1 Discrete I: Electric Field due to an Electric Dipole (Parallel on the Dipole Axis)

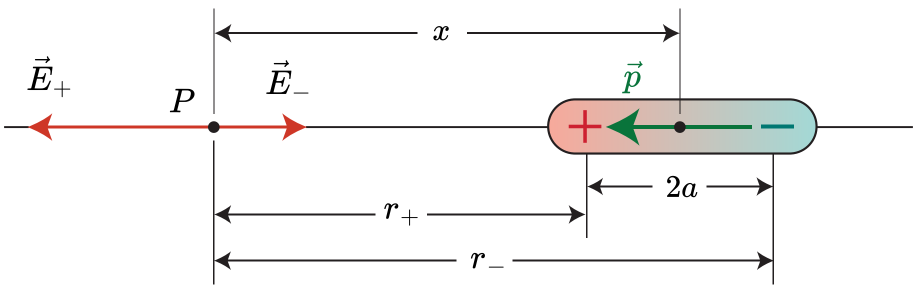

Figure 1: An electric dipole. The dipole's two charges result from the electric field vectors at point P on the dipole axis. Point P is at distances $r_+ = x - a$ and $r_- = x + a$ from the individual charges that make up the dipole. Here, $x \gg a$.

Measuring the field x distance away from the bisecting point of the dipole:

$$
\begin{aligned}
\vec{E}_{\text{dipole}} &= \vec{E}_+ + \vec{E}_- \\
&= \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{r_+^2} + \frac{q}{r_-^2}\right)\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right\}\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{(x-a)^2-(x+a)^2}{\{(x+a)(x-a)\}^2}\right]\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{4xa}{(x^2-a^2)^2}\right]\hat{i} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{2\cdot 2xa}{x^4}\right]\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\cdot 2\cdot\frac{q\cdot 2a}{x^3}\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2p}{x^3}\hat{i} \\
&= \frac{1}{2\pi\epsilon_0}\frac{p}{x^3}\hat{i}.
\end{aligned}
\tag{1}
$$

The field points from the direction of the dipole moment. i.e., toward the positive charge.

To account for the arbitrary direction of the observation point P along the dipole axis. We replace x with r, giving us:

$$
\vec{E}_{\text{dipole}} = \frac{1}{2\pi\epsilon_0}\frac{\vec{p}}{r^3}\hat{r}_{(-)\to(+)} = \frac{1}{2\pi\epsilon_0}\frac{p}{r^3}\hat{p} \tag{2}
$$

The electric field of a dipole thus points in the direction of the electric dipole moment, regardless of whatever orientation we try. It further establishes the *why* behind the convention of $\vec{p}$'s direction.

2 Discrete II: Electric Field due to an Electric Dipole (Perpendicular to the Dipole Axis)

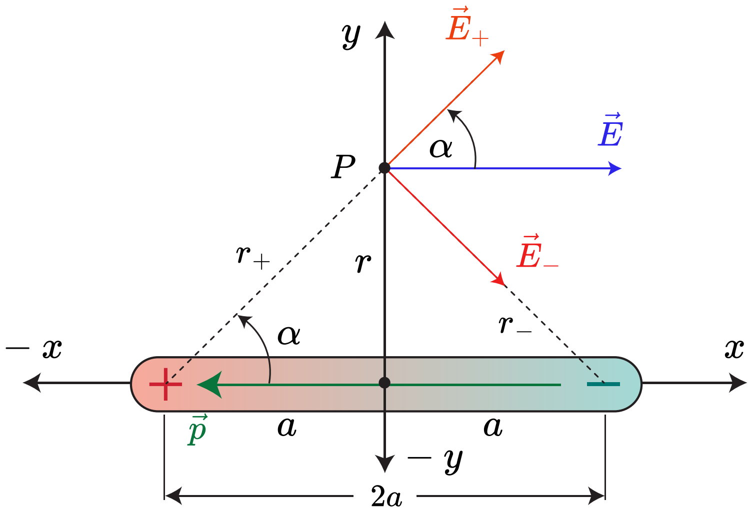

Figure 2: An electric dipole produces an electric field at point P off the dipole axis. Point P is at a distance r perpendicularly off the dipole axis. Here, $r \gg a$.

Measuring the field r distance, perpendicularly away from the bisecting point of the dipole:

$$\begin{aligned}
\vec{E}_{\text{dipole}} &= \vec{E}_+ + \vec{E}_- \\
&= \{E_+(x) + E_-(x)\}\,\hat{i} + \{E_+(y) + E_-(y)\}\,\hat{j} \\
&= \{E_+(x) + E_-(x)\}\cos\alpha\hat{i} + 0\hat{j} \\
&= \frac{2}{4\pi\epsilon_0}\cdot\frac{q}{y^2+a^2}\cdot\frac{a}{\sqrt{y^2+a^2}}\hat{i} \\
&= \frac{q\cdot 2a}{4\pi\epsilon_0}\frac{1}{(y^2+a^2)^{\frac{3}{2}}}\hat{i} \\
&= \frac{1}{4\pi\epsilon_0}\frac{p}{(y^2+a^2)^{\frac{3}{2}}}\{-\hat{p}\}\,.
\end{aligned} \tag{3}$$

The field points from the direction of the dipole moment. i.e., toward the positive charge.

To account for the arbitrary direction of the observation point P along the dipole axis. We replace y with r, giving us:

$$\vec{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{p}{(r^2+a^2)^{\frac{3}{2}}}\{-\hat{p}\} \tag{4}$$

In the limit $r \gg a$, we get the following:

$$\vec{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\{-\hat{p}\}\,. \tag{5}$$

We can see that Eqs. (5) and (2) are identical except for a factor of two, which shows up only when the field is measured along the dipole axis.

In both cases, it is evident that the electric field due to an electric dipole falls off as $\frac{1}{r^3}$. This is called the *Inverse Cube Law*, unlike the well-known *Inverse Square Law* we encountered for a single-point charge.

Solutions to Exercise Problems — Class-3 — Summer-2025

Incepting Ideas (1)

Given parameters:

$$
\begin{aligned}
E &= 5.0 \times 10^5\,\mathrm{N\,C^{-1}} \\
q &= 1.6 \times 10^{-19}\,\mathrm{C} \\
d &= 0.125\,\mathrm{nm} = 0.125 \times 10^{-9}\,\mathrm{m} \\
\theta &= 145^\circ
\end{aligned}
$$

(a) Net force on dipole:
Equal and opposite forces act on both charges $\Rightarrow$ net force is zero.

$$F_{\text{net}} = 0$$

(b) Dipole moment:

$$
\begin{aligned}
p = qd &= (1.6 \times 10^{-19})(0.125 \times 10^{-9}) \\
&= 2.0 \times 10^{-29}\,\mathrm{C\,m}
\end{aligned}
$$

Direction: 145° clockwise from the electric field (x-axis).
(c) Torque on dipole:

$$
\begin{aligned}
\tau &= pE\sin\theta \\
&= (2.0 \times 10^{-29})(5.0 \times 10^5)\sin(145^\circ) \\
&= 1.0 \times 10^{-23} \cdot \sin(145^\circ)\,\mathrm{N\,m} \\
&= 1.0 \times 10^{-23} \cdot 0.5736 \\
&= 5.74 \times 10^{-24}\,\mathrm{N\,m}
\end{aligned}
$$

Direction: Out of the page (using right-hand rule for torque).
(d) Potential energy:

$$
\begin{aligned}
U &= -pE\cos\theta \\
&= -(2.0 \times 10^{-29})(5.0 \times 10^5)\cos(145^\circ) \\
&= -1.0 \times 10^{-23} \cdot \cos(145^\circ) \\
&= -1.0 \times 10^{-23} \cdot (-0.8192) \\
&= 8.19 \times 10^{-24}\,\mathrm{J}
\end{aligned}
$$

Incepting Ideas (2)

Given parameters:

$$
\begin{aligned}
p &= 1.1 \times 10^{-30}\,\mathrm{C\,m} \\
E &= 8.0 \times 10^8\,\mathrm{N\,C^{-1}}
\end{aligned}
$$

Torque on a dipole:

$$\tau = pE\sin\theta$$

Maximum torque: occurs when $\sin\theta = 1$ (i.e., $\theta = 90°$)

$$\begin{aligned}\tau_{\max} &= (1.1 \times 10^{-30})(8.0 \times 10^{8}) \\ &= 8.8 \times 10^{-22}\,\mathrm{N\,m}\end{aligned}$$

Minimum torque: occurs when $\sin\theta = 0$ (i.e., $\theta = 0°$ or $180°$)

$$\tau_{\min} = 0\,\mathrm{N\,m}$$

Incepting Ideas (3)

Given parameters:

$$\begin{aligned}p &= 1.1 \times 10^{-30}\,\mathrm{C\,m} \\ E &= 8.0 \times 10^{8}\,\mathrm{N\,C^{-1}}\end{aligned}$$

Potential energy of a dipole:

$$U = -pE\cos\theta$$

(a) When dipole is <u>parallel</u> to field: $\theta = 0° \Rightarrow \cos\theta = 1$

$$\begin{aligned}U_{\parallel} &= -(1.1 \times 10^{-30})(8.0 \times 10^{8}) \cdot \cos(0°) \\ &= -8.8 \times 10^{-22}\,\mathrm{J}\end{aligned}$$

(b) When dipole is <u>perpendicular</u> to field: $\theta = 90° \Rightarrow \cos\theta = 0$

$$\begin{aligned}U_{\perp} &= -(1.1 \times 10^{-30})(8.0 \times 10^{8}) \cdot \cos(90°) \\ &= 0\,\mathrm{J}\end{aligned}$$

Incepting Ideas (4)

Given angles:

$$\begin{aligned}\theta_A &= 0° \\ \theta_B &= 180° \\ \theta_C &= 90° \\ \theta_D &= 45°\end{aligned}$$

(1) Net force on dipoles in a uniform field:

$$F_{\text{net}} = 0 \quad \text{(for all dipoles in uniform } \vec{E}\text{)}$$

Ranking:

$$F_{\text{net},A} = F_{\text{net},B} = F_{\text{net},C} = F_{\text{net},D}$$

(2) Net torque on dipoles:

$$\begin{aligned}
\tau &= pE\sin\theta \\
\tau_A &= pE\sin(0^\circ) = 0 \\
\tau_B &= pE\sin(180^\circ) = 0 \\
\tau_C &= pE\sin(90^\circ) = pE \\
\tau_D &= pE\sin(45^\circ) \\
&= pE \cdot \frac{\sqrt{2}}{2} \\
&\approx 0.707pE
\end{aligned}$$

Ranking:

$$\tau_C > \tau_D > \tau_A = \tau_B$$

(3) Dipole potential energy:

$$\begin{aligned}
U &= -pE\cos\theta \\
U_A &= -pE\cos(0^\circ) = -pE \\
U_B &= -pE\cos(180^\circ) = +pE \\
U_C &= -pE\cos(90^\circ) = 0 \\
U_D &= -pE\cos(45^\circ) \\
&= -pE \cdot \frac{\sqrt{2}}{2} \\
&\approx -0.707pE
\end{aligned}$$

Ranking (from highest to lowest):

$$U_B > U_C > U_D > U_A$$

3 Equilibrium of an Electric Dipole

Equilibrium Type	**Orientation (θ)** w.r.t $\vec{E}$	**Potential Energy (U)**	**Torque on Deviation** (τ)	**Stability**
Stable	0° (parallel)	$-pE$ (minimum)	Restoring	Returns to equilibrium
Unstable	180° (antiparallel)	pE (maximum)	Destabilizing	Moves away from equilibrium
Neutral	Any (no field, $\mathbf{E} = 0$)	0 (constant)	None	No change in position