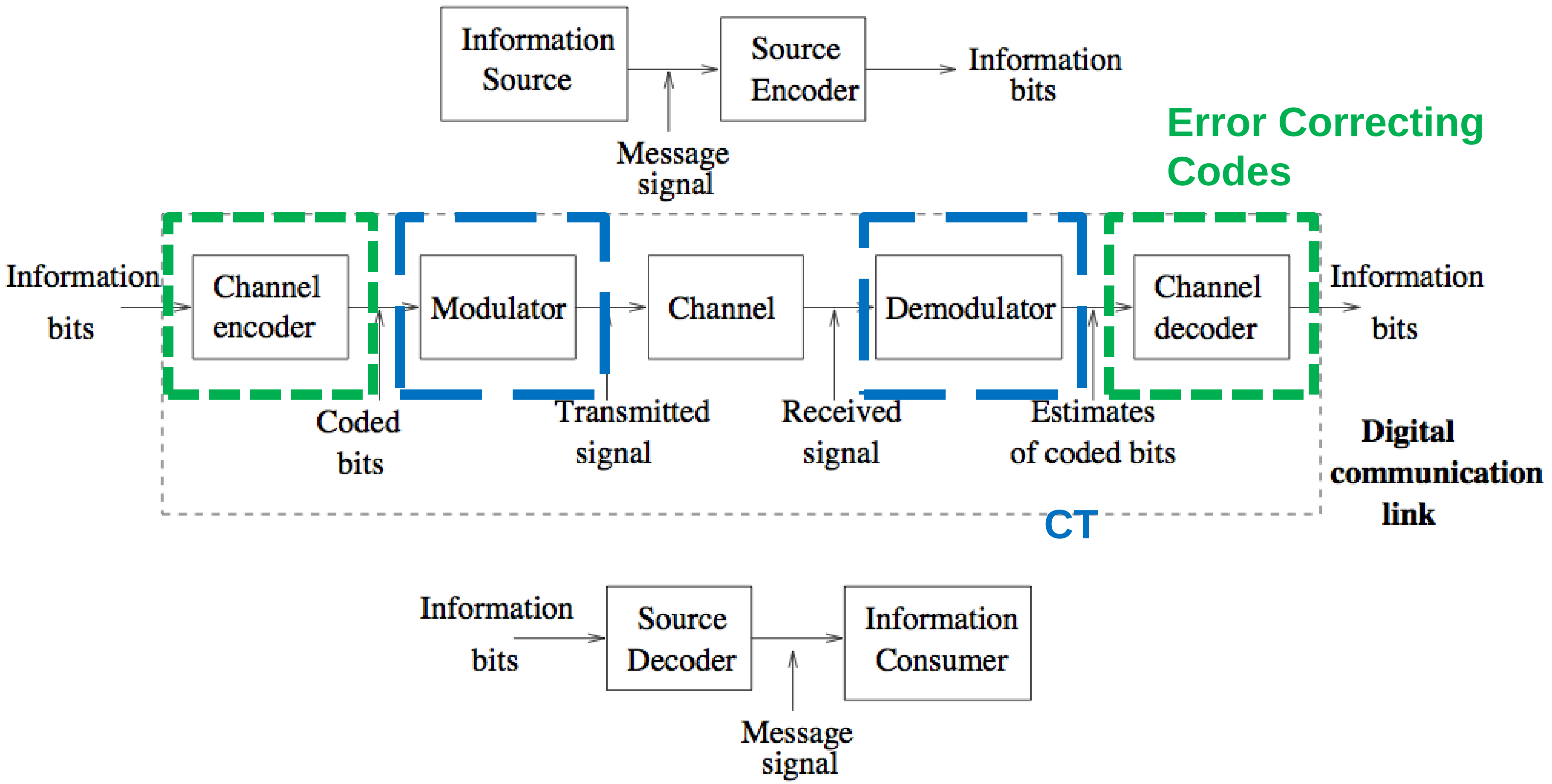
- Core subject
  - Form foundation for communication theory
  - Prerequisite for higher level subjects
    - Wireless Communications
    - Error Correcting Codes
    - Information Theory
    - Machine Learning for Communications

Analog Communication Systems

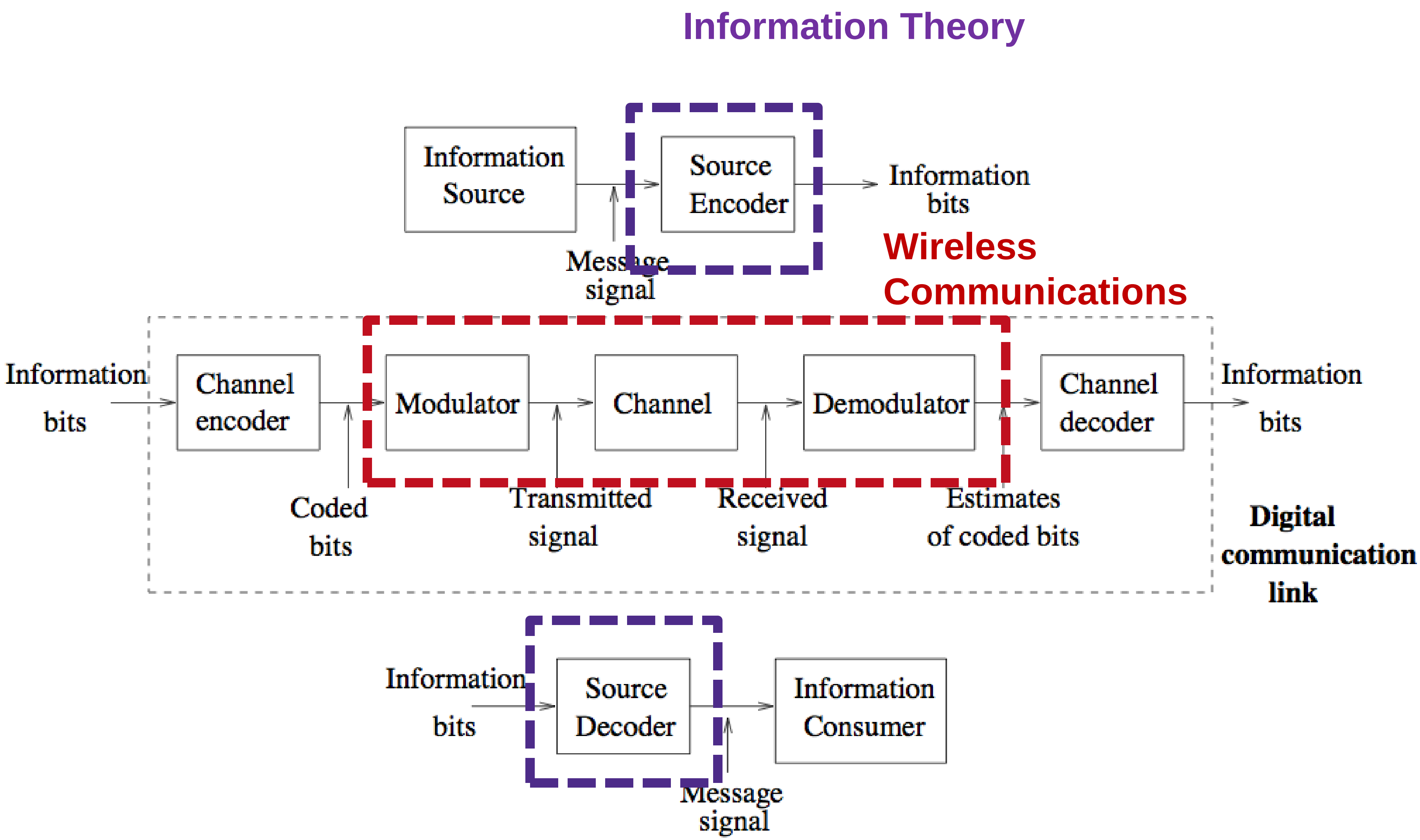


CT

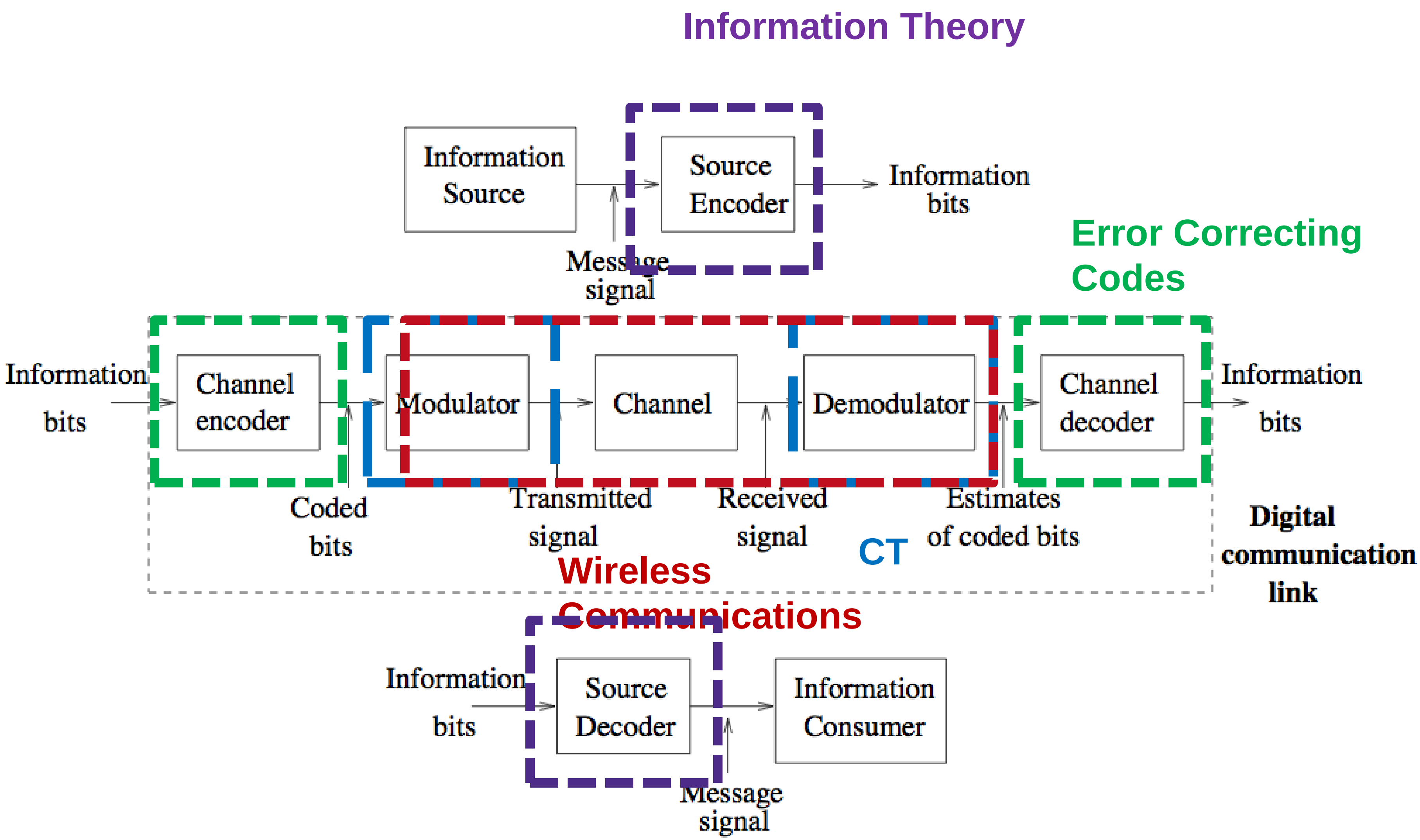
Digital Communication Systems



Digital Communication Systems



Digital Communication Systems



Importance of this subject

- Core subject
  - Form foundation for communication theory
  - Prerequisite for higher level subjects
    - Wireless Communications
    - Error Correcting Codes
    - Information Theory
    - Machine Learning for Communications
- Industry and Academic
  - Decades of ECE: 5G, 6G, IoT, ML for Comm., mmWave, Tactile internet (Autonomous vehicles, remote surgery, AR/VR tours)
- Problem solving and analytical skills
- Opportunity to work in SPCRC

Course Details/ Logistics

Syllabus (Tentative)

- Representation of bandpass signals and systems
  - lowpass equivalent of bandpass signals and systems
- Analog Communication Methods
  - AM-DSB and SSB; FM-narrowband and wideband; Demodulation of AM and PM/FM, Phased locked loop (PLL);
  - Brief overview of Line Coding and PWM
- Digital Modulation
  - Representation of Digitally Modulated Signals; Memoryless modulation methods: PAM, PSK, QAM, Orthogonal Multi-Dimensional Signals
- Random Processes
  - Review of Correlation, ESP and PSD; Noise Modelling, Thermal Noise, AWGN
- Performance of Digital methods in the presence of AWGN
  - Hypothesis testing, Signal Space Concepts, Performance analysis of ML reception, Bit error probability, Link budget analysis

Resources

BOOKS

- **U. Madhow: Introduction to Communication Systems**
  - **http://www.ece.ucsb.edu/wcsl/Publications/intro_comm_systems_madhow_jan2014b.pdf**
- B.P.Lathi, “Modern Digital and Analog Communication Systems”, 3rd Edition, Oxford University Press, 2007.
- J.G.Proakis, M.Salehi, “Fundamentals of Communication Systems”, Pearson Education 2006

VIDEOS

- National Programme on Technology Enhanced Learning (NPTEL)
  - Analog Communication http://nptel.ac.in/courses/117102059/
  - Digital Communication: http://nptel.ac.in/courses/117101051/

VIRTUAL LAB

- Virtual lab on Systems, Communications and Control, IIT Guwahati

  http://iitg.vlab.co.in/?sub=59&brch=163

Course Portal

MOODLE: https://courses.iiit.ac.in/

Under Spring 2025

If you are not already enroled, email me.

- Assignments
- News
- Discussion Forum

Approach

- Slides + Blackboard
- Active participation in class
- Regular Attendance
- Expected: Read-Attend-Revise-Assignments

Guidelines

- No disturbance in class
- No mobile phones/tablets in class
- No recording

Exams and Evaluation

- Mark Distribution
  - Quiz 1 (10)
  - Quiz 2 (10)
  - MidSem (20)
  - Assignments + Quiz (20)
  - Final Exam (40-50)

- Grading: TBD
  - Mostly Gaussian with fixed cut/off for A (>=85) and F(<40) grades)

Assignments

- 6-8 handwritten assignments + 1-2 Matlab
- Due in one week
- Firm Deadlines
  - One late homework assignment allowed without penalty
  - 2 marks will be deducted on other late assignments
- Quiz based on Assignment!
  - There are 2-2.5 marks for each assignment
  - These marks will be given only if the assignment is completed and quiz answers are correct.
- Cooperative learning is encouraged not copying!!!
  - Discussion of concepts in homework is encouraged
  - Sharing of homework or code is not permitted and will be penalized

Tutorials

- Time: Tuesdays 2-3.30 pm
- Venue: H204
- Attendance: Mandatory if informed in advance!
- Quizes, Problem Solving, Queries
- Use of discussion forum
- TAs: Naganjani, Nikhil, Manasa and Vishnu Priya

EC5.203 Communication Theory I (3-1-0-4):

Lecture 2
Overview of Signals And Systems

Instructor: Dr. Sachin Chaudhari

Email: sachin.chaudhari@iiit.ac.in

Jan. 9, 2025

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Reference

- Chapter 2 (Madhow)
  - Sec. 2.1-2.5 : Overview of Signals and Systems, Notations
  - Sec. 2.6-2.8: Energy Spectral Density, Bandwidth, Structure of passband signal

Quick Review of Signals and Systems and Notations

Indicator Function

- S&S: The step function was given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

- *CT-1*: $u(t)$ represent CT signal and $u[n]$ represent DT signal while step function is used for

$$I_{[0,\infty)}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

- Indicator function

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

Sinusoidal Signal

- Sinusoids

$$s(t) = A\cos\left(2\pi f_0 t + \theta\right) \quad \textbf{Polar Form}$$

  where $A > 0$ is the amplitude, f_0 is the frequency, and $\theta \in (0, 2\pi]$ is the phase.

- Sinusoids with known A, f_0, and θ cannot carry information.

- Modulation varies one or more of these parameters to convey information.

Sinusoidal Signal

- Sinusoids

$$s(t) = A\cos(2\pi f_0 t + \theta)$$

Polar Form

where $A > 0$ is the amplitude, f_0 is the frequency, and $\theta \in (0, 2\pi]$ is the phase.

- Sinusoid can also be written as

Rectangular form

$$s(t) = A_c \cos 2\pi f_0 t - A_s \sin 2\pi f_0 t$$

where $A_c = A\cos\theta$ and $A_s = A\sin\theta$ are real numbers. Using Euler's formula

Complex number

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta = A_c + jA_s$$

where $A = \sqrt{A_c^2 + A_s^2}$ and $\theta = \tan^{-1}(A_s/A_c)$.

Complex Exponential

- Complex exponentials

$$s(t) = Ae^{j(2\pi f_0 t + \theta)} = \alpha e^{j2\pi f_0 t}$$

where $\alpha = Ae^{j\theta}$ a complex number that contains both the amplitude and phase information.

Inner Product

- The inner product for two $m \times 1$ complex vectors $\mathbf{s} = (s[1], \ldots, s[m])^T$ and $\mathbf{r} = (r[1], \ldots, r[m])^T$ is given by

$$< \mathbf{s}, \mathbf{r} >= \sum_{i=1}^{m} s[i] r^*[i] = \mathbf{r}^H \mathbf{s}.$$

  where $(.)^H$ denotes Hermitian operation with vector $(\mathbf{r})^H$ being conjugate transpose of vector $\mathbf{r}$.

- Similarly, we define the inner product of two possibly complex-valued signals $s(t)$ and $r(t)$ as follows

$$< s, r >= \int_{-\infty}^{\infty} s(t) r^*(t) \, dt$$

Energy and Norm

- The energy E_s of a signal

$$E_s = ||s||^2 \; = < s, s > = \int_{-\infty}^{\infty} |s(t)|^2 \; dt$$

- If $E_s = 0$, then it means that $s(t)$ must be zero *almost everywhere* which is also true for the norm of a vector.

Example 1: Solve!

- Find energy for signal

  (a) $s(t) = 2I_{[0,T]} + jI_{[T/2,2T]}$

  (b) $s(t) = e^{-3|t| + j2\pi t}$

Power

- The power of a signal $s(t)$ is defined as the time average of its energy computed over a large time interval

$$P_s = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |s(t)|^2 dt$$

- Finite energy signals have zero power.

Time average

- Time average of signal $g(t)$ is denoted by

$$\bar{g} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t)dt$$

- Using the above notation, power of signal is given by

$$P_s = \overline{|s(t)|^2}$$

while the DC value of $s(t)$ is $\overline{s(t)}$.

Example 2

- Compute E_s, P_s, and DC value for $s(t) = Ae^{j(2\pi f_0 t + \theta)}$ where $A > 0$ is the amplitude and $\theta \in [0, 2\pi]$ and f_0 is the real-valued frequency.

- Next, compute P_s and DC value for a real valued sinusoid $s(t) = A\cos(2\pi f_0 t + \theta)$, where $A > 0$ is the amplitude and $\theta \in [0, 2\pi]$ and f_0 is the real-valued frequency.

Convolution

- Convolution of two signals $u_1(t)$ and $u_2(t)$ is given by

$$v(t) = u_1(t) * u_2(t) = (u_1 * u_2)(t)$$

Fourier Series

- A periodic signal $x(t)$ can be represented in terms of Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where Fourier series coefficients are given by

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

- A periodic signal $u(t)$ can be represented in terms of Fourier series as

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n f_0 t}$$

where Fourier series coefficients are given by

$\omega = 2\pi f$

$$u_k = \frac{1}{T_0} \int_{T_0} u(t) e^{-j2\pi k f_0 t} dt$$

Fourier Transform

- Any aperiodic and finite duration signal $x(t)$ can be represented using Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$

Inverse Fourier Transform

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

Fourier Transform

- A aperiodic signal $x(t)$ can be represented in terms of Fourier transform as

$$u(t) = \int_{-\infty}^{\infty} U(f)e^{j2\pi ft} df$$

where Fourier transform is given by

$\omega = 2\pi f$

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-j2\pi ft} dt$$

Properties of Fourier Transform

- Linearity: For arbitrary complex numbers α and β,

$$\alpha u(t) + \beta v(t) \longleftrightarrow \alpha U(f) + \beta V(f)$$

- Time delay

$$u(t - t_0) \longleftrightarrow U(f) e^{-j2\pi f t_0}$$

- Frequency shift

$$U(f - f_0) \longleftrightarrow u(t) e^{j2\pi f_0 t}$$

- Differentiation in time domain

$$x(t) = \frac{du(t)}{dt} \longleftrightarrow j2\pi f U(f)$$

Properties of Fourier Transform

- Step Function: For $v(t) = I_{[0,\infty)}$

$$V(f) \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

- Integration:

$$u(t) = \int_{-\infty}^{t} x(t)\ dt = x(t) * v(t)$$

$$U(f) = X(f)V(f) = \frac{X(f)}{j2\pi f} + \overline{u}\delta(f)$$

Properties of Fourier Transform

- Parseval's identity:

$$< u, v > = \int_{-\infty}^{\infty} u(t)v^*(t)dt = \int_{-\infty}^{\infty} U(f)V^*(f)df$$

- For $u = v$,

$$||u||^2 = < u, u > = \int_{-\infty}^{\infty} |u(t)|^2 dt \ = \int_{-\infty}^{\infty} |U(f)|^2 df$$

Properties of Fourier Transform

- Convolution in time domain:

$$y(t) = u(t) * v(t) = (u * v)(t) \longleftrightarrow Y(f) = U(f)H(f)$$

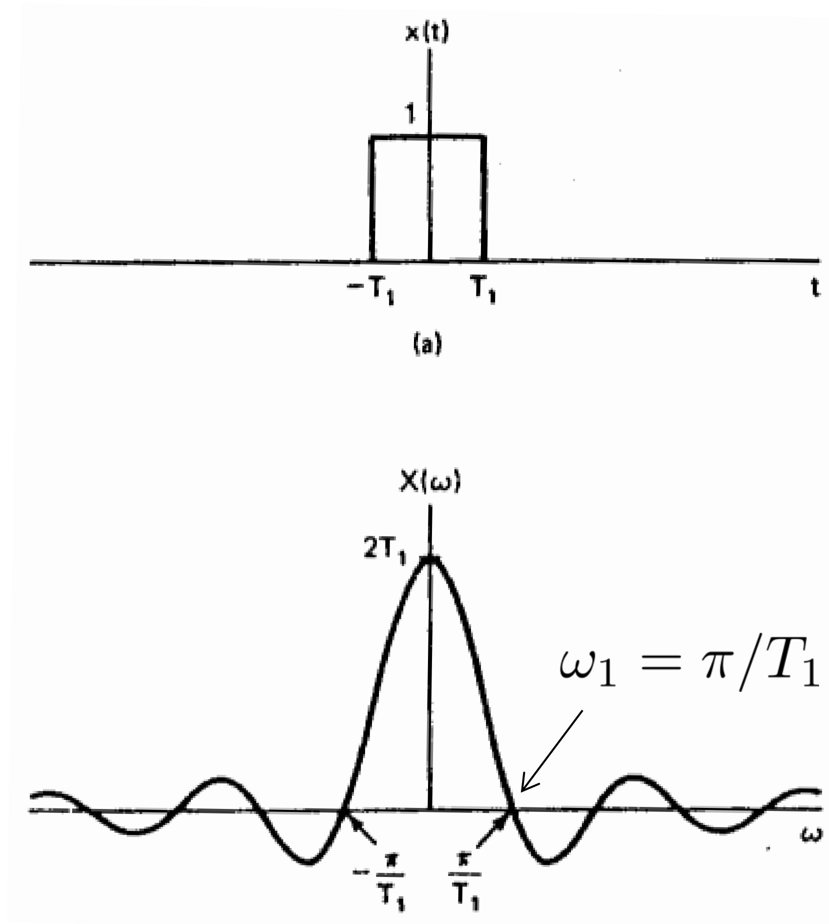
- Multiplication in time domain:

$$y(t) = u(t)v(t) \longleftrightarrow Y(f) = (U * H)(f) = U(f) * V(f)$$

Rectangular pulse and Sinc function: *S&S*

- Find Fourier transform for the signal

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$



$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega}$$

Rectangular pulse and Sinc function

- For a rectangular pulse $u(t) = I_{[-T/2,T/2]}(t)$ of duration T. Its Fourier transform is given by

$$U(f) = T\mathrm{sinc}(fT) = T\frac{\sin(\pi f T)}{\pi f T}$$

$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$

- The Fourier transform is denoted by

$$I_{[-T/2,T/2]}(t) \longleftrightarrow T\mathrm{sinc}(fT)$$

- Using duality, the other Fourier transform pair is

$$I_{[-W/2,W/2]}(f) \longleftrightarrow W\mathrm{sinc}(Wt)$$

Summary of Properties: S&S

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Summary of Properties: S&S

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

EC5.203 Communication Theory I (3-1-0-4):

Lecture 3
Baseband and Passband Representations

Instructor: Dr. Sachin Chaudhari

Email: sachin.chaudhari@iiit.ac.in

Jan. 16, 2025
and Jan. 20, 2025

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Reference

- Chapter 2 (Madhow)
  - Sec. 2.6-2.8: Energy Spectral Density, Bandwidth, Structure of passband signal

Recap

Sinusoidal Signal

- Sinusoids

$$s(t) = A\cos(2\pi f_0 t + \theta)$$ **Polar Form**

where $A > 0$ is the amplitude, f_0 is the frequency, and $\theta \in (0, 2\pi]$ is the phase.

- Sinusoids with known A, f_0, and θ cannot carry information.
- Modulation varies one or more of these parameters to convey information.
- Sinusoid can also be written as

Rectangular form

$$s(t) = A_c \cos 2\pi f_0 t - A_s \sin 2\pi f_0 t$$

where $A_c = A\cos\theta$ and $A_s = A\sin\theta$ are real numbers. Using Euler's formula

$$Ae^{j\theta} = A_c + jA_s$$ **Complex number**

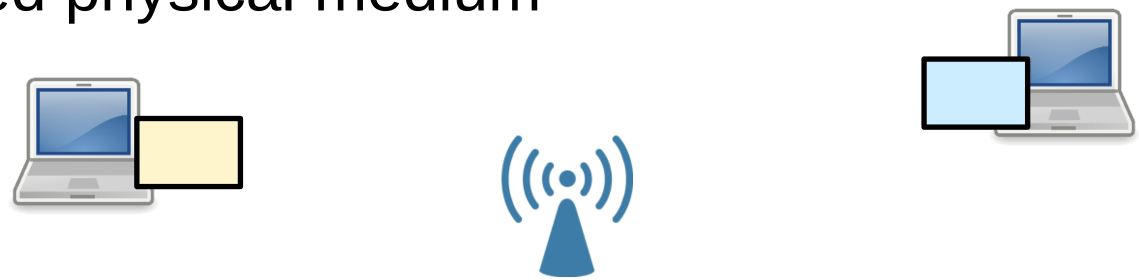
where $A = \sqrt{A_c^2 + A_s^2}$ and $\theta = \tan^{-1}(A_s/A_c)$.

Today's Class

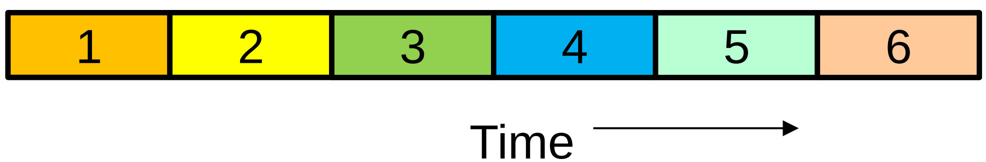
Energy Spectral Density and Bandwidth

Motivation

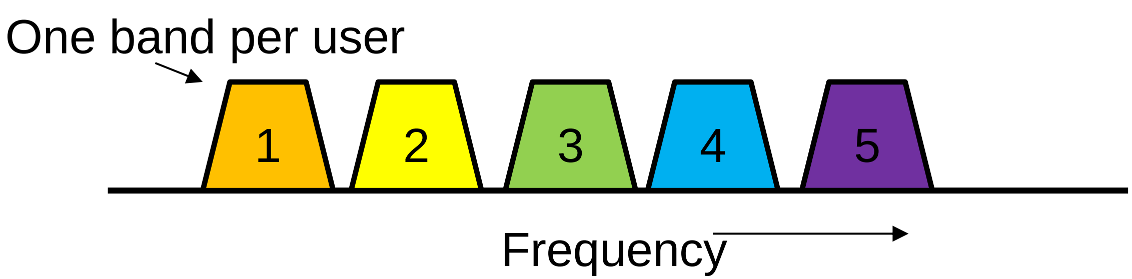
- Shared physical medium



- Example of Time Division Multiple Access (TDMA)



- Example of Frequency Division Multiple Access (FDMA)

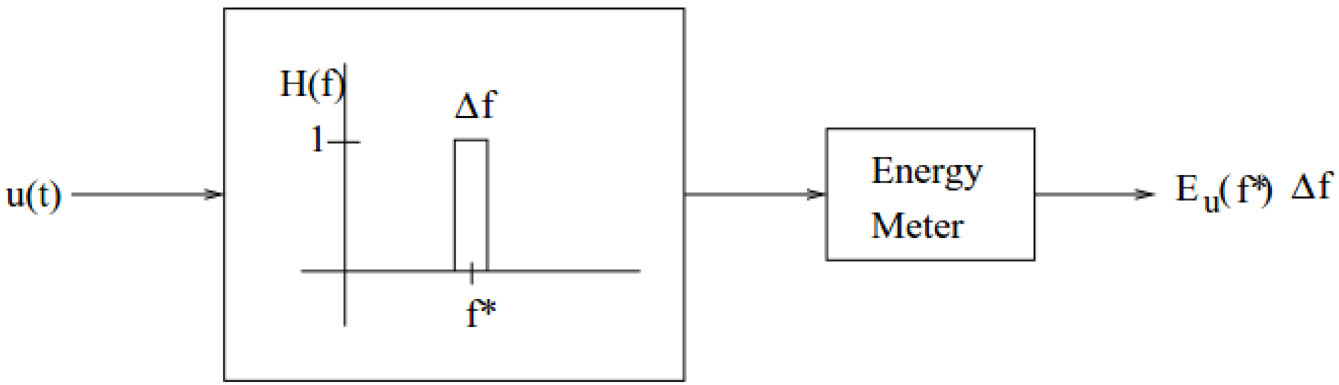


Motivation

- Communication channels have frequency-dependent characteristics
- Wireless spectrum is precious
- A broadcast and a shared medium
- Spectral characterization of the transmitted signal is important
  - Time limited signal has infinite bandwidth and vice-versa.

Energy Spectral Density

- Energy spectral density $E_u(f)$ of a signal $u(t)$ is defined to be the energy at the output of the filter divided by the width Δf in the limit as $\Delta(f) \to 0$.



- Prove that $E_u(f) = |U(f)|^2$

Energy Spectral Density

- Consider an ideal narrowband filter with transfer function

$$H_{f^*}(f) = \begin{cases} 1 & f^* - \Delta f < f < f^* + \Delta f \\ 0 & \text{otherwise} \end{cases}$$

- The frequency response of output $y(t)$

$$Y(f) = U(f)H(f) = \begin{cases} U(f) & f^* - \Delta f < f < f^* + \Delta f \\ 0 & \text{otherwise} \end{cases}$$

- By Parseval's identity, the energy at the output of the filter is

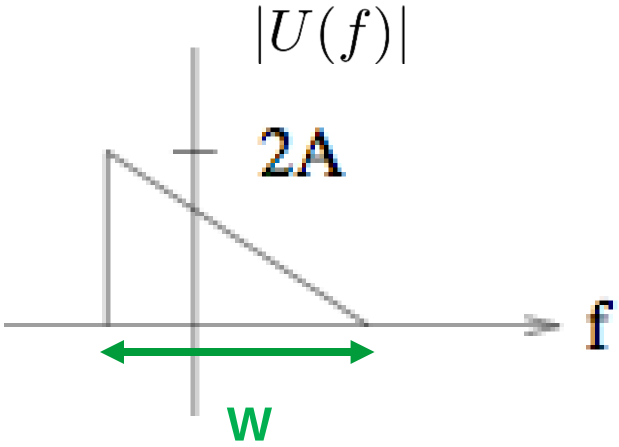
$$\int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{f^*-\Delta/2}^{f^*+\Delta/2} |U(f)|^2 \, df \approx |U(f)|^2 \Delta f$$

- Equating this to output of energy meter, we get the desired result.

Bandwidth for bandlimited signals

Several definitions depending on the application

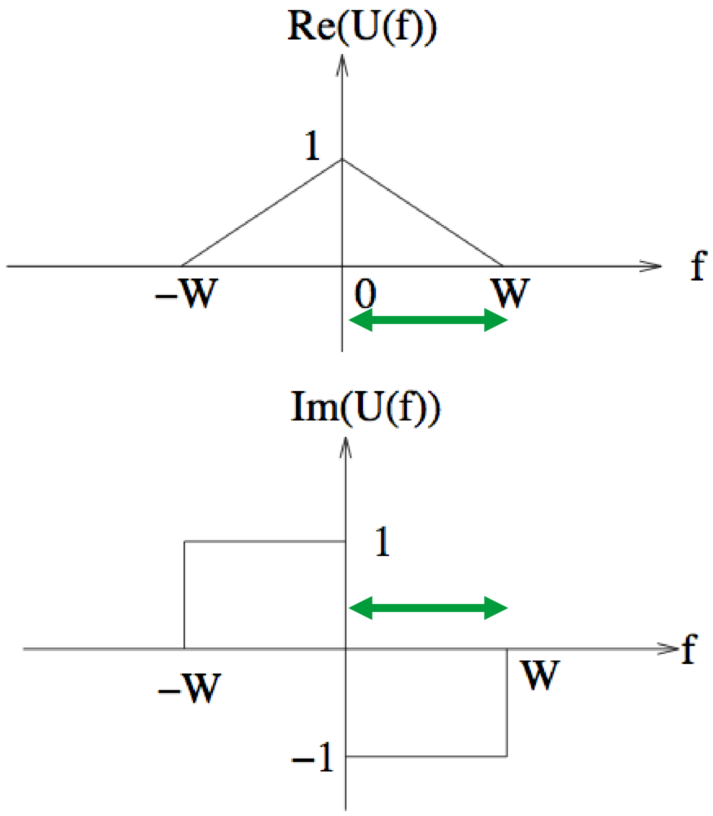
- The bandwidth of a strictly bandlimited signal $u(t)$ is defined to be the size of the band of frequencies occupied by $U(f)$.



Bandwidth for bandlimited signal: *Real Signal*

Several definitions depending on the application

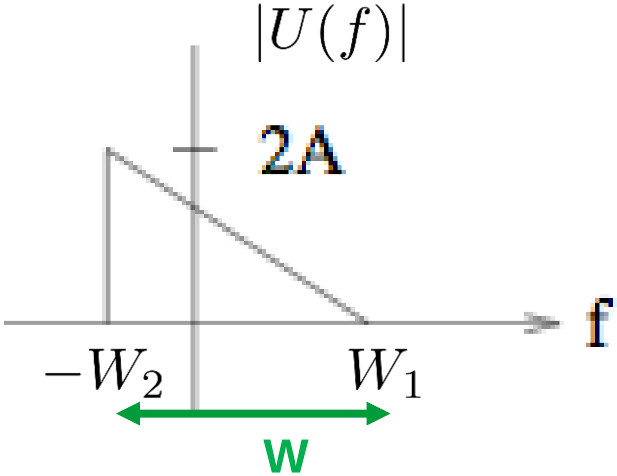
- For physical signals (real-valued), $U(-f) = U^*(f)$.
- Only positive frequencies count when computing bandwidth for physical (real-valued) signals. Also called one-sided bandwidth.



Bandwidth for bandlimited signals: complex

Several definitions depending on the application

- The bandwidth of a strictly bandlimited signal $u(t)$ is defined to be the size of the band of frequencies occupied by $U(f)$.

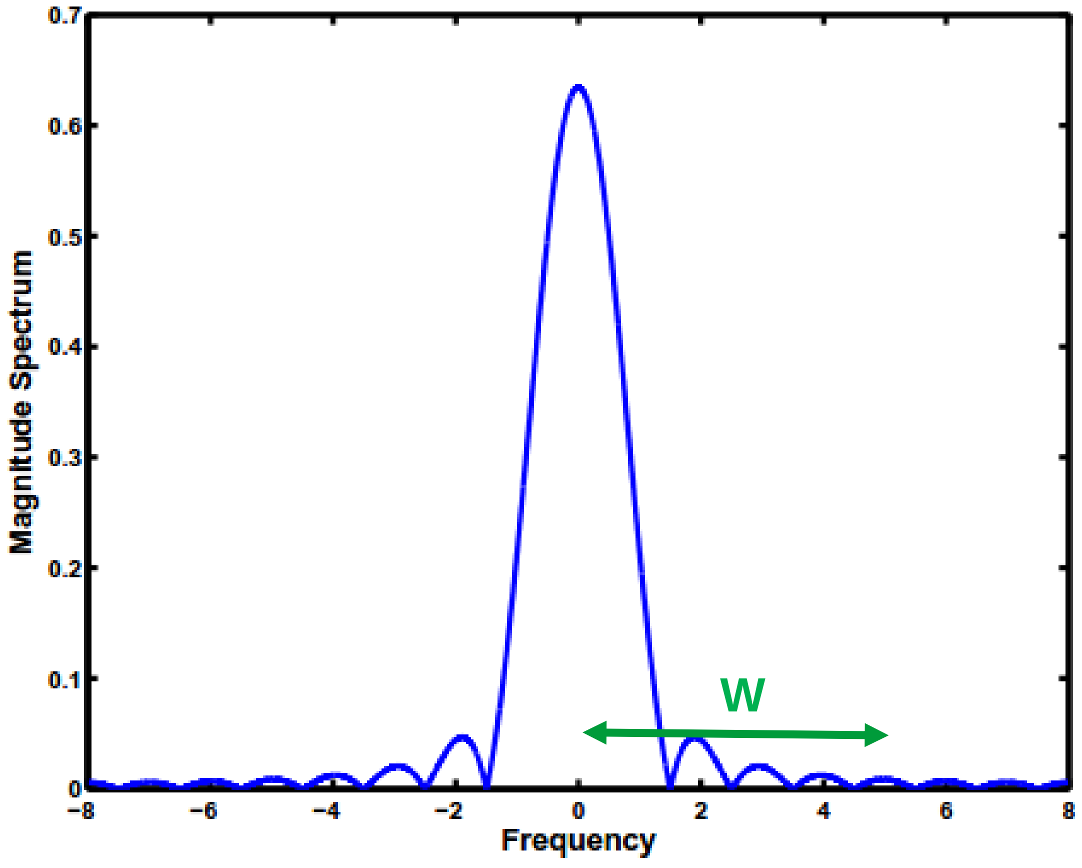


- The bandwidth of complex-valued signals is defined as the size of the frequency band it occupies over both positive and negative frequencies.

Bandwidth for not-bandlimited signal

Several definitions depending on the application

- The bandwidth of signal $u(t)$ which is not bandlimited is defined in terms of energy-containment bandwidth: the size of the smallest band which contains specified fraction of the signal energy.



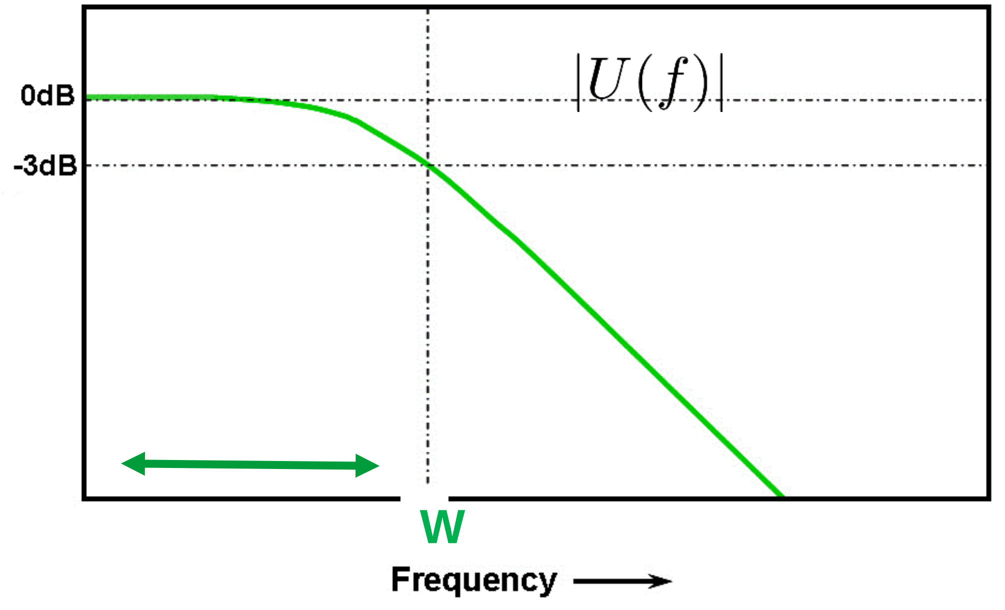
99% Energy

Bandwidth for not-bandlimited signal

Several definitions depending on the application

- The bandwidth of signal $u(t)$ which is not bandlimited can be defined in terms of band over which $|U(f)|^2$ is within some fraction of its peak value.

- For example, if the fraction is 0.5, the bandwidth is called 3-dB bandwidth.



Baseband and Passband

The Complex Baseband Representation

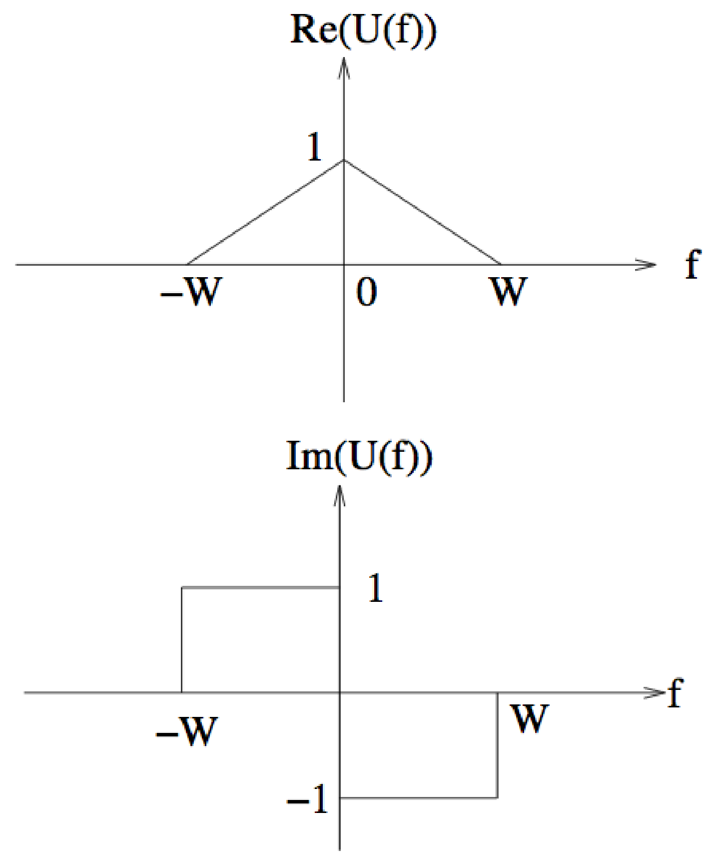
Baseband Signal

- A signal $u(t)$ is said to be baseband if the signal energy is concentrated in a band around DC and

$$U(f) \approx 0, \quad |f| > W$$

  for some W.

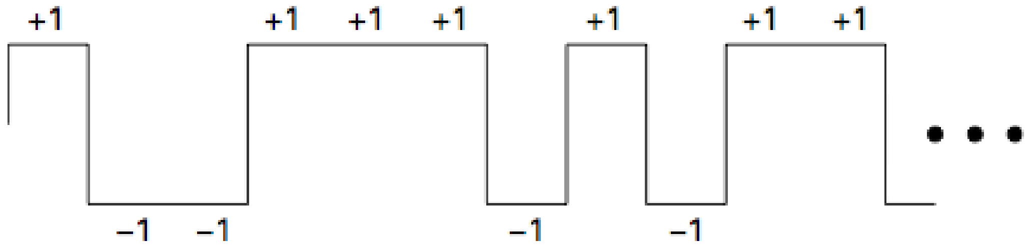
- Example: Information sources



Examples of baseband signals

- Speech and audio are baseband signals

- Two level digital signals over wired connection without modulation (also called line coding)



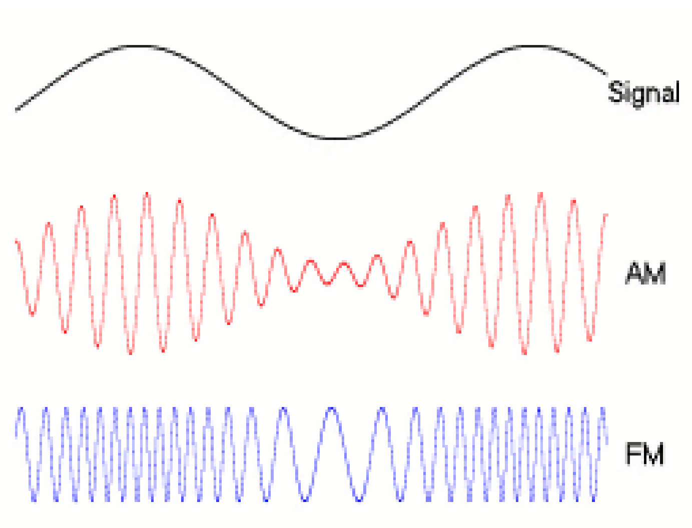
Passband Signal

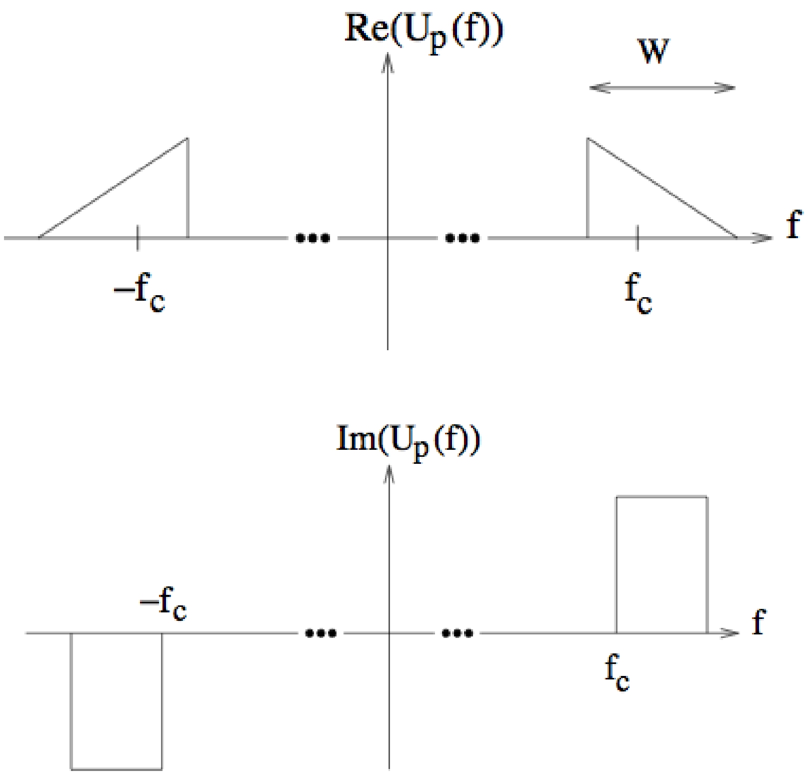
- A signal $u(t)$ is said to be passband if the signal energy is concentrated in a band away from DC with

$$U(f) \approx 0, \quad |f \pm f_c| > W$$

for some $f_c > W > 0$. Typically $f_c \gg W$

- Example: Wireless signal



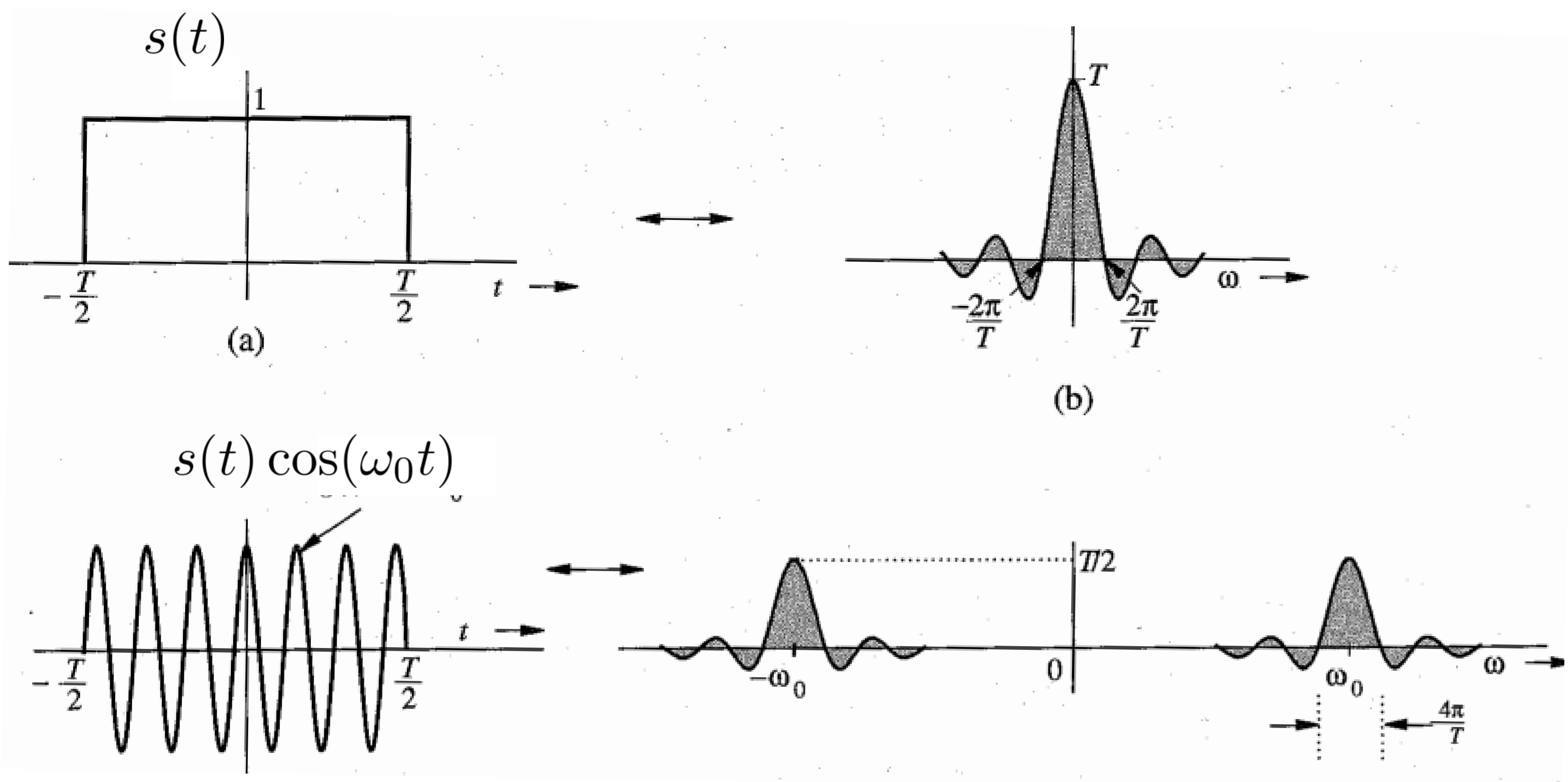


Example of Passband

- Consider this signal

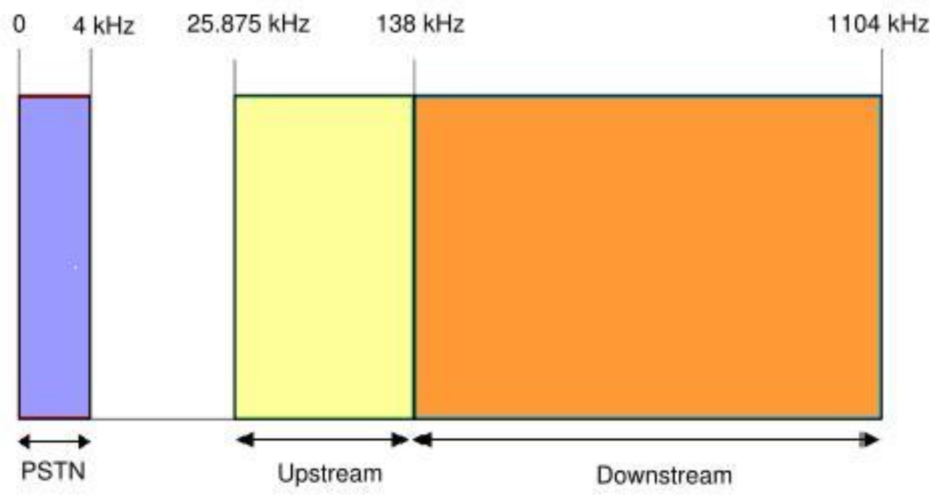
$$u_{\mathrm{p}}(t) = I_{[-T/2,T/2]}(t)\cos(2\pi f_0 t)$$

where $\omega_0 = 2\pi f_0 t$



Example of Passband in Wired Systems

- Digital Subscriber Line (DSL) is a technology, where high-speed data transmission coexists with voice transmissions in 0-4 KHz band
- Physical media is twisted pair copper wire



https://en.wikibooks.org/wiki/Communication_Networks/DSL

Differences between Baseband and Passband

- Baseband
    1. Original signal generated from the message source without any modulation of high frequency carrier
    2. Concentrated near zero frequency
    3. Can carry only one message signal at a time
    4. Examples: Ethernet, landline cables, network cables, coaxial cables.
- Passband
    1. Refers to modulated signal mostly
    2. Away from DC and concentrated around carrier frequency
    3. Multiple signals can be multiplexed simultaneously
    4. Examples: Wireless signal

Baseband Channel

- A channel modeled as a LTI system is said to be baseband if its transfer function $H(f)$ has support concentrated around DC and satisfies

$$H(f) \approx 0, \quad |f| > W$$

  for some W.

- Example: Wired channels

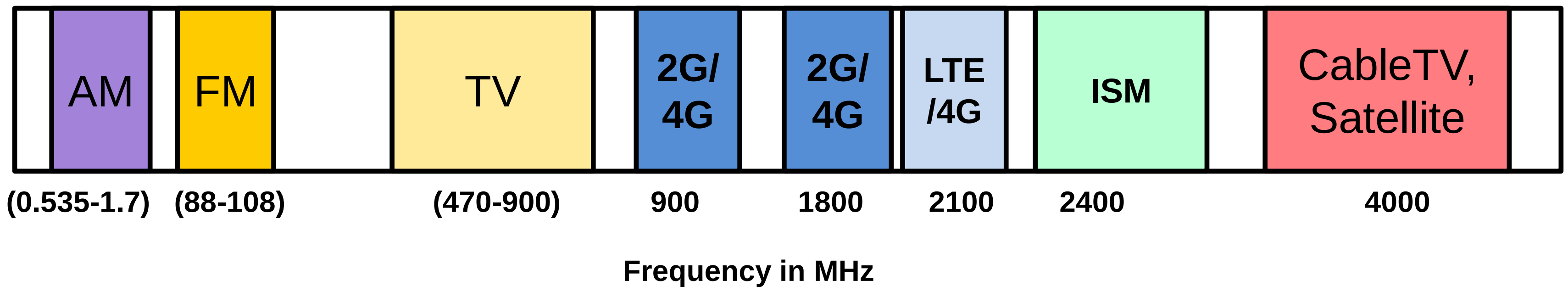
Passband Channel

- A channel modeled as a LTI is said to be passband if its transfer fuction $H(f)$ satisfies

$$H(f) \approx 0, \quad |f \pm f_c| > W$$

for some $f_c > W > 0$.

- Example: Wireless Channel



Baseband and Passband Signals/Channels

- Channels are often modeled as LTI systems: Signal passes through channel and then noise is added
- Channels allocated/described typically in terms of frequency bands: Signals have to be designed for corresponding frequency band
- Complex baseband representation of passband systems: Unified treatment of baseband and passband systems.

Structure of Passband Signal

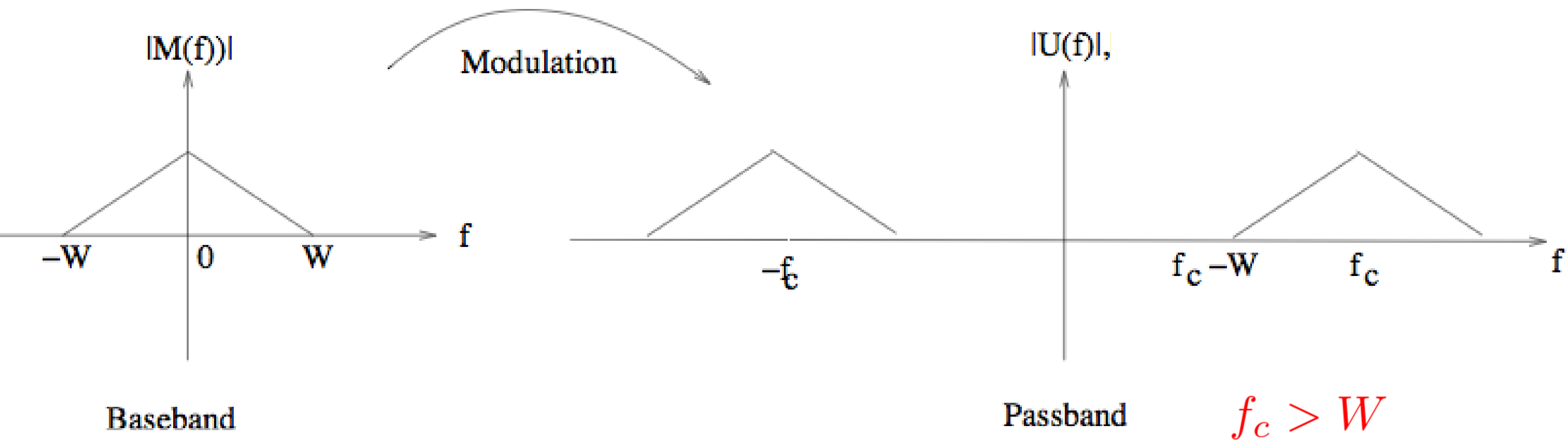
Motivation

- Passband is important for different communication systems
- Need to understand how to design transmitted signal from the information and recover the same from the received signal

Modulation or Upconversion

- Let $m(t)$ be the message signal of bandwidth W to be sent over passband channel around f_c called as carrier frequency.
- One method: multiply by a sinusoid at f_c

$$u_{\mathrm{p}}(t) = m(t)\cos(2\pi f_c t) \longleftrightarrow \frac{1}{2}\left(M(f - f_c) + M(f + f_c)\right)$$



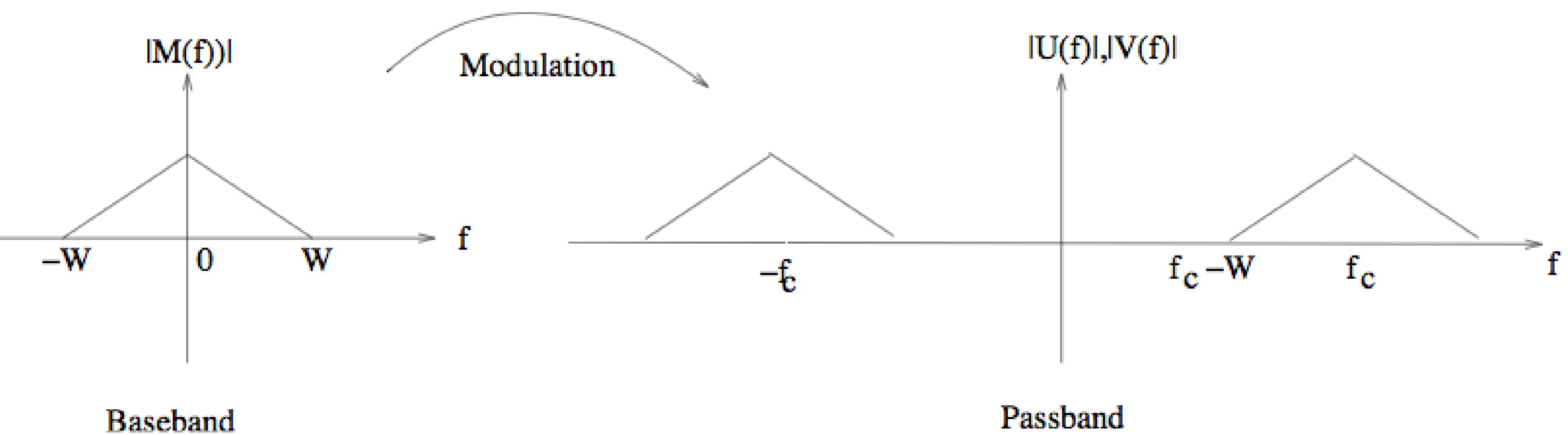
$f_c > W$

Modulation or Upconversion

- You can also use a sine

$$v_{\mathrm{p}}(t) = m(t)\sin(2\pi f_c t) \longleftrightarrow \frac{1}{2j}\left(M(f - f_c) - M(f + f_c)\right).$$

- Magnitude spectrum for $v_{\mathrm{p}}(t)$ will be same as that for $u_{\mathrm{p}}(t)$.



Modulation: *I and Q components*

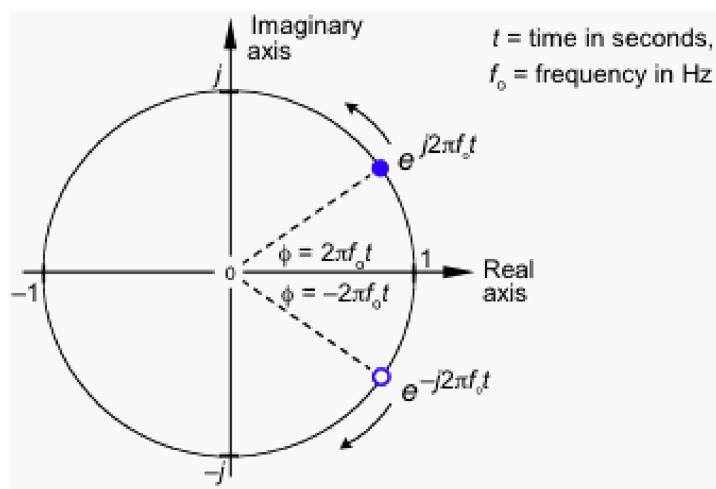
- If we use both cosine and sine carriers, we can construct a passband signal of the form

$$u_{\mathrm{p}}(t) = \boxed{u_{\mathrm{c}}(t)}\cos(2\pi f_c t) - \boxed{u_{\mathrm{s}}(t)}\sin(2\pi f_c t)$$

where $u_c(t)$ and $u_s(t)$ are real baseband signals of bandwidth at most W, with $f_c > W$.

In-phase or I component

Quadrature or Q component



Modulation: *I and Q components*

- If we use both cosine and sine carriers, we can construct a passband signal of the form

$$u_p(t) = \boxed{u_c(t)}\boxed{\cos(2\pi f_c t)} - \boxed{u_s(t)}\boxed{\sin(2\pi f_c t)}$$

where $u_c(t)$ and $u_s(t)$ are real baseband signals of bandwidth at most W, with $f_c > W$.

Carry all the information

Carry NO information

Orthogonality of I and Q channels

- Show that the passband waveforms $a_{\mathrm{p}}(t) = u_{\mathrm{c}}(t)\cos(2\pi f_c t)$ and $u_{\mathrm{b}}(t) = u_{\mathrm{s}}(t)\sin(2\pi f_c t)$ are orthogonal

$$< a_{\mathrm{p}}, b_{\mathrm{p}} >= 0$$

  i.e., I and Q components of $u_{\mathrm{p}}(t)$ are orthogonal.

- Advantage: we can send separate information on I and Q at the same time!.

Demodulation or Downconversion

- If we use both cosine and sine carriers, we can construct a passband signal of the form

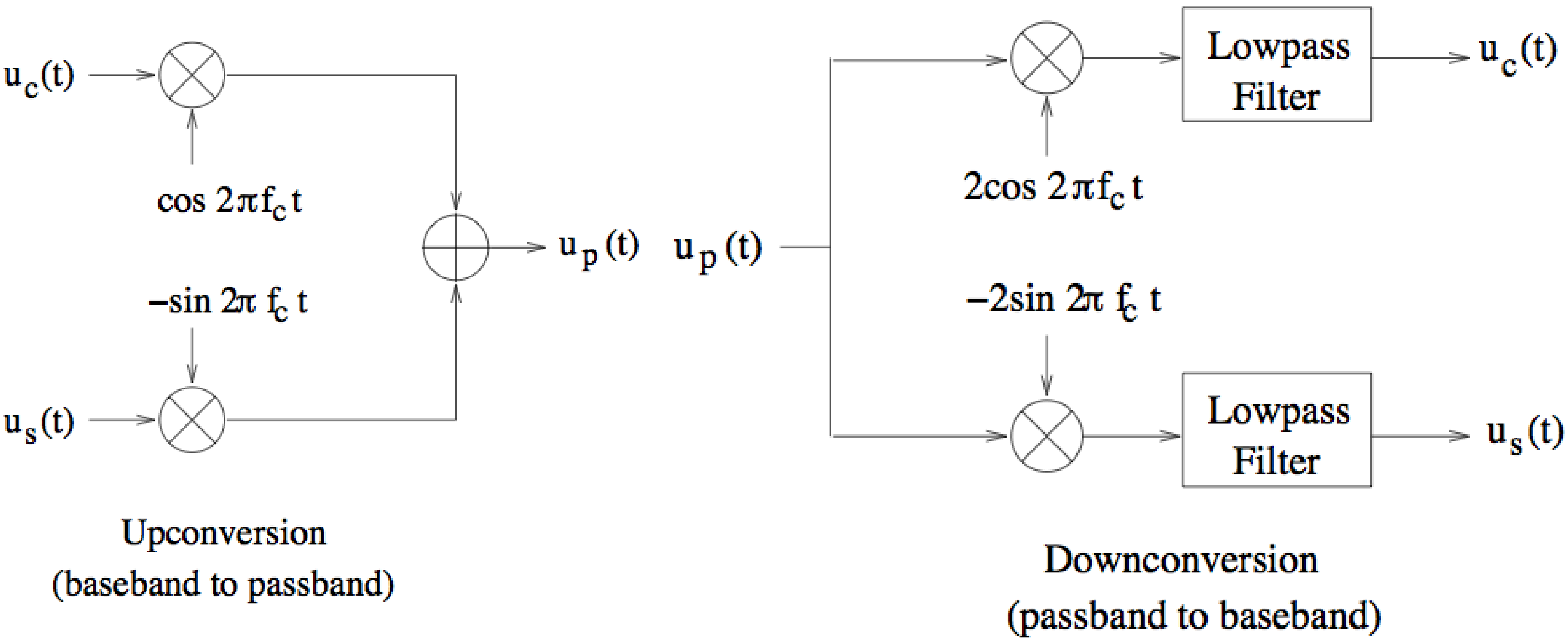
$$u_{\mathrm{p}}(t) = u_{\mathrm{c}}(t)\cos(2\pi f_c t) - u_{\mathrm{s}}(t)\sin(2\pi f_c t)$$

  where $u_c(t)$ and $u_s(t)$ are real baseband signals of bandwidth at most W, with $f_c > W$.

- Show that downconversion from passband to baseband can be done by lowpass filtering of these two components

$$\begin{aligned} &2u_{\mathrm{p}}(t)\cos(2\pi f_c t) \\ &-2u_{\mathrm{p}}(t)\sin(2\pi f_c t) \end{aligned}$$

Upconversion and Downconversion: *Block Diagrams*



Envelope and Phase

- $u_{\mathrm{p}} \equiv (u_{\mathrm{c}}, u_{\mathrm{s}})$ is called two-dimensional modulation.
- I and Q components corresponds to two dimensional waveforms in rectangular coordinates.
- Converting them to polar coorinates, we get

$$e(t) = \sqrt{u_c^2(t) + u_s^2(t)}$$

$$\theta(t) = \tan^{-1}\left(\frac{u_{\mathrm{s}}(t)}{u_{\mathrm{c}}(t)}\right)$$

where $e(t) \geq 0$ is the envelope and $\theta(t)$ is the phase.

- Here

$$u_{\mathrm{c}}(t) = e(t)\cos\theta(t)$$

$$u_{\mathrm{s}}(t) = e(t)\sin\theta(t)$$

Complex Envelope

- A two dimensional point can be mapped to a complex number.
- The complex envelope $u(t)$ of the passband signal $u_p(t)$ is defined as

$$u(t) = u_c(t) + ju_s(t) = e(t)e^{j\theta(t)}$$

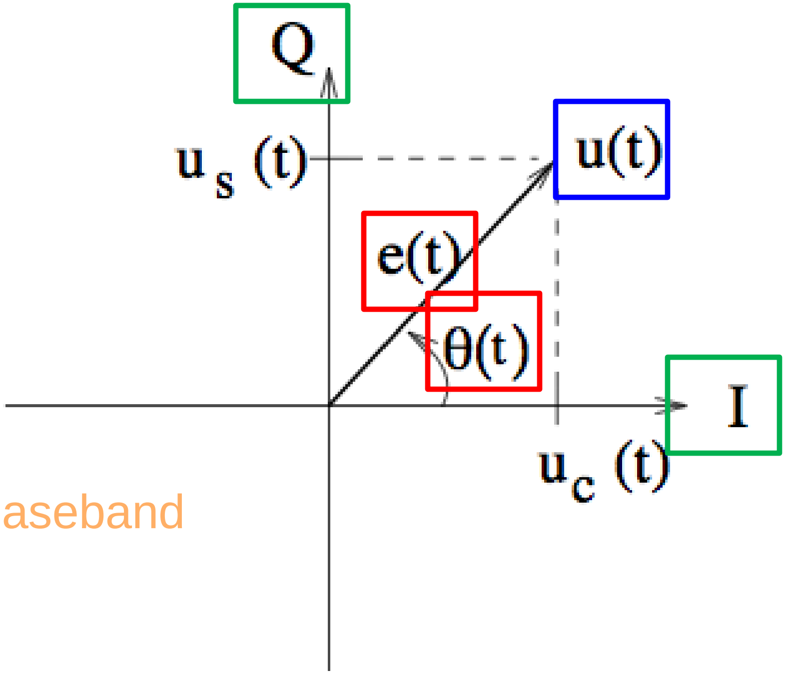
- Show that the passband signal $u_p(t)$ can be expressed in terms of the complex envelope $u(t)$ as

$$u_p(t) = \mathrm{Re}(u(t)e^{j2\pi f_c t})$$

Already proved in previous class

Relationship between three representations

- Two-dimensional modulation: Passband signal u_p can be mapped to a pair of real baseband signals.
- The three representations are
  - Rectangular Coordinates I and Q
  - Envelope and Phase $e(t)$ and $\theta(t)$
  - Complex number or envelope $u(t)$



Information resides in complex baseband

Time domain expressions for a passband signal

- In terms of I and Q components

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$

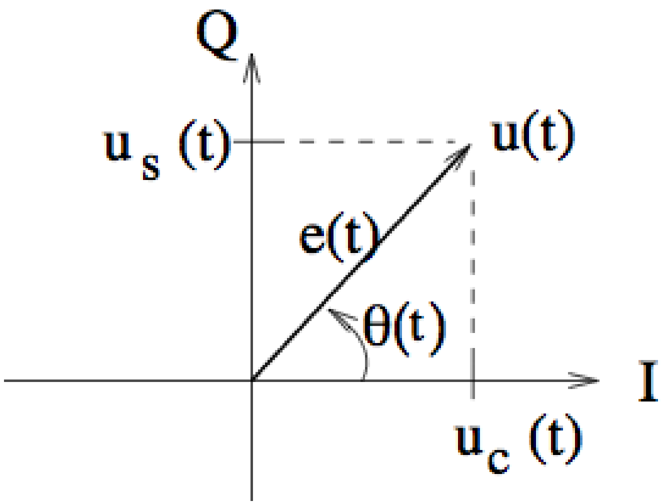
- In terms of envelope and phase

$$u_p(t) = e(t)\cos(2\pi f_c t + \theta(t))$$

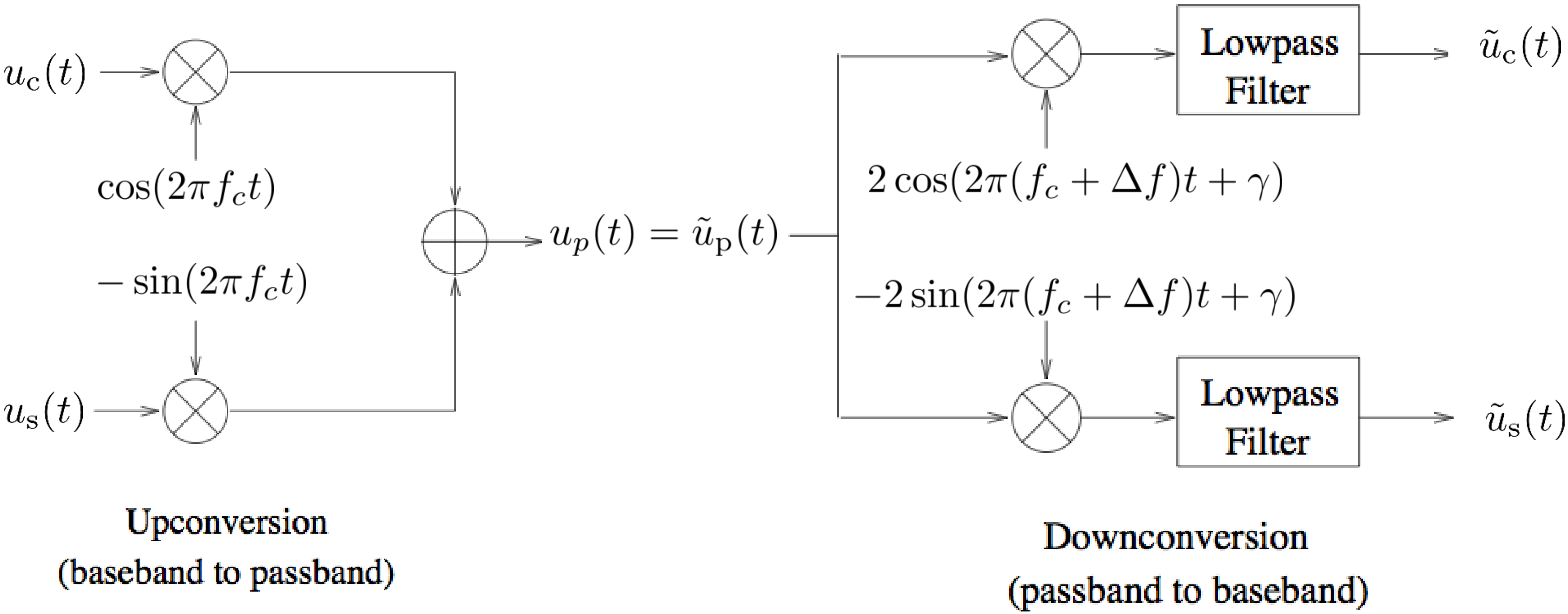
- In terms of complex envelope

$$u_p(t) = \mathrm{Re}(u(t)e^{j2\pi f_c t})$$

Starting from one representation, can derive the rest based on the relations depicted in the figure



Effect of Frequency and Phase Offset



- Show that in this case

$$\tilde{u}_c(t) = u_c(t)\cos\phi(t) + u_s(t)\sin\phi(t)$$
$$\tilde{u}_s(t) = -u_c(t)\sin\phi(t) + u_s(t)\cos\phi(t)$$

where $\phi(t) = 2\pi\Delta f t + \gamma$ is the phase offset resulting from frequency offset Δf and the phase offset γ.

Coherent Detection: Synchronization

- Frequency offset and phase offset cause cross-interference between I and Q components

$$\tilde{u}_c(t) = u_c(t)\cos\phi(t) + u_s(t)\sin\phi(t)$$
$$\tilde{u}_s(t) = -u_c(t)\sin\phi(t) + u_s(t)\cos\phi(t)$$

  where $\phi(t) = 2\pi\Delta t + \gamma$ is the phase offset resulting from frequency offset Δf and the phase offset γ.

- Either have tight synchronization, i.e., $\Delta f \approx 0$ and $\gamma \approx 0$.

- Compensate for the offset $u(t) = \tilde{u}(t)e^{j\phi}$.

Summary of Properties: S&S

- If $u(t)$ is complex with Fourier transform is $U(f)$, then
  - Time reversal: $u(-t) \longleftrightarrow U(-f)$
  - Time conjugation: $u^*(t) \longleftrightarrow U^*(-f)$
  - Frequency shift: $u(t)e^{j2\pi f_0 t} \longleftrightarrow U(f - f_0)$
- If $u(t)$ is real
  - Conjugate symmetry in frequency domain $U(f) = U^*(-f)$
  - Even symmetry for real part of $U(f)$: $\Re\{U(f)\} = \Re\{U(-f)\}$
  - Odd symmetry for imaginary part of $U(f)$: $\Im\{U(f)\} = -\Im\{U(-f)\}$
  - Magnitude has even symmetry: $|U(f)| = |U(-f)|$
  - Angle has odd symmetry: $\angle U(f) = -\angle U(-f)$

$U(f) = a + jb \Rightarrow$ $U^*(f) = a - jb$

$Re(U(f)) = Re(U^*(-f))$

Frequency Domain Relationship

- Show that the frequency response for

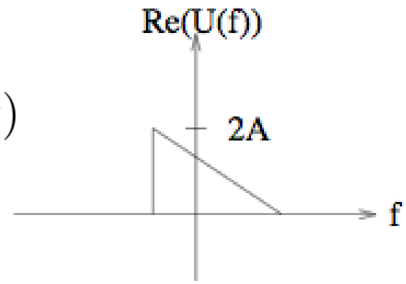
$$u_{\mathrm{p}}(t) = \mathrm{Re}\{u(t)e^{j2\pi f_c t}\}$$

is given by

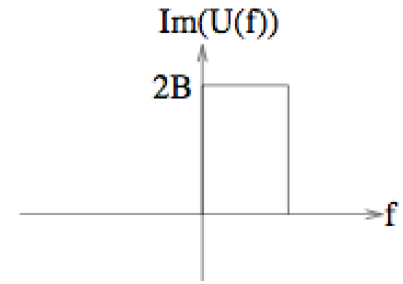
$$U_{\mathrm{p}}(f) = \frac{U(f - f_c) + U^*(-f - f_c)}{2}$$

Construct Passband from Baseband

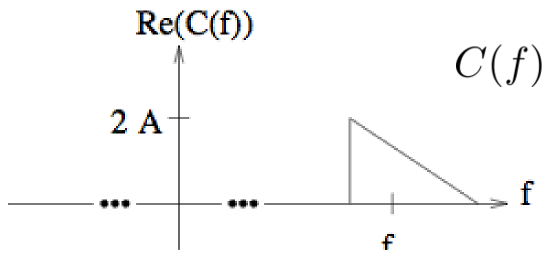
Complex signal $u(t)$



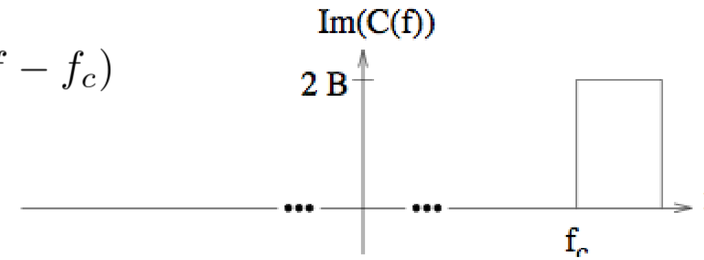
Complex $U(f)$



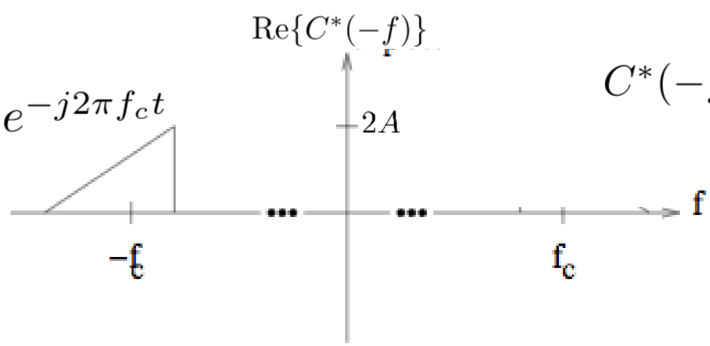
$c(t) = u(t)e^{j2\pi f_c t}$



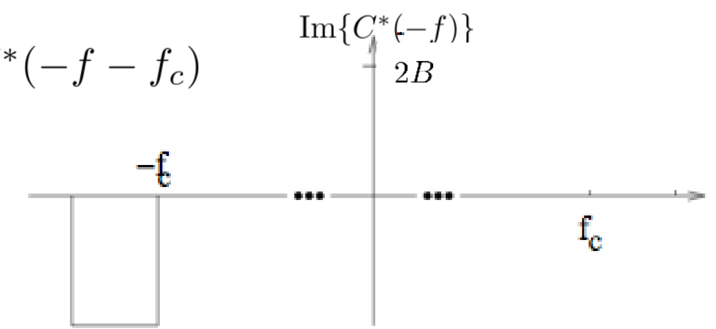
$C(f) = U(f - f_c)$



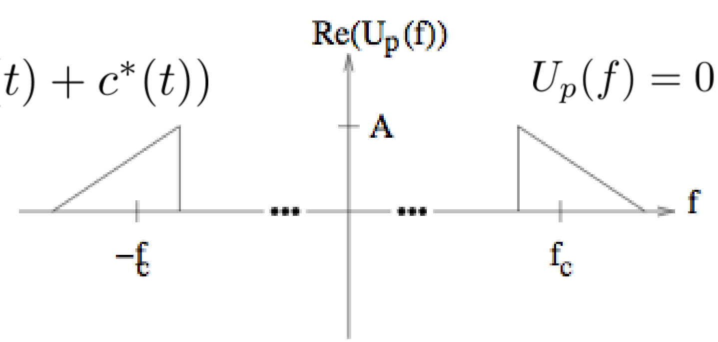
$c^*(t) = u^*(t)e^{-j2\pi f_c t}$



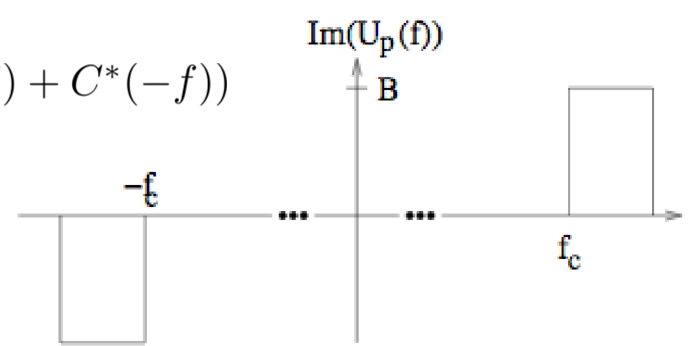
$C^*(-f) = U^*(-f - f_c)$



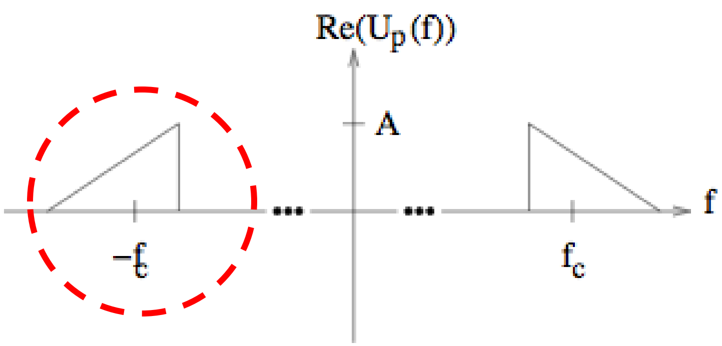
$u_p(t) = 0.5(c(t) + c^*(t))$

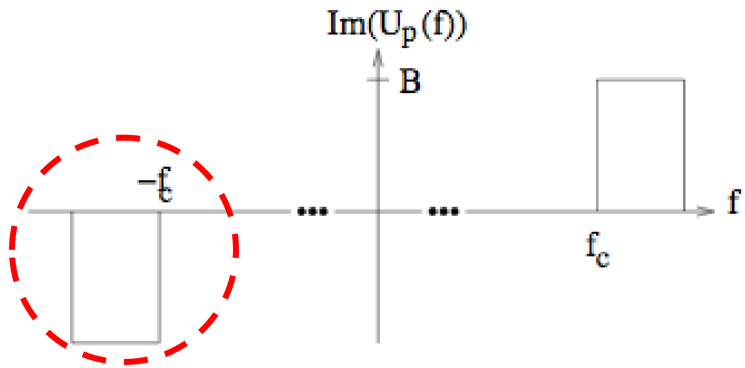


$U_p(f) = 0.5(C(f) + C^*(-f))$

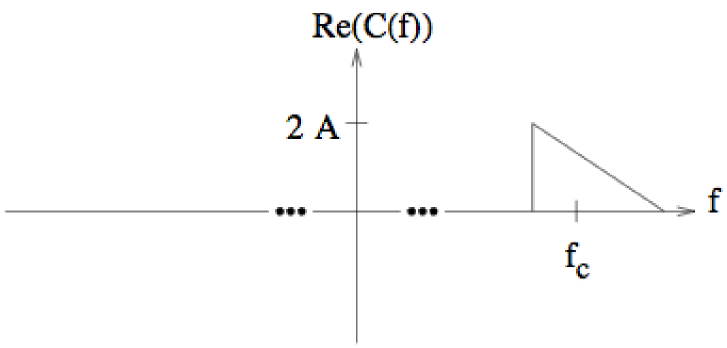


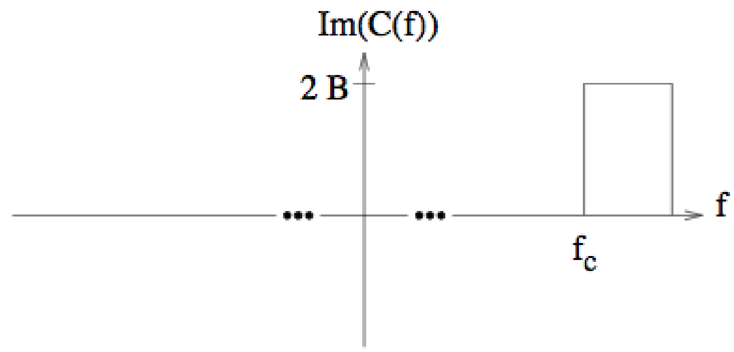
Reconstruct Baseband from Passband



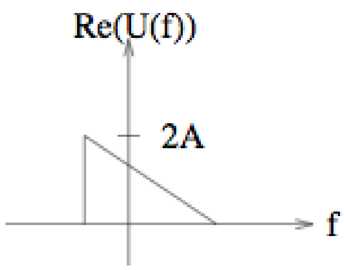


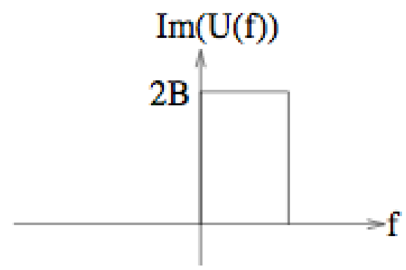
$$C(f) = 2U_p^+(f) = \begin{cases} 2U_p(f)\ , & f > 0 \\ 0\ , & f < 0 \end{cases}$$





$$u(t) = c(t)e^{-j2\pi f_c t} \leftrightarrow U(f) = C(f + f_c) = 2U_p^+(f + f_c)$$





I and Q components in frequency domain

- I and Q in time domain

$$u_{\mathrm{c}}(t) = \frac{1}{2}(u(t) + u^*(t))$$
$$u_{\mathrm{s}}(t) = \frac{1}{2j}(u(t) - u^*(t))$$

$$\updownarrow$$

- I and Q in frequency domain

$$U_{\mathrm{c}}(f) = \frac{1}{2}(U(f) + U^*(-f))$$
$$U_{\mathrm{s}}(f) = \frac{1}{2j}(U(f) - U^*(-f))$$

Passband filtering = complex baseband filtering

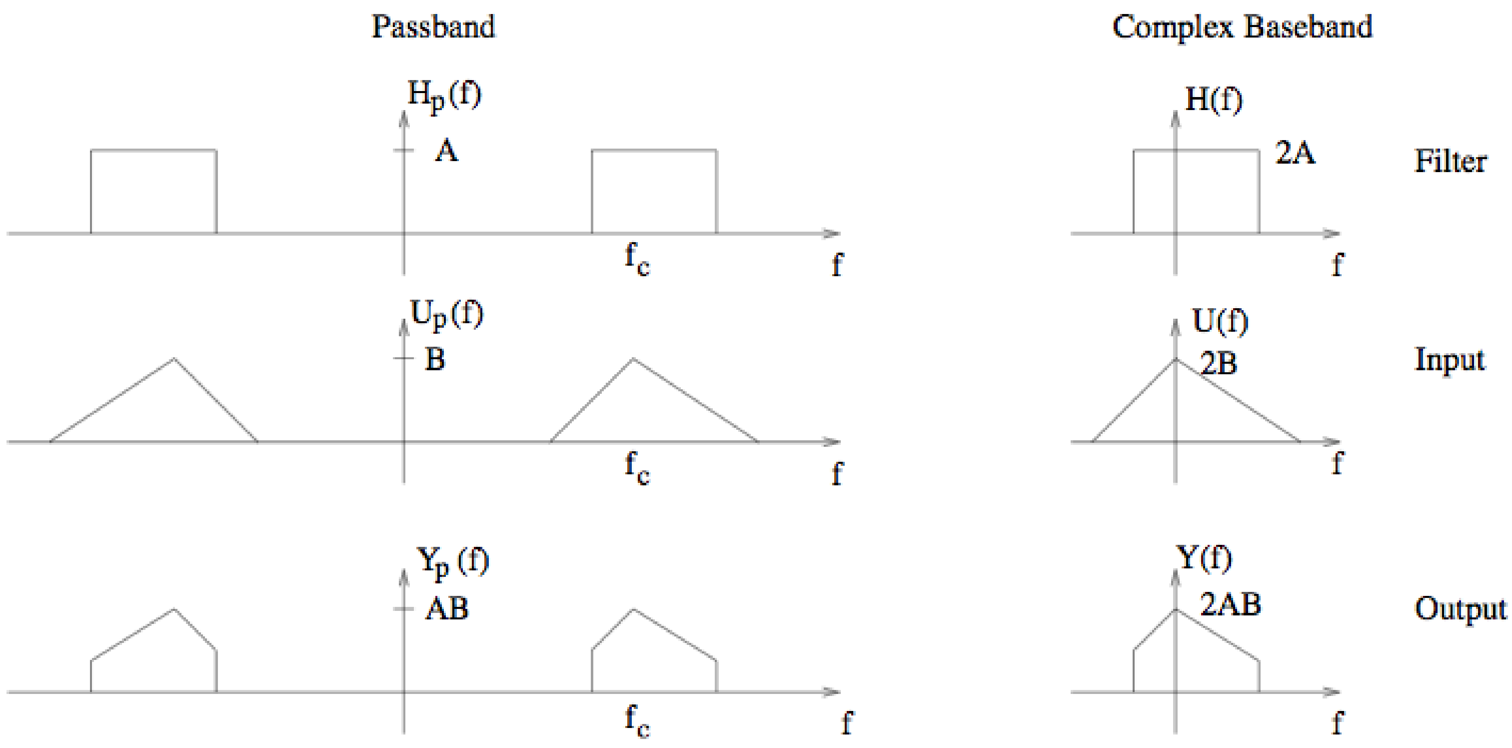
$$u_p(t) \longrightarrow \boxed{h_p(t)} \longrightarrow y_p(t)$$

EQUIVALENT TO
(except for a scale factor of 2)

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$u_{\mathrm{p}}(t) = \Re\{u(t)e^{j2\pi f_c t}\}$$
$$h_{\mathrm{p}}(t) = \Re\{h(t)e^{j2\pi f_c t}\}$$
$$y_{\mathrm{p}}(t) = \Re\{y(t)e^{j2\pi f_c t}\}$$

What is complex baseband representation of passband filtering?



$$Y(f) = 2Y_p^+(f+f_c) = 2U_p^+(f+f_c)H_p^+(f+f_c) = \frac{1}{2}U(f)H(f)$$

$$\boxed{Y_p(f) = U_p(f)H_p(f) \quad \equiv \quad Y(f) = \frac{1}{2}U(f)H(f)}$$

$$\boxed{y_p(t) = u_p(t) * h_p(t) \quad \equiv \quad y(t) = \frac{1}{2}u(t) * h(t)}$$

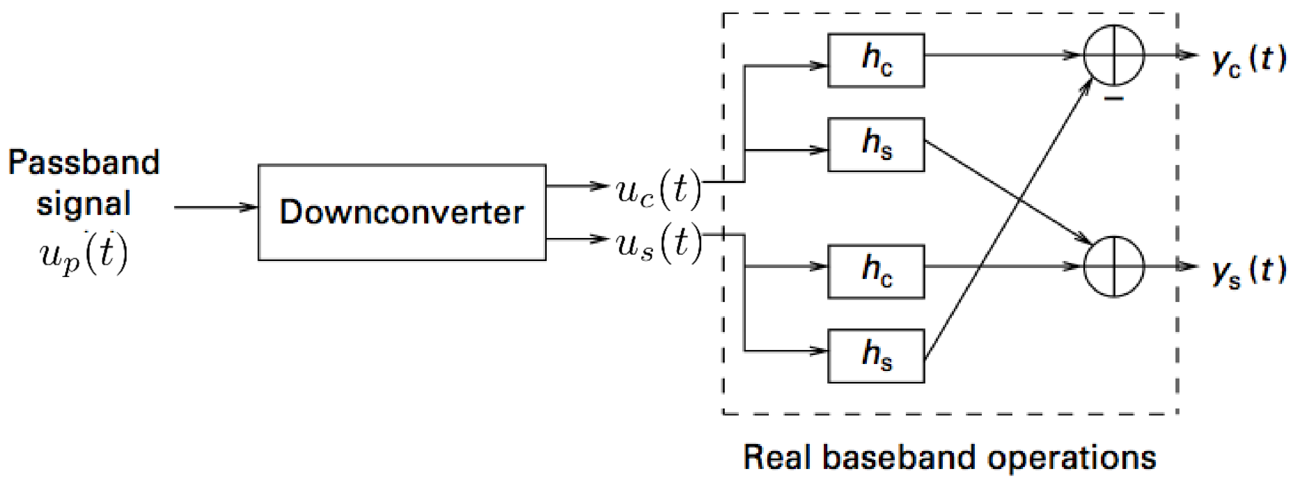
Filtering in complex baseband

- Complex-valued convolution to implement equivalent of passband filtering operation

$$y(t) = \frac{1}{2}(u * h)(t)$$

- Requires four real-valued convolutions

$$y_c = \frac{1}{2}(u_c * h_c - u_s * h_s); \qquad y_s = \frac{1}{2}(u_s * h_c + u_c * h_s);$$



$$x_{\mathrm{p}}(t) = x_{\mathrm{c}}(t)\cos(2\pi f_c t) - x_{\mathrm{s}}(t)\sin(2\pi f_c t)$$
$$x(t) = x_c(t) + jx_s(t)$$

Energy and Power

- Solve: Show that the energy of a passband signal equals that of its complex envelope, i.e.,

$$||u_p||^2 = \frac{1}{2}\left(||u_c||^2 + ||u_s||^2\right) = \frac{1}{2}||u||^2$$

- Since power is computed as time average of energy, similar relation exists for finite-power signals.

Correlation between two signals

- Correlation or inner product of two real-valued passband signals u_p and v_p is defined as

$$< u_p, v_p >= \int_{-\infty}^{\infty} u_p(t) v^*{}_{\mathrm{p}}(t)\ dt$$

- Show that

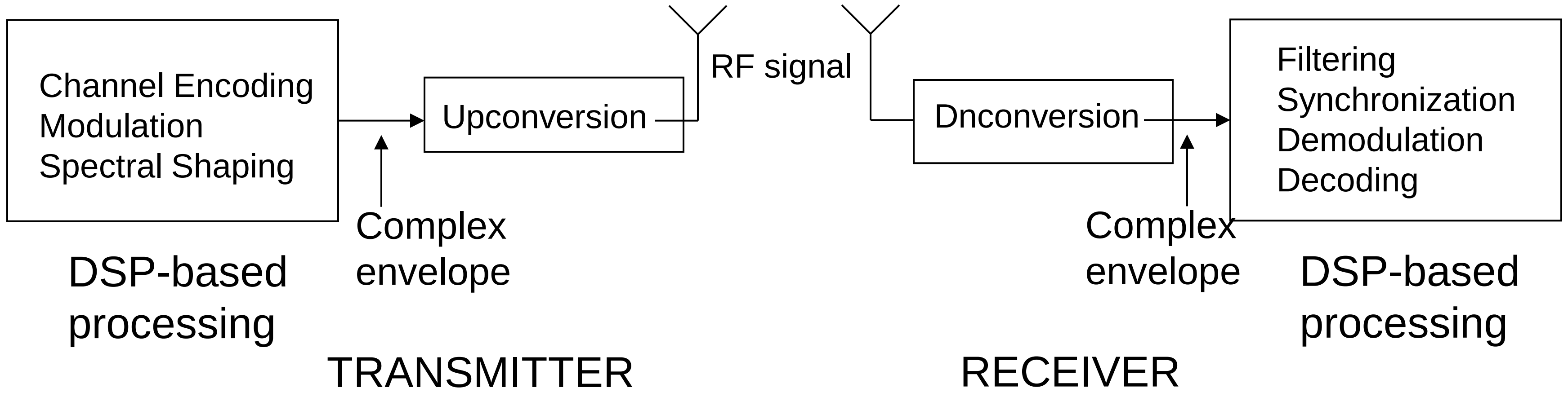
$$< u_p, v_p >= \frac{1}{2} \Re\{< u, v >\}$$

Modern transceiver architectures based on complex baseband

Modern transceiver work with the complex envelope rather than with the passband signal

- Complex baseband signals can be represented accurately by samples at a reasonable sampling rate
- Inexpensive to perform complicated DSP on the samples
- This architecture has been responsible for economics of scale in cellular and WiFi devices.
- Most of the research simulations happen in baseband.

All the action is in complex baseband for a typical wireless transceiver



Complex baseband representation: summary

- Any real-valued passband signal can be represented by a baseband signal which is in general complex-valued. This is called its complex envelope, or complex baseband representation.
- The complex envelope carries all the information in the passband signal
- Passband filtering operations can be equivalently performed in complex baseband
- Two-dimensional representation of complex envelope
  - Cartesian coordinates: A pair of real-valued baseband waveforms called the in-phase (I) and quadrature (Q) components
  - Polar coordinates: Envelope and phase waveforms
- Most of the sophisticated signal processing action in modern transceivers happens in complex baseband

EC5.203 Communication Theory I (3-1-0-4):

Lecture 5
Analog Communication Techniques: Amplitude Modulation

Instructor: Dr. Sachin Chaudhari

Email: sachin.chaudhari@iiit.ac.in

Jan. 27, 2025

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

References

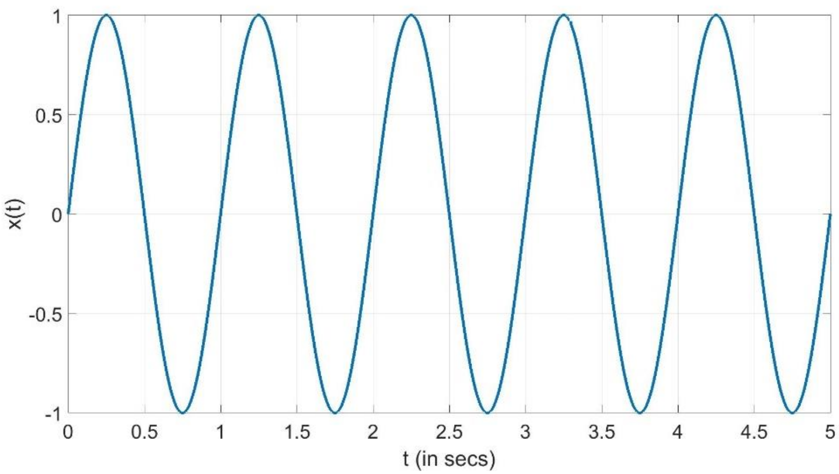
- Chap. 3 (Madhow)

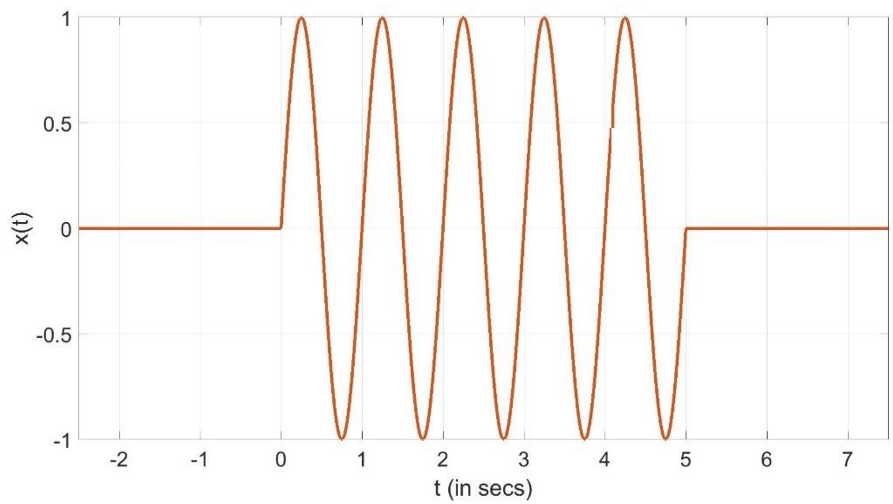
Analog comm techniques: Motivation

- Why bother?
  - After all, the world is going digital
  - Modern comm system designers focused mainly on DSP algorithms for digital comm
- But need to understand the underlying physical analog signals
  - Establishes common language with circuit designers
  - Analog-centric techniques become critical when pushing the limits of carrier frequency, bandwidth and/or power consumption
- Focus of these techniques is on baseband to passband conversion, and back

Terminology and Notations

- Let $m(t)$ denote the message signal with frequency response $M(f)$.
- For a real signal $m(t)$, $M(f) = M^*(-f)$.
- Based on our convenience, we will consider it to be a power or an energy signal.





Terminology and Notations

- Let $m(t)$ denote the message signal with frequency response $M(f)$.

- For a real signal $m(t)$, $M(f) = M^*(-f)$.

- Based on our convenience, we will consider it to be a power or an energy signal.

- Power of the signal is given by

$$\overline{m^2} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} m^2(t)\ dt$$

- DC value of the signal is

$$\overline{m} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} m(t)\ dt$$

Terminology and Notations

- Let $u_p(t)$ denote the signal transmitted over the channel. Also called passband signal given in term of cartesian coordinates as

$$u_p(t) = \underbrace{\boxed{u_c(t)}}_{\mathrm{I}}\cos(2\pi f_c t) - \underbrace{\boxed{u_s(t)}}_{\mathrm{Q}}\sin(2\pi f_c t)$$

- In Polar coordinates

$$u_p(t) = e(t)\cos(2\pi f_c t + \theta(t))$$

where $e(t)$ is magnitude of the envelope and $\theta(t)$ is the phase.

Key Concepts

- Two ways of encoding info in complex envelope
  - I and Q: amplitude modulation (several variants)
  - Envelope and phase: angle modulation (constant envelope)
- For example, for a sinusoid carrier

$$A_c(t)\cos(2\pi f_c(t)t + \theta_c(t))$$

where $A_c(t)$, $f_c(t)$, $\theta_c(t)$ are the amplitude, frequency, and the phase of the carrier respectively.

Amplitude Modulation

Frequency Modulation

Phase Modulation

Key Concepts

- Up/Down conversion
  - Multiple stages or single stage (superhet or direct conversion)
- Phase locked loop
  - Feedback-based synchronization and tracking

Amplitude Modulation

Example: sinusoidal message

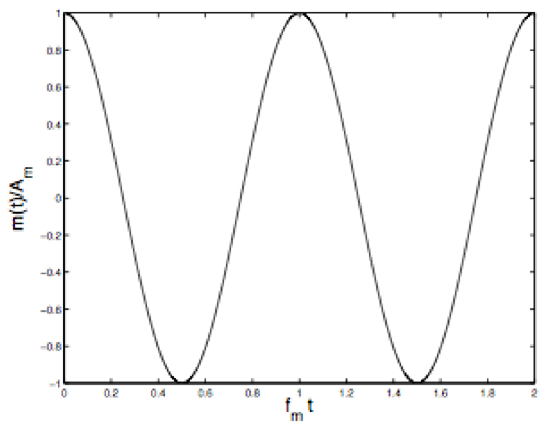
- Consider a sinusoidal message given by

$$m(t) = A_m \cos(2\pi f_m t)$$

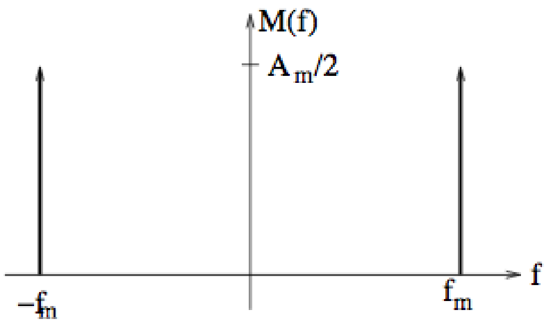
where A_m is magnitude of the envelope and f_m is the signal frequency.

- Fourier transform is given by

$$M(f) = \frac{A_m}{2}(\delta(f + f_m) + \delta(f - f_m))$$



(a) Sinusoidal message waveform



(b) Sinusoidal message spectrum

- Find power and average for this signal. (Assignment)

AM: Double Sideband Suppressed Carrier

- Here the message $m(t)$ modulates the I component of the pass-band signal $u(t)$ and is given by

$$u_{DSB}(t) = m(t) \cdot A\cos(2\pi f_c t)$$

while the Fourier transform is given by

$$U_{DSB}(f) = \frac{A}{2}(M(f - f_c) + M(f + f_c))$$

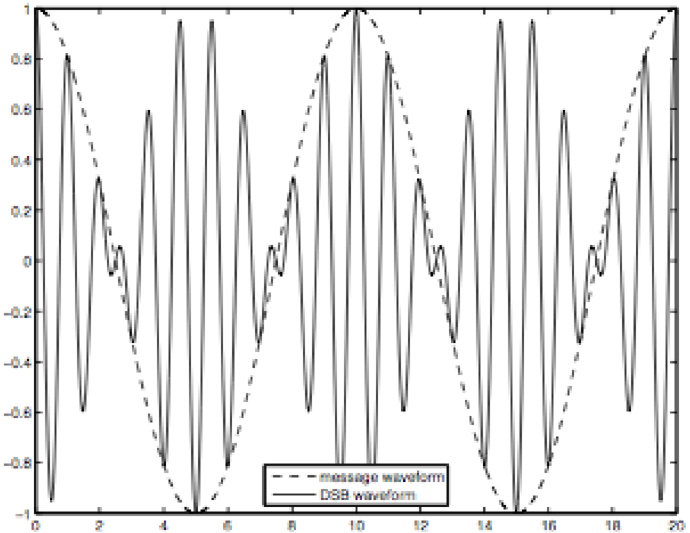
DSB-SC signal for sinusoidal message

Here the signal is given by

$$u_{DSB}(t) = A_m \cos(2\pi f_m t) \cdot A \cos(2\pi f_c t)$$

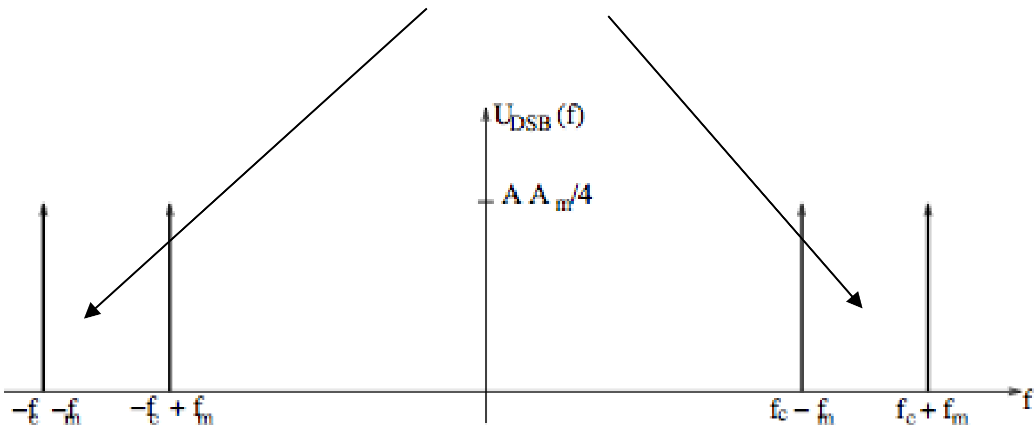
while the Fourier transform is given by

$$U_{DSB}(f) = \frac{AA_m}{4}\{\delta(f - f_c - f_m) + \delta(f - f_c + f_m) + \delta(f + f_c + f_m) + \delta(f + f_c - f_m)\}$$



(a) DSB time domain waveform

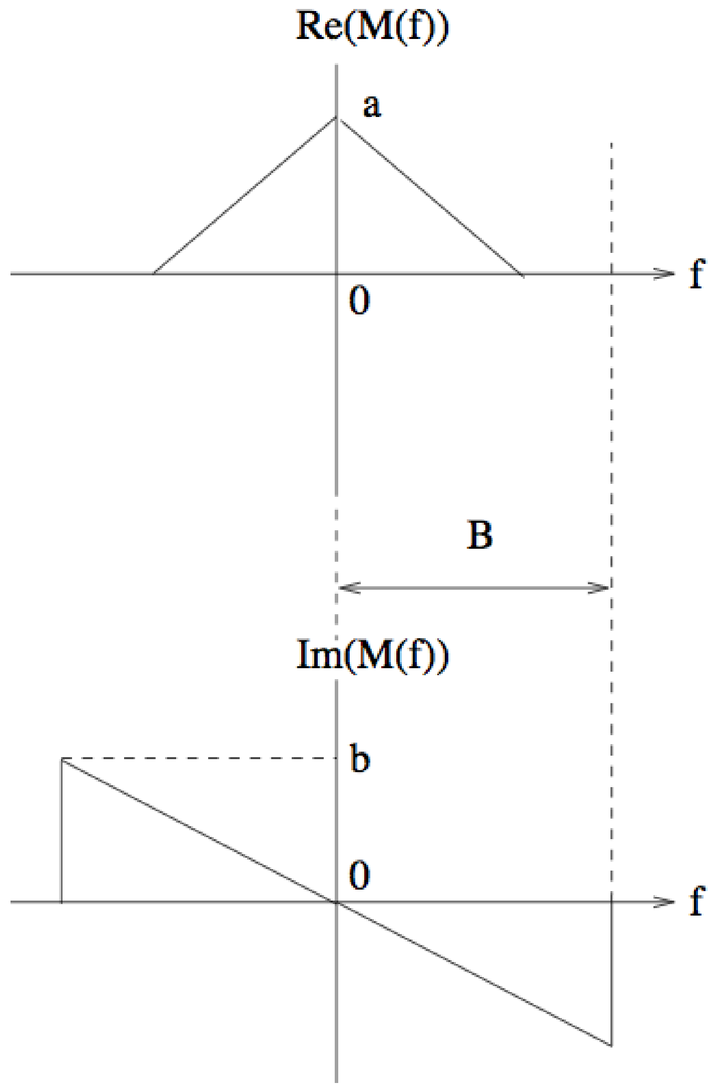
No impulses at f_c or $-f_c$!



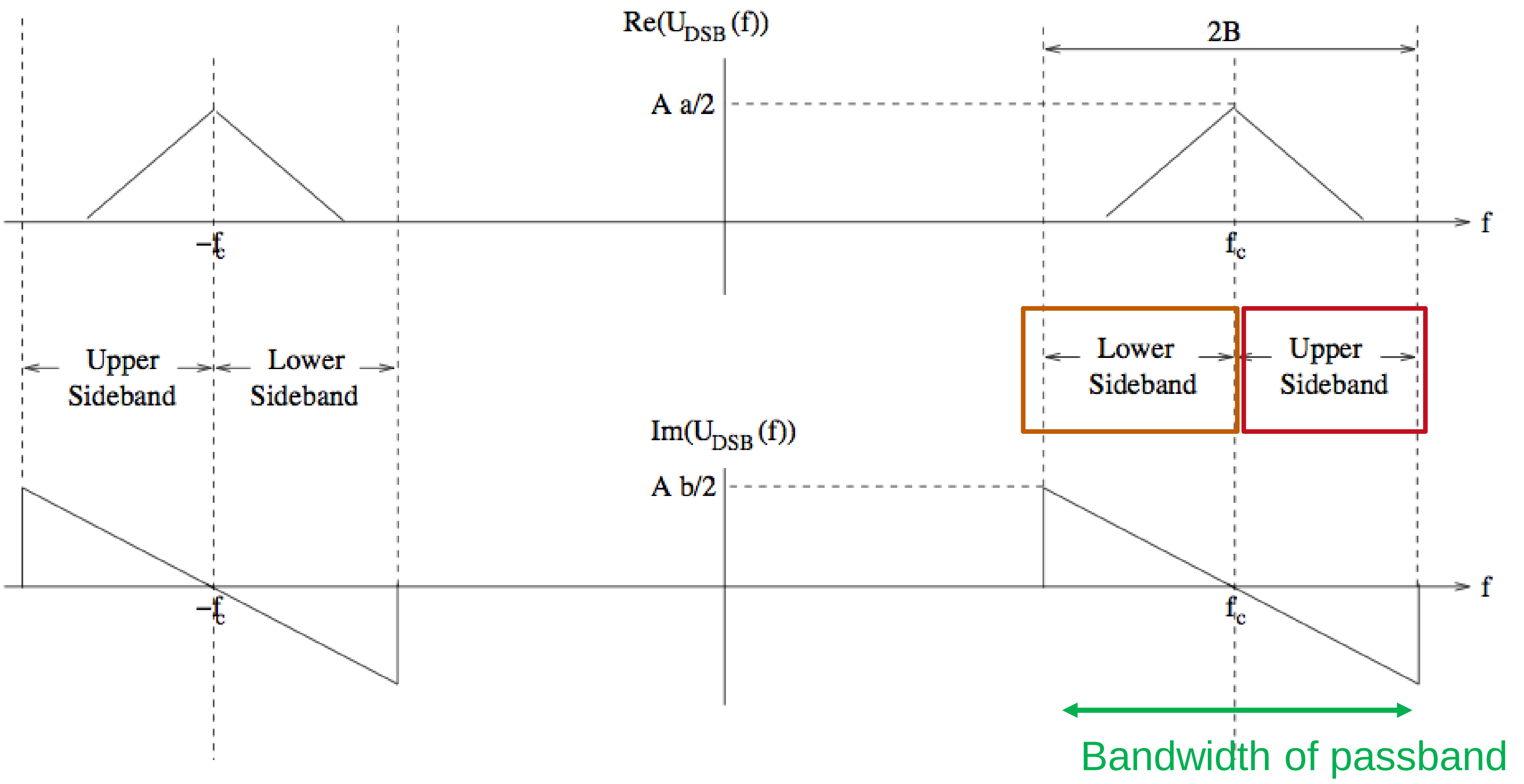
(b) DSB spectrum

Example 2

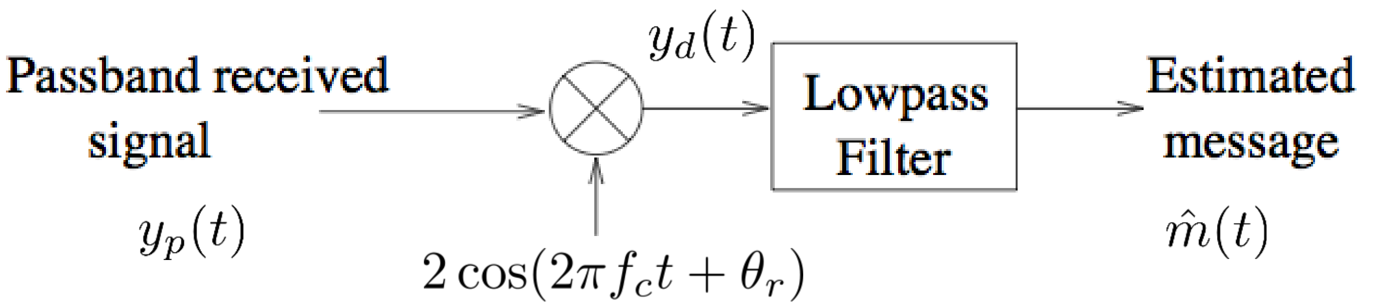
- Consider a message signal $m(t)$ with following frequency response $M(f)$



DSB-SC spectrum for Example 2



Demodulation of DSB-SC



- Standard downconverter to recover I component
- The received signal in the passband is

$$y_p(t) = Am(t)\cos(2\pi f_c t)$$

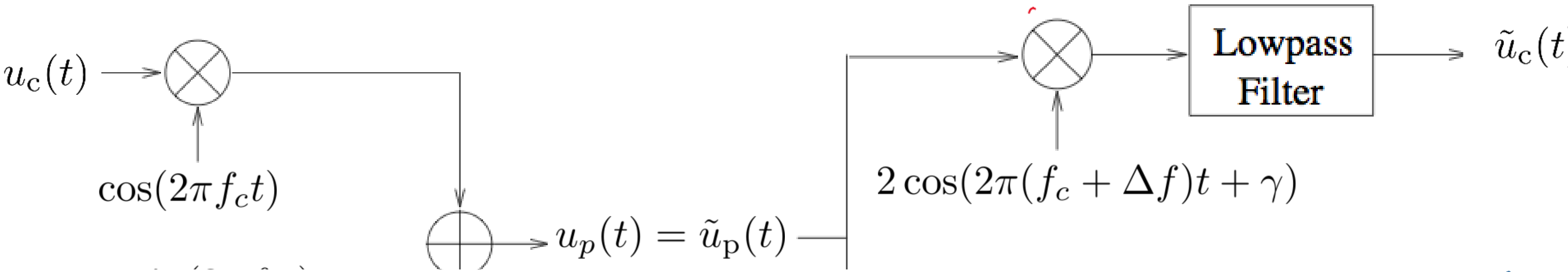
where θ_r is the phase difference arising from the phase offset with respect to local carrier at Rx.

- The output of multiplier followed by low pass filter is

$$\hat{m}(t) = Am(t)\cos\theta_r$$

Try this as a assignment!

Recap: Chapter 2
Effect of Frequency and Phase Offset



Focus in this chapter mostly on I component!

Upconversion (baseband to passband)

Downconversion (passband to baseband)

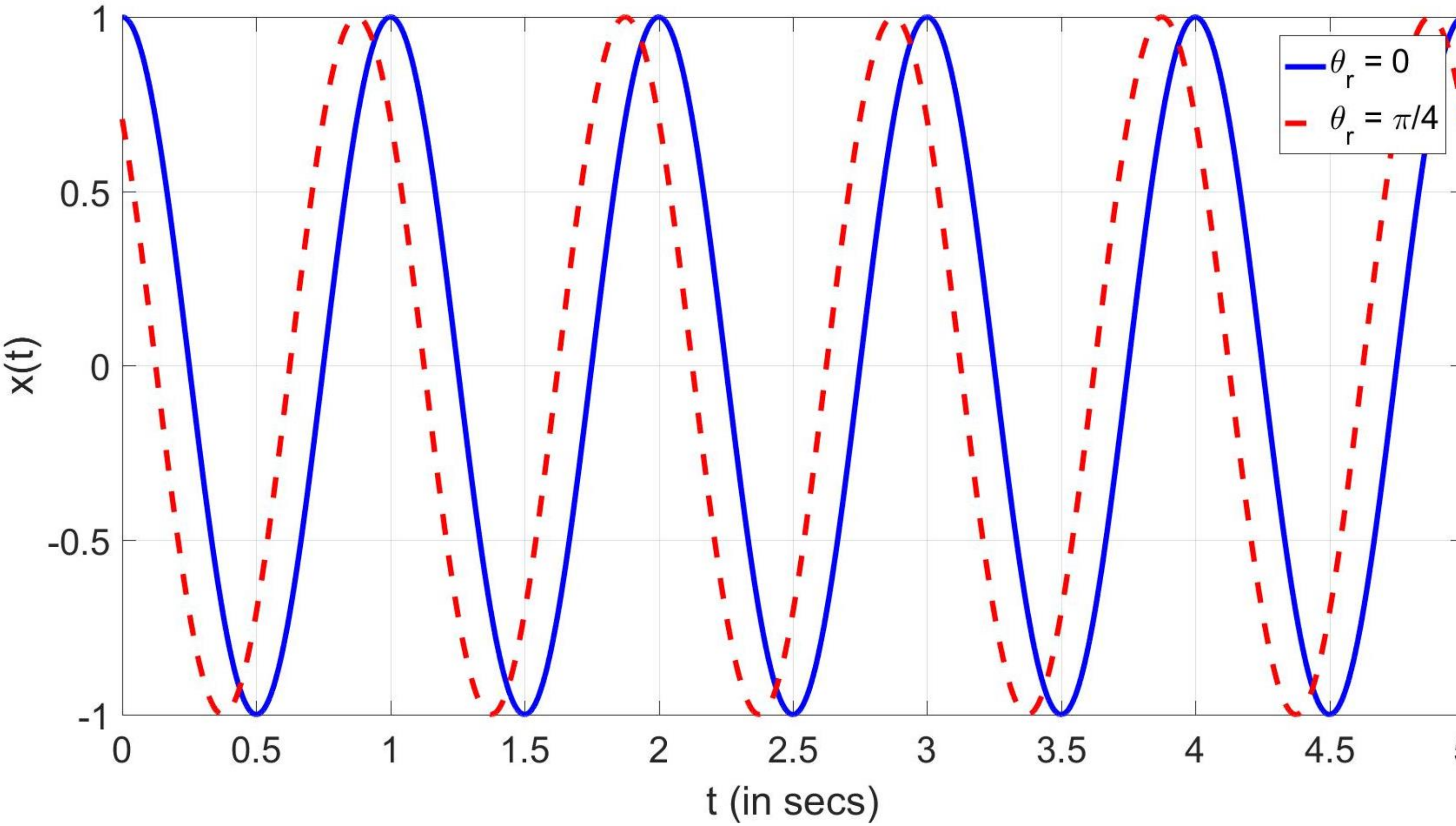
- Show that in this case

$$\tilde{u}_c(t) = u_c(t)\cos\phi(t) + u_s(t)\sin\phi(t)$$

where $\phi(t) = 2\pi\Delta f t + \gamma$ is the phase offset resulting from frequency offset Δf and the phase offset γ. Here $\theta_r = \phi(t)$

Example: Phase Offset

$$x(t) = \cos(2\pi f_c t + \theta_r)$$



Here $\theta = \gamma$

Causes of Phase Offset

- Frequency offset: The local oscillator at the receiver is generating frequency at $f_c + \Delta f$

  $$\theta_r = 2\pi \Delta f t$$

  This happens as the two physical devices cannot be exactly same resulting in slight differences. Here there will be phase difference even if they are same place.

- Timing offset: The transmitter and receiver have slightly different time references or they are separated by distance d resulting in time offset of δt.

  $$\theta_r = 2\pi f_c \Delta t$$

Need of Coherent Detection

- The output of multiplier followed by low pass filter is

$$\hat{m}(t) = Am(t)\cos\theta_r$$

- For $\theta_r = 0$, $\hat{m}(t) = Am(t)$
- For $\theta_r = \pi/2$, $\hat{m}(t) = 0$
- For $\theta_r(t) = 2\pi\Delta f t + \phi$, time varying signal degradation in amplitude and unwanted sign changes.
- Need of synchronization of phase of the local oscillator with the phase of incoming signal:
  - Phased locked loop (PLL)
- Switch to other AM techniques which do not need synchronization
  - Conventional AM or DSB (with carrier)

Conventional AM

Conventional AM

- Add a large carrier component to a DSB-SC signal so that the passband has the following form

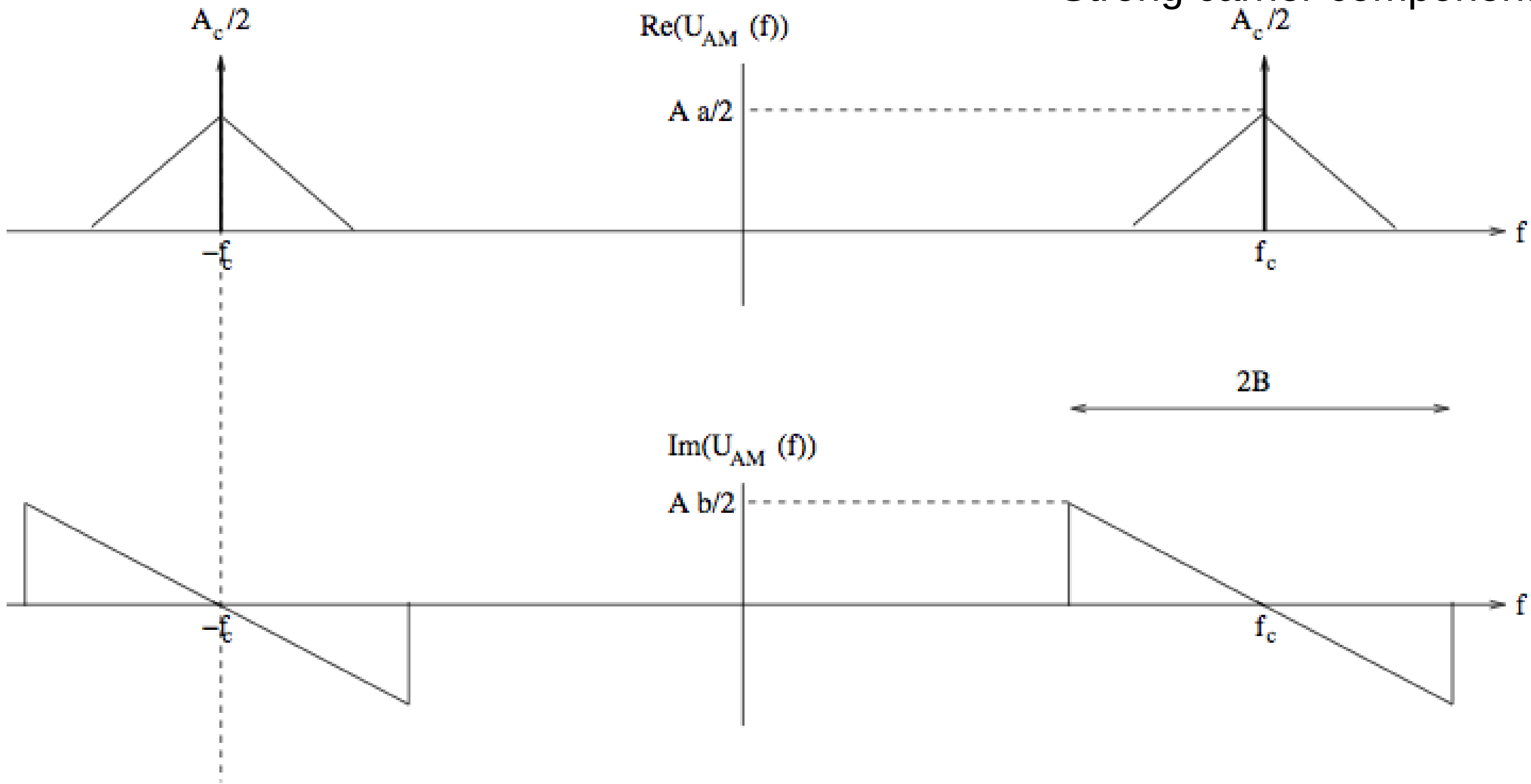
$$
\begin{aligned}
u_{\mathrm{AM}}(t) &= (Am(t) + A_c)\cos(2\pi f_c t) \\
&= Am(t)\cos(2\pi f_c t) + A_c \cos(2\pi f_c t)
\end{aligned}
$$

- Taking Fourier transform

$$
U_{\mathrm{AM}}(f) = \frac{A}{2}(M(f-f_c)+M(f+f_c))+\frac{A_c}{2}(\delta(f-f_c)+\delta(f+f_c))
$$

Conventional AM: spectrum

Strong carrier component



Envelope and its importance

- Add a large carrier component to a DSB-SC signal so that the passband has the following form

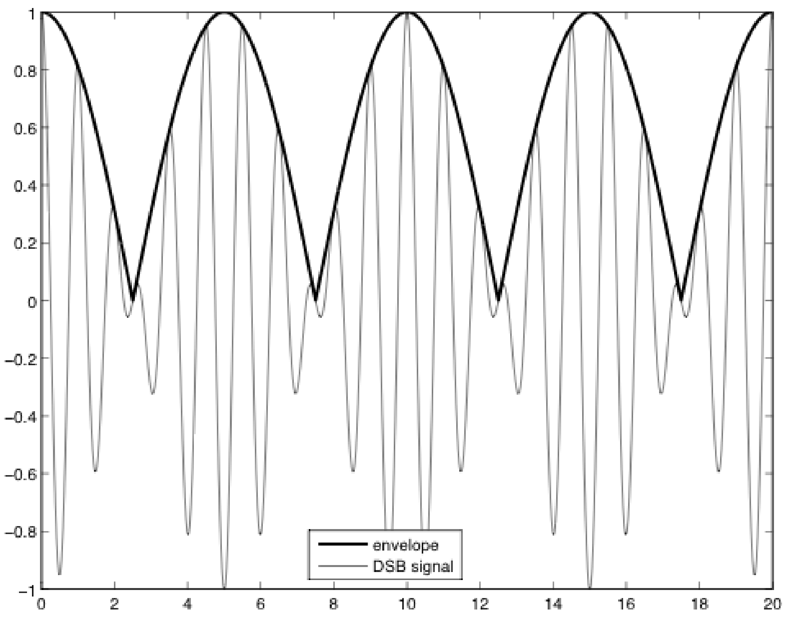
$$
\begin{aligned}
u_{\mathrm{AM}}(t) &= \boxed{(Am(t) + A_c)}\cos(2\pi f_c t) \\
&= Am(t)\cos(2\pi f_c t) + A_c\cos(2\pi f_c t)
\end{aligned}
$$

- Envelope is given by $e(t) = |Am(t) + A_c|$.
- If $Am(t) + A_c > 0$, then $e(t) = Am(t) + A_c$. In this case, message $m(t)$ can be recovered from $e(t)$.

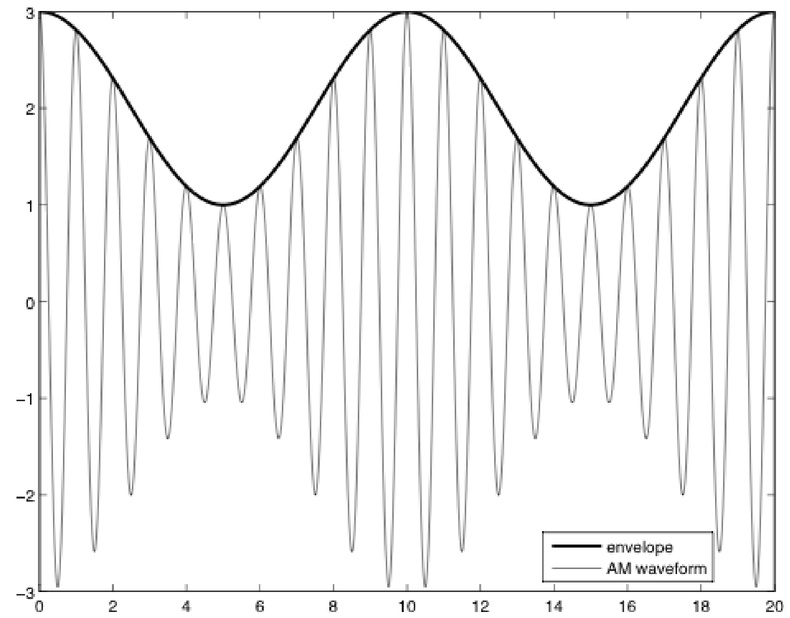
What does the envelope tell us?

- Example: sinusoidal message signal

$$m(t) = A_m \cos(2\pi f_m t)$$



DSB-SC signal
Envelope = message magnitude
→ Envelope detection loses info in message sign.



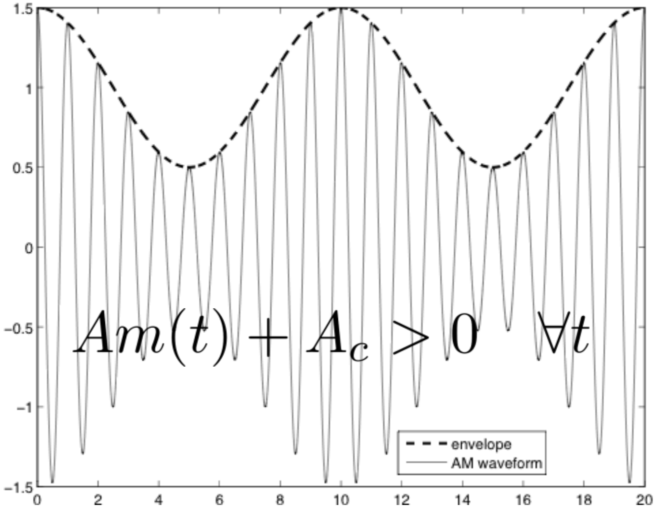
DSB + strong carrier component
Envelope = message + DC
→ Envelope detector + DC block recovers message info

Sidestepping sync requirement

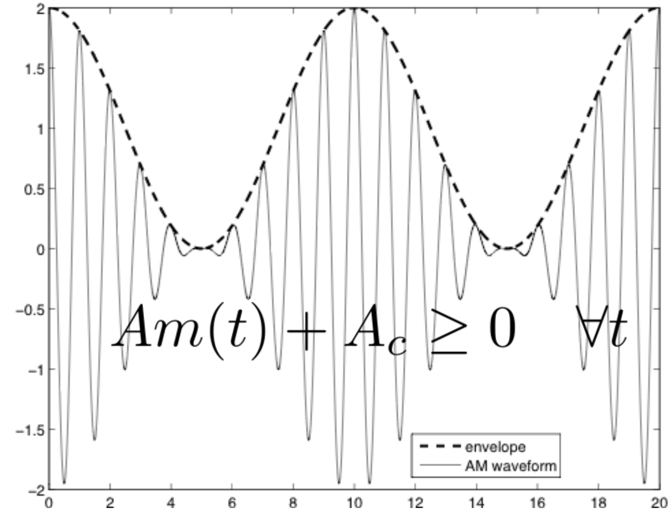
- The envelope (or magnitude of complex envelope) does not depend on carrier phase
- Suppose we can extract the envelope of a passband signal (will soon see a simple circuit for this purpose)
  - Does not require carrier sync
- Can we recover the message?

Constraint for recovering message from envelope

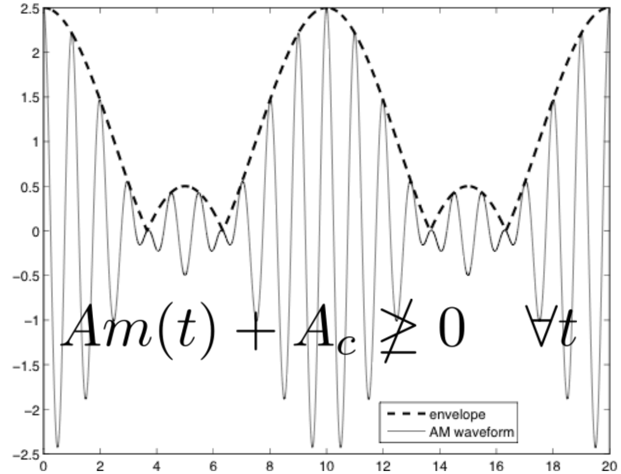
Example of sinusoidal message



Envelope = message + DC

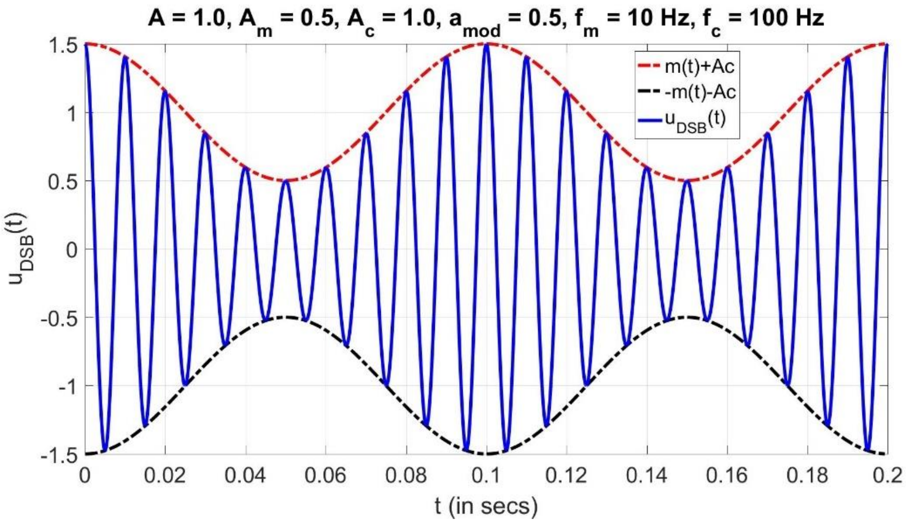


Envelope = message



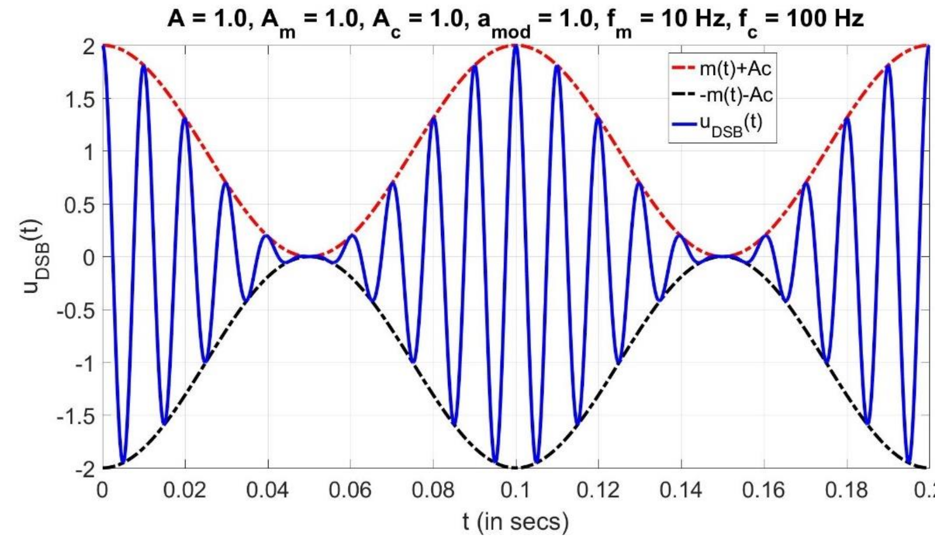
Message info not preserved in envelope

Example of Sinusoidal Message



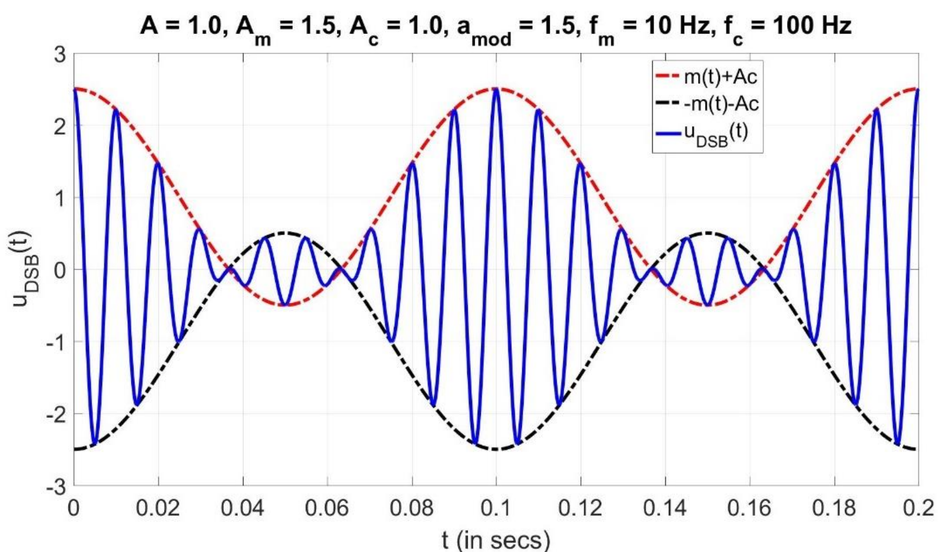
$$Am(t) + A_c > 0 \quad \forall t$$

Envelope = message + DC



$$Am(t) + A_c \geq 0 \quad \forall t$$

Envelope = message + DC



Message info not preserved in envelope

$$Am(t) + A_c \ngeq 0 \quad \forall t$$

Modulation Index

- Condition needed for envelope to preserve message info

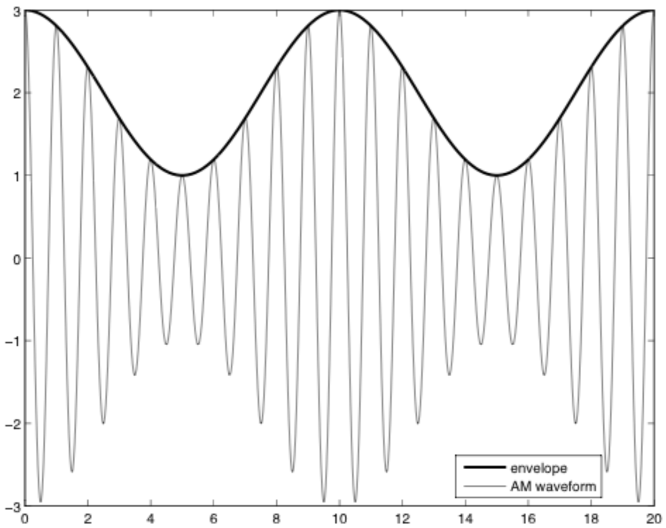
$$A\ m(t) + A_c > 0 \quad \forall t$$

$$A \min_t m(t) + A_c > 0$$

- Can be expressed in terms of modulation index

$$a_{\text{mod}} = \frac{AM_0}{A_c} = \frac{A|\min_t m(t)|}{A_c}$$

- For signal to be recoverable, $a_{\text{mod}} \leq 1$.



AM signal in terms of modulation index

- Convenient to normalize message so that the largest negative swing is -1

$$m_n(t) = \frac{m(t)}{M_0} = \frac{m(t)}{|\min_t m(t)|}$$

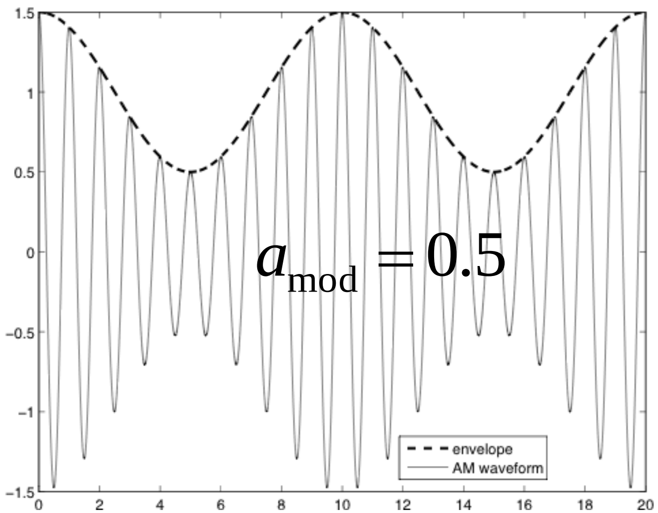
$$\min_t m_n(t) = \frac{\min_t m(t)}{M_0} = -1$$

- AM signal in terms of modulation index and normalized message

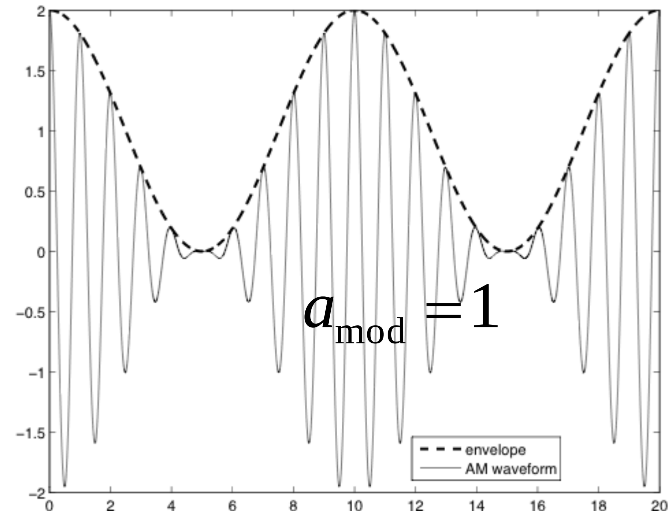
$$y_p(t) = B(1 + a_{\text{mod}} m_n(t)) \cos(2\pi f_c t + \theta_r)$$

Effect of modulation index

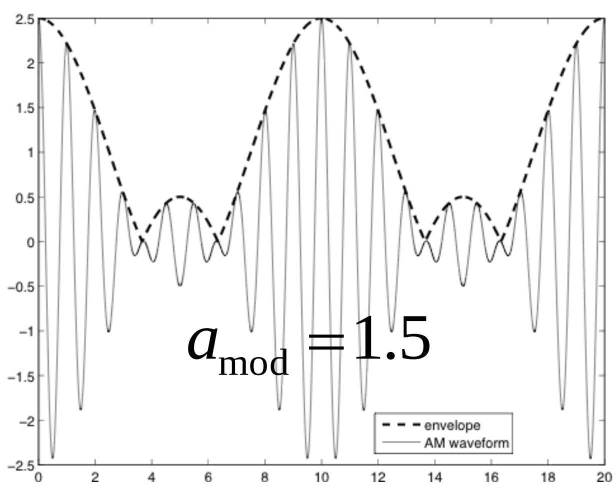
Example of sinusoidal message



Envelope = message + DC

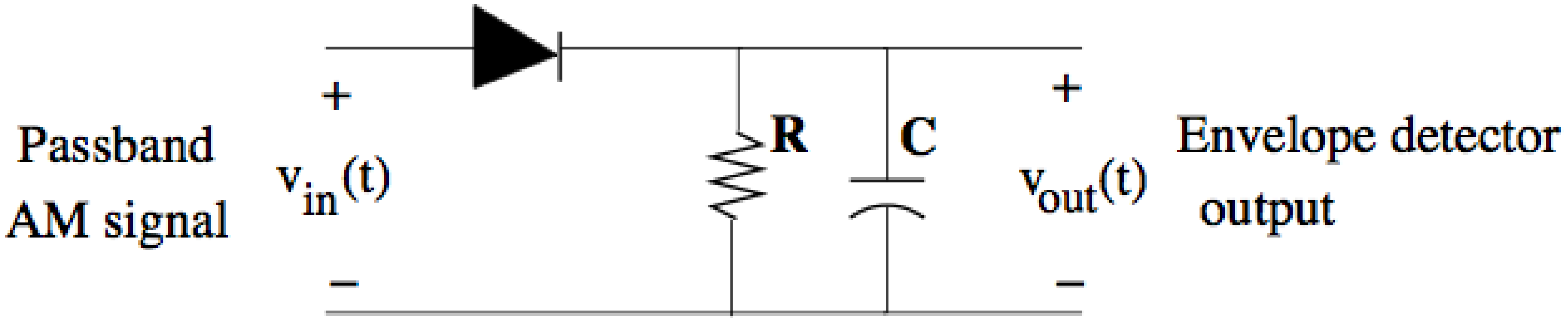


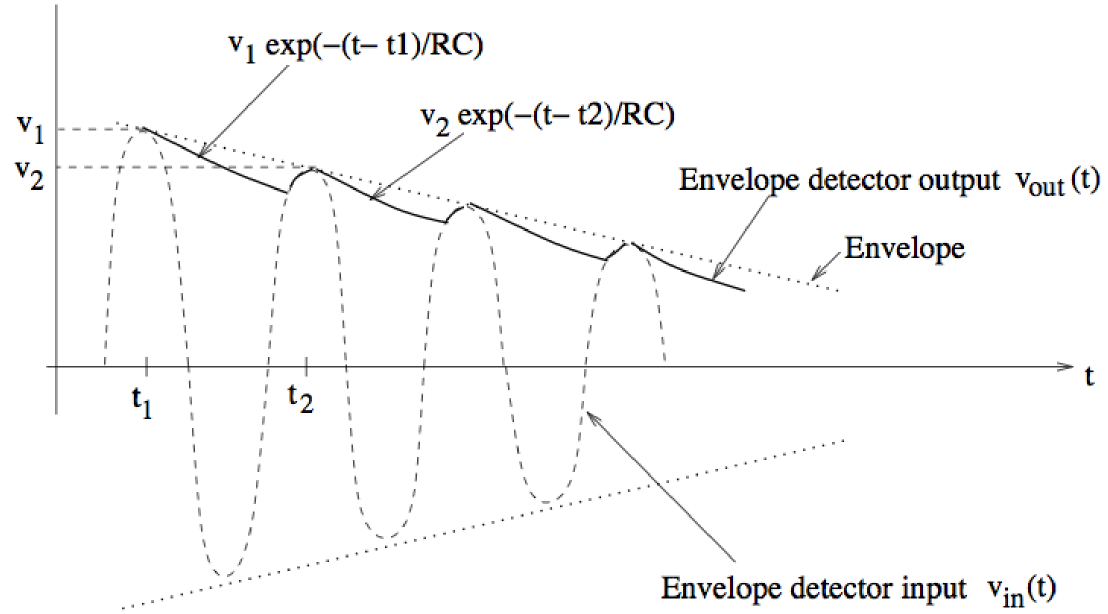
Envelope = message



Message info not preserved in envelope

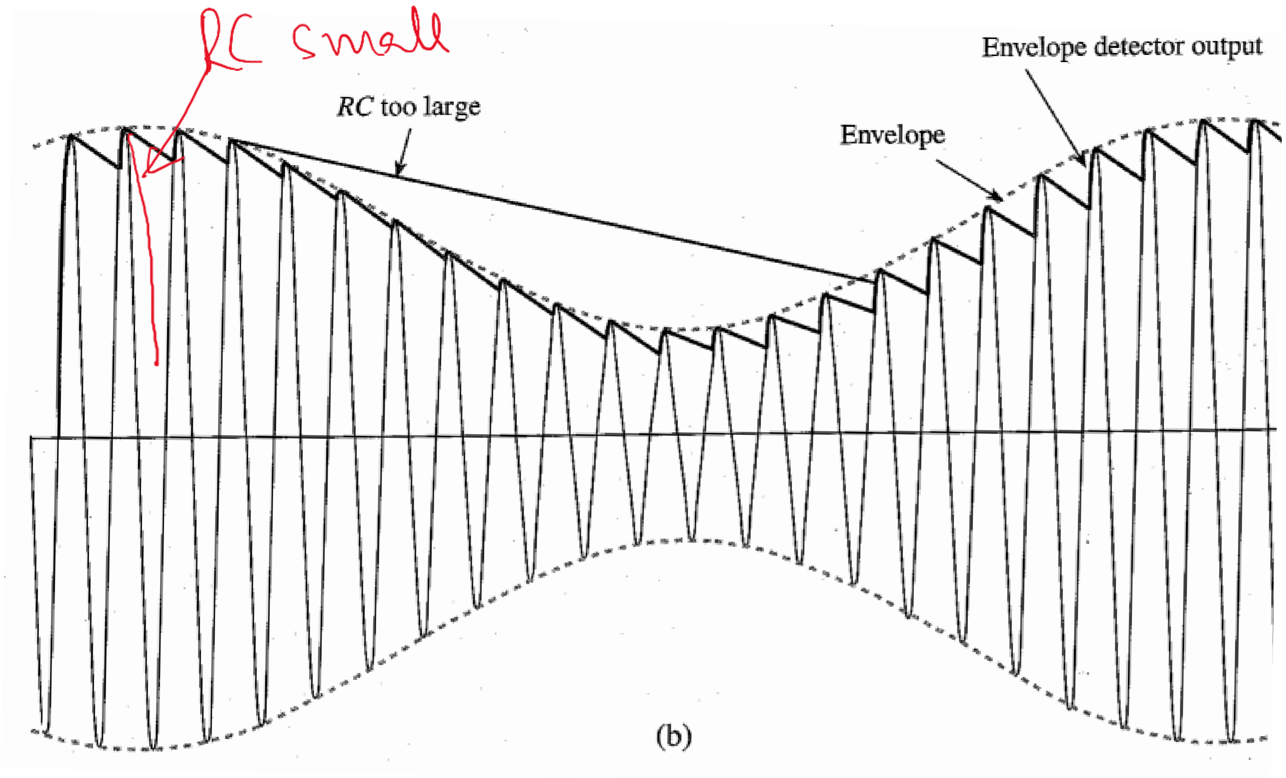
Envelope Detectors





Positive carrier cycle ➔ capacitor charges up (reaches value of envelope)
Negative carrier cycle ➔ capacitor discharges with RC time constant

Envelope detector operation



Positive carrier cycle ➔ capacitor charges up (reaches value of envelope)
Negative carrier cycle ➔ capacitor discharges with RC time constant
Should not discharge too fast during negative cycle
Should react fast enough to follow variations in envelope (which depend on message bandwidth B)

$$\frac{1}{f_c} \ll RC \ll \frac{1}{B}$$

EC5.203 Communication Theory I (3-1-0-4):

Lecture 6:

Analog Communication Techniques: Amplitude Modulation

Feb. 03, 2025

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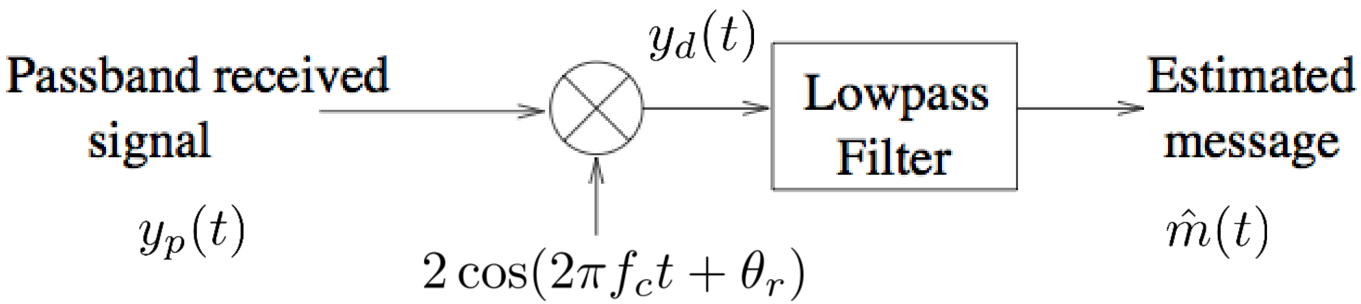
Demodulation of DSB-SC



- Standard downconverter to recover I component
- The received signal in the passband is

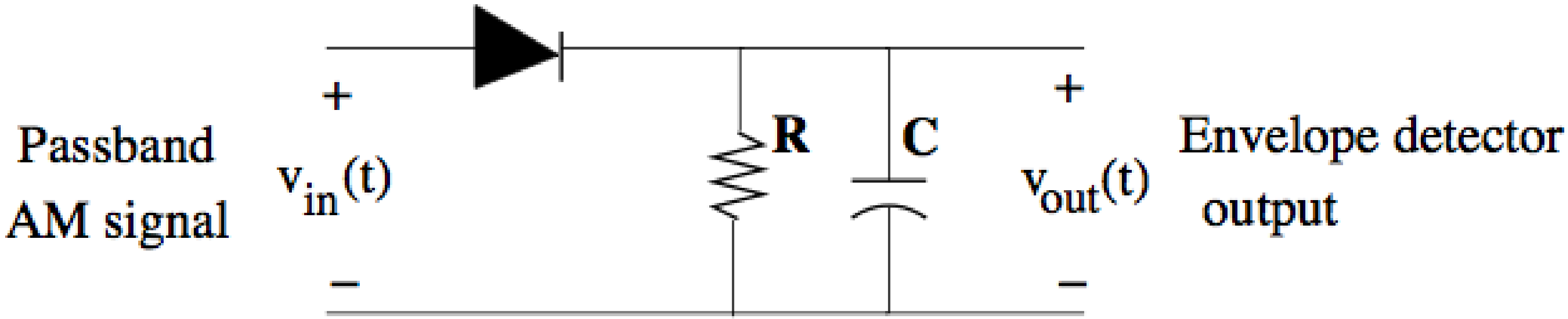
$$y_p(t) = Am(t)\cos(2\pi f_c t)$$

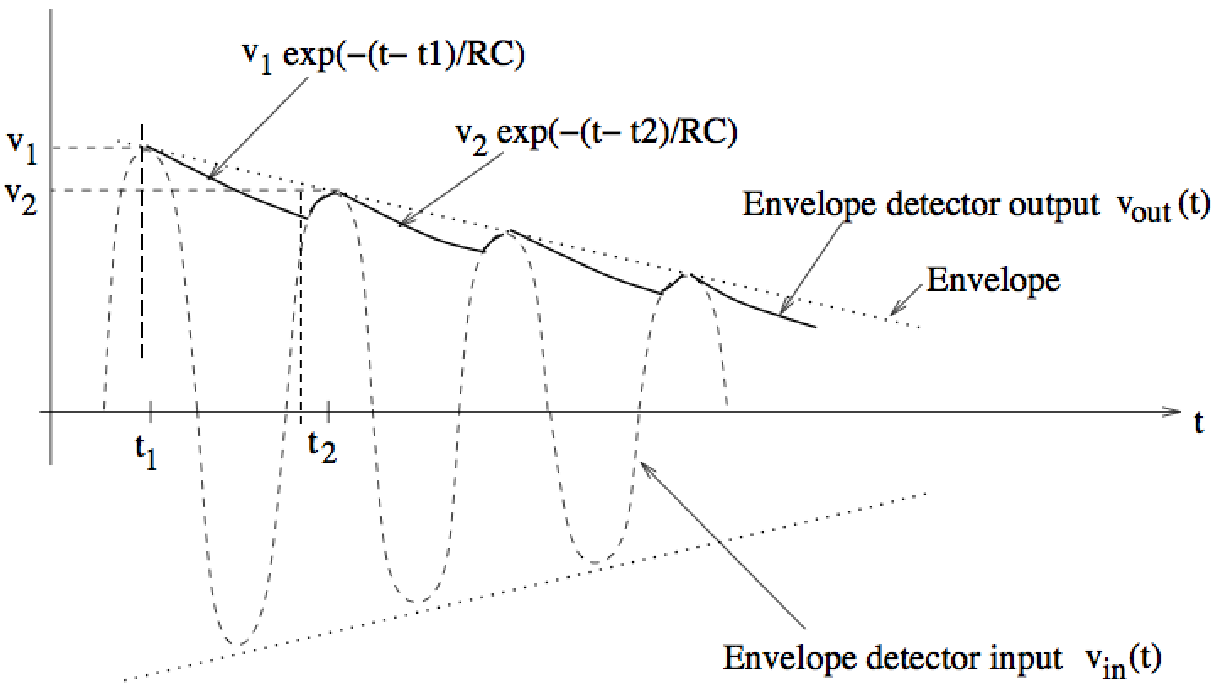
where θ_r is the phase difference arising from the phase offset with respect to local carrier at Rx.

- The output of multiplier followed by low pass filter is

$$\hat{m}(t) = Am(t)\cos\theta_r$$

Envelope Detectors

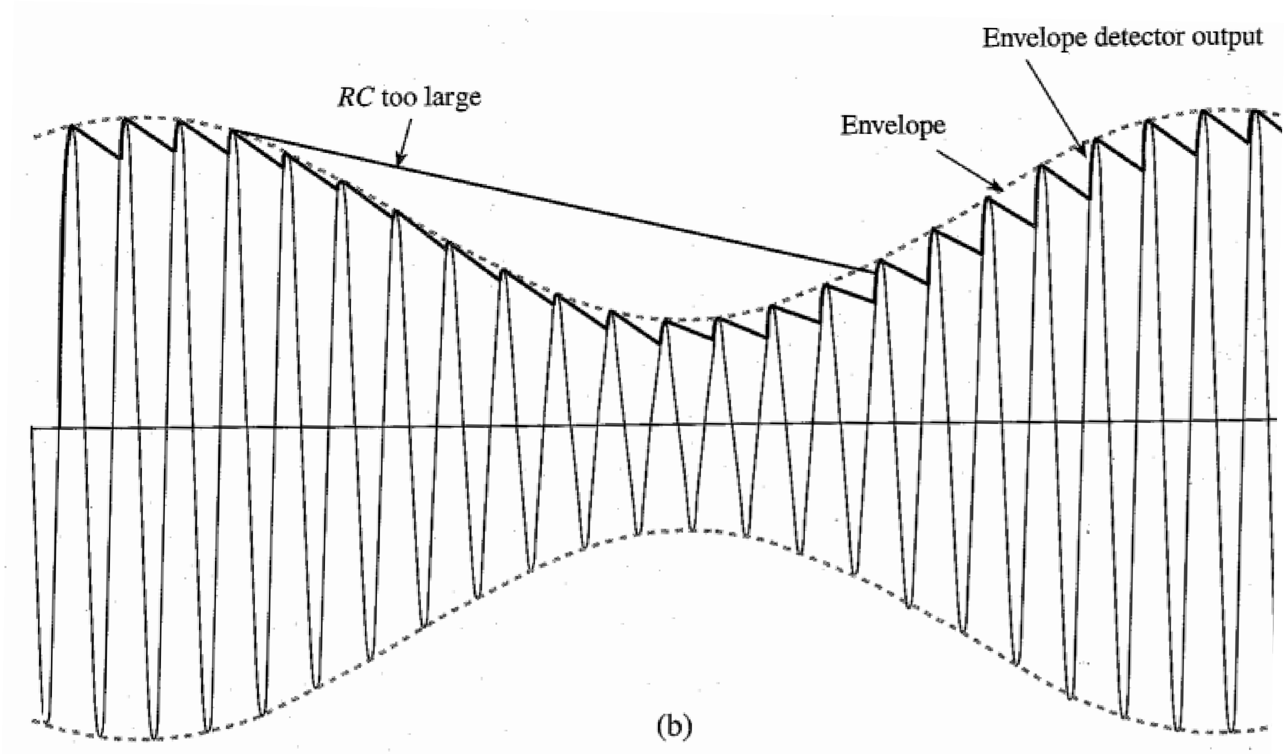




- At $t = t_1$, diode reverse biases and capacitor starts discharging.
- When $t = t_2'$, diode forward biases and capacitor starts charging.

Positive carrier cycle ➔ capacitor charges up (reaches value of envelope)
Negative carrier cycle ➔ capacitor discharges with RC time constant

Envelope detector operation



Positive carrier cycle ➔ capacitor charges up (reaches value of envelope)
Negative carrier cycle ➔ capacitor discharges with RC time constant

Should not discharge too fast during negative cycle
Should react fast enough to follow variations in envelope (which depend on message bandwidth B)

$$\frac{1}{f_c} \ll RC \ll \frac{1}{B}$$

Todays' Class

References

- Chap. 3 (Madhow)
- BP Lathi

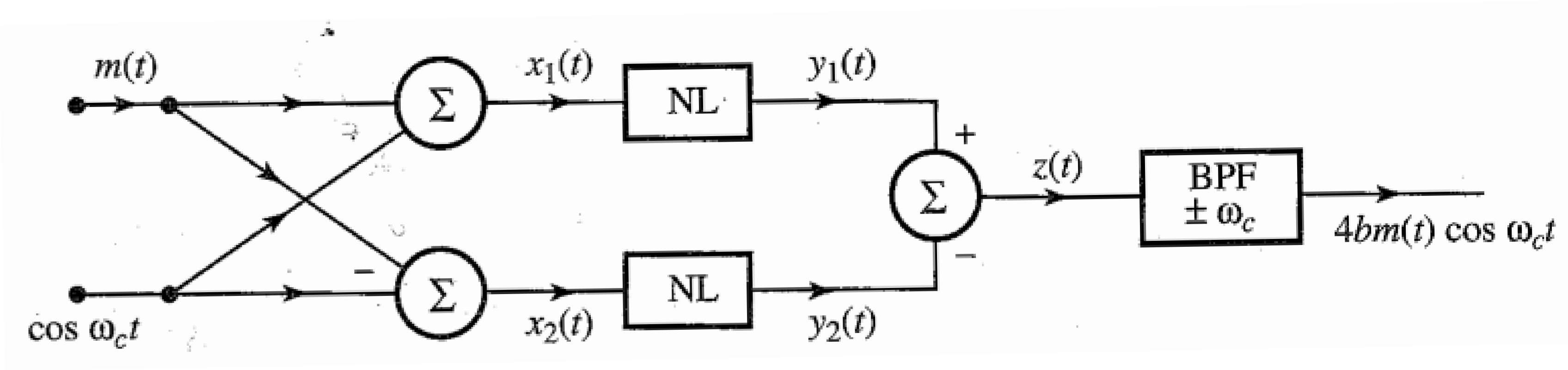
Conventional AM Modulators

B.P.Lathi pages: 155-159

How do we do conventional AM modulation?

- Use of multiplier
  - Several ways: Analog multiplier such as Sheingold, Variable gain amplifier, etc
  - It is rather difficult to maintain linearity in this kind of amplifier
  - They are expensive
- Few of other simple yet practical methods
  - Non-linear modulators
  - Switching modulators

Non-linear Modulators

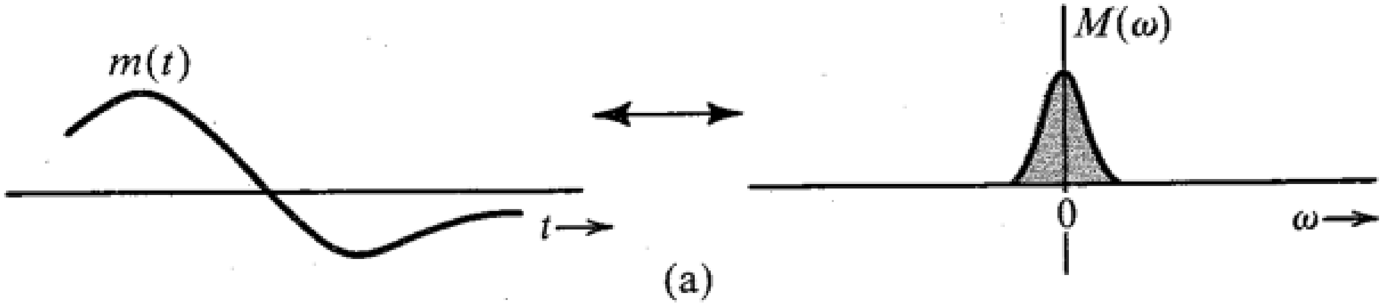


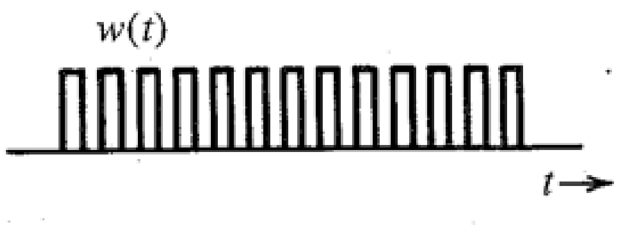
- Assuming the input-output characteristics of the nonlinear (NL) elements be approximated by a power series

$$y(t) = ax(t) + bx^2(t)$$

show that the output of the above ciccuit is $4bm(t)\cos\omega_c t$.

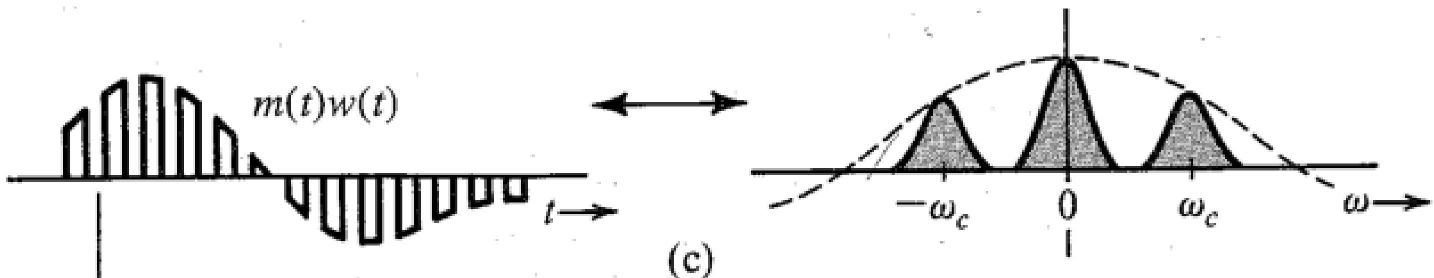
Switching Modulators

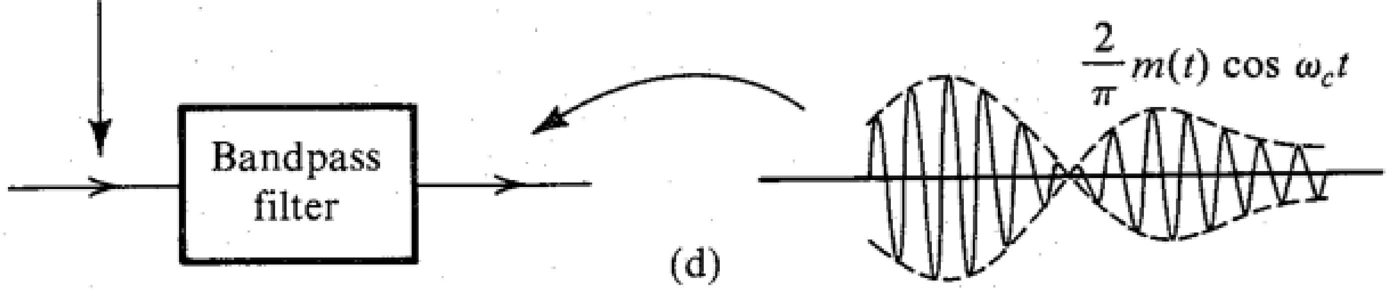




$$w(t) = \sum c_n \cos(n\omega_c t + \theta_n)$$

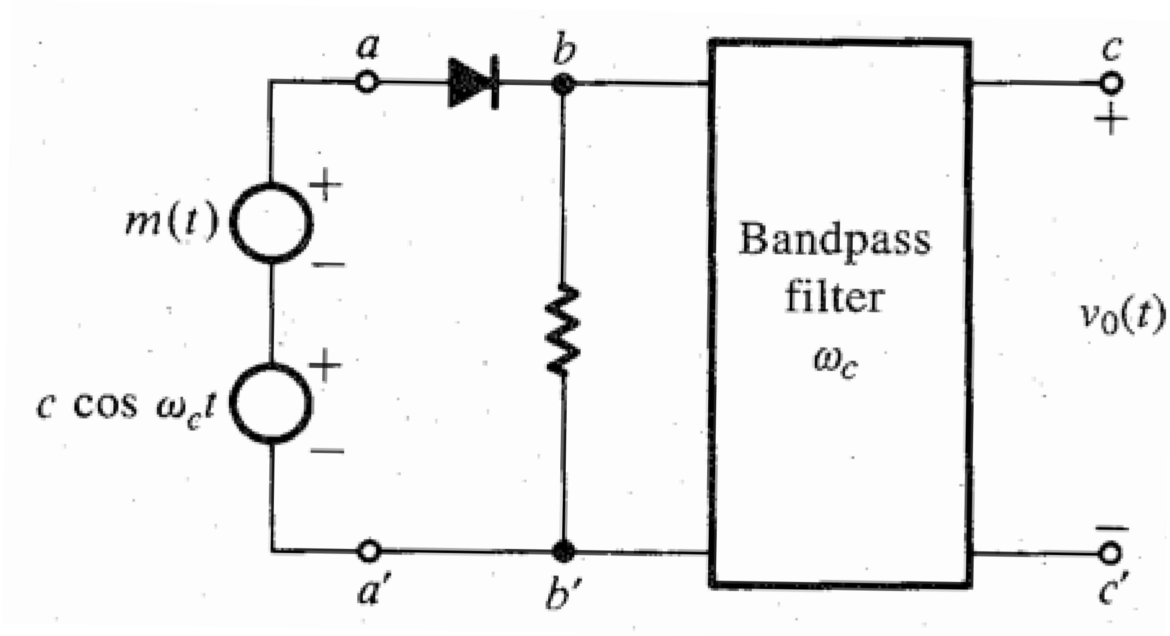
$$= \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \dots\right)$$





DSB-SC

Switching Modulators



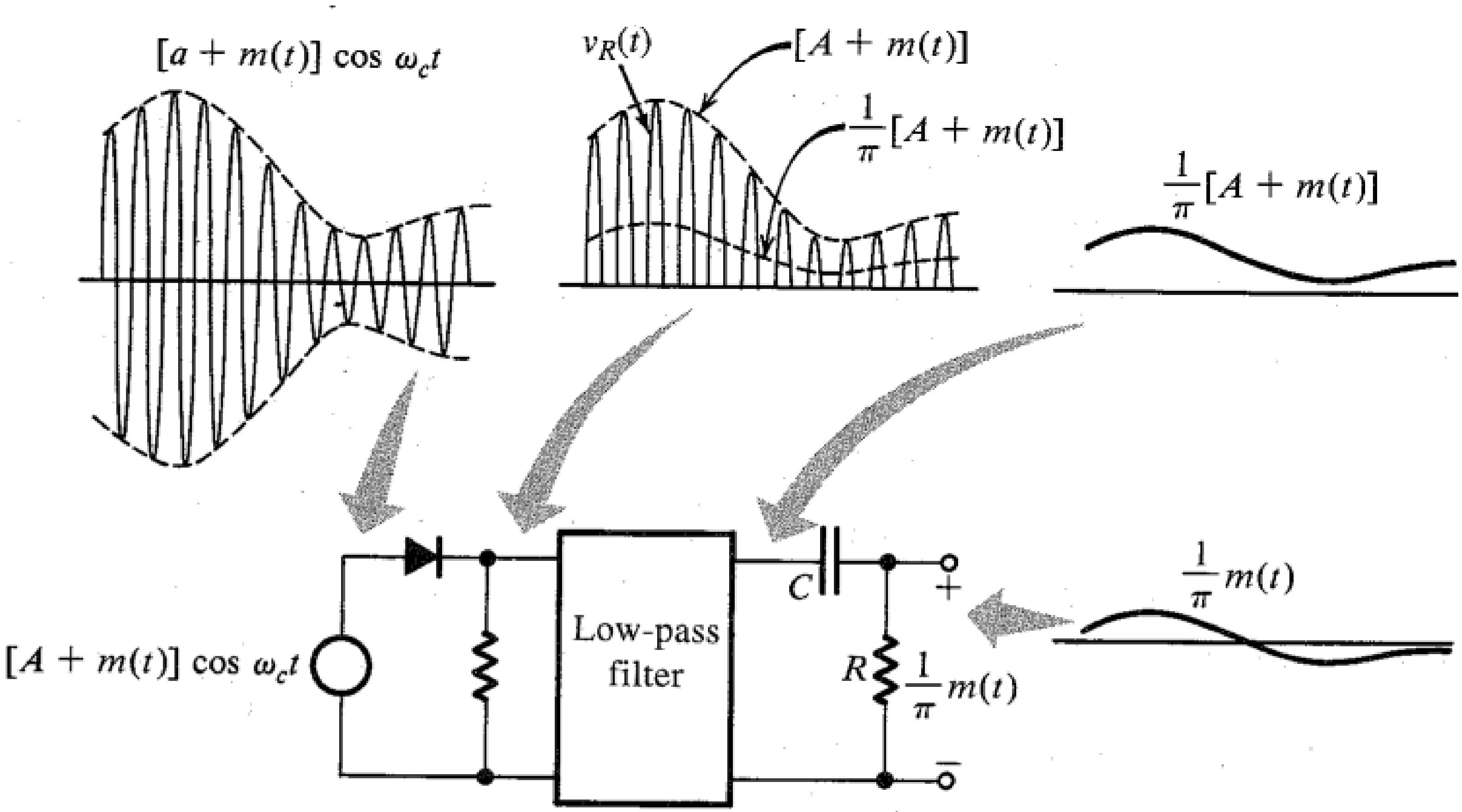
Conventional DSB

$$w(t) = \sum c_n \cos(n\omega_c t + \theta_n) \; = \; \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \dots\right)$$

$$\begin{aligned} v_{bb'} &= (c\cos\omega_c t + m(t))w(t) \\ &= (c\cos\omega_c t + m(t))\{\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \dots\right)\} \\ &= \frac{c}{2}\cos\omega_c t + \frac{2}{\pi}m(t)\cos\omega_c(t) + \boxed{\text{Other terms}} \end{aligned}$$

Suppressed by BPF

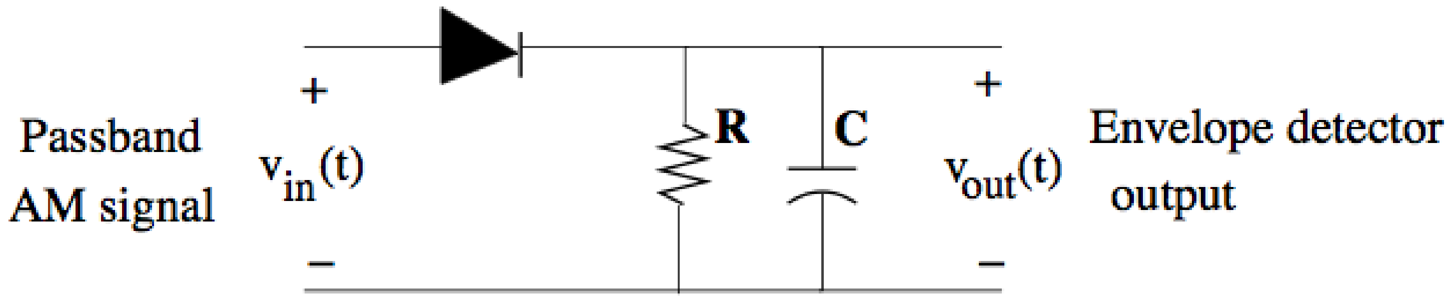
Rectifier Detector

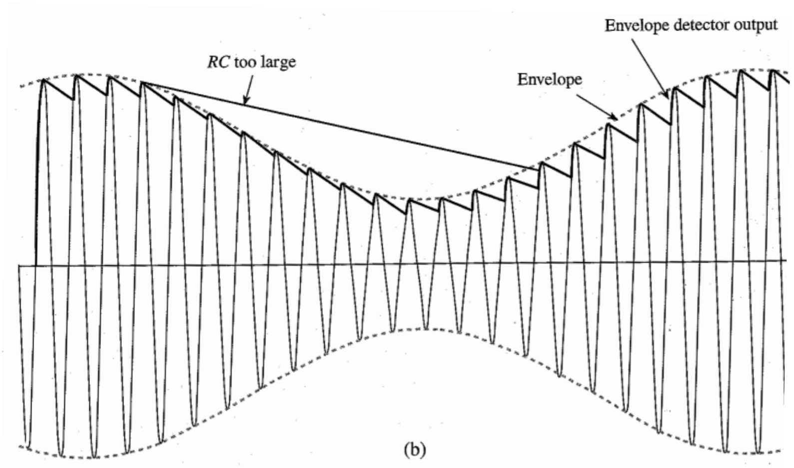


Lathi

$$
\begin{aligned}
v_{bb'} &= \{(A + m(t))\cos\omega_c t\}w(t) \\
&= \{(A + m(t))\cos\omega_c t\}\{\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \dots\right)\} \\
&= \frac{1}{\pi}[A + m(t)] + \text{Other terms of higher frequencies}
\end{aligned}
$$

Envelope detector operation





(b)

$$\frac{1}{f_c} \ll RC \ll \frac{1}{B}$$

Madhow

Positive carrier cycle ➔ capacitor charges up (reaches value of envelope)
Negative carrier cycle ➔ capacitor discharges with RC time constant
Should not discharge too fast during negative cycle
Should react fast enough to follow variations in envelope (which depend on message bandwidth B)

Power efficiency of conventional AM

Power efficiency of conventional AM

- DSB expression

$$u_{\mathrm{AM}}(t) = Am(t)\cos(2\pi f_c t) + \boxed{A_c \cos(2\pi f_c t)}$$

Extra Non-information carrying component

- Power efficiency is given by

$$\eta = \frac{\text{Power in information carrying signal}}{\text{Power in total signal}}$$

- Prove that power efficiency for conventional AM is given by

$$\eta_{\mathrm{AM}} = \frac{a_{\mathrm{mod}}^2 \overline{m_n^2}}{1 + a_{\mathrm{mod}}^2 \overline{m_n^2}}$$

- Further prove that

$$\eta_{\mathrm{AM}} \leq 50\%$$

- Solve: Find η_{AM} for sinusoidal message signal $m(t) = A_m \cos(2\pi f_m t)$

Comments on Conventional AM

- Conventional AM trades-off synchoronous requirement with power efficiency.
- Suitable for broadcasting application
- Note that coherent detector is also possible for this!

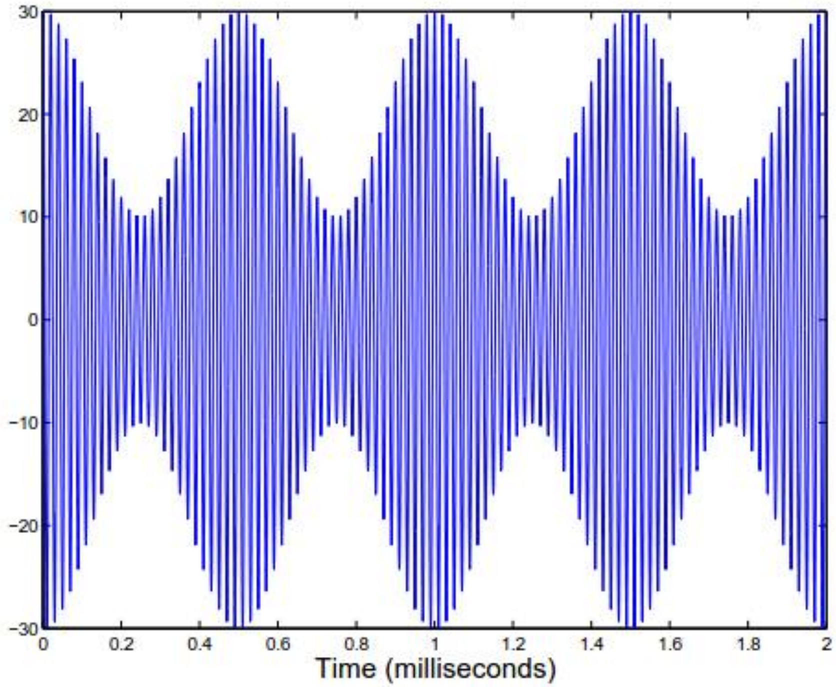
Example on power efficiency computation

The message $m(t) = 2\sin(2000\pi t) - 3\cos(4000\pi t)$ is used in AM system with a modulation index of 70% and carrier frequency of 580 KHz.

- What is the power efficiency?
- If the net transmitted power is 10 W, find magnitude spectrum of the transmitted signal.

Tutorial

Example: Tutorial!



$$u_{\text{AM}}(t) = (A_c + m(t)) \cos 2\pi f_c t$$

Fig. above shows a signal obtained after amplitude modulation by a sinusoidal message. The carrier frequency is difficult to determine from the figure and is not required for answering following questions

- Find the modulation index
- Find the signal power
- Find the bandwidth of the AM signal

EC5.203 Communication Theory I (3-1-0-4):

Lecture 7:

Analog Communication Techniques: Amplitude Modulation - 3

Feb. 06, 2025

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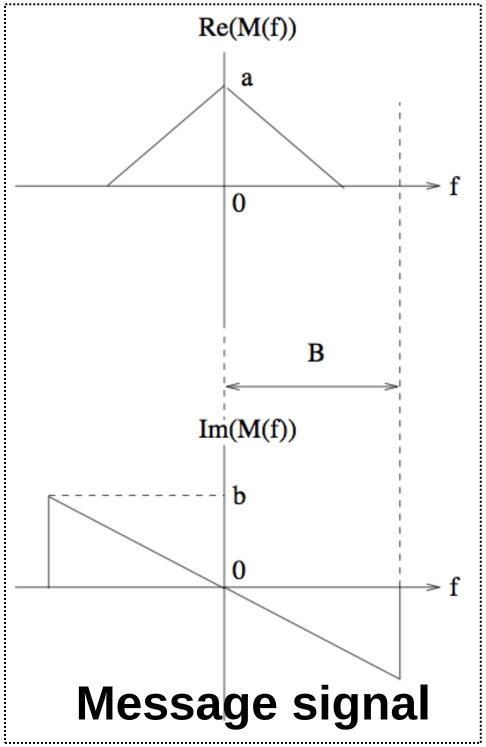
Conventional AM modulation

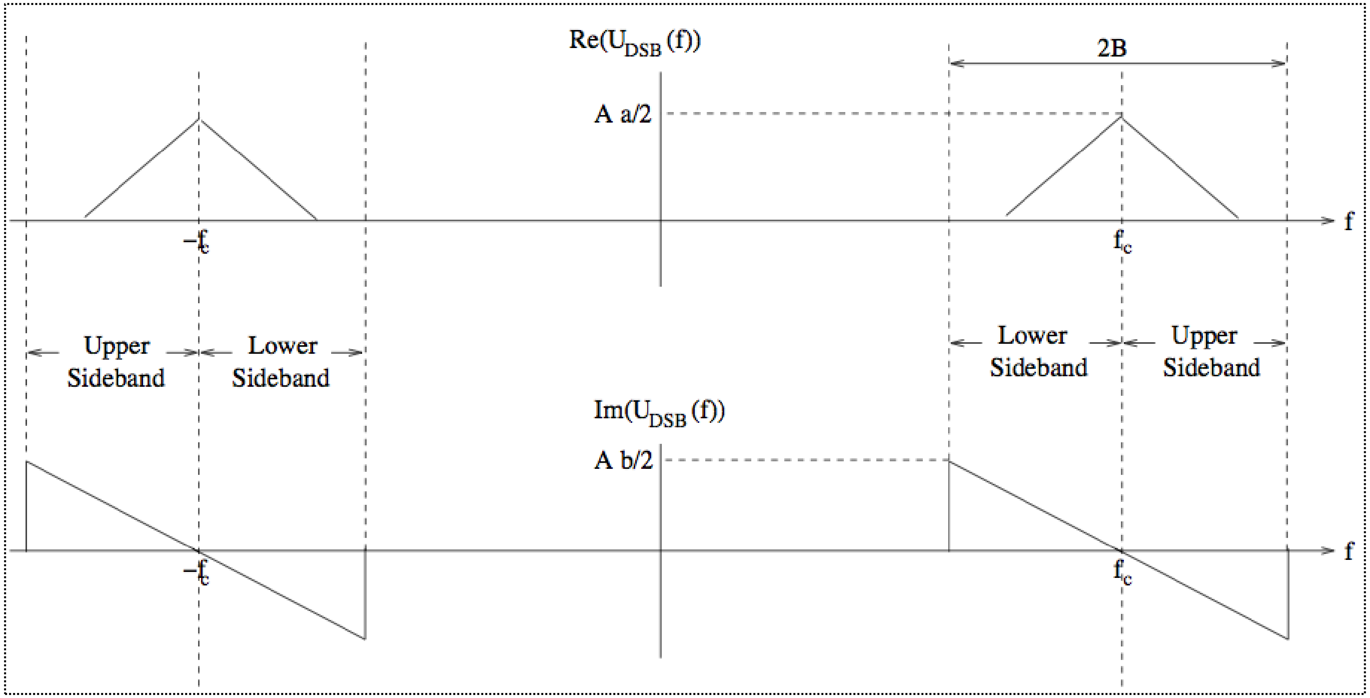
- Use of multiplier
  - Several ways: Analog multiplier such as Sheingold, Variable gain amplifier, etc
  - It is rather difficult to maintain linearity in this kind of amplifier
  - They are expensive
- Few of other simple yet practical methods
  - Non-linear modulators
  - Switching modulators

B.P.Lathi pages: 155-159

Amplitude Modulation: Single Side Band

SSB: Motivation



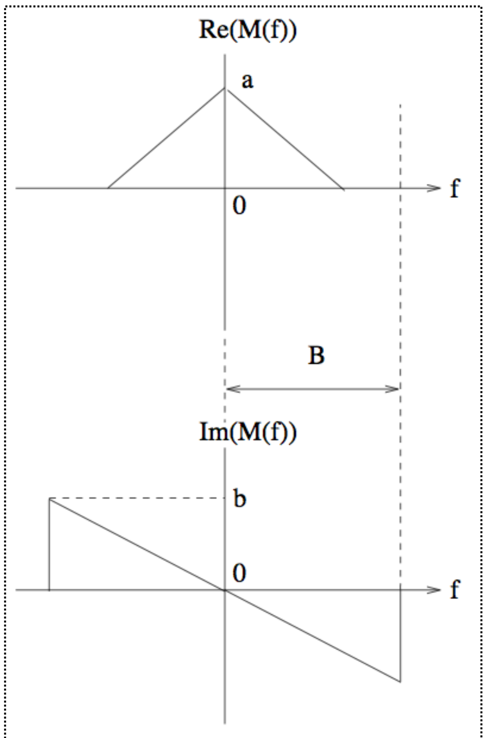


- Each sideband has enough information to extract the original message.

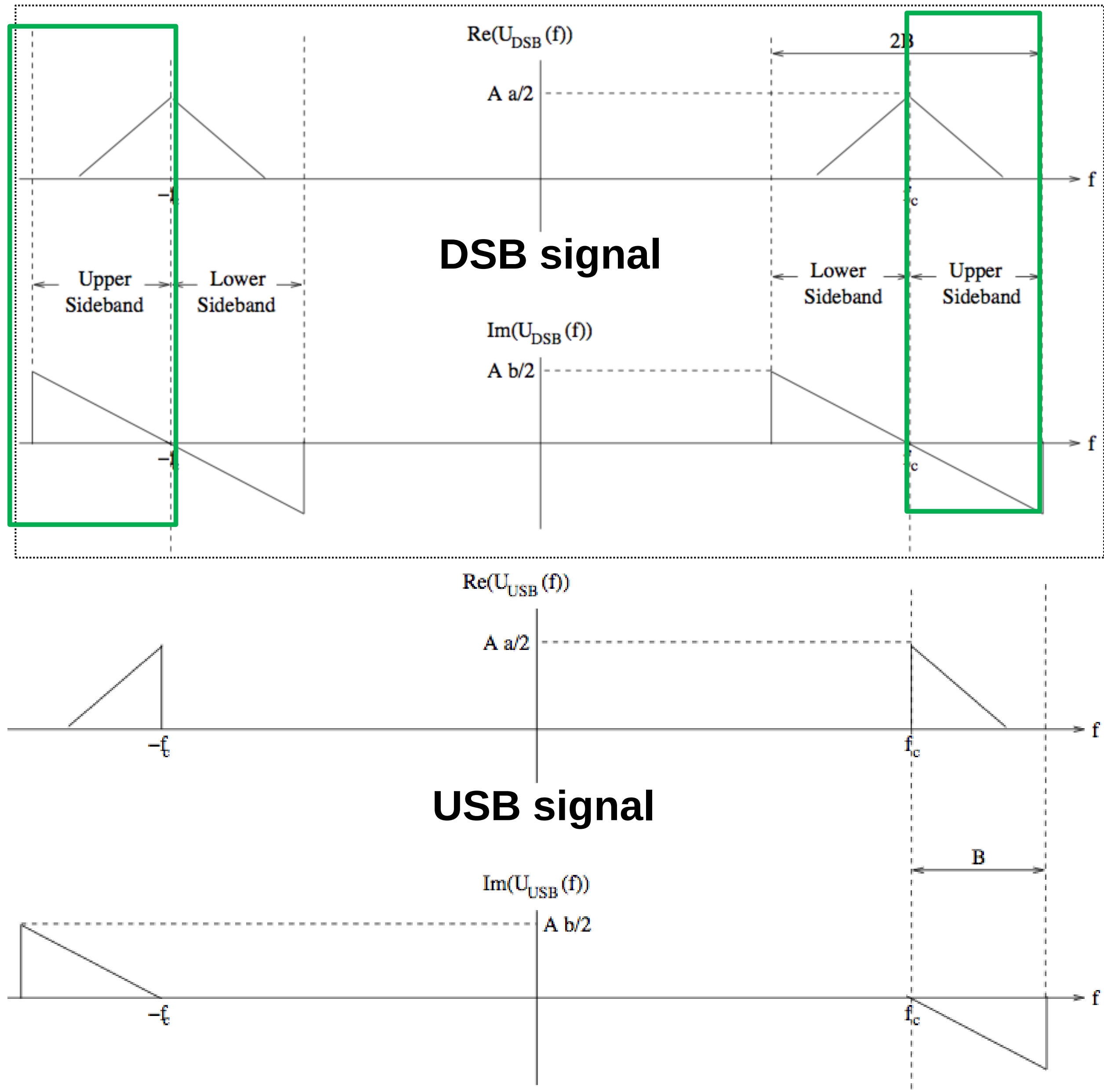
m(t) is complex envelope of DSB

- Message $m(t)$ is the I component of an DSB signal.
- Sending only one sideband reduces our bandwidth requirement by 50%.

DSB ➔ USB (SSB)



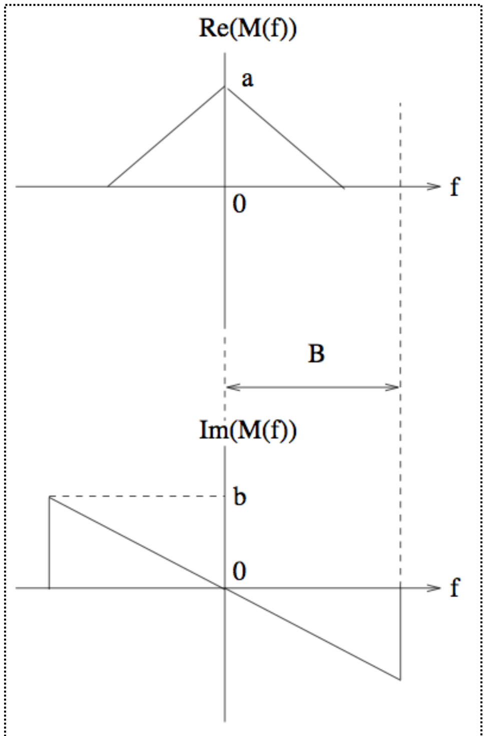
Message signal



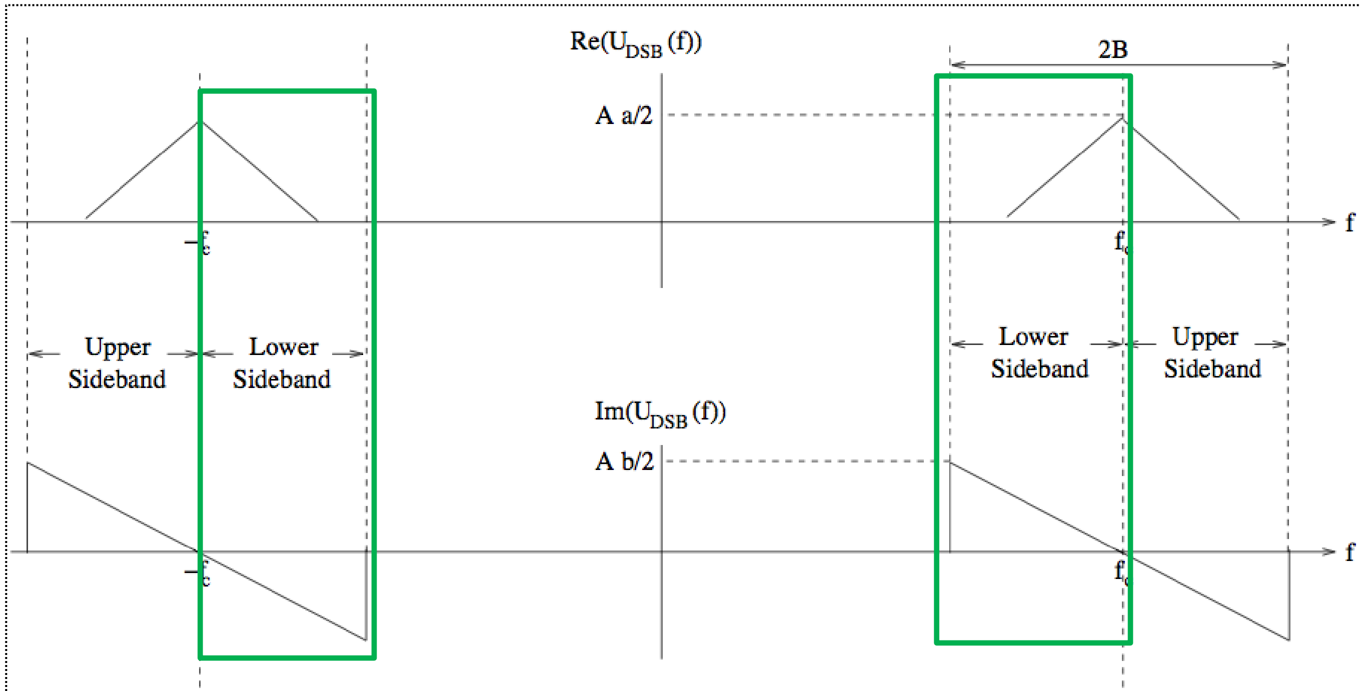
DSB signal

USB signal

DSB ➔ LSB (SSB)

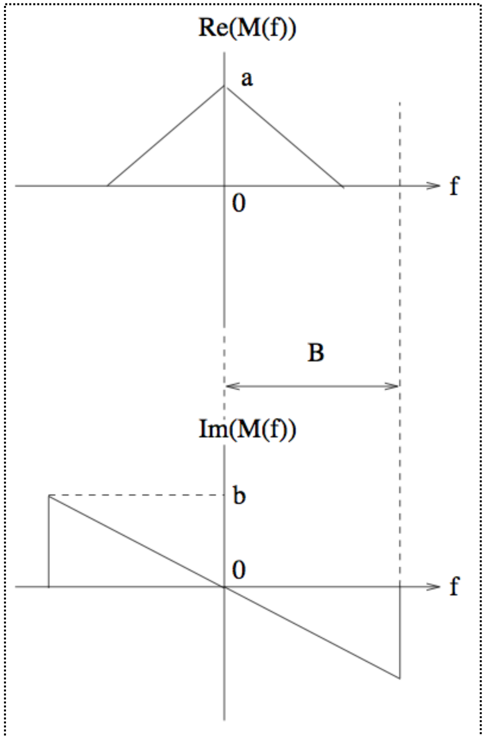


Message signal

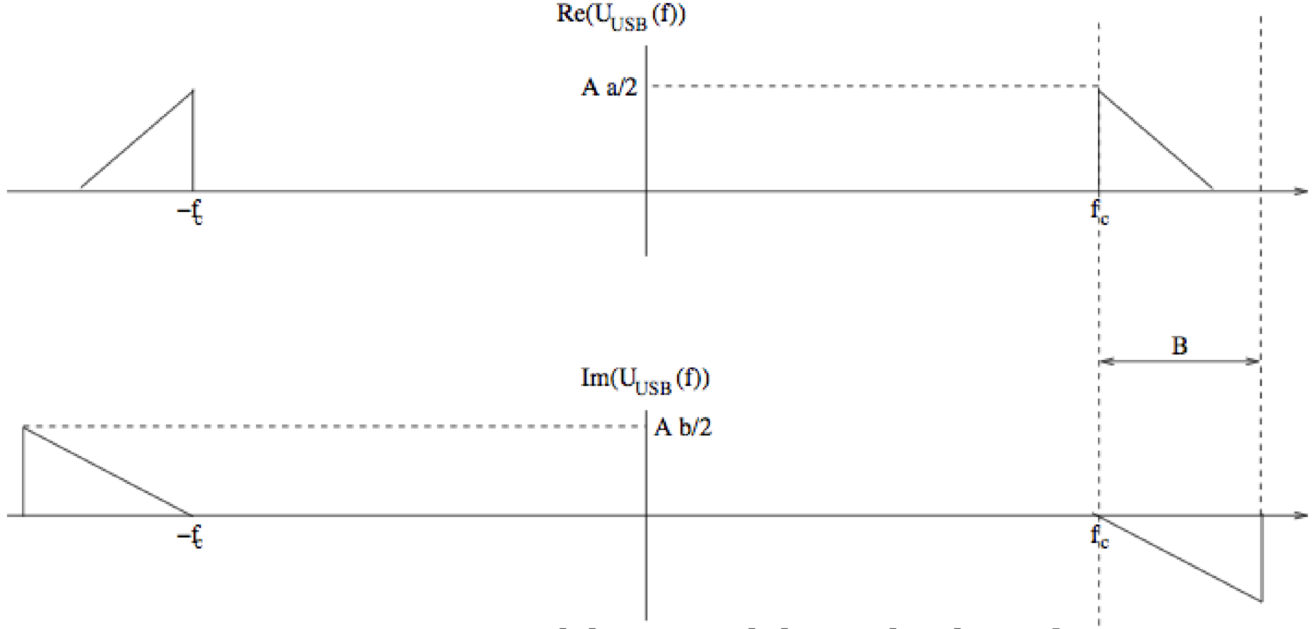


DSB signal

Recover Signal from USB Signal? Solve!

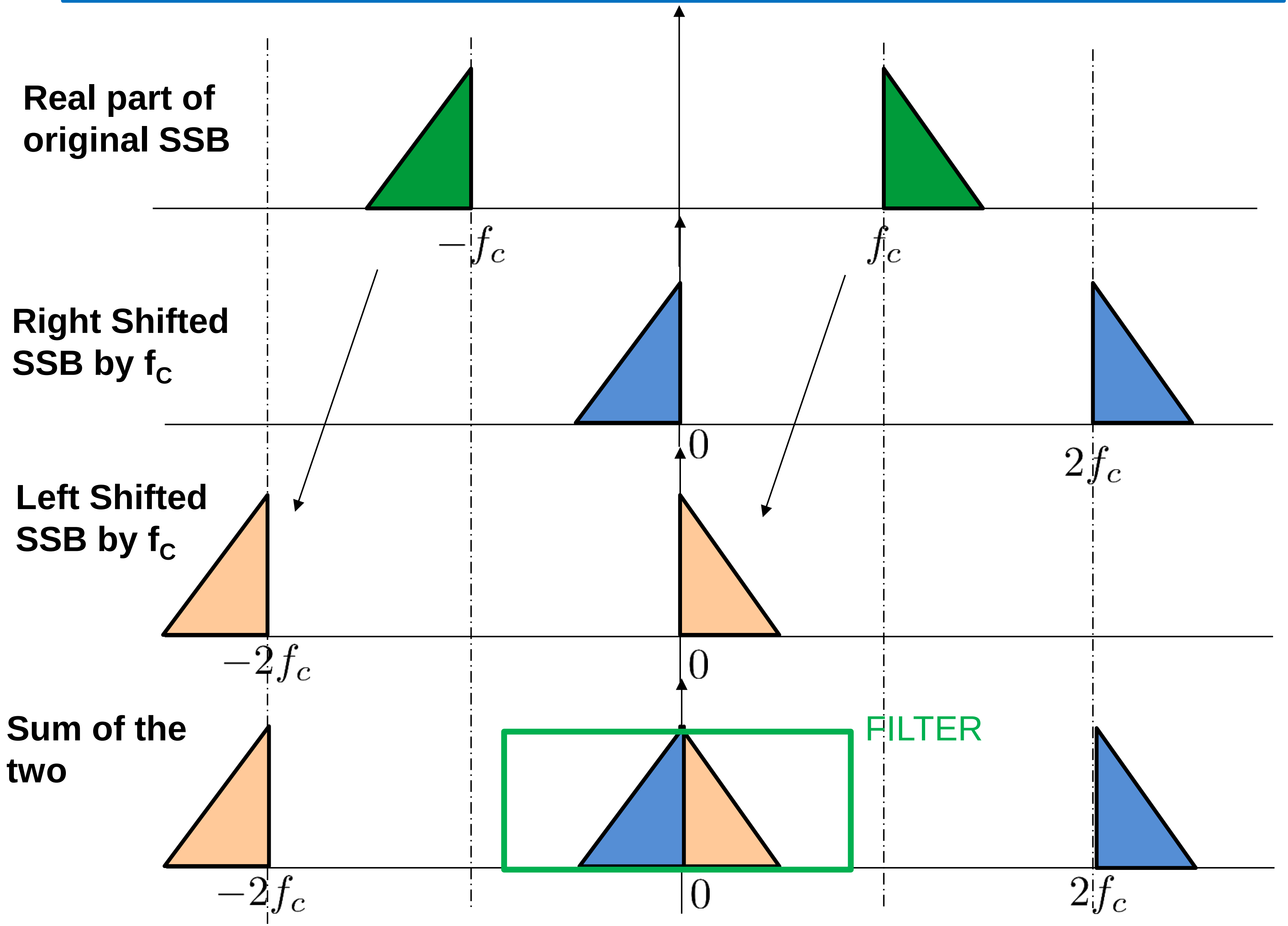


Message signal

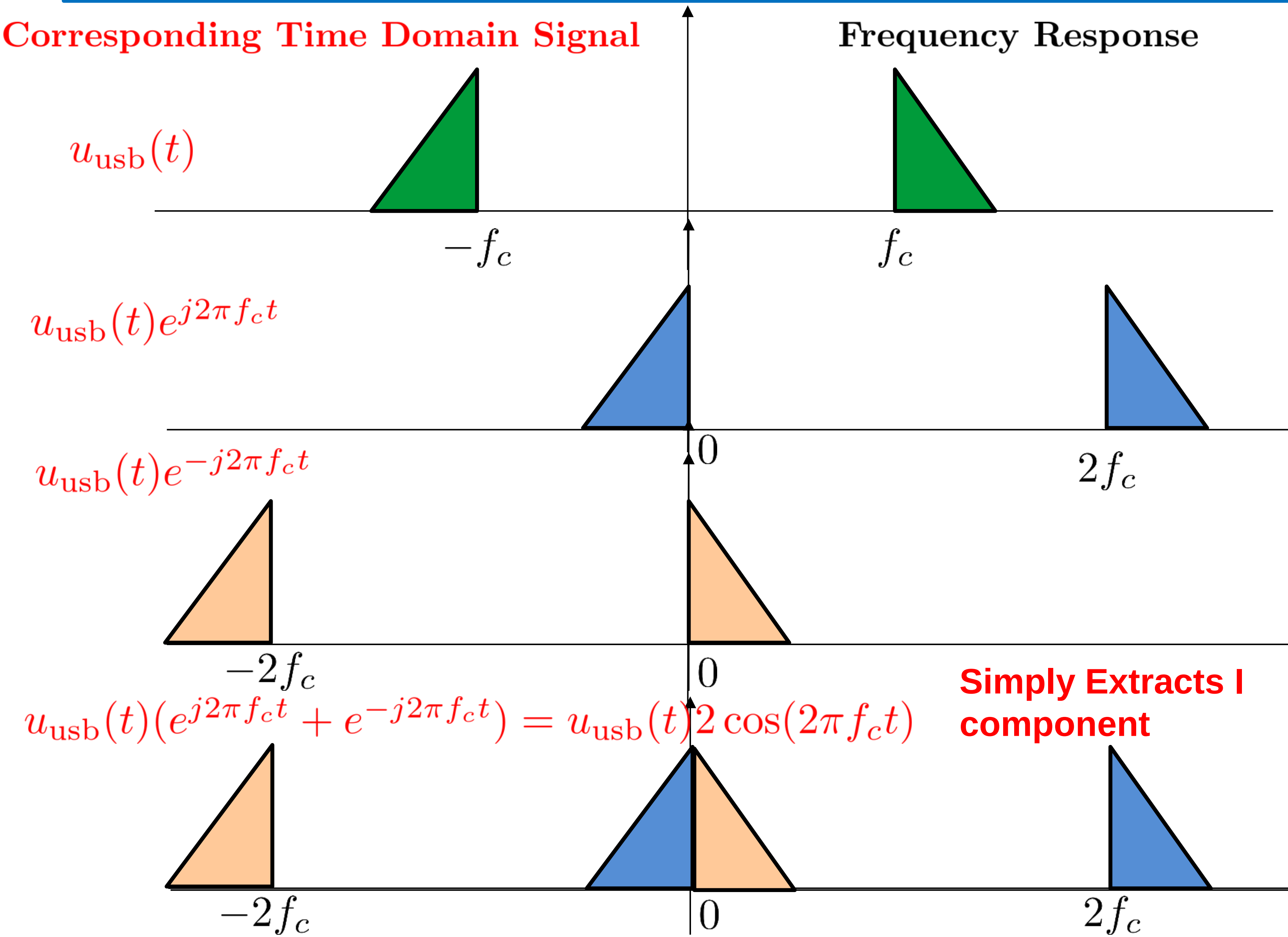


Upper Side Band (USB) Signal

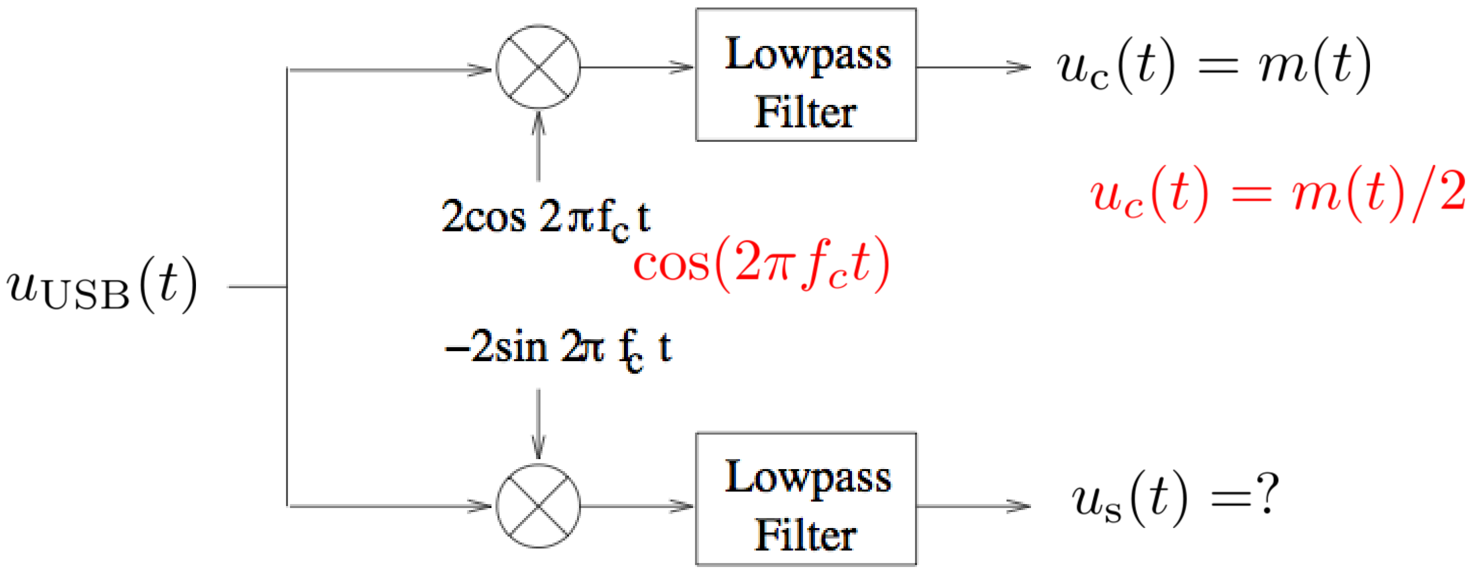
Solution!


Real part of original SSB
Right Shifted SSB by f_C
Left Shifted SSB by f_C
Sum of the two
$-f_c$
f_c
0
$2f_c$
$-2f_c$
0
FILTER
$-2f_c$
0
$2f_c$

Corresponding Time Domain Equations

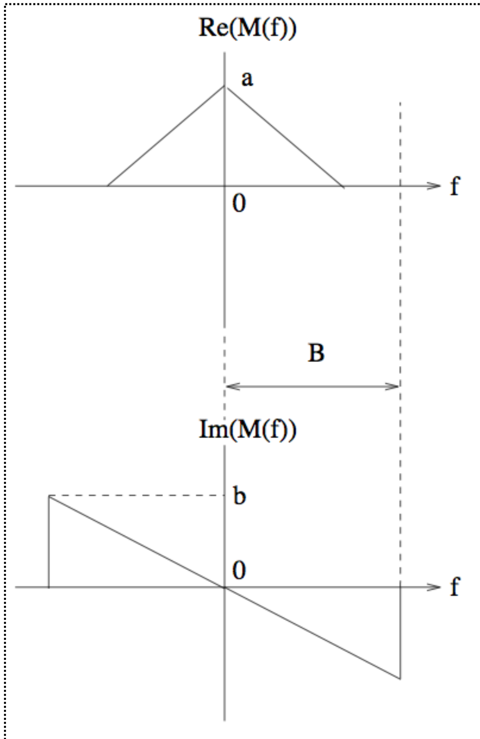


Message signal is I component of filter o/p

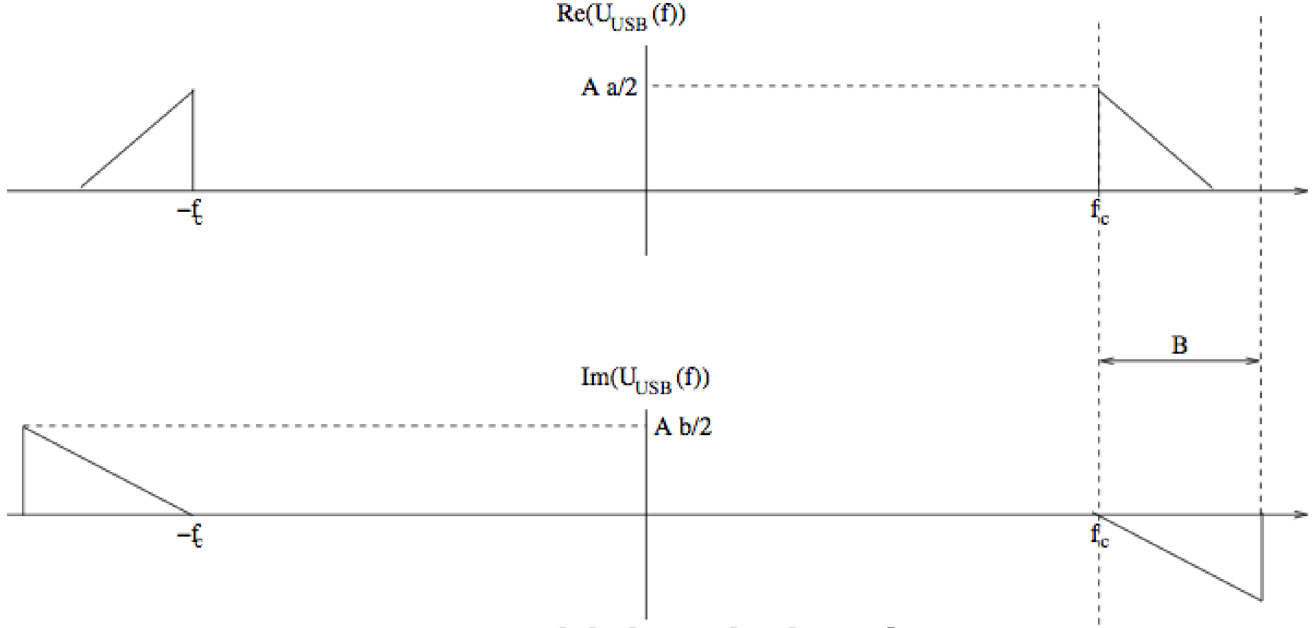


Downconversion
(passband to baseband)

Recover Signal from USB Signal!



Message signal

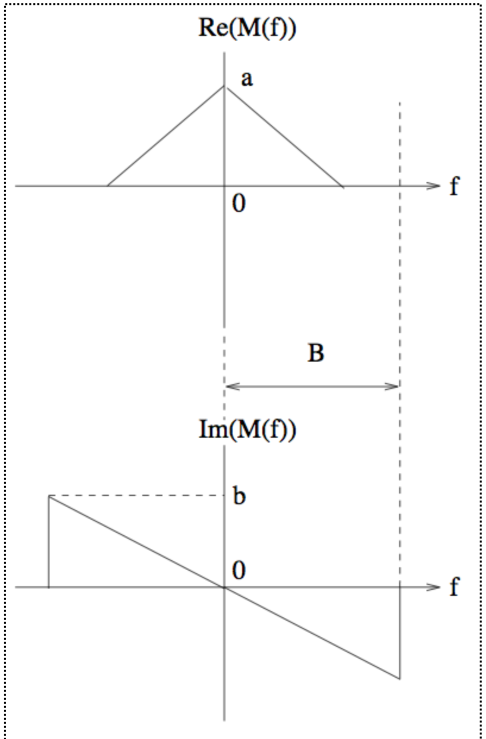


Upper Sideband Signal

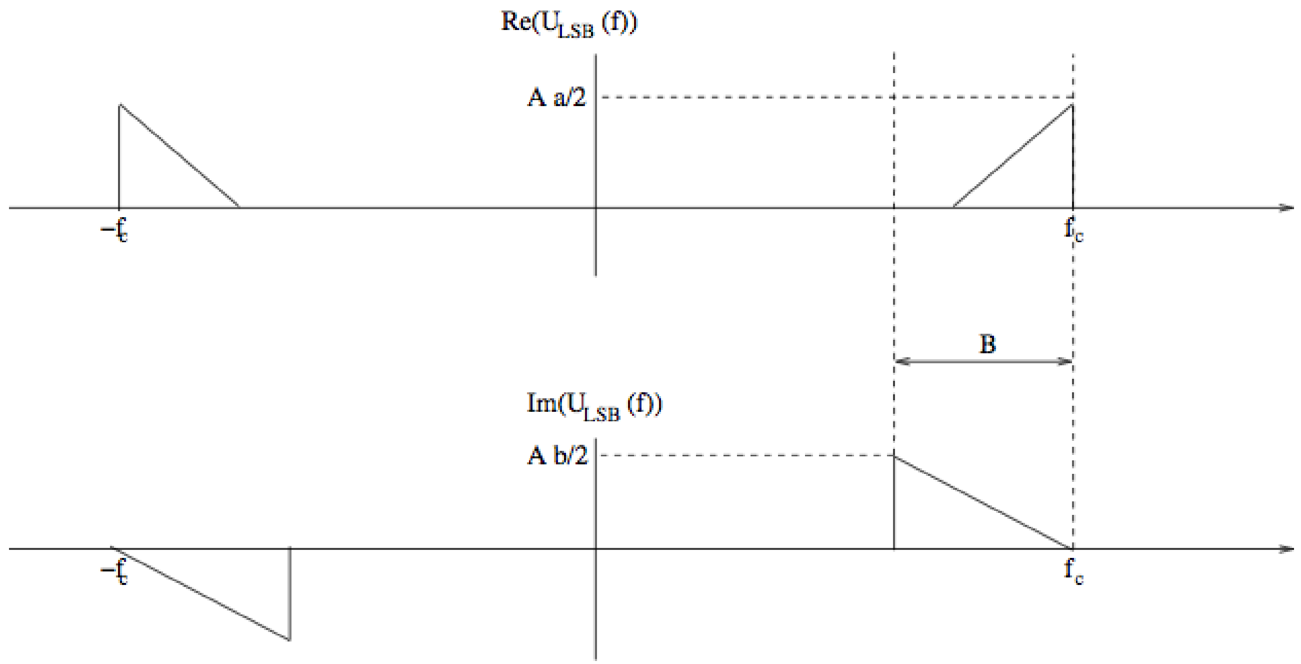
The original signal can be obtained from the USB signal by

1. Shifting the USB signal to right by f_c
2. Shifting the USB signal to left by f_c
3. Add the two signals
4. Pass the resultant signal through low pass filter to filter $2f_c$ component

Recover Signal from LSB Signal?



Message signal



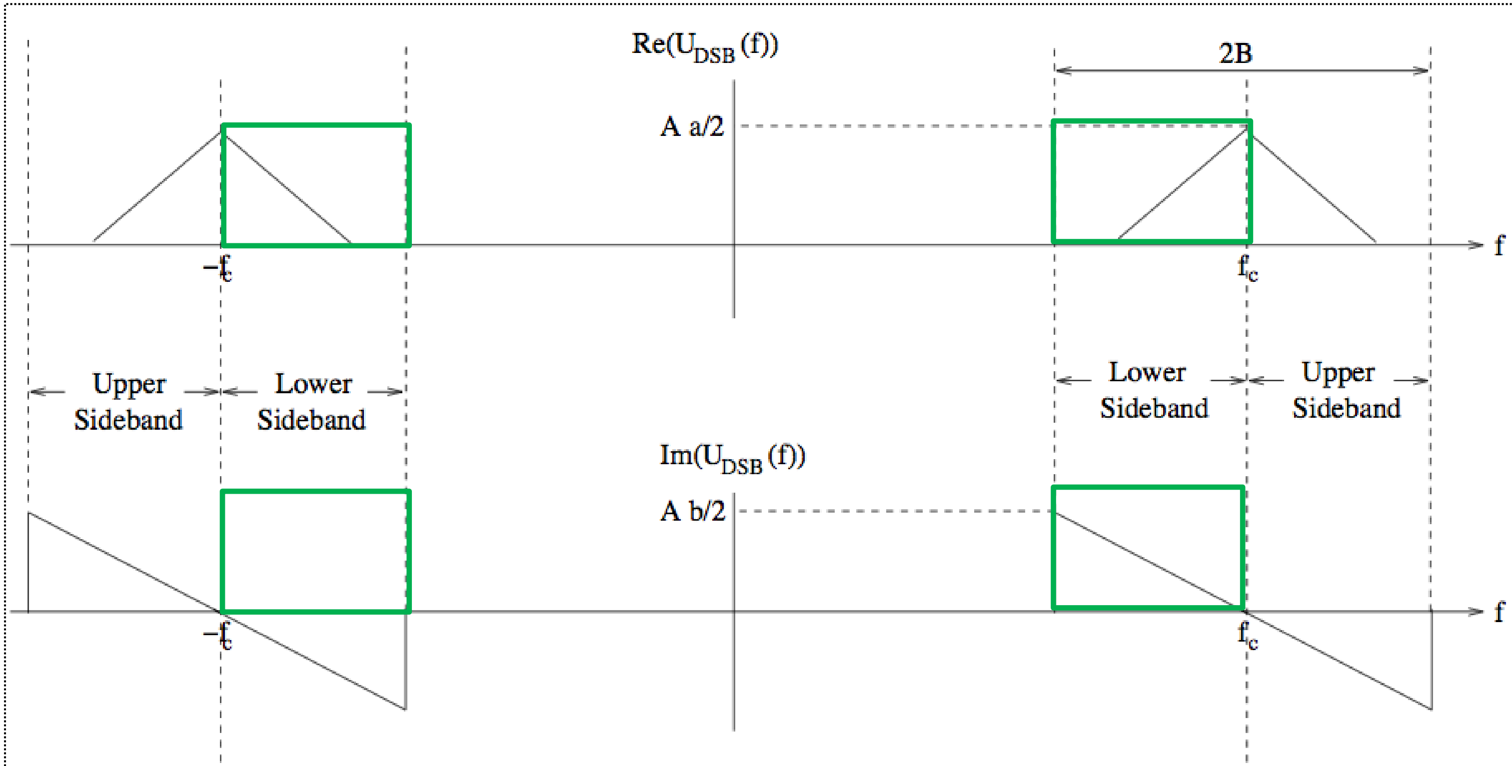
Lower Sideband (LSB) Signal

The original signal can be obtained from the USB signal by

1. Shifting the LSB signal to right by f_c
2. Shifting the LSB signal to left by f_c
3. Add the two signals
4. Pass the resultant signal through low pass filter to filter $2f_c$ component

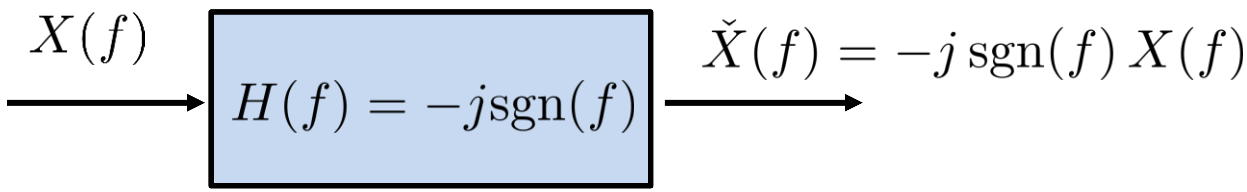
Use of Hilbert Transform for SSB Generation

Motivation: *Requirement of ideal filters*

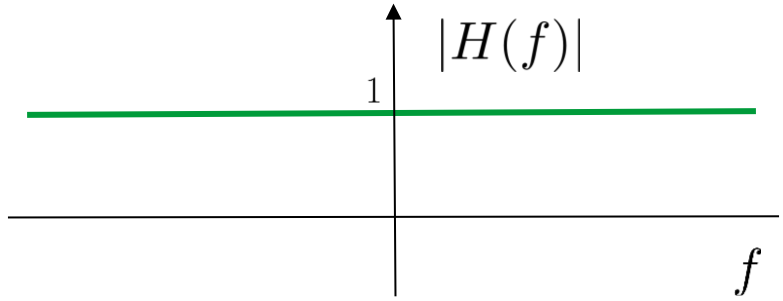


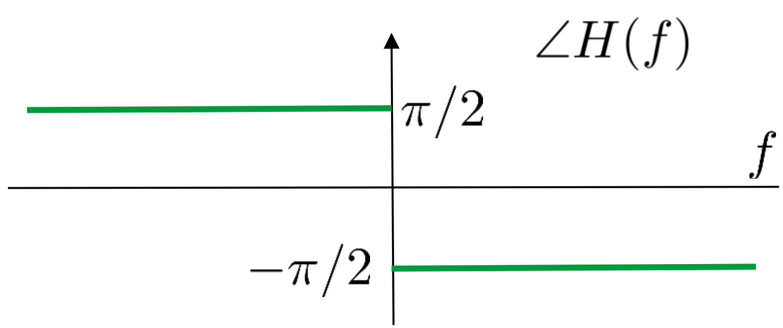
- Logical approach: Filtering one of the sideband requires rectangular filters with sharp cut-off!
- Practically infeasible!!!
- Implementing Hilbert transform in baseband avoids need for sharp filtering at passband!

Hilbert transform

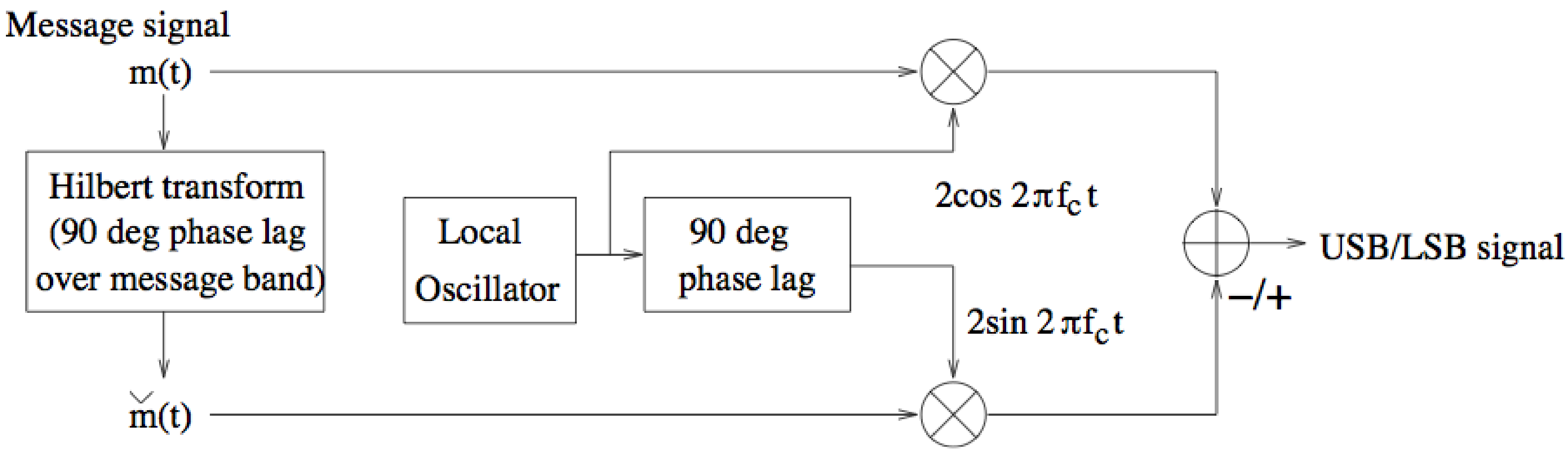


$$H(f) = -j\,\mathrm{sgn}(f) \longleftrightarrow h(t) = \frac{1}{\pi t}$$



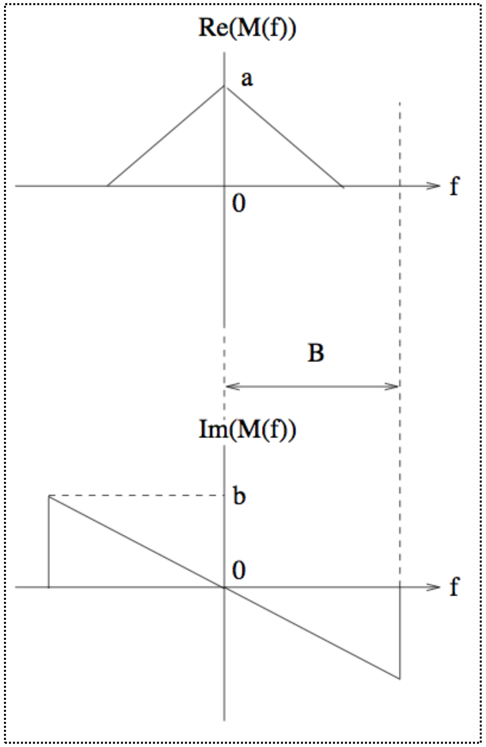


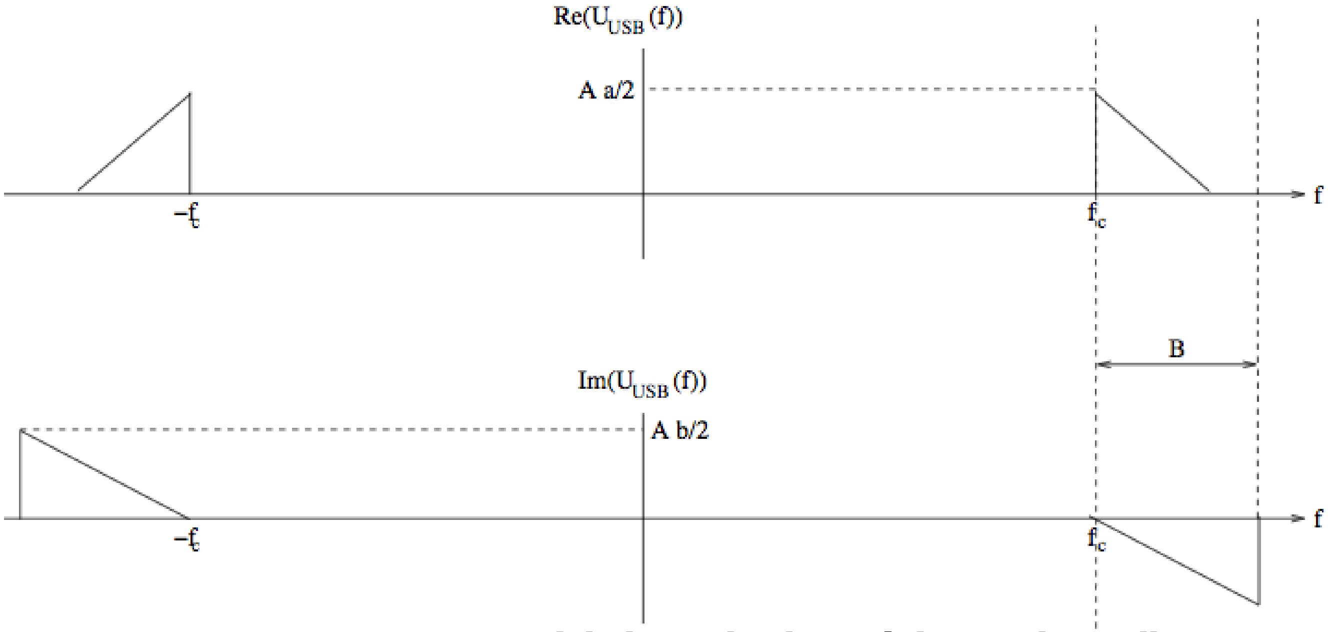
SSB in baseband using Hilbert Transform



- Implementing Hilbert transform in baseband avoids need for sharp filtering at passband!
- In next few slides, we will see why it works!

USB passband signal is real!





Upper Sideband Signal (Passband)

Baseband Message signal

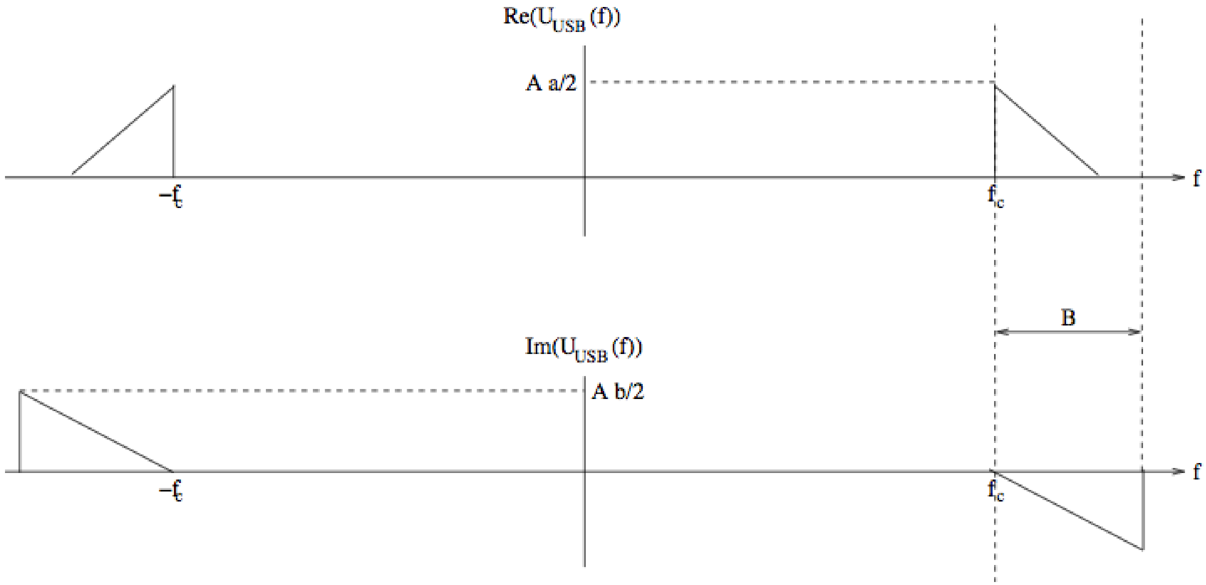
m(t) will be real!

For a real signal $x(t)$

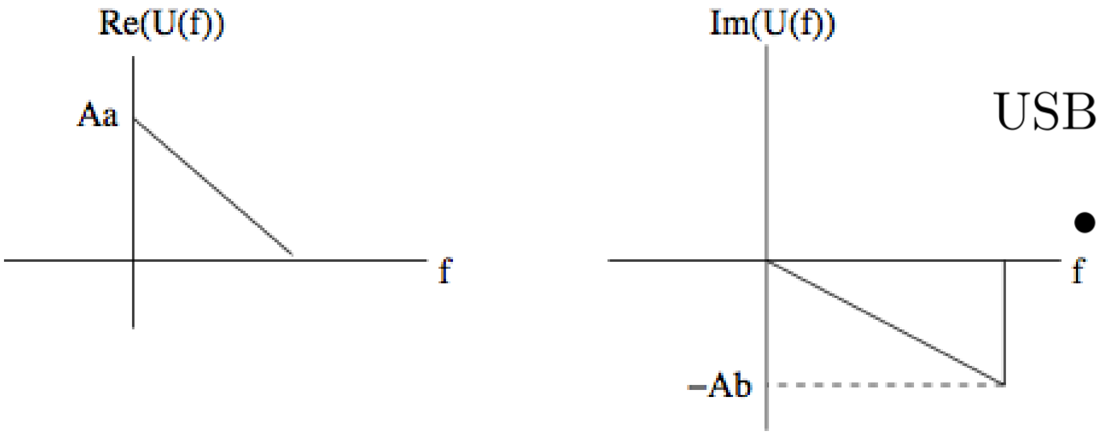
- $X(f) = X^*(-f)$ Even symmetry for magnitude spectrum
- $\mathrm{Re}\{X(f)\} = \mathrm{Re}\{X^*(-f)\}$, i.e., Even Symmetry
- $\mathrm{Im}\{X(f)\} = -\mathrm{Im}\{X^*(-f)\}$ Odd Symmetry

m(t) is complex envelope of DSB
In SSB discussion, u(t) will be complex envelope of USB

Complex envelope for USB signal



USB passband

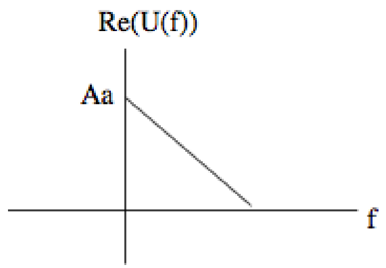


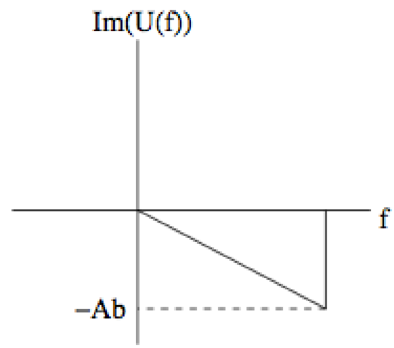
USB baseband signal $u(t)$ is complex

- No odd or even symmetry!

In SSB discussion, u(t) will be complex envelope of USB

I and Q components for SSB

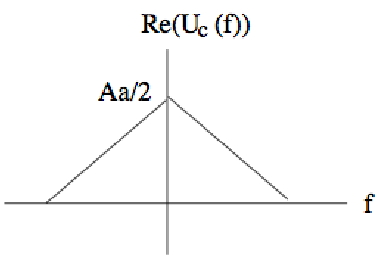


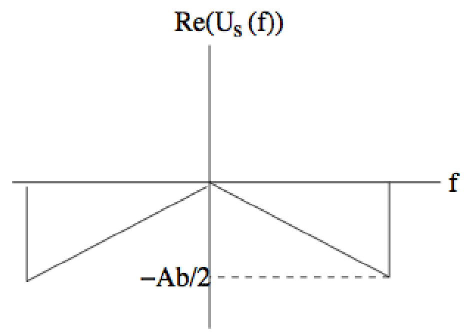


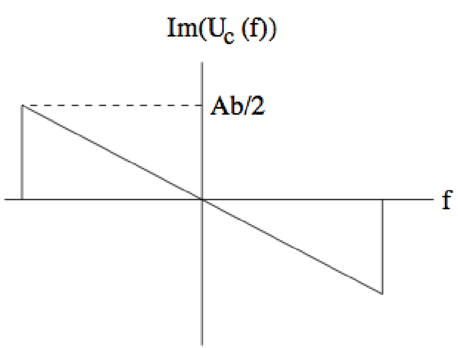
USB Complex envelope $U(f)$

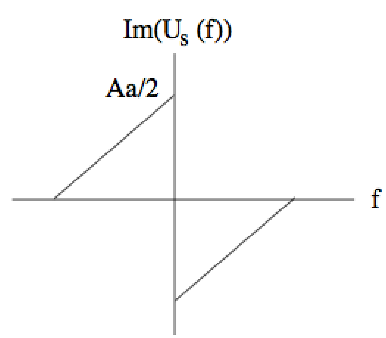
I component

Q component









USB I and Q components

$$U_c(f) = \frac{U(f) + U^*(-f)}{2}$$

$$U_s(f) = \frac{U(f) - U^*(-f)}{2j}$$

Prove

$$U_c(f) = A\,M(f)/2$$

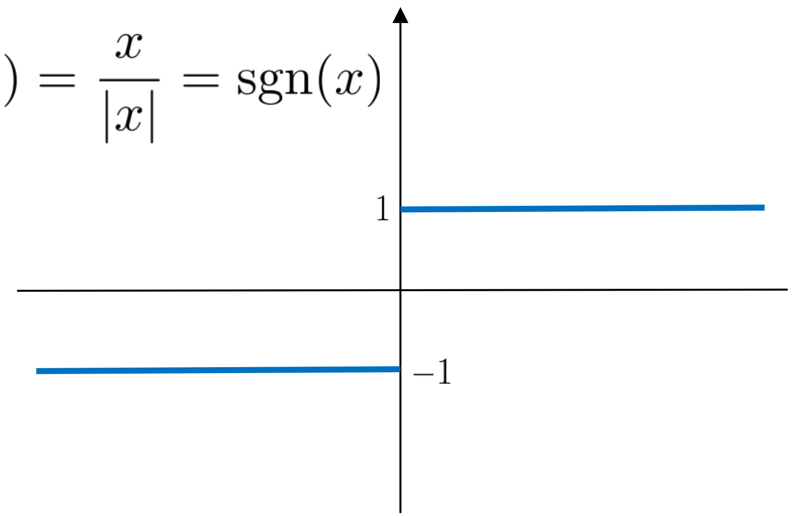
$$U_s(f) = -j\,\mathrm{sgn}(f)\,U_c(f)$$

Sign Function

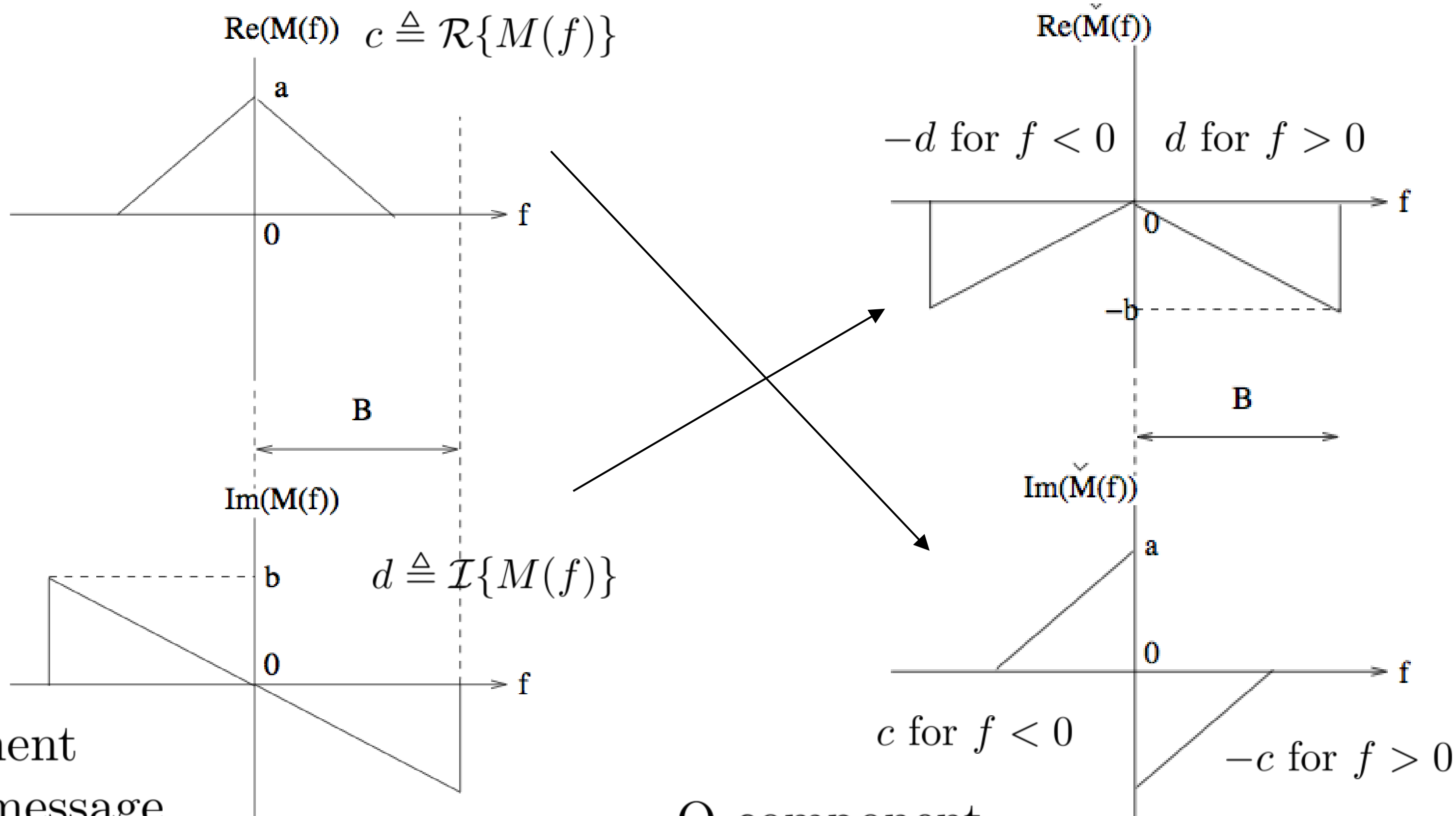
$$U_s(f) = -jU_c(f) \quad f > 0$$
$$U_s(f) = jU_c(f) \quad f < 0$$

$$\Longrightarrow \quad U_s(f) = -j\,\mathrm{sgn}(f)U_c(f)$$

$$y(x) = \frac{x}{|x|} = \mathrm{sgn}(x)$$



SSB and Hilbert Transform

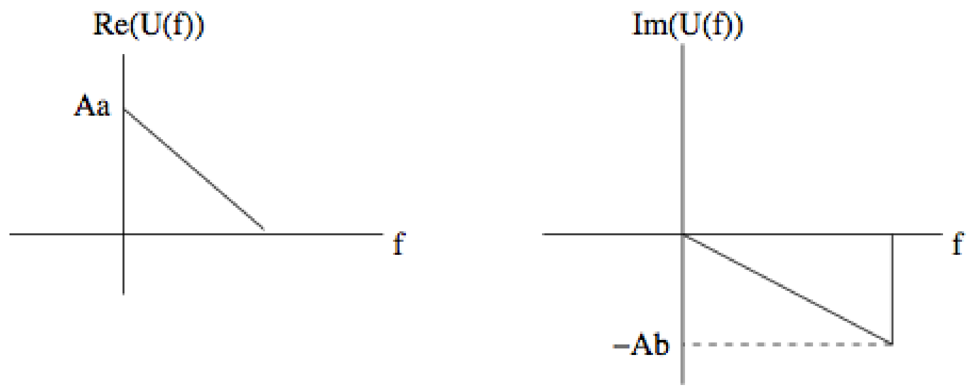


I component
original message

Q component
Hilbert transform of original message

- let $M(f) \triangleq c + jd$, then
  for $f > 0$, $\check{M}(f) = -jM(f) = -jc + d \quad \rightarrow \quad \mathcal{R}\{\check{M}(f)\} = d$ and $\mathcal{I}\{\check{M}(f)\} = -c$
  for $f < 0$, $\check{M}(f) = jM(f) = jc - d \quad \rightarrow \quad \mathcal{R}\{\check{M}(f)\} = -d$ and $\mathcal{I}\{\check{M}(f)\} = c$

Complex envelope for SSB signal



- USB baseband complex envelope in terms of message is given as

$$
\begin{aligned}
U(f) &= U_c(f) + jU_s(f) \\
&= M(f) + j\check{M}(f)
\end{aligned}
$$

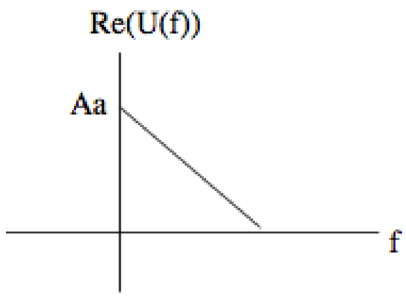
  Taking inverse FT, we get

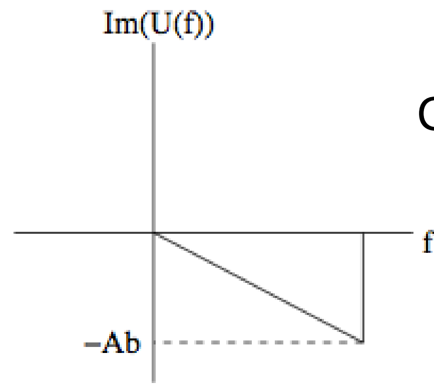
$$u(t) = m(t) + j\check{m}(t)$$

  where $\check{m}(t) = m(t) * \frac{1}{\pi t}$.

- Thus for USB, $u_c(t) = m(t)$ and $u_s(t) = \check{m}(t)$.

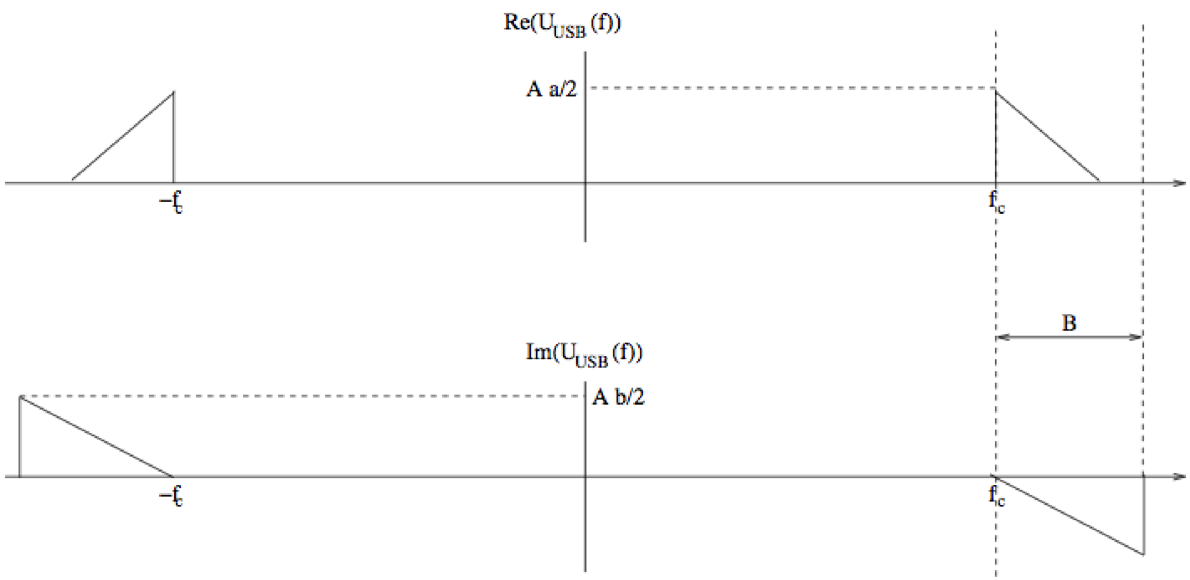
Complex envelope for SSB signal





Complex baseband

$$u_{\text{USB}}(t) = \text{Re}\{u(t)e^{j2\pi f_c t}\}$$
$$= m(t)\cos(2\pi f_c t) - \check{m}(t)\sin(2\pi f_c t)$$

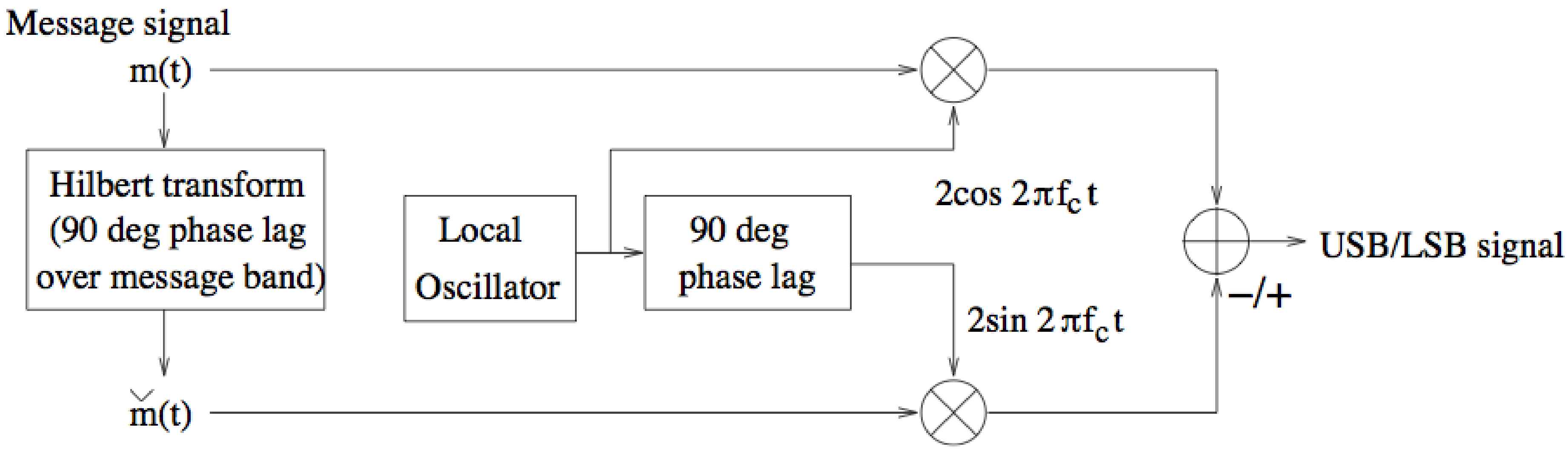


Real USB passband

Real in time
Even and Odd
Symmetric in frequency domain

Implementing SSB in baseband

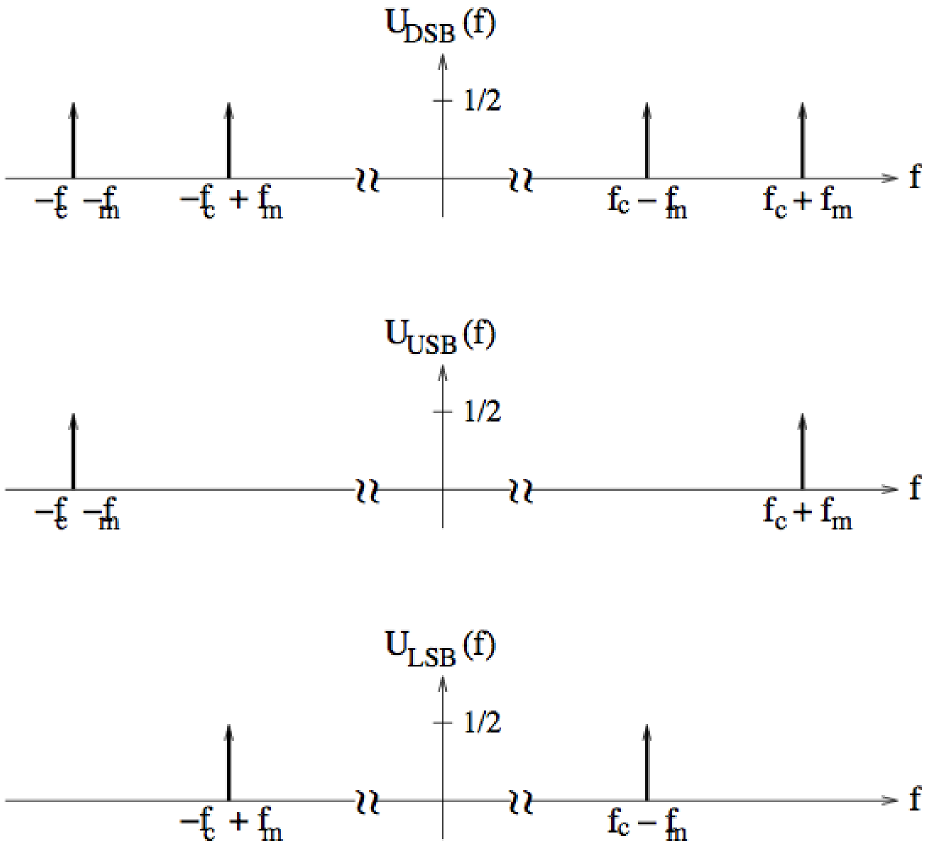
$$
\begin{aligned}
u_{\mathrm{USB}}(t) &= \mathrm{Re}\{u(t)e^{j2\pi f_c t}\} \\
&= m(t)\cos(2\pi f_c t) - \check{m}(t)\sin(2\pi f_c t) \\
u_{\mathrm{LSB}}(t) &= \mathrm{Re}\{l(t)e^{j2\pi f_c t}\} \\
&= m(t)\cos(2\pi f_c t) + \check{m}(t)\sin(2\pi f_c t)
\end{aligned}
$$



Implementing Hilbert transform in baseband avoids need for sharp filtering at passband

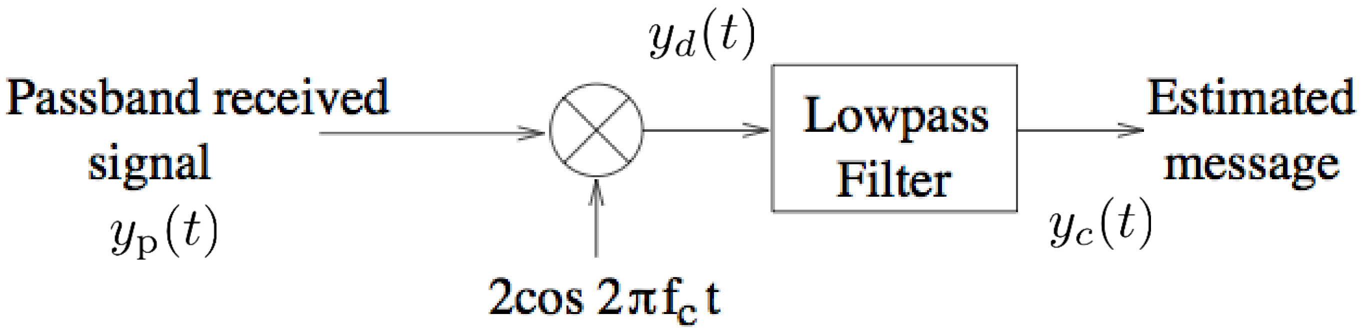
SSB for sinusoidal message

- Consider message $m(t) = \cos(2\pi f_m t)$.
  - Find $\check{m}(t)$.
  - Find $u_{\text{DSB}}(t)$, $u_{\text{USB}}(t)$, $u_{\text{LSB}}(t)$ assuming $\overline{u_{\text{DSB}}^2} = 1$.
  - Plot spectrum for DSB, USB, and LSB.



SSB demodulation: Coherent

- Synchronous demodulation to extract I component



- Prove that for SSB signal,

$$y_c(t) = m(t)\overset{\text{Attenuation}}{\boxed{\cos\theta_r}} - \overset{\text{Interference}}{\boxed{\check{m}\sin\theta_r}}$$

  where θ_r is the phase difference between the received signal and local oscillator. Assignment!

- Vulnerable to carrier phase offset!!

SSB demodulation: Noncoherent

- Add strong carrier component and employ envelope detection
- The expression for the received signal with strong carrier component

$$y_p(t) = (A + m(t)) \cos 2\pi f_c t + \theta_r \pm \check{m}(t) \sin(2\pi f_c t + \theta_r)$$

- Message info preserved in envelope

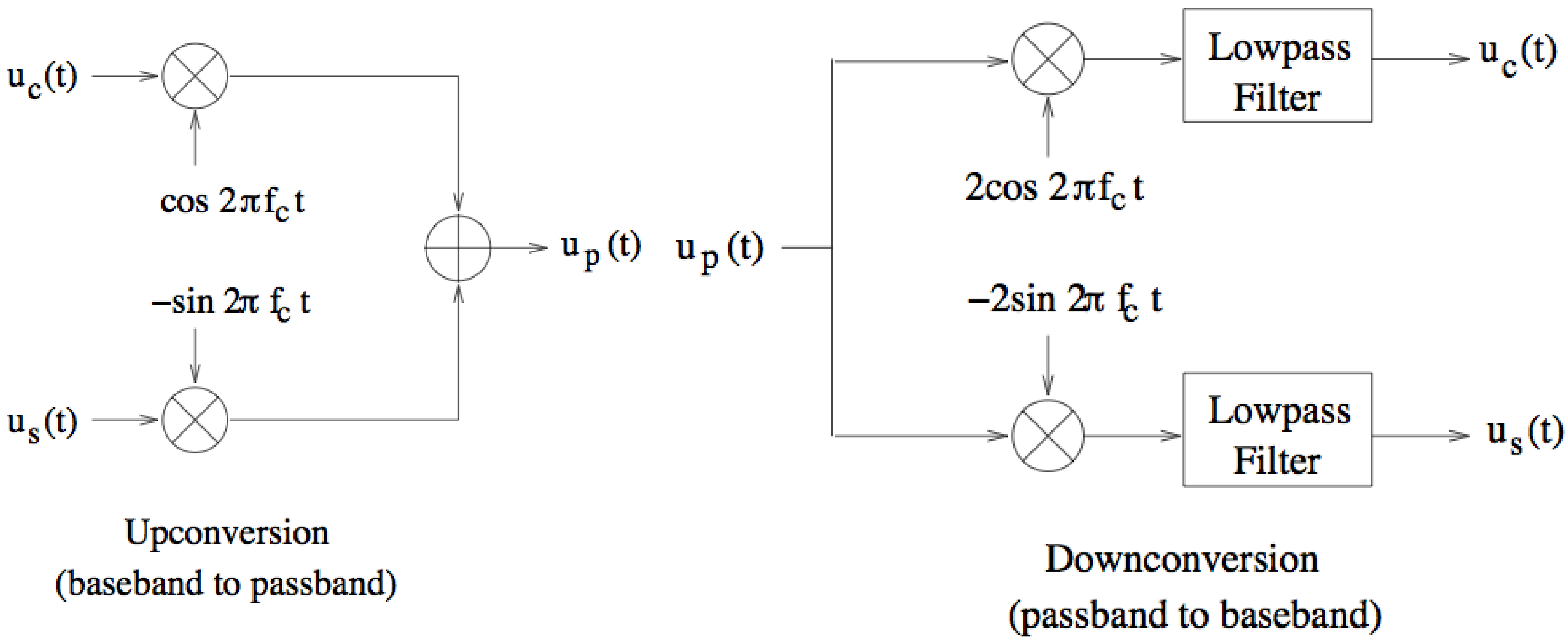
$$e(t) = \sqrt{(A + m(t))^2 + \check{m}^2(t)} \approx A + m(t)$$

as long as $|(A + m(t))| \gg |\check{m}(t)|$.

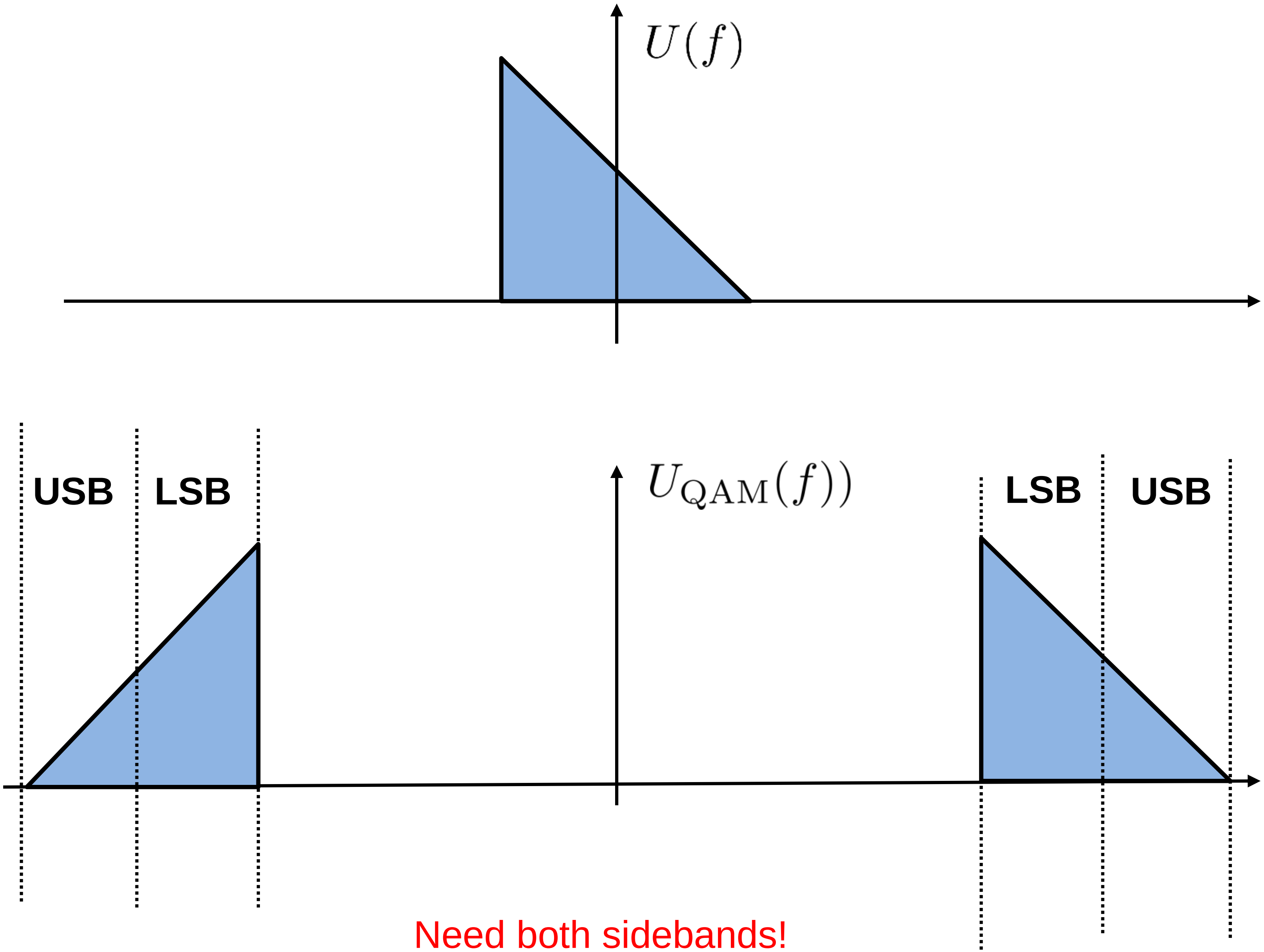
Quadrature Amplitude Modulation

QAM

$$
\begin{aligned}
u(t) &= u_c(t) + j u_s(t) \\
u_{\mathrm{QAM}}(t) &= \mathrm{Re}\{u(t) e^{j2\pi f_c t}\} \\
&= u_{\mathrm{c}}(t)\cos(2\pi f_c t) - u_{\mathrm{s}}(t)\sin(2\pi f_c t)
\end{aligned}
$$

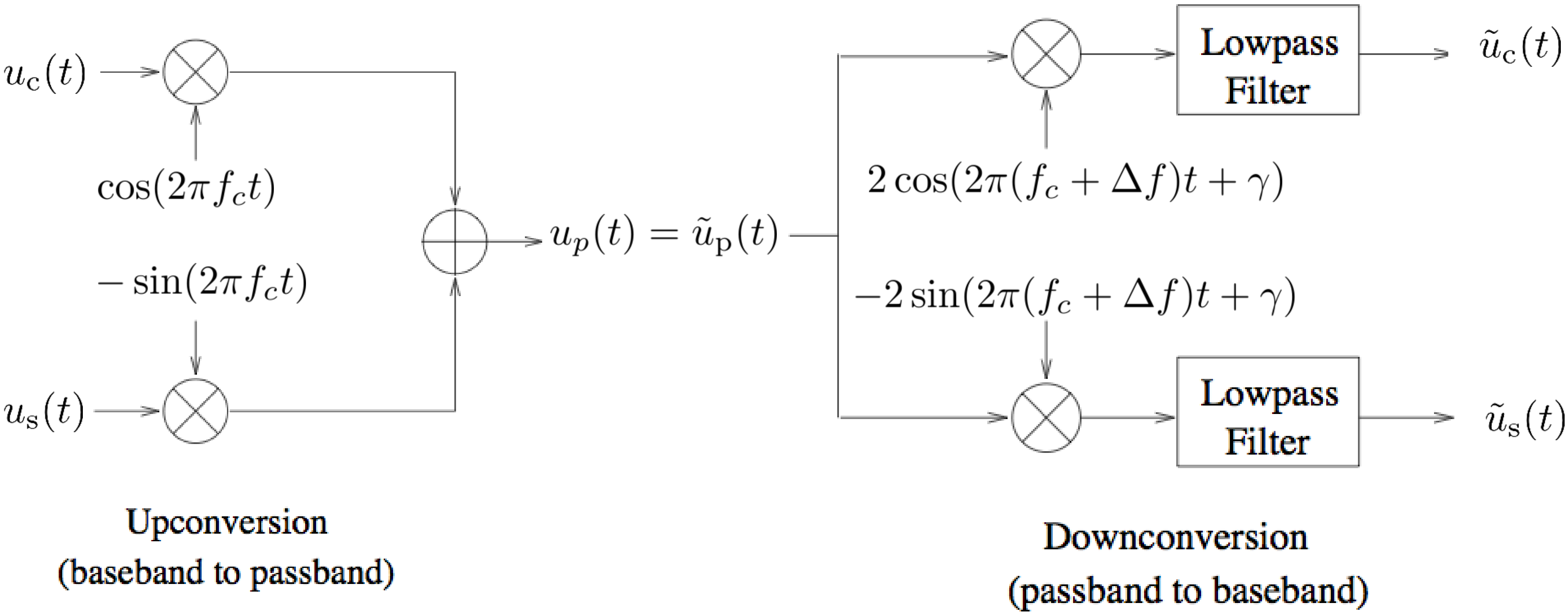


QAM



Need both sidebands!

Effect of Frequency and Phase Offset



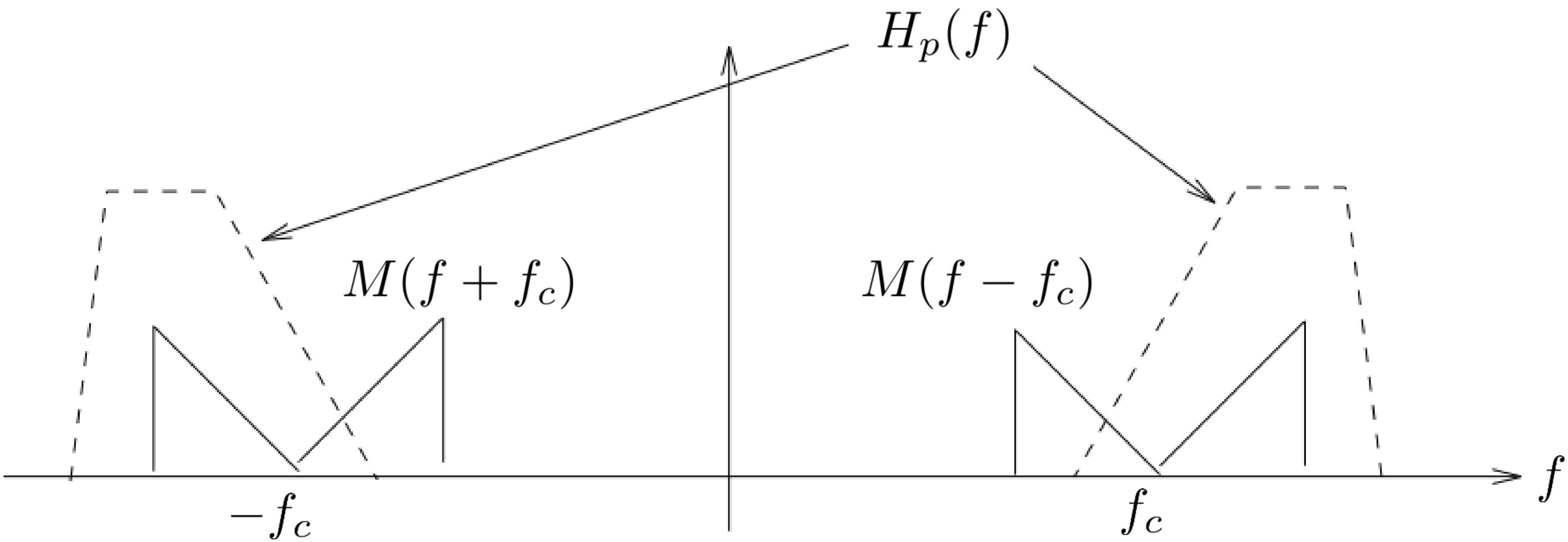
- We have already seen this in Ch. 2: In this case

$$\tilde{u}_c(t) = u_c(t)\cos\phi(t) + u_s(t)\sin\phi(t)$$
$$\tilde{u}_s(t) = -u_c(t)\sin\phi(t) + u_s(t)\cos\phi(t)$$

where $\phi(t) = 2\pi\Delta f t + \gamma$ is the phase offset resulting from frequency offset Δf and the phase offset γ.

Amplitude Modulation:
Vestigial Side Band (VSB)

VSB signaling



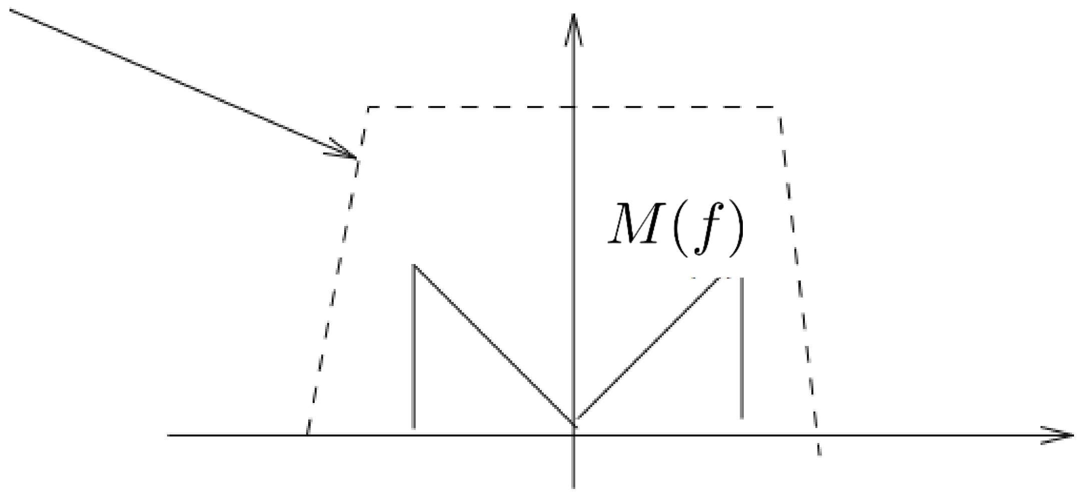
- Dictionary meaning of vestige: *a trace or remnant of something that is disappearing or no longer exists*

- VSB is general form of SSB: Filter DSB signal so as to leave vestige of one sideband

- Trade-off of ease of filtering requirements and bandwidth

How to choose VSB filter?

$H_p(f - f_c) + H_p(f + f_c)$ constant over message band (Prove!)



I component = message
Q component = filtered version of message that cancels portion of spectrum

Frequency Modulation

Recap: *Different Modulations*

- Two ways of encoding info in complex envelope
  - I and Q: amplitude modulation (several variants)
  - Envelope and phase: angle modulation (constant envelope)
- For example, for a sinusoid carrier

$$A_c(t)\cos(2\pi f_c(t)t + \theta_c(t))$$

where $A_c(t)$, $f_c(t)$, $\theta_c(t)$ are the amplitude, frequency, and the phase of the carrier respectively.

Amplitude Modulation

Frequency Modulation

Phase Modulation

Frequency Modulation

- The transmitted signal is given as

$$u_{\mathrm{FM}}(t) = A_c \cos(2\pi\underline{(f_c + f(t))}t + \phi)$$

- Here $f(t) = k_f m(t)$ is the frequency offset relative to the carrier, $m(t)$ is the message signal while k_f (design constant), A_c (amplitude) and ϕ (phase) are constants.

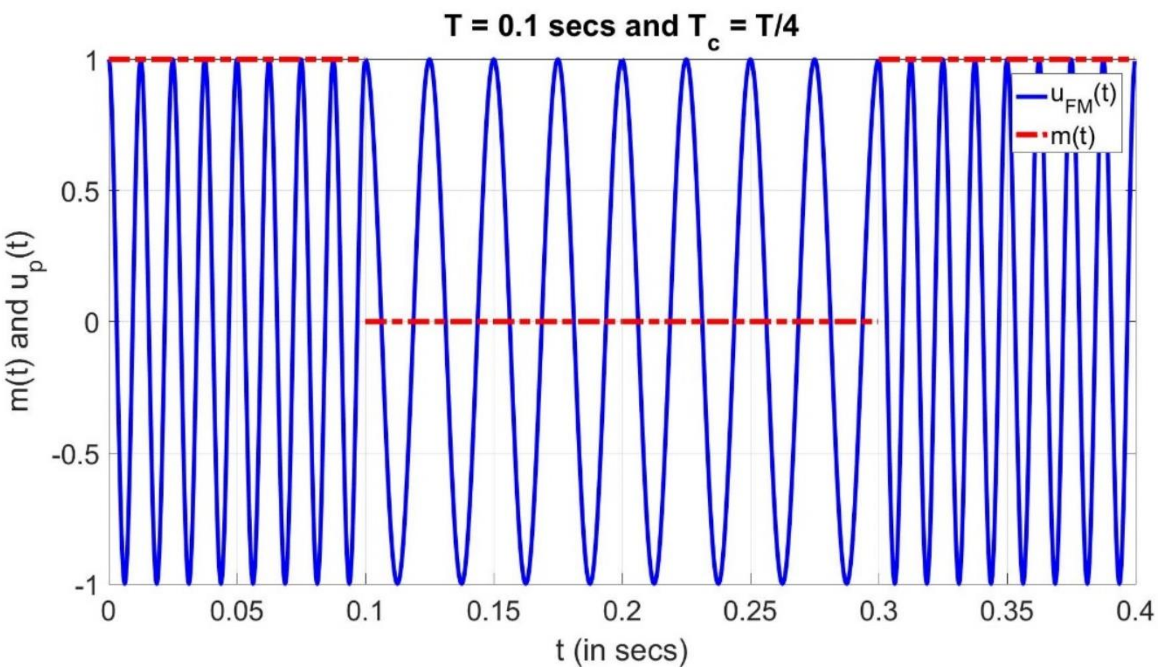
- The instantaneous frequency is given by

$$f_i(t) = f_c + f(t) = f_c + k_f m(t)$$

Example of FM Wave

- The instantaneous frequency is given by

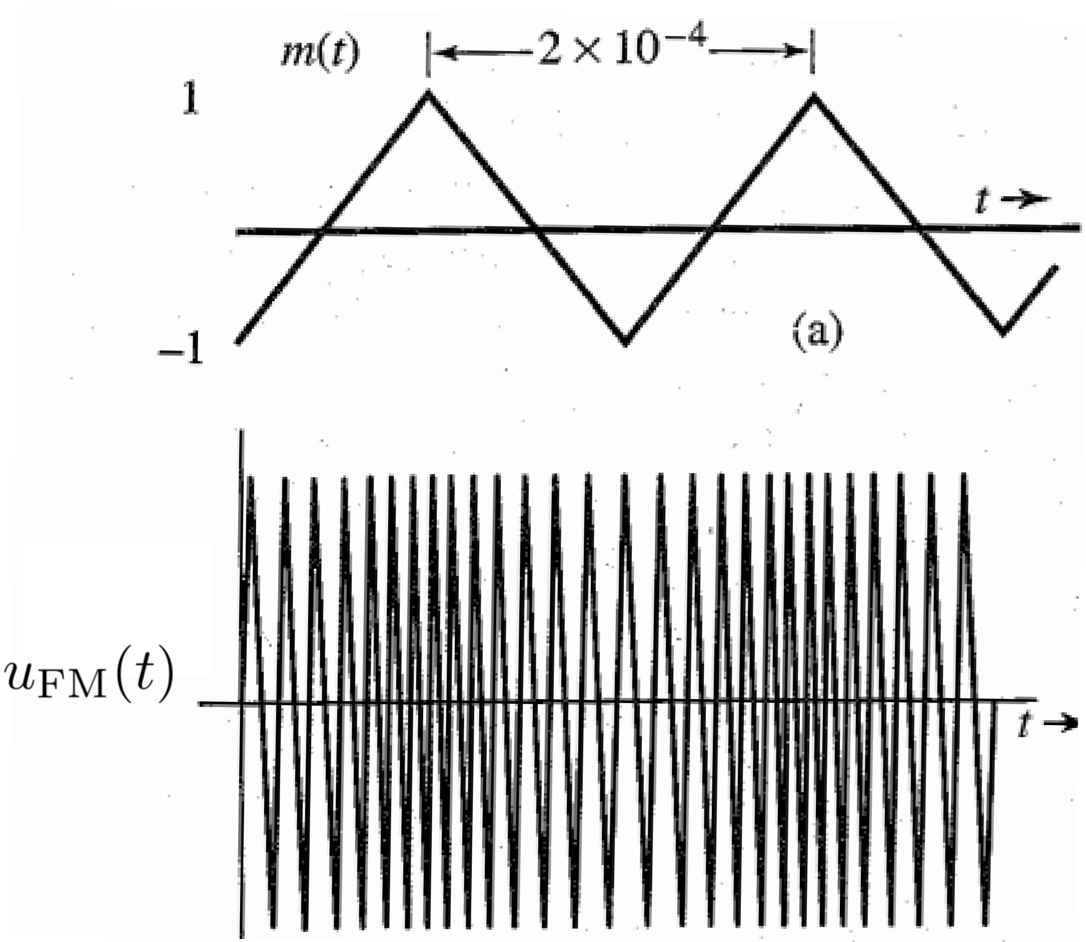
$$f_i(t) = f_c(1 + m(t))$$



Example 2 of FM Wave

- The instantaneous frequency is given by

$$f_i(t) = f_c + f(t) = f_c + k_f m(t)$$



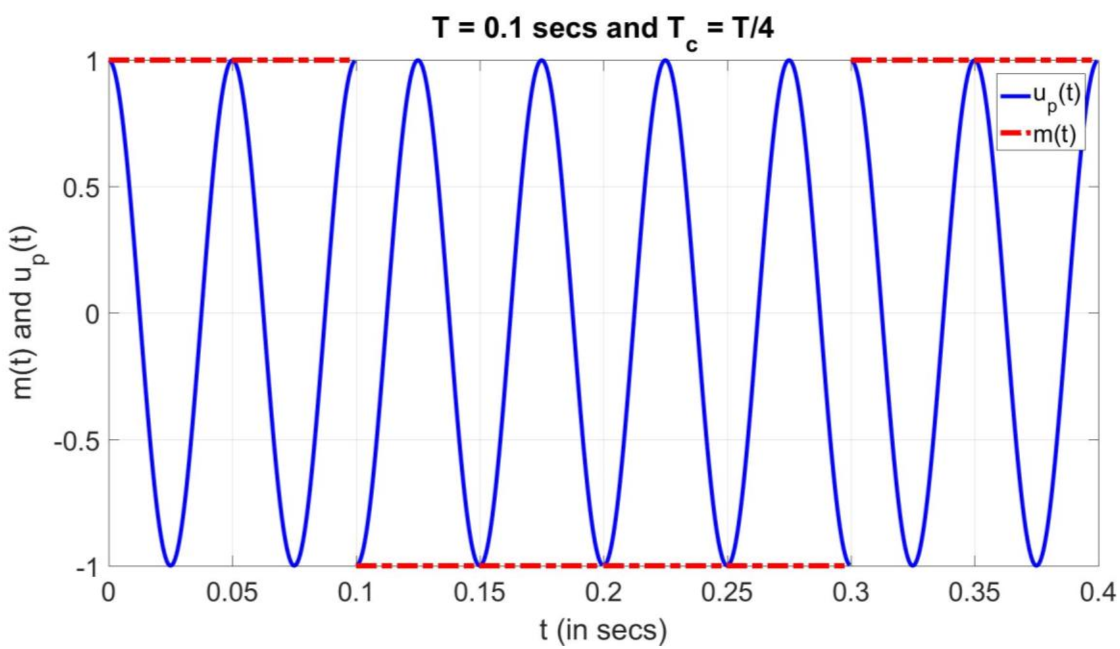
Phase Modulation

- The transmitted signal is given as

$$u_{\mathrm{PM}}(t) = A_c \cos(2\pi f_c t + \boxed{\theta(t)} + \phi)$$

- Here $\theta(t) = k_p m(t)$ while k_p, A_c, ϕ and f_c are constants.

Example of PM Wave



$$
\begin{aligned}
u(t) &= A_c \cos(2\pi f_c t + 0.5(m(t) - 1)\pi) \\
&= A_c \cos(2\pi f_c t) \qquad \color{red}{m(t) = 1} \\
&= A_c \cos(2\pi f_c t - \pi) \quad \color{red}{m(t) = -1}
\end{aligned}
$$

Generalized Model: Angle Modulation

- The transmitted signal is given as

$$u_{\mathrm{p}}(t) = A_c \cos(2\pi f_c t + \boxed{\theta(t)})$$

$$\theta(t) = g(m(t))$$

- Angle modulation is a general form
  - Phase modulation

$$\theta(t) = \theta(0) + k_p m(t)$$

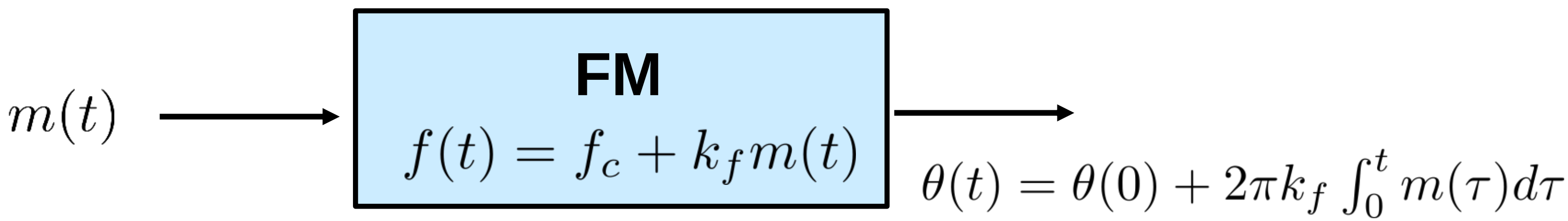
  - Frequency modulation

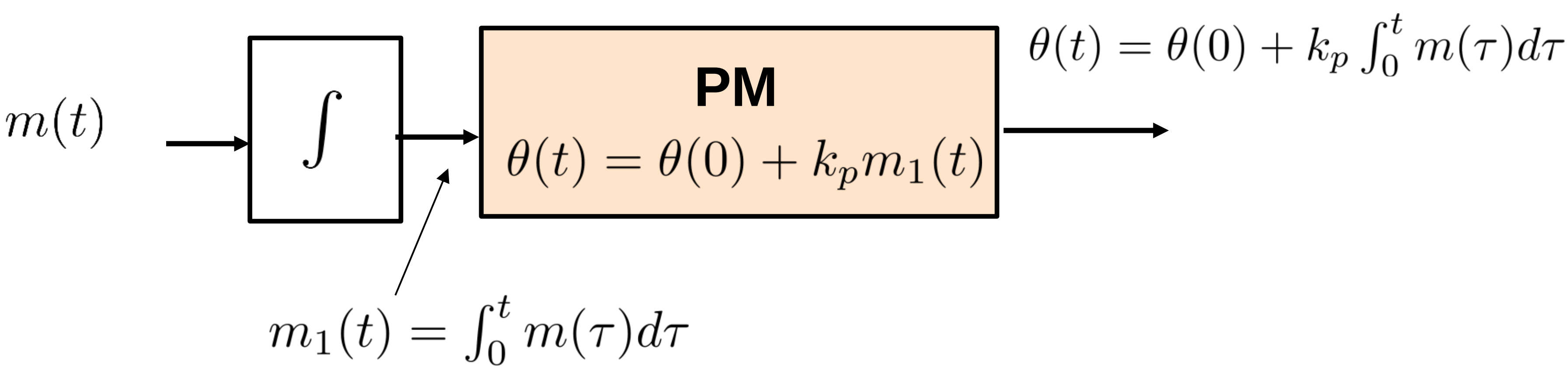
$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = f(t) = k_f m(t)$$

$$\theta(t) = \theta(0) + 2\pi k_f \int_0^t m(\tau)\, d\tau$$

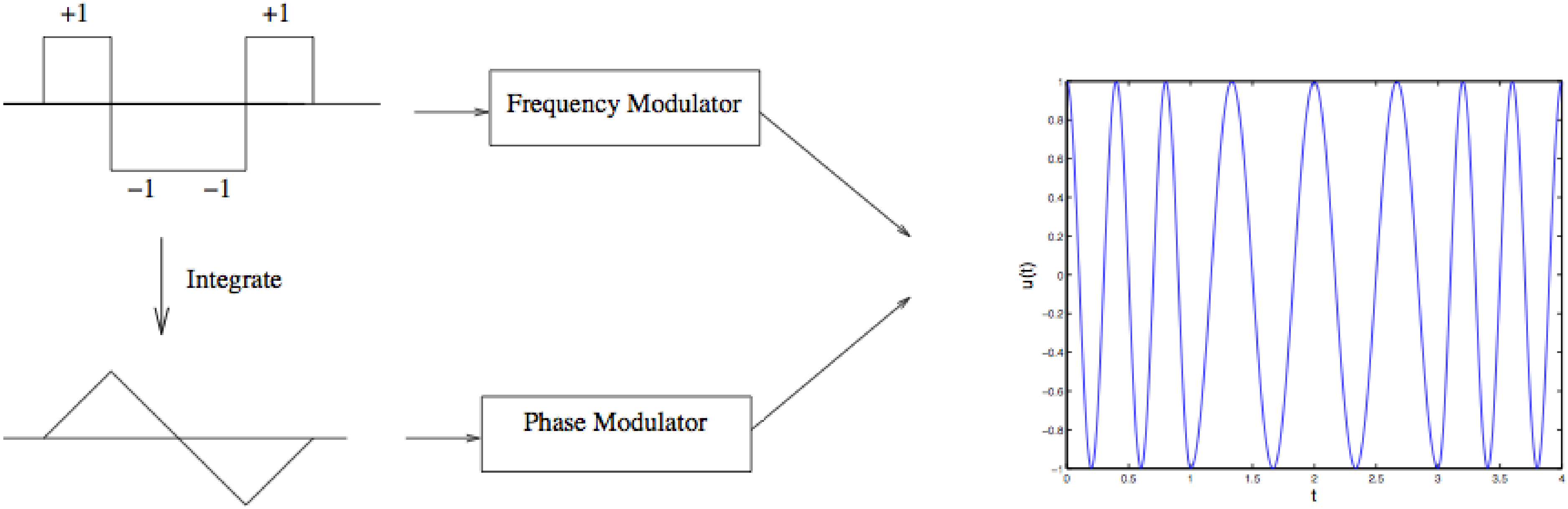
where k_p and k_f are constants while $f(t)$ is the frequency offset relative to the carrier. Also $\phi = \theta(0)$ where $t = 0$ is used as reference point.

Equivalence of PM and FM: *FM using PM*

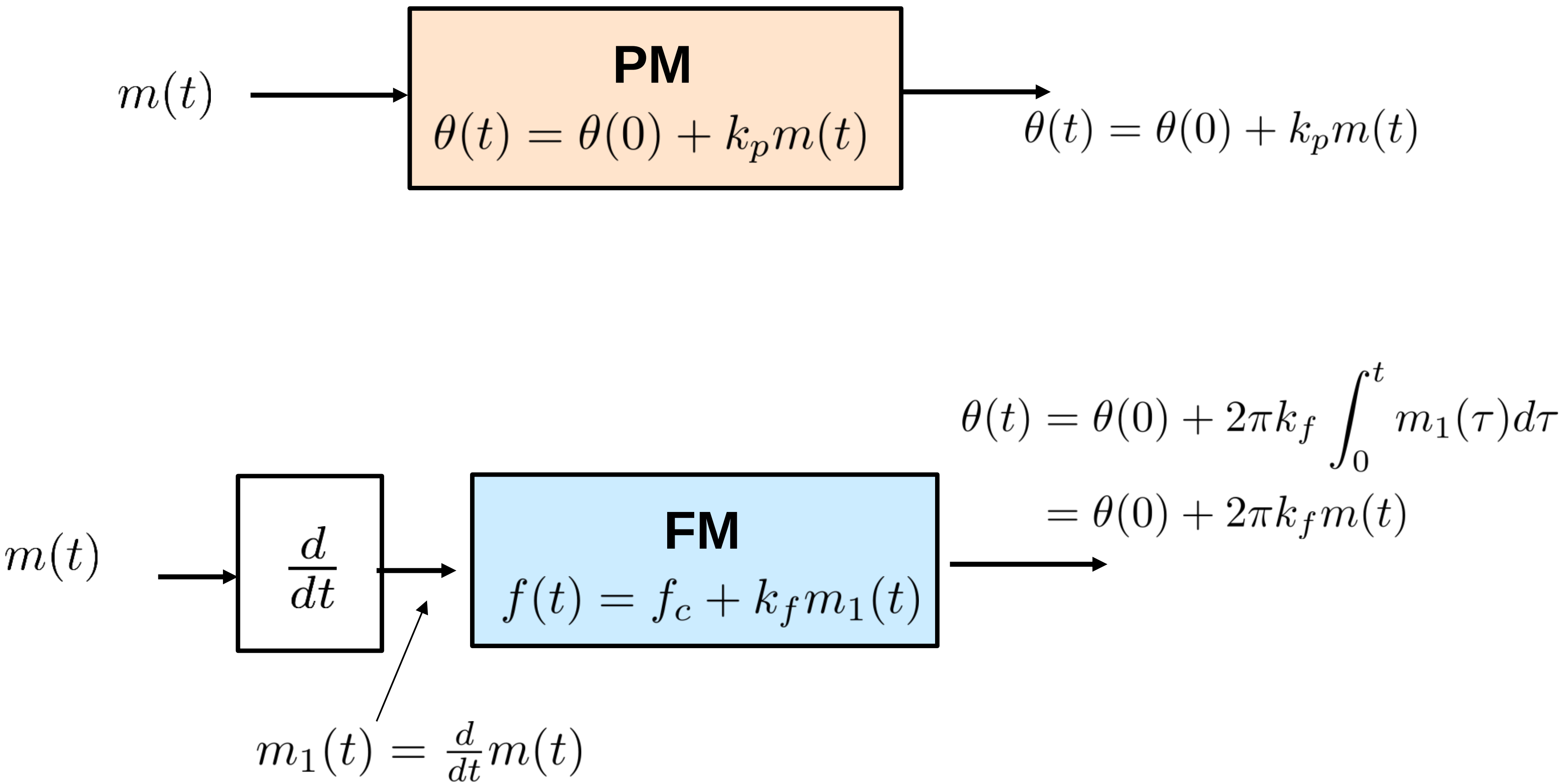




Equivalence of FM and PM: *FM using PM*



Equivalence of PM and FM: *PM using FM*



Poll

Which of the following are non-linear modulations?

- AM
- PM
- FM
- None of the above

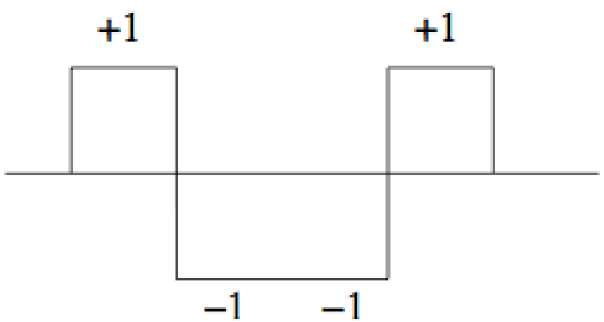
Non-linearity of Angle or Phase Modulation

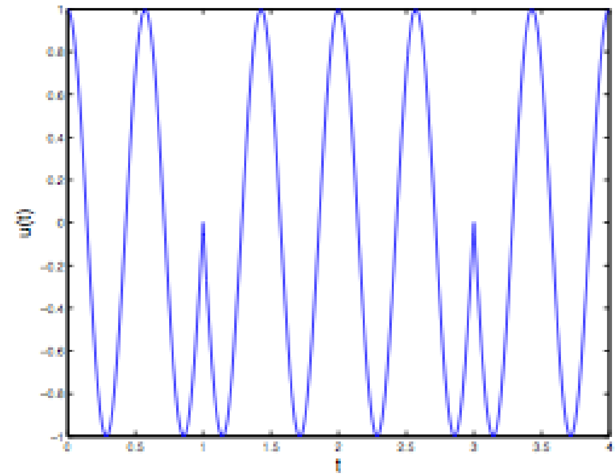
- Prove that angle modulation is a non-linear operation while amplitude modulation is a linear operation.

PM versus FM

PM
(PSK for digital
messages)

Digital message

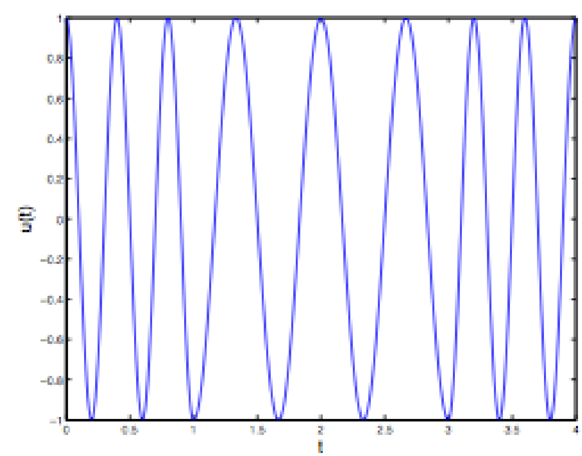




Discontinuous phase ➔ vulnerable to nonlinearities,
poorer frequency containment

Continuous phase ➔ more robust to nonlinearities,
better frequency containment

FM



PM versus FM in practice

- Legacy analog communication $\rightarrow$ no control over message signal $\rightarrow$ FM preferred
  - Integration of message prior to phase modulation leads to smooth phase which leads to better bandwidth containment.
  - Most famous application: radio broadcasting
  - FM has been used in 2G GSM (Gaussian MSK, a form of FM); Optimal demodulation more complicated
  - Lately being used in power limited systems: FSK is used in LoRaWAN
- Digital communication $\rightarrow$ can design message signal $\rightarrow$ PM (PSK specifically) often preferred
  - Easier to implement optimal demodulator
  - Use bandwidth-efficient pulses rather than rectangular pulses to create smoother signals with better frequency containment
  - Used in modern digital communication systems

Focus on FM in this chapter. PSK studied in Chapter 4 and beyond.

FM Modulation: Effect on Phase

- The transmitted signal is given as

$$u_{\mathrm{p}}(t) = A_c \cos(2\pi f_c t + \theta(t))$$

- Instantaneous phase $\theta(t)$

$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = f(t) = k_f m(t)$$

$$\theta(t) = \theta(0) + 2\pi k_f \int_0^t m(\tau)\, d\tau$$

where k_f are constants while $f(t)$ is the frequency offset relative to the carrier.

FM Modulation Index

- Modulation index for FM is given by

$$\beta = \frac{\Delta f_{\max}}{B}$$

  where the frequency deviation $\Delta f_{\max} = k_f \max_t |m(t)|$ and B is the bandwidth of the signal..

- Narrowband FM: $\beta < 1$
- Wideband FM: $\beta > 1$

- Solve: For sinusoidal message $m(t) = A_m \cos(2\pi f_m t)$, find β. Also find $\theta(t)$ in terms of β assuming $\theta(0) = 0$.

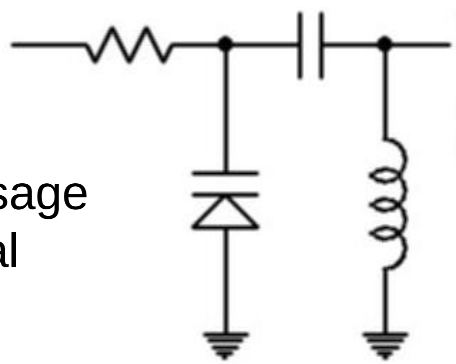
Modulation

- Direct method
  - Voltage controlled oscillator (VCO)
  - Use of varacter diode which provides voltage controlled capacitance in LC tuned circuits
  - Directly generates passband
  - Both narrow and wideband
- Indirect method
  - An alternative method for wideband FM signal generation when direct method is infeasible or costly
  - First generate narrowband signal (using PM modulation) and then increase the frequency shift and frequency by using several stages of multipliers (non-linearity)
  - Not used nowadays as direct FM methods are now feasible and cost-effective.

Example of VCO

- Use of varacter diode which provides voltage controlled capacitance in LC tuned circuits
- Directly generates passband
- Both narrow and wideband

Message signal



Passband Sinusoid with varying freq.

Source: Globe Collector - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=18036362

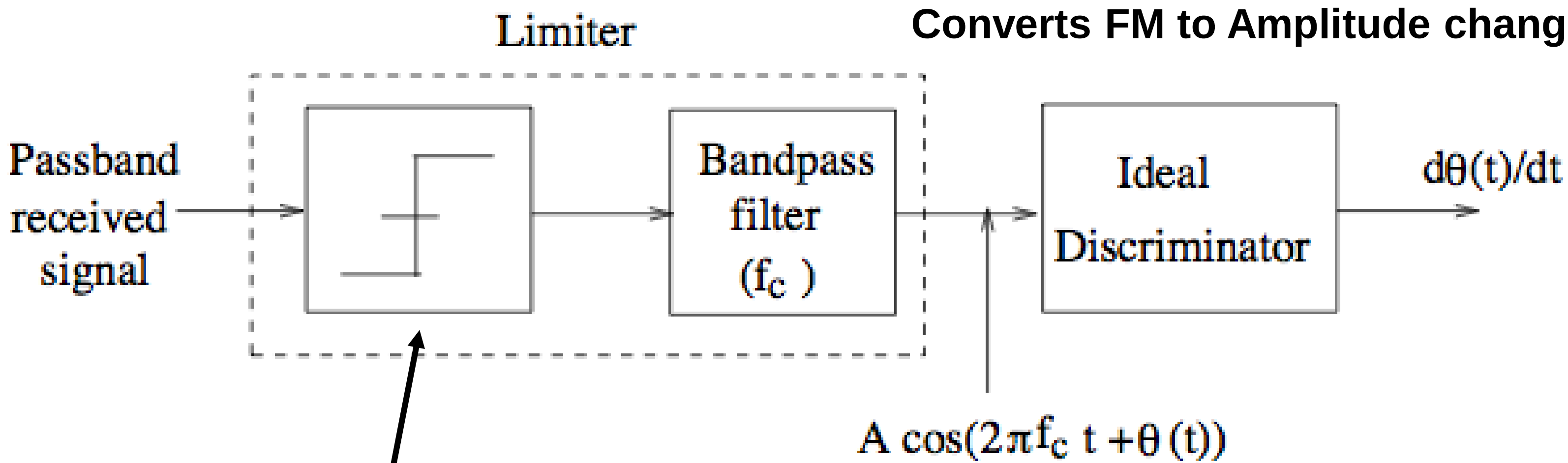
FM Demodulation

- There are several methods
  - Limiter discriminator
  - Phase locked loop (PLL) (in detail later in this chapter)

Limiter Discriminator

Enforces constant envelope

Converts FM to Amplitude changes

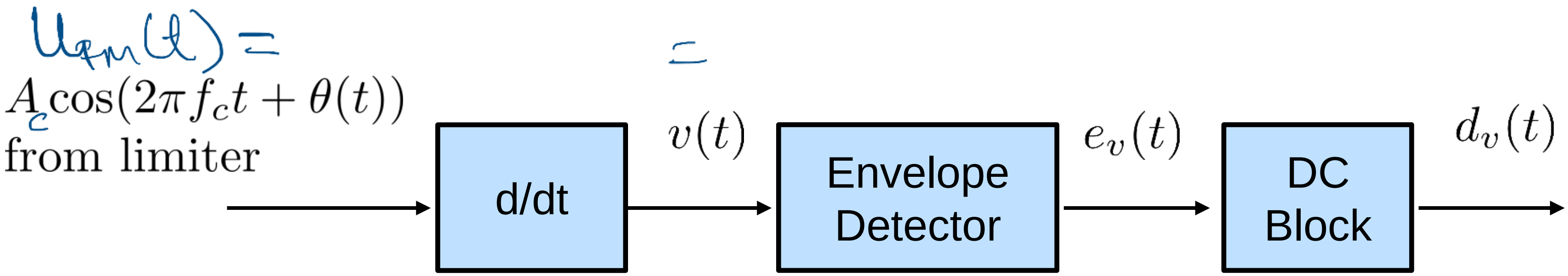


Removes amplitude fluctuations caused by noise and channel

Limiting enduces harmonics since it is a non-linear operation

$y(t) = a x(t) + b x^2(t) + c x^3(t) + \ldots$

A crude discriminator



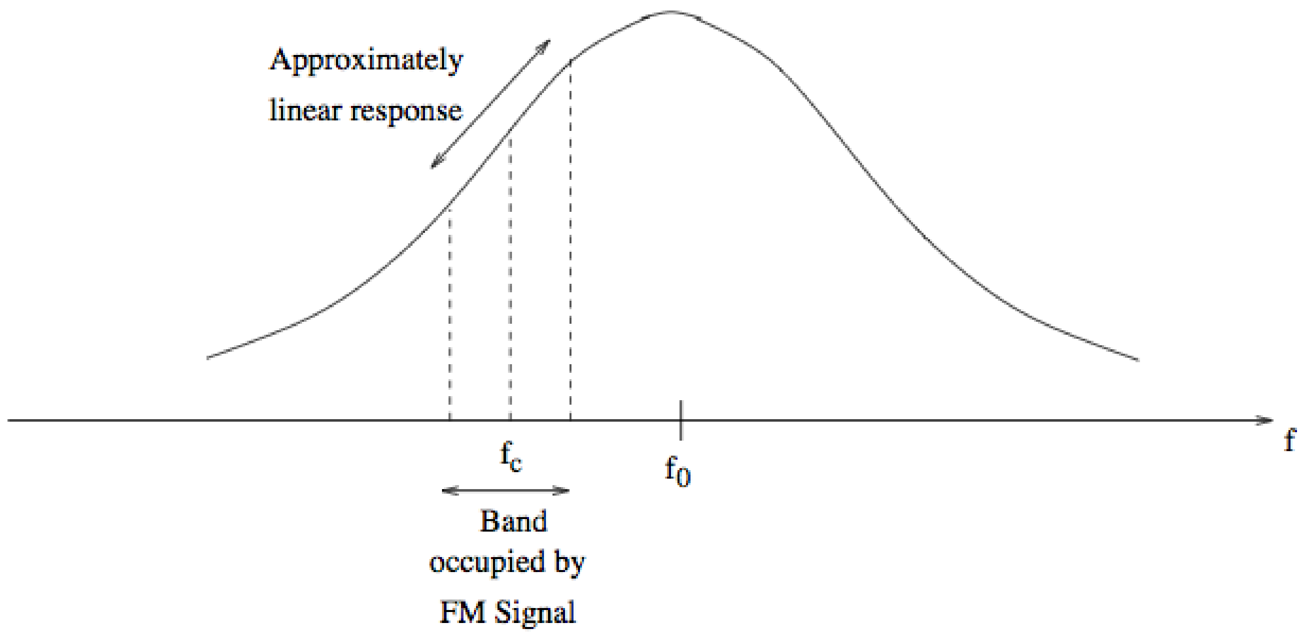
- Here $f(t) = \frac{d\theta(t)}{dt} = 2\pi k_f m(t)$.
- Show that the output of the crude discriminator shown above is a scaled version of the message signal $m(t)$.

Approximate differentiation

- For Differentiation, Fourier transform pair is $\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j2\pi f X(f)$

$$X(f) \longrightarrow \boxed{H(f) = j2\pi f} \longrightarrow j2\pi f X(f)$$

- Can use linear slope region of filter response



Today's Class

Ref Books: U. Madhow and B. P. Lathi

FM Spectrum

FM spectrum

- Narrowband FM
  - Similar to DSB
  - Bandwidth = 2B (where B=message bandwidth)
- Wideband FM
  - Bandwidth dominated by max frequency deviation
- Carson's formula: adds the two components

Recap: *Time domain expressions for a passband signal*

- In terms of I and Q components

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$

- In terms of envelope and phase

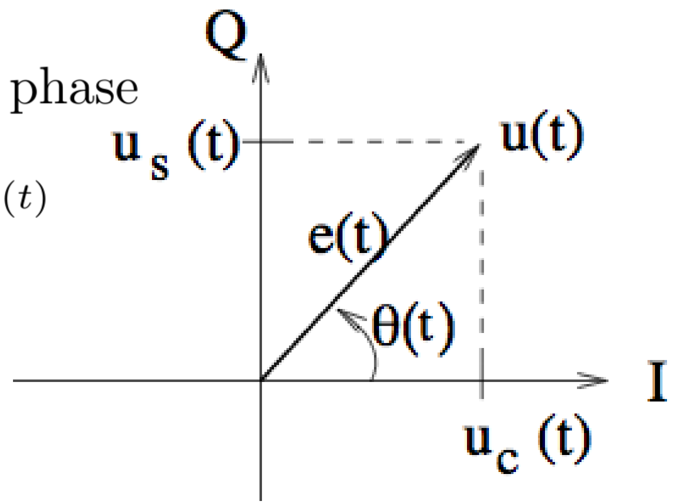
$$u_p(t) = e(t)\cos(2\pi f_c t + \theta(t))$$

- In terms of complex envelope

$$u_p(t) = \mathrm{Re}(u(t)e^{j2\pi f_c t})$$

- Complex baseband in terms of Envelope and phase

$$u(t) = u_c(t) + ju_s(t) = e(t)e^{j\theta(t)}$$



Starting from one representation, can derive the rest based on the relations depicted in the figure

Narrowband FM

- Show that the bandwidth of narrowband of FM is 2B where B is the bandwidth of the signal $m(t)$.

Narrowband FM: Example

- Example: find the bandwidth of FM signal corresponding to a sinusoidal message $m(t) = \cos 2\pi f_m t$.

Wideband FM

- Bandwidth is dominated by frequency deviation

$$\Delta f = k_f m(t)$$

- Frequency will swing between $\pm\Delta f_{\max}$ assuming equal positive and negative swings in message.

$$\Delta f_{\max} = k_f \max_t |m(t)|$$

- Bandwidth $B_{\mathrm{FM}} = 2\Delta f_{\max}$

Carson's rule

- Add up estimates for narrowband and wideband FM

$$B_{\mathrm{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

Bandwidth of Angle Modulated Waveforms

- Prove that Angle Modulated Waveforms have infinite bandwidth theoretically!
- As a special case, derive narrowband FM expression and its bandwidth!

Ref: B.P.Lathi

FM spectrum for periodic messages

- Complex envelope is periodic for periodic messages $\rightarrow$ Fourier series
  - Spectrum of complex envelope is discrete with impulses at integer multiples of fundamental frequency
- Standard example: sinusoidal message
  - But approach is quite general
- Somewhat artifical since most messages (such as speech) are not periodic

(Approximate) FM spectrum

- The bandwidth of narrowband FM ($\beta < 1$) is 2B where B is the bandwidth of the signal $m(t)$.

Assumption $\theta(t)$ is small for narrowband FM! Not valid for general case.

- Add up estimates for narrowband and wideband FM

$$B_{\mathrm{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

Exact FM spectrum for sinusoidal message

- Prove that FM spectrum for sinusoidal message is given by

$$U(f) = \sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f - nf_m)$$

where $J_n(\beta)$ is the n^{th} order Bessel function given by

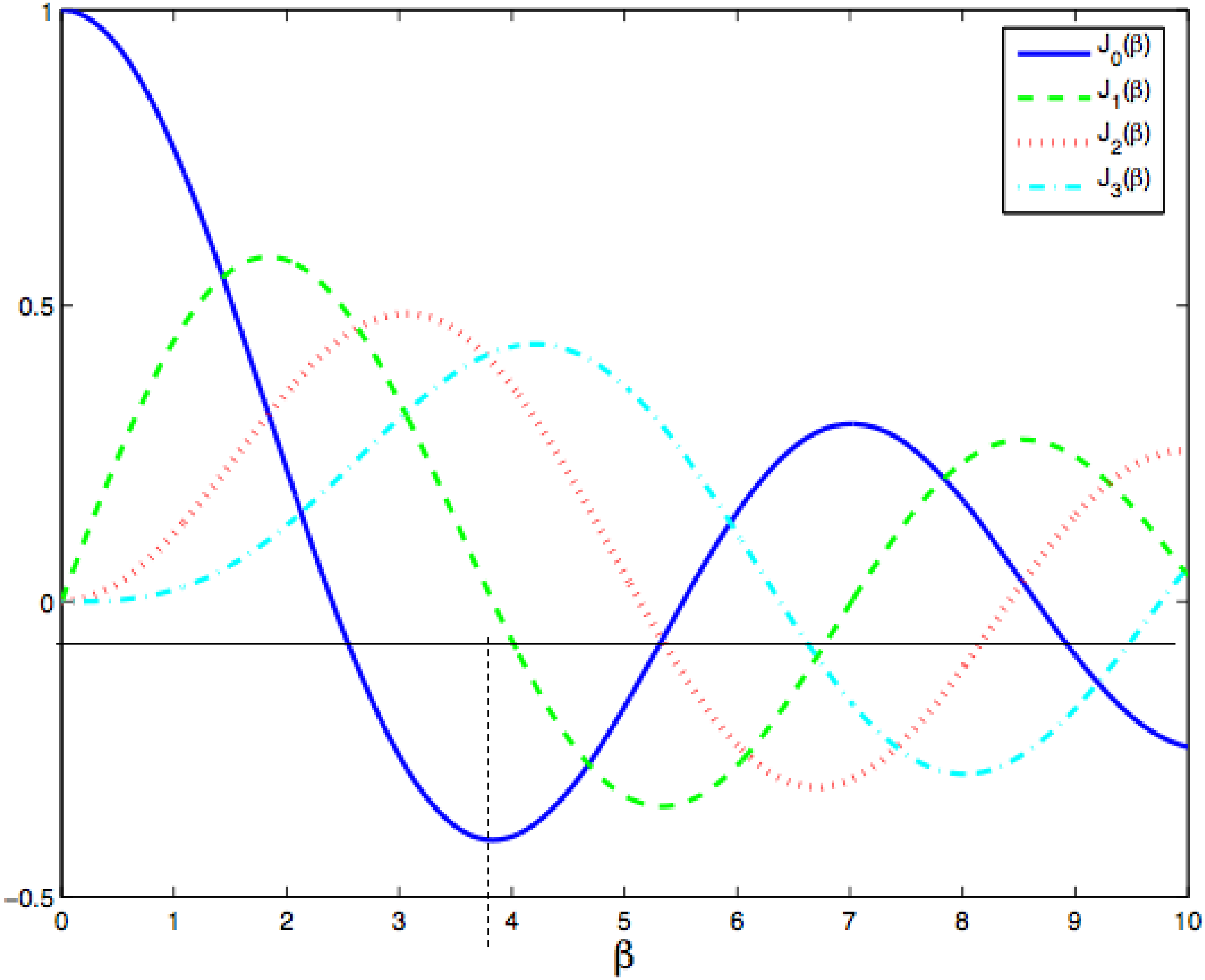
$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)\, dx}$$

Complex-valued integral

$$= \frac{1}{\pi} \cos(\beta \sin x - nx)\, dx$$

Real Valued

Bessel function plots



$$J_n(-\beta) = (-1)^n J_n(\beta)$$

Bessel function properties

- Note that the Bessel function $J_n(\beta)$ is real

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)}\,dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(\beta \sin x - nx)\,dx$$

Properties of Bessel function

- $J_n(\beta) = (-1)^n J_{-n}(\beta) = (-1)^n J_n(-\beta)$
- For fixed β, $J_n(\beta) \to 0$ as $n \to \infty$.
  Generally, $J_n(\beta) \approx 0$ for $|n| > \beta + 1$
  $B_{\mathrm{FM}} \approx 2(\beta + 1) f_m$
  This is consistent with Carson's rule.

This is only an approximation

- For fixed n, $J_n(\beta)$ vanishes for specific values of β. This is useful in spectral shaping.

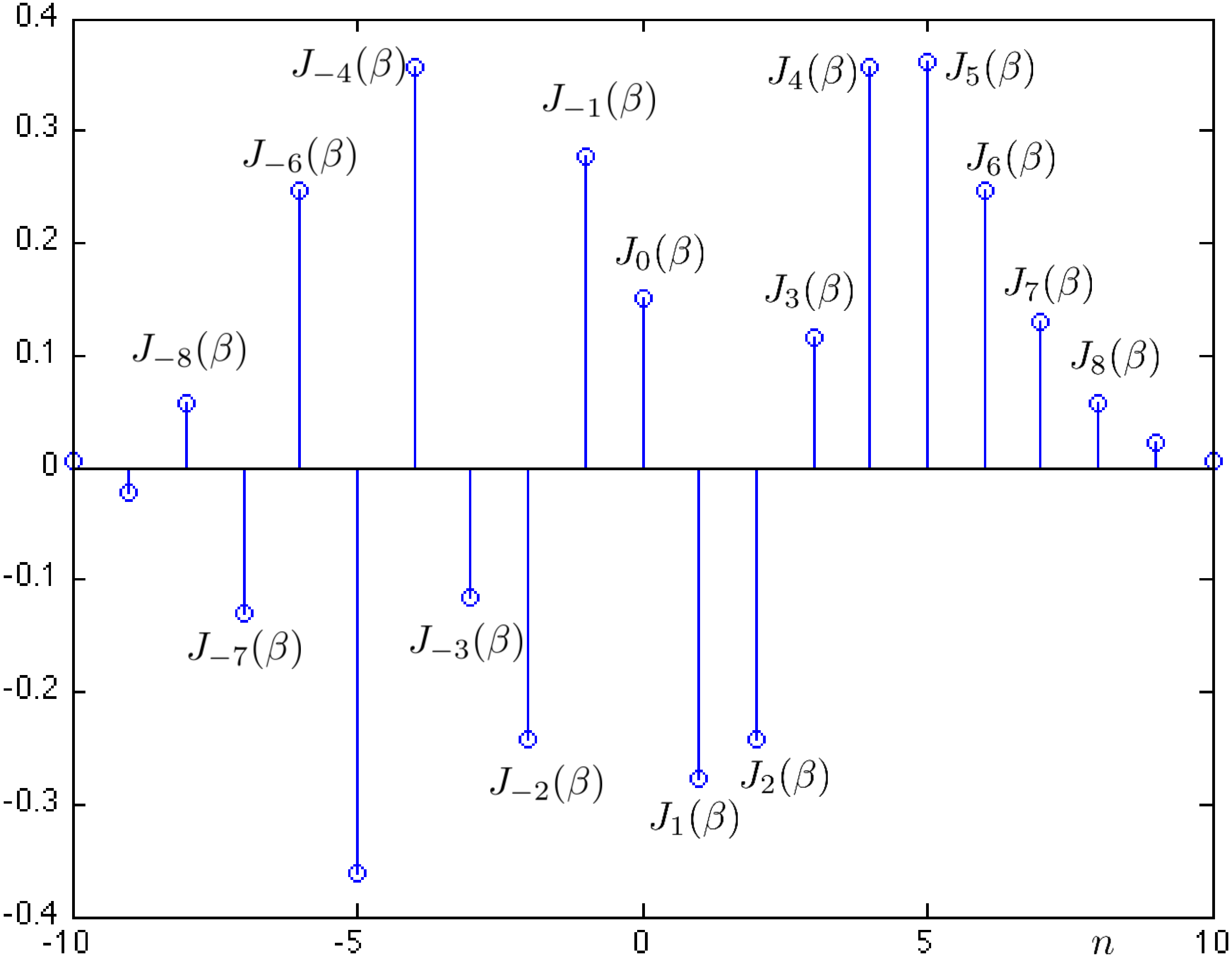
Modulation index and Power in Sidebands

Modulation index	Sideband																
	Carrier	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0.00	1.00																
0.25	0.98	0.12															
0.5	0.94	0.24	0.03														
1.0	0.77	0.44	0.11	0.02													
1.5	0.51	0.56	0.23	0.06	0.01												
2.0	0.22	0.58	0.35	0.13	0.03												
2.41	0	0.52	0.43	0.20	0.06	0.02											
2.5	−0.05	0.50	0.45	0.22	0.07	0.02	0.01										
3.0	−0.26	0.34	0.49	0.31	0.13	0.04	0.01										
4.0	−0.40	−0.07	0.36	0.43	0.28	0.13	0.05	0.02									
5.0	−0.18	−0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02								
5.53	0	−0.34	−0.13	0.25	0.40	0.32	0.19	0.09	0.03	0.01							
6.0	0.15	−0.28	−0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02							
7.0	0.30	0.00	−0.30	−0.17	0.16	0.35	0.34	0.23	0.13	0.06	0.02						
8.0	0.17	0.23	−0.11	−0.29	−0.10	0.19	0.34	0.32	0.22	0.13	0.06	0.03					
8.65	0	0.27	0.06	−0.24	−0.23	0.03	0.26	0.34	0.28	0.18	0.10	0.05	0.02				
9.0	−0.09	0.25	0.14	−0.18	−0.27	−0.06	0.20	0.33	0.31	0.21	0.12	0.06	0.03	0.01			
10.0	−0.25	0.04	0.25	0.06	−0.22	−0.23	−0.01	0.22	0.32	0.29	0.21	0.12	0.06	0.03	0.01		
12.0	0.05	−0.22	−0.08	0.20	0.18	−0.07	−0.24	−0.17	0.05	0.23	0.30	0.27	0.20	0.12	0.07	0.03	0.01

https://en.wikipedia.org/wiki/Frequency_modulation#Bessel_functions

Fourier coefficients for complex envelope

Plot of $J_n(\beta)$ as a function of n for a given $\beta = 6$



$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

Passband Bandwidth (frequency unit = message frequency f_m)

Fractional power containment BW

- Parseval's theorem: Power = sum of magnitude of Fourier series coefficients

$$1 = |u(t)|^2 = \overline{|u(t)|^2} = \sum_{n=-\infty}^{\infty} J_n^2(\beta) = J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta)$$

- Fractional power containment bandwidth for fraction α is $2Kf_m$ with K given by

$$J_0^2(\beta) + 2\sum_{n=1}^{K} J_n^2(\beta) \geq \alpha$$

Carson's rule

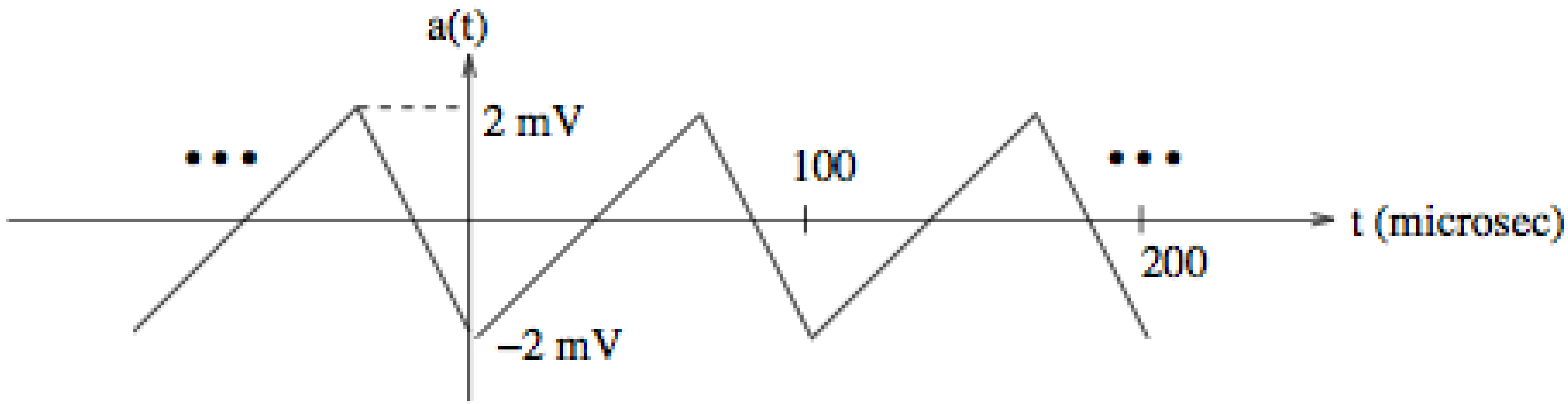
- Add up estimates for narrowband and wideband FM

$$B_{\text{FM}} \approx 2B + 2\Delta f_{\max} = 2B(\beta + 1)$$

  where $\beta = \Delta f_{\max}/B$ is the FM modulation index or the deviation ration.

- Carson's rule uses $\alpha = 0.98$.

Example 3.3.1: Tutorial



- The signal $a(t)$ is fed to a VCO with quiescent frequency 5 MHz and frequency deviation of 25 KHz/mV.
- Give an estimate of the bandwidth of y(t), which is output of VCO. Use only the first harmonic for bandwidth calculation.
- If signal $y(t)$ is passed through an ideal passband filter of bandwidth 5 KHz, centered at 5.005 MHz, then provide the simplest possible expression for the power at the filter output.

Features of Angle Modulated Non-linearities

- Exchanging signal power with bandwidth
  - Bandwidth for AM cannot be changed while it can be changed changed based on Δf.
  - SNR is roughly proportional to square of transmission signal bandwidth.
- Immunity of angle modulation to non-linearities.
  - Non-linearity does not affect FM signal while it does affect AM signal.(Proofs)

Ref: B.P.Lathi

EC5.203 Communication Theory (3-1-0-4):

Lecture 10:
Analog Communication Techniques:
Superheterodyne Receiver

Feb. 22, 2025

INTERNATIONAL INSTITUTE OF
INFORMATION TECHNOLOGY
HYDERABAD

Bandwidth of Angle Modulated Waveforms

- Prove that Angle Modulated Waveforms have infinite bandwidth theoretically!
- As a special case, derive narrowband FM expression and its bandwidth!
- Note that Angle modulation is non-linear but NBFM and NBPM are (incrementally) linear modulation schemes.

Ref: B.P.Lathi

Features of Angle Modulated Non-linearities

- Exchanging signal power with bandwidth
  - Bandwidth for AM cannot be changed while it can be changed based on Δf.
  - SNR is roughly proportional to square of transmission signal bandwidth.
- Immunity of angle modulation to non-linearities.
  - Non-linearity does not affect FM signal while it does affect AM signal.(Already proved)
  - Amplitude changes do not affect the FM signal

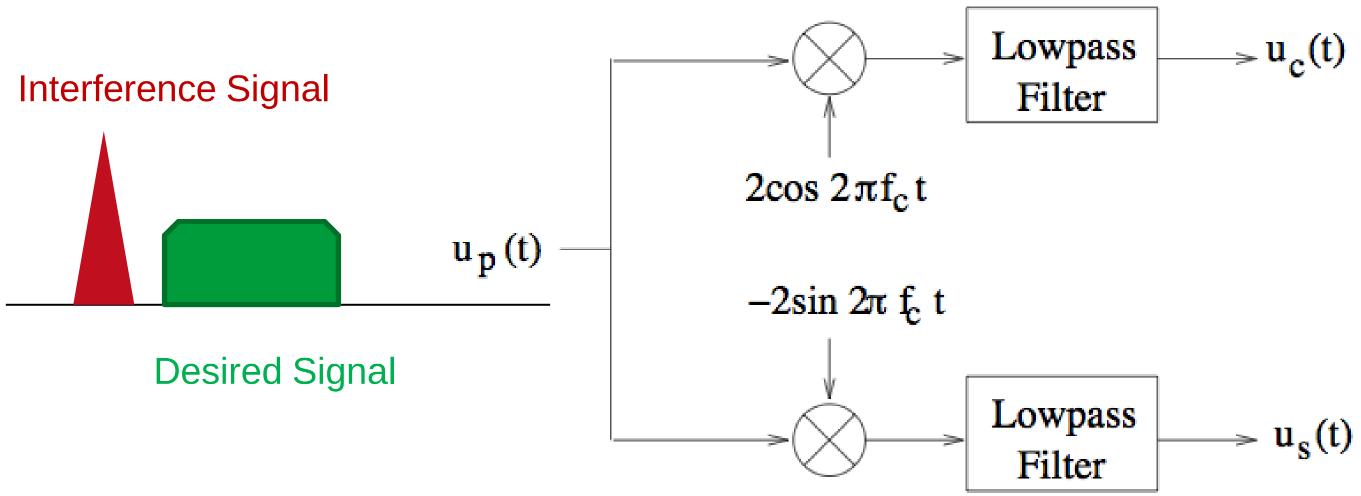
Ref: B.P.Lathi

Today’s Class

Superheterodyne Receiver:

Applicable to AM/FM/PM

Downconversion: How it is done?



Downconversion
(passband to baseband)

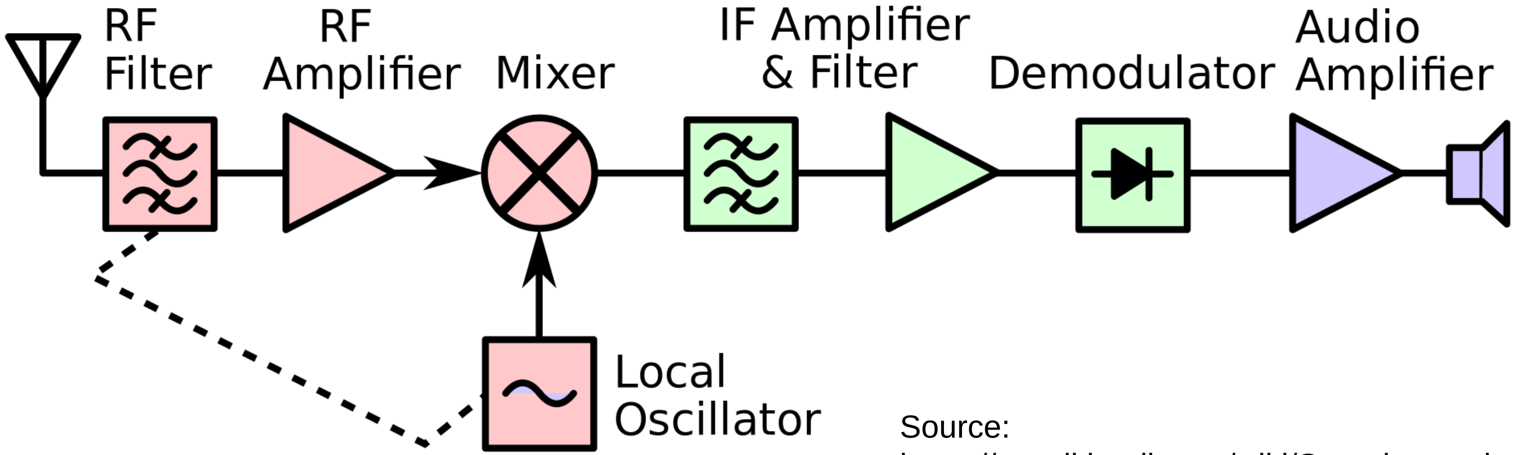
Not so simple as in the figure!

- Need to separate desired signal from interference
- Need to amplify the desired signal

Different Methods for Receiver

- Direct Conversion (Popular nowadays)
- Superheterodyne (Historical importance): Indirect conversion (convert to intermediate frequencies (IF) and then to baseband)

Superheterodyne Receiver



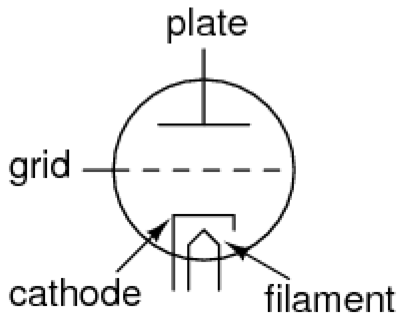
Source:
https://en.wikipedia.org/wiki/Superheterodyne_receiver

- Invented by Edwin Armstrong in 1918 during World War I.
- Superheterodyne is a contraction of *supersonic heterodyne*
  - supersonic indicates frequencies above the range of human hearing.
  - The word heterodyne is derived from the Greek roots hetero meaning *different*, and dyne meaning *power*.

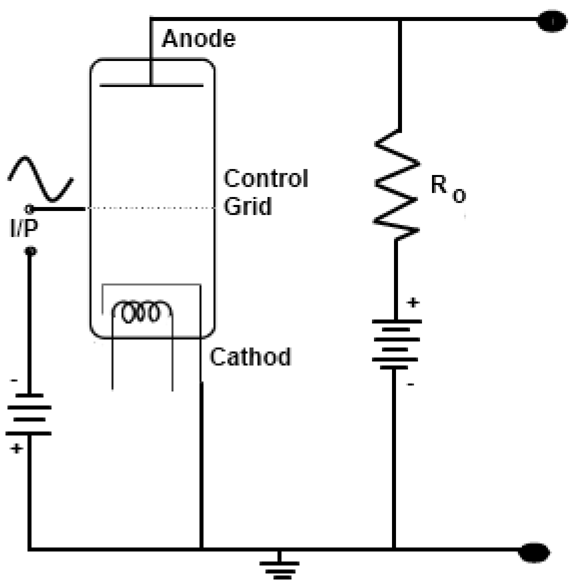
Motivation for Superheterodyne Receivers

- Difficulty in amplifying the received signal at frequecies beyond few MHz while using vacuum tube diodes and triodes.
- However higher frequencies are desired because of larger bandwidth and the smaller antennas required
- Still true: Difficult to provide large gains at high carrier frequencies. Gain is easier to provide at lower frequencies. Same is true with filtering.
- It becomes possible to optimize the processing at fixed IF in terms of filter design and amplification while permitting a tunable RF front end with more relaxed operation, which is important for design of radios that operate over a wide range of carrier frequencies
- Superhet architecture uses multiple stages of mixing to alleviate these problems.

Vacuum Tube Triode



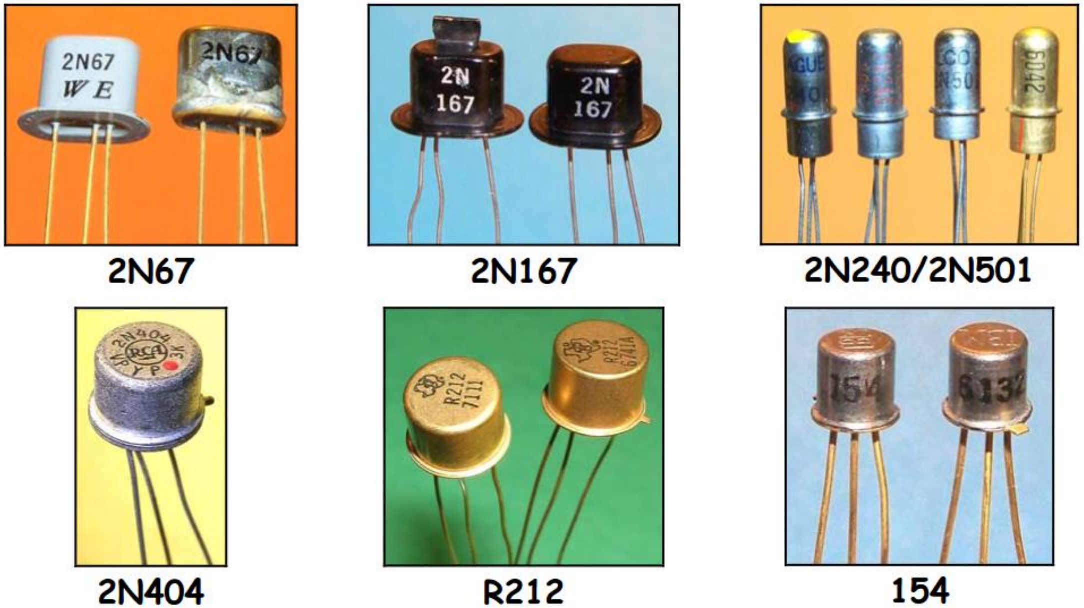
Symbol: http://www.allaboutcircuits.com/textbook/semiconductors/chpt-13/the-triode/

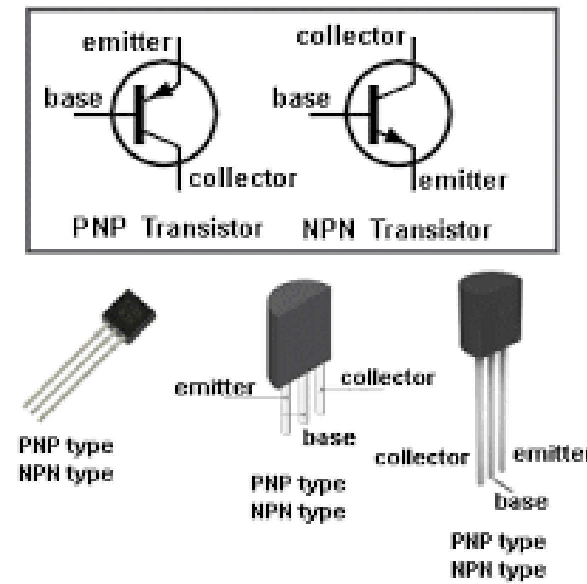


Amplifier Circuit: http://www.daenotes.com/electronics/devices-circuits/vacuum-tube-triode

http://www.oneillselectronicmuseum.com/largephotos/tubes/yel/yel23.jpg

Transistors





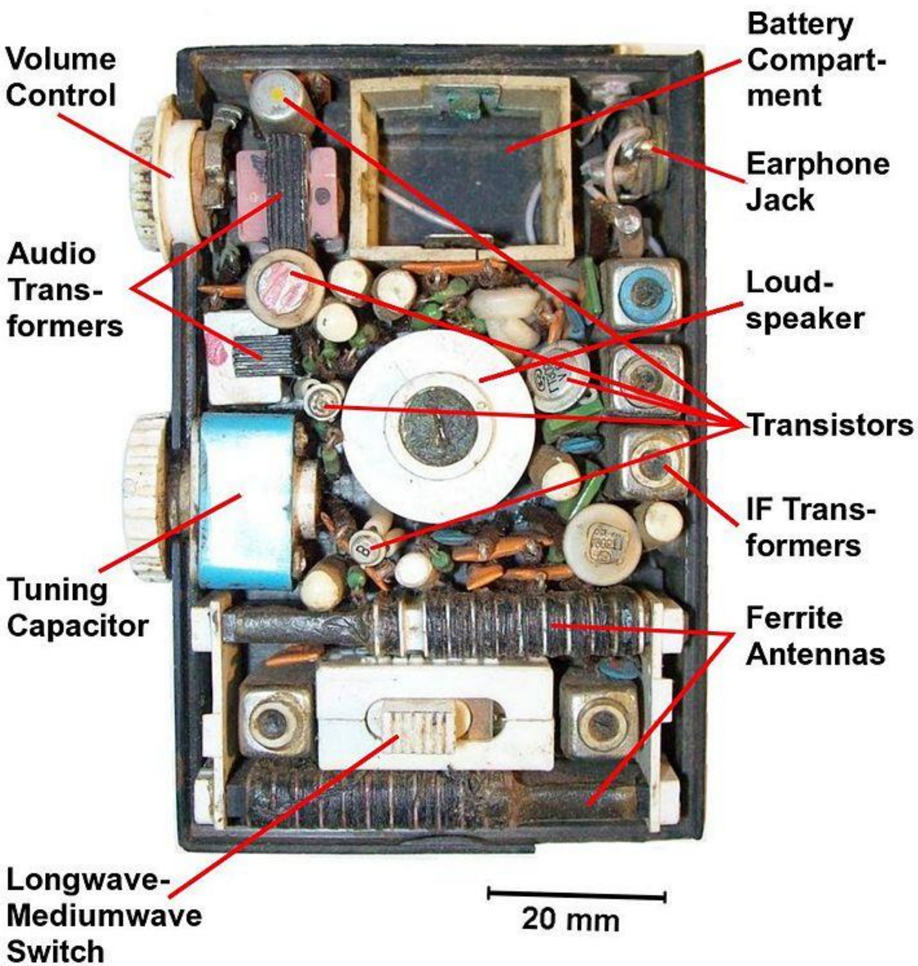
Old Germanium

Modern (Silicon)

http://ibm-1401.info/VintageGermaniumComputerTransistorsDavid%20Laws.pdf

http://www.talkingelectronics.com/projects/BasicElectronics-1A/BasicElectronics-1A_Page2.html

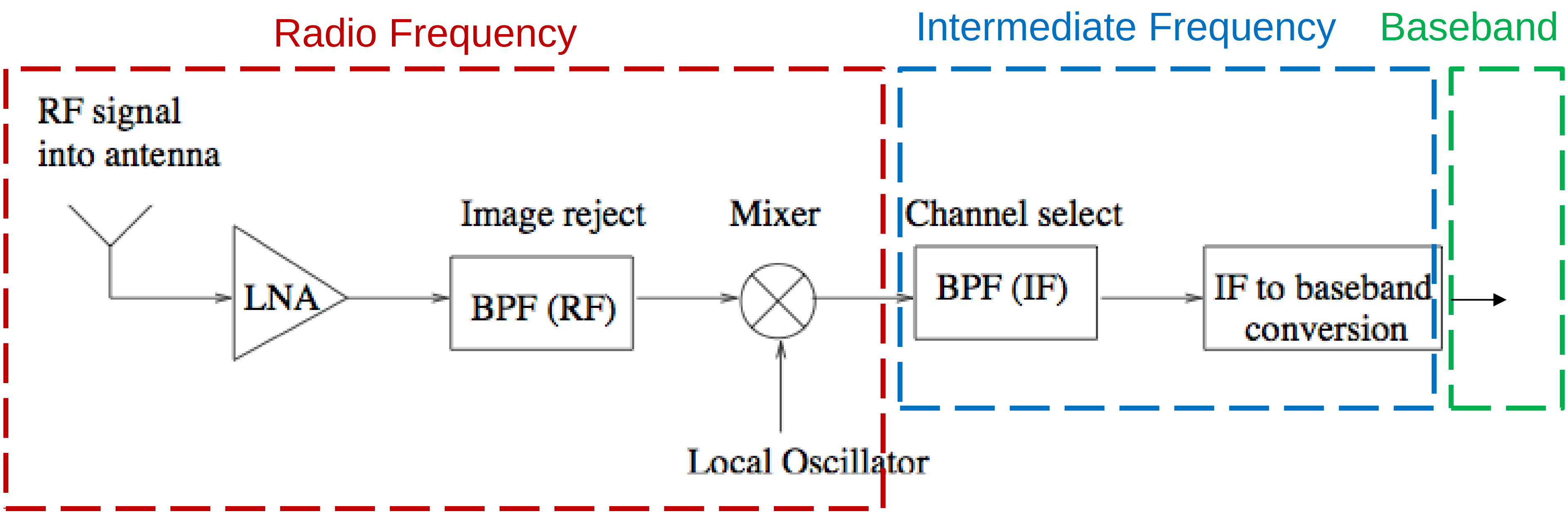
Example of Superheterodyne Transistor Receiver



Pocket transistor radio, late 1960s to 1970s, with 5 germanium transistors, that received long wave and medium wave (AM broadcast) bands. View with back open, showing parts. Powered by 2 button cells supplying 3V

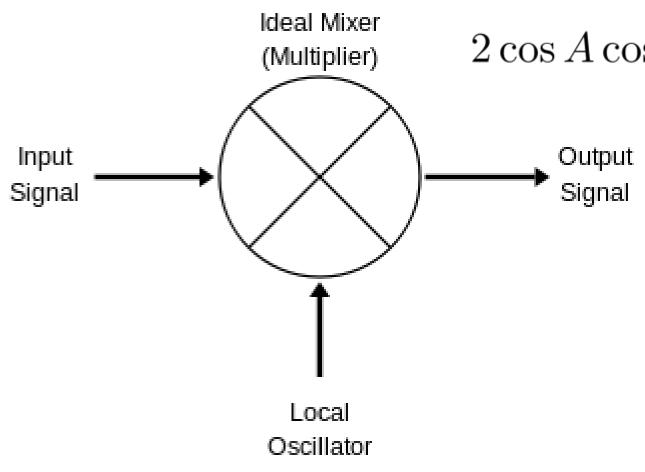
By Pocket_radio_open.jpg: Ulfbastelderivative work: Chetvorno (talk) - Pocket_radio_open.jpg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=12819355

Superheterodyne receiver: *Principle*



- Sloppy RF filtering at the RF
- Careful processing at fixed IF: IF bandpass filter and amplifier provide most of the gain and narrowband filtering for the radio.
- RF front end often tunable (e.g., multiple bands in WiFi and cellular, multiple stations in AM or FM)

Mixer



$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$

https://en.wikipedia.org/wiki/Frequency_mixer

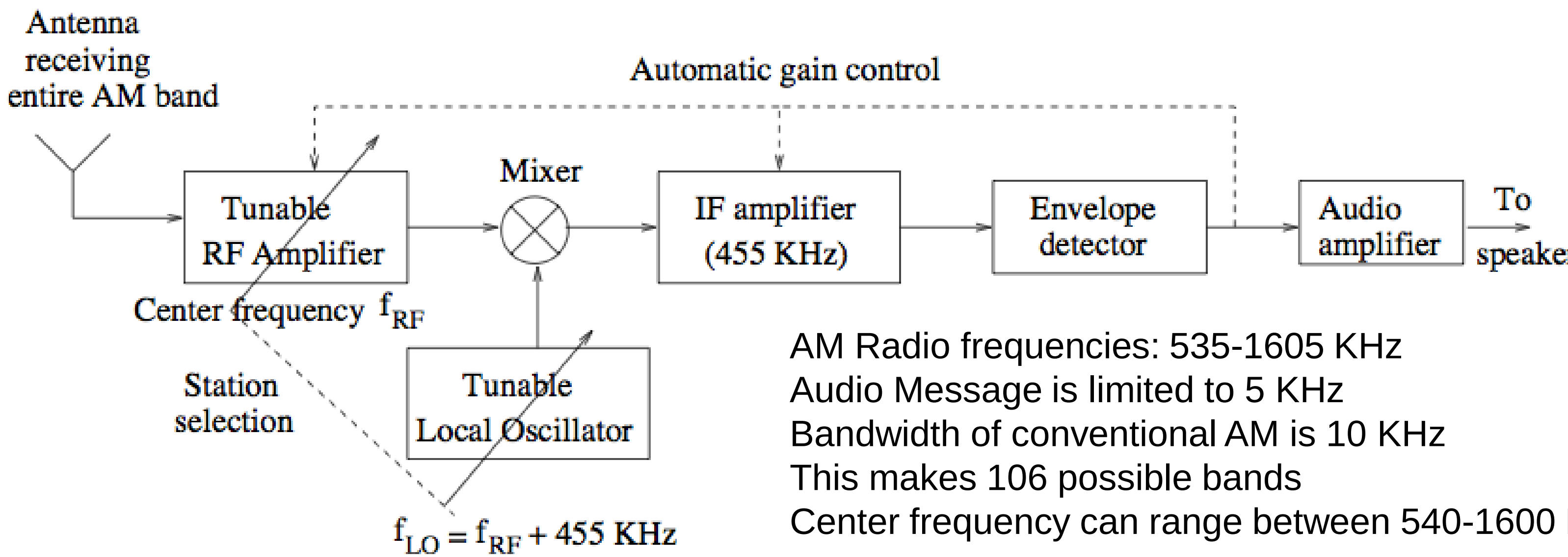
- In electronics, a mixer or frequency mixer is a nonlinear electrical circuit that creates new frequencies from two signals applied to it. In its most common application, two signals at frequencies f_1 and f_2 are applied to a mixer and it produces new signal at the sum $f_1 + f_2$ and difference $|f_1 - f_2|$ of the original frequencies called heterodynes.

- Nonlinear electronic components that are used as mixers include diodes, transistors biased near cutoff,

Mixer: Use of Non-linearity

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) \longrightarrow \boxed{y(t) = x(t) + x^2(t)} \longrightarrow z(t)?$$

Example: superhet for AM radio

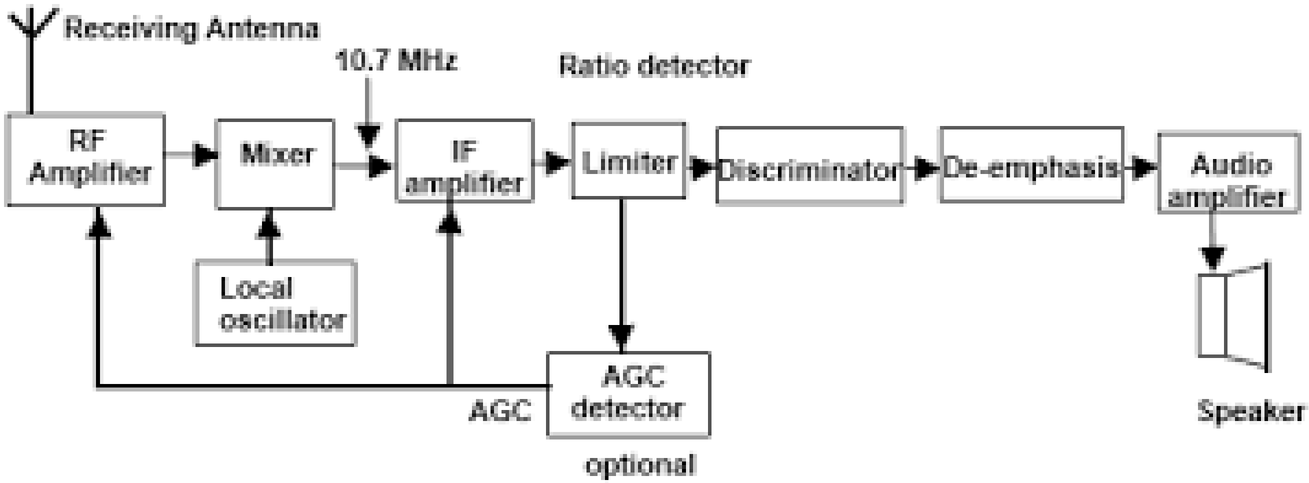


AM Radio frequencies: 535-1605 KHz
Audio Message is limited to 5 KHz
Bandwidth of conventional AM is 10 KHz
This makes 106 possible bands
Center frequency can range between 540-1600 KHz

- Two possibilities for LO frequency: Given IF, how to set LO frequency?
- Example for AM radio: RF freq. ranges from 540 to 1600 KHz
  - $f_{\mathrm{IF}} = 455$ KHz
  - $f_{\mathrm{LO}} = f_{\mathrm{RF}} + f_{\mathrm{IF}} \rightarrow$ LO freq. ranges from 995 to 2055 KHz (less variation in tuning range) ~ 2
  - $f_{\mathrm{LO}} = f_{\mathrm{RF}} - f_{\mathrm{IF}} \rightarrow$ LO freq. ranges from 85 to 1145 KHz (huge variation in tuning range) > 10

Example: superhet for FM radio

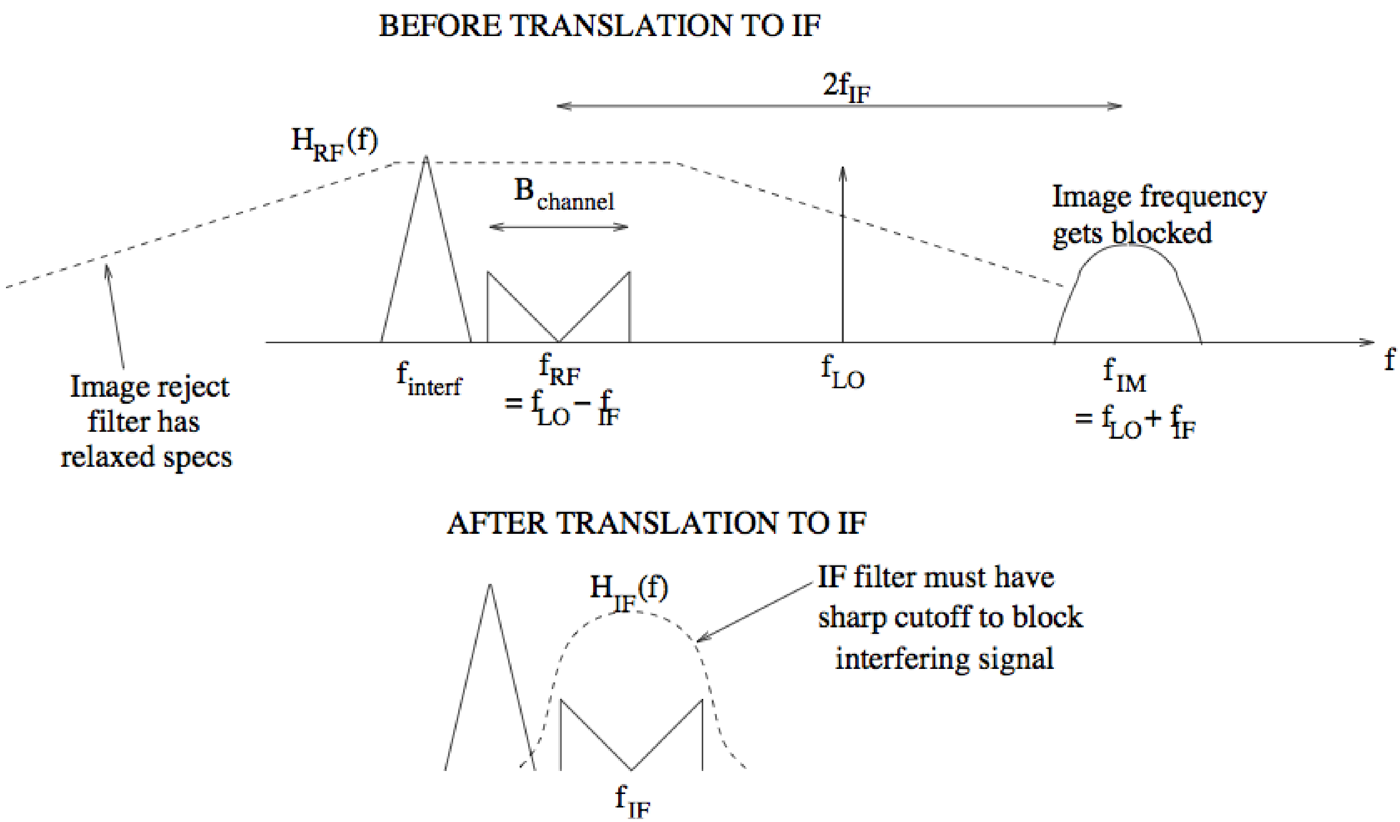
http://www.daenotes.com/electronics/communication-system/superheterodyne-fm-receiver



FM Radio frequencies: 88-108 MHz
Bandwidth of 150 KHz + 25 KHz guard band
200KHz spacing makes 100 stations
Center frequency assigned starting at 88.1 MHz

- What frequency should we choose for LO?

Superhet: freq domain operations



- Both $f_{\mathrm{LO}} - f_{\mathrm{IF}}$ and $f_{\mathrm{LO}} + f_{\mathrm{IF}}$ lead to an IF component $\rightarrow$ one of these is the undesired image frequency must be filtered out by RF front end.
- The IF filter filters out interference from adjacent channels.
- Sloppy requirement for RF but tight requirement for IF.

Disadvantages of SHR

- Issue of image frequency!
- Need of extra IF stage.
- There is a trade-off between rejecting image frequency and rejecting interference signal while choosing IF. This may necessitate multiple IF stages.
- Cannot use IC implementation and has to use bulky and costly filters such as surface-acoustic-wave (SAW).

Direct Conversion Receiver

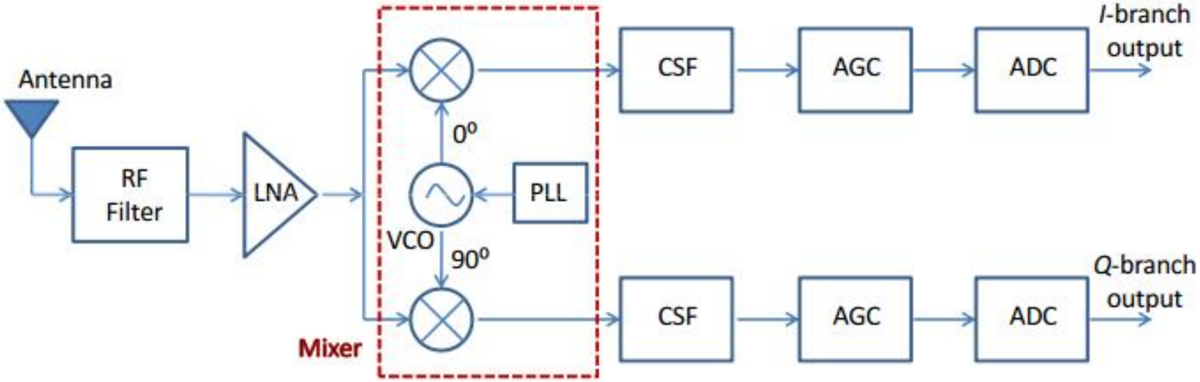


Figure 3.2. Typical RF front end of a direct conversion receiver. The components of the analog front end: wideband antenna, RF filter, low noise amplifier (LNA), voltage controlled oscillator (VCO), phase locked loop (PLL), channel select filter (CSF), automatic gain control (AGC), and analog-to-digital converter (ADC).

Source: Sachin Chaudhari, PhD Thesis; "Sensing for Cognitive Radios: Algorithms, Performance, and Limitations", Aalto University School of Electrical Engineering, Finland, 2012

- Direct conversion (*zero IF* or *homodyne* or *synchrodyne*, i.e., mixing down to baseband)
- Advantages: No image frequency, No IF stage, Cost benefit, Use of IC for compactness, Use of microprocessors for baseband processing of signal such as filtering.
- *Obvious thing to do*!

Disadvantages of DCR

- To match the performance of the superheterodyne receiver, a number of the functions normally addressed by the IF stage must be accomplished at baseband.
- Inability to implement envelope detection of AM signals. Therefore use of PLL is needed!
- Suffer from LO leakage to mixer input $\rightarrow$ DC component that can swamp later circuit components.
- Other issues such as Frequency offset, IQ imbalance, Narrowband interference.

Receiver for mmWave communication?

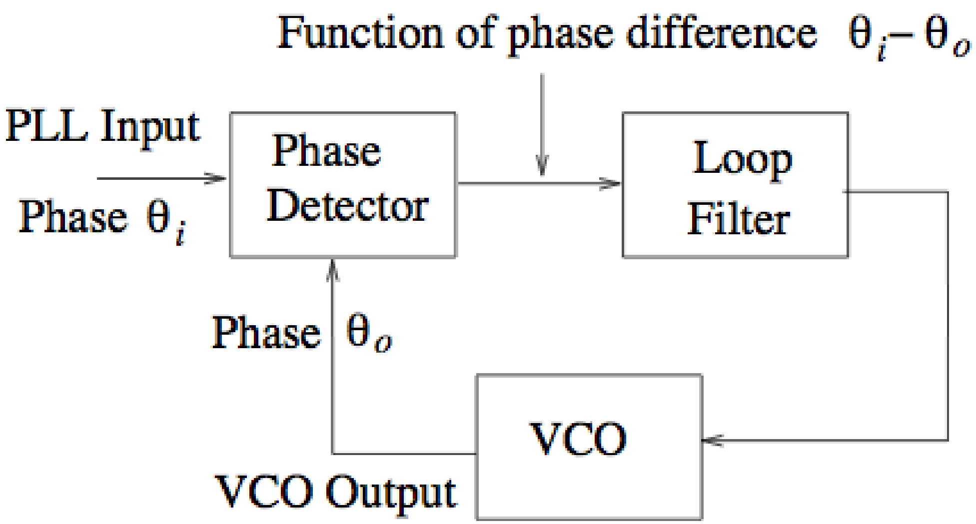
- Currently we are using mostly frequencies less than 5 GHz for communication
  - WLAN is using frequencies 2.4 and 5 GHz.
  - Cellular communication is using 900 and 1800 MHz bands.
- For the evergrowing need, we are looking at using frequencies much beyond 5 GHz (mmWave communication).
- For example, 60 GHz of carrier frequency can give us bandwidth of 5GHz.
- However direct receiver conversion is still a challenge there.
- Superheterodyne principle can be used with IF upto 5 GHz.

Phase Locked Loop (PLL)

PLL intro

- Legacy applications in analog communication and analog front-end of digital communication
  - FM demodulation
  - Carrier synchronization
- Key application in communication today: LO frequency synthesis
- Canonical structure for using feedback for continuous tracking

High-level view of PLL

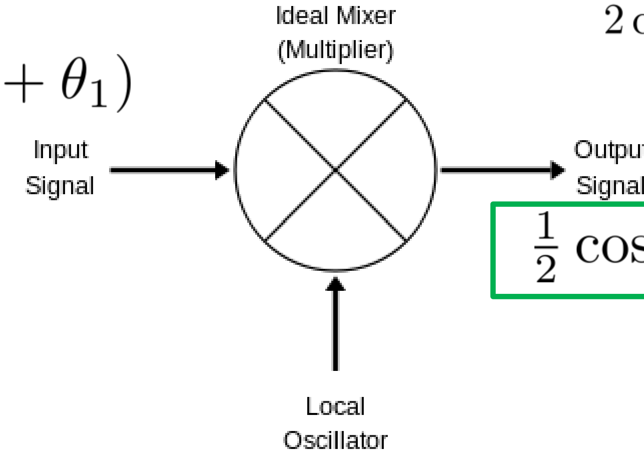


- Aim is to lock on the phase of the PLL input
- Phase detector compares input phase with locally generated phase at the VCO output
- Phase detector output is smoothed through loop filter and fed back to VCO input
  - If VCO output is ahead of the input phase, then retard the VCO phase
  - If VCO output is behind of the output phase, then advance VCO phase

Mixers can extract phase differences

$$2\cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$\cos(2\pi f_c t + \theta_1)$$



$$\boxed{\frac{1}{2}\cos(\theta_1 - \theta_2)} + \frac{1}{2}\cos(4\pi f_c t + \theta_1 + \theta_2)$$

$$\cos(2\pi f_c t + \theta_2)$$

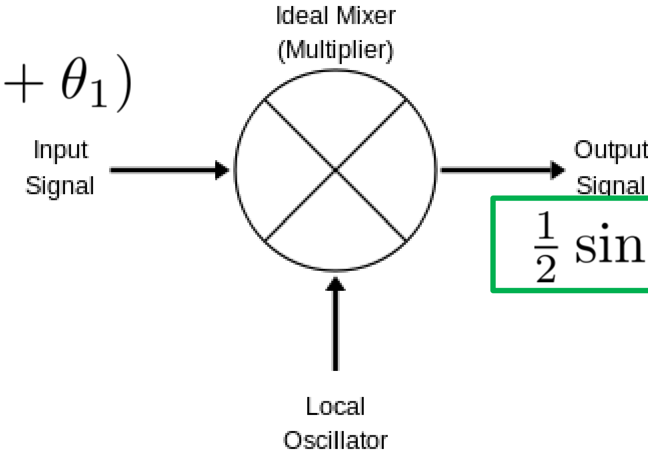
Phase lock condition: The term $\cos(\theta_1 - \theta_2) = 0$, i.e., $\theta_1 - \theta_2 = \pi/2$

For a natural phase lock condition of $\theta_1 - \theta_0 = 0$, change one of the sinusoids to sine wave.

Mixers can extract phase differences

$$-2\cos A \sin B = \sin(A-B) - \sin(A+B)$$

$$\cos(2\pi f_c t + \theta_1)$$

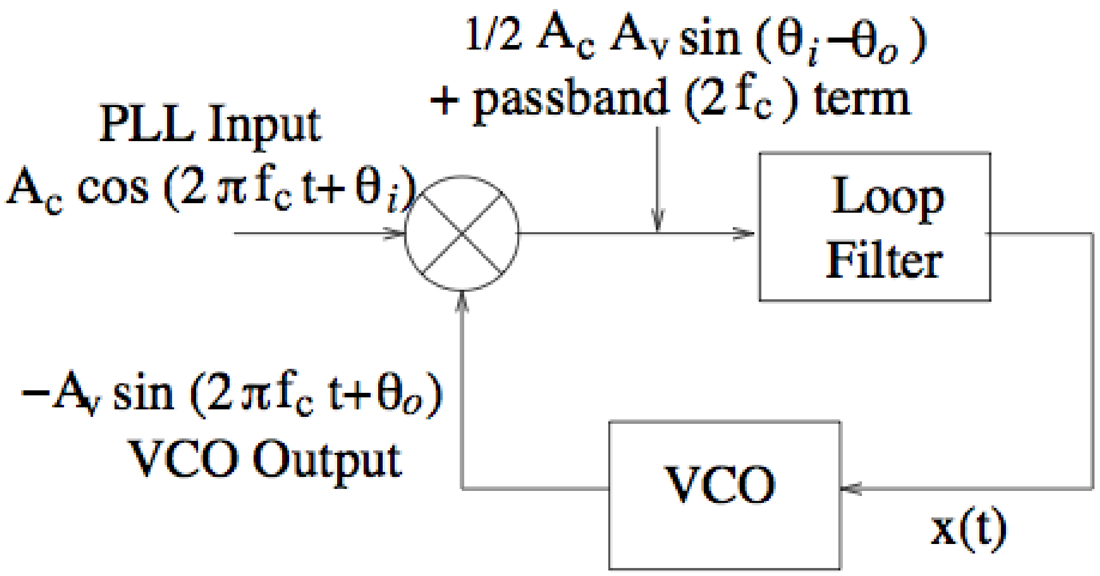


$$\boxed{\tfrac{1}{2}\sin(\theta_1 - \theta_2)} - \cancel{\tfrac{1}{2}\sin(4\pi f_c t + \theta_1 + \theta_2)}$$

$$-\sin(2\pi f_c t + \theta_2)$$

Phase lock condition: The term $\sin(\theta_1 - \theta_2) = 0$, i.e., $\theta_1 - \theta_2 = 0$

Mixer-based phase detector



- With this convention, the mixer output is given by

$$
\begin{aligned}
& - A_c A_v \cos(2\pi f_c t + \theta_i(t)) \sin(2\pi f_c t + \theta_o(t)) \\
& = \frac{A_c A_v}{2} \sin(\theta_i(t) - \theta_o(t)) - \frac{A_c A_v}{2} \sin(4\,\pi f_c t + \theta_i(t) + \theta_o(t))
\end{aligned}
$$

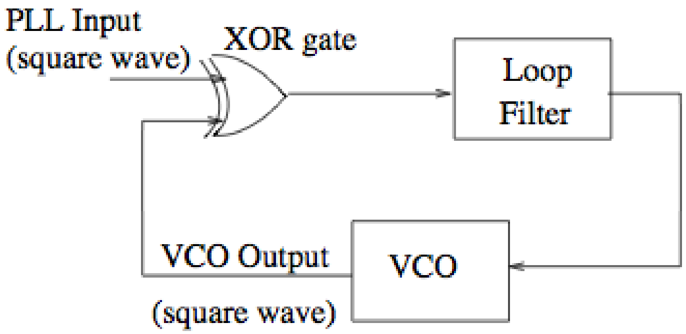
Phase difference term driving feedback loop

Double frequency term (filtered out)

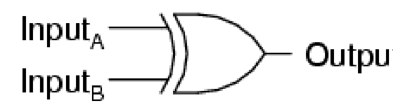
Mixed signal phase detection

- Modern PLL implementation often make heavy use of digital logic
- For frequency synthesis application, VCO output is often a square wave (filtered later to extract sinusoid as a harmonic)
- Phase detector can also be implemented using digital logic, example, XOR gate

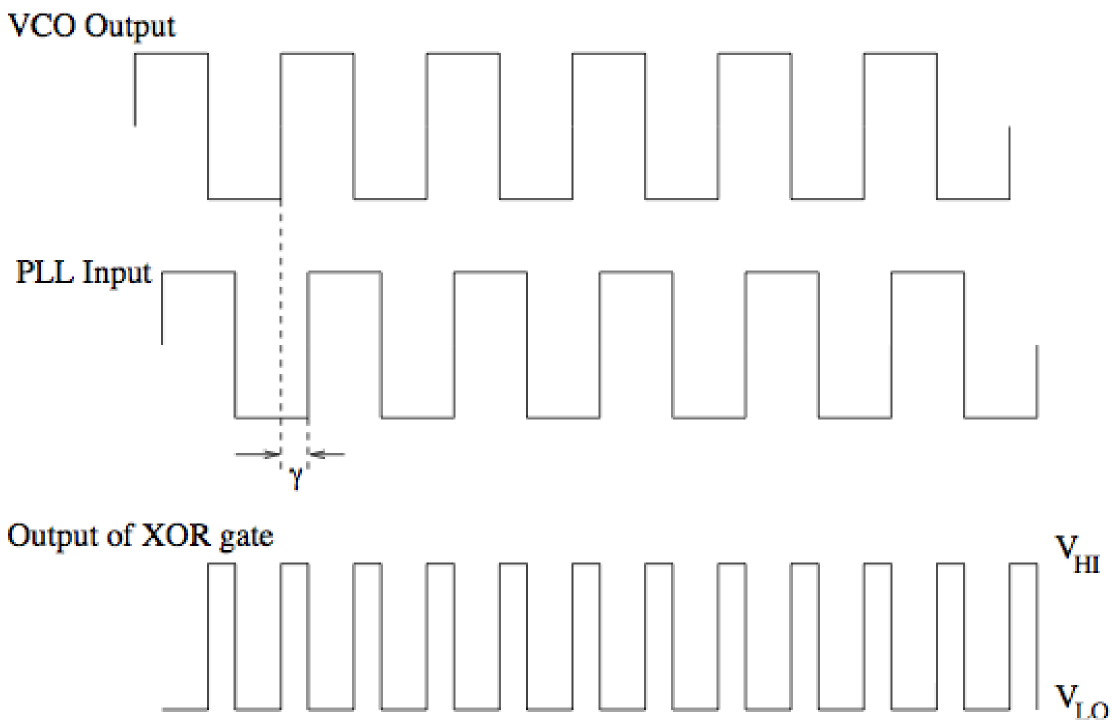
XOR-based phase detector



Exclusive-OR gate

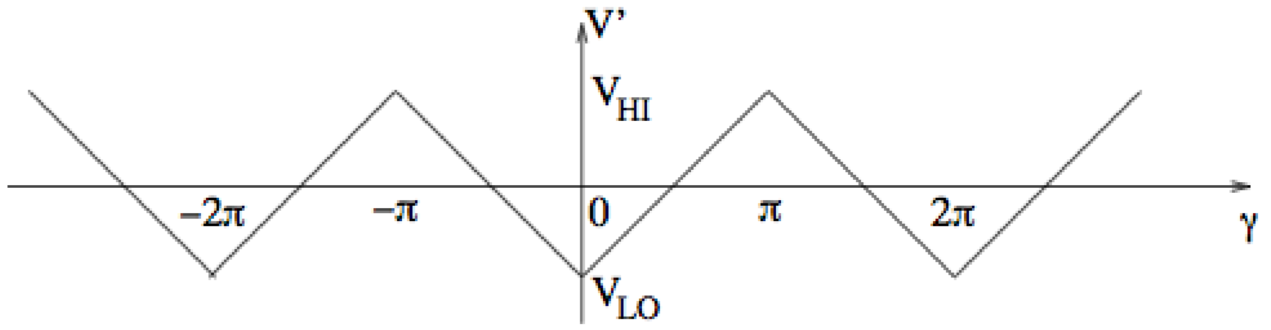


A	B	Output
0	0	0
0	1	1
1	0	1
1	1	0



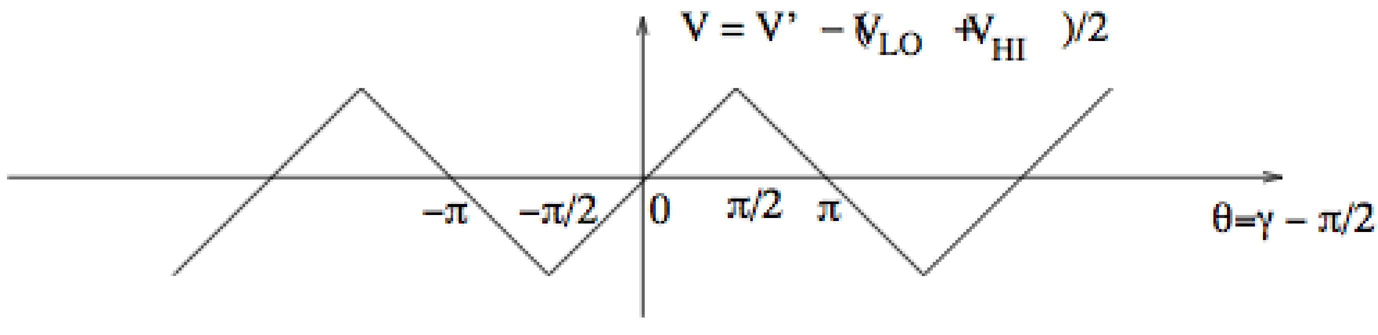
Average of XOR output is related to phase difference

XOR-based phase detector output



(a) DC value of output of XOR gate.

- Translate along both axes to get a more natural curve, i.e., phase detector output is zero for zero phase difference
- Response symmetric around origin



(b) XOR phase detector output after axes translation.

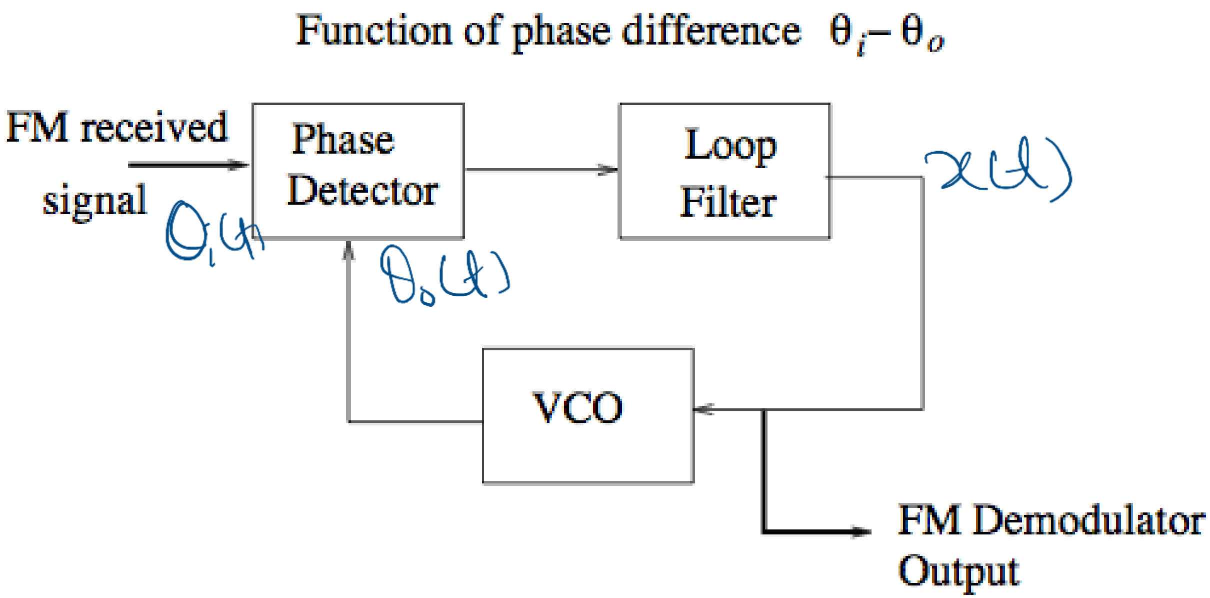
PLL Applications:

FM Demodulation

and

Frequency Synthesis

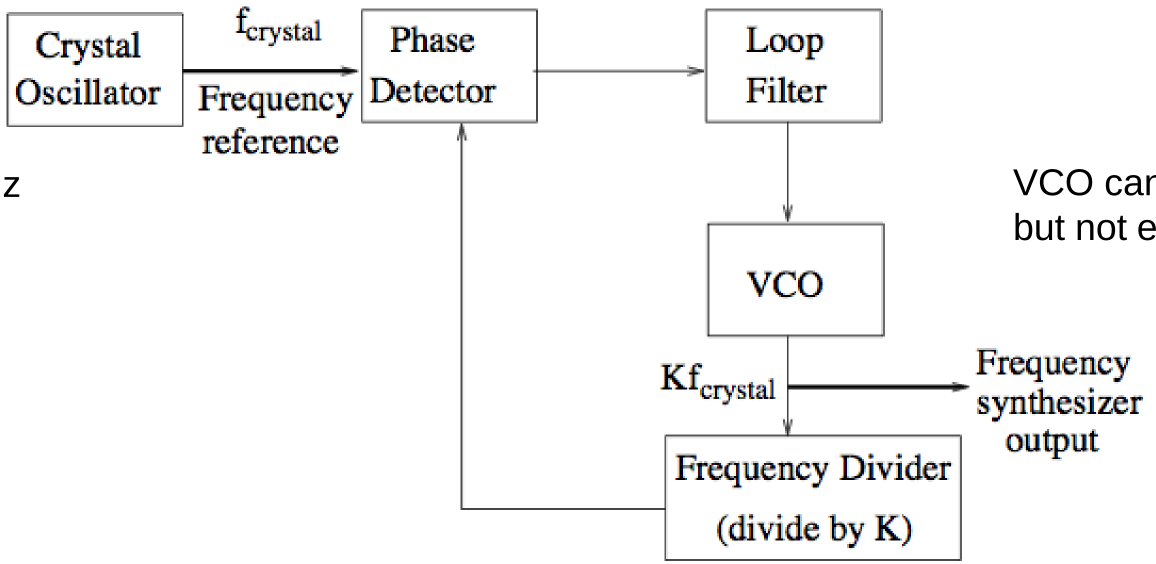
PLL for FM demodulation



- Show that if the PLL is tracking and PLL input is FM signal, then the input to VCO is a scaled version of the message.

PLL for frequency synthesis

- Typical application in communication transceivers
- Synthesize LO (e.g. at 2.4 GHz) from a lower reference frequency source (such as crystal oscillator)



40 MHz

VCO can produce 2.4 GHz but not exact value

- VCO quiescent frequency is around the desired frequency. Precise lock to multiple of crystal frequency is enabled by the PLL.
- Divide frequency of VCO (e.g., by skipping clock cycles in a digital implementation).
- Compare with crystal phase/freq to drive VCO

ECE335 Communication Theory I (3-1-0-4):

Lecture 11:
Analog Communication Techniques:
Phase Locked Loop (PLL)

Feb. 22, 2025

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Today’s Class

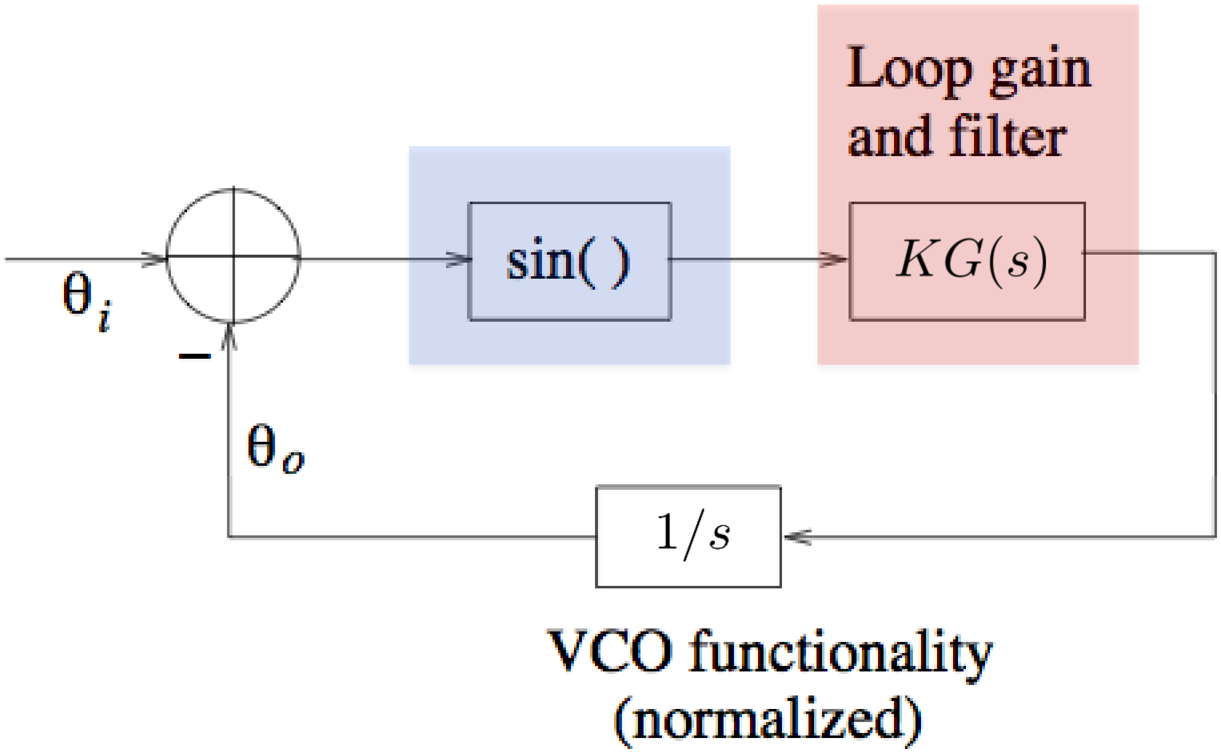
PLL Applications:

FM Demodulation and Frequency Synthesis

For mixer-based phase detector

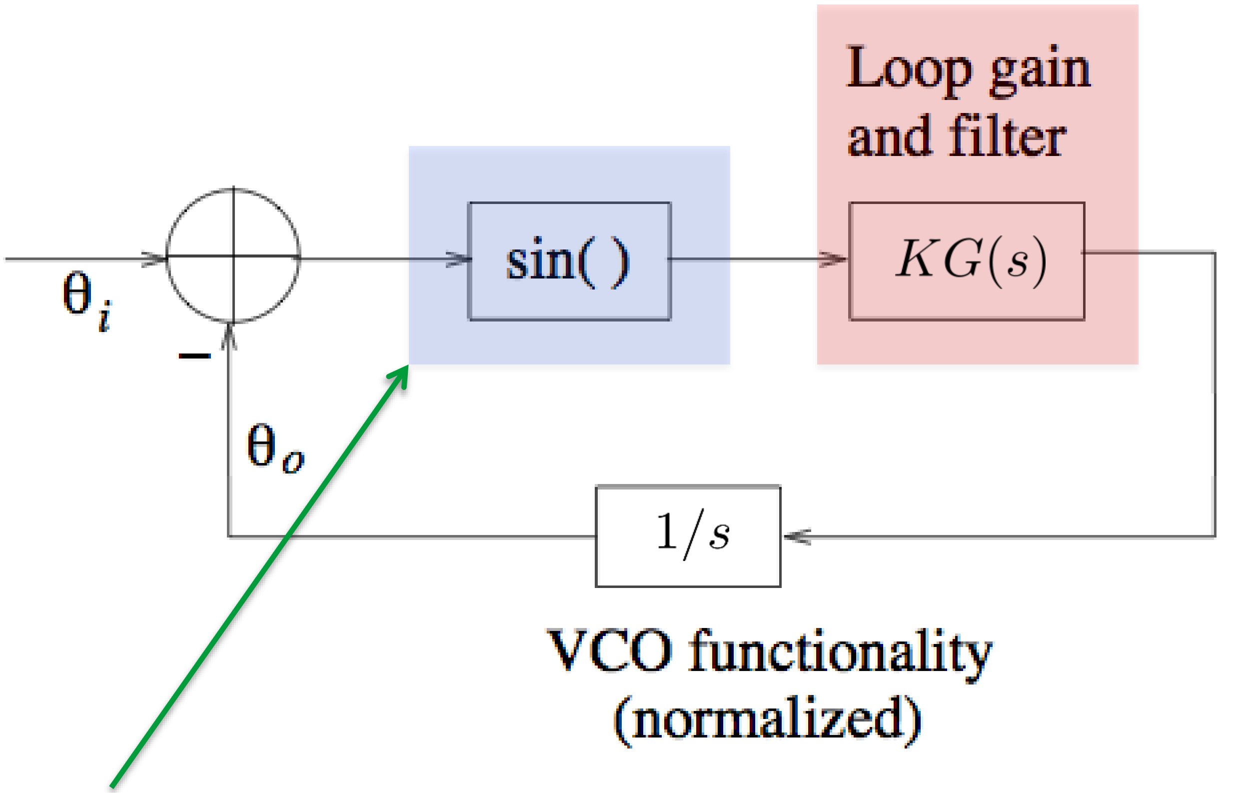
Output of mixer as phase detector

$$= \frac{A_c A_v}{2} \sin(\theta_i(t) - \theta_o(t)) - \frac{A_c A_v}{2} \sin(4\,\pi f_c t + \theta_i(t) + \theta_o(t))$$



$$\theta_o(t) = K_v \int_0^t x(\tau)d\tau$$

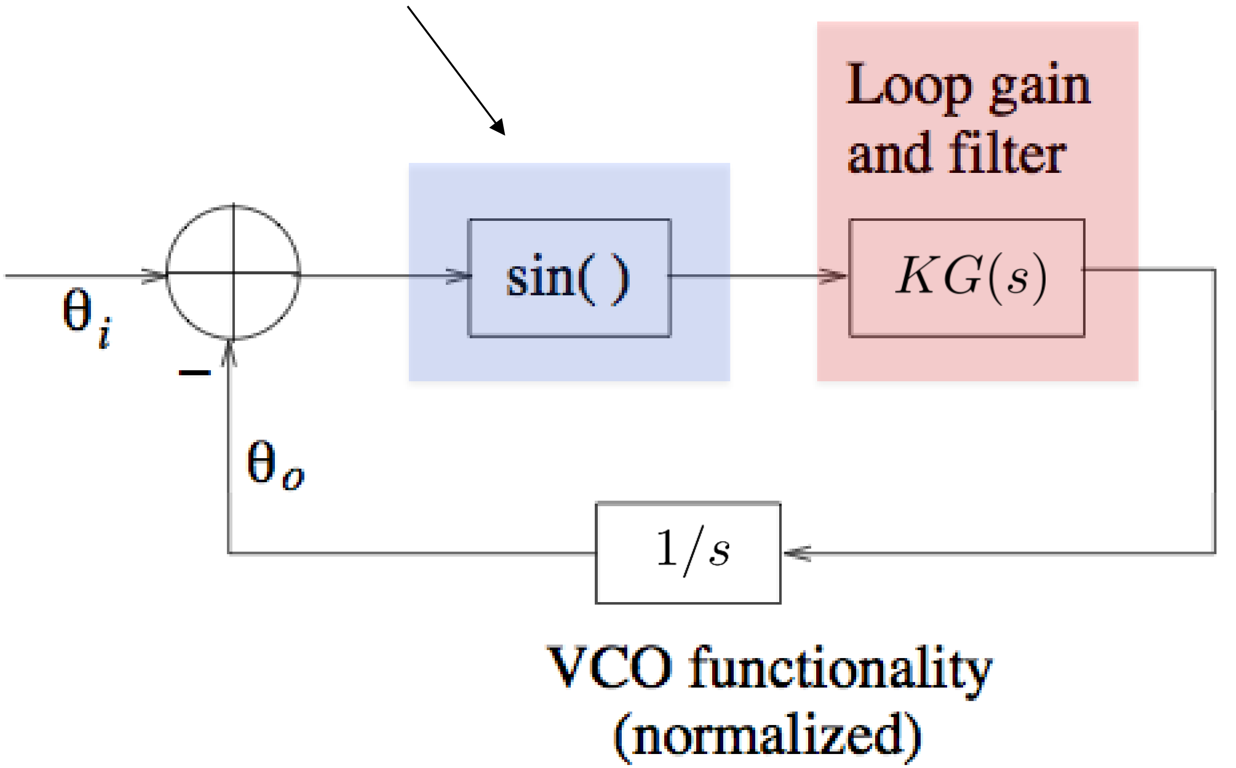
For mixer-based phase detector



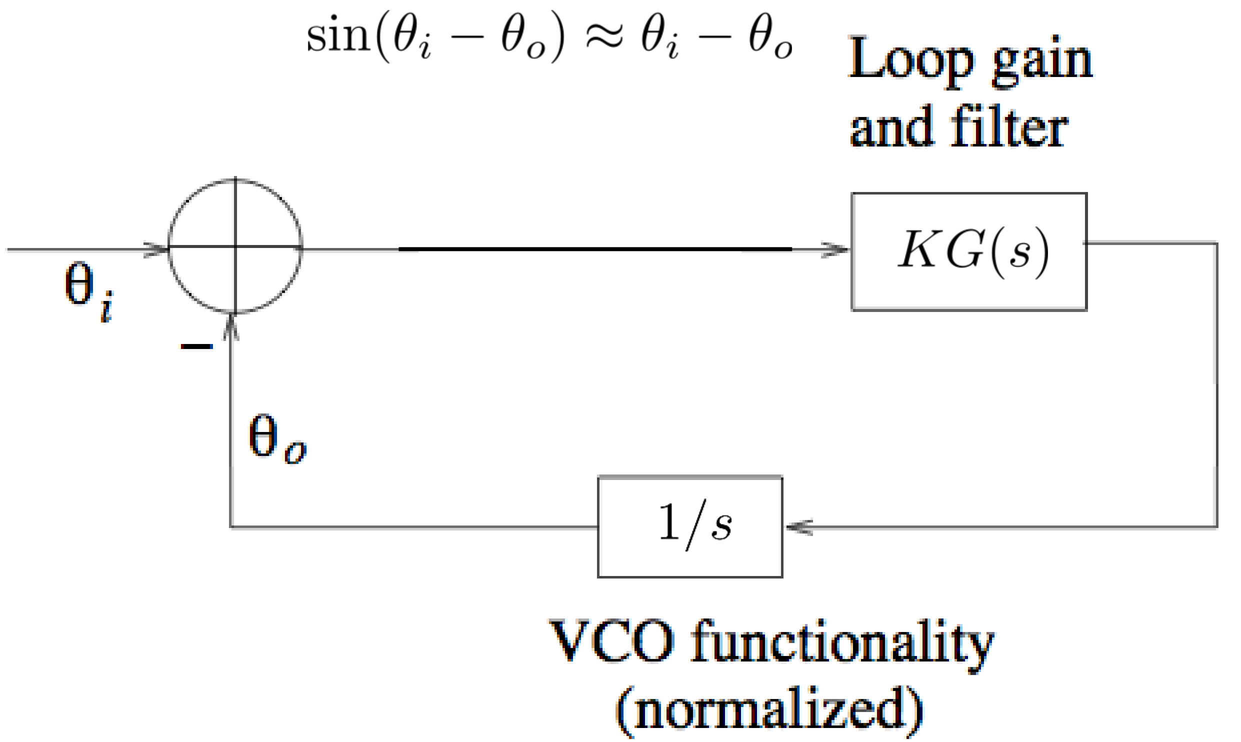
Replace sine by triangular wave for XOR-based phase detector

For mixer-based phase detector

This is difficult to analyze as system is non-linear

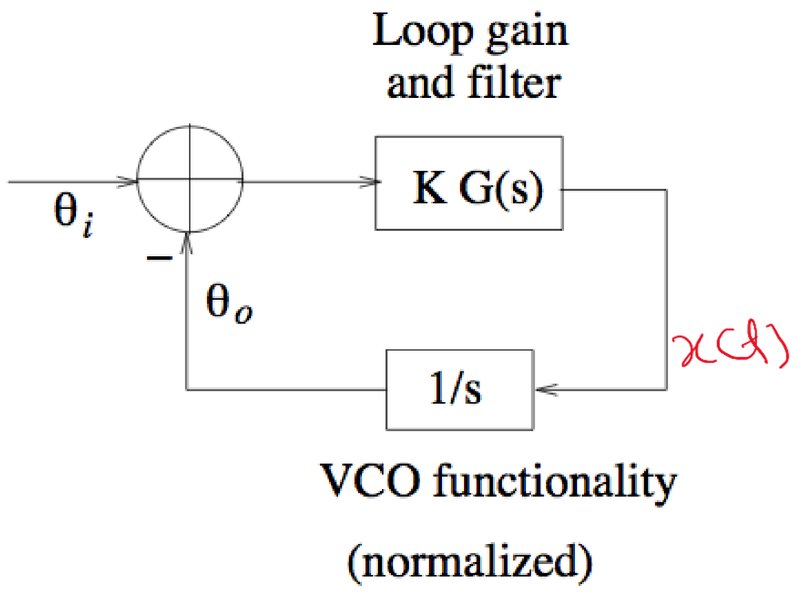


Linearized Model



- Can analyze using Laplace transform.
- Linearized approximation applies even better to triangular response corresponding to XOR-based detector since response is exactly linear for small phase error.

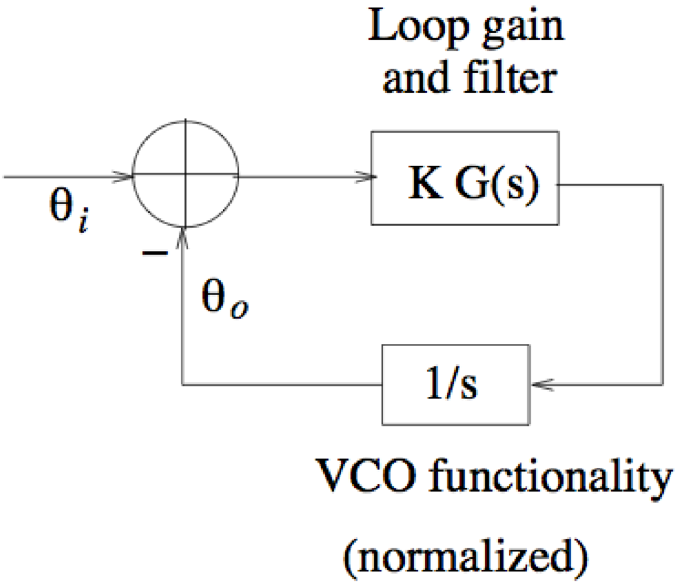
Linearized analysis



- Show that the transfer function between input and error

$$H_e(s) = \frac{\Theta_i(s) - \Theta_o(s)}{\Theta_i(s)} = \frac{s}{s + KG(s)}$$

First order PLL



$$H(s) = \frac{KG(s)}{s+KG(s)} \quad H_e(s) = \frac{s}{s+KG(s)}$$

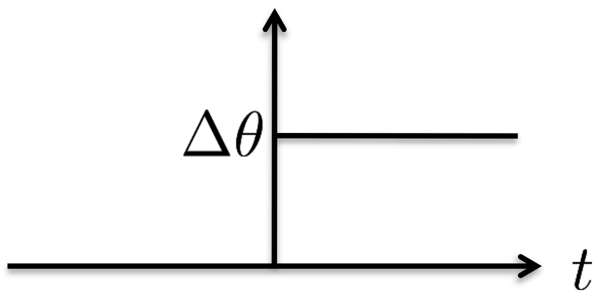
$$\Big\downarrow \; G(s) = 1$$

$$H(s) = \frac{K}{s+K} \quad H_e(s) = \frac{s}{s+K}$$

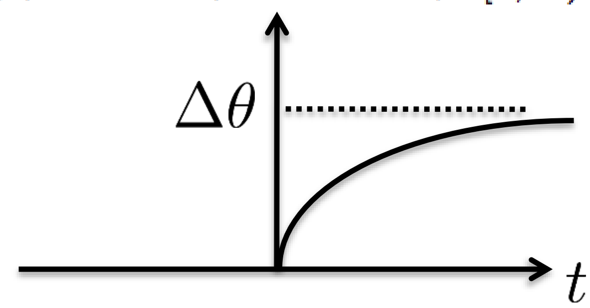
Single pole at $s = -K$
Stable when loop gain $K > 0$

1st order PLL: phase step response

$$\theta_i(t) = \Delta\theta I_{[0,\infty)}(t)$$



$$\theta_o(t) = \Delta\theta(1 - e^{-Kt})I_{[0,\infty)}(t)$$

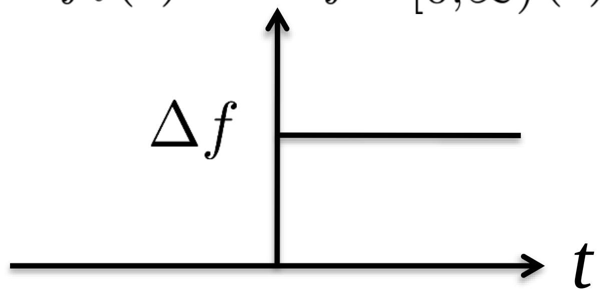


- Assume PLL is tracking well. If the input phase suddenly jumps, then would PLL be able to track down this change? To answer this, we need to answer following question
  - Find $\theta_o(t)$ in terms of $\theta_i(t)$.
  - Also find steady state error.

$$\lim_{t\to\infty} \theta_e(t) = 0$$

1st order PLL: freq step response

$$f_i(t) = \Delta f \ I_{[0,\infty)}(t)$$



- Assume PLL is tracking well. If the input frequency suddenly jumps, then would PLL be able to track down this change? To answer this, we need to answer following question
  - Find $f_o(t)$ and $\theta_o(t)$ in terms of $f_i(t)$ and $\theta_i(t)$.
  - Also find steady state error.

Second order PLL

- Can we track frequency step using a more complex loop filter?
- What if we feed both the phase error and the integral of the phase error (Proportional + Integral Feedback)

$$G(s) = 1 + \frac{a}{s} \quad a > 0$$

- For the above $G(s)$, the frequency responses $H(s)$ and $H_e(s)$become

$$H(s) = \frac{KG(s)}{s + KG(s)} = \frac{K(s+a)/s}{s + (s+a)/s}$$

$$= \frac{K(s+a)}{s^2 + Ks + Ka}$$

$$H_e(s) = \frac{s}{s + KG(s)}$$

$$= \frac{s^2}{s^2 + Ks + Ka}$$

- Poles at $s = \frac{-K \pm \sqrt{K^2 - 4Ka}}{2}$
- Stable for $K > 0$
- Oscillations in response if poles have an imaginary components.
  Happens if $K^2 - 4Ka < 0$ or $K < 4a$

2^{nd} order PLL: freq step response

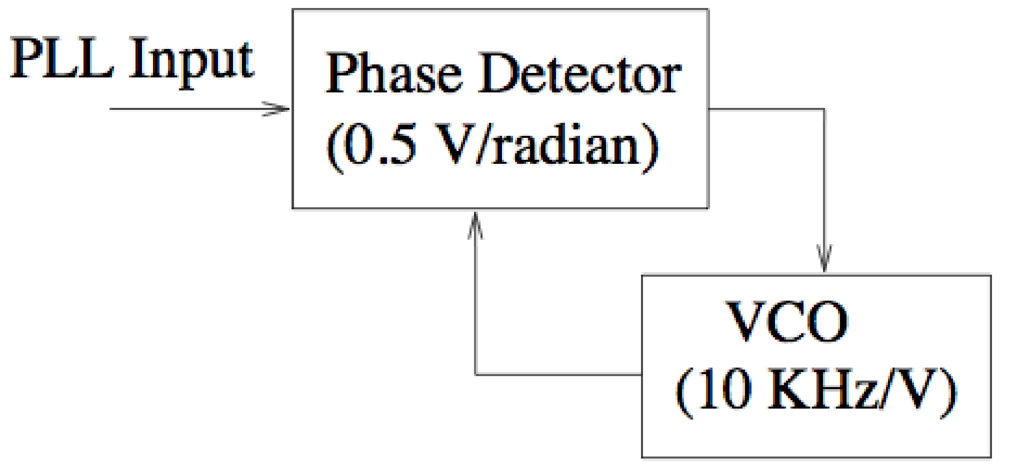
- Similar to first order PLL, we can find expressions for phase and frequency!

- However we are more interested if the phase error goes to zero in this case (Show!).

Choosing PLL order

- First order PLL can track step in phase
- Second order PLL can track step in frequency (i.e., linear ramp in phase)
- Third order PLL can track step in frequency derivative (i.e., linear ramp in frequency)
- Typical design choice is 2nd and 3rd order PLL.

Example



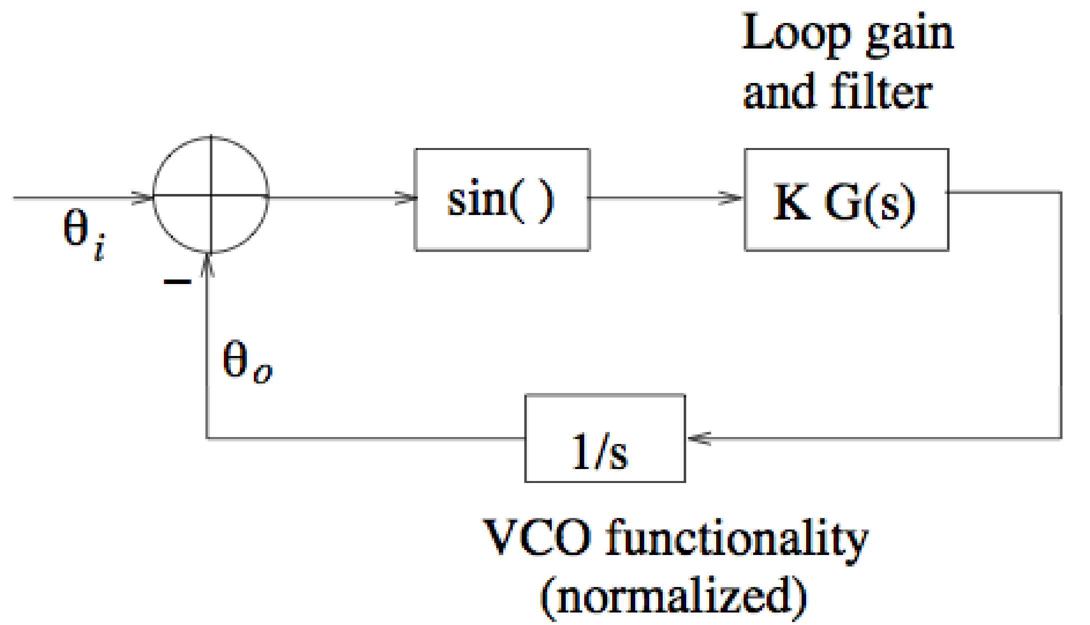
Consider the PLL shown in Fig. assumed to be locked at time zero

1. Suppose that the input phase jumps by $e = 2.72$ radians at time zero assuming $\theta_i(0) = 0$. How long does it take for the difference between the PLL input phase and the VCO output phase to shrink to 1 radian.

2. Find the limiting value of the phase error (in radians) if the frequency jumps by 1 KHz just after time zero.

Actual nonlinear PLL model

Original nonlinear model



- Assuming $G(s) = 1$ (i.e., first order PLL) and for frequency step input, show that

$$\frac{d\theta_e(t)}{dt} = 2\pi\Delta f - K\sin\theta_e(t)$$

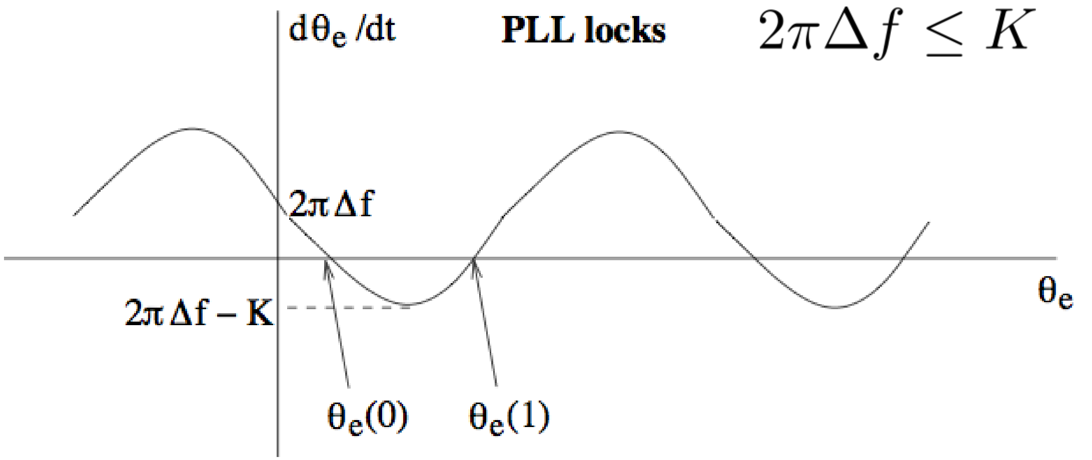
No closed form solution

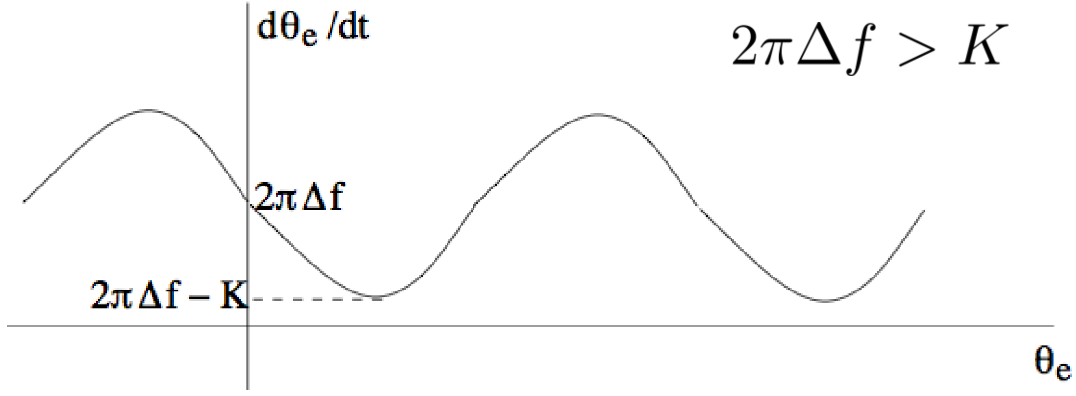
Phase plane plot

$$\frac{d\theta_e(t)}{dt} = 2\pi\Delta f - K\sin\theta_e(t)$$

$$\Downarrow \quad \sin\theta_e(t) \leq 1$$

$$\frac{d\theta_e(t)}{dt} \geq 2\pi\Delta f - K$$



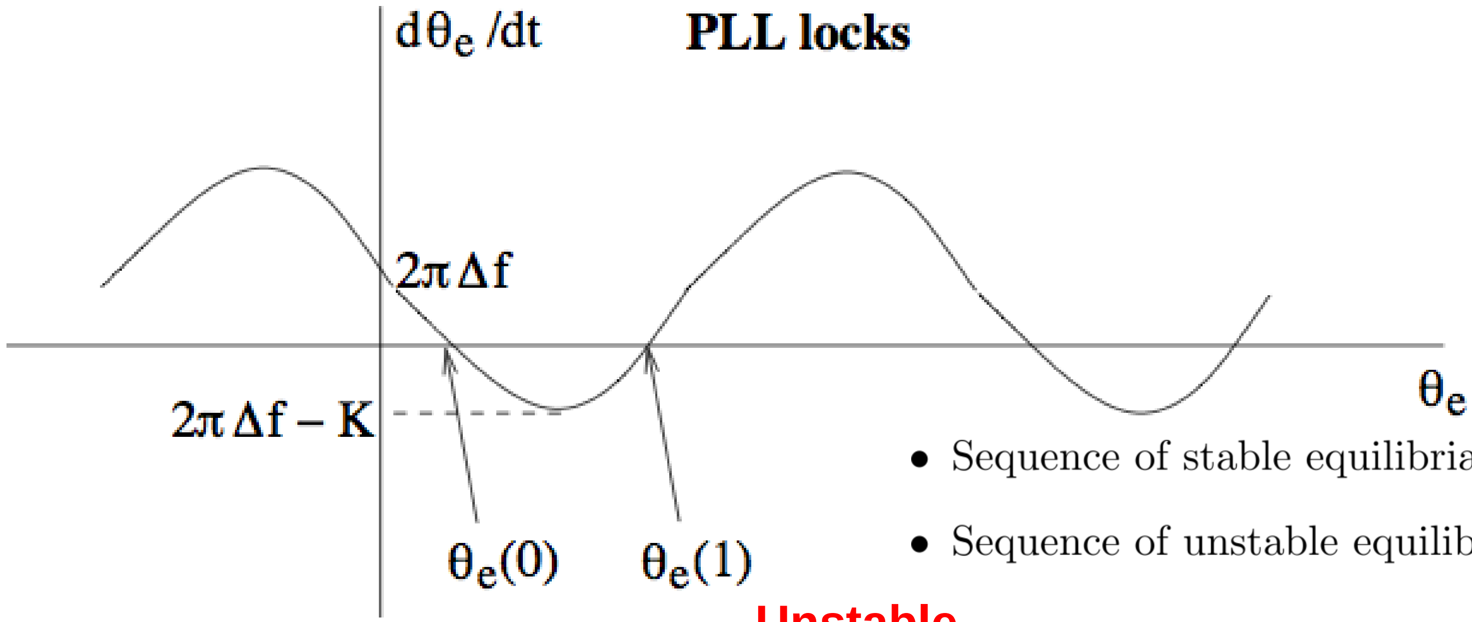


PLL locking

- Find the values of θ_e for which $\frac{d\theta_i}{dt} = 0$

$$2\pi\Delta f \leq K$$

- Investigate their stability.



- Sequence of stable equilibria spaced by 2π
- Sequence of unstable equilibria spaced by 2π

Stable equilibrium

Unstable equilibrium

Derivative and error have opposite signs around this, so error is driven back towards zero as it deviates from this equilibrium

Derivative and error have same sign around this so error is driven away from zero as it deviates from this equilibrium.

Nonlinear vs linearized PLL analysis

- Linearized analysis just says that the phase converges to a non-zero value, which gets larger with frequency step.

$$\theta_e(t) \to \frac{2\pi\Delta f}{K}$$

- For first order PLL, non-linear analysis tells us that the phase does not converge if the frequency step is too large. while there is equilibrium if the frequency step is small enough.

- Linearized analysis gives accurate answers when things are going well, but may miss critical behavior

EC5.203 Communication Theory (3-1-0-4):

Lecture 12:
Digital Modulation

03 March 2025

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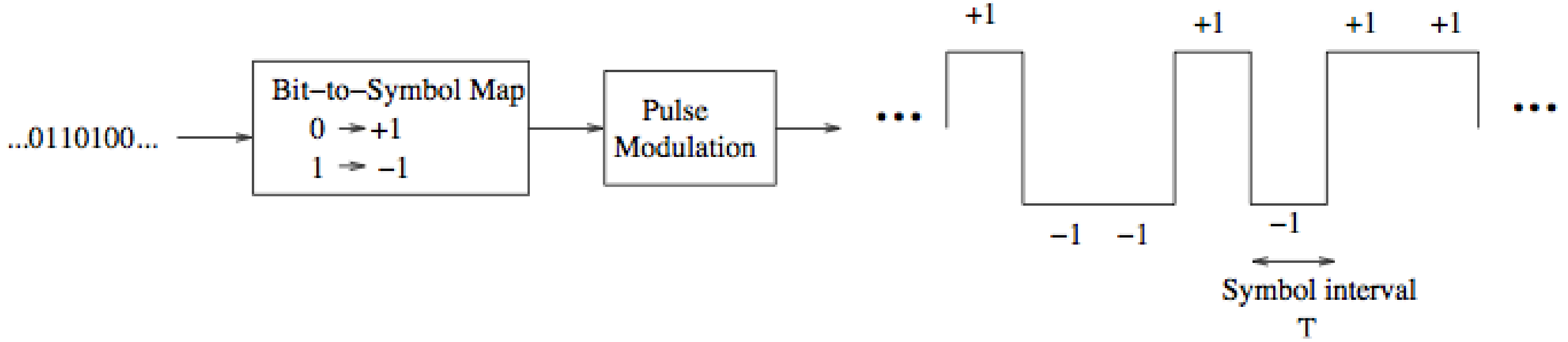
Digital modulation

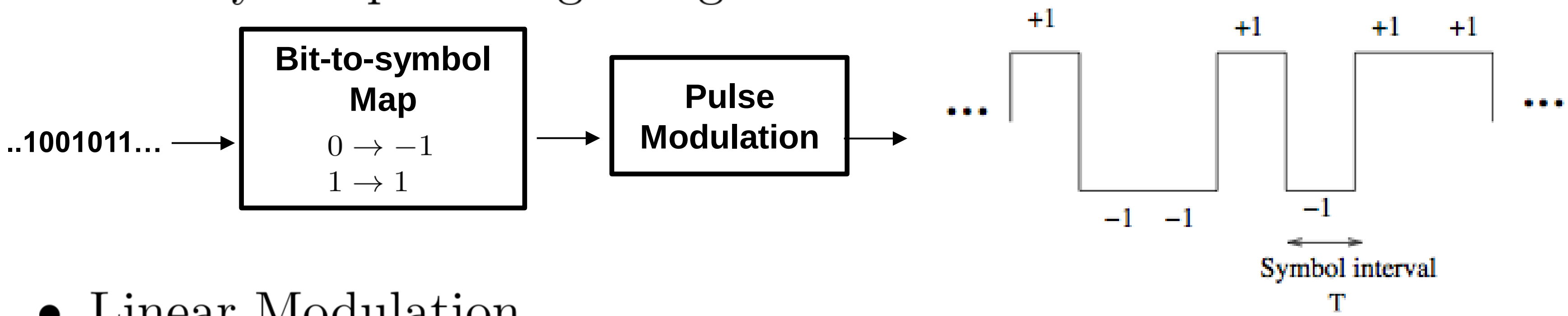
- Digital modulation is the process of translating bits to analog wave-forms that can be sent over a physical channel.

- Baseband example: Binary antipodal Signaling



Digital modulation: baseband example

- Binary antipodal Signaling



- Linear Modulation

$$u(t) = \sum_n b[n]p(t - nT)$$

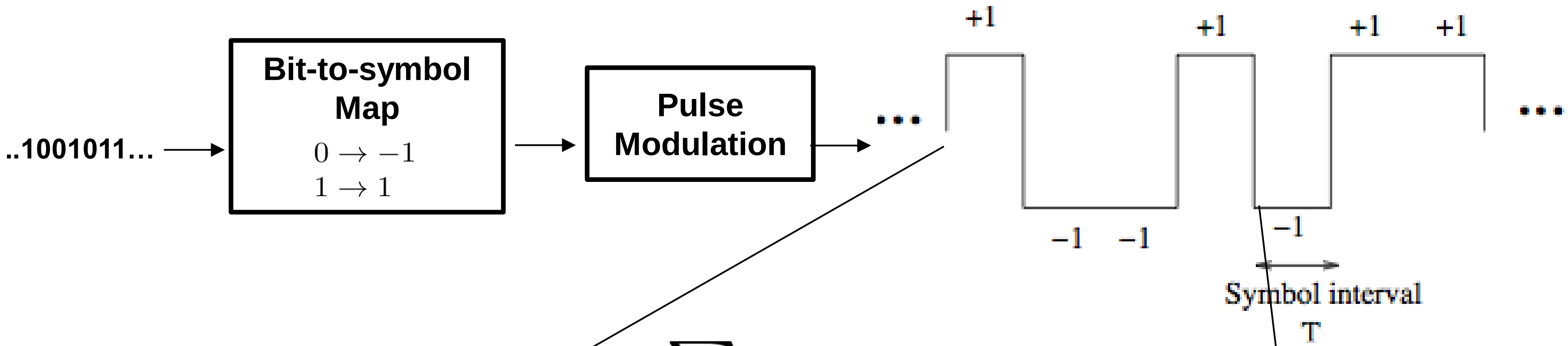
where $\{b[n]\}$ is sequence of symbols and $p(t)$ is modulating pulse for T seconds. For this example

$$p(t) = I_{[0,T]}(t)$$

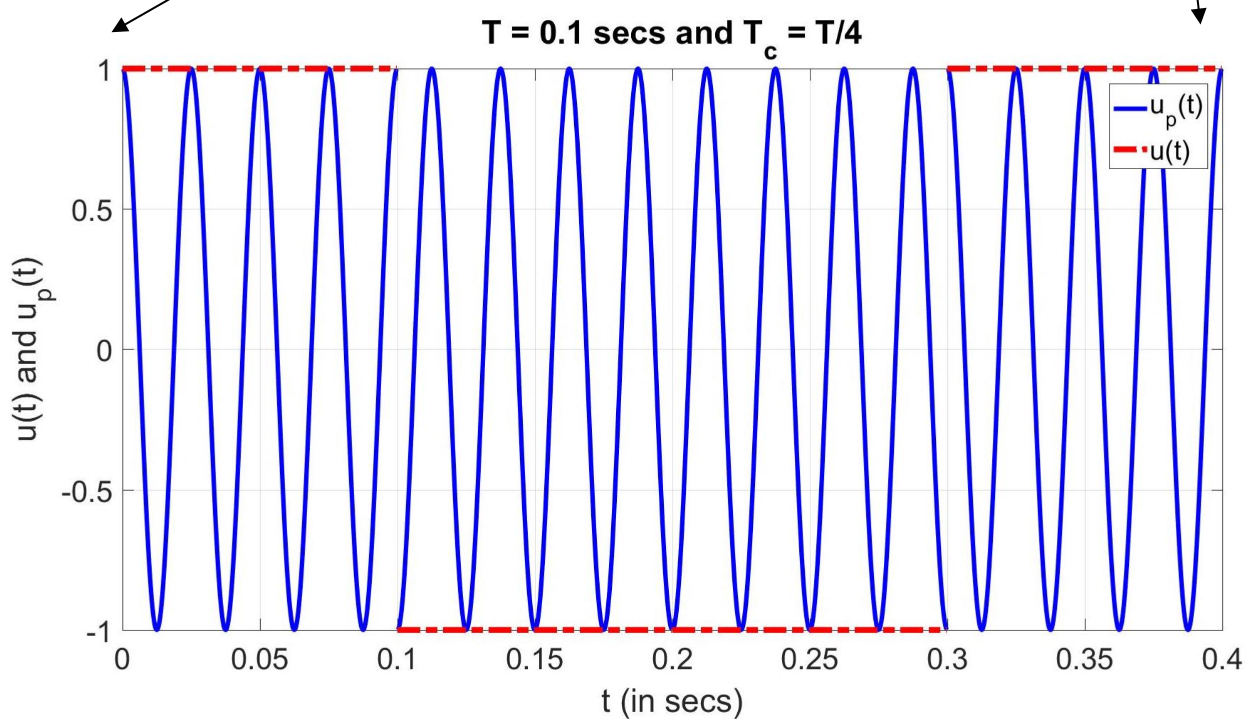
is rectangular window in time domain.

- Baseband signal sent directly over the physical baseband channel.

Digital modulation: passband example



$$u_p(t) = u(t)\cos(2\pi f_c t) = \sum_n b[n]p(t - nT)\cos(2\pi f_c t)$$



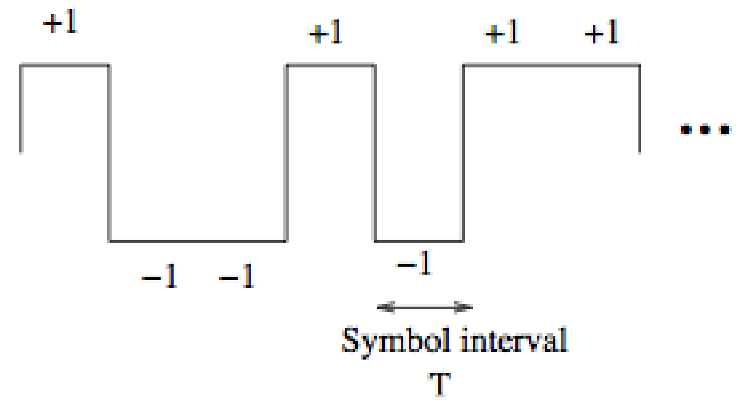
BPSK (also called 2-PAM)

Baseband

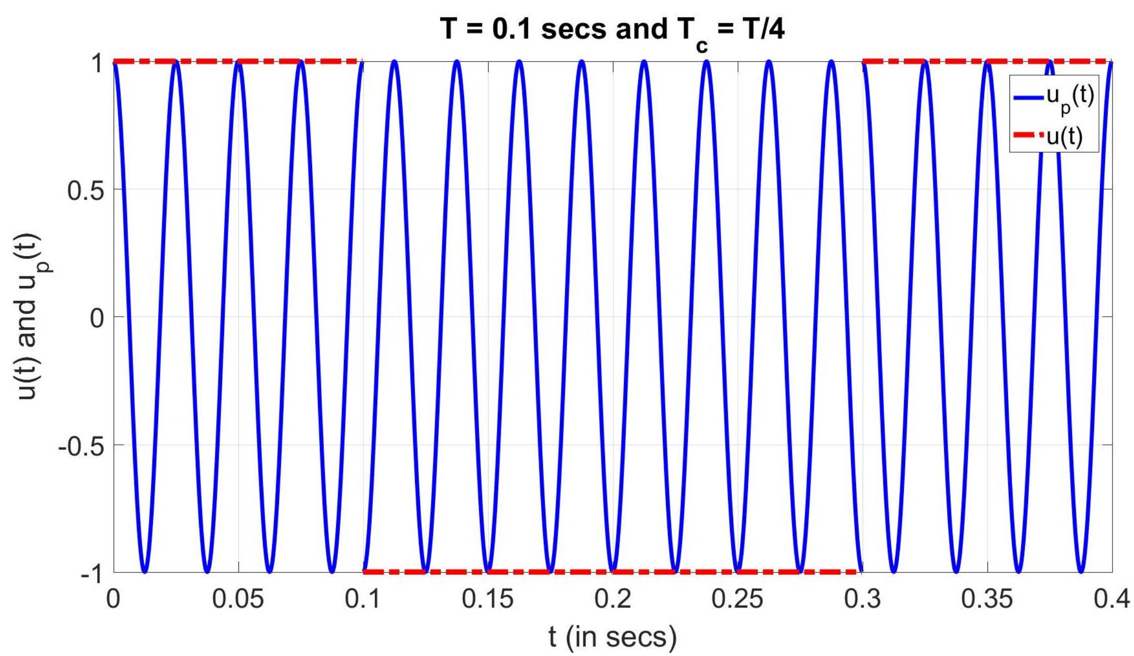
$\{b[n]\}$
1 0 0 1 0 1 1 $\longrightarrow$

$$\boxed{\begin{matrix} 0 \to -1 \\ 1 \to 1 \end{matrix}}$$

$\longrightarrow$



$$u_p(t) = u(t)\cos(2\pi f_c t) = \sum_n b[n]p(t - nT)\cos(2\pi f_c t)$$

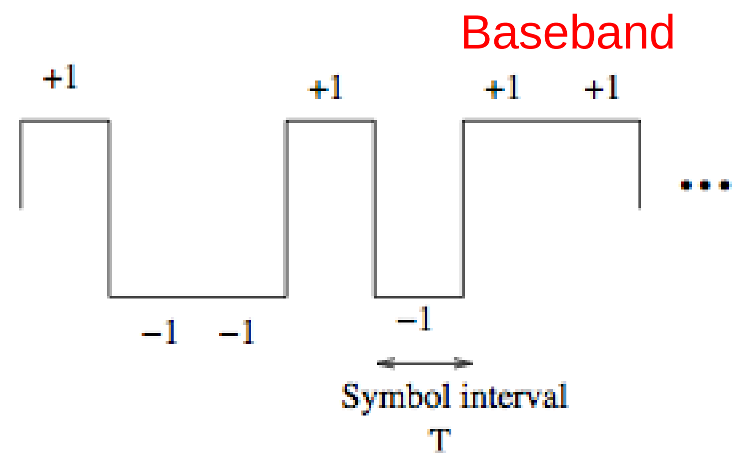


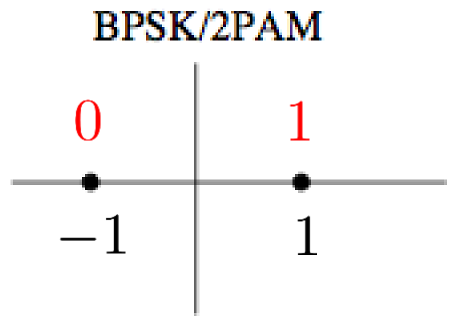
Passband

Signal Constellation for BPSK

$\{b[n]\}$

1 0 0 1 0 1 1 $\longrightarrow$ $\boxed{\begin{array}{c} 0 \rightarrow -1 \\ 1 \rightarrow 1 \end{array}}$ $\longrightarrow$





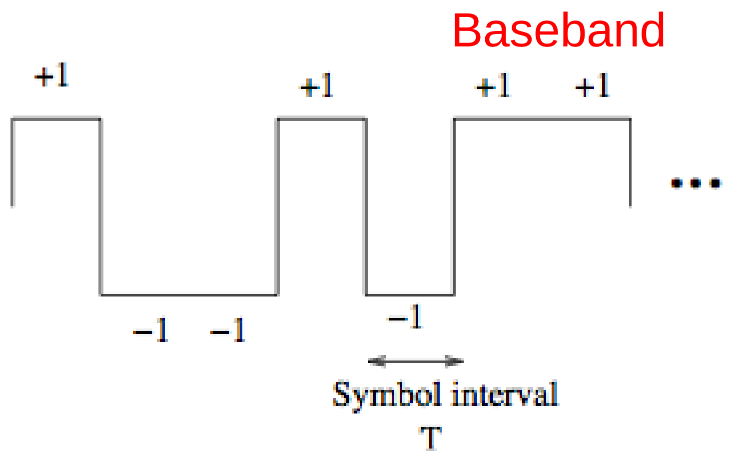
- *Constellation* or *Alphabet*: The set of values each symbol can take

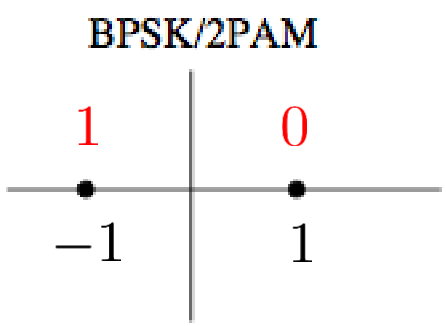
BPSK (also called 2-PAM)

- For BPSK, it does not matter which bits maps to which phase.

$\{b[n]\}$
0 1 1 0 1 0 0

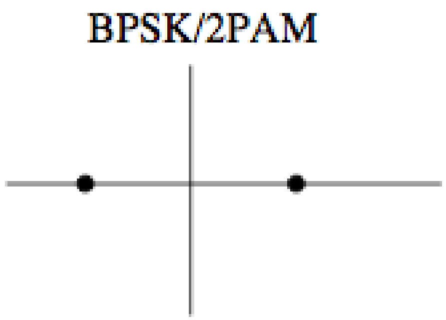
$$0 \to 1$$
$$1 \to -1$$



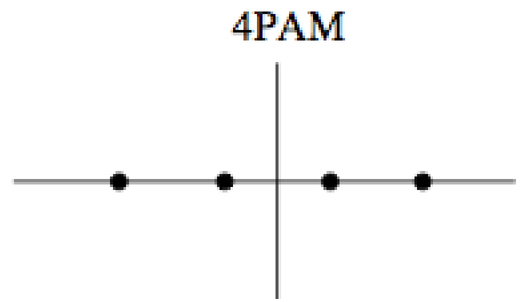


Can we load more bits per symbol?

1 bit per symbol of T secs

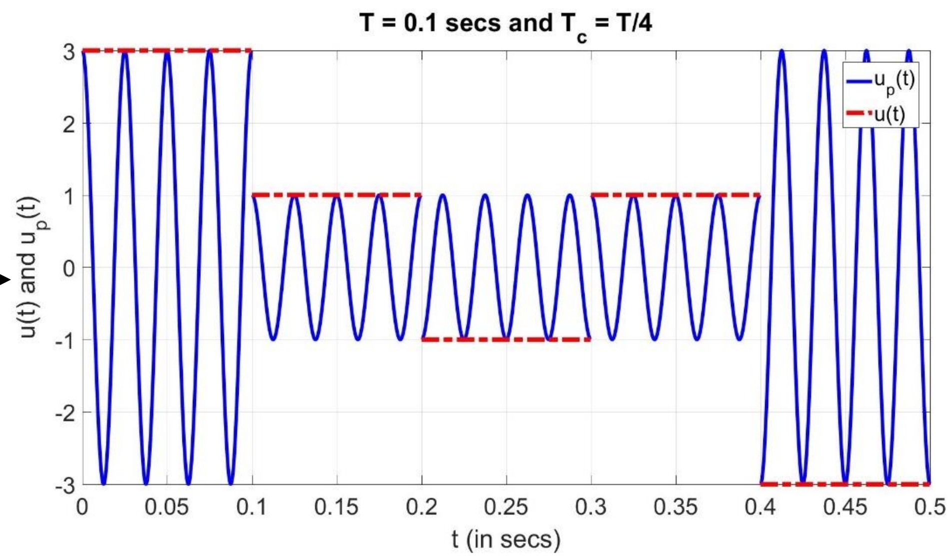


2 bits per symbol



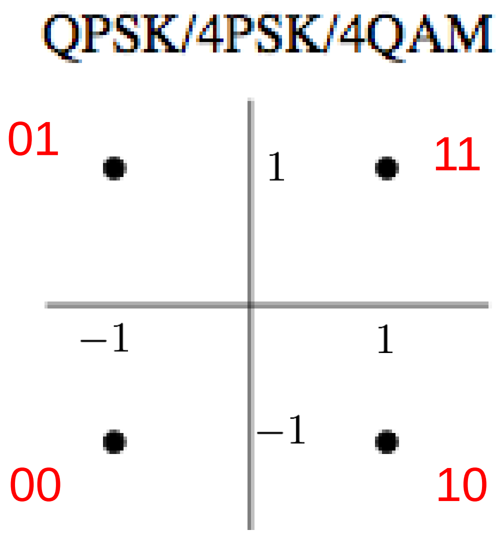
1 1 1 0 0 1 1 0 0 0

$$
\begin{aligned}
00 &\to -3\\
01 &\to -1\\
10 &\to +1\\
11 &\to +3
\end{aligned}
$$

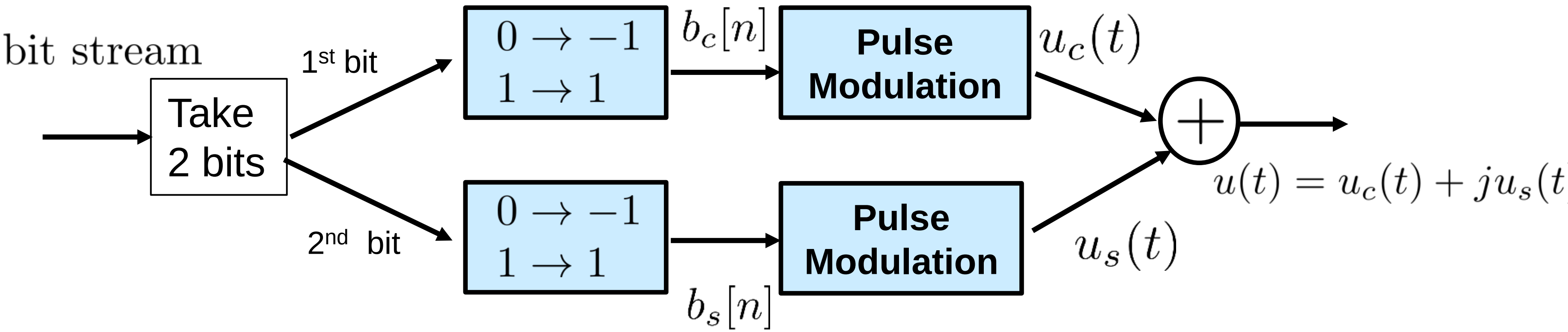


Another way for 2 bits/symbol: QPSK

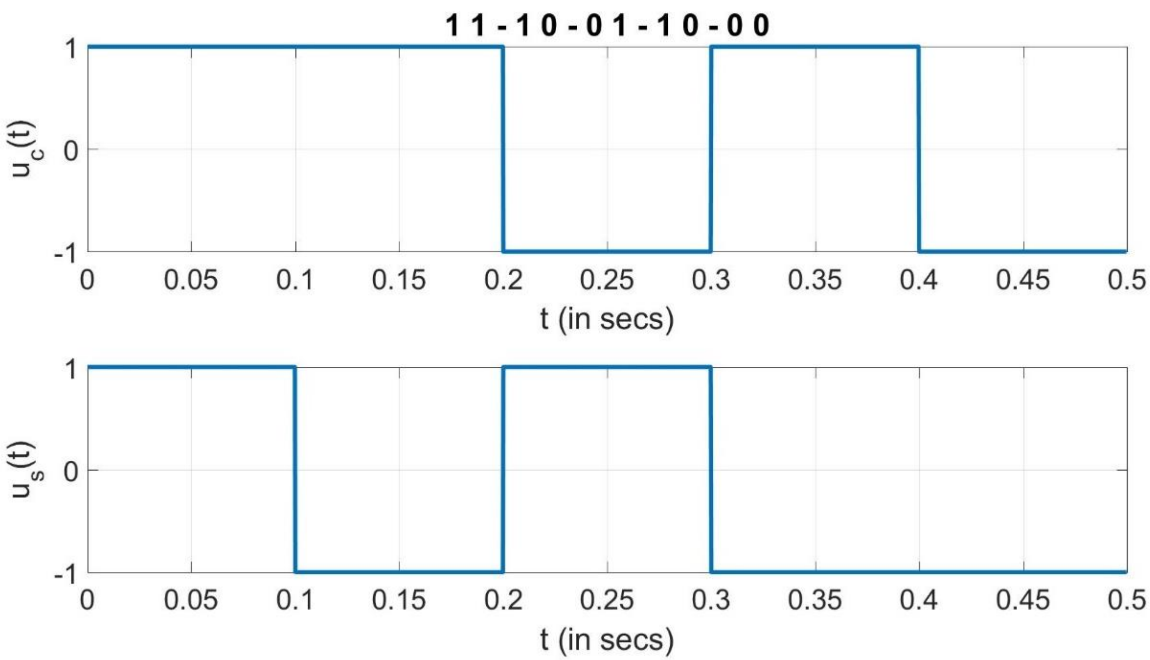
- Load Q component also giving rise to QAM and PSK modulation schemes



QPSK baseband



$$b[n] = b_c[n] + jb_s[n]$$



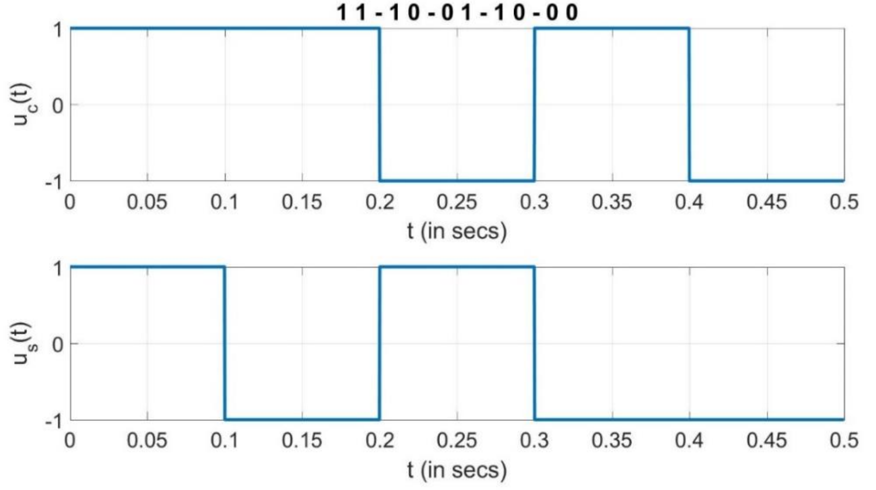
$$u_c(t) = \sum_n b_c[n]p(t - nT)$$

$$u_s(t) = \sum_n b_s[n]p(t - nT)$$

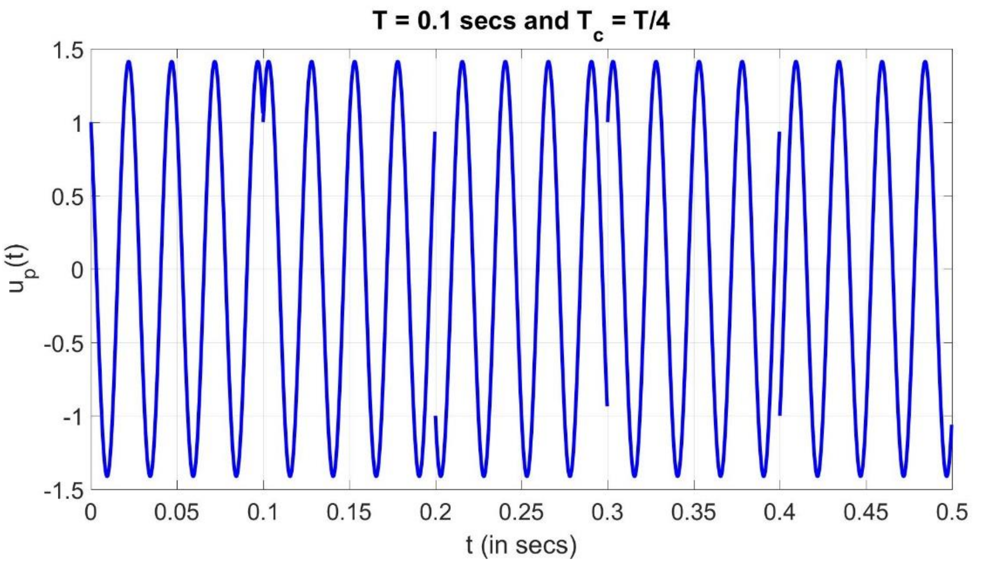
- For single physical baseband channel, QPSK is not possible; simply set $b_s[n] = 0$

QPSK: Passband

1 1 1 0 0 1 1 0 0 0



$$u_p(t) = \Re\{u(t)e^{j2\pi f_c t}\} = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$



QPSK: Passband

$$u_p(t) = \Re\{u(t)\cos(2\pi f_c t)\} = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$$

$$u(t) = u_c(t) + ju_s(t) = \pm 1 \pm j1 \quad nT < t < (n+1)T$$

- When $b_c[n] = 1$ and $b_s[n] = 1$:

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t + \pi/4)$$

- When $b_c[n] = -1$ and $b_s[n] = 1$:

$$u_p(t) = -u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t + 3\pi/4)$$

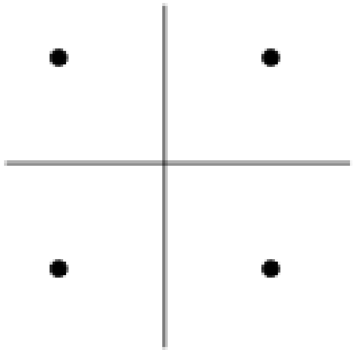
- When $b_c[n] = -1$ and $b_s[n] = -1$:

$$u_p(t) = -u_c(t)\cos(2\pi f_c t) + u_s(t)\sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t - 3\pi/4)$$

- When $b_c[n] = 1$ and $b_s[n] = -1$:

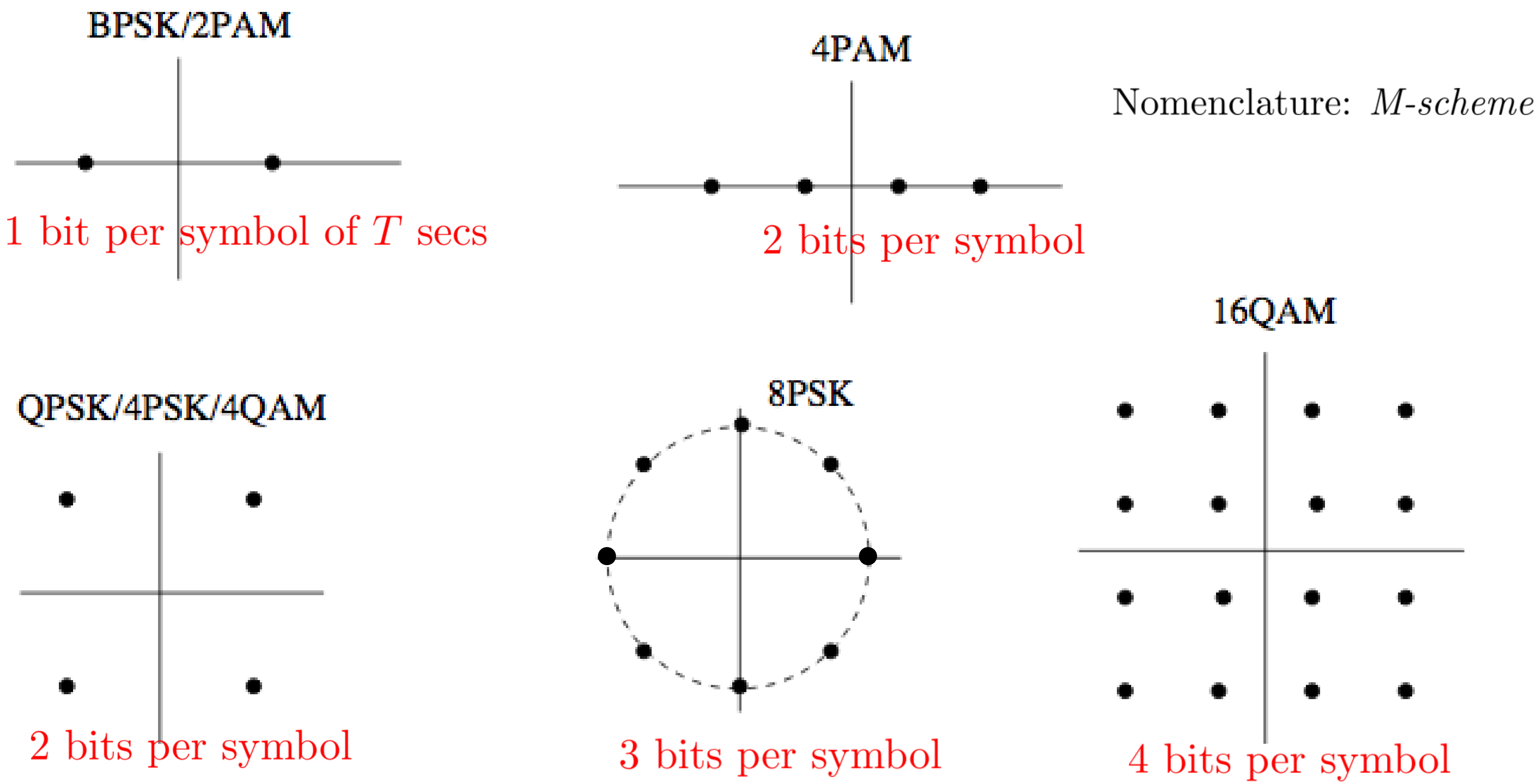
$$u_p(t) = u_c(t)\cos(2\pi f_c t) + u_s(t)\sin(2\pi f_c t) = \sqrt{2}\cos(2\pi f_c t - \pi/4)$$

QPSK/4PSK/4QAM



$$b[n] = \sqrt{2}e^{j\theta[n]}$$
$$\theta[n] \in \{\pm\pi/4, \pm 3\pi/4\}$$

Can we load even more bits?

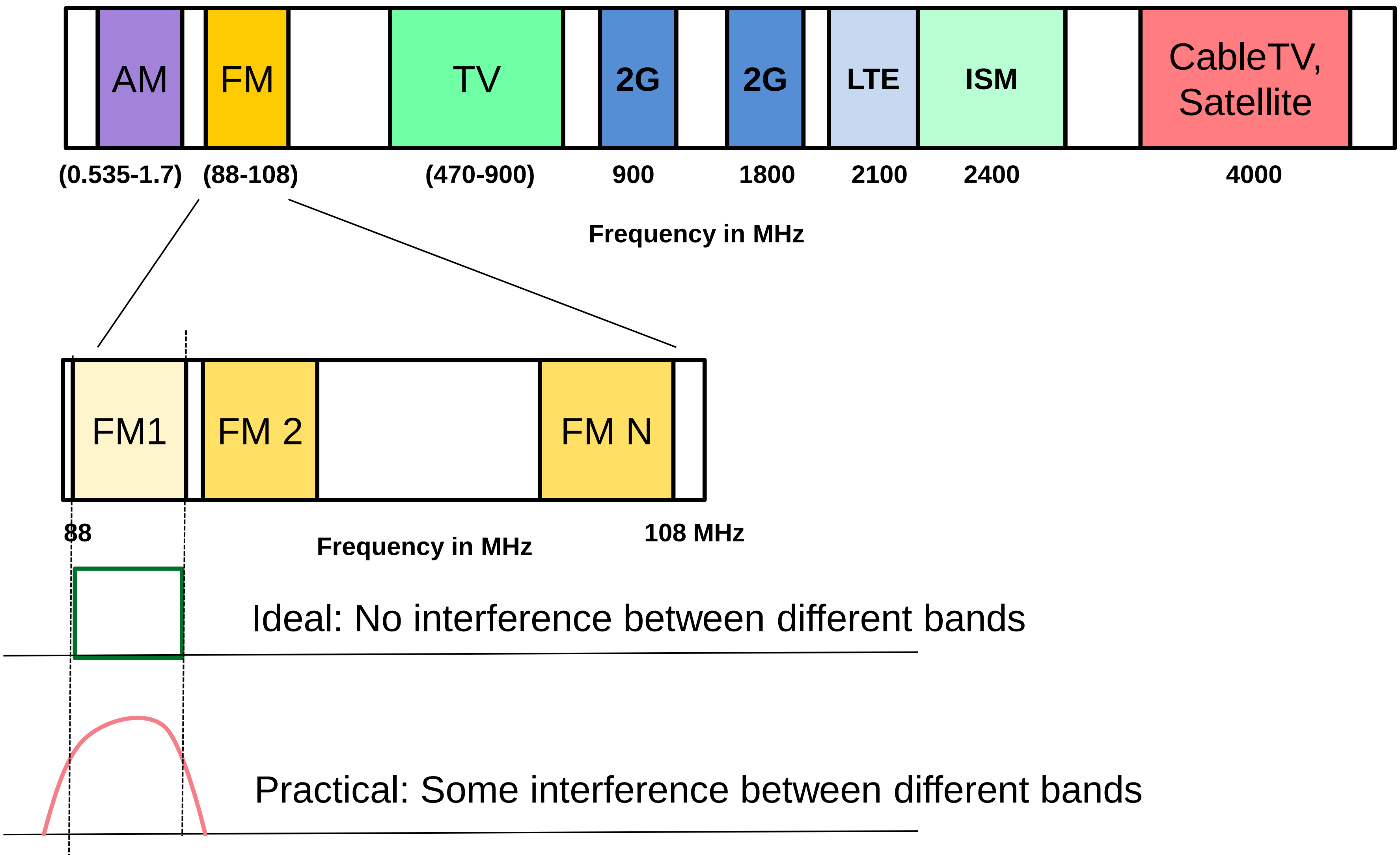


COMMON CONSTELLATIONS

- In general, M-ary modulation scheme can transmit $\log_2 M$ bits per symbol.
- Information rate $= \frac{\log_2 M}{T}$ bits/sec.

Bandwidth Occupancy

Motivation


AM
FM
TV
2G
2G
LTE
ISM
CableTV,
Satellite
(0.535-1.7)
(88-108)
(470-900)
900
1800
2100
2400
4000
Frequency in MHz
FM1
FM 2
FM N
88
108 MHz
Frequency in MHz
Ideal: No interference between different bands
Practical: Some interference between different bands

Modeling bandwidth occupancy

- Consider the complex envelope of a linearly modulated signal

$$u(t) = \sum_n b[n]p(t - nT)$$

  where $\{b[n]\}$ is sequence of symbols and $p(t)$ is modulating pulse for T seconds.

- $\{b[n]\}$ is modeled as random at the transmitter as well as receiver.

- However for characterizing the bandwidth occupancy of digitally modulated signal u, we define the quantities of interest in terms of average across time.

- We treat $u(t)$ as a finite power signal that can be modeled as a deterministic sequences once $\{b[n]\}$ is fixed.

- Bandwidth is then defined in terms of power spectral density.

Power Spectral Density (PSD)

- Power spectral density $S_x(f)$ for signal $x(t)$ specifies how the power in a signal is distributed in different frequency bands.
- PSD is defined as power per unit frequency.
- Units of watts/hertz or joules (since power/frequency = energy)
- Total power in $x(t)$ is given by

$$P_x = \int_{-\infty}^{\infty} S_x(f)\, df = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{T_0}^{T_0} |x(t)|^2\, dt$$

- PSD is generally used for random or periodic signals.
  - PSD for random process: We are interested in modeling the complete random process instead of a particular realization of it in a given time.

Periodogram-based PSD estimation

- Limit the signal $x(t)$ to a finite observation interval
$$x_{T_0} = x(t) I_{[-T_0/2, T_0/2]}(t)$$
where T_0 is the length of the observation interval. Since T_0 is finite and $x(t)$ has finite power, $x_{T_0}(t)$ has finite energy. So its Fourier transform is given by
$$X_{T_0} = \mathcal{F}(x_{T_0}(t))$$
- The energy spectral density of x_{T_0} is given by $|X_{T_0}(f)|^2$.
- Therefore the estimated PSD is given by
$$\hat{S}_x(f) = \frac{|X_{T_0}(f)|^2}{T_0}$$
- Formally, the PSD is in the limit of large time windows as follows
$$\hat{S}_x(f) = \lim_{T_0 \to \infty} \frac{|X_{T_0}(f)|^2}{T_0}$$

PSD of Linearly Modulated Signal

- Theorem 4.2.1: Consider a linearly modulated signal where the symbol sequence $\{b[n]\}$ is zero mean and uncorrelated with average symbol energy

$$\sigma_b^2 = \overline{|b[n]|^2} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |b[n]|^2,$$

then the PSD is given by

$$S_u(f) = \frac{|P(f)|^2}{T} \sigma_b^2$$

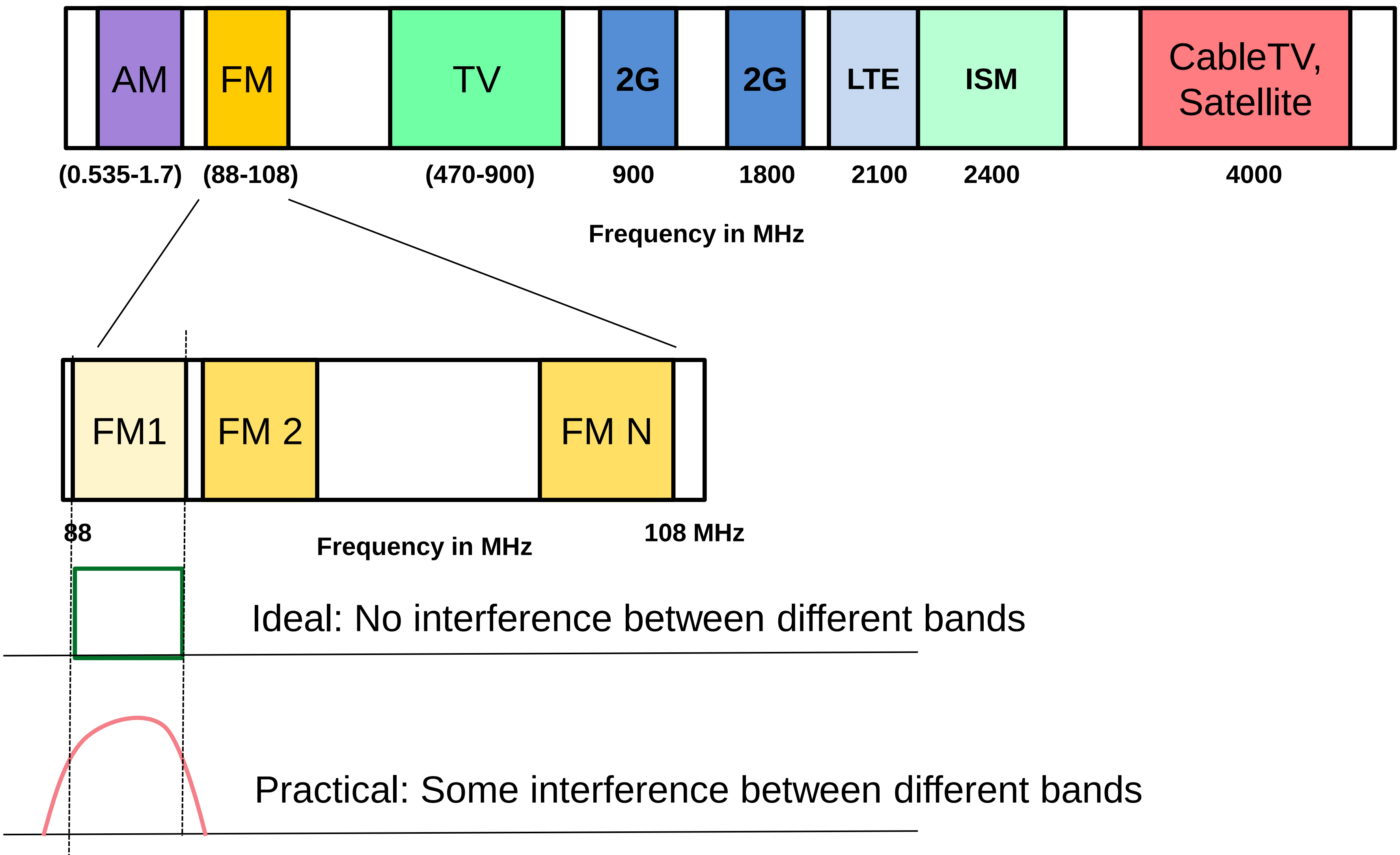
and the power of the modulated signal is

$$P_u = \frac{\sigma_b^2 ||p||^2}{T}$$

where $||p||^2$ denotes the energy of the modulating pulse. Proof.

- Assumptions:
  - The symbols have zero DC value: $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} b[n] = 0$.
  - The symbols are uncorrelated: $\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} b[n]b^*[n-k] = 0 \quad \text{for } k \neq 0$.

Motivation for Bandwidth Occupancy



* Intuitive interpretation: Every T time units, we

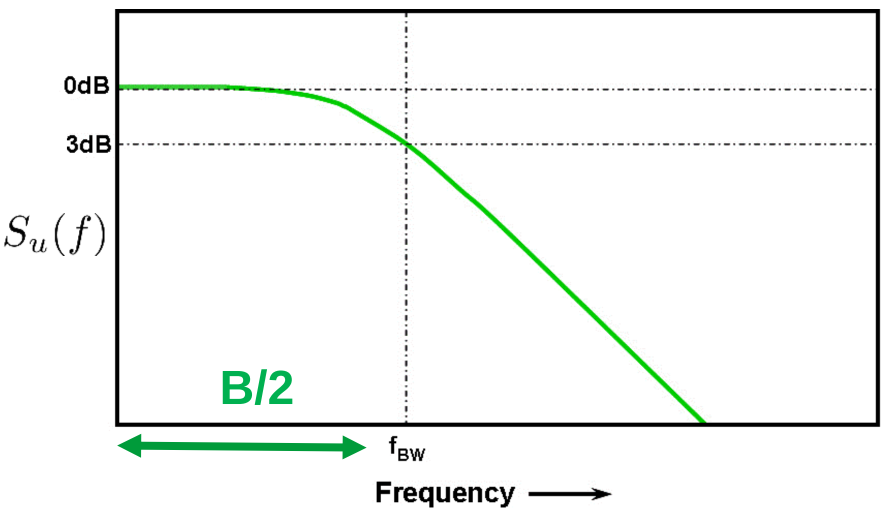
send a pulse of form $b[n]p(t-nT)$ with average energy spectral density $\sigma_b^2 |P(f)|^2$ so that the PSD is obtained as ESD$/T$

* Same reasoning applies for power in time domain. Every T time units, we send a pulse $b[n]p(t-nT)$ with average $\sigma_b^2 \|p\|^2$ so that the PSD is $\dfrac{\sigma_b^2 \|p\|^2}{T}$

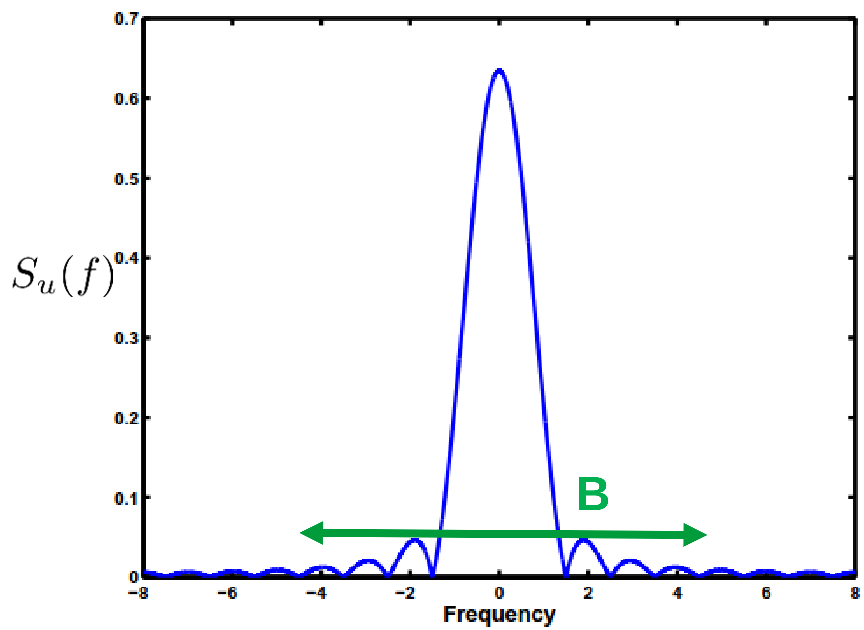
Bandwidth based on PSD

- 3-dB bandwidth
- Fractional power-containment bandwidth: This is the smallest interval that contains a given fraction of the power

$$\int_{-B/2}^{B/2} S_u(f)df = \gamma P_u = \gamma \int_{-\infty}^{\infty} S_u(f)df$$



$S_u(B_{3dB}/2) = S_u(-B_{3dB}/2) = S_u(0)/2$



We will focus mostly on this!

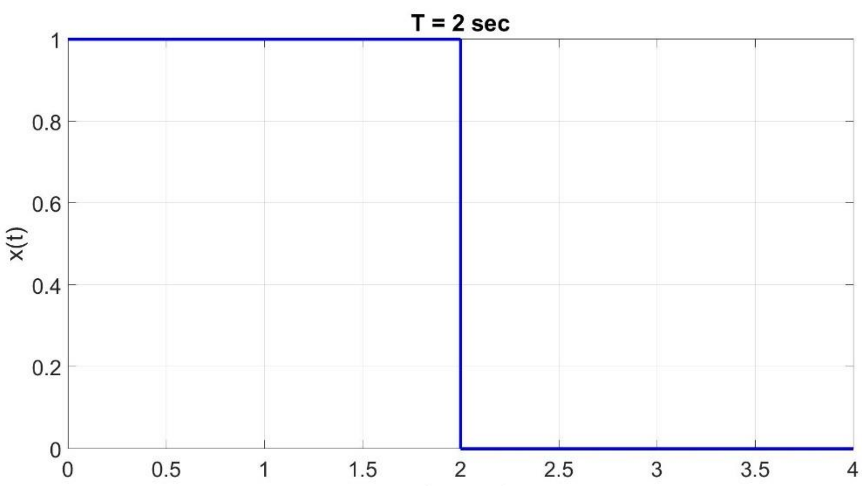
Time Frequency Normalization: *Time Domain*

- If we are sending one symbol every T time units, then the symbol rate is $1/T$ in units of symbols/time unit.

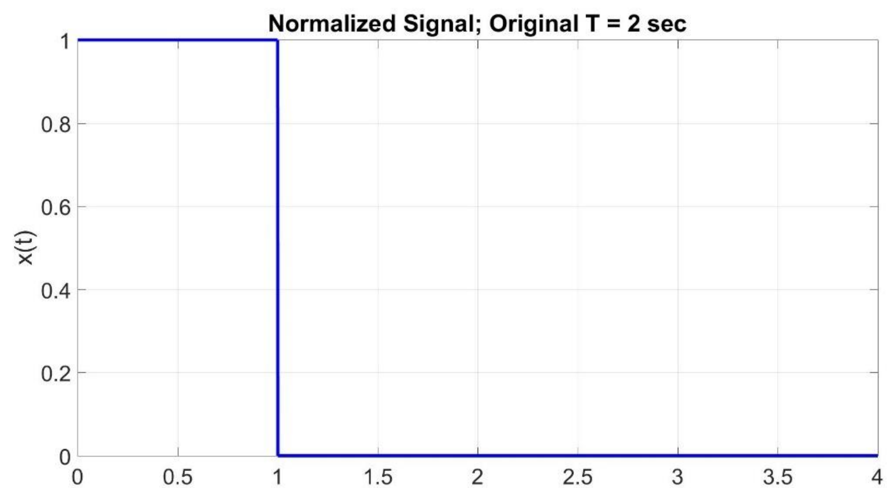
- If we normalize the system for the symbol rate of 1, where the unit of time is T. This implies unit of frequency is $1/T$. In terms of new unit, the linearly modulated signals can be written as

$$u_1(t) = \sum_n b[n]p_1(t-n)$$

where $p_1(t)$ is the modulation pulse for the normalized system.



t in secs

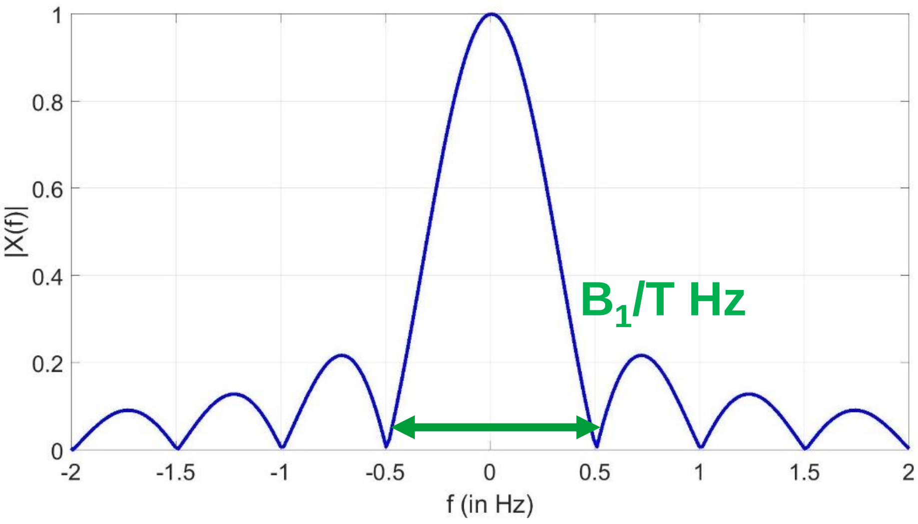


$t_0 = t/T$ Time Unit = T secs

Time Frequency Normalization: *Freq. Domain*

- Let us denote B_1 to be the bandwidth of the normalized system, then the bandwidth of the original system is B_1/T.
- In terms of determining the bandwidth occupancy, we can work, without loss of generality, with the normalized system.
- In essesence, we are working in the original system with the normalized time domain t/T and normalized frequency fT.

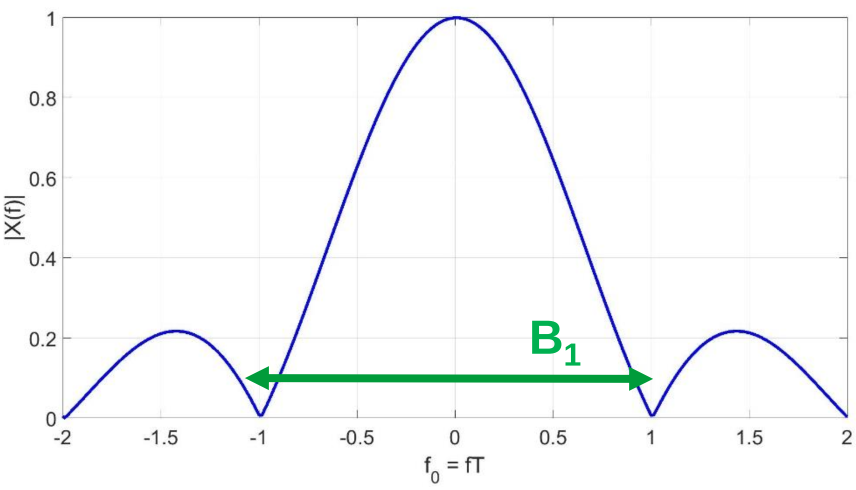
$$|P(f)| = |\mathrm{sinc(fT)}|$$



f

Frequency unit = 1 cycle/sec

$$|P(f)| = |\mathrm{sinc(f)}|$$ **normalized**



f_0 = fT

Frequency unit = 1/T

Bandwidth computation for Rectangular Pulse

- Consider normalized system with $p_1(t) = I_{[0,1]}(t)$ for which

$$P_1(f) = \text{sinc}(f)e^{j\pi f}$$

- For $\{b[n]\}$ iid, taking values between ± 1 with equal probability, $\sigma_b^2 = 1$, we get $S_{u_1}(f) = \text{sinc}^2(f)$. Real and Positive

- For a fractional power-containment bandwidth with fraction γ

$$\int_{-B_1/2}^{B_1/2} S_{u_1}(f)\,df = \int_{-B_1/2}^{B_1/2} \text{sinc}^2(f)\,df = \gamma \int_{-\infty}^{\infty} \text{sinc}^2(f)\,df$$

$$= \gamma \int_0^1 1^2(t)\,dt = \gamma \qquad \text{Parseval}$$

$$\int_0^{B_1/2} S_{u_1}(f)\,df = \gamma/2 \qquad \text{Symmetry of PSD}$$

- For $\gamma = 0.99$, we obtain $B_1 = 10.2$ while for For $\gamma = 0.9$, we obtain $B_1 = 0.85$.

Bandwidth Computation: *Example*

- Consider a passband system operating at a carrier frequency of 2.4 GHz at a bit-rate of a 20 Mbps. A rectangular modulation pulse timelimited to the symbol interval is employed.
  - Find the 99% and 90% power-containment bandwidths if the constellation used is 16 QAM.
  - Find the 99% and 90% power-containment bandwidths if the constellation used is QPSK.

Poll!

- Consider the two statements:

  1. Transmission of $M_1 > 2$-PSK baseband signal is possible over a single physical channel
  2. Transmission of $M_2 > 4$-QAM passband signal is possible over a single physical channel

  Which of the statement is true?
  (a) Both are true (b) Both are false (c) First is false but second is true (d) First is true but second is false

Bandwidth computation for Sine Pulse

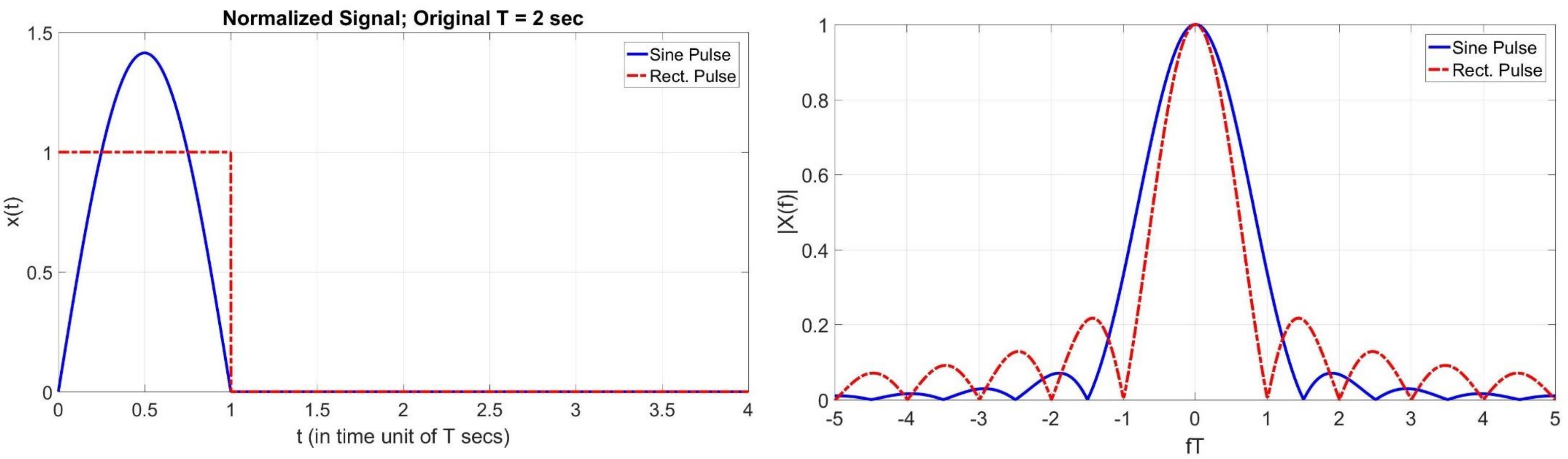
- Consider normalized sine pulse with $p_1(t) = \sqrt{2}\sin(\pi t) I_{[0,1]}(t)$ for which Fourier transform is given by

$$P_1(f) = \frac{2\sqrt{2}}{\pi}\frac{\cos(\pi f)}{1 - 4f^2}$$

- Corresponding PSD

$$S_{u_1}(f) = \frac{8}{\pi^2}\frac{\cos^2(\pi f)}{(1 - 4f^2)^2}$$

- Corresponding $\gamma = 0.99$, we obtain $B_1 = 1.2$.



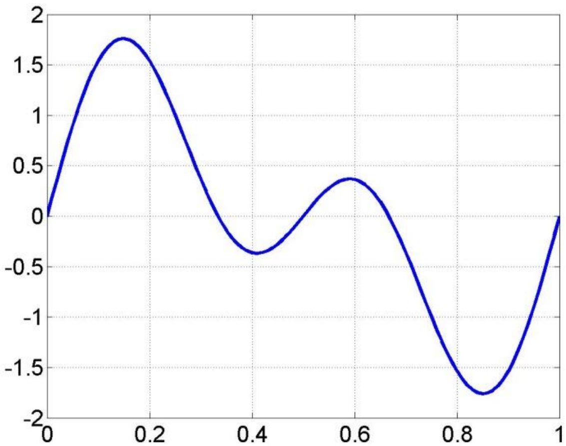
Design for Bandlimited Channels: Nyquist Sampling Criteria

Recap of S&S Oppenheim Chapter 7
[Not in syllabus]

Relation between CT and DT signals

$$x(t) \longrightarrow \boxed{\text{Sampling}} \longrightarrow x(nT_s) = x[n]$$

Continuous-time signal



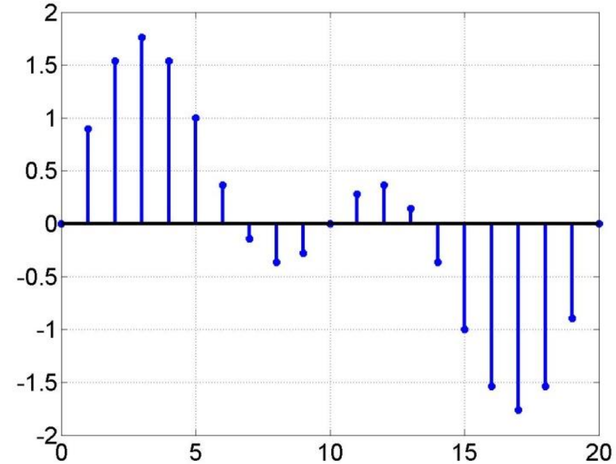
time t

Discrete-time sequence

Discrete-time signal

$x(nT_s)$

$x[n]$



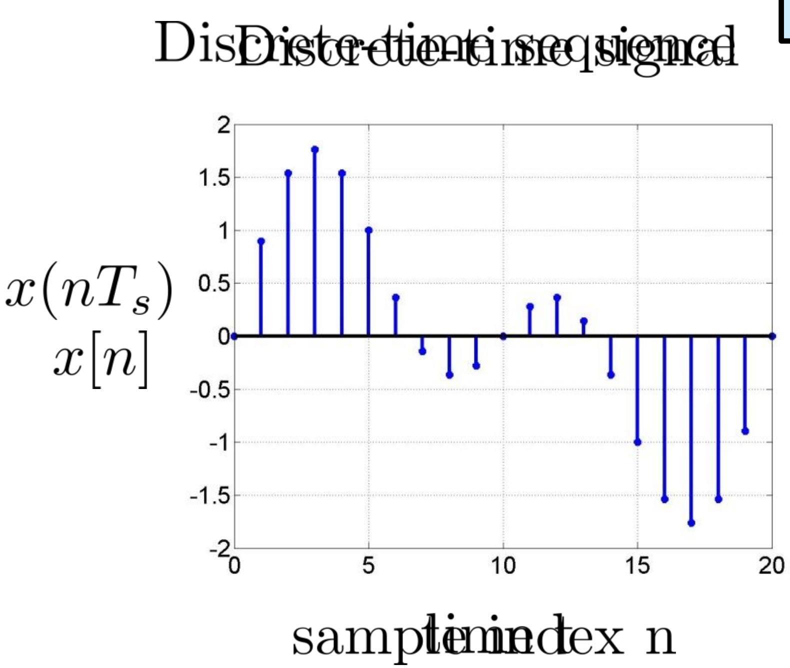
sample index n

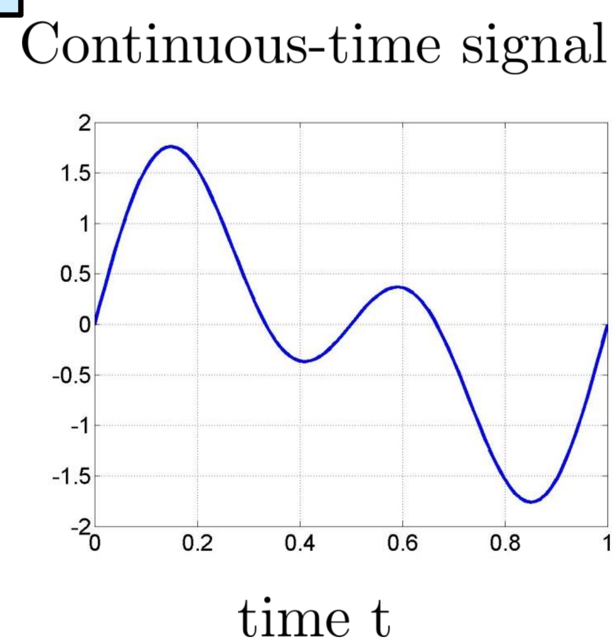
time t

Relation between CT and DT signals

- Reconstruction

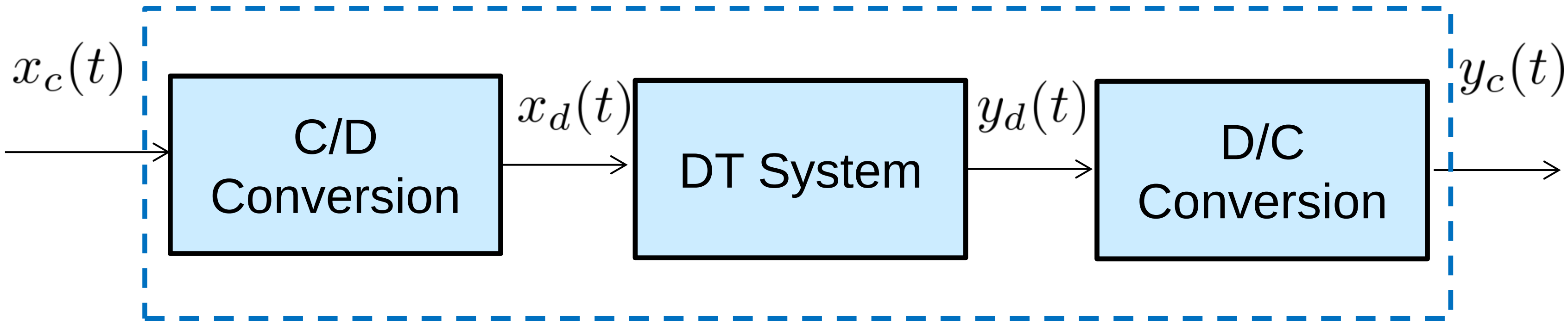
$$x(nT_s) = x[n] \longrightarrow \boxed{\text{Interpolation}} \longrightarrow x(t)$$



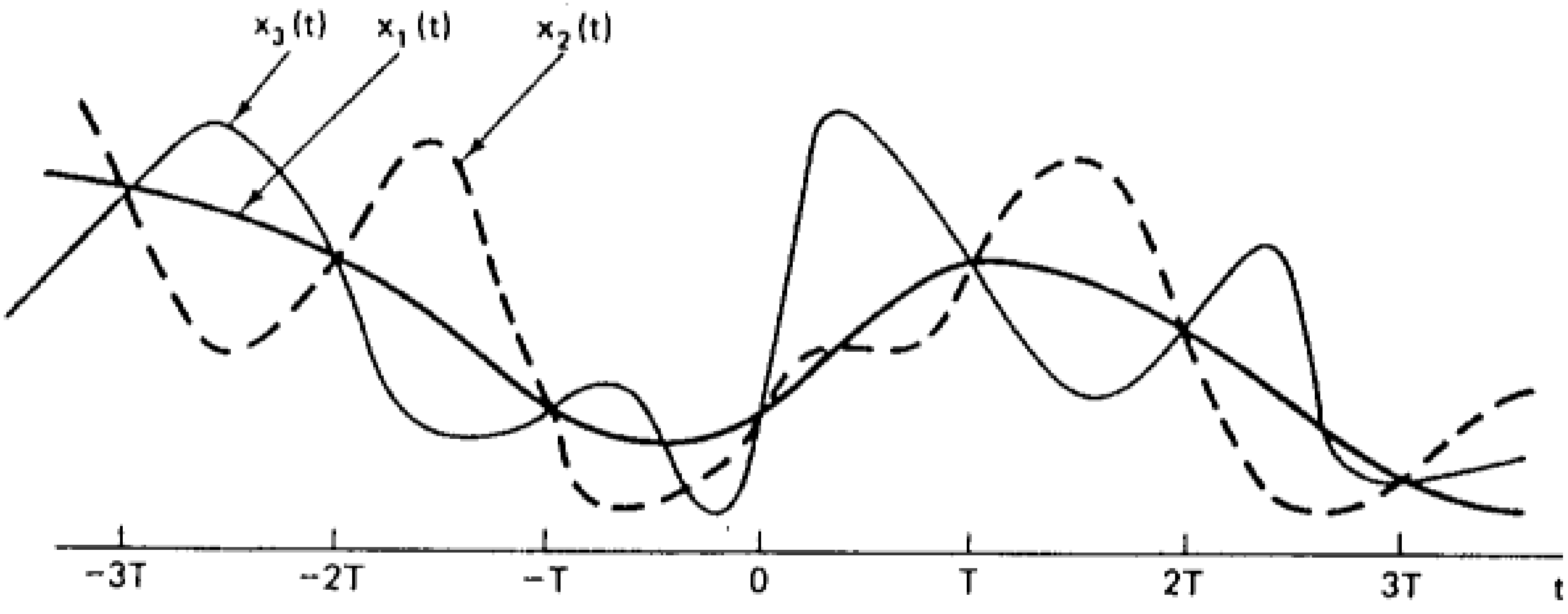


Importance of Sampling

- Bridge between CT and DT signals
- CT Signals can be represented by DT sequence (Sampling Theorem)
- DT signal processing of CT signals
  - DT systems are inexpensive, flexible, reliable, programmable, and efficient than CT systems



Issue with sampling

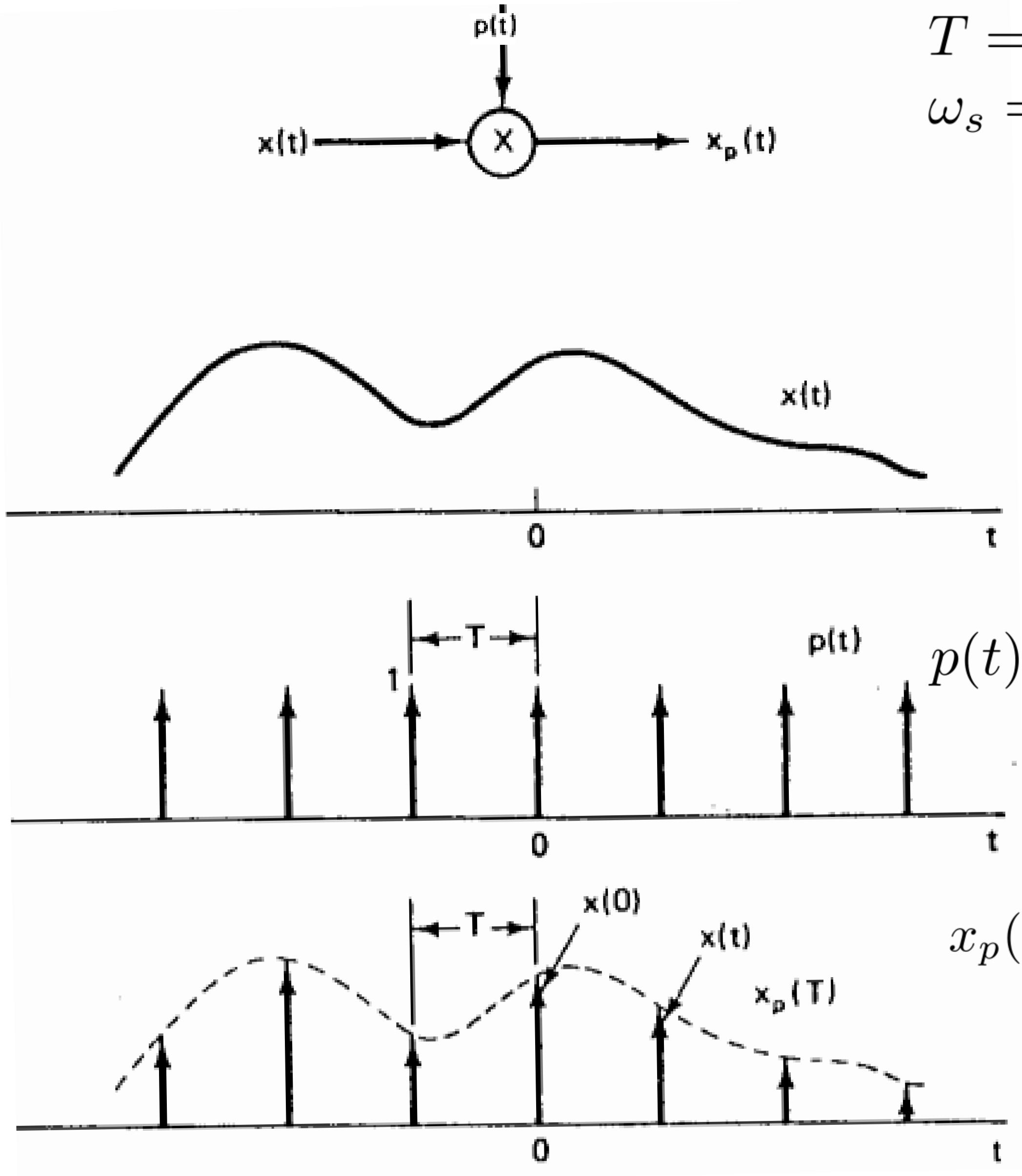


Sampling Theorem

- If $x(t)$ be a bandlimited signal with $X(j\omega) = 0$ for $\omega > \omega_M$, and
$$w_s > 2w_M$$
then $x(t)$ is uniquely determined by its samples $x(nT)$, for $n = 0, \pm 1, \pm 2, \ldots,$. Here $w_s = 2\pi/T$.

- To develop sampling theorem, we will have a look at impulse train sampling and reconstruction.

S&S Recap: *Impulse Train Sampling (Time)*



T = sampling time

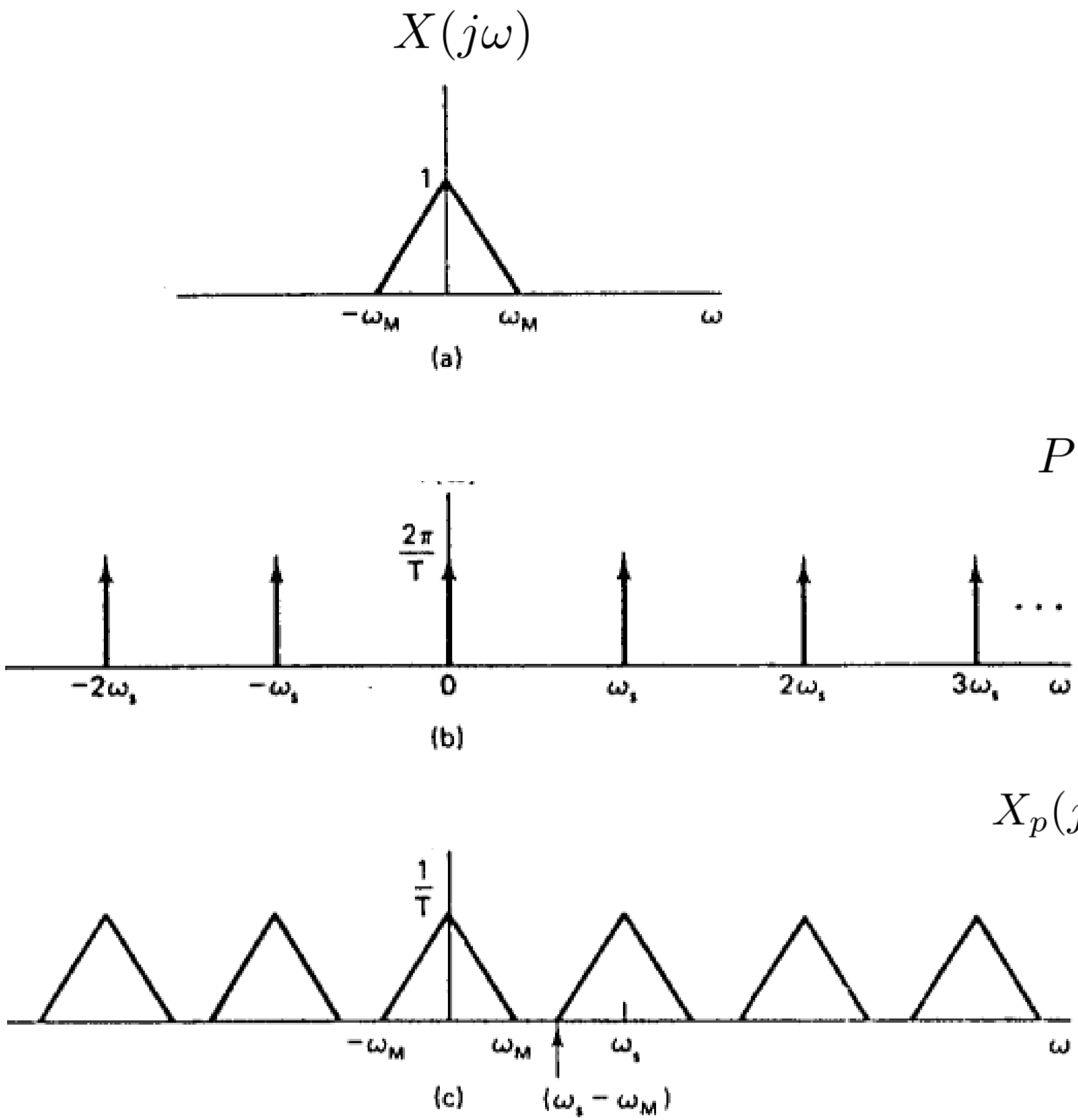
$\omega_s = 2\pi/T$ = sampling frequency

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

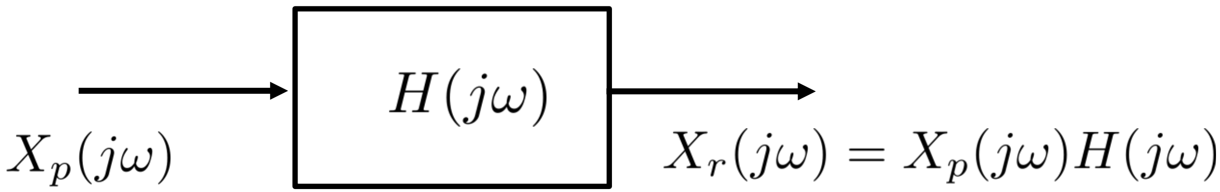
S&S Recap: *Impulse Train Sampling (Freq.)*

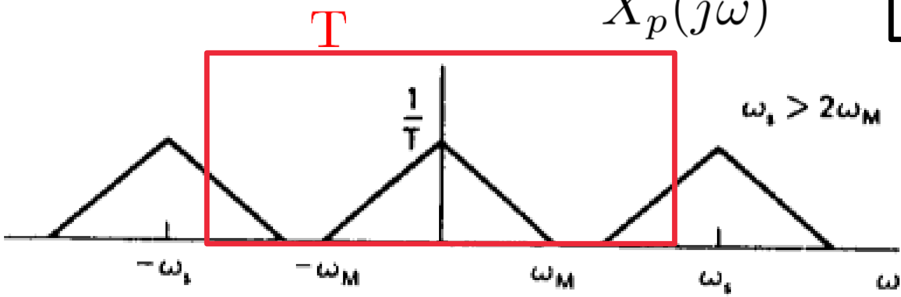


$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{k=\infty} \delta(\omega - k\omega_s)$$

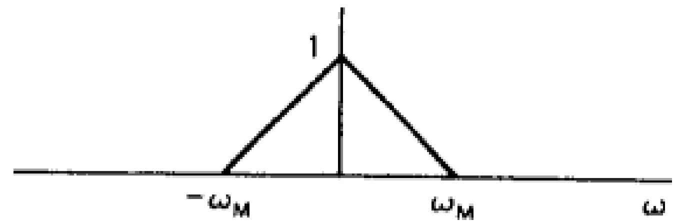
$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} X(j\omega) * P(j\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} P(j\theta) X(j(w - \theta))\, d\theta \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

Reconstruction of signal: Frequency Domain

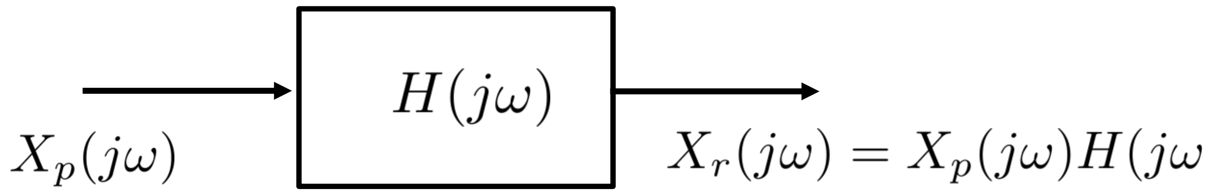


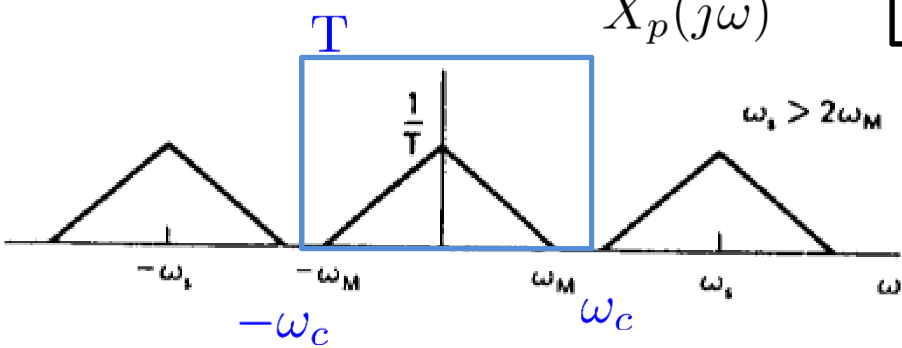


What should be $H(j\omega)$
if $X_r(j\omega) = X(j\omega)$?

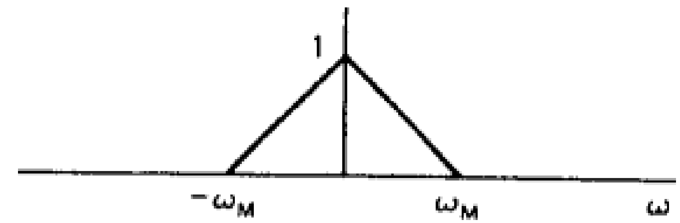


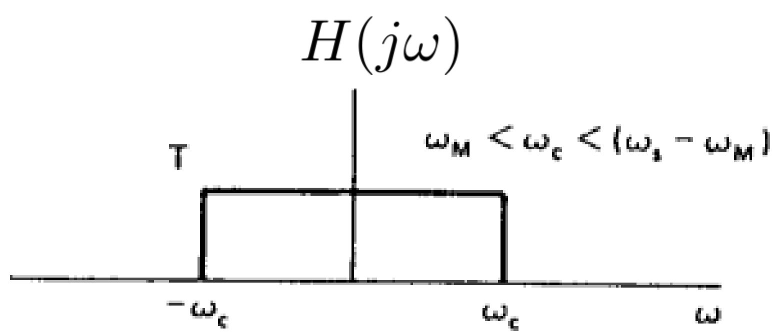
S&S Recap: *Frequency Domain*



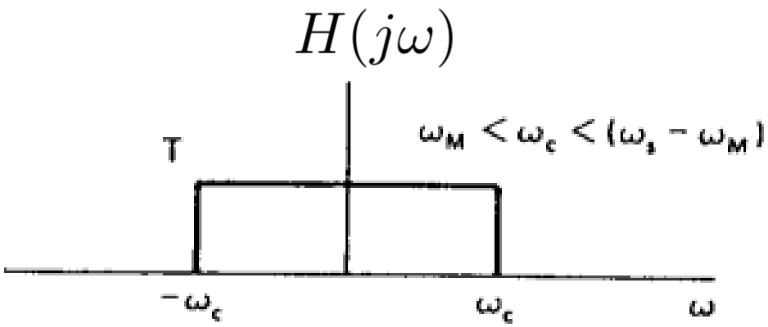


What should be $H(j\omega)$
if $X_r(j\omega) = X(j\omega)$?

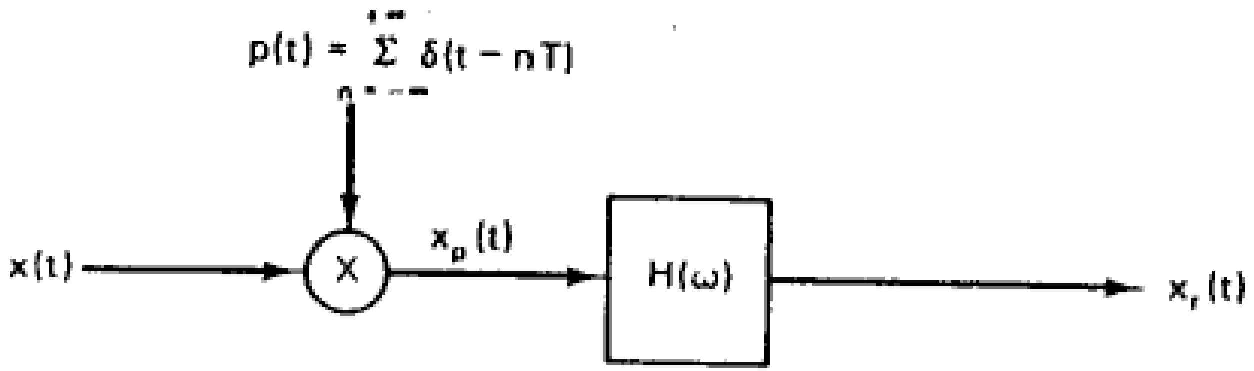




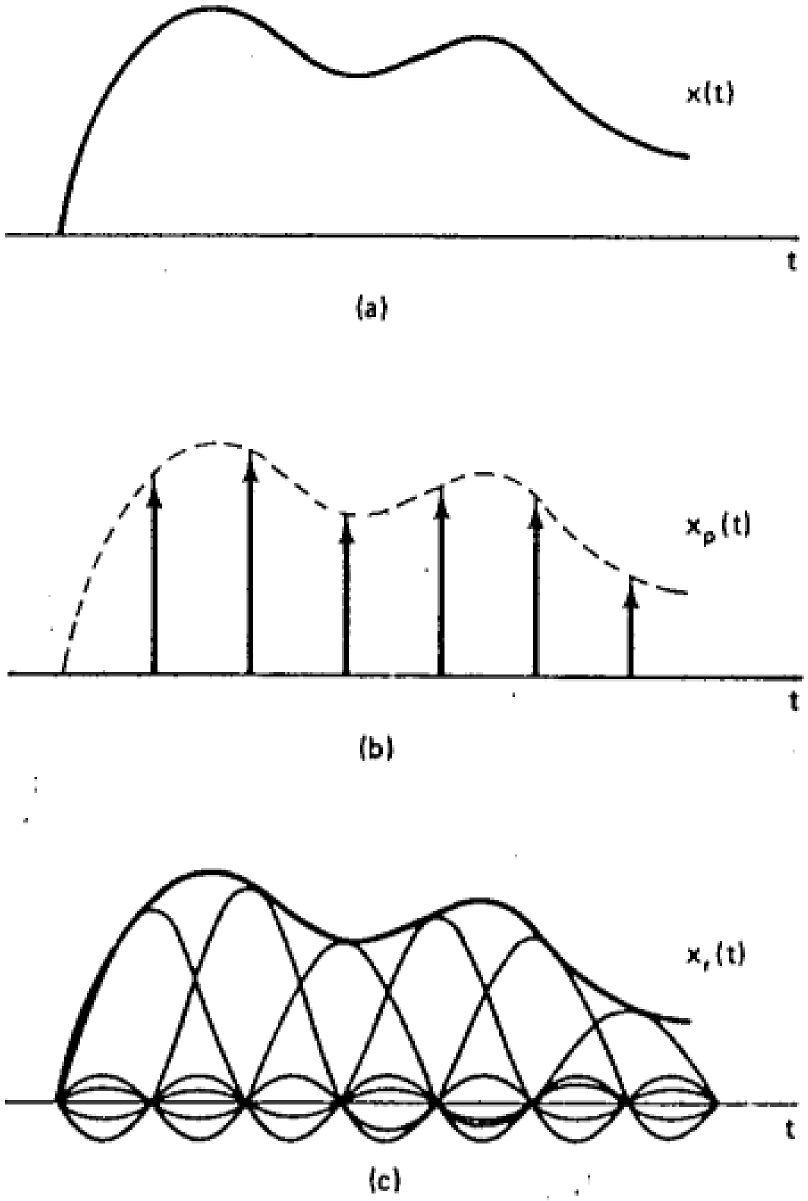
S&S Recap: *Reconst. of signal (Freq. Domain)*



$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t} = \frac{\omega_c T}{\pi} \mathrm{sinc}(\omega_c t)$$



S&S Recap: *Reconst. of signal (Time Domain)*



$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \Big(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\Big) * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\mathrm{sinc}(\omega_c(t-nT))
\end{aligned}
$$

Questions?

Design for Bandlimited Channels:

Nyquist Sampling Criteria

[Madhow]

Nyquist Sampling Theorem (Book Notations!!!)

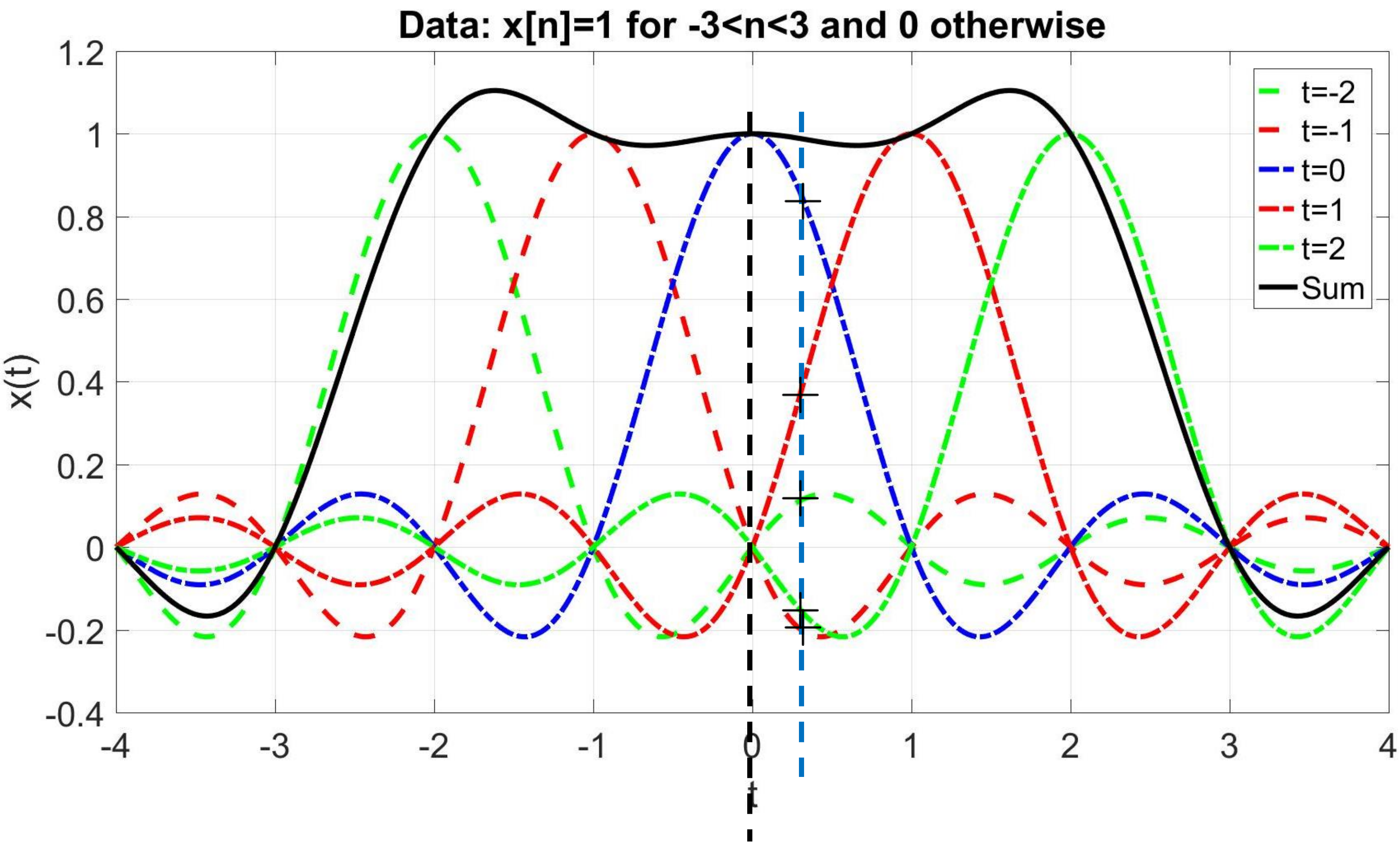
- Any signal $s(t)$ bandlimited to $[-W/2, W/2]$ can be described completely by its samples $\{s(n/W)\}$ at rate W. The signal $s(t)$ can be recovered from its samples using the following interpolation formula

$$s(t) = \sum_{n=-\infty}^{\infty} s\Big(\frac{n}{W}\Big) g\Big(t - \frac{n}{W}\Big)$$

  where $g(t) = \text{sinc}(Wt)$.

- Book uses $p(t)$ here but I have used $g(t)$ on purpose to differentiate it from the modulating pulse $p(t)$.

Problem With Sinc Pulse



Time-offset=0.3

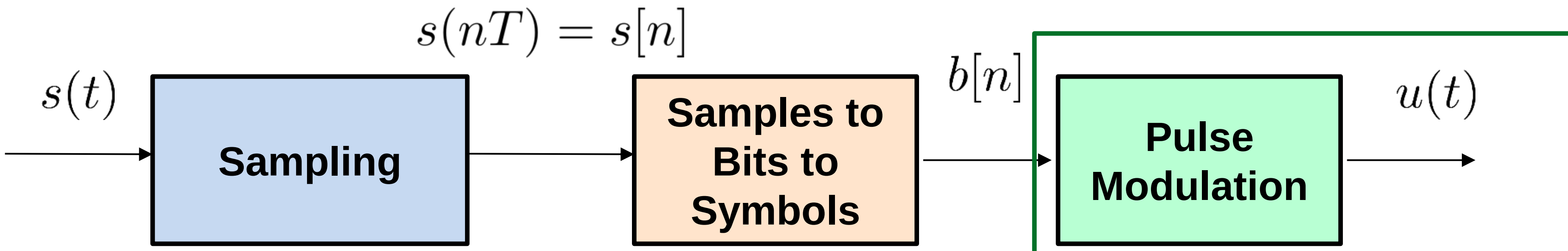
Design for Bandlimited Channels:

Nyquist Criteria for pulse shaping!

Bandlimited Channels

- Consider that we are given 20 MHz bandwidth at a carrier frequency of 2.4 GHz.
- Any signal that we send on this passband has a complex envelop from -10 MHz to 10 MHz.
- In general, for a passband signal of bandwidth W, we have corresponding complex-baseband signal spanning $[-W/2, W/2]$.
- Note that *sinc* and *sine* pulses are not strictly bandlimited.

Design for Bandlimited Channels



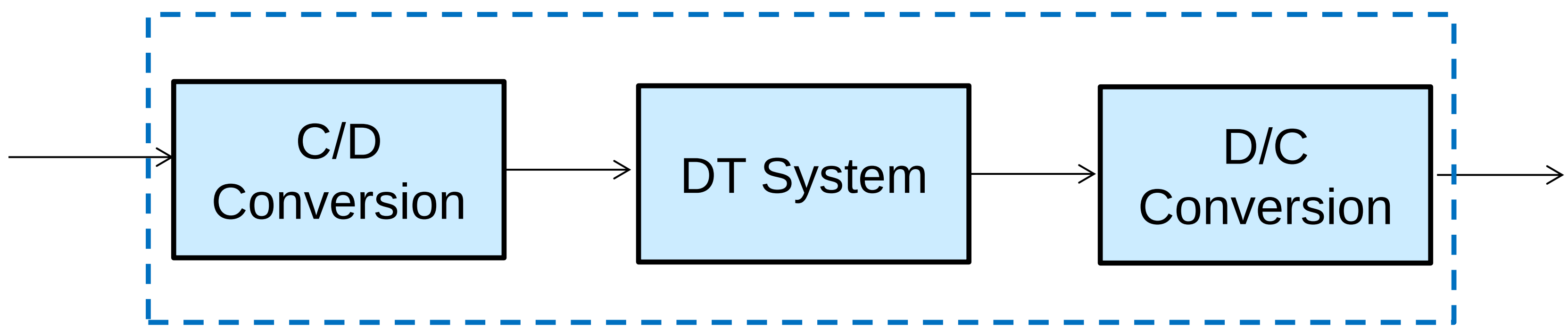
$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) g\left(t - \frac{n}{W}\right)$$

$$u(t) = \sum_{n} b[n] p(t - nT)$$

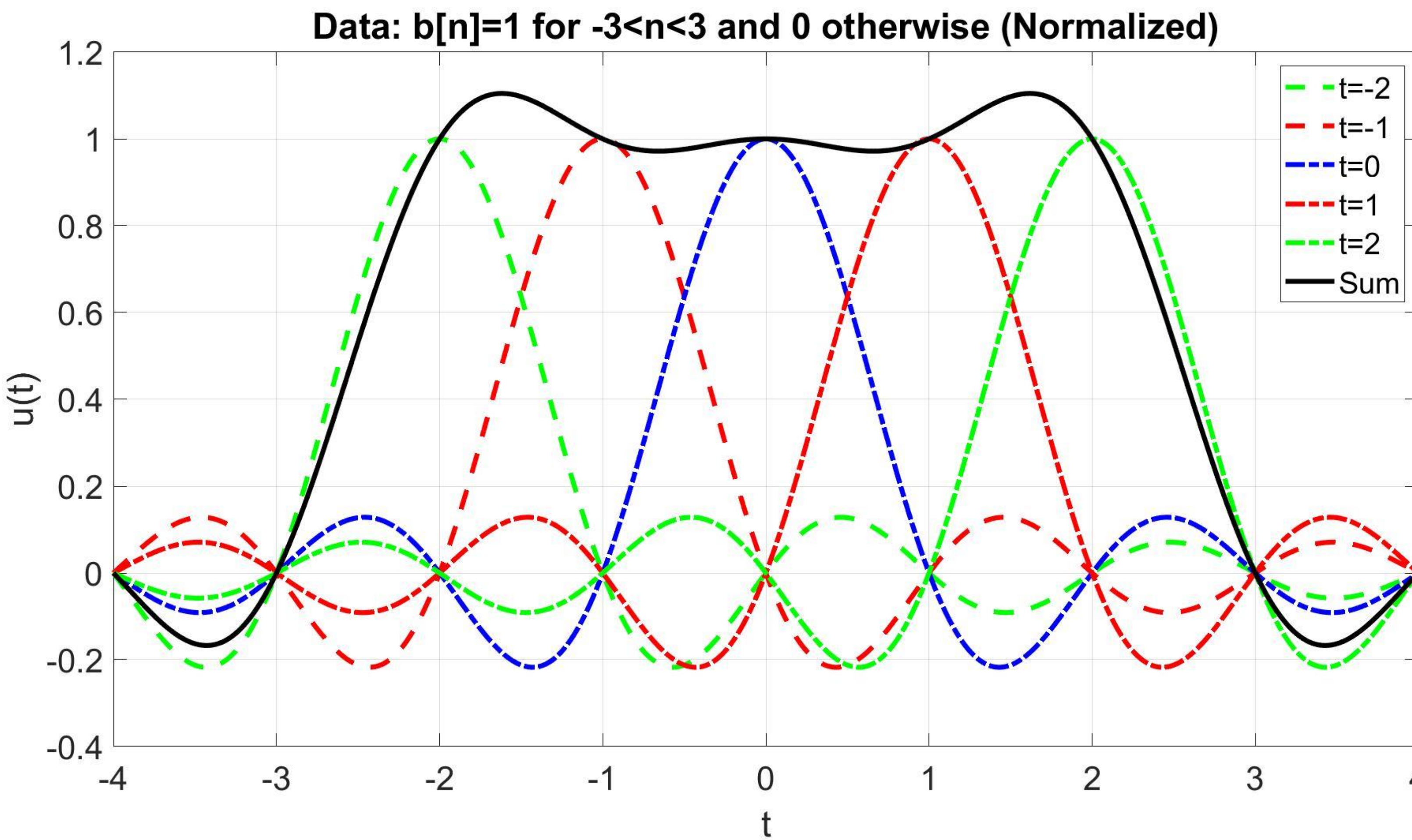
$$S_u(f) = \frac{|P(f)|^2}{T} \sigma_b^2$$

Minimum bandwidth dictated by Sampling
W=1/T

Bandwidth occupancy can be designed based on P(f) and independent of S(f) (FT of s(t)).



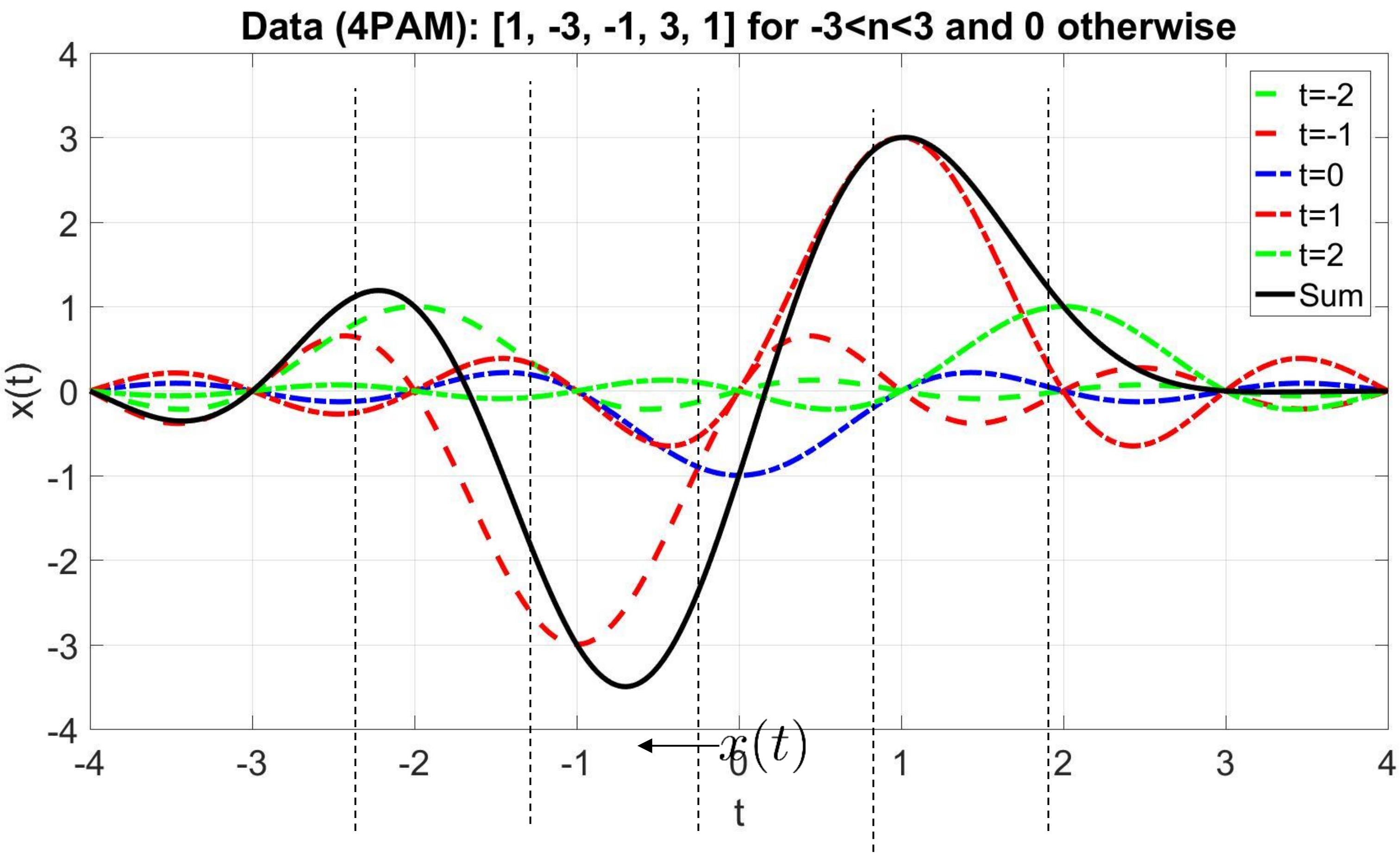
Sum of Sinc Pulses: Ex. 1



$$u(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{W}\right)\mathrm{sinc}\left(t - \frac{n}{W}\right) = \sum_{n=-\infty}^{\infty} b[n]\mathrm{sinc}\left(t - \frac{n}{W}\right)$$

- Normalized W=1=1/T
- *p(t)* is sinc function

Signal as Sum of Sinc Pulses: Ex. 2



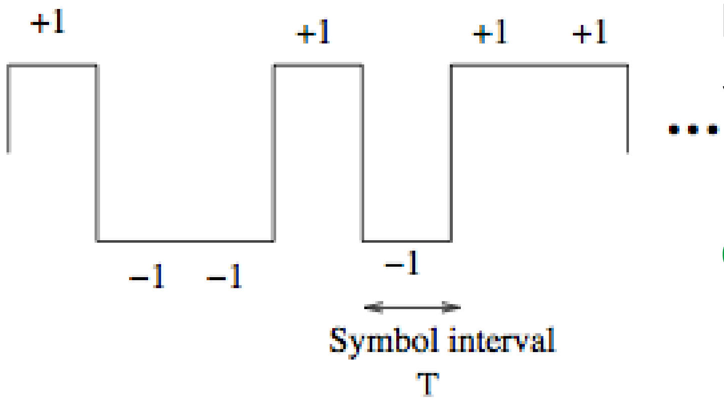
$$u(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{W}\right)\mathrm{sinc}\left(t - \frac{n}{W}\right) = \sum_{n=-\infty}^{\infty} b[n]\mathrm{sinc}\left(t - \frac{n}{W}\right)$$

- Normalized W=1=1/T
- *p(t)* is sinc function

What is Inter Symbol Interference (ISI)?

Time Domain *p(t)*

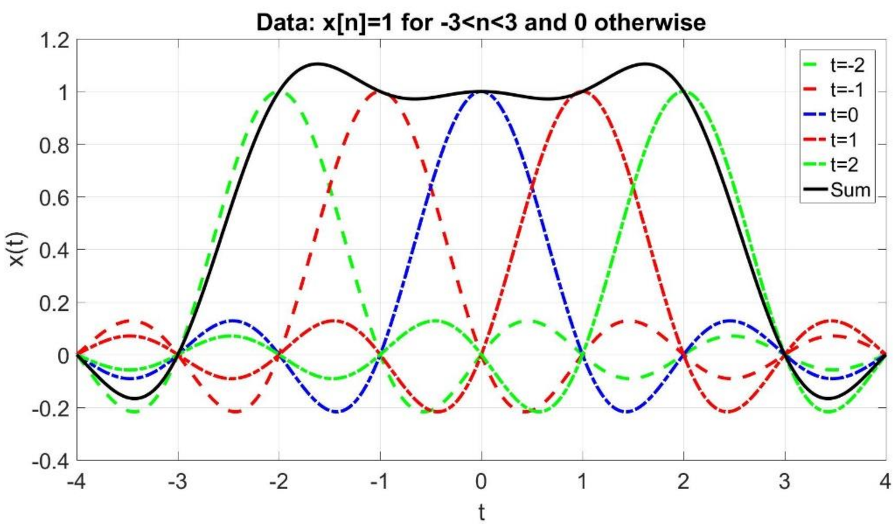
Rectangular Pulse



Freq. Domain *P(f)*
Sinc

No ISI=> SmallTiming offset does not cause issues

Sinc Pulse



Freq. Domain *P(f)*
Rect. Pulse

ISI=> SmallTiming offset does not cause issues

No ISI at sampling instances though!

Nyquist Criterion for ISI avoidance

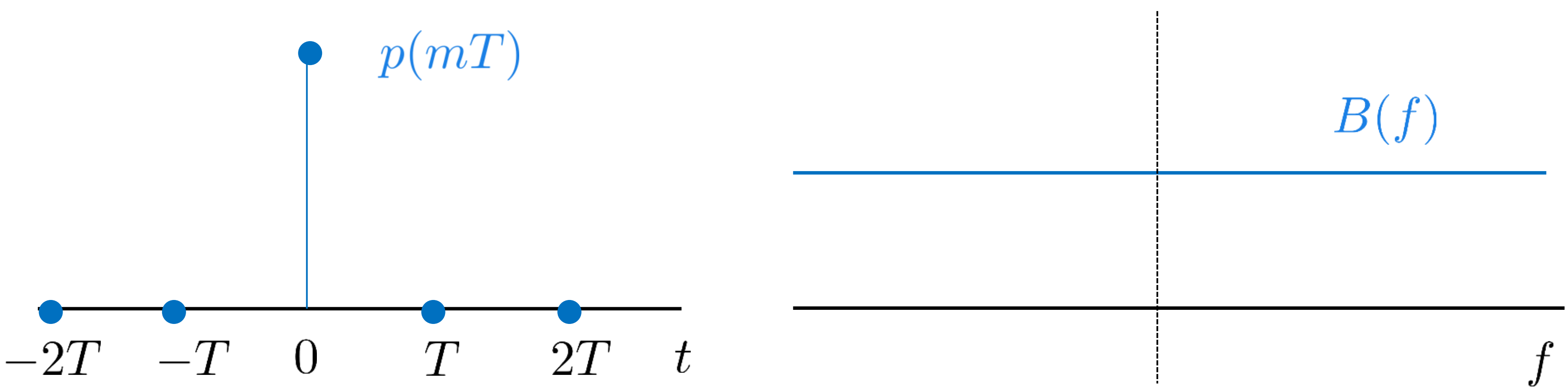
- The pulse $p(t) \leftrightarrow P(f)$ is Nyquist for sampling rate $1/T$ if

$$p(mT) = \delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

or equivalently

DT Fourier Transform Pair

$$B(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} P\left(f + \frac{k}{T}\right) = 1 \quad \forall f$$



Nyquist Criterion for ISI avoidance

- The pulse $p(t) \leftrightarrow P(f)$ is Nyquist for sampling rate $1/T$ if

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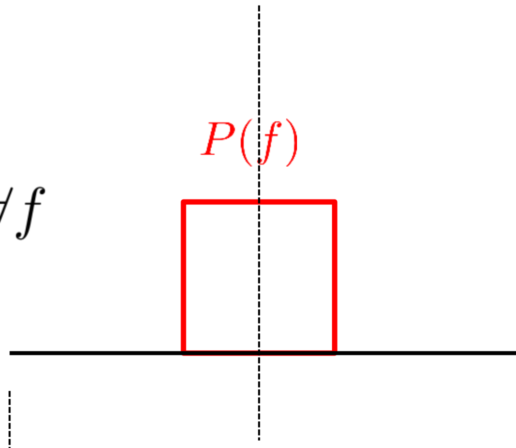
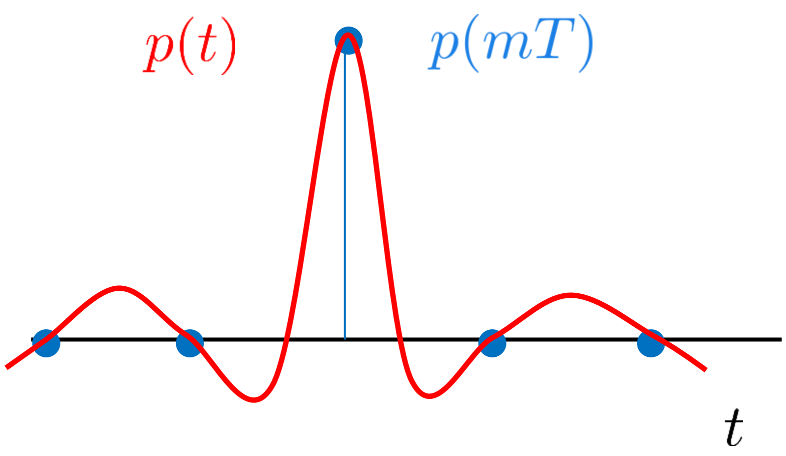
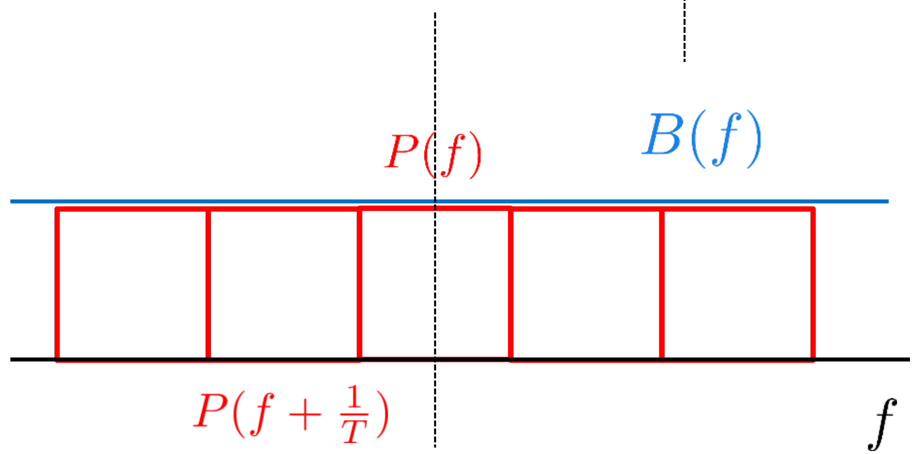






Theorem 4.5.1: Sampling

- Theorem (Sampling): Consider a signal $s(t)$, sampled at rate $1/T_s$. Let $S(f)$ denote the spectrum of $s(t)$, and let

$$B(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} S(f + \frac{k}{T_s})$$

denotes the sum of translates of the spectrum. Then the following observations hold

1. $B(f)$ is periodic with period $1/T_s$.
2. The samples $s(nT_s)$ are Fourier series for $B(f)$, satisfying

$$s(nT_s) = T_s \int_{-\frac{1}{2T_s}}^{\frac{1}{2T_s}} B(f) e^{j2\pi f n T_s} df$$

$$B(f) = \sum_{n=-\infty}^{\infty} s(nT_s) e^{-j2\pi f n T_s}$$

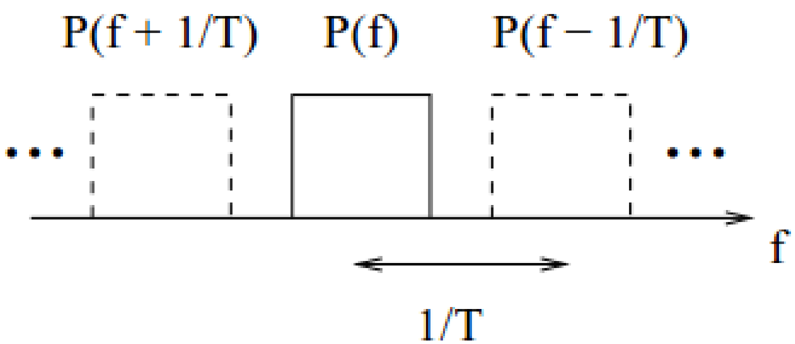
Significance of Nyquist Criterion for ISI avoidance

- It provides freedom to expand the modulation in time beyond the symbol duration so that bandwidth containment is better in frequency domain while ensuring that there is no ISI at appropriately chosen sampling intervals despite the significant overlap between successive pulses.

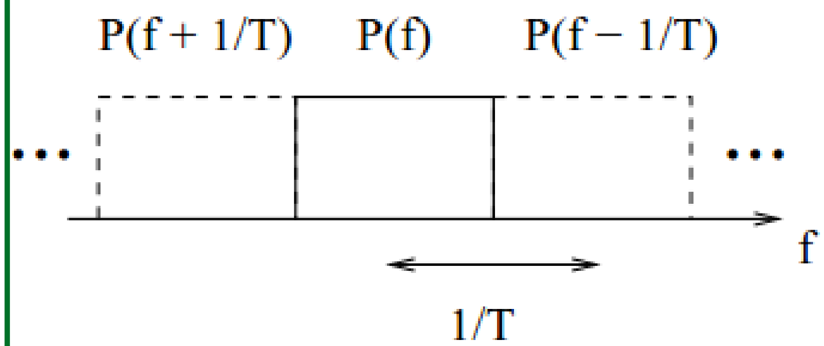
Nyquist Pulses?

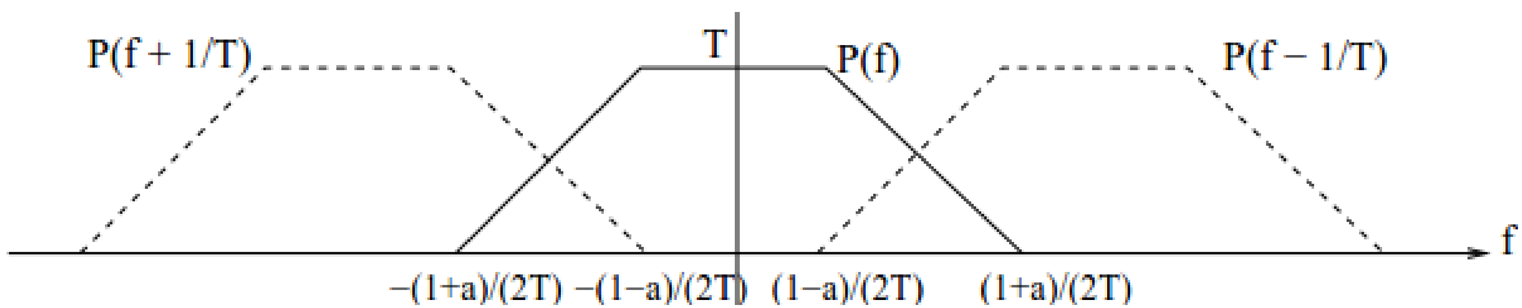
Corresponds to sinc in time domain

Not Nyquist



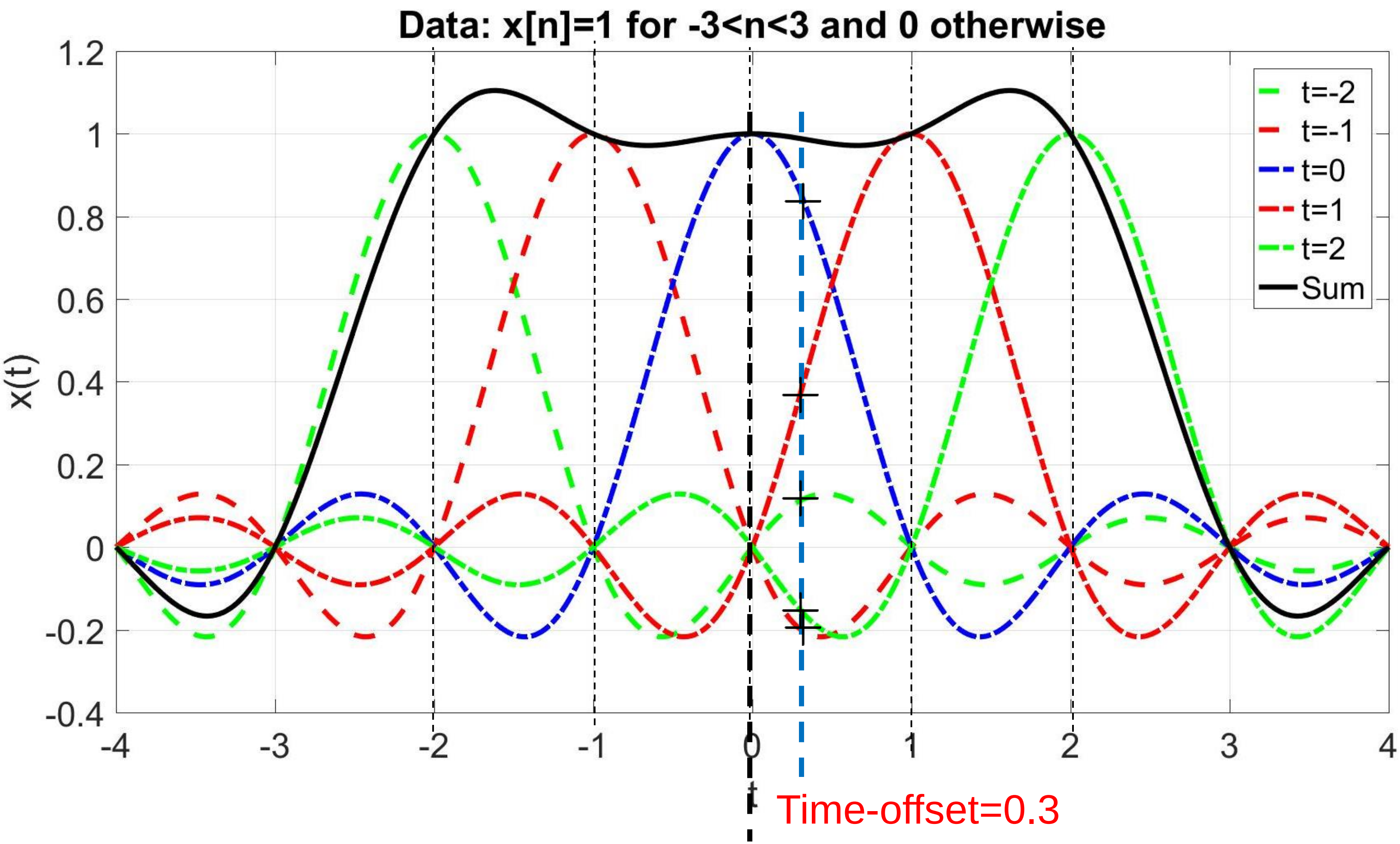
Nyquist pulse with minimum bandwidth 1/T





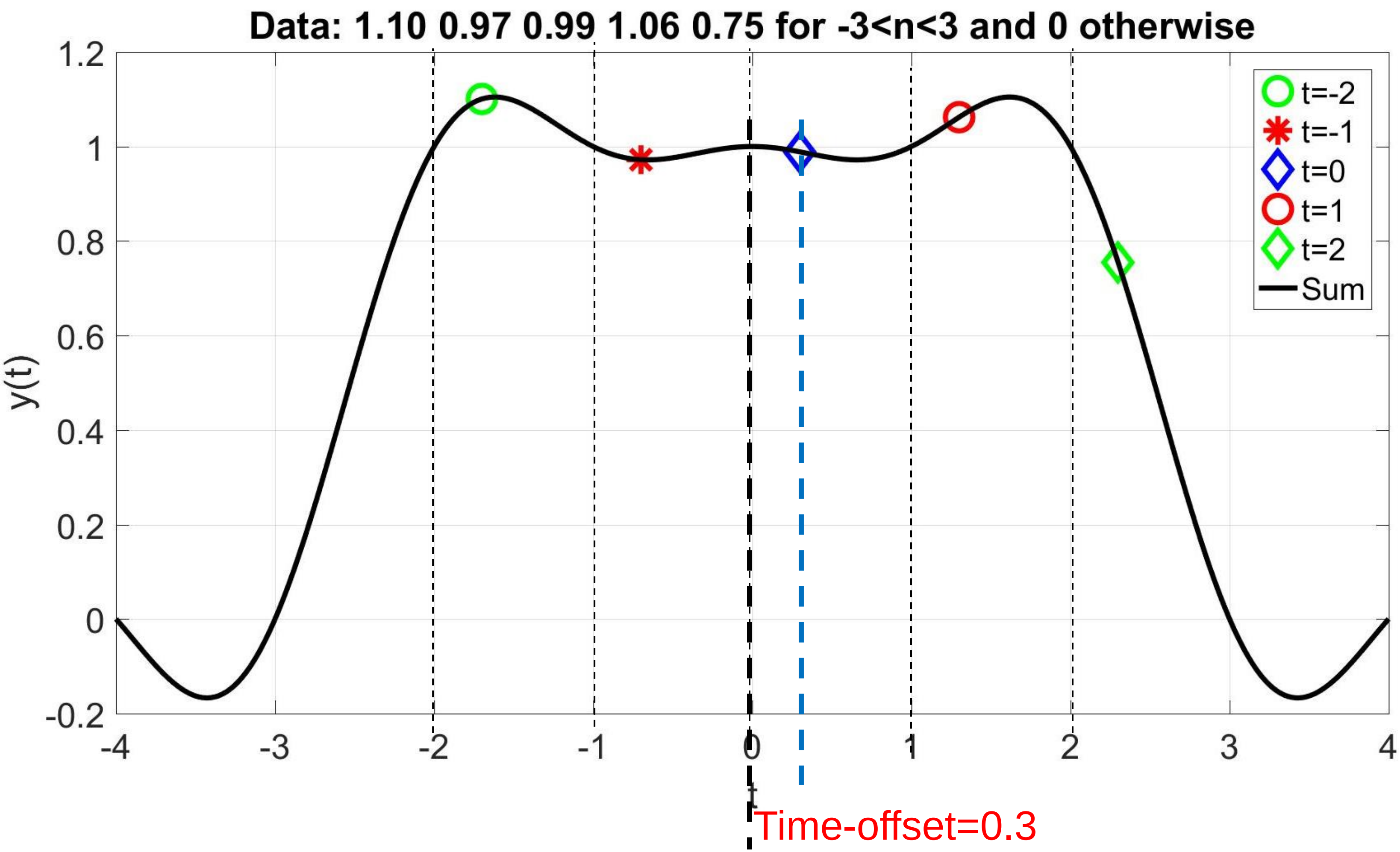
Nyquist

Problem With Sinc Pulse: *ISI*



- Sinc pulse decays as $1/|t|$ and the divergence of harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies significant interference from distant symbols.

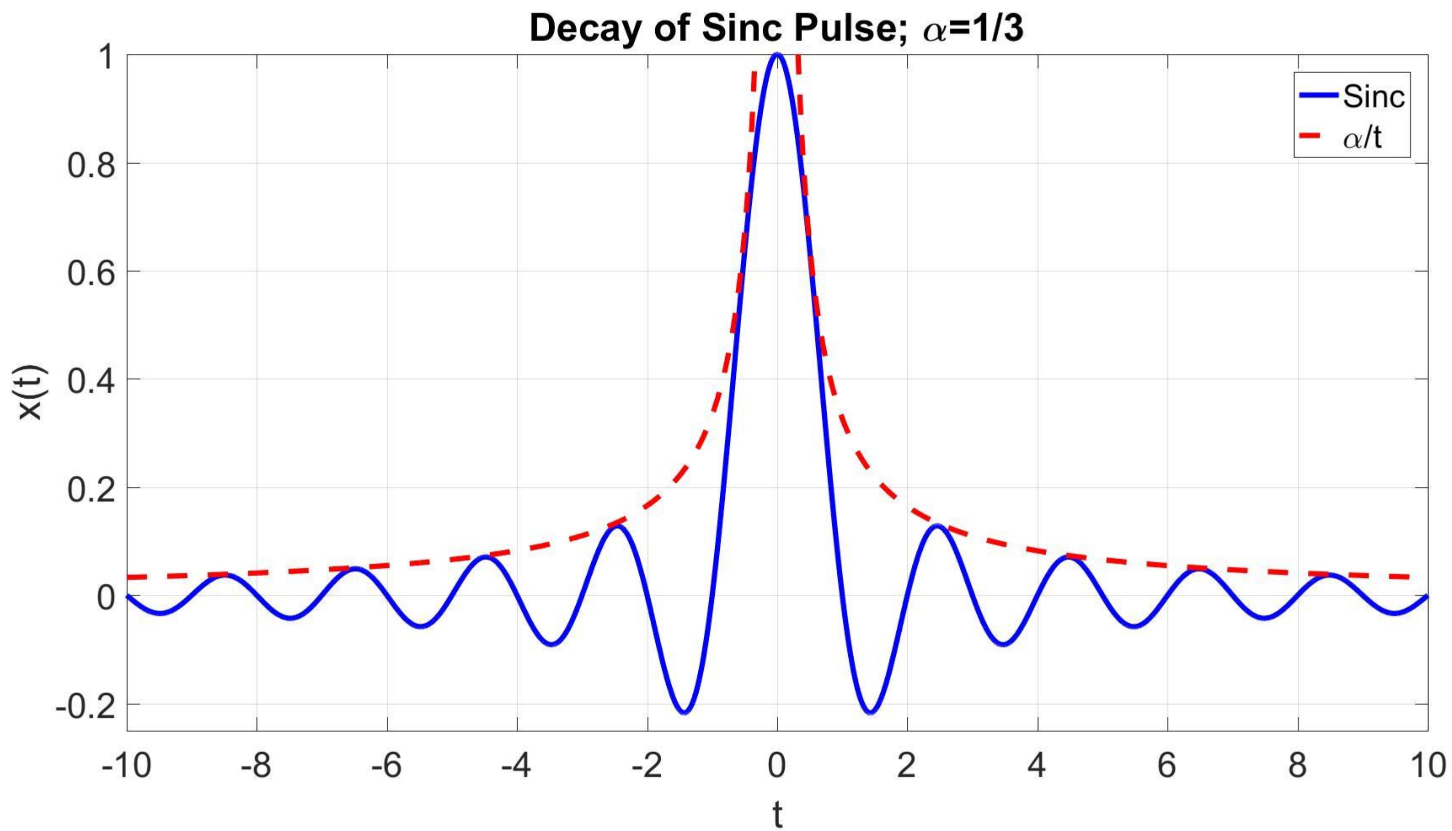
Problem with Sinc Pulse



Data: All ones were sent for -3<n<3 and 0 otherwise.

Here interference considered only from neighbouring 4 pulses

Problems with Sinc Pulse: *Slow Decay*



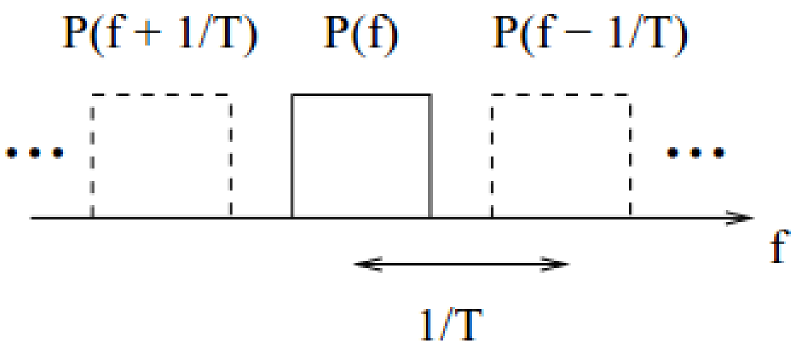
- Sinc pulse is given by

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$

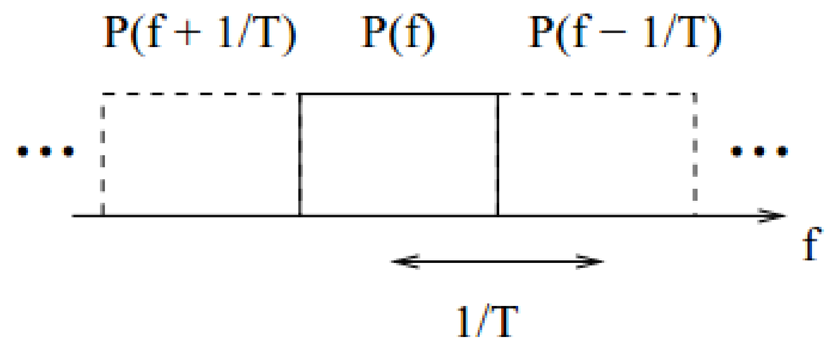
- Sinc pulse decays as $1/|t|$.

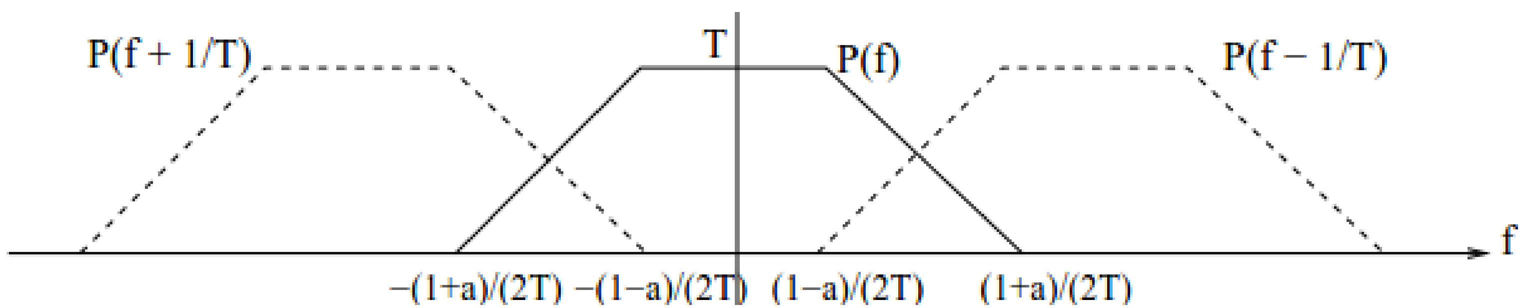
Nyquist Pulses: Excess Bandwidth

Not Nyquist



Nyquist pulse with minimum bandwidth





Nyquist pulse with excess bandwidth

Need of Excess Bandwidth!

- Sinc pulse decays as $1/|t|$ and the divergence of harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ implies significant interference from distant symbols.

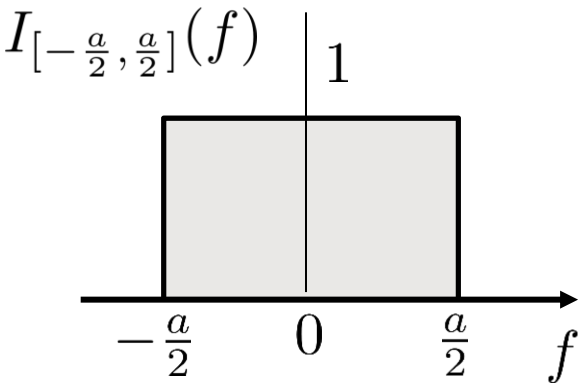
- However a pulse decaying as $1/|t|^b$ with $b > 1$ should work as $\sum_{n=1}^{\infty} \frac{1}{n^b}$ converges for $b > 1$.

- A faster decay in time requires slower decay in frequency. Thus we need excess bandwidth.

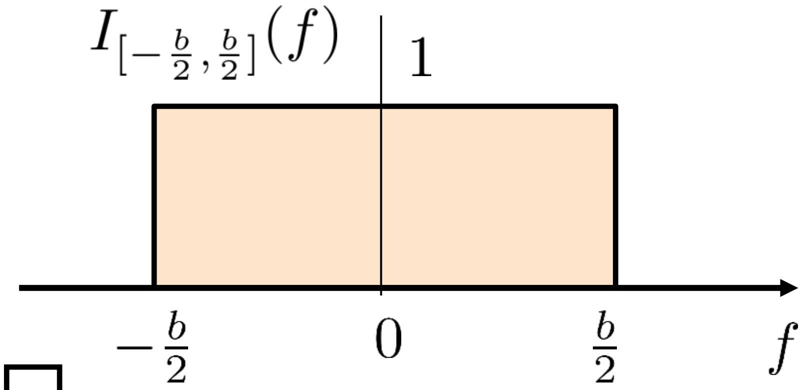
- Excess bandwidth is defined as the fraction of the bandwidth over the minimum required for ISI avoidance at a given symbol rate.

Trapezoidal Pulse

- Trapeziodal pulse is convolution of two rectangular pulses!

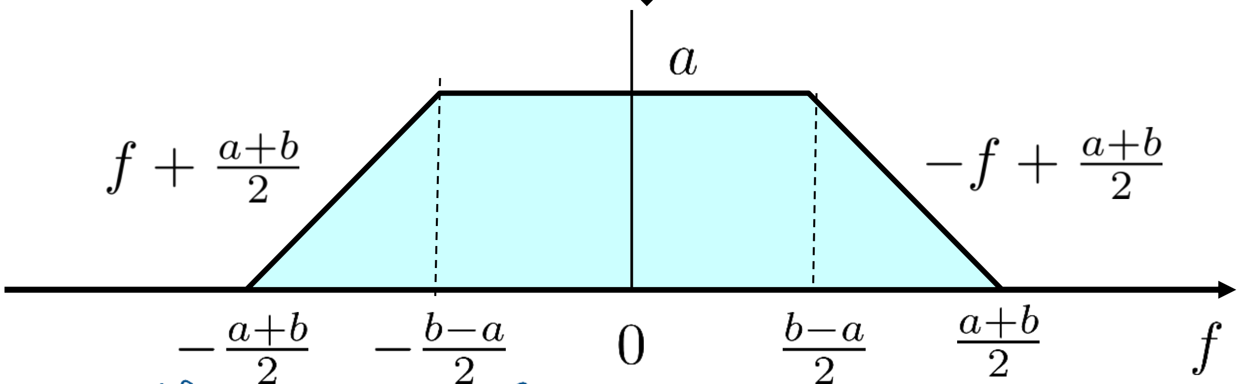


$*$



$I_{[-\frac{a}{2},\frac{a}{2}]}(f) \leftrightarrow a\,\mathrm{sinc}(at)$

Assumption: $b \geq a$

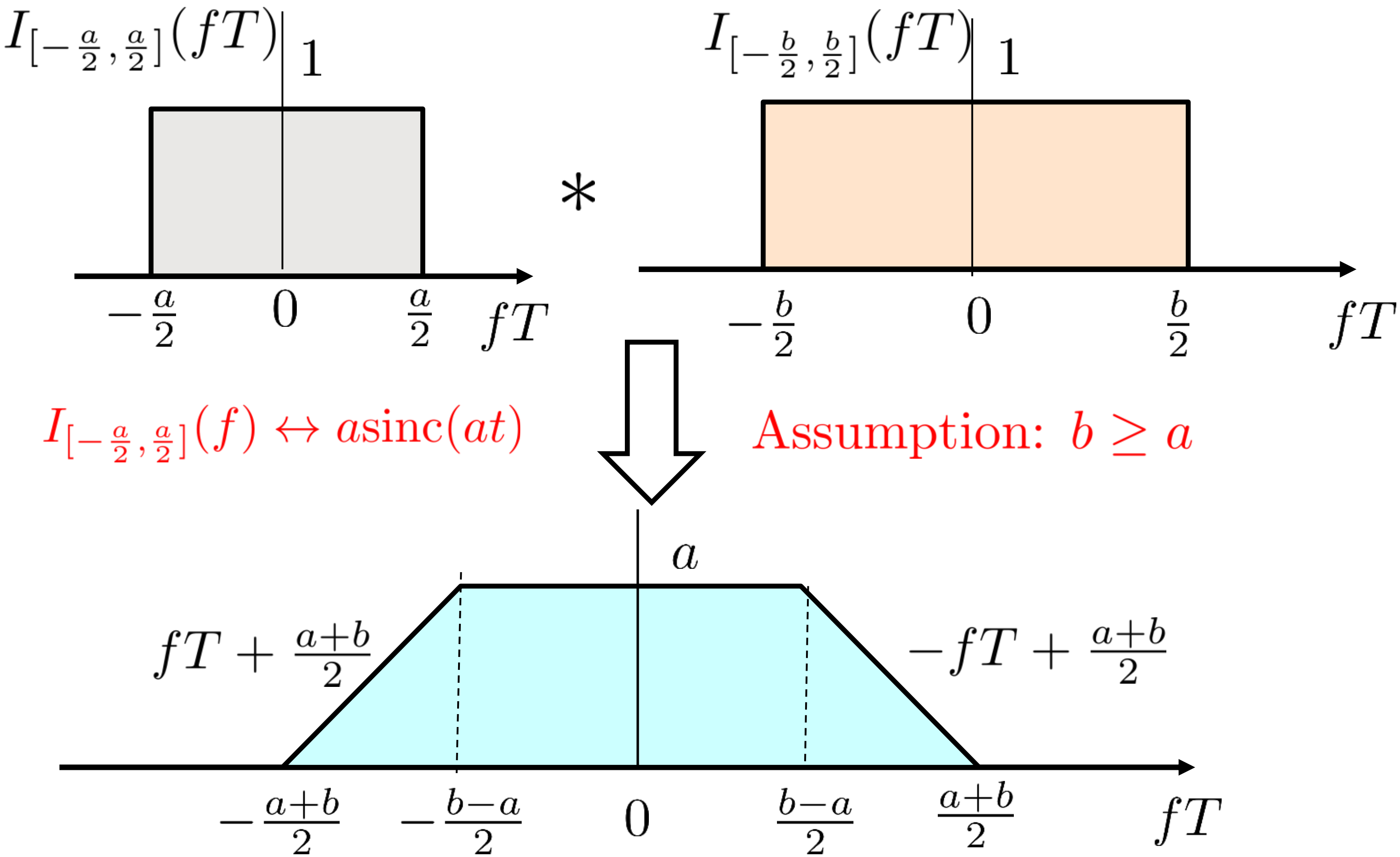


X(f)　　Y(f)

Z(f)　$I_{[-\frac{a}{2},\frac{a}{2}]}(f) * I_{[-\frac{b}{2},\frac{b}{2}]}(f) \longleftrightarrow a\,b\,\mathrm{sinc}(at)\,\mathrm{sinc}(bt)$

Trapezoidal Pulse: Normalized Freq.

- Trapeziodal pulse is convolution of two rectangular pulses!

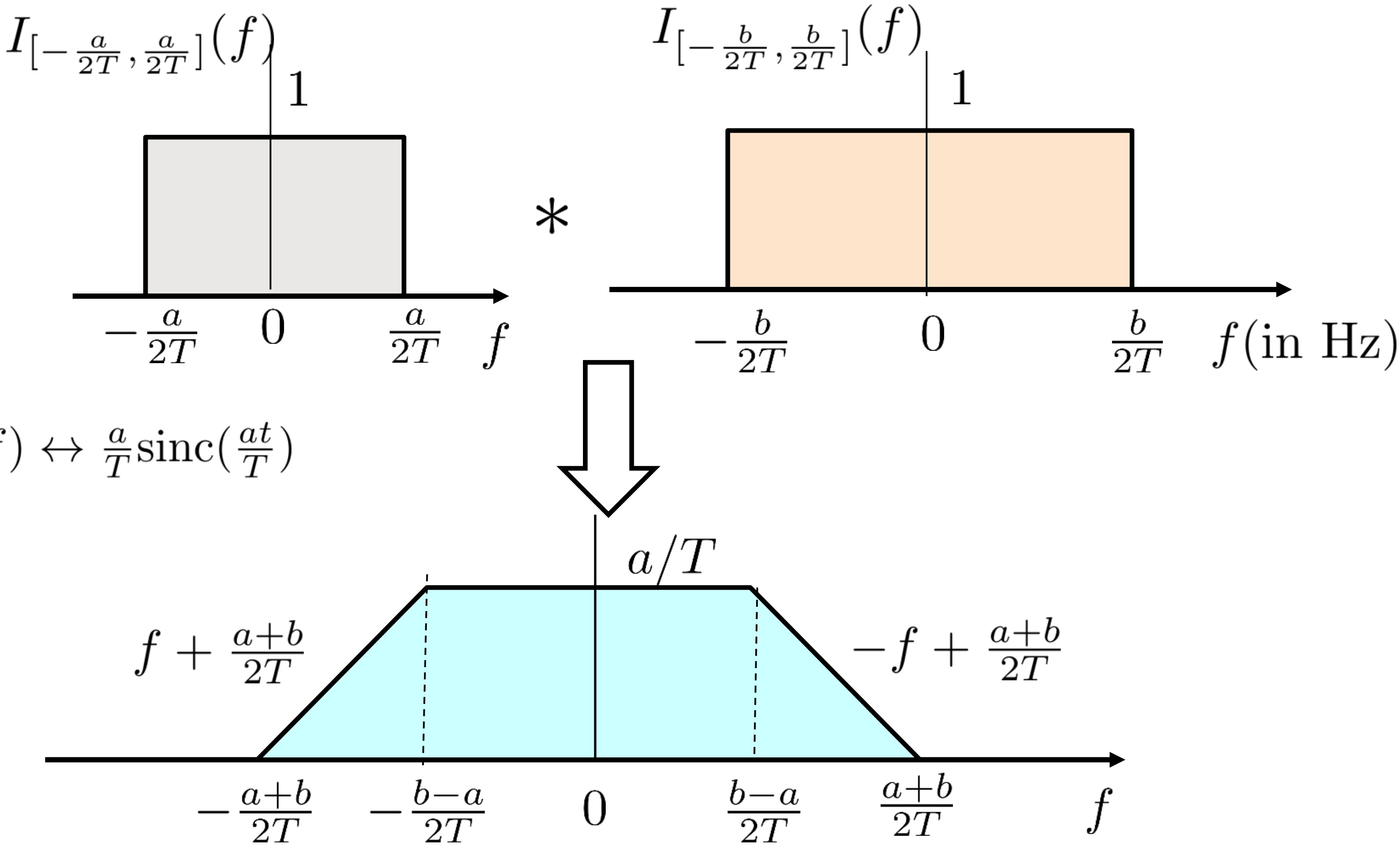


$I_{[-\frac{a}{2},\frac{a}{2}]}(f) \leftrightarrow a\mathrm{sinc}(at)$

Assumption: $b \geq a$

$$I_{[-\frac{a}{2},\frac{a}{2}]}(fT) * I_{[-\frac{b}{2},\frac{b}{2}]}(fT) \longleftrightarrow a\, b\, \mathrm{sinc}(at)\, \mathrm{sinc}(bt)$$

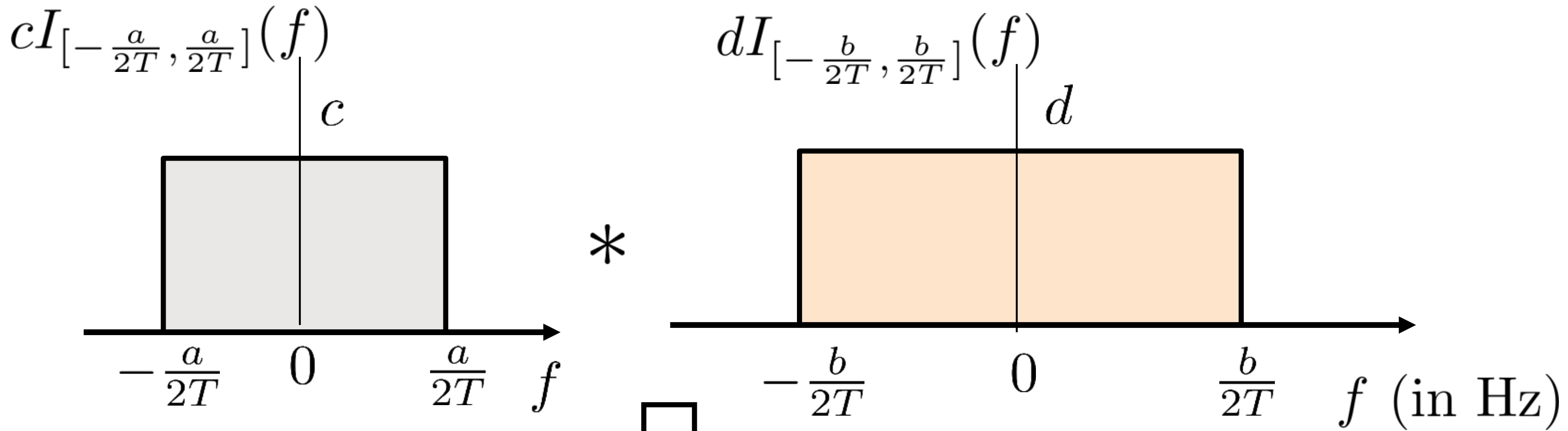
Trapezoidal Pulse: Non-normalized



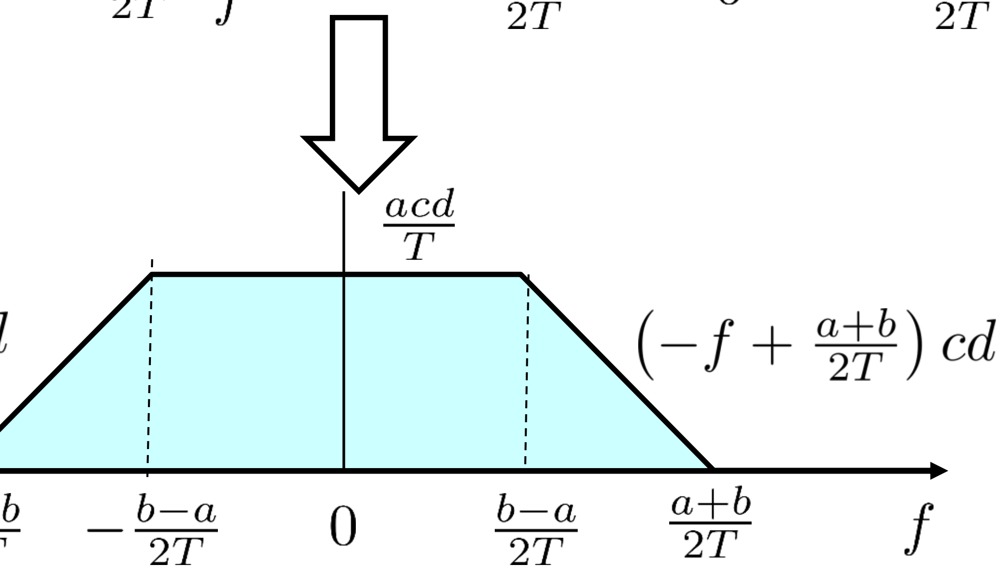
$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\mathrm{sinc}\left(\frac{at}{T}\right)$$

$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * I_{[-\frac{b}{2T},\frac{b}{2T}]}(f) \longleftrightarrow \frac{a\,b}{T^2}\,\mathrm{sinc}\left(\frac{at}{T}\right)\mathrm{sinc}\left(\frac{bt}{T}\right)$$

Trapezoidal Pulse: General Expressions with scaling



$$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\mathrm{sinc}\left(\frac{at}{T}\right)$$



$$c\, I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * d\, I_{[-\frac{b}{2T},\frac{b}{2T}]}(f) \longleftrightarrow \frac{a\,b\,c\,d}{T^2}\,\mathrm{sinc}\left(\frac{at}{T}\right)\mathrm{sinc}\left(\frac{bt}{T}\right)$$

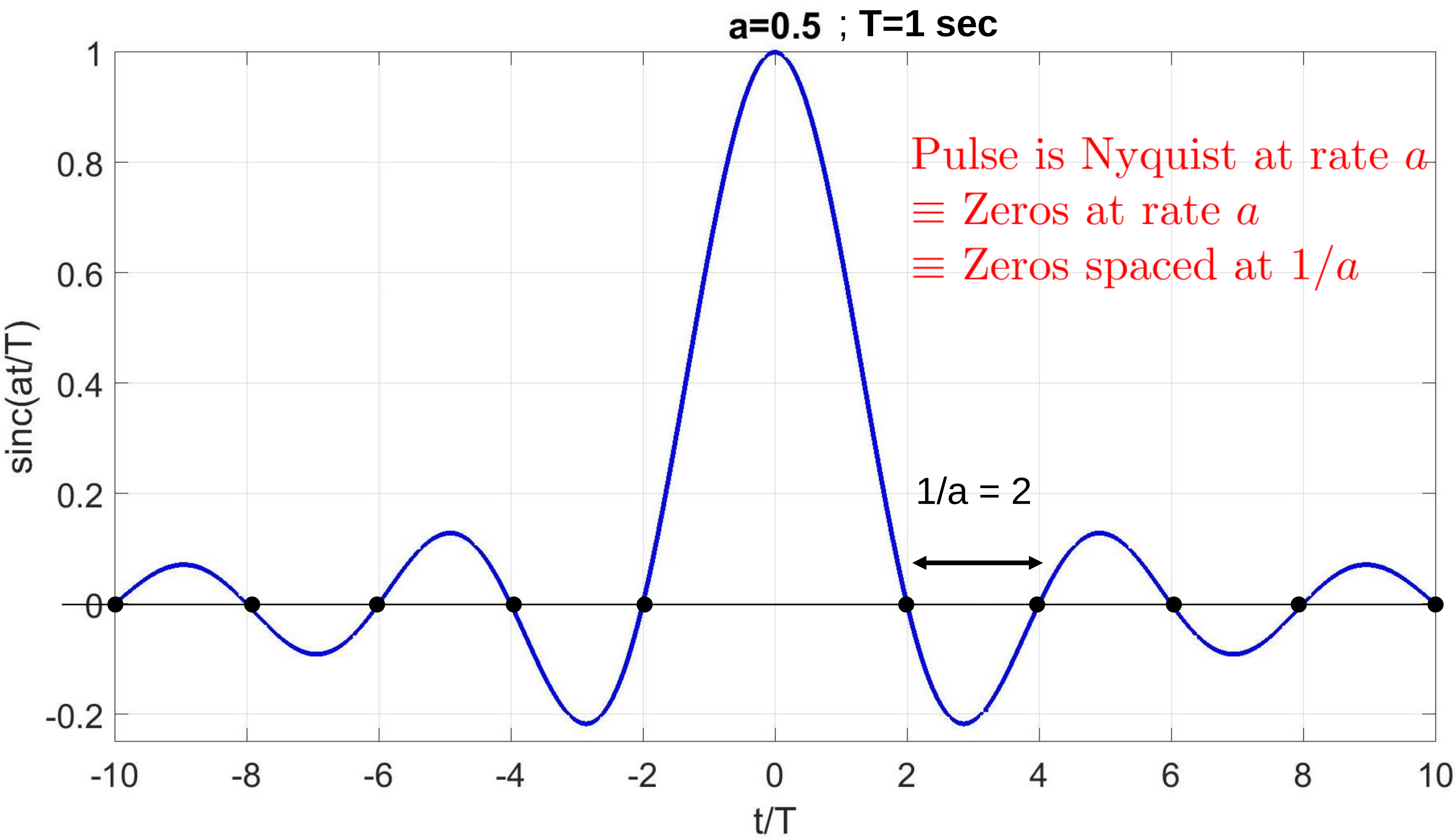
Some Interesting Properties of Nyquist Pulses

- For trapezoidal and the raised-cosine waveforms, the time-domain pulse has a $\text{sinc}(at)$ term that provides zeros at the integer multiples of $1/a$. This means that pulse is Nyquist at rate a. In other words, a time domain factor that provides *zeros at rate a* enables Nyquist signaling at rate a.

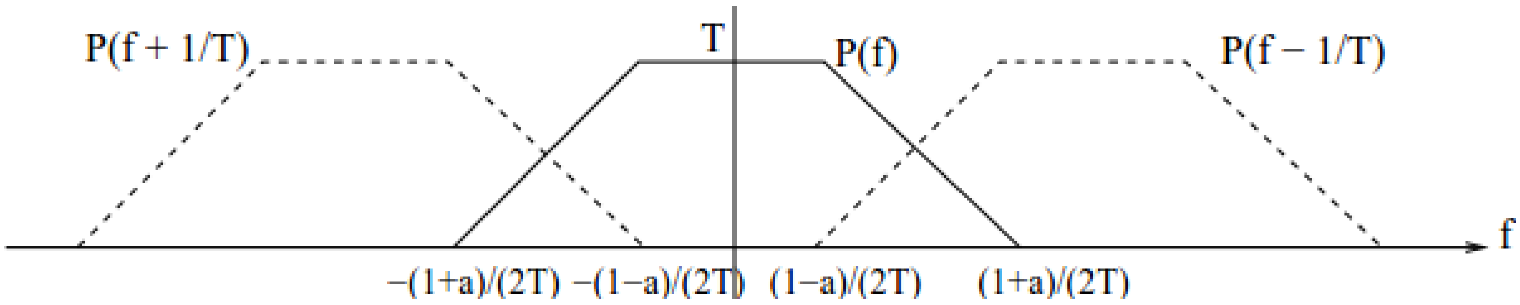
- A pulse that is trapedoizal has a time-domain pulse of the form $\text{sinc}(at)\text{sinc}(bt)$, which provides zeros at rate a and b. Thus this is Nyquist at rate a and rate b.

- Once we have zeros at integer multiples of T, we also have zeros at integer multiples of KT where K is any positive integer. In other words, if a pulse is Nyquist at rate $1/T$, then it is also Nyquist at integer submultiples of this rate, i.e., $1/KT$.

Sinc Pulse and Nyquist Rate

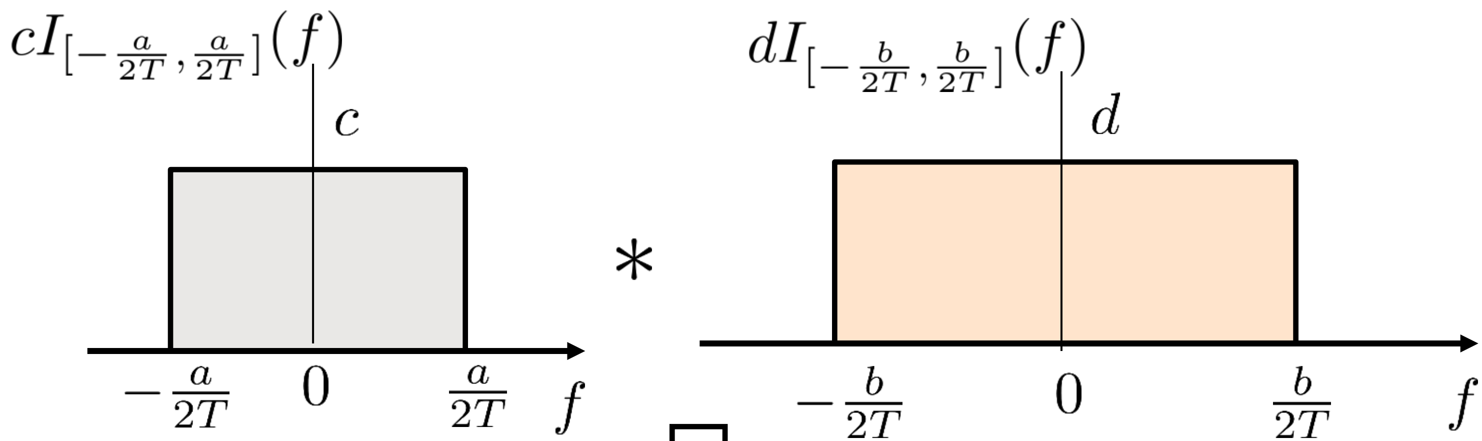


Example: Tutorial



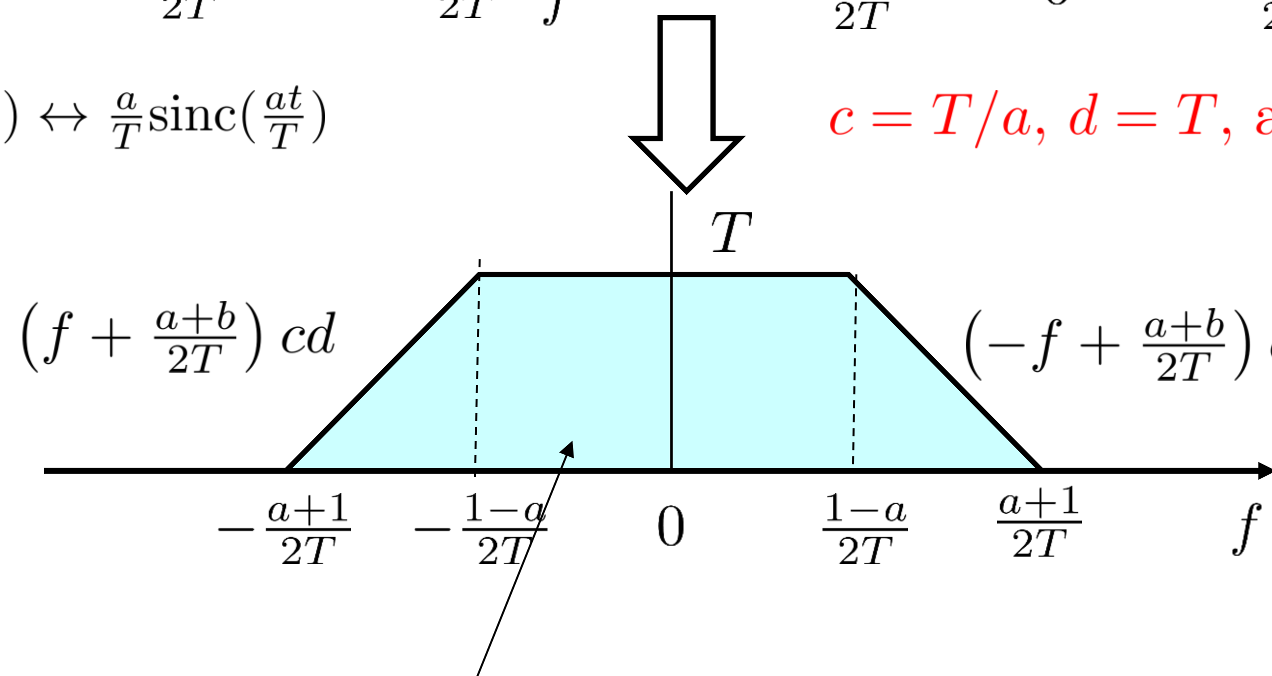
- Consider the trapezoidal pulse of excess bandwidth a shown in figure above.
  - Find an explicit expression for the time-domain pulse $p(t)$.
  - What is the bandwidth required for a passband system using this pulse operating at 120 Mbps using 64 QAM with an excess bandwidth of 25%?

Trapezoidal Pulse: Example 4.1



$I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) \leftrightarrow \frac{a}{T}\mathrm{sinc}(\frac{at}{T})$

$c = T/a$, $d = T$, and $b = 1$



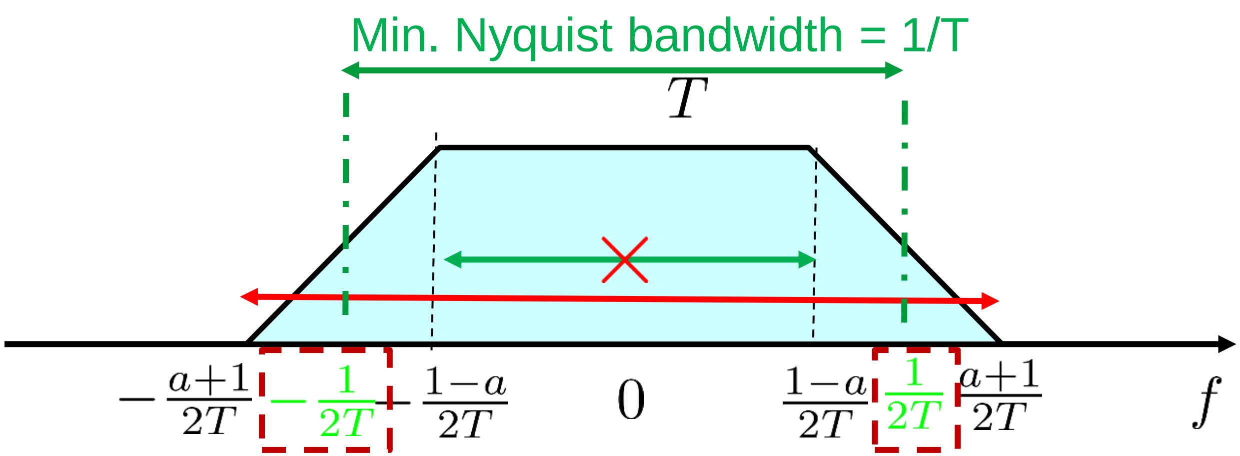
$$\frac{T^2}{a} I_{[-\frac{a}{2T},\frac{a}{2T}]}(f) * I_{[-\frac{1}{2T},\frac{1}{2T}]}(f) \longleftrightarrow \boxed{\mathrm{sinc}(\tfrac{at}{T})\,\mathrm{sinc}(\tfrac{t}{T})}$$

Trapezoidal Pulse: Example 4.1

- Given bit rate is 120 Mbps and M=64 corresponding to to 64 QAM.

- Symbol rate is given by

$$\frac{1}{T} = \frac{\text{Bit Rate}}{\log_2 M} = \frac{120}{6} = 20 \text{ MHz}$$



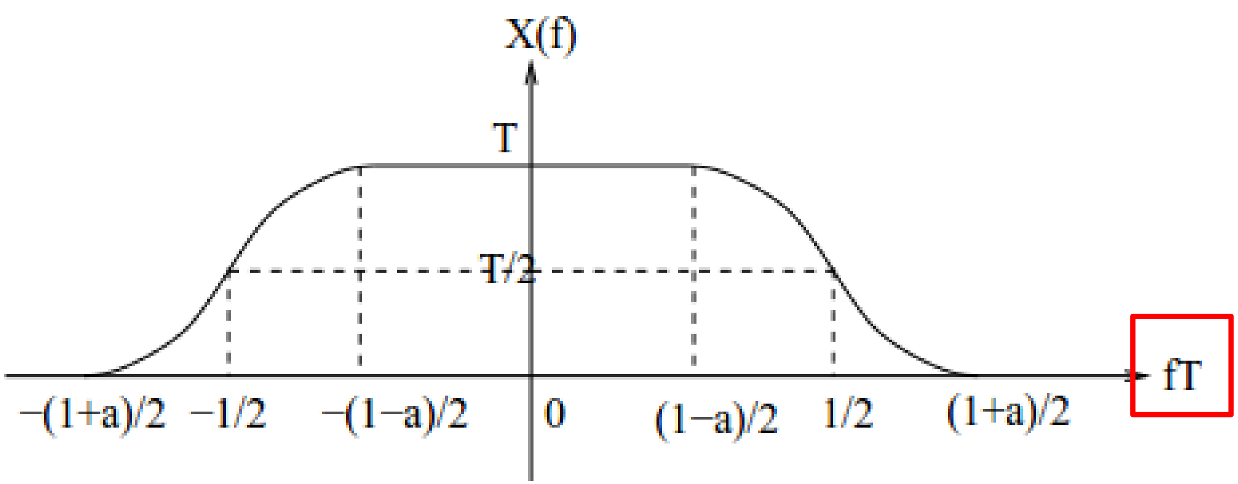
Excessive Nyquist bandwidth $= \frac{1+a}{T}$ with a being fraction of excessive bandwidth, i.e., $a < 1$

- With excess bandwidth of 25%,i.e., $a = 0.25$, we need 1.25*20 = 25 MHz.

Raised-cosine pulse: *Freq. Domain*

- Raise cosine pulse, which has a decay rate of $1/t^3$ in time domain, is given by

$$P(f) = \begin{cases} T, & |f| \leq \frac{1-a}{2T} \\ \frac{T}{2}\left(1 + \cos\left(\left(|f| - \frac{1-a}{2T}\right)\frac{\pi T}{a}\right)\right), & \frac{1-a}{2T} \leq |f| \leq \frac{1+a}{2T} \\ 0, & |f| > \frac{1+a}{2T} \end{cases}$$

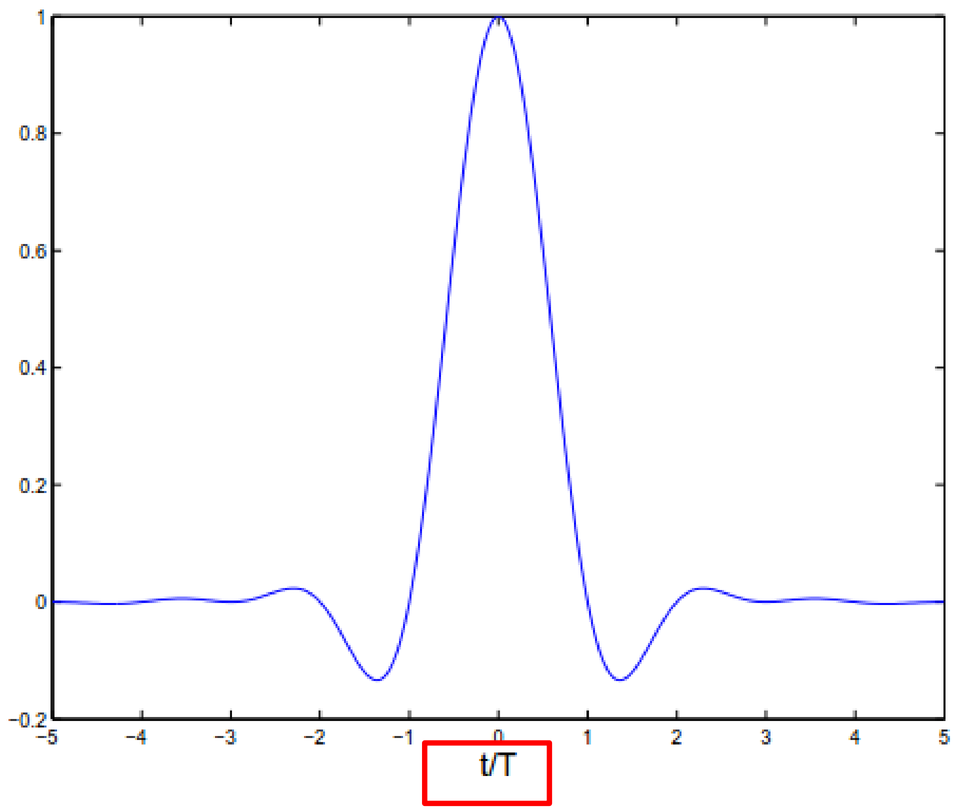


(a) Frequency domain raised cosine

Raised-cosine pulse: *Time Domain*

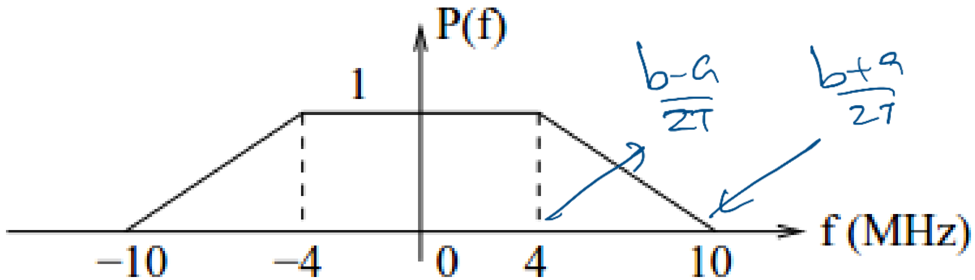
- The time domain pulse $p(t)$ is given by

$$p(t) = \operatorname{sinc}\left(\frac{t}{T}\right)\frac{\cos(\pi a t/T)}{1-(2at/T)^2} \qquad \text{overall decay of } 1/t^3$$



(b) Time domain pulse (excess bandwidth $a = 0.5$)

Example: Tutorial

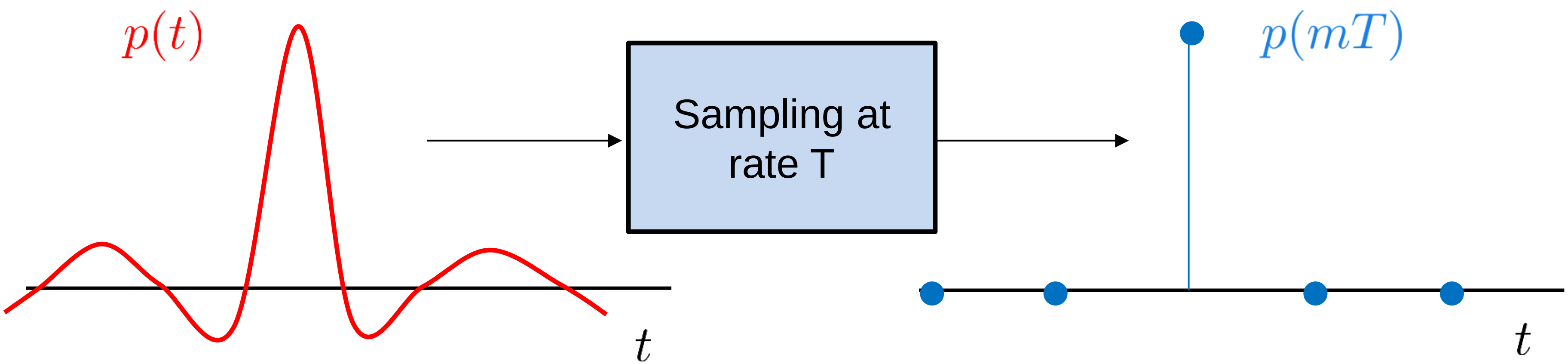


$\frac{b-a}{2T}$ $\frac{b+a}{2T}$ $\frac{acd}{T} = 1 \Rightarrow c = T/a$

$d = T.$

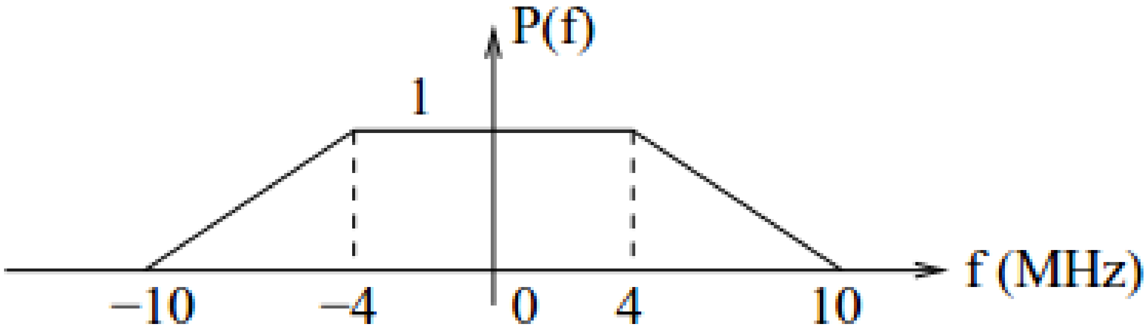
- Consider passband linear modulation using the bandlimited pulse shown in Fig. above.

- Can the pulse $p(t)$ be used for Nyquist sampling while using bit rate of 56 Mbps and 16 QAM constellation?



- The question is if we sample $p(t)$ at sample rate of $1/T$ or sampling intervals of T, would the resulting sequence $p(mT)$ be Nyquist with $p(mT) = 1$ for $m = 0$ and $p(mT) = 0$ $m \neq 0$.

Example continues: Tutorial



- Can the pulse $p(t)$ be used for Nyquist sampling while using the combination of given bit rate and constellation?
  - 21 Mbps and 8 PSK
  - 18 Mbps and 8 PSK
  - 25 Mbps and QPSK

Bandwidth Efficiency

- The bandwidth efficiency of linear modulation with an $M-$ary alphabet is given by

$$\eta_B = \log_2 M \text{ bits/symbol}$$

- Knowing the bit rate R_b and bandwidth efficiency we can determine symbol rate and hence the minimum required bandwidth $B_{\min}$

$$B_{\min} = \frac{R_b}{\eta_B}$$

- In terms of excessive bandwidth a,

$$B = (1+a)B_{\min} = (1+a)\frac{R_b}{\eta_B}$$

Recap: Design for Bandlimited Channels:

Nyquist Criteria for pulse shaping!

Today's Class
Orthogonal Modulation

Orthogonal Modulation

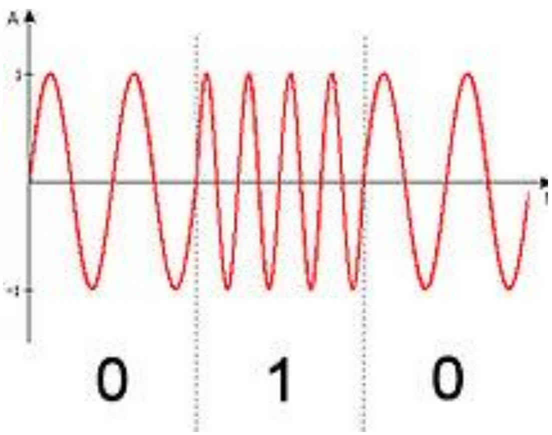
- Power efficient as opposed to bandwidth efficient
- Use of M orthogonal waveforms for sending messages
- Note that $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ are also orthogonal
- Example: Frequency shift keying (FSK)
- Used in satellite communication

Frequency Shift Keying (FSK)

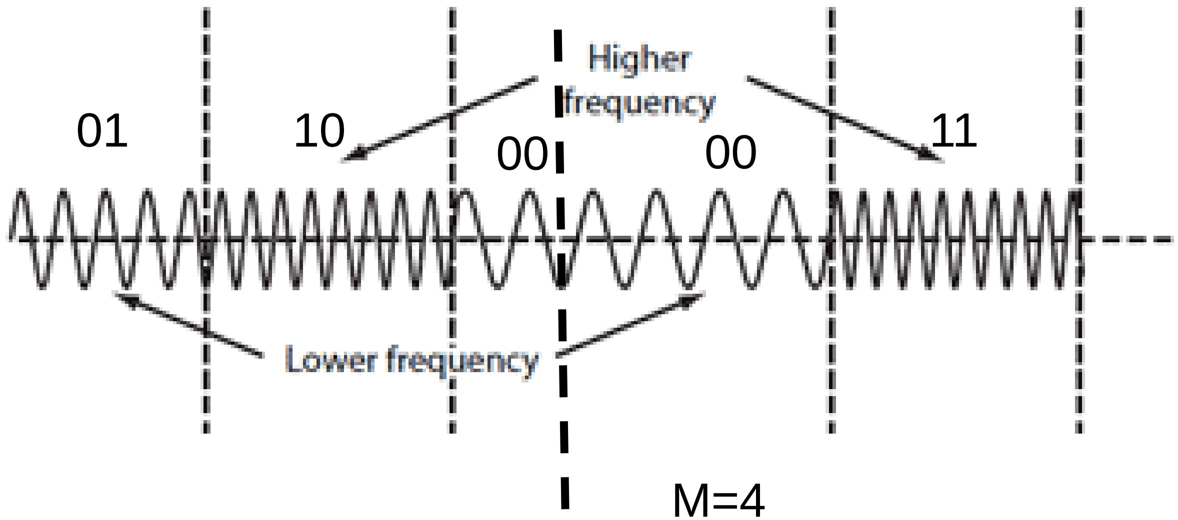
- Use M sinusoidal tones separated by Δf to send M messages (or $\log_2 M$ bits)
- The k^{th} sinusoidal tone (passband signal) is given by

$$u_{p,k} = \cos(2\pi(f_c + k\Delta f)t) \quad 0 \leq t \leq T$$

for $k = 0, 1, \ldots, M-1$.



M=2



M=4

FSK

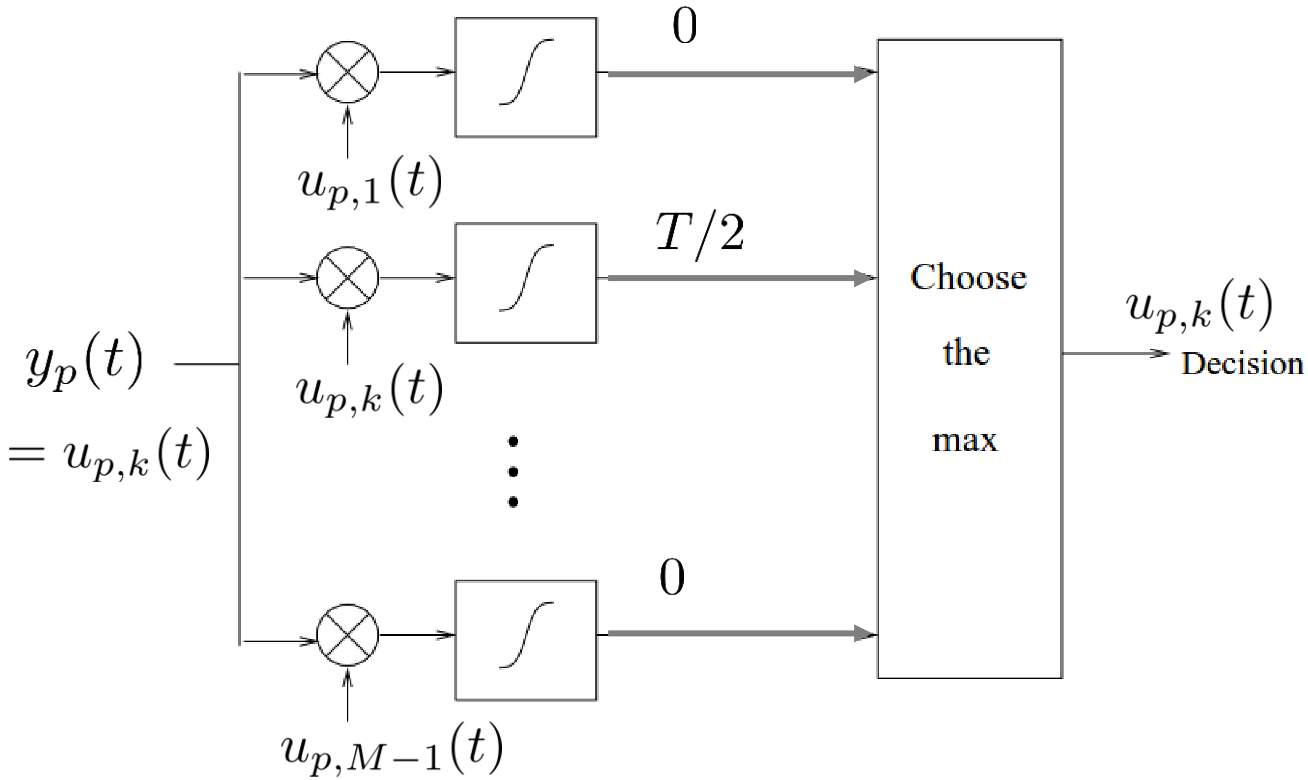
- Show that the minimum frequency spacing between two adjacent tones for FSK to be orthogonal is

$$\Delta f = \frac{1}{2T}$$

- Each tone correspond to one dimension so we have M dimensions instead of 2 dimensions as in PAM, PSK and QAM.

- The minimum bandwidth required in this case is $M\Delta f = \frac{M}{2T}$.

FSK: coherent demodulation (passband)



- Correlate the incoming signal with all M reference sinusoidal tones (passband signal) given by

$$u_{p,k} = \cos(2\pi(f_c + k\Delta f)t) \quad 0 \leq t \leq T$$

for $k = 0, 1, \ldots, M-1$.

- Choose $u_{p,m}$ for which the output is maximum among all the correlated outputs.

FSK: Issue with coherent demodulation

- Consider that $u_{p,m}$ was sent and the corresponding received signal is
$$y_p(t) = \cos(2\pi(f_c + m\Delta f)t + \frac{\pi}{2}) \quad 0 \leq t \leq T$$
- If there is phase offset between the incoming (received) signal $y_p(t)$ and the correct reference signal $u_{p,m}(t)$, then
$$< y_p, u_{p,m} >= \int_0^T y_p(t) u_{p,m}(t)\, dt = 0$$

$$\int_0^T \cos(2\pi(f_c + m\Delta f)t + \pi/2) \cdot \cos(2\pi(f_c + m\Delta f)t)\, dt$$
$$= 0.$$

Noncoherent Reception

- Solution to this problem is to use $|\langle u_k, u_l \rangle|$ instead of $\mathrm{Re}\{\langle u_k, u_l \rangle\}$.
- Show that the design requirement for non-coherent reception is $\Delta f = 1/T$.

Summarizing the concept of Orthogonality

- For a signal set $\{s_k(t)\}$, orthogonality requires that for $k \neq l$, we have

$$\begin{aligned} \mathrm{Re}\{\langle s_k, s_l \rangle\} = 0 &\qquad \text{coherent orthogonality criteria} \\ \langle s_k, s_l \rangle = 0 &\qquad \text{non-coherent orthogonality criteria} \end{aligned}$$

- Bandwidth required for M-ary orthogonal modulation with coherent detection criteria is $\frac{M}{2T}$ while that for orthogonal modulation with non-coherent detection is $\frac{M}{T}$.

- Similarly the bandwidth efficiency for the two schemes is $\eta_B = (\log_2 2M)/M$ for coherent criteria while $\eta_B = (\log_2 M)/M$ for non-coherent criteria.

Biorthogonal Modulation

- If a signal set $\{s_k(t)\}$ for $k = 0, \ldots, M-1$ denote orthogonal modulation, then set $\{\pm s_k(t)\}$ containing $2M$ waveforms represents biorthogonal modulation.

- Bandwidth efficiency of biorthogonal modulation is twice that of orthogonal modulation, i.e., $\eta_B = \frac{\log_2 4M}{M}$.

- Note that this is only applicable to coherent systems.

EC5.203 Communication Theory (3-1-0-4)

Lectures 16 and 17:

Noise Modelling,
And
Linear Operations on Random Process

17 and 20 March 2025

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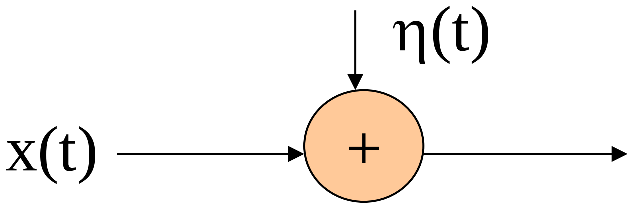
References

- Chap. 5 (Madhow)
  - Sections 5.1-5.7: Probability theory and Random Process (Self-Study, Not part of syllabus)
  - Section 5.8: Noise Modeling
  - Section 5.9: Linear operation on Random Processes

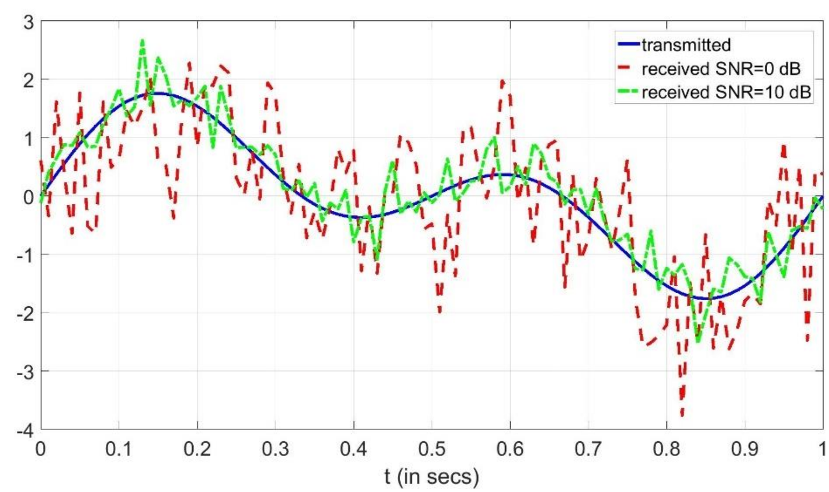
Noise Modeling

Noise in Communication Systems

- Noise is any undesired signal corrupting the desired signal
- It can be man-made or naturally occuring
  - Naturally occuring: thermal noise and shot noise in the receiver, EMI, atmospheric noise, cosmic noise
  - Man-made: Microwave, power-line interference, electric motors, ignition systems, interference from other signal in same band
- Mostly thermal noise is dominant!
- Generally additive nature is assumed

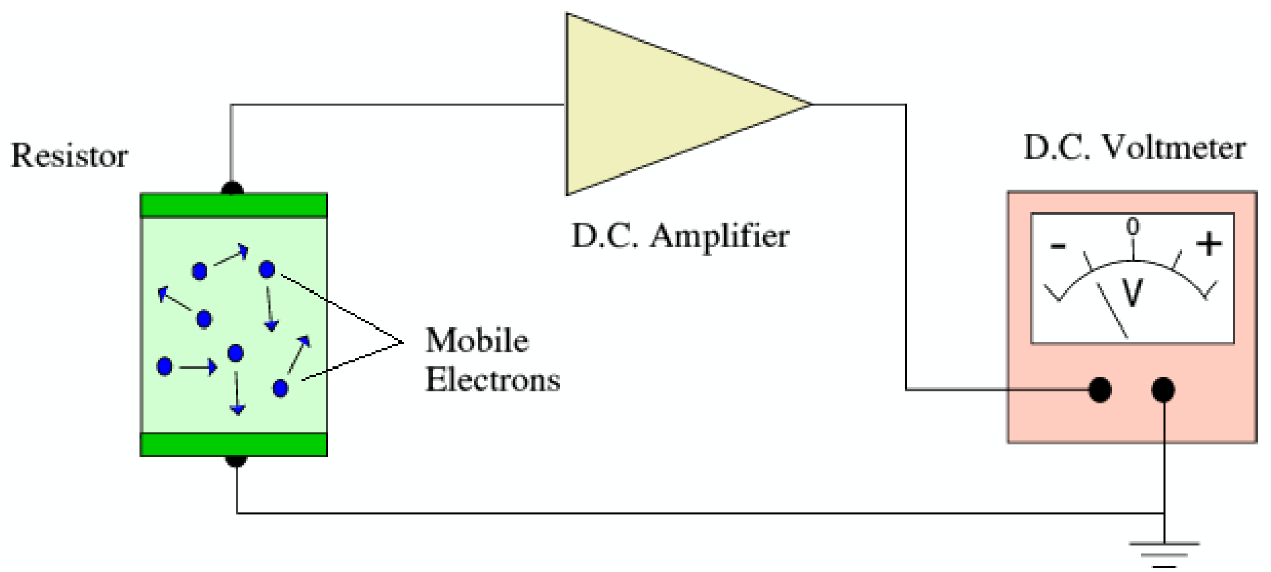


$$y(t) = x(t) + \eta(t)$$



Thermal Noise

- It is also called as Johnson-Nyquist noise
- It is the electronic noise generated by the thermal agitation of the charge carriers (usually the electrons) inside an electrical conductor at equilibrium, which happens regardless of any applied voltage
- This is type of intrinsic noise



https://www.st-andrews.ac.uk/~www_pa/Scots_Guide/iandm/part3/page1.html

Thermal Noise..

- The mean square voltage across a resistor R at temperature $\mathcal{T}$ degree corresponding to bandwidth B is

$$\overline{v_n^2} = 4Rk_B\mathcal{T}B$$

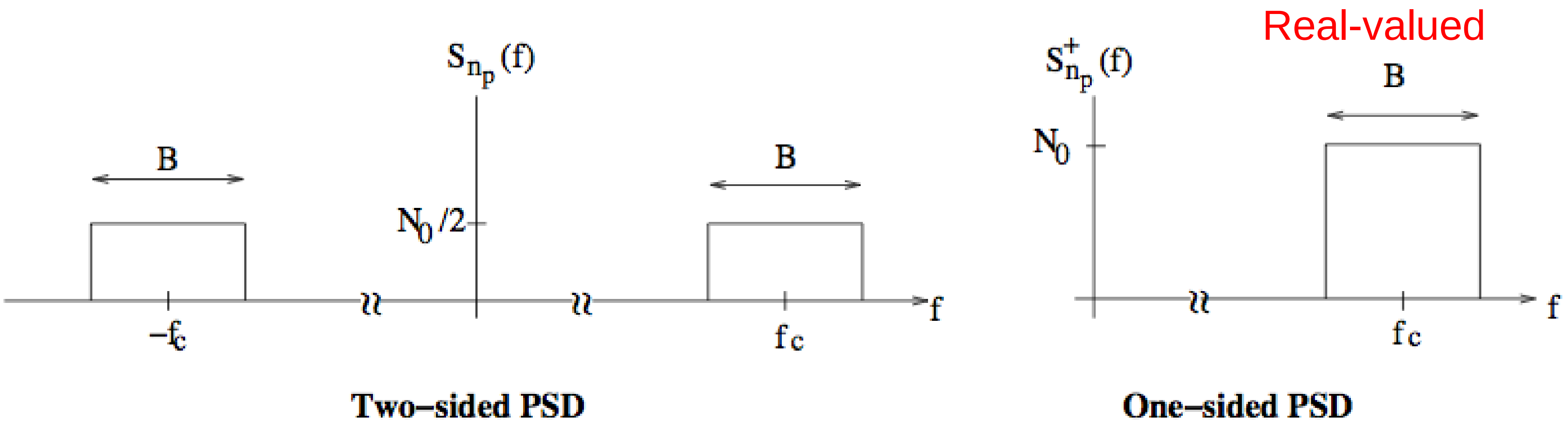
where $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann's constant

- If we connect the noise source to a matched load of impedance R, the mean squared power delivered is

$$\overline{P_n^2} = \frac{\overline{(v_n/2)^2}}{R} = k_B\mathcal{T}B = N_0B$$

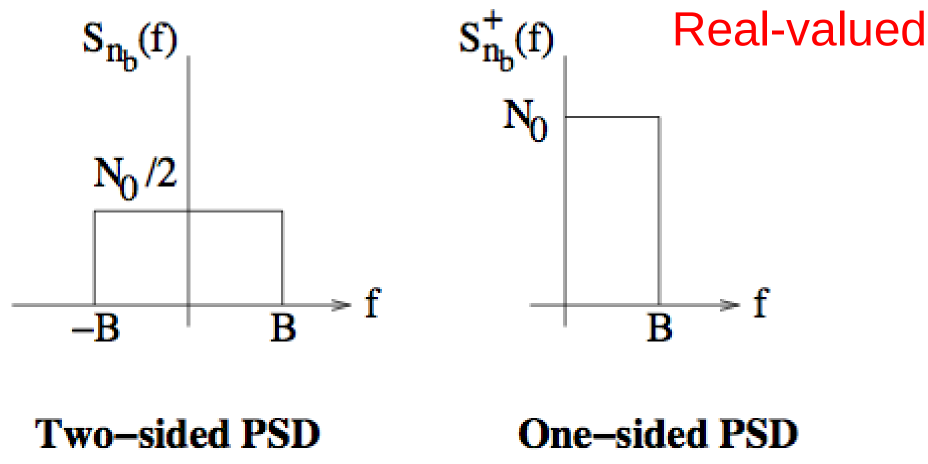
- PSD is given by $N_0 = k_B\mathcal{T}$, a constant for a given $\mathcal{T}$

Modeling Noise: *Passband model*



- Receiver noise is modeled as random process with zero DC value and with a flat PSD or white over a band of interest.
- Two-sided PSD is $N_0/2$ while one-sided is N_0 so that noise power in Bandwidth is N_0B.
- Key noise mechanisms in communication systems such as thermal and shot noise are both white.

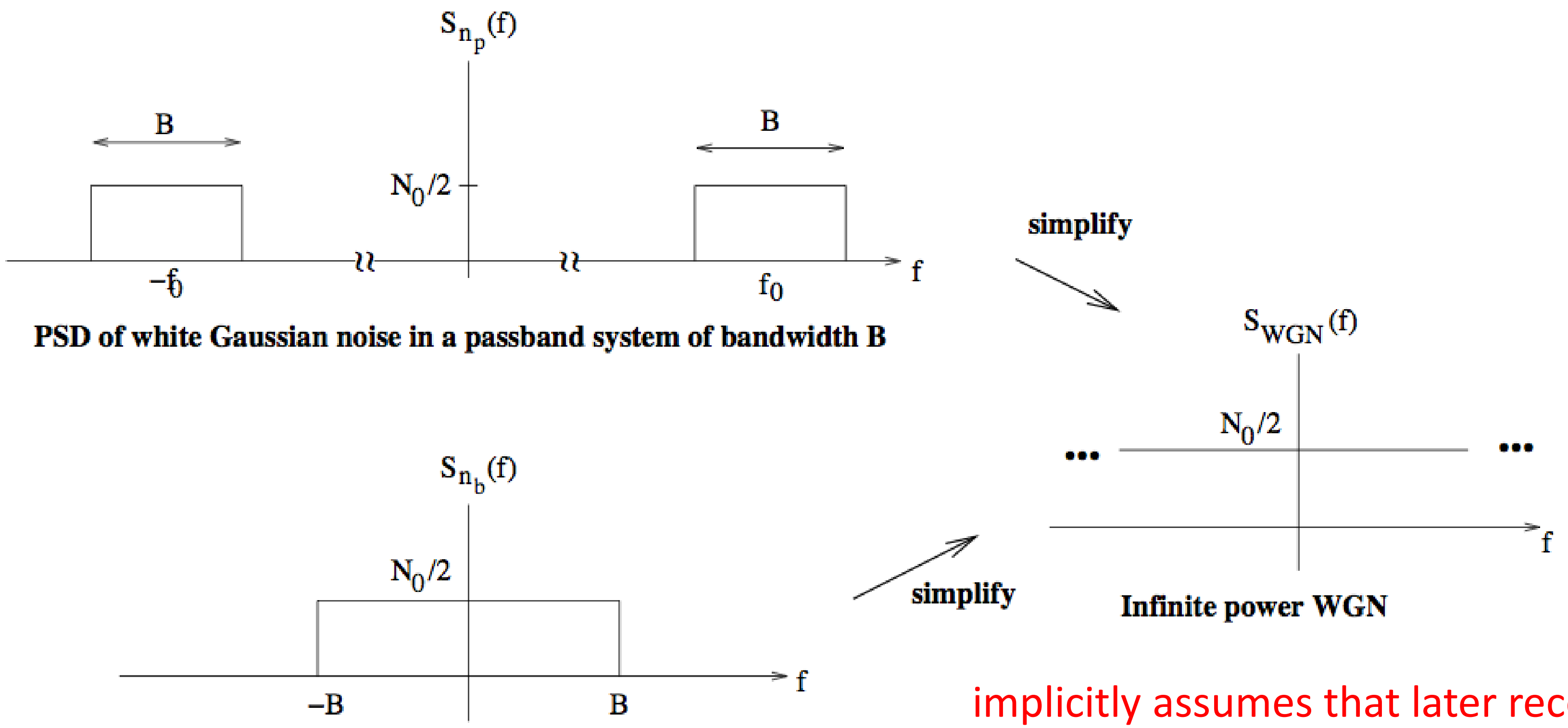
Modeling Noise: *Baseband model*



- Receiver noise is modeled as random process with zero DC value and with a flat PSD or white over a band of interest.
- Two-sided PSD is $N_0/2$ while one-sided is N_0 so that noise power in Bandwidth is N_0B.

Unified Model: *White Noise*

- Simplify the model by removing band limitation so that white applies to all systems, passband and baseband.



PSD of white Gaussian noise in a passband system of bandwidth B

PSD of white Gaussian noise in a baseband system of bandwidth B

implicitly assumes that later receiver processing will limit the bandwidth

Modeling Noise as Gaussian

- Assume that the noise is Gaussian
- For example, thermal noise can be modeled as Gaussian: random motion of large number of charge carriers $\rightarrow$ Gaussianity due to central limit theorem

White Gaussian Noise (WGN)

- Combining Gaussian and white assumptions, we get white Gaussian noise (WGN) model.
- One of the widely used models in communications
- Also called additive white Gaussian noise (AWGN) since WGN is additive in nature.
- Real-valued WGN is a zero mean process, WSS Gaussian random process

$$S_n(f) = \frac{N_0}{2} = \sigma^2 \leftrightarrow R_n(\tau) = \frac{N_0}{2}\delta(\tau) = \sigma^2\delta(\tau)$$

where σ^2 is two-sided noise PSD (also called noise variance per dimension)

Example computation: Tutorial

5 GHz WLAN with receiver bandwidth 20 MHz and receiver noise figure 6 dB. What is the noise power?

$$P_n = N_0 B = kT_0 10^{F/10} B = (1.38 \times 10^{-23})(290)(10^{6/10})(20 \times 10^6)$$
$$= 3.2 \times 10^{-13} \text{ Watts} = 3.2 \times 10^{-10} \text{ milliWatts (mW)}$$

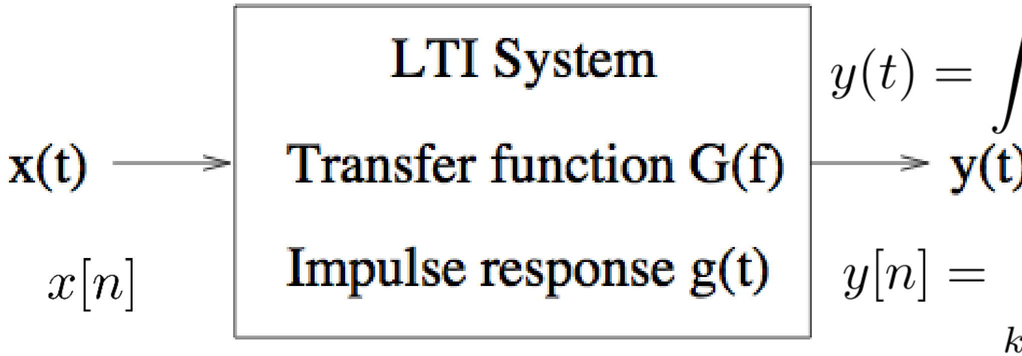
$\downarrow$ convert to dBm

$$P_{n,\text{dBm}} = 10 \log_{10} P_n(\text{mW}) = -95\text{dBm}$$

Can do this computation directly in the dB domain.

Filtering

Filtered Random Processes

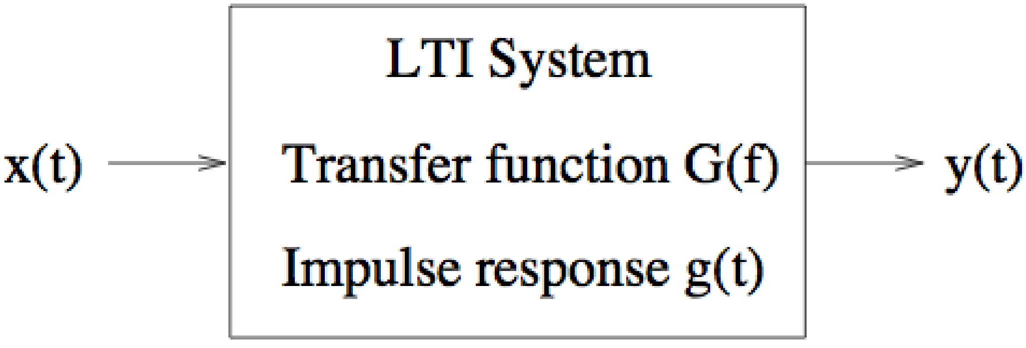


$$y(t) = \int_{-\infty}^{\infty} x(t-\tau]h(\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

- What is distribution of $y(t)$ if $x(t)$ is Gaussian distributed?

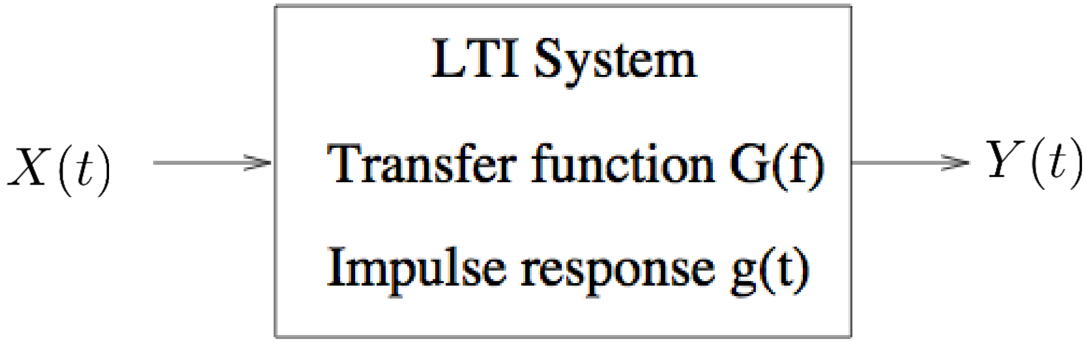
Filtered Random Processes



- Show that the output PSD is given by

$$S_y(f) = S_x(f)|G(f)|^2$$

WSS Filtered Random Processes



- For random processes $X(t)$ and $Y(t)$, show that $Y(t)$ is WSS if $X(t)$ is WSS.
- Also, show that

$$S_Y(f) = S_X(f)|G(f)|^2$$
$$\updownarrow$$
$$R_Y(\tau) = R_X(\tau) * g(\tau) * g^*(-\tau)$$

Wide Sense Stationarity

- A random process X is said to be WSS if

$$m_X(t) = \text{constant} \quad \forall t$$
$$R_X(t_1, t_2) = R_X(t_1 - t_2, 0) \quad \forall t_1, t_2$$

- A WSS random process has shift-variant second order statistics

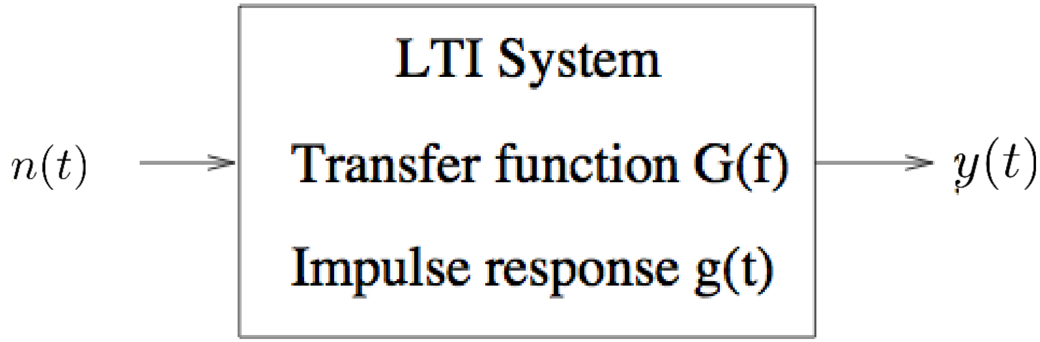
- Expressing the autocorrelation function as a function of $\tau = t_1 - t_2$,

$$R_X(\tau) = E[X(t)X^*(t + \tau)]$$

while the autocovariance is given by

$$C_X(\tau) = R_X(\tau) - |m_X|^2$$

Example

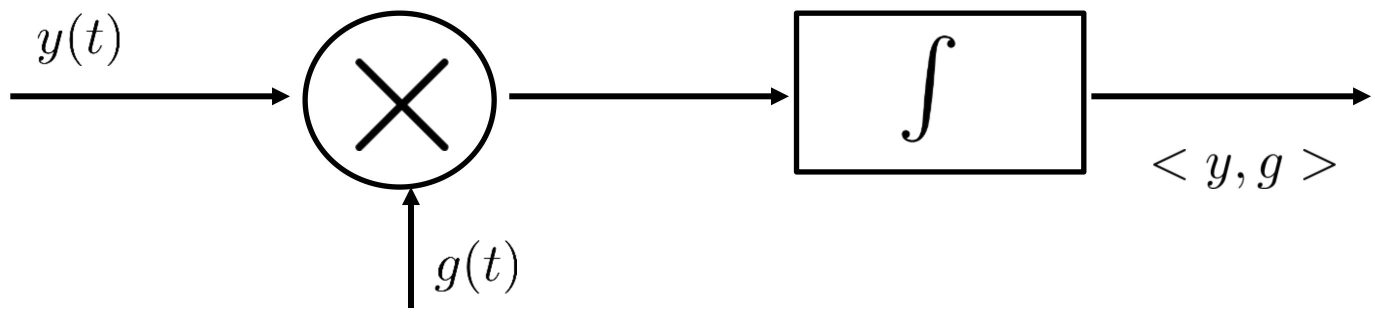


- White noise $n(t)$ with PSD $S_n(f) = \frac{N_0}{2}$ is passed through an LTI system with impulse response $g(t)$. Find the PSD, autocorrelation function and power of the output $y(t) = n(t) * g(t)$.

Example: Tutorial

- Suppose that WGN $n(t)$ with PSD $\sigma^2 = \frac{N_0}{2} = \frac{1}{4}$ is passed through an LTI system with impulse response $g(t) = I_{[0,2]}(t)$ to obtain the output $y(t) = n(t) * g(t)$.
  - Find the autocorrelation function and PSD of y.
  - Find $E[y^2(100)]$.
  - Is y a stationary random process?
  - Are $y(100)$ and $y(101)$ independent random variables?
  - Are $y(100)$ and $y(102)$ independent random variables?

Definition and Motivation

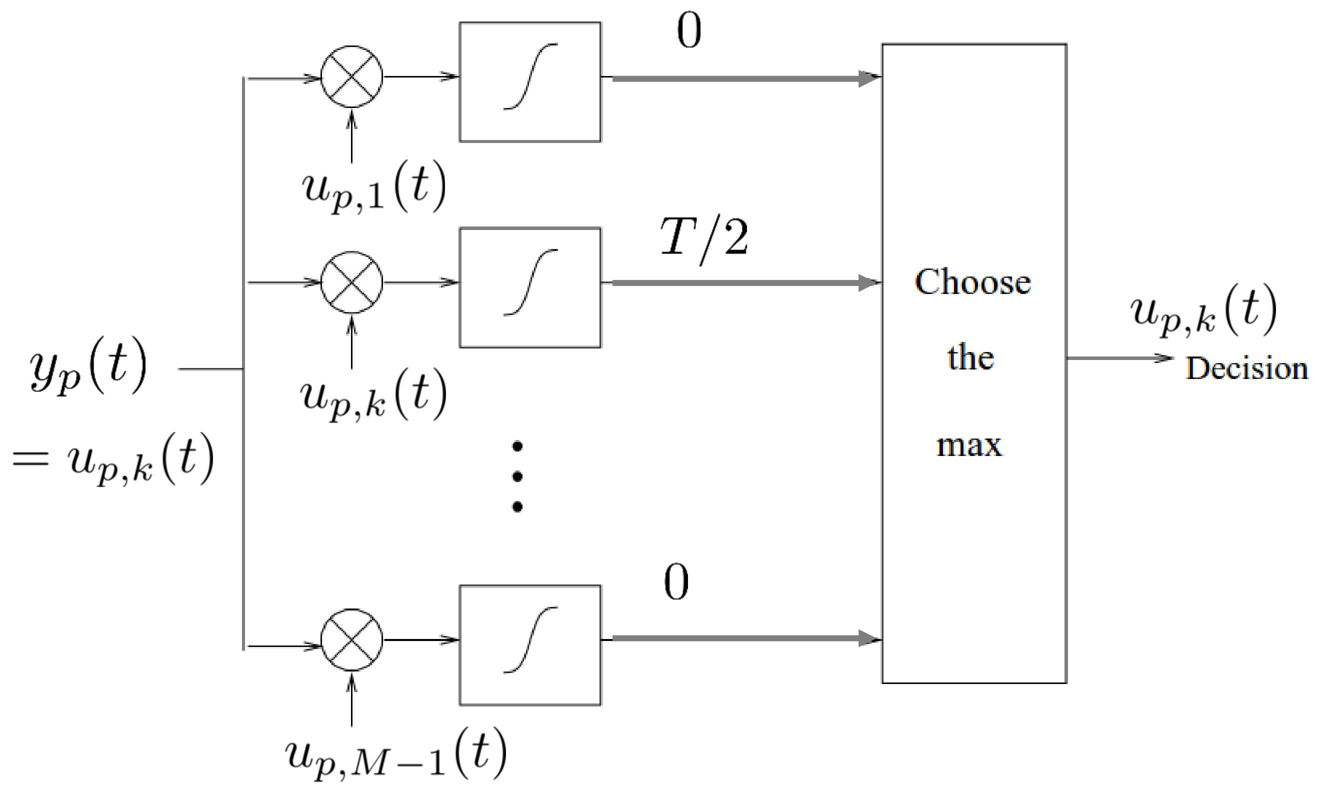


- Correlation between signals $y(t)$ and $g(t)$ is the inner product between the two and is given by

$$< y, g >= \int_{-\infty}^{\infty} y(t) g^*(t) dt$$

- One of the most common operations in communications.
- Used in demodulation of orthogonal waveforms.

Recap: *FSK coherent demodulation*



- Correlate the incoming signal with all M reference sinusoidal tones (passband signal) given by

$$u_{p,k} = \cos(2\pi(f_c + k\Delta f)t) \quad 0 \le t \le T$$

for $k = 0, 1, \ldots, M-1$.

- Choose $u_{p,m}$ for which the output is maximum among all the correlated outputs.

SNR

- Consider that the received signal

$$y(t) = s(t) + n(t)$$

  where $s(t)$ is a deterministic signal, corresponding to specific choice of transmitted symbols and $n(t)$ is zero-mean white noise with PSD $S_n(f) = \frac{N_0}{2}$.

- Considering real-valued signals, show that SNR at the output of correlator is given by

$$\text{SNR} = \frac{|< s, g >|^2}{\frac{N_0}{2}||g||^2}$$

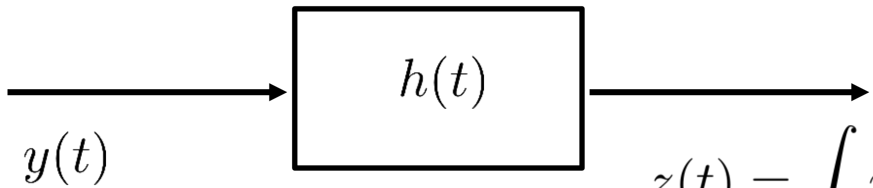
- Find $g(t)$ which will maximize the output SNR? Also find the maximum SNR!

- Also valid for complex signal and noise scenarios with $g(t) = cs^*(t)$

Theorem 5.9.1

- For linear processing of a signal $s(t)$ corrupted by white noise, the output SNR is maximized by correlating against $s(t)$. The resulting SNR is given by
$$\mathrm{SNR}_{\max} = \frac{2||s||^2}{N_0}$$

Filter as Correlator



$$z(t) = \int y(\tau)h(t-\tau)dt$$

- Any correlation example can be implemented using a filter and a sampler:
  - For $h(\tau) = g^*(-\tau)$,

$$z(t) = \int y(\tau)h(t-\tau)d\tau = \int y(\tau)g^*(\tau - t)d\tau$$

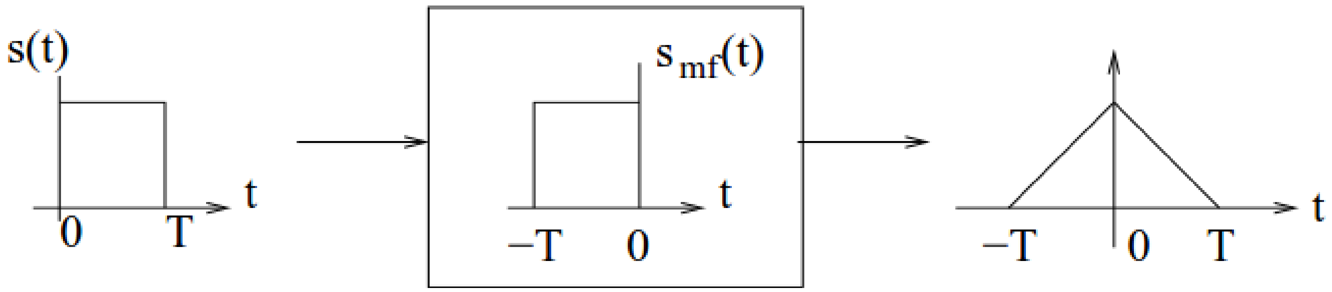
  - Sampled at $t = 0$, we get

$$z(0) = \int y(\tau)g^*(\tau)d\tau = < y, g >$$

- If $y(t) = s(t) + n(t)$, then choosing $g(t) = s^*(-t)$ maximizes SNR at the output of filter.

Matched Filter

- **Theorem**: For linear processing of a signal $s(t)$ corrupted by white noise, the output SNR is maximized by employing a matched filter with impulse response $s_{MF}(t) = s^*(-t)$, sampled at $t = 0$.



- When the received signal $y(t) = s(t) + n(t)$, optimum sampling time is $t = 0$. When the signal is delayed by $t = t_0$, the peak occurs at $t = t_0$, which now becomes the sampling time.

- Thus matched filter enables us to implement an infinite bank of correlators, each corresponding to a version of our signal template at a different delay.

EC5.203 Communication Theory I (3-1-0-4):

Lecture 18:
Optimal Demodulation

24 March 2025

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Optimal Demodulation

- In 16 QAM, one of 16 passband waveforms corresponding to 16 symbols is sent, where each passband waveform is given by

$$s_i(t) = s_{b_c,c_s} = b_c p(t) \cos(2\pi f_c t) - b_s p(t) \sin(2\pi f_c t)$$

  where b_c, b_s each takes value in $\{\pm 1, \pm 3\}$

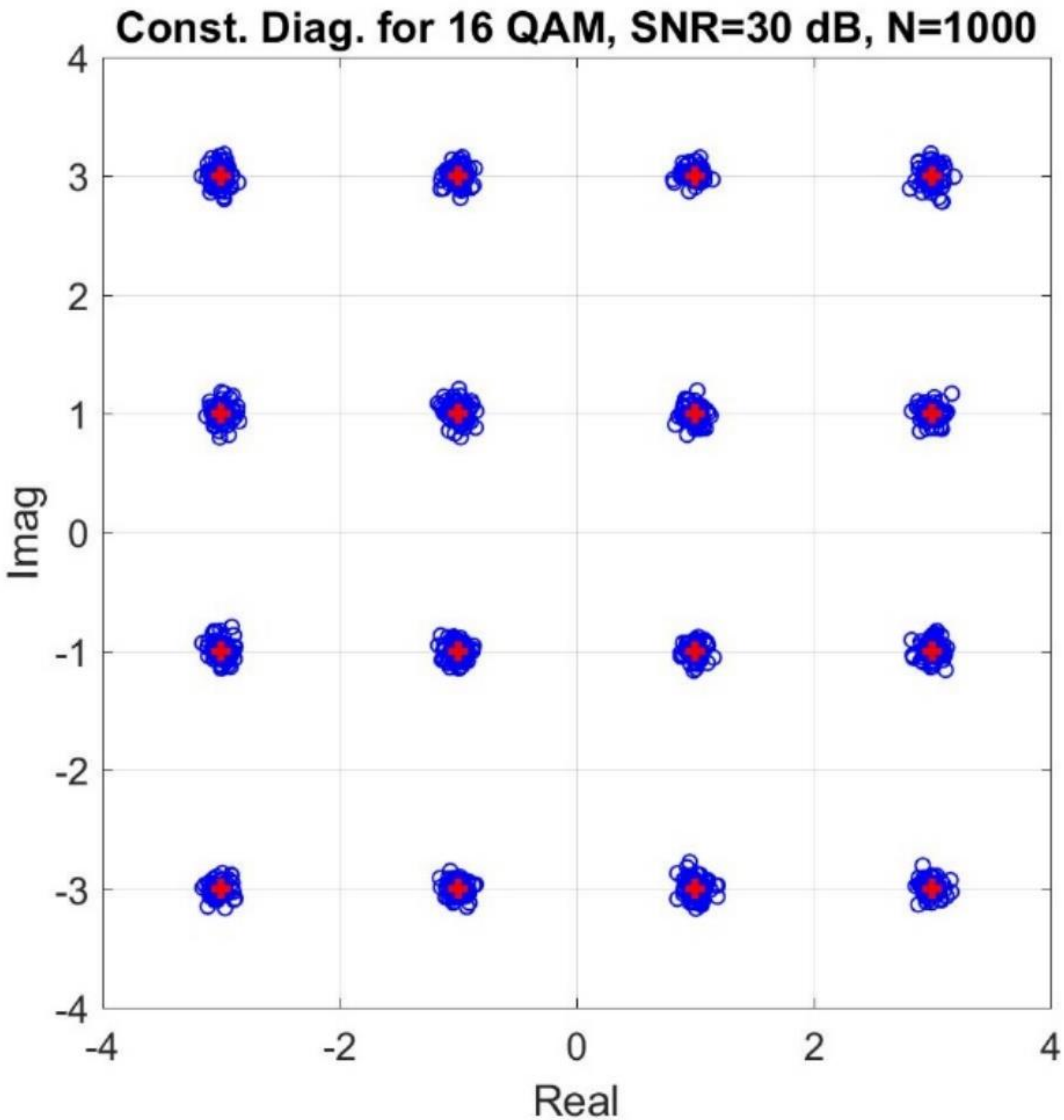
- At the receiver, we have noisy observations

$$y(t) = s_i(t) + n(t)$$

- Now, we are faced with a hypothesis testing problem at the receiver: we have M possible hypotheses about which signal was sent.

- Hypothesis: Possible cause of an event.

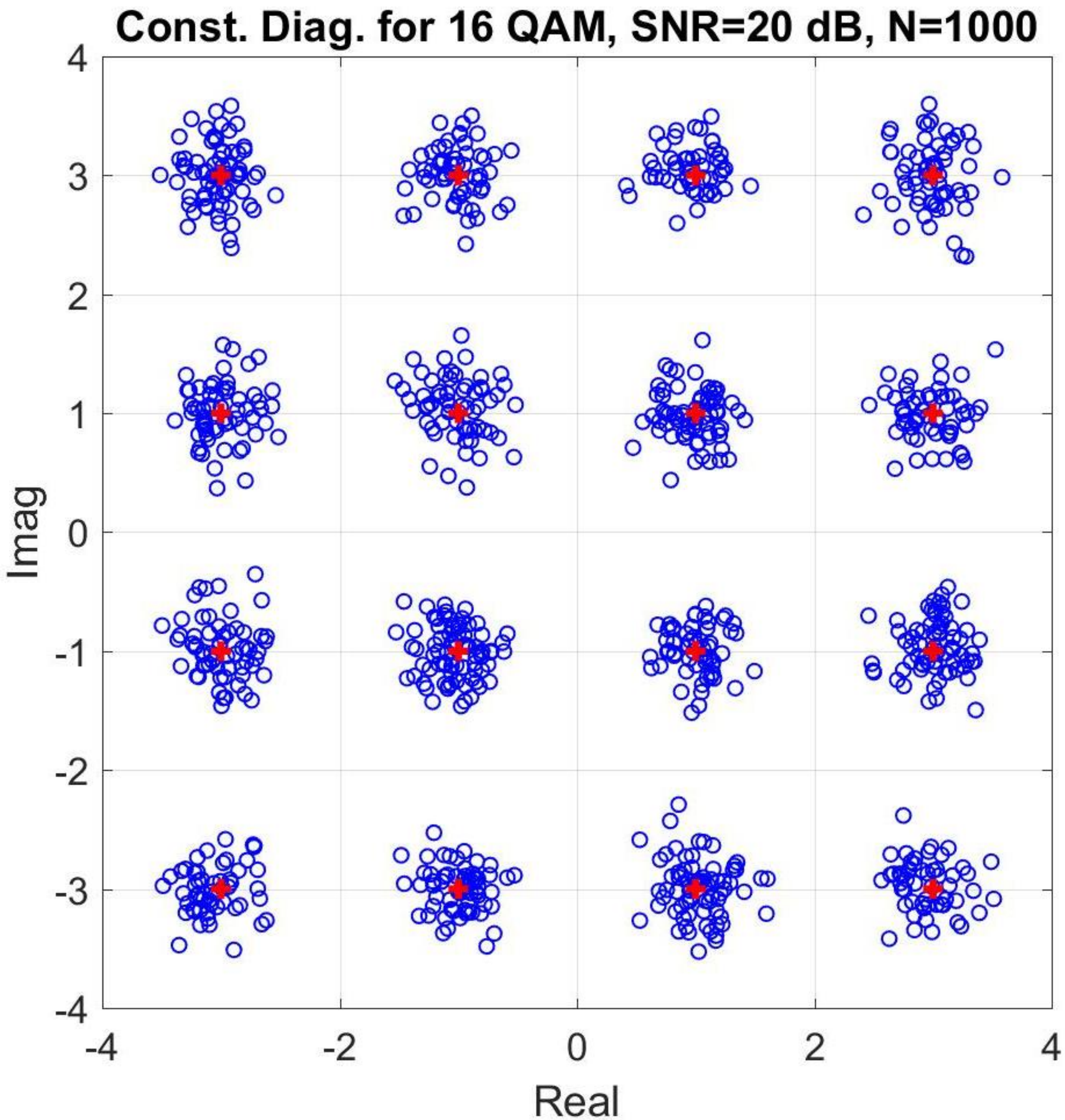
Received Data at SNR = 30 dB



$y(t) = s_i(t) + n(t)$

$n(t)$ is AWGN

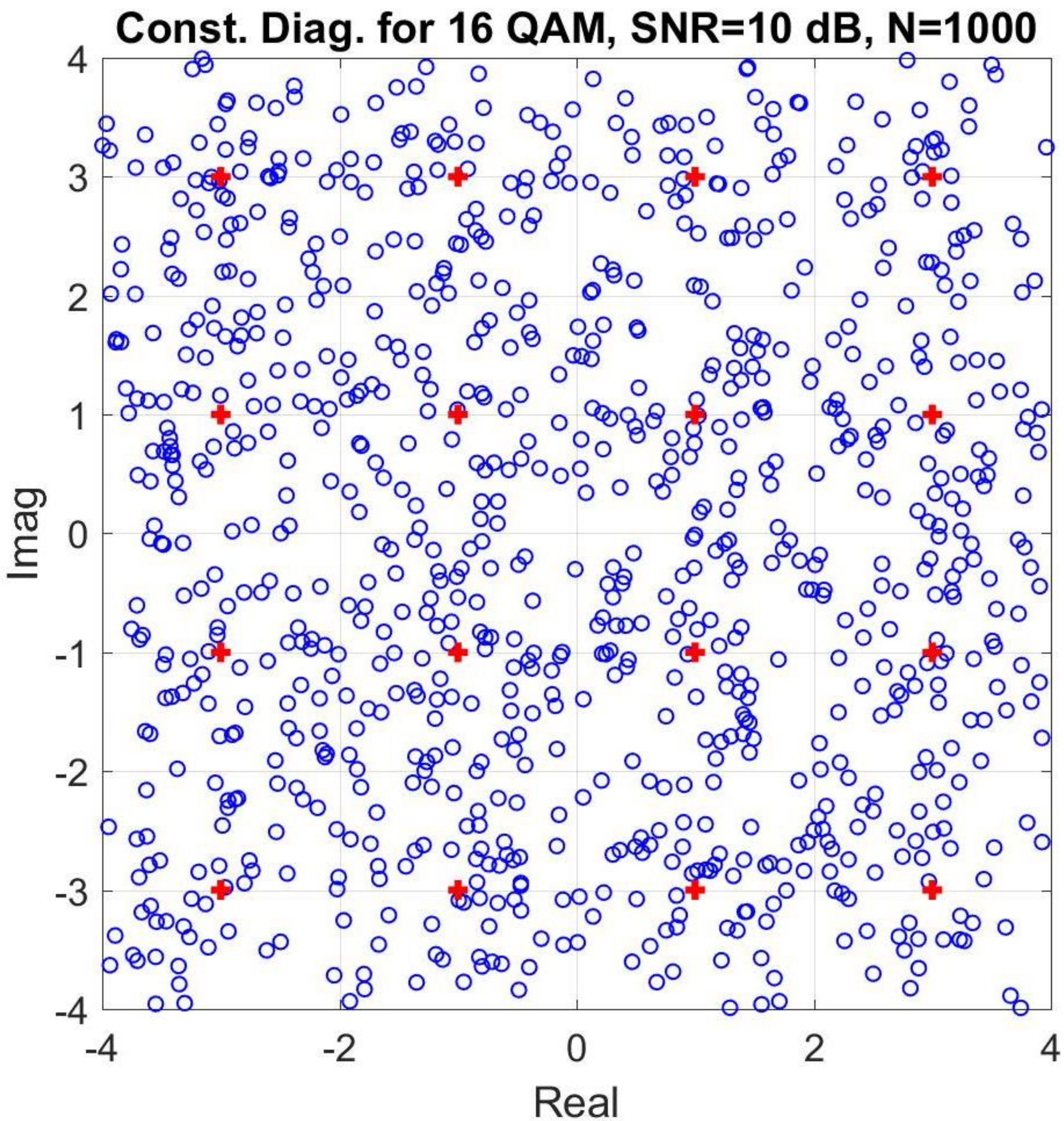
Received Data at SNR = 20 dB



$$y(t) = s_i(t) + n(t)$$

$n(t)$ is AWGN

Received Data at SNR=10dB



$$y(t) = s_i(t) + n(t)$$

$n(t)$ is AWGN

Optimal Demodulation

- In 16 QAM, one of 16 passband waveforms corresponding to 16 symbols is sent, where each passband waveform is given by

$$s_i(t) = s_{b_c,c_s} = b_c p(t)\cos(2\pi f_c t) - b_s p(t)\sin(2\pi f_c t)$$

  where b_c, b_s each takes value in $\{\pm 1, \pm 3\}$.

- At the receiver, we are faced with a hypothesis testing problem: we have M possible hypotheses about which signal was sent.

- Based on the observations

$$y(t) = s_i(t) + \boxed{n(t)} \quad \text{AWGN}$$

  we are interested in finding a decision rule to make a best guess which hypothesis was sent.

- For communications applications, performance criteria is to minimize the probability of error (i.e., the probability of making a wrong guess).

S&S Recap: Signal Energy

- The energy in a CT signal $x(t)$ over time interval (t_1, t_2)

$$\int_{t_1}^{t_2} |x(t)|^2 \; dt$$

  where $x(t)$ is a complex signal.

- The energy in a DT signal $x[n]$ over sample interval $[n_1, n_2]$ is

$$\sum_{n_1}^{n_2} |x[n]|^2$$

- The energy in a CT $x(t)$ and a DT signal $x[n]$ over infinite time interval respectively are

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \; dt \qquad \text{and} \qquad E_\infty = \sum_{-\infty}^{\infty} |x[n]|^2$$

S&S Recap: Signal Power

- The power in a continuous-time signal $x(t)$ over time interval (t_1, t_2)

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, \mathrm{d}t$$

- The power in a discrete-time signal $x[n]$ over samples (n_1, n_2) is

$$\frac{1}{n_2 - n_1 + 1} \sum_{n_1}^{n_2} |x[n]|^2$$

- The power in $x(t)$ and $x[n]$ over infinite time interval respectively are

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \qquad \text{and} \qquad P_\infty = \lim_{N \to \infty} \frac{1}{2N} \sum_{-N}^{N} |x[n]|^2$$

S&S Recap: Note on Dimension of Energy and Power Definitions

- Consider an example where $v(t)$ and $i(t)$ are the instantaneous voltage and current across a resistor R, then the instantaneous power across the resistor is

$$p(t) = v(t)i(t) = \frac{v^2(t)}{R}$$

- The total energy dissipated over the time interval (t_1, t_2) is

$$E = \int_{t_1}^{t_2} p(t)\ dt$$

- The average power dissipated over the time interval (t_1, t_2) is

$$P = \frac{E}{t_2 - t_1}$$

The definitions used earlier are generic and may have wrong dimensions and scaling. The advantage is the convenience and wide applicability irrespective of where the signal is coming from.

Example 5.6.3

- **Binary on-off keying in Gaussian noise**

$$Y = m + n \quad \text{if 1 is sent}$$
$$Y = n \quad \text{if 0 is sent}$$

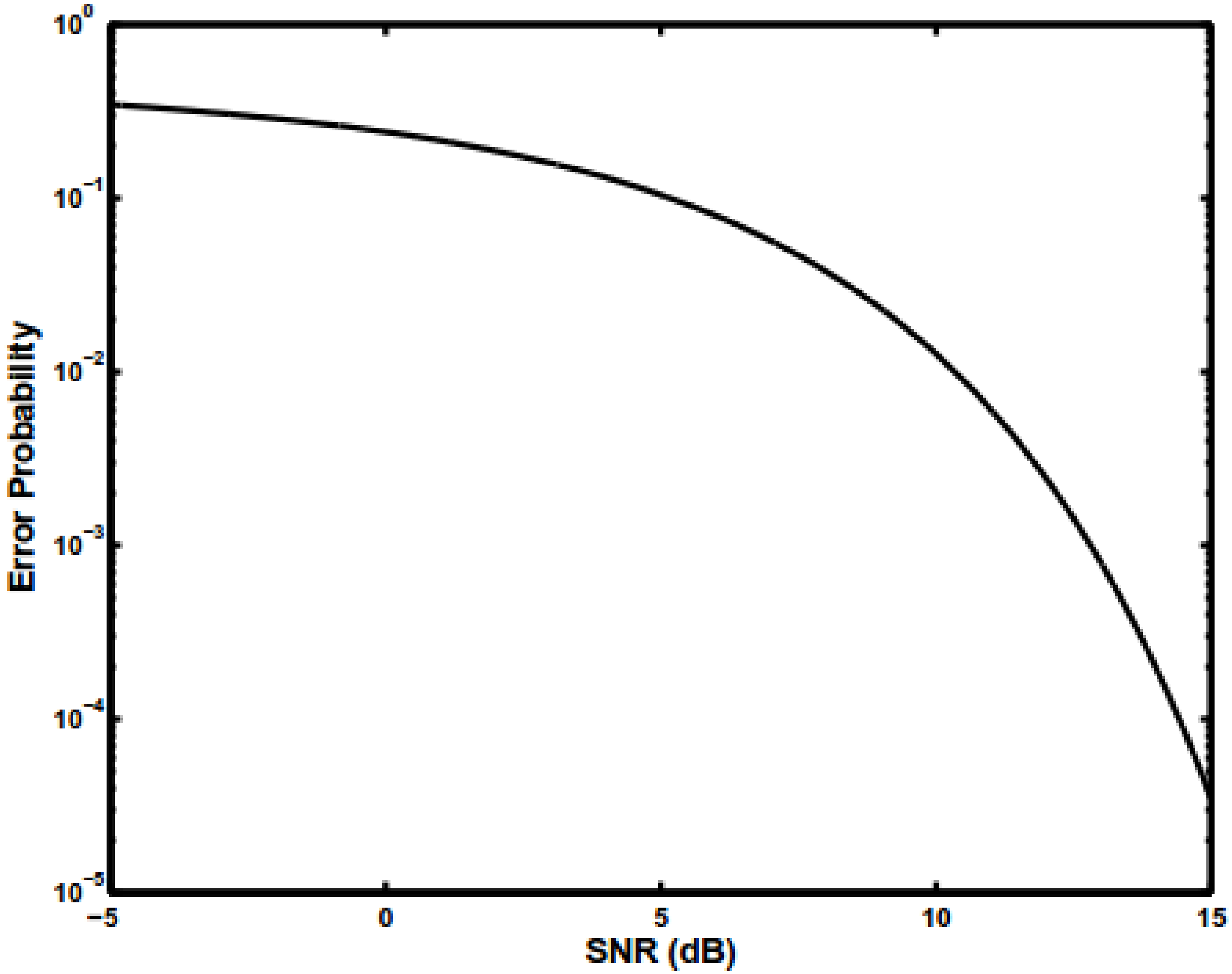
  Here Y is the received sample, $m > 0$ is some constant and n is AWGN sample with $\mathcal{N}(0, v^2)$.

- At the receiver, the detection strategy is

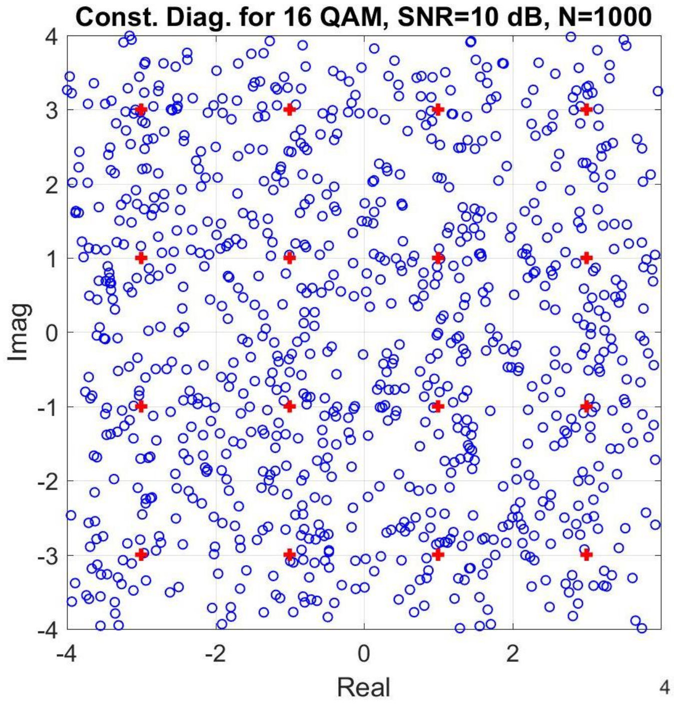
$$Y > m/2 \quad \text{Decide 1 is sent}$$
$$Y \leq m/2 \quad \text{Decide 0 is sent}$$

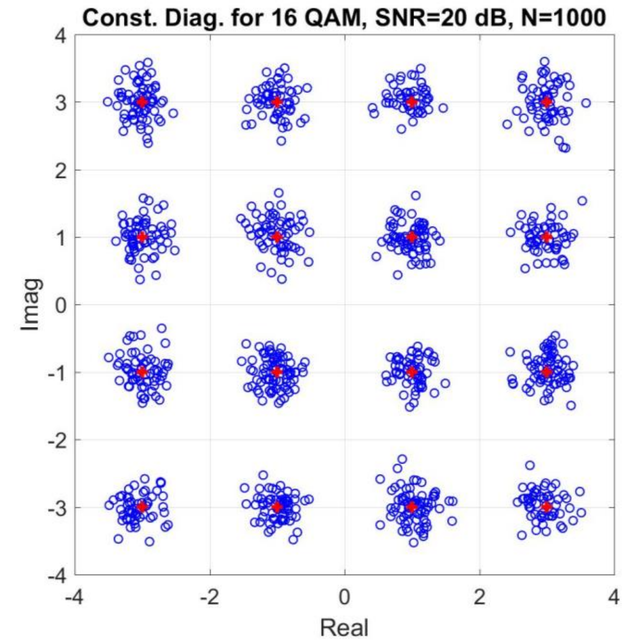
- Assuming that both 0 and 1 are equally likely,
  - Find the average signal power
  - Find the conditional probability of error conditioned on 0 being sent
  - Find the conditional probability of error conditioned on 1 being sent
  - Find average error probability
  - Find the probability of error for SNR of 13 dB?

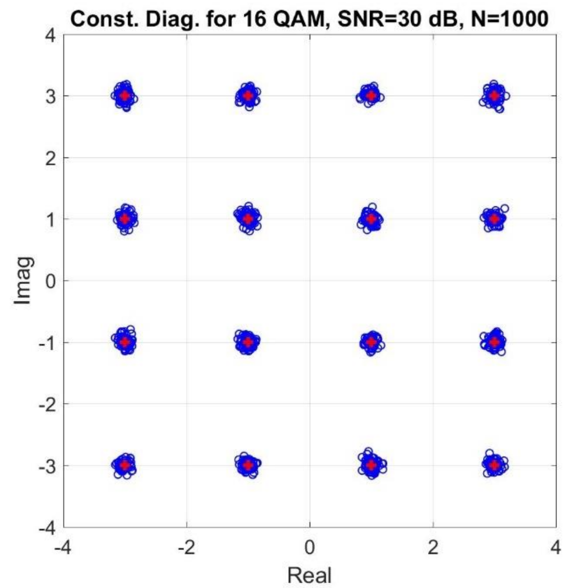
(Bit) Error Probability vs SNR for Example



Example: 16 QAM in AWGN



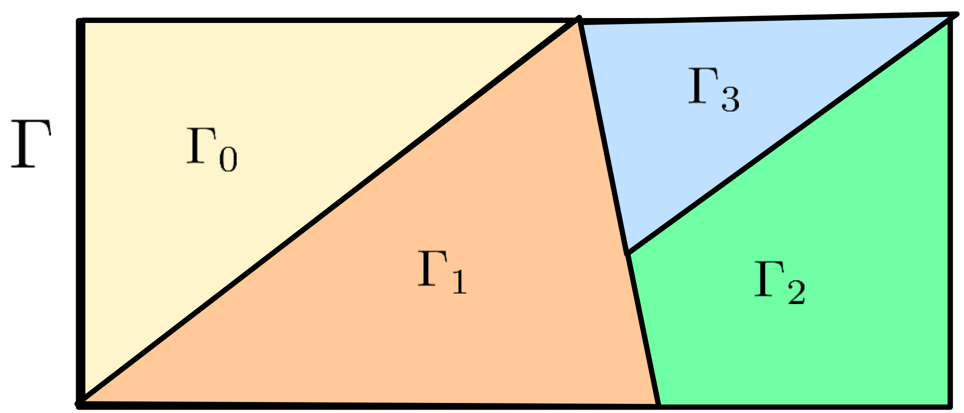




Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$

Example of Decision regions for M=4



Error Probabilities

- Conditional error probabilities, conditioned on H_i, is

$$
\begin{aligned}
P_{e|i} &= P(\text{decide } j \text{ for some } j \neq i | H_i \text{ is true}) \\
&= \sum_{j \neq i} P(Y \in \Gamma_j | H_i) \\
&= 1 - P(Y \in \Gamma_i | H_i)
\end{aligned}
$$

- Conditional probabilities of correct detection, conditioned on H_i, is

$$
\begin{aligned}
P_{c|i} &= P(Y \in \Gamma_i | H_i) \\
&= 1 - P_{e|i}
\end{aligned}
$$

- Average error probability

$$P_e = \sum_{i=1}^{M} \pi_i P_{e|i}$$

- Average probability of correct detection

$$P_c = \sum_{i=1}^{M} \pi_i P_{c|i}$$

Ingredients of Hypothesis Testing Framework

- Hypotheses $H_0, H_1, \ldots, H_{M-1}$
- Observation $Y \in \Gamma$
- Conditional densities $p(y|i)$ for $i = 0, 1, \ldots, M-1$
- Prior probabilities $\pi_i = P(H_i)$ with $\sum_i \pi_i = 1$
- Decision rule $\delta : \Gamma \to \{0, 1, M-1\}$
- Decision region $\Gamma_i : \{y \in \Gamma_i : \delta(y) = i\}$ for $i = 0, 1, M-1$

- In earlier example
  - Hypotheses H_0, H_1, Observation $Y \in \Gamma = \mathcal{R}$
  - Conditional densities $p(y|0)$ and $p(y|1)$
  - Prior probabilities π_0 and π_1
  - Decision rule δ:

$$\delta(y) = \begin{cases} 0, & y \leq m/2 \\ 1, & y > m/2 \end{cases}$$

  - Decision regions: $\Gamma_0 = (-\infty, m/2]$ and $\Gamma_1 = (m/2, \infty)$

MAP rule

- Definitions:
  - *A priori* probability: Before the data is observed : $P(H_i) = \pi_i$
  - *A posteriori* probability: After the data is observed: $P(H_i|y)$
- Maximum *a posteriori* probability (MAP) rule:

$$\delta_{\mathrm{MAP}}(y) = \arg\max_i P(H_i|Y = y)$$

where $i = 0, 1, \ldots, M-1$

- Using Bayes rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$, MAP rule can be rewritten as

$$\begin{aligned}
\delta_{\mathrm{MAP}}(y) &= \arg\max_i \frac{P(Y = y|H_i)P(H_i)}{P(Y = y)} \\
&= \arg\max_i \frac{p(y|i)\pi_i}{p(y)} \\
&= \arg\max_i p(y|i)\pi_i \\
&= \arg\max_i \log \pi_i + \log p(y|i)
\end{aligned}$$

Optimality of MAP (or MPE) rule

- Optimality of MAP rule: The MAP rule minimizes the probability of error. Proof!

ML rule

- Definitions:
  - Likelihood function: $p(y|i) = P(Y = y|H_i)$
- Maximum likelihood (ML) rule

  $$\delta_{\mathrm{ML}}(y) = \arg\max_i p(y|i)$$

  where $i = 0, 1, \ldots, M-1$
- Equivalently,

  $$\delta_{\mathrm{ML}}(y) = \arg\max_i \log p(y|i)$$

- ML is equivalent of MAP for equal prior probabilities, i.e., $\pi_i = \frac{1}{M}$, we have

  $$\begin{aligned}\delta_{\mathrm{MAP}}(y) &= \arg\max_i \log \pi_i + \log p(y|i) \\ &= \arg\max_i \log \frac{1}{M} + \log p(y|i) \\ &= \arg\max_i \log p(y|i)\end{aligned}$$

Binary Hypothesis Testing Problem

- For two hypotheses case, ML decision rule is

$$\delta_{\mathrm{ML}}(y) = \arg\max_i p(y|i)$$
$$= \arg\max_i \{p(y|0), p(y|1)\}$$

- Equivalently,

$$p(y|0) > p(y|1) \quad \rightarrow \quad \delta_{\mathrm{ML}}(y) = 0$$
$$p(y|1) > p(y|0) \quad \rightarrow \quad \delta_{\mathrm{ML}}(y) = 1$$

- This can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Similarly, MAP or MPE rule for binary hypothesis testing problem can be written as

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

$$P(H_1|Y = y) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0|Y = y)$$

Example of exponential distribution

- A binary hypothesis testing problem is specified as follows

$$H_0 : Y \sim \mathcal{E}(1)$$
$$H_1 : Y \sim \mathcal{E}(1/4)$$

  where $\mathcal{E}(\mu)$ denotes an exponential density $\mu e^{-\mu y}$ and CDF $1 - e^{-\mu y}$ where $y \geq 0$. Note that the mean of $\mathcal{E}(\mu)$ is $1/\mu$.

- Find the ML rule and the corresponding error probabilities.

- Find the MAP rule when the prior probability of H_1 is $1/5$. Also find the conditional and average error probabilities.

Likelihood Ratio

- For two hypotheses case, ML decision rule can be written as

$$p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} p(y|0)$$

- Equivalently,

$$\frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

  This ratio of likelihood functions is called likelihood ratio(LR) and denoted by $L(y)$ and the test is called as likelihood ratio test (LRT)

- Taking log of both sides

$$\log \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

  The test statistic in this case is Log of LR and is called log-likelihood ratio (LLR) while the test is called as LLRT

Likelihood Ratio: MAP

- For two hypotheses case, MAP decision rule is

$$\pi_1 p(y|1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 p(y|0)$$

- In terms of LR, the LRT is

$$L(y) = \frac{p(y|1)}{p(y|0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

- Taking log of both sides, the test is given in terms of LLR

$$\log L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1}$$

Signal Space Concepts

Receiver design as hypothesis testing

- Consider the multiple hypothesis testing

$$H_i \;:\; y(t) = s_i(t) + n(t) \quad i = 0, 1, \ldots, M-1$$

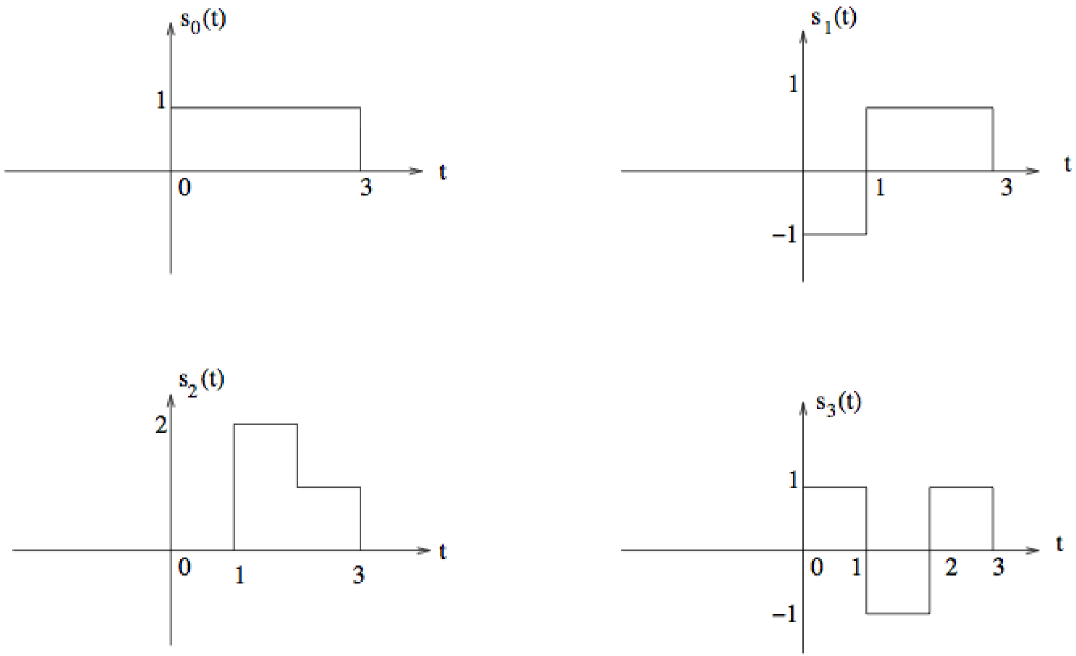
  where $n(t)$ is WGN with $S_n(f) = \frac{N_0}{2} = \sigma^2$

- Strategy:
  - Show that we can reduce the continuous time received signal to a finite-dimensional vector without losing information
  - Derive the optimal receiver based on the finite-dimensional vector observation
  - Map the optimal receiver back to continuous time

- This approach is based on signal space concepts: even though the received signal lives in an infinite-dimensional space, we can restrict attention to the subspace spanned by the signals that could have been transmitted

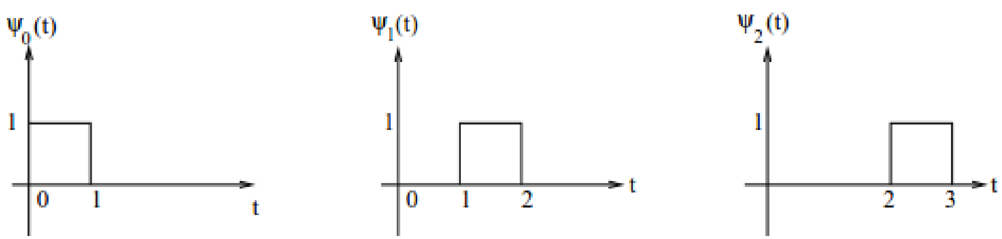
Signal space concepts: an example

- Four possible transmitted signals living in a 3-dimensional space

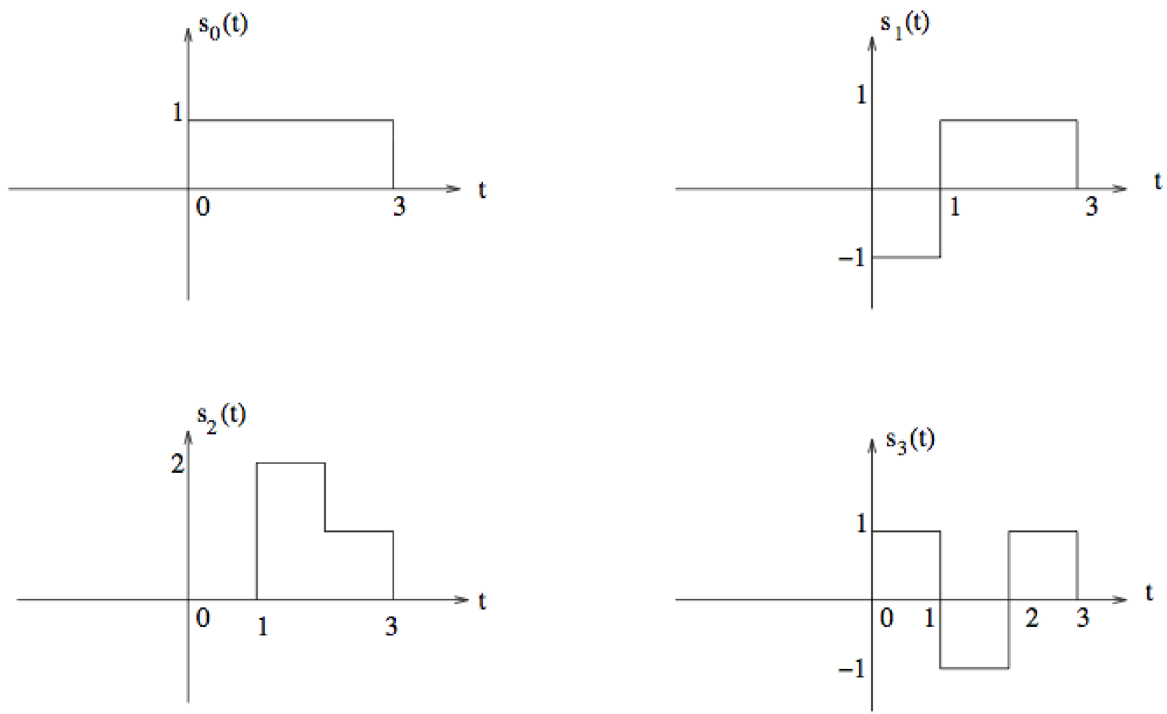


- Orthonormal basis (by inspection)

$$\psi_0(t) = I_{[0,1]}(t),\ \psi_1(t) = I_{[1,2]}(t),\ \psi_2(t) = I_{[2,3]}(t),$$



Example (continued)



- Expand with respect to basis $s_i(t) = \sum_{k=0}^{2} s_i[k]\psi_k(t)$
- Vectors of basis coefficient

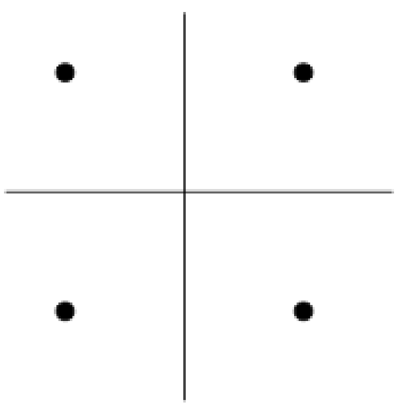
$$\mathbf{s}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \mathbf{s}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{s}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix};$$

Another example: 2D modulation

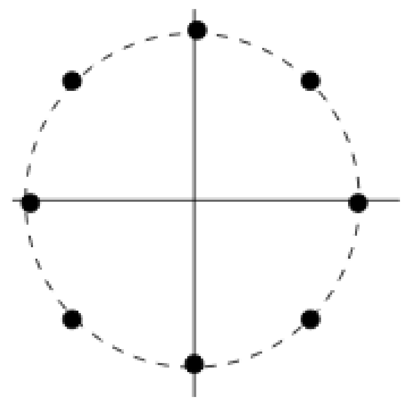
- 2D constellation for passband signaling: signal transmitted over a single symbol interval

$$s_i(t) = s_{b_c,b_s}(t) = b_c p(t) \cos 2\pi f_c t - b_s p(t) \sin 2\pi f_c t$$

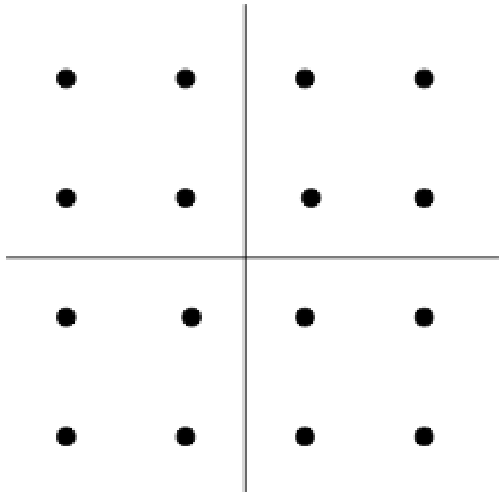
QPSK/4PSK/4QAM



8PSK



16QAM



- Signal is spanned by the following two signals

$$\phi_c(t) = p(t) \cos 2\pi f_c t; \quad \phi_s(t) = -p(t) \sin 2\pi f_c t$$

Another example: 2D modulation…

- Signal is spanned by the following two orthogonal signals

$$\phi_c(t) = p(t)\cos 2\pi f_c t; \quad \phi_s(t) = -p(t)\sin 2\pi f_c t$$

- Noting that $||\phi_c||^2 = ||\phi_s||^2 = \frac{1}{2}||p||^2$, corresponding orthonormal basis functions are

$$\psi_c(t) = \frac{\phi_c(t)}{||\phi_c||}; \quad \psi_s(t) = \frac{\phi_s(t)}{||\phi_s||};$$

- Expansion with respect to orthonormal basis is

$$s_{b_c,b_s}(t) = \frac{1}{\sqrt{2}}||p||b_c\psi_c(t) + \frac{1}{\sqrt{2}}||p||b_s\psi_s(t)$$

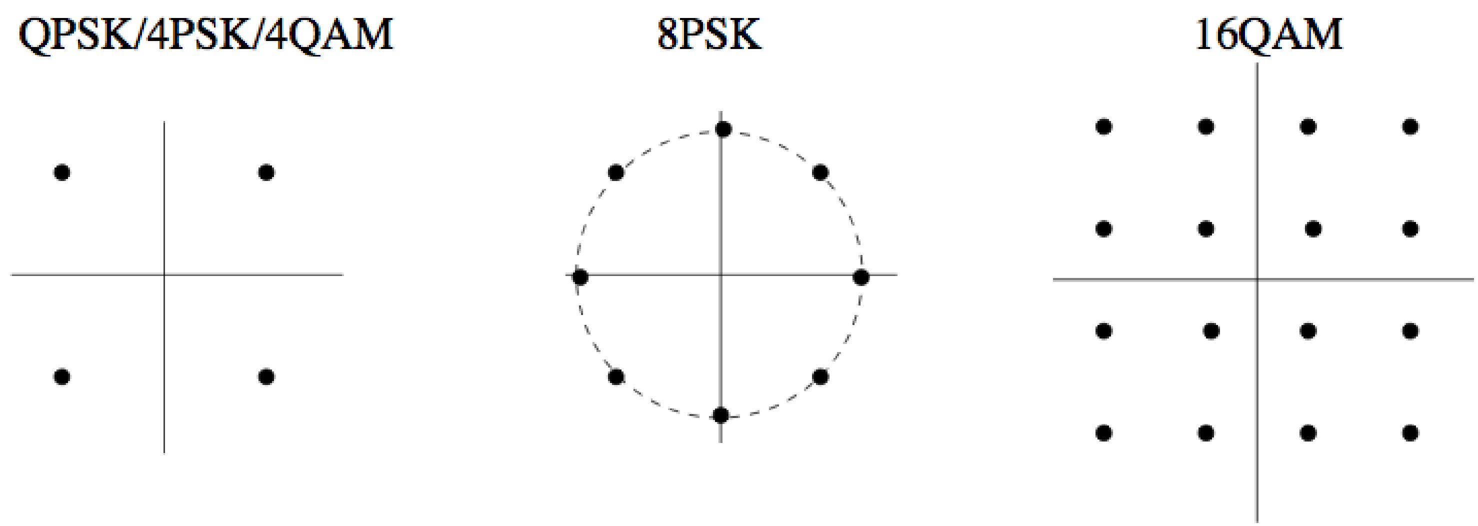
Basis expansion ⇔ constellation

- 2D constellation for passband signaling: signal transmitted over a single symbol interval

$$s_i(t) = s_{b_c,b_s}(t) = b_c p(t) \cos 2\pi f_c t - b_s p(t) \sin 2\pi f_c t$$

- Vector basis coefficients are given by

$$\mathbf{s}_i = \frac{1}{\sqrt{2}} ||p|| \begin{pmatrix} b_c \\ b_s \end{pmatrix}$$



From Signals to Vectors

- Signal space consists of all possible linear combinations of $s_0(t), s_1(t), \ldots,$ $s_{M-1}(t)$.
- We can always find an orthonormal basis for the signal space, i.e., $\psi_0(t)$, $\psi_1(t)$, $\ldots$, $\psi_{n-1}(t)$ where $n \leq M$
- We can express each signal as a vector of basic coefficients

$$s_i(t) = \sum_{k=0}^{n-1} s_i[k]\psi_k(t)$$

  where $s_i[k] =< s_i, \psi_k >$

- Using $s_i[k]$, we can now express $s_i(t)$ as a $n \times 1$ vector $\mathbf{s}_i = [s_i[0] \; s_i[1] \; \ldots s_i[n-1]]^T$
- Note that finite-dimensional basis always exists
- If it is not possible to find natural basis by inspection, then we can always use Gram-Schmidt orthoginalization procedure

Gram-Schmit Orthogonalization Process

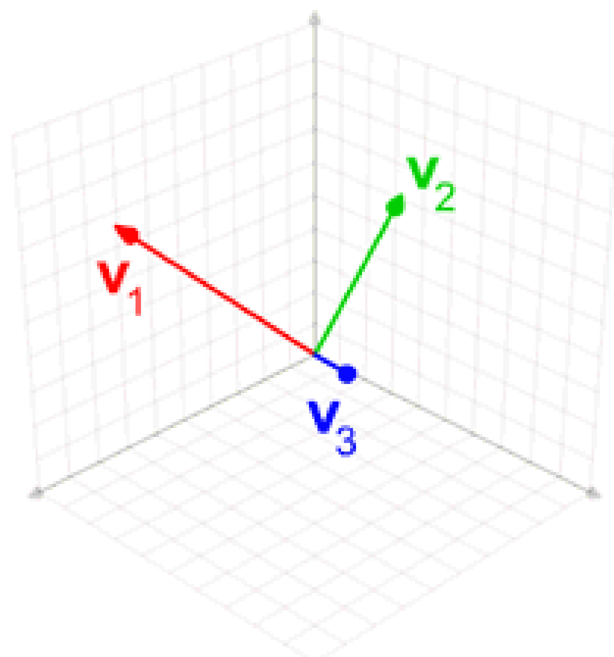
Orthogonal Vectors	Orthonormal Vectors
$\mathbf{u}_1 = \mathbf{v}_1,$	$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$
$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2),$	$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$
$\mathbf{u}_3 = \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3),$	$\mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}$
$\mathbf{u}_4 = \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4),$	$\mathbf{e}_4 = \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|}$
$\vdots$	$\vdots$
$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k),$	$\mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.$

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

This operator projects the vector **v** orthogonally onto the line spanned by vector **u**.

Source: Wikipedia http://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

Gram-Schmit Orthogonalization Process



$$\mathrm{proj}_{\mathbf{u}}\,(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u},$$

Source: Wikipedia http://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

Gram-Schmidt Orthogonalization Process

- **Step 0 (Initialization)**: Let $\phi_0 = s_0$. If $\phi_0 \neq 0$, then set $\psi_0 = \frac{\phi_0}{||\phi_0||}$. Note that ψ_0 provides a basis function for $\mathcal{S}_0$.

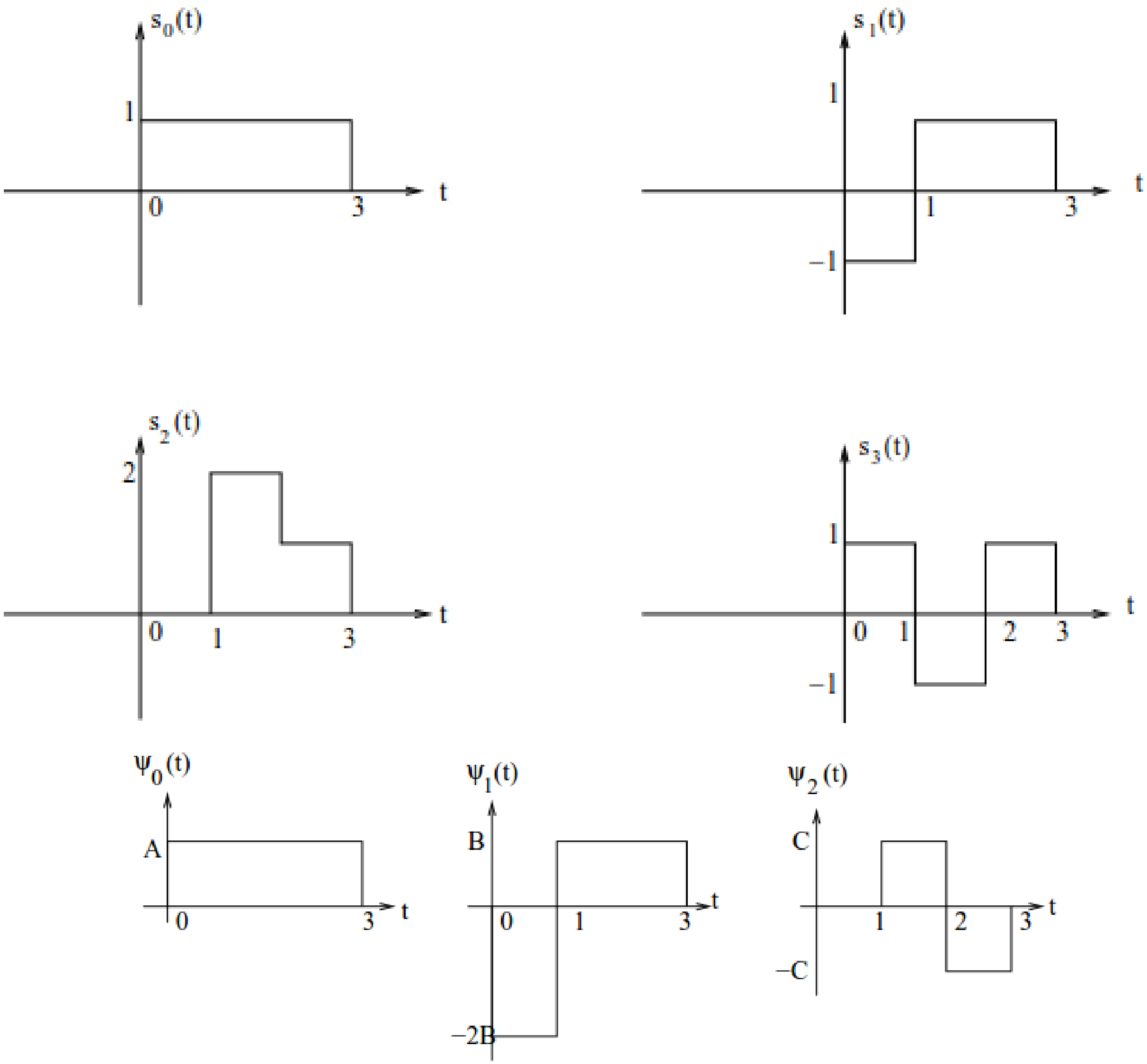
- **Step k**: Suppose that we have constructed an orthornormal basis $\mathcal{B}_{k-1} = \{\psi_0, \ldots, \psi_{m-1}\}$ for subspace $\mathcal{S}_{k-1}$ spanned by the first k signals $s_0, \ldots, s_{k-1}$. Note that $m \leq k$. Define

$$\phi_k(t) = s_k(t) - \sum_{i=0}^{m-1} < s_k, \psi_i > \psi_i(t)$$

- The signal $\phi_k(t)$ is the component of $s_k(t)$ orthogonal to the subspace $\mathcal{S}_{k-1}$. If $\phi_k \neq 0$, define a new basis function $\psi_m = \frac{\phi_k(t)}{||\phi_k||}$ and the basis as $\mathcal{B}_k = \{\psi_0, \ldots, \psi_m\}$. If $\phi_k = 0$, then $s_k \in \mathcal{S}_{k-1}$ and it is not necessary to update the basis. In this case $\mathcal{B}_k = \mathcal{B}_{k-1} = \{\psi_0, \ldots, \psi_{m-1}\}$.

Example: Gram-Schmidt Orthogonalization

- Find orthonormal basis set for these signals.



From signals to vectors

- Signal space consists of all possible linear combinations of $s_0(t)$, $s_1(t)$, …, $s_{M-1}(t)$.
- We can always find an orthonormal basis for the signal space, i.e., $\psi_0(t)$, $\psi_1(t)$, …, $\psi_{n-1}(t)$ where $n \leq M$

We can express each signal as a vector of basis coefficients

$$s_i(t) = \sum_{l=0}^{n-1} s_i[k]\psi_k(t), \text{ where } s_i[k] = \langle s_i, \psi_k \rangle \quad \leftrightarrow \mathbf{s}_i = \begin{pmatrix} s_i[0] \\ s_i[1] \\ . \\ . \\ . \\ s_i[n-1] \end{pmatrix}$$

Inner products are preserved

- Performance of M-ary signaling in AWGN depends only on the inner products between the signals if the noise PSD is fixed.
- When mapping the CT hypothesis testing problem to DT, it is important to check that these inner products are preserved when projecting them to signal space.
- Check the following

$$\langle s_i, s_j \rangle = \int s_i(t)s_j(t)dt = \sum_{k=0}^{n-1} s_i[k]s_j[k] = \langle \mathbf{s}_i, \mathbf{s}_j \rangle$$

Inner products are preserved from CT to DT and vice versa
➔ norms, energies, distances are preserved

Modeling WGN in signal space

- What about projection of noise on the signal subspace?
- The noise projection onto the i^{th} basis function as

$$N[i] = \langle \boxed{n}, \psi_i \rangle = \int n(t)\psi_i(t)$$

for $i = 0, 1, \ldots, \boxed{n} - 1$

Noise

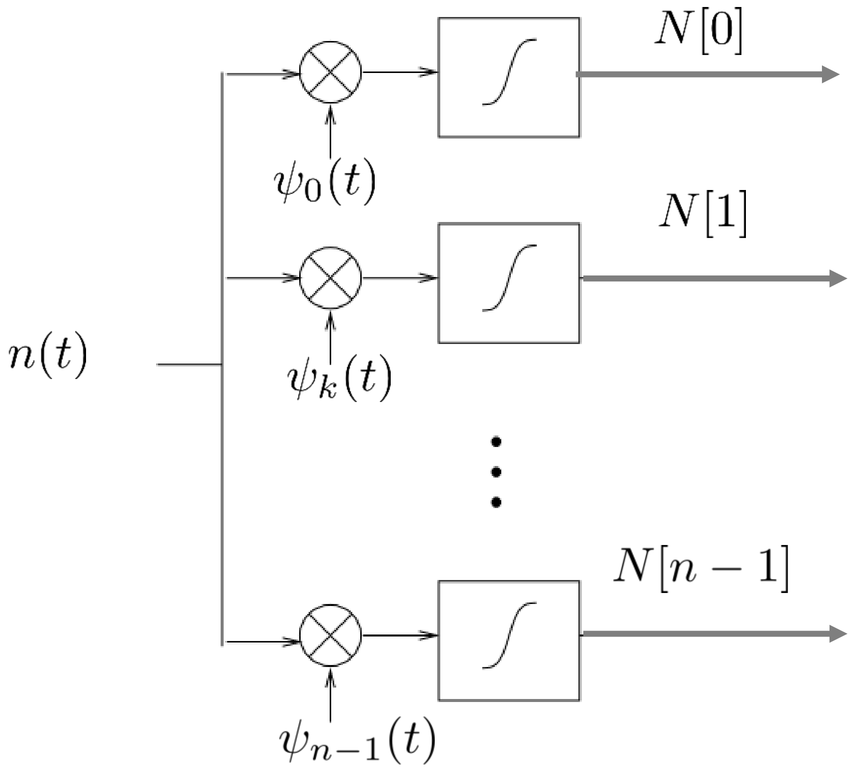
Dimension of signal space

- We can write noise $n(t)$ as follows

$$n(t) = \boxed{\sum_{i=0}^{n-1} N[i]\psi_i(t)} + \boxed{n^{\perp}(t)}$$

where $n^{\perp}(t)$ is the projection of noise orthogonal noise subspace.

Noise projection on Signal Subspace



- Prove that noise projection on signal space is discrete-time WGN.
- For this we will need theorem on the next slide!

Noise projection on Signal Subspace

- **WGN through correlators** The random variables $Z_1 = \langle n, u_1 \rangle$ and $Z_2 = \langle n, u_2 \rangle$ are zero-mean, jointly Gaussian, with

$$\text{cov}(Z_1, Z_2) = \sigma^2 \langle u_1, u_2 \rangle$$

so that $\mathbf{Z} = (Z_1, Z_2)^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ with covariance matrix

$$\mathbf{C} = \begin{pmatrix} \sigma^2 ||u_1||^2 & \sigma^2 \langle u_1, u_2 \rangle \\ \sigma^2 \langle u_1, u_2 \rangle & \sigma^2 ||u_2||^2 \end{pmatrix}$$

Proof!

- Prove that noise projection on signal space is discrete-time WGN.

Geometric interpretation of WGN

- The projection of WGN in any direction in signal space is an $\mathcal{N}(0, \sigma^2)$ random variable

- Projections in orthogonal directions are independent

- Projections along an orthonormal basis are iid $\mathcal{N}(0, \sigma^2)$ random variables

Hypothesis testing in signal space

- For the H_i hypothesis, the received signal is given by

$$y(t) = s_i(t) + n(t)$$

for $i = 0, 1, \dots, M-1$

- By projecting this on signal space, we get

$$Y[k] = \langle y, \psi_k \rangle = s_i[k] + N[k] \quad k = 0, 1, \dots, n-1$$

- On collecting these into an $n-$dimensional vector, we get

$$\mathbf{Y} = \mathbf{s}_i + \mathbf{N}$$

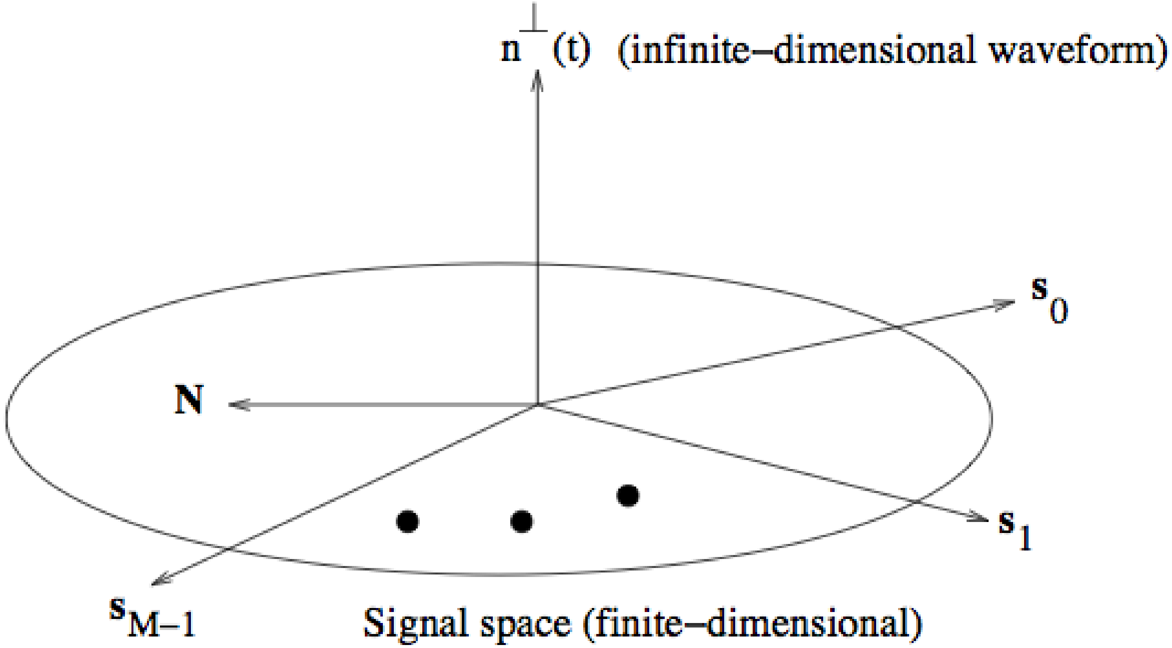
with $\mathbf{N} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$

- Note that the vector $\mathbf{Y}$ completely describes the component of the received signal $y(t)$ in the signal space given by

$$y_{\mathcal{S}} = \sum_{j=0}^{n-1} \langle y, \psi_j \rangle \psi_j(t) = \sum_{j=0}^{n-1} Y[j] \psi_j(t)$$

Irrelevance of orthogonal-noise component

- The noise component orthogonal to signal space, denoted by $n^\perp$,
  - does not give any information about signal
  - does not give any information about noise components in the signal space



Optimal Reception in AWGN

- Prove that the MAP/MPE rule for optimum demodulation in AWGN is

$$\delta_{\mathrm{MAP}}(\mathbf{y}) = \arg\max_i ||\mathbf{y} - \mathbf{s}_i||^2 - 2\sigma^2 \log \pi_i$$

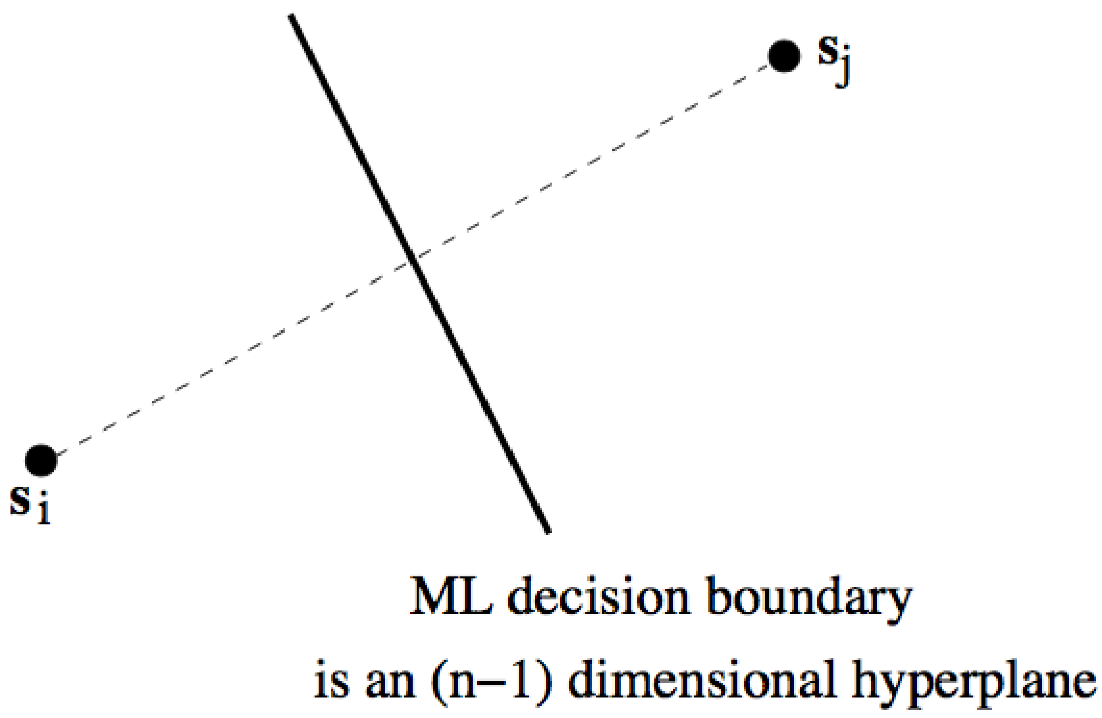
- For equiprobable prior probabilities, MAP is equivalent to ML

$$\begin{aligned}\delta_{\mathrm{ML}}(\mathbf{y}) &= \delta_{\mathrm{MAP}}(\mathbf{y}) \\ &= \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2 - 2\sigma^2 \log \frac{1}{M} \\ &= \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2\end{aligned}$$

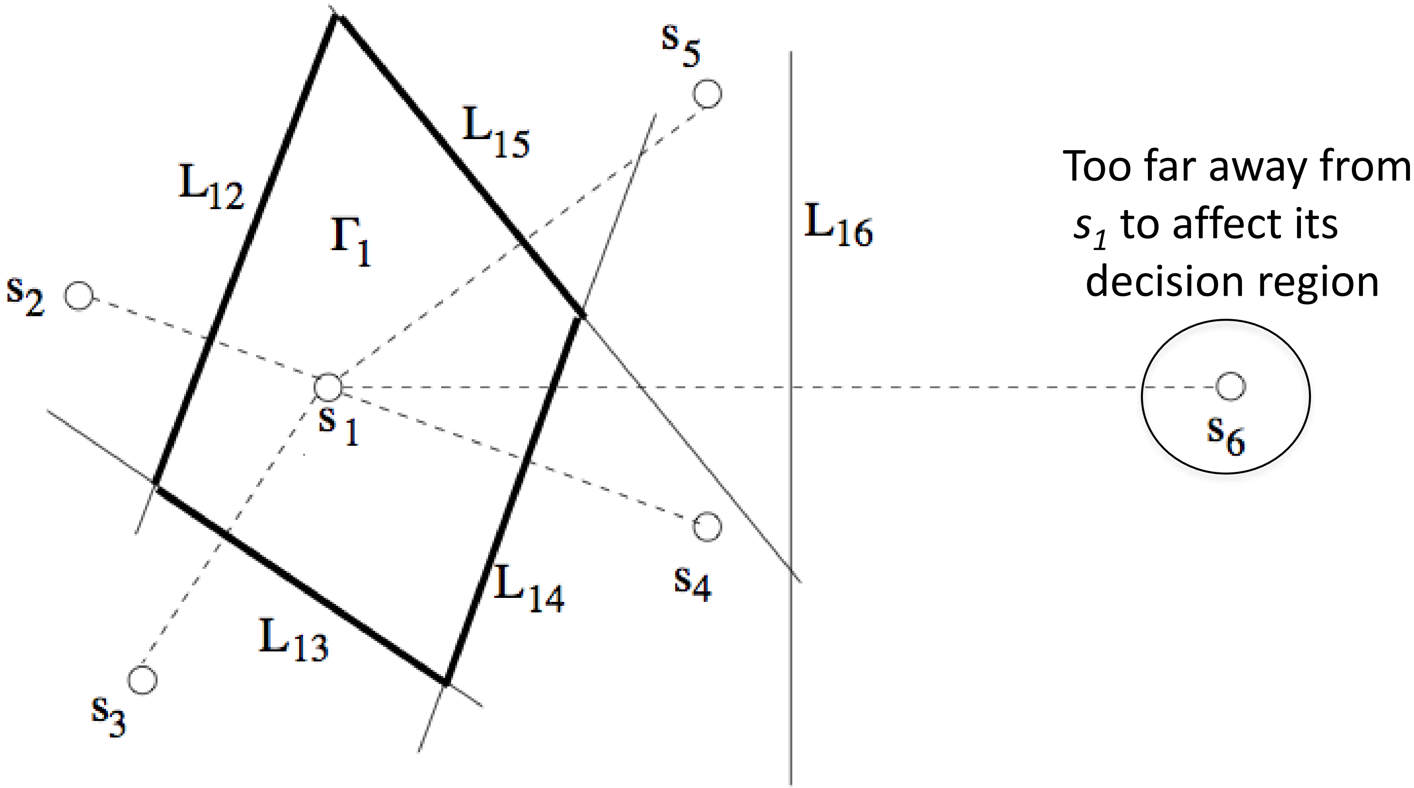
This is **minimum distance rule**!

Min distance rule between 2 signals

- Draw line between signal points.
- The ML decision boundary between them is the line bisecting this in 2D
- For two signal points in n dimensions, the ML decision boundary is $(n-1)$ dimensional hyperplane bisecting the line connecting the two points

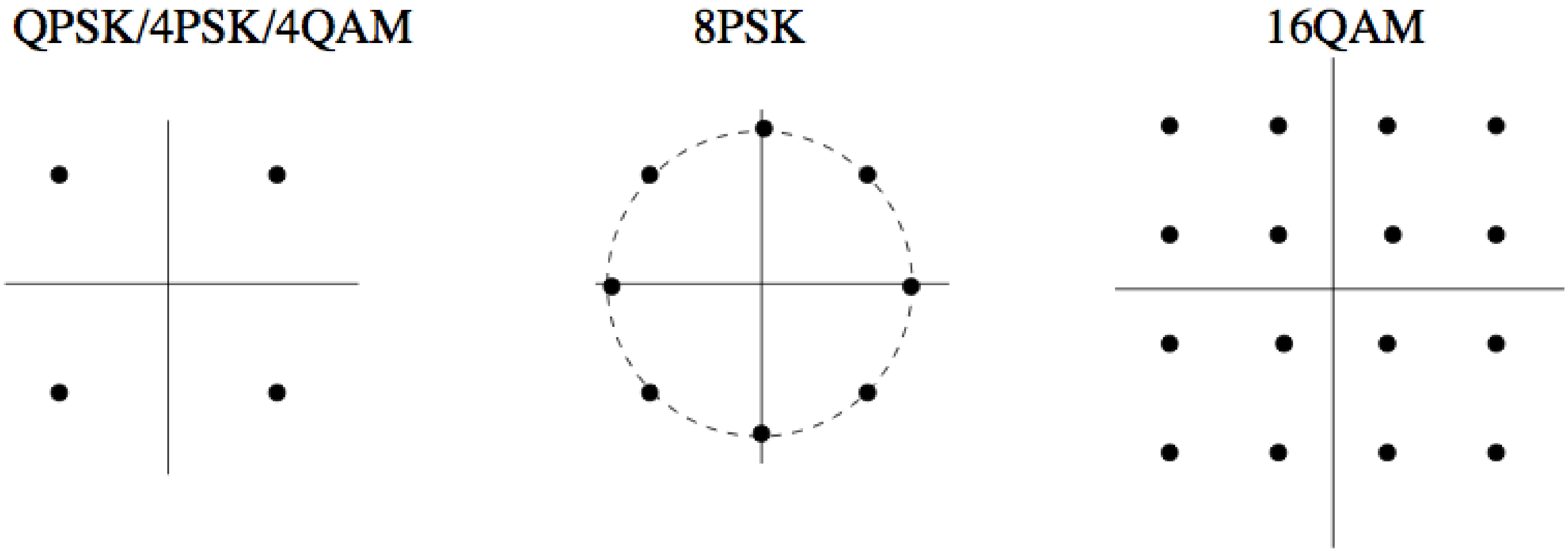


Geometry of min distance rule



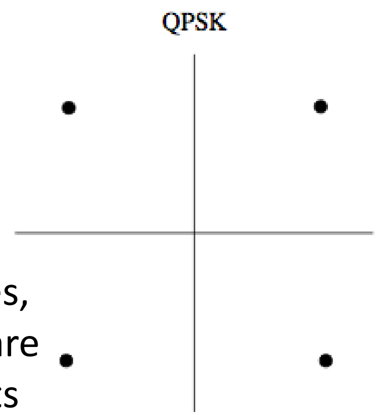
- Decision region for a signal is where it wins compared to all other signals
- Intersect the half-planes corresponding to all the pairwise comparisons
- Note that the decision region is typically determined by nearby neighbors

2 Dimensional Examples

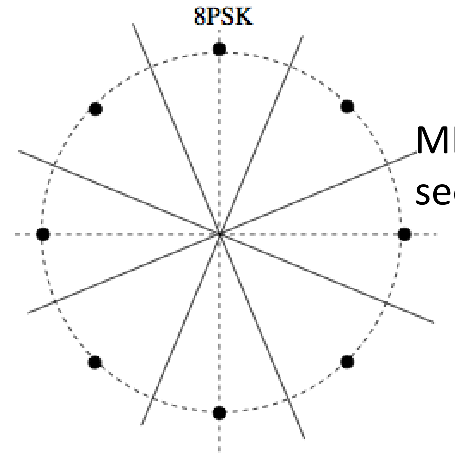


Solve: Find decision regions for these constellations for ML demodulation

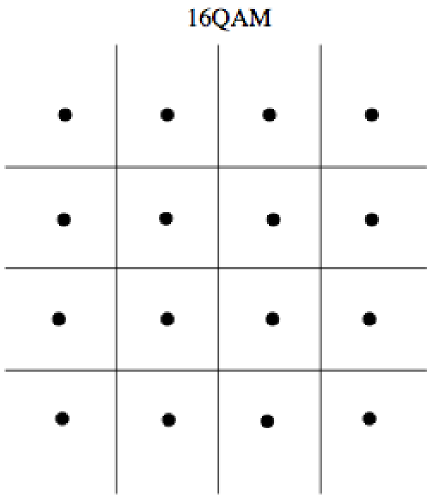
ML decisions for common 2D constellations



ML boundaries parallel to I/Q axes, decision regions are the four quadrants



ML regions are sectors



ML boundaries parallel to I/Q axes

MAP Demodulation In CT

- Remember that optimal modulation for signaling in DT AWGN is the MAP/MPE rule given by

DT(Finite Dimensional)

$$\begin{aligned}\delta_{\mathrm{MAP}}(\mathbf{y}) &= \arg\min_i ||\mathbf{y}-\mathbf{s}_i||^2 - 2\sigma^2\log\pi_i \\ &= \arg\max_i \langle \mathbf{y}, \mathbf{s}_i\rangle - ||\mathbf{s}_i||^2/2 + 2\sigma^2\log\pi_i\end{aligned}$$

- Equivalently, optimal demodulation for signaling in CT AWGN is

$$\begin{aligned}\delta_{\mathrm{MAP}}(\mathbf{y}) &= \arg\max_i \langle y, s_i\rangle - ||s_i||^2/2 + \sigma^2\log\pi_i \\ &= \arg\max_i \langle y_{\mathcal{S}}, s_i\rangle - ||s_i||^2/2 + \sigma^2\log\pi_i\end{aligned}$$

CT(Infinite Dimensional)

Recap: Receiver design as hypothesis testing

- Consider the multiple hypothesis testing

$$H_i \ : \ y(t) = s_i(t) + n(t) \quad i = 0, 1, \ldots, M-1$$

where $n(t)$ is WGN with $S_n(f) = \frac{N_0}{2} = \sigma^2$

- Strategy:
  - Show that we can reduce the continuous time received signal to a finite-dimensional vector without losing information
  - Derive the optimal receiver based on the finite-dimensional vector observation
  - Map the optimal receiver back to continuous time

- This approach is based on signal space concepts: even though the received signal lives in an infinite-dimensional space, we can restrict attention to the subspace spanned by the signals that could have been transmitted

ML Demodulation In CT

- For equiprobable prior probabilities, MAP is equivalent to ML

$$\delta_{\text{ML}}(\mathbf{y}) = \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2$$

DT(Finite Dimensional)

$$= \arg\max_i \langle y, s_i \rangle - ||s_i||^2/2$$

CT(Infinite Dimensional)

- Here we have used

$$||\mathbf{s}_i||^2 = ||s_i||^2$$

$$\begin{aligned}\langle y, s_i \rangle &= \langle y_{\mathcal{S}} + y^{\perp}, s_i \rangle \\ &= \langle y_{\mathcal{S}}, s_i \rangle + \langle y^{\perp}, s_i \rangle \\ &= \langle y_{\mathcal{S}}, s_i \rangle \\ &= \langle \mathbf{y}, \mathbf{s}_i \rangle\end{aligned}$$

- Note that we don't have minimum distance rule in CT as the squares of distance in CT will be

$$\begin{aligned}||y - s_i||^2 &= ||y_{\mathcal{S}} - s_i||^2 + ||y^{\perp}||^2 \\ &= ||y_{\mathcal{S}} - s_i||^2 + \boxed{||n^{\perp}||^2}\end{aligned}$$

Infinite Power

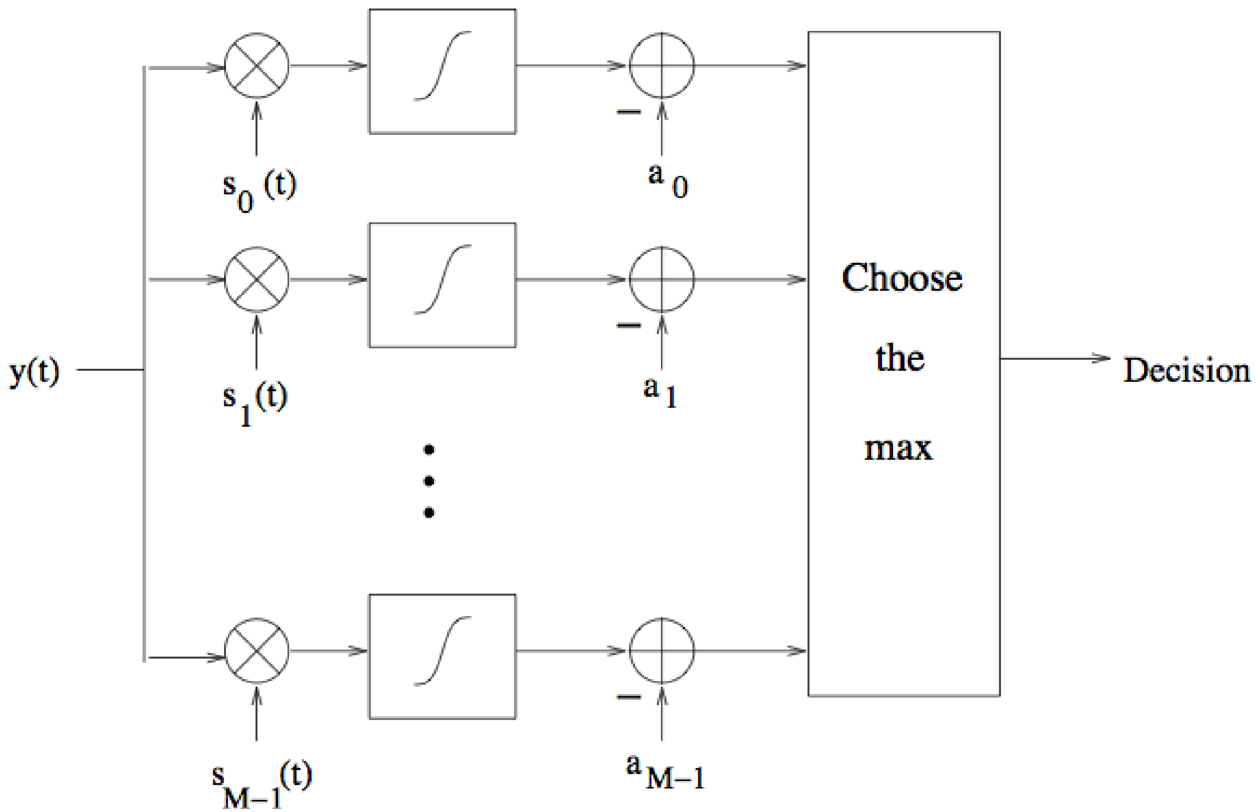
Implementation using correlation

- For equiprobable prior probabilities, MAP is equivalent to ML

$$\delta_{\mathrm{ML}}(\mathbf{y}) = \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2 \quad \text{DT(Finite Dimensional)}$$

$$= \arg\max_i \langle y, s_i \rangle - ||s_i||^2/2$$

CT(Infinite Dimensional)

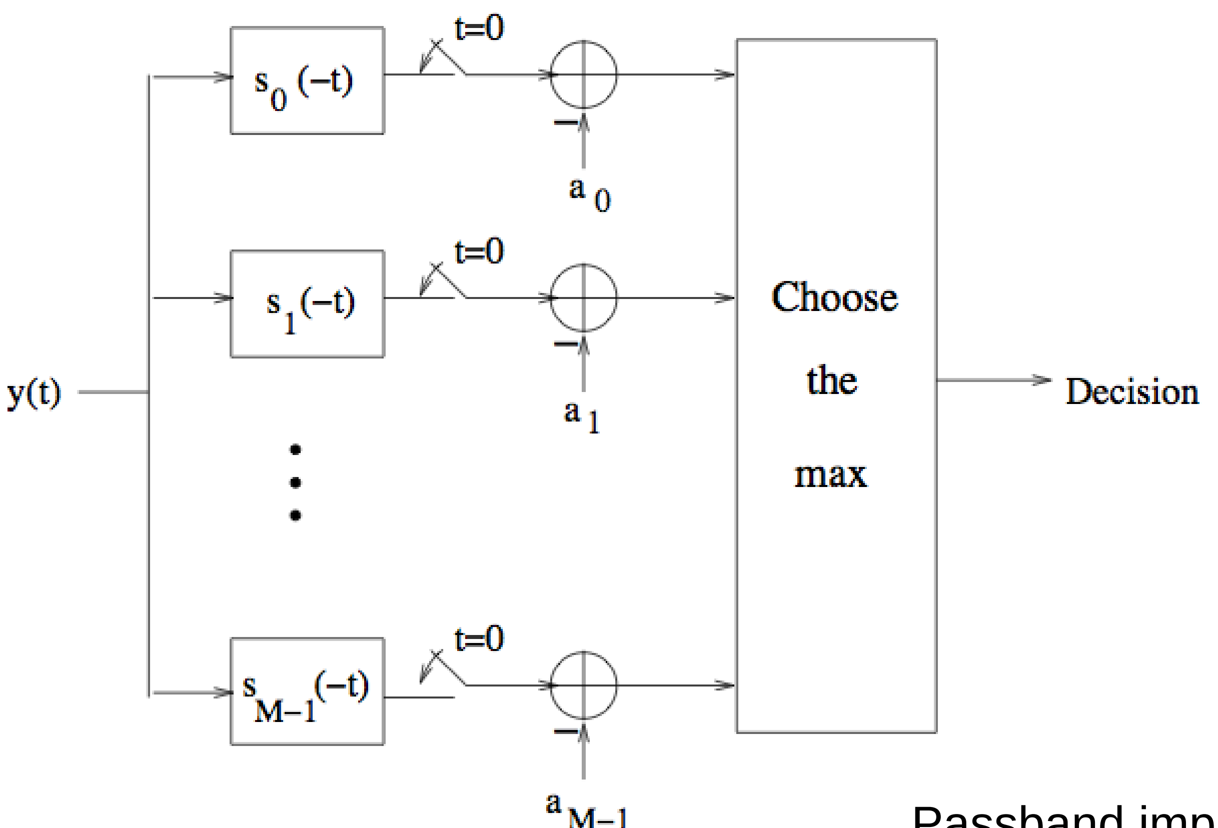


Equivalent Match Filter Implementation

- For equiprobable prior probabilities, MAP is equivalent to ML

$$\delta_{\mathrm{ML}}(\mathbf{y}) = \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2 \qquad \text{DT(Finite Dimensional)}$$
$$= \arg\max_i \langle y, s_i \rangle - ||s_i||^2/2$$

CT(Infinite Dimensional)



Passband implementation

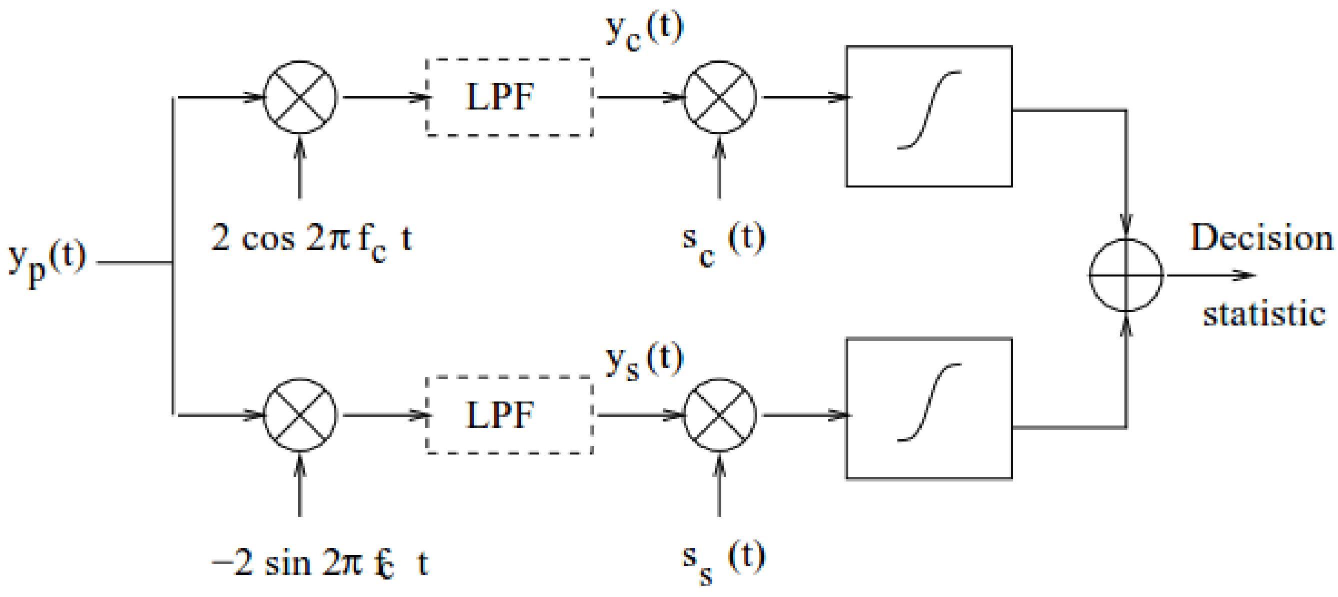
Implementation in Complex Baseband

- For equiprobable prior probabilities, MAP is equivalent to ML

$$\delta_{\mathrm{ML}}(\mathbf{y}) = \arg\min_i ||\mathbf{y} - \mathbf{s}_i||^2$$
$$= \arg\max_i \langle y, s_i \rangle - ||s_i||^2/2$$

- Note the relation between correlation in passband and baseband is given by

$$\langle u_p, v_p \rangle = \frac{1}{2}\mathrm{Re}\langle u, v \rangle = \frac{1}{2}(\langle u_c, v_c \rangle + \langle u_s, v_s \rangle)$$



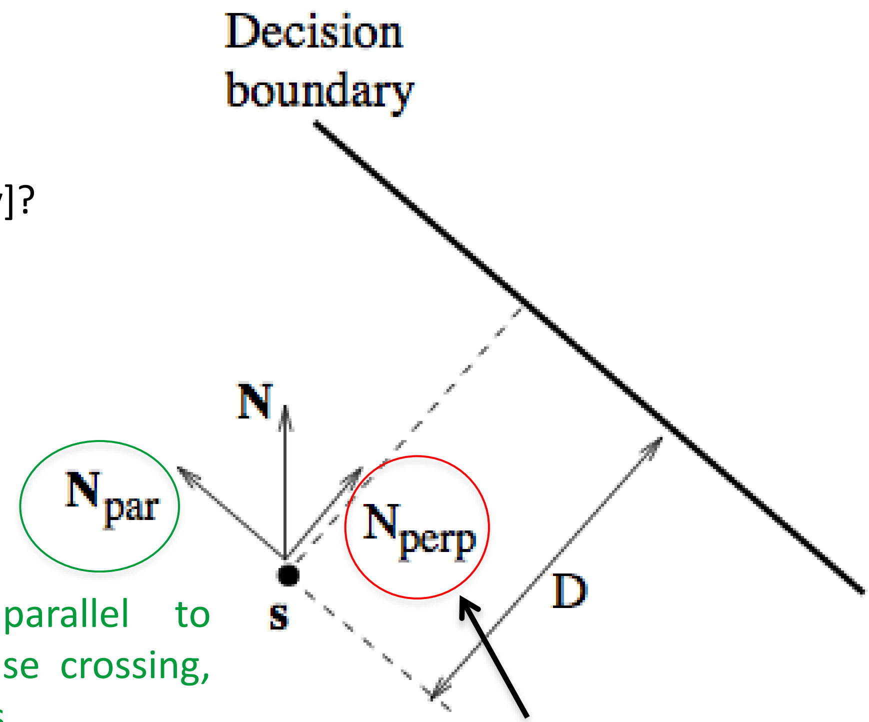
Performance Analysis of ML Reception

Performance Analysis of ML Reception

- Performance analysis of ML under the assumption of equiprobable priors.
- Although performance analysis of MPE or MAP is skipped, it is simple extension of ML case.

The Geometry of Errors

Basic building block:
Send signal **s.**
Noise **N** gets added.
P[crossing a given boundary]?



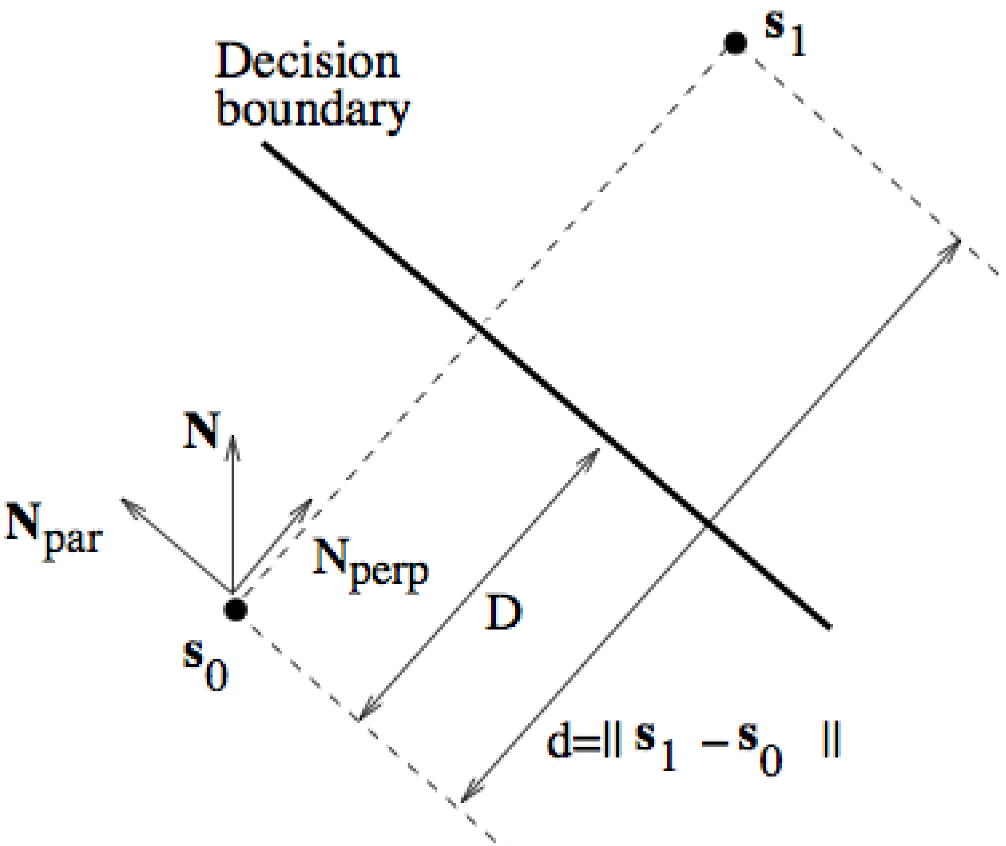
Noise component parallel to boundary cannot cause crossing, no matter how big it is.
In *n*-dimensional space, this component is *(n-1)*-dimensional

Boundary crossing determined by length and sign of noise component perpendicular to boundary. 1-dimensional, regardless of the signal space dimension

$$N_{perp} \sim N(0, \sigma^2)$$

$$\boxed{P[\text{cross a boundary at distance } D] = P[N_{perp} > D] = Q\left(\frac{D}{\sigma}\right)}$$

Geometry for binary signaling



Distance between signals
$d = ||\mathbf{s}_1 - \mathbf{s}_0||$

Distance from boundary
(starting from either signal)
$D = d/2 = ||\mathbf{s}_1 - \mathbf{s}_0||/2$

$$P[\text{cross ML boundary between } \mathbf{s}_0 \text{ and } \mathbf{s}_1] = Q\left(\frac{||\mathbf{s}_1 - \mathbf{s}_0||}{2\sigma}\right) = \boxed{Q\left(\frac{||s_1 - s_0||}{2\sigma}\right)}$$

(starting from either signal)

$$P_{e|0} = P_{e|1} = P_e$$

Important Note

- As the equivalence between CT signal s_i and vector $\mathbf{s}_i$ has been already established, we drop the boldface notation, using y, s_i, and n to denote the received signal, the transmitted signal, and the noise, respectively, in both the settings

Performance with binary signaling: *Algebraic*

- For the binary hypothesis test

$$H_0 : y(t) = n(t)$$
$$H_1 : y(t) = s(t) + n(t)$$

show that

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

- Moreover for following binary hypothesis test

$$H_0 : y(t) = s_0(t) + n(t)$$
$$H_1 : y(t) = s_1(t) + n(t)$$

show that

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

Importance of Scale Invariance

- The error probability is given by

$$P_e = P_{e|1} = P_{e|0} = Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) = Q\left(\frac{d}{2\sigma}\right)$$

- Same scaling in the signal and noise does not change the performance
- The performance depends on ratio rather than individually on the signal and noise strengths

Some Standard Measures

- Energy Bit E_b: For binary signaling it is given by

$$E_b = \frac{1}{2}(||s_0||^2 + ||s_1||^2)$$

  assuming that 0 and 1 are equally likely

- Scale invariant parameter

$$\eta_p = \frac{d^2}{E_b}$$

  Note that this does not change if we scale both s_1 and s_0 by A

- In terms of energy per bit and scale-invariant parameter, the probability of error for binary signaling is given by

$$P_e = Q\left(\frac{d}{2\sigma}\right) = Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

  where $\sigma^2 = N_0/2$

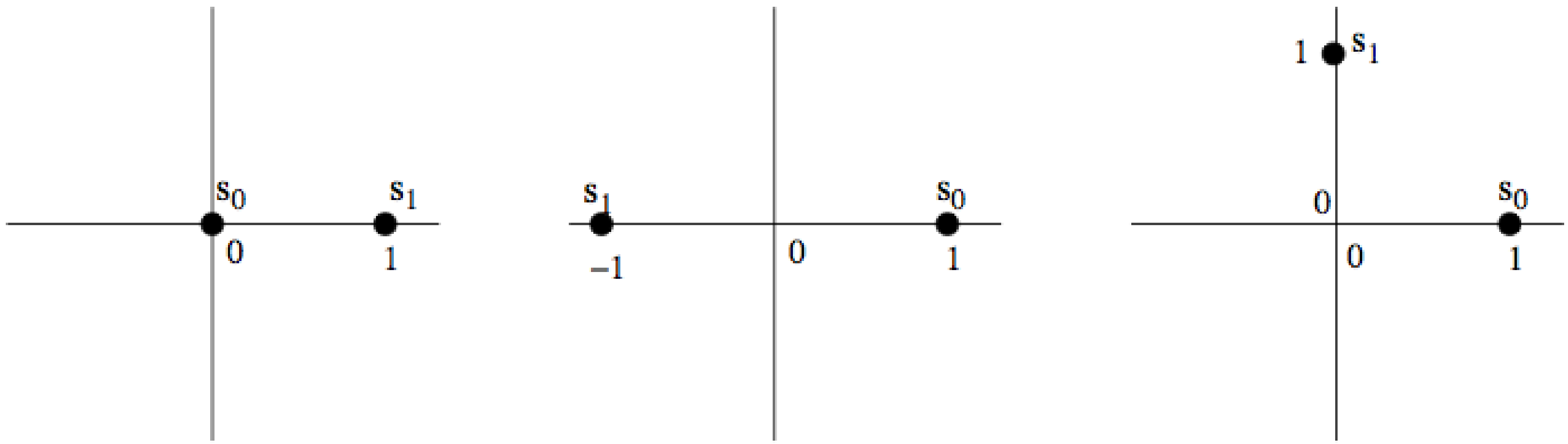
ML Binary Signaling Performance..

- In terms of energy per bit and scale-invariant parameter, the probability of error for binary signaling is given by

$$P_e = Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

- Important observations
  - Performance depends on the signal-to-noise ratio
  - For a fixed E_b/N_0, the performance is better for higher for higher value of η_p. The parameter η_p is also called as power efficiency of the constellation

Performance for different binary schemes



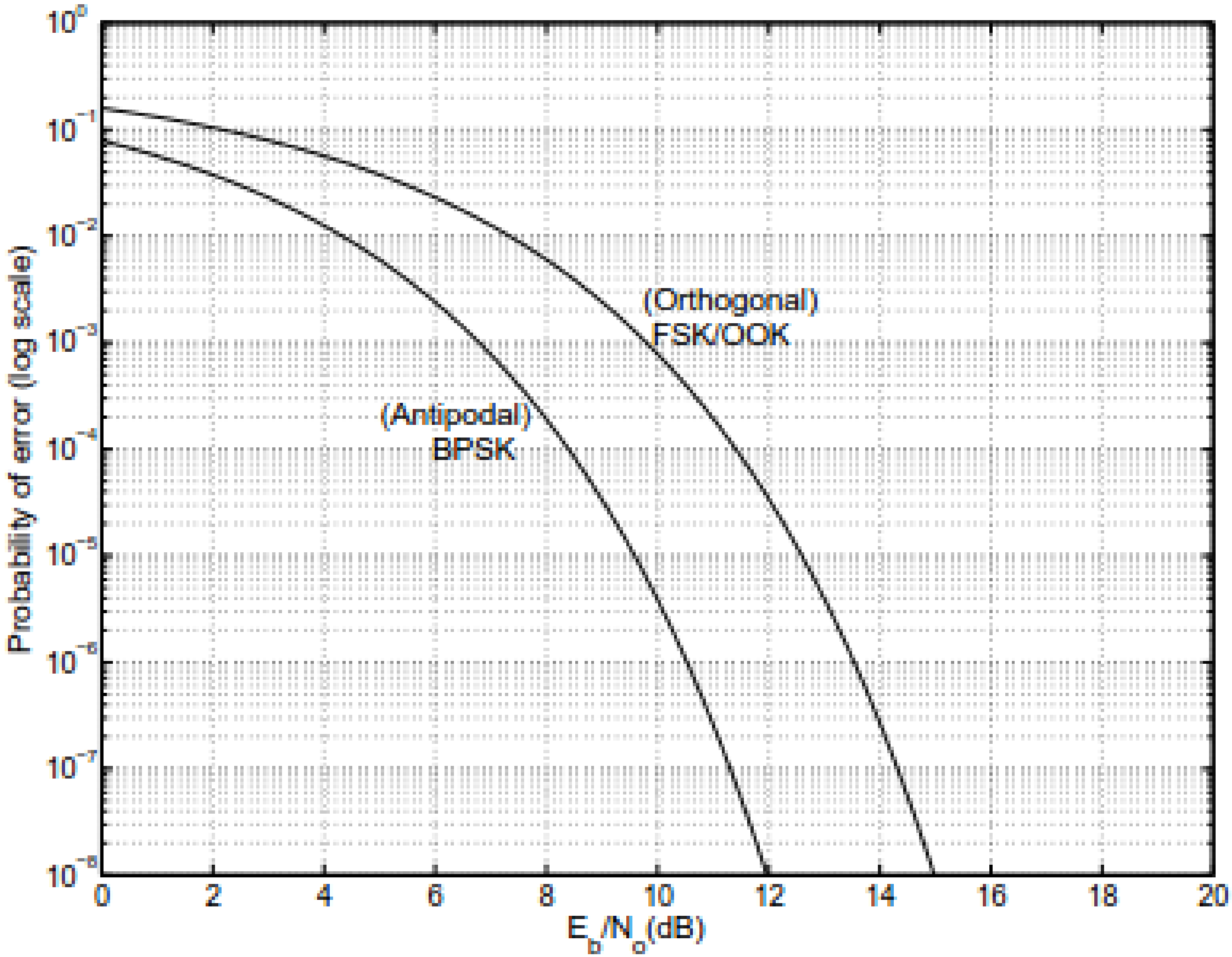
$$d = 1$$
$$E_b = (0^2 + 1^2)/2 = 1/2$$
$$\eta_P = d^2 / E_b = 2$$
$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$d = 2$$
$$E_b = ((-1)^2 + 1^2)/2 = 1$$
$$\eta_P = d^2 / E_b = 4$$
$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$d = \sqrt{2}$$
$$E_b = (1^2 + 1^2)/2 = 1$$
$$\eta_P = d^2 / E_b = 2$$
$$P_{e,ML} = Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- OOK and equal energy orthogonal signaling have same power efficiency. Antipodal is 3 dB better.

Performance for different binary schemes..



- OOK and equal energy orthogonal signaling have same power efficiency. Antipodal is 3 dB better.

Performance analysis for larger constellations

M-ary signaling in AWGN

- For the multiple hypothesis testing problem given by

$$H_i : y(t) = s_i(t) + n(t)$$

where $i = 0, 1, \ldots, M-1$, the ML rule is given by

$$\delta_{\mathrm{ML}} = \arg\max_i Z_i$$

with decision statistics

$$Z_i = \langle y, s_i \rangle - \frac{1}{2}||s_i||^2$$

and corresponding decision region is

$$\Gamma_i = \{y : Z_i > Z_j \text{ for all } \; j \neq i\}$$

- The conditional error probability, conditioned on H_i is given by

$$P_{e|i} = P[y \notin \Gamma_i | i \text{ sent}] = P[Z_i < Z_j \text{ for some } j \neq i \mid i \text{ sent}]$$

Some important parameters

- Energy per symbol is given by

$$E_s = \frac{1}{M}\sum_{i=1}^{M} ||s_i||^2$$

- Energy per bit is given by

$$E_b = \frac{E_s}{\log_2 M}$$

- Bandwidth efficiency

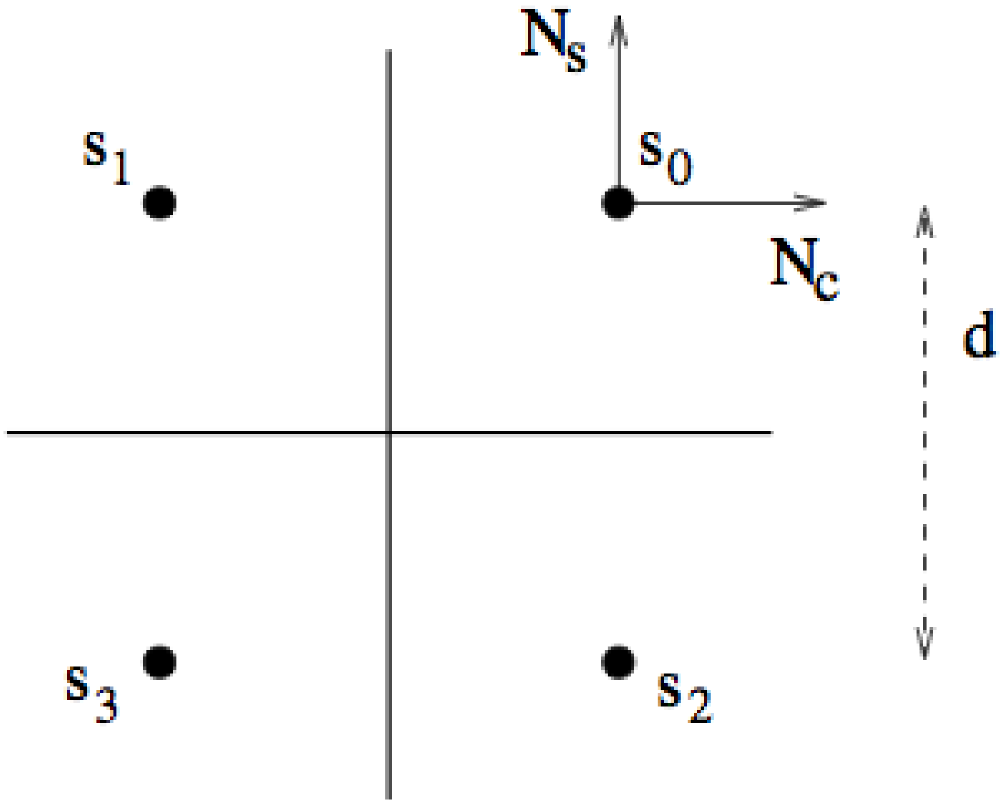
$$\begin{aligned}\eta_B = \frac{R}{B_{\min}} &= \frac{\text{Number of Bits per sec}}{\text{Min. Bandwidth}} \\ &= \frac{\text{Number of Bits per symbol} \times \text{Symbol Rate}}{\text{Number of Complex Dimensions} \times \text{Symbol Rate}} \\ &= \frac{\log_2 M}{N_D}\end{aligned}$$

- Show that performance of M-ary signaling depends only on E_b/N_0 and on constellation shape

Exact Error Probability of QPSK

- Show that the exact error probability of QPSK constellation shown below is given by

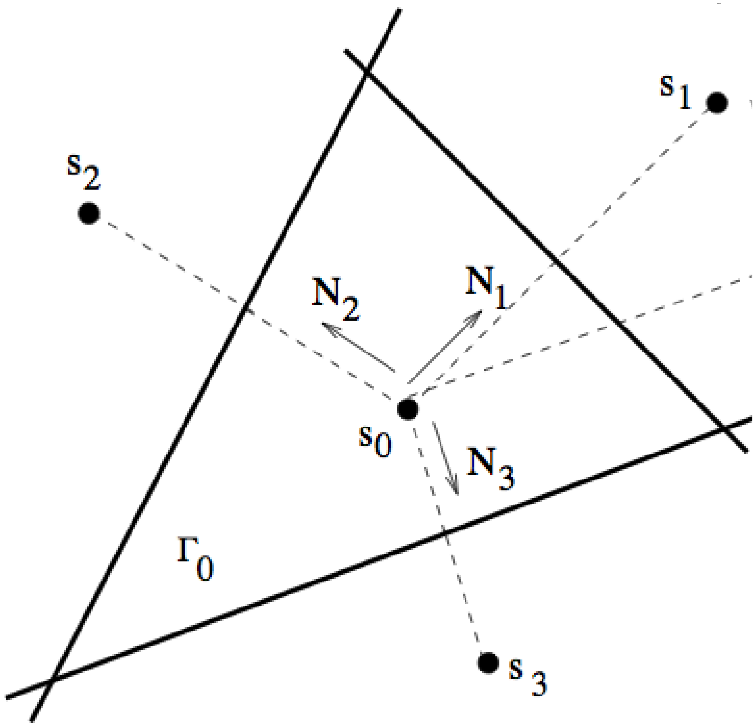
$$P_e = P_{e|1} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - 2Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

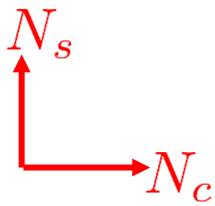


Performance analysis for larger constellations:

Union Bounds and Variants

Exact analysis might be difficult!





- Conditioned on s_0 sent, error occurs if we cross one of the 3 boundaries

$$P_{e|0} = P\left[N_1 > \frac{||s_1 - s_0||}{2} \text{ or } N_2 > \frac{||s_2 - s_0||}{2} \text{ or } N_3 > \frac{||s_3 - s_0||}{2}\right]$$

- Problem: The three noise variances, which cause crossing of boundary, are not independent! $P(A \cap B \cap C) \neq P(A)P(B)P(C)$

- Exact evaluation would require integrating a 3D Gaussian density function!

Approximation using Union Bound

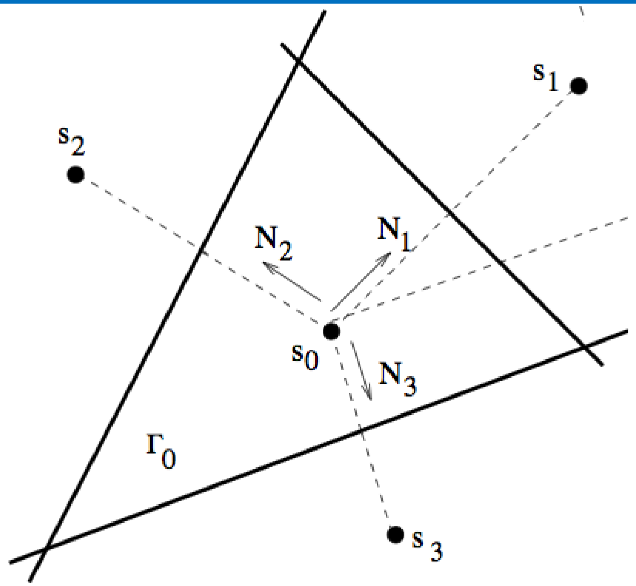
- Union bound is given by

  $$P\left[A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_2\right] = P[A_1 \cup A_2 \ldots \cup A_n] \leq P[A_1] + P[A_2] + \ldots + P[A_n]$$

  The bound is satisfied with equalilty only for mutually exclusive events

- This allows us to estimate the union (OR) of events which need not be independent!

Applying the union bound



- Conditioned on s_0 sent, error occurs if we cross one of the 3 boundaries

$$P_{e|0} = P\left[N_1 > \frac{||s_1 - s_0||}{2} \text{ or } N_2 > \frac{||s_2 - s_0||}{2} \text{ or } N_3 > \frac{||s_3 - s_0||}{2}\right]$$

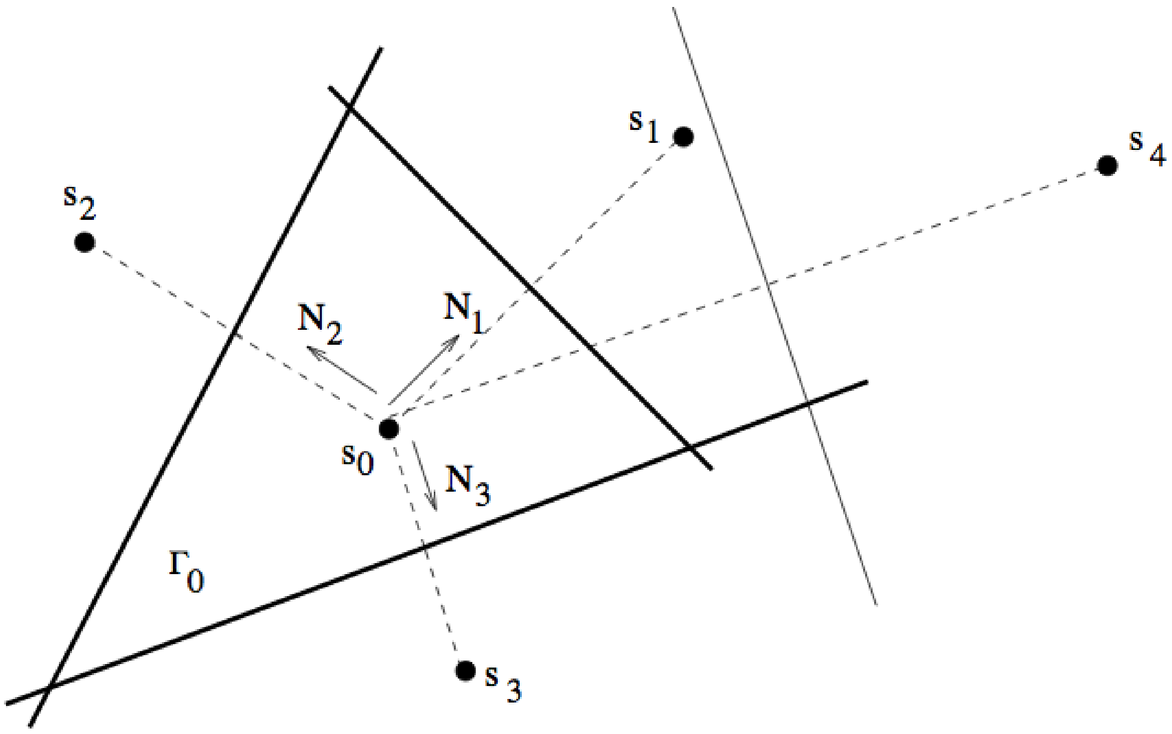
$$\leq P\left[N_1 > \frac{||s_1 - s_0||}{2}\right] + P\left[N_2 > \frac{||s_2 - s_0||}{2}\right] + P\left[N_3 > \frac{||s_3 - s_0||}{2}\right]$$

$$= Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) + Q\left(\frac{||s_2 - s_0||}{2\sigma}\right) + Q\left(\frac{||s_3 - s_0||}{2\sigma}\right)$$

Simple evaluation using Q-functions

Exact Analysis for higher dimensionals difficult!

Can’t draw pictures like this in a 20-dimensional signal space!



General Union Bound for M-ary Signal

- Conditional error probability is a union of events involving pairwise comparison

$$P_{e|i} = P\left(\cup_{j \neq i}\{Z_i < Z_j\} \mid i \text{ sent}\right)$$

Error occurs if the decision statistic for the right hypothesis loses to any other decision statistic.

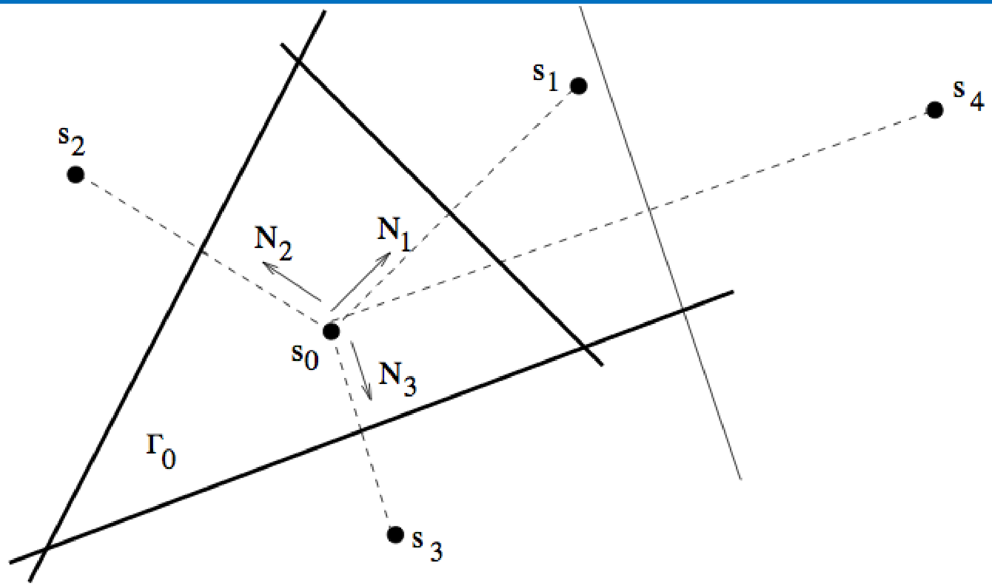
- Note that union bound is sum of pairwise error probabilities

$$P_{e|i} \leq \sum_{j \neq i} P[Z_i < Z_j \mid i \text{ sent}]$$

where the error probability for binary hypothesis test: H_i vs H_j

$$P[Z_i < Z_j \mid i \text{ sent}] = Q\left(\frac{||s_j - s_i||}{2\sigma}\right)$$

Applying the general union bound



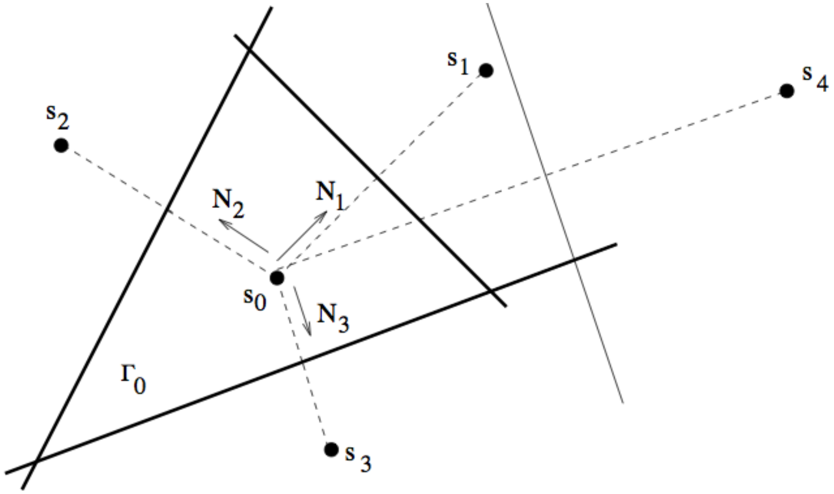
- Conditioned on s_0 sent, error occurs if we cross one of the 4 boundaries

$$P_{e|0} \leq \boxed{Q\left(\frac{||s_1 - s_0||}{2\sigma}\right) + Q\left(\frac{||s_2 - s_0||}{2\sigma}\right) + Q\left(\frac{||s_3 - s_0||}{2\sigma}\right)} + \boxed{Q\left(\frac{||s_4 - s_0||}{2\sigma}\right)}$$

Intelligent union bound

Intelligent Union Bound

- Denote by $N_{\mathrm{NL}}(i)$ the indices of the set of neighbors of signal s_i that characterizes the ML decision region Γ_i.
- For example below, $N_{\mathrm{NL}}(i) = \{1, 2, 3\}$

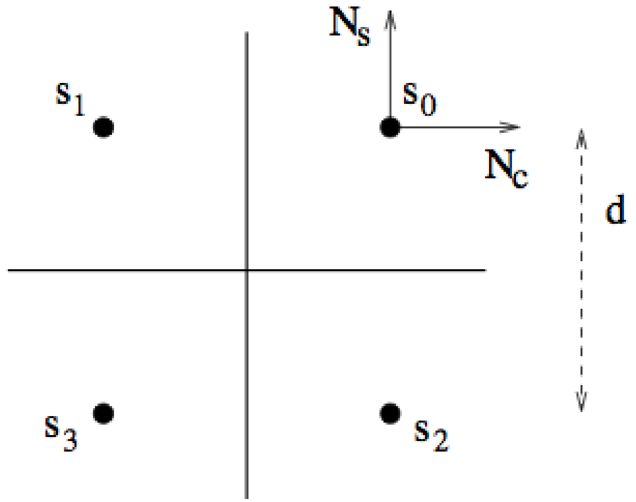


- Intelligent union bound in general is given by

$$P_{e|i} \leq \sum_{j \in N_{\mathrm{ML}}(i)} Q\left(\frac{||s_j - s_i}{2\sigma}\right)$$

Example: QPSK

- Find union bound and intelligent union bound for QPSK



- Union bound for QPSK is given by

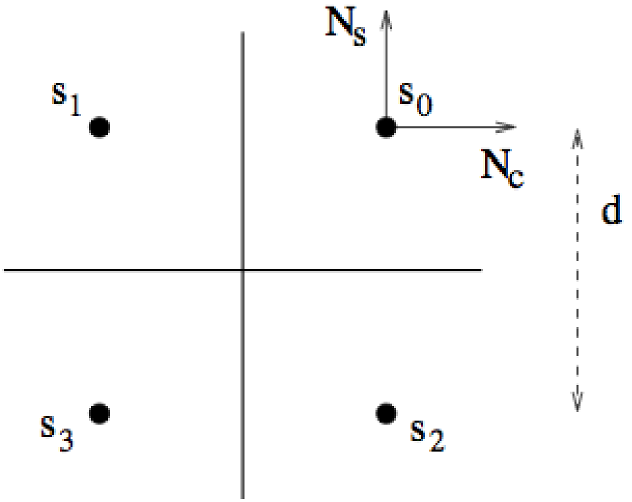
$$P_e = P_{e|0} \leq Q\left(\frac{d_{01}}{2\sigma}\right) + Q\left(\frac{d_{02}}{2\sigma}\right) + Q\left(\frac{d_{03}}{2\sigma}\right) = 2Q\left(\frac{d}{2\sigma}\right) + \boxed{Q\left(\frac{\sqrt{2}d}{2\sigma}\right)}$$

- Using $\frac{d^2}{E_b} = \frac{d^2 \log_2 M}{E_s} = 4$ with $E_s = 2(\frac{d}{2})^2 = \frac{d^2}{2}$, we get

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \boxed{Q\left(\sqrt{\frac{4E_b}{N_0}}\right)}$$

Example: QPSK ..

- Find union bound and intelligent union bound for QPSK



- Intelligent union bound for QPSK is given by

$$P_e = P_{e|0} \leq Q\left(\frac{d_{01}}{2\sigma}\right) + Q\left(\frac{d_{02}}{2\sigma}\right) = 2Q\left(\frac{d}{2\sigma}\right)$$

- Using $\frac{d^2}{E_b} = \frac{d^2 \log_2 M}{E_s} = 4$ with $E_s = 2(\frac{d}{2})^2 = \frac{d^2}{2}$, we get

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Nearest Neighbor Approximation

- Rationale: $Q(x)$ decays rapidly so that the terms in the union bound corresponding to the smallest arguments for the Q dominate at high SNR.

$$P_{e|i} \approx N_{d_{\min}}(i) Q\left(\frac{d_{\min}}{2\sigma}\right)$$

- By averaging over i, we obtain that

$$P_e \approx \bar{N}_{d_{\min}} Q\left(\frac{d_{\min}}{2\sigma}\right) = \bar{N}_{d_{\min}} Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

where $\bar{N}_{d_{\min}}$ denotes the average number of nearest neighbors for a signal point and is given by

$$\bar{N}_{d_{\min}} = \frac{1}{M} \sum_{i=1}^{M} N_{d_{\min}}(i)$$

is the average number of nearest neighbors for the signal points in the constellation.

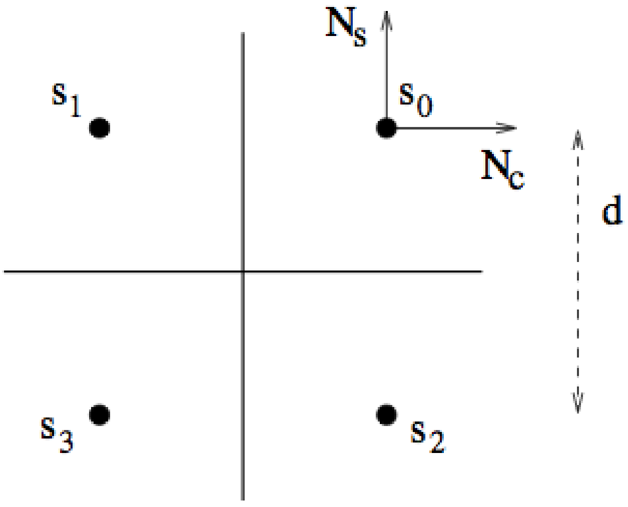
Example: QPSK ...

- For QPSK

$$N_{d_{\min}}(i) = 2 = \bar{N}_{d_{\min}}$$

so that

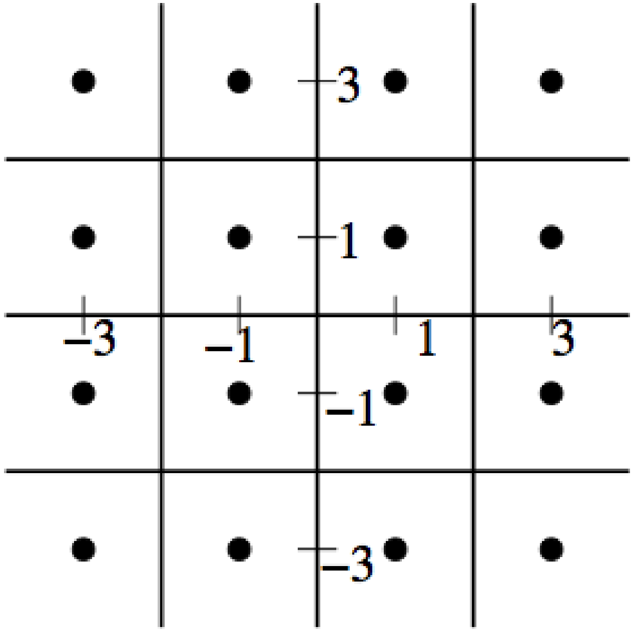
$$P_e \approx 2Q\left(\frac{d_{\min}}{2\sigma}\right)$$



- Using $\frac{d^2}{E_b} = \frac{d^2 \log_2 M}{E_s} = 4$ with $E_s = 2(\frac{d}{2})^2 = \frac{d^2}{2}$, we get

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Performance Analysis of 16 QAM

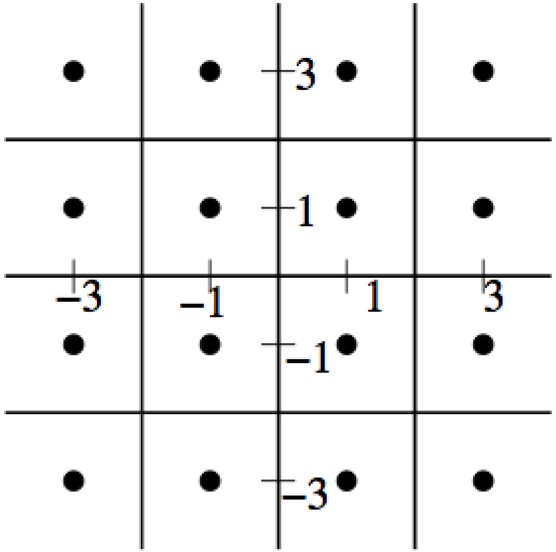


- Intelligent union bound coincides with nearest neighbors approximation since nearest neighbors define decision boundaries

- Need to compute power efficiency and average number of nearest neighbors

- Since we are computing scale invariant quantities, choose any convenient scaling

16QAM performance…

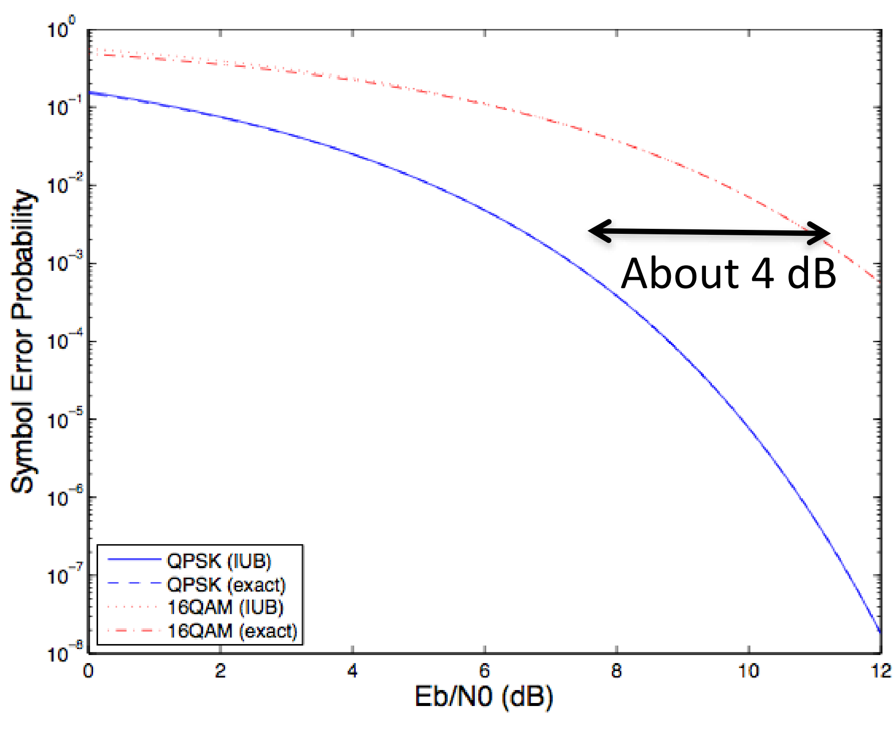


- Using nearest neighbor approximation, prove that the probability of error for 16 QAM is given by

$$P_e \approx 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

QPSK vs 16QAM

- Find bandwidth efficiency of 4 QPSK and 16 QAM
- Find power efficiency of 4 QPSK and 16 QAM
- Which out of the two will perform better for same E_b/N_0?



Note: Intelligent union bound is very close to the exact error prob in both cases

Performance Analysis of M-ary orthogonal modulation

- Using union bound, show that the probability of error for M-ary orthogonal modulation can be approximated by

$$P_e = (M-1)Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

Included in course

- Exact probability of correct reception for M-ary orthogonal signaling

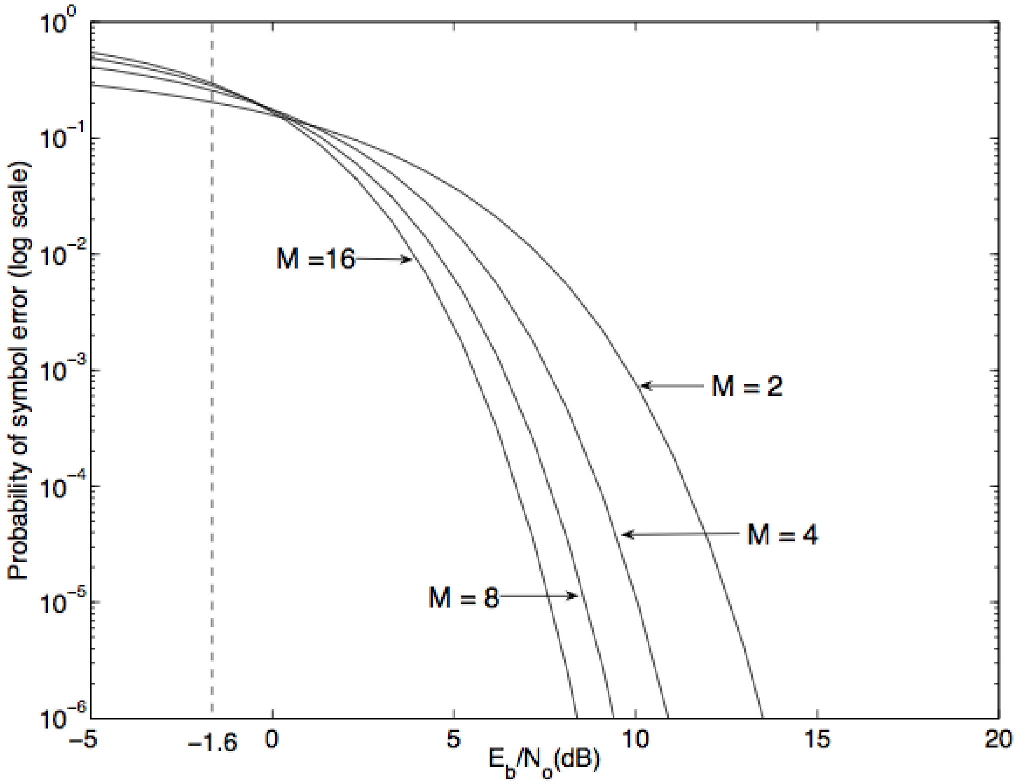
Not included in course

$$P_c = P_{c|i} = \int_{-\infty}^{\infty} [\Phi(x)]^{M-1} \frac{1}{\sqrt{2\pi}} exp(-\frac{(x-m)^2}{2})dx$$

where $\Phi(x)$ is the CDF of the standard Gaussian random variable while

$$m = \sqrt{\frac{2E_s}{N_0}} = \sqrt{\frac{2E_b \log_2 M}{N_0}}$$

Symbol Error Probability for M-ary Orthogonal

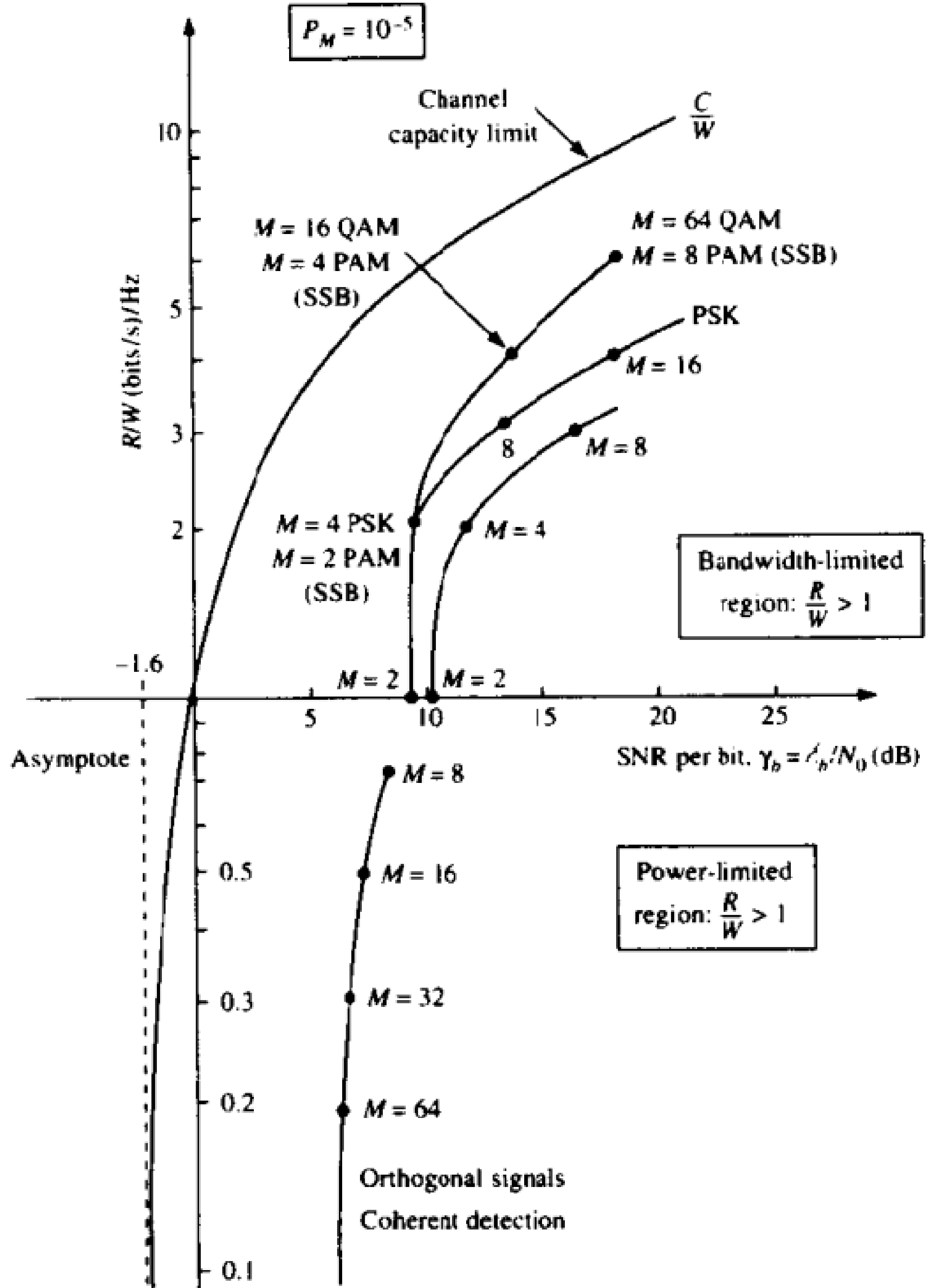


- Performance improves as M increases!
- Asymptotic performance

$$\lim_{M\to\infty} P_e = \begin{cases} 0, & E_b/N_0 > \ln 2 \\ 1, & E_b/N_0 < \ln 2 \end{cases}$$

Power-Bandwidth Tradeoff for M-ary orthogonal!

- Power efficiency $= \log_2 M$
- Bandwidth effificiency is $\eta_B = \frac{\text{No. of Bits per symbol}}{\text{Number of Complex Dimensions}} = \frac{\log_2 M}{M}$
- Trade-off between power and bandwidth efficiency!



$P_M = 10^{-5}$
Channel capacity limit
$\frac{C}{W}$
M = 16 QAM
M = 4 PAM
(SSB)
M = 64 QAM
M = 8 PAM (SSB)
PSK
M = 16
8
M = 8
M = 4 PSK
M = 2 PAM
(SSB)
M = 4
M = 2
M = 2
Bandwidth-limited region: $\frac{R}{W} > 1$
R/W (bits/s)/Hz
10
5
3
2
−1.6
0.5
0.3
0.2
0.1
5
10
15
20
25
SNR per bit, $\gamma_b = \mathcal{E}_b/N_0$ (dB)
Asymptote
M = 8
M = 16
M = 32
M = 64
Power-limited region: $\frac{R}{W} > 1$
Orthogonal signals
Coherent detection