Permutations and Combinations

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Permutations

Key Concepts:

1. **Permutation Formula:**
   - **General Permutation Formula:**

$$P(n, k) = \frac{n!}{(n-k)!}$$

     - n is the total number of items.
     - k is the number of items to arrange.
   - **Example:**
     If you have 5 distinct objects and you want to arrange 3 of them, the number of permutations is:

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{120}{2} = 60$$

     There are 60 different permutations of selecting 3 objects from a set of 5 distinct objects.
2. **Permutations of n Distinct Objects:**
   - When all objects in a set are distinct, the number of permutations is simply the factorial of the total number of objects:

$$P(n) = n!$$

   - **Example:**
     For 4 distinct objects:

$$P(4) = 4! = 24$$

     There are 24 different ways to arrange 4 distinct objects.
3. **Circular Permutations:**

- In circular permutations, the arrangement forms a closed loop, meaning the first and last elements are adjacent.
- The formula is:

$$P\_\text{circular}(n) = (n-1)!$$

- **Example:**
  For 5 distinct objects arranged in a circle:

$$P\_\text{circular}(5) = 4! = 24$$

  There are 24 different circular permutations of arranging 5 distinct objects.

4. **Permutations with Repetition:**
   - If some objects are repeated, the permutation formula is adjusted to account for the repetitions:

$$P\_\text{repetition}(n, k_1, k_2, \ldots, k_r) = \frac{n!}{k_1! \times k_2! \times \ldots \times k_r!}$$

   - **Example:**
     Arranging the letters in the word "BOOK":

$$P\_\text{repetition}(4, 1, 2, 1) = \frac{4!}{1! \times 2! \times 1!} = \frac{24}{2} = 12$$

     There are 12 different ways to arrange the letters in "BOOK".

5. **Permutation with All Items:**
   - When all items are taken ($k = n$), the permutation simplifies to:

$$P(n, n) = n!$$

   - **Example:**
     For 4 objects:

$$P(4, 4) = 4! = 24$$

Edge Cases in Permutations

Permutations with Constrained Positions:

- **Scenario:** When certain items must remain together or in specific positions.

- **Example:** Arranging 4 vowels and 3 consonants in a row, where the consonants must be together.
  - **Steps:**
    a. Treat the group of consonants as a single unit.
    b. Arrange the units, then arrange the items within the unit.
  - **Calculation:**

$$\text{Total Arrangements} = (n - k + 1)! \times k!$$

    - For 4 vowels and 3 consonants (consonants together):

$$5! \times 3! = 120 \times 6 = 720$$

      So, there are 720 different ways to arrange the letters.

Combinations

Key Concepts:

1. **Combination Formula:**
   - **General Combination Formula:**

$$C(n, k) = \frac{n!}{k! \times (n - k)!}$$

     - n is the total number of items.
     - k is the number of items to select.
   - **Example:**
     If you have 6 different books and want to select 3 to read:

$$C(6, 3) = \frac{6!}{3! \times (6 - 3)!} = \frac{720}{6 \times 6} = 20$$

     There are 20 different combinations of selecting 3 books from a set of 6 books.
2. **Combination with All Items:**
   - When all items are selected ($k = n$), the combination simplifies to:

$$C(n, n) = 1$$

- **Example:**
  If selecting all 4 objects from a set of 4:

$$C(4,4) = 1$$

Edge Cases in Combinations

1. **Selecting Items with One Predefined Position:**
   - **Scenario:** Selecting a subset where one item must be in a specific position, such as a chairperson in a committee.
   - **Example:** Selecting 4 people from a group of 12, where one person must be the chairperson.
     - **Steps:**
       a. Select the chairperson first.
       b. Select and arrange the remaining people.
     - **Calculation:**

$$\text{Total Selections} = n \times C(n-1, k-1)$$

       - For selecting 4 people with a chairperson:

$$12 \times C(11,3) = 12 \times 165 = 1980$$

So, there are 1,980 different ways to form the committee.

How to Distinguish Between Permutations and Combinations

Permutation Keywords (Order Matters)

1. **"Arrange"**: When the problem involves arranging items, the order is important, so it's a permutation problem.
   - **Example**: "How many different ways can you arrange the letters in the word 'UNIQUE'?"
2. **"Order"**: If the order in which the items are arranged matters, it's a permutation.
   - **Example**: "How many different orders can you create with these 5 people?"
3. **"Line up" or "Position"**: If people or items are being lined up or placed in a specific position, the problem is about permutations.
   - **Example**: "In how many ways can you line up 4 people?"

4. **"Permutations"**: If the problem directly mentions "permutations," then it's obviously a permutation problem.
    - **Example**: "Calculate the permutations of selecting 3 items out of 5."
5. **"Sequence"**: When the order in which items appear is important.
    - **Example**: "Find the number of possible sequences of letters."
6. **"Rank"**: When the problem asks for a specific rank or position within an ordered list.
    - **Example**: "What is the rank of the word 'MATH' in all possible arrangements?"
7. **"Placement"**: When placing items or people in specific positions or slots.
    - **Example**: "Determine the number of placements of 4 people in 6 chairs."
8. **"Line" or "Row"**: When items or people are arranged in a line or row, indicating that order is important.
    - **Example**: "In how many ways can you arrange these items in a row?"
9. **"Schedule"**: When events or tasks are arranged in a particular order.
    - **Example**: "How many ways can you schedule 5 meetings?"
10. **"Permutations of"**: Often used to indicate that the problem is asking for the number of permutations of a certain number of items.
    - **Example**: "Find the number of permutations of 3 out of 8 objects."
11. **"Line-up"**: Specifically when people or objects are lined up in a particular order.
    - **Example**: "In how many ways can you line-up these players?"
12. **"Combination lock"**: Despite the term "combination," the concept of a combination lock often involves permutations because the order of numbers matters.
    - **Example**: "How many different combination locks can be made with 4 digits?"
13. **"Tournament"**: When the order of games or matches matters.
    - **Example**: "In how many ways can a tournament of 8 teams be organized?"

Combination Keywords (Order Doesn't Matter)

1. **"Select"**: If the problem involves selecting items without regard to order, it's a combination problem.
    - **Example**: "In how many ways can you select 3 students from a group of 10?"
2. **"Choose"**: Like "select," if you are choosing items and the order doesn't matter, it's a combination.
    - **Example**: "How many ways can you choose 4 books from a shelf of 12?"
3. **"Committee"**: Problems involving forming a committee where the positions are not important usually refer to combinations.
    - **Example**: "In how many ways can a committee of 5 be formed from 20 people?"
4. **"Subset"**: If you are forming a subset from a larger set, and the order doesn't matter, it's a combination problem.

- **Example**: "How many different subsets of size 3 can be created from a set of 8 elements?"

5. **"Combinations"**: If the problem mentions "combinations," it's a combination problem.
   - **Example**: "Calculate the combinations of selecting 4 items from a set of 10."
6. **"Group"**: Forming a group where the arrangement within the group doesn't matter.
   - **Example**: "How many ways can a group of 3 be formed from 7 people?"
7. **"Team"**: Selecting a team where positions within the team don't matter.
   - **Example**: "How many ways can you form a team of 5 from 10 people?"
8. **"Sample"**: When drawing a sample from a larger population where the order doesn't matter.
   - **Example**: "How many different samples of 4 can be drawn from a group of 20?"
9. **"Partition"**: Dividing a set into groups or parts where the order of parts doesn't matter.
   - **Example**: "In how many ways can you partition a set of 10 objects into two groups?"
10. **"Collection"**: Gathering items where the order doesn't matter.
    - **Example**: "How many ways can you form a collection of 5 items from 12 available items?"
11. **"Choose from"**: When you are asked to choose from a set, often implying combinations.
    - **Example**: "In how many ways can you choose from 10 different books?"
12. **"Lottery"**: Often involves selecting numbers where the order doesn't matter.
    - **Example**: "How many different lottery combinations can be made from 49 numbers?"
13. **"Combination of"**: Similar to "choose from," explicitly asks for combinations.
    - **Example**: "Find the number of combinations of 5 out of 15 objects."
14. **"Selection"**: When selecting items from a set where the order doesn't matter.
    - **Example**: "How many ways can you make a selection of 3 items from 10?"
15. **"Draw"**: When drawing items from a set, implying combinations if the order doesn't matter.
    - **Example**: "How many ways can you draw 4 cards from a deck of 52?"

Summary of Key Identifiers

- **Permutation** (Order Matters): arrange, order, line up, position, sequence, rank, placement, line, row, schedule, line-up, tournament, combination lock.
- **Combination** (Order Doesn't Matter): select, choose, committee, subset, combinations, group, team, sample, partition, collection, choose from, lottery, selection, draw.