PhD Micro (Part 2)
General Equilibrium Under Uncertainty

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1 Framework

1.1 Background

In this section, we extend what we learned about general equilibrium to a world of uncertainty. We'll start with the simplest case, and the one that we will work with the most.

In this world, there are I consumers and two time periods: $t = 0$ (today) and $t = 1$ (tomorrow). At $t = 0$, there is uncertainty about what will happen in the next period. This is represented by S possible states of the world that could occur at $t = 1$. All consumers have the same beliefs about the probabilities of each state occurring: $\pi_s \in [0, 1]$ that state $s \in \{1, ..., S\}$ occurs, with $\sum_{s=1}^{S} \pi_s = 1$. These probabilities are what consumers believe at $t = 0$; at $t = 1$ the true state of the world is revealed and there is no more uncertainty.

An easy example to understand this is an election. Before the election ($t = 0$), we don't know who will win, it could be either the Democrat ($s = 1$) or the Republican ($s = 2$). This means there are $S = 2$ possible states of the world. Say we also have objective beliefs about the probabilities, e.g. $p_1 = 0.6$ says that the Democrat has a 60% chance of winning (and so $p_2 = 0.4$). However, after the election ($t = 1$), there is no more uncertainty. We know who won the election, so we know whether we are in either state 1 (D won) or state 2 (R won).

Why do care about uncertainty? Not only does this add some more element of realism, it also allows to think about another dimension of trades. For example, uncertainty allows you to have a bet with your friend, e.g. if the Republican wins, you owe me \$50 but if the Democrat wins, then I owe you. There could also be uncertainty in terms of endowments, e.g. tomorrow you could be healthy ($s = 1$) or sick ($s = 2$) and this could affect the amount of time you are able to devote to leisure/labor (i.e. your endowment of time).

We could do a lot more to this. Some possible extensions would be to allow for subjective probabilities about the states, allowing for firms and uncertainty over their technologies, uncertainty over private ownership of firms and a lot more. There can also be a lot more than 2 time periods (e.g. last period is $T > 1$) where, instead of all uncertainty being resolved tomorrow, there could be a gradual release of information and at each period we would be updating our beliefs about what state we will end up in.

The way we'll think about this is to have consumer trading on state-contingent commodities. This is effectively a promise to receive x units of a good if and only if a state occurs. Think back to the election

betting example: you could either win or lose money depending on the state. Here, the same idea will hold so we will allow x to be positive (someone owes you) or negative (you owe someone).

As we are trading state-contingent commodities, our preferences need to be over these. But these commodities are being traded *before* the uncertainty is resolved. For example, say your friend offers you two bets: (1) you get \$20 if D wins and you pay \$20 if R wins, or (2) you get \$80 if D wins but you pay \$100 if R wins. How should you compare between these two bets? In essence, this is just a comparison of lotteries, and as we learned in Part 1, it is useful to make consumers be *expected utility maximizers*. The key idea here is that we will always be making comparisons *ex ante* (before the uncertainty has been resolved).

1.2   Notation

I will break down the notation by each time period. To keep things simple, I will only consider an economy with I consumers, L goods, and two time periods (i.e. $T = 1$). We will also allow for preferences to change over time/states, i.e. they will have different <u>Bernoulli</u> utilities in each state.

At $t = 0$:

- *Endowments:* A consumer i's endowment is a $L \times 1$ vector denoted as $\omega_{0i} = (\omega_{10i}, ..., \omega_{L0i}) \in \mathbb{R}^L$

- *Allocations:* A consumer i's allocation is a $L \times 1$ vector denoted as $x_{0i} = (x_{10i}, ..., x_{L0i}) \in \mathbb{R}^L$

- *Prices:* Prices for goods are a $L \times 1$ vector denoted as $p_0 = (p_{10}, ..., p_{L0}) \in \mathbb{R}^L_+$, where p_{l0} is the price for x_{l0i}

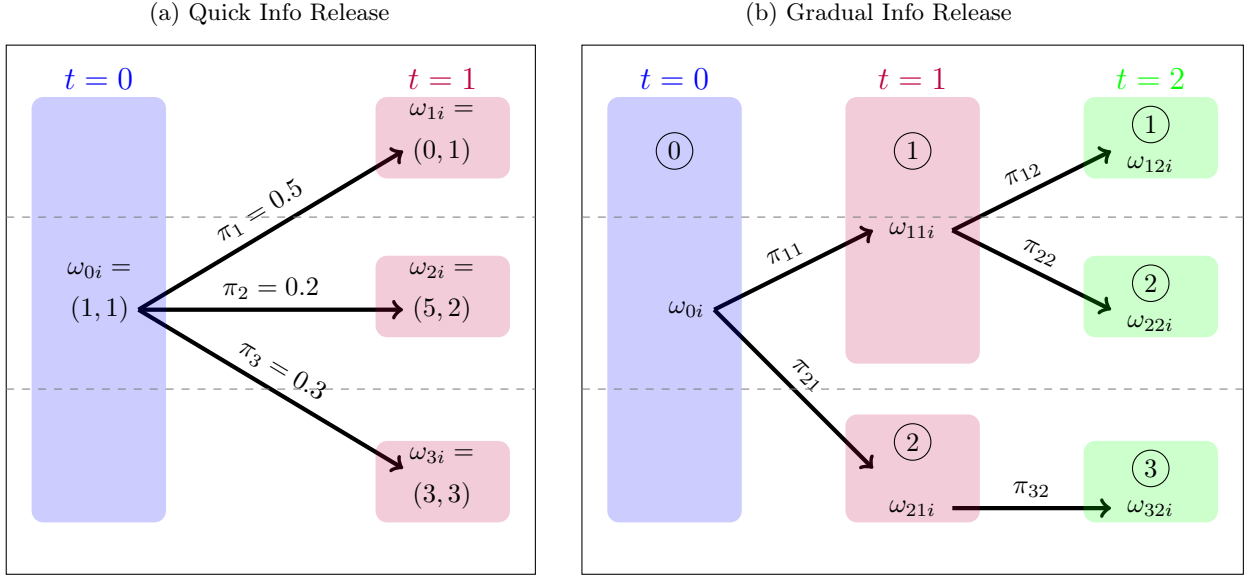- *Utility:* Consumer i's utility in this period is $u_{0i}(\cdot)$

At $t = 1$:

- *States:* At time $t = 1$, a state $s \in \{1, ..., S\}$ will occur. The probability that state s occurs is $\pi_s \in [0, 1], \forall s \in \{1, ..., S\}$ (i.e. this is the belief about states in $t = 0$)

- *Endowments:* A consumer i's state-contingent endowment is a $L \times S$ vector denoted as $\omega_i = (\omega_{1i}, ..., \omega_{Si}) \in \mathbb{R}^{LS}$. Each component is a $L \times 1$ vector $\omega_{si} = (\omega_{1si}, ..., \omega_{Lsi})$, where ω_{lsi} represents the endowment of good l that consumer i receives if state s occurs.

- *Allocations:* A consumer i's state-contingent allocation is a $L \times S$ vector denoted as $x_i = (x_{1i}, ..., x_{Si}) \in \mathbb{R}^{LS}$. Each component is a $L \times 1$ vector $x_{si} = (x_{1si}, ..., x_{Lsi})$, where x_{lsi} represents the allocation of good l that consumer i receives if state s occurs.

- *Prices:* Prices for goods are a $L \times S$ vector denoted as $p = (p_1, ..., p_S) \in \mathbb{R}^{LS}_+$, where $p_s = (p_{1s}, ..., p_{Ls}) \in \mathbb{R}^L_+$ is the price vector for state s and p_{ls} is the price for good x_{lsi}

- *Utility:* Consumer i's utility in state s is $u_{si}(\cdot)$. Since this is being evaluated at $t = 0$, this period will also be discounted by consumer i at a rate $\delta_i \in [0, 1]$

To give an example of the notation, consider Figure 1a. At $t = 0$, we are unsure of the three possible states that could occur at $t = 1$, but we are not sure which one. We do know the probabilities though: state 1 has a 50% chance of happening, state 2 has 20% chance, and state 3 has 30% chance. Consumer i's endowment also changes in each state too.

We could also generalize this to more time periods. An example of this is in Figure 1b, where there are now three time periods but there are still 3 true states of the world (all revealed at $t = 2$). At $t = 0$, we don't know which of the 3 we are in. At $t = 1$, if we are in the top box (red 1), then we could either be

Figure 1: Uncertainty Example

(a) Quick Info Release

$t = 0$

$\omega_{0i} = (1,1)$

$\pi_1 = 0.5$ $\omega_{1i} = (0,1)$

$\pi_2 = 0.2$ $\omega_{2i} = (5,2)$

$\pi_3 = 0.3$ $\omega_{3i} = (3,3)$

$t = 1$

(b) Gradual Info Release

$t = 0$ $t = 1$ $t = 2$

(0)   (1)   (1) ω_{12i}

ω_{0i} π_{11} ω_{11i} π_{12}

π_{22} (2) ω_{22i}

π_{21} (2)   (3)

ω_{21i} π_{32} ω_{32i}

in state 1 or 2. If we are in the bottom box (red 2), then we know for sure that we will end up in state 3. The shaded boxes represent an *information set*: what do we know up at to that point. A state is formally a complete trajectory or path from $t = 0$ to $t = T$. So state 1 is actually $\{0,1,1\}$, state 2 is $\{0,1,2\}$ and state 3 is $\{0,2,3\}$. At $t = 1$ if we are at red box 1, the trajectory so far is $\{0,1\}$; this means we can't distinguish between states 1 and 2 (and hence in the diagram the box covers both worlds). You will see these type of diagrams a lot next semester in game theory. For notation, we can have $\omega_{s_t t i}$ and $x_{s_t t i}$ represent endowments and allocation for consumer i in time t and information set s_t, e.g. $x_{l s_t t i}$ represents the allocation of good l that consumer i would receive in time t at information set $s_t \in S_t$.

Now that we understand the notation, the next step is study equilibria. The above information is what you would get in a question (uncertainty structure, probabilities, and endowments). Your goal (as in the usual equilibrium) will be to find allocations and prices. However, we have to also define how agents trade (the market structure) as that will define the type of equilibrium we will have. In the next section we will consider three different market structures (and hence three types of equilibria)

2 Market Structures and Equilibria

2.1 Arrow Debreu Equilibrium

In this setup, agents trade state-contingent commodities at $t = 0$. In other words, a consumer i agrees to get $x_{lsi} \in \mathbb{R}_+$ units of good l if and only if state s occurs. To get this, the agent pays a price p_{ls} for each unit in $t = 0$. This is very important: the payments will be made regardless of what state ends up happening at $t = 1$. Once we get to $t = 1$, the uncertainty is resolved and the promised state-contingent commodities are traded - but no other trading occurs.

> **Market Structure for Arrow-Debreu Equilibrium:**
> There are LS markets for every contingent commodity x_{lsi} that open *before* the uncertainty is resolved. All payments are made before knowing the state, but once the state s^* is known, you must deliver *only* on the promises x_{ls^*i}

We want to find the analogue of a Walrasian equilibrium in this setting, i.e. consumers are utility maximizing and markets are clearing. Everything is essentially the same except for two key differences:

1. Consumers maximize *expected* utility subject to a *intertemporal* budget constraint

2. Markets need to clear in *every* state

Under the described market structure, we will have an *Arrow-Debreu equilibrium*.

> **Arrow-Debreu Equilibrium:**
> An Arrow-Debreu Equilibrium is
> - An allocation $x^* = ((x_{01}^*, ..., x_{0I}^*), (x_1^*, ..., x_I^*))$, where $x_i^* = (x_{11i}^*, ..., x_{L1i}^*, ..., x_{1Si}^*, ..., x_{LSi}^*)$
> - A system of prices $(p_0^*, p^*) = (p_0^*, p_1^*, ..., p_S^*) \in \mathbb{R}^{L(S+1)}$, where $p_s^* = (p_{1s}^*, ..., p_{Ls}^*) \in \mathbb{R}^L$
>
> that satisfy the following conditions:
>
> 1. Every consumer i maximizes their expected utility subject to $t = 0$ budget constraint under prices p^*
>
> $$\max_{(x_{0i}, x_i)} \left[u_{0i}(x_{0i}) + \delta_i \sum_{s=1}^{S} \pi_s u_{si}(x_{si}) \right]$$
>
> $$\text{s.t. } p_0^* \cdot x_{0i} + \sum_{s=1}^{S} p_s^* \cdot x_{si} \leq p_0^* \cdot \omega_{0i} + \sum_{s=1}^{S} p_s^* \cdot \omega_{si}$$
>
> 2. Feasibility constraint at $t = 0$
>
> $$\sum_{i=1}^{I} x_{l0i}^* \leq \sum_{i=1}^{I} \omega_{l0i}, \forall l = 1, ..., L$$
>
> 3. Feasibility constraint at $t = 1$ (all states)
>
> $$\sum_{i=1}^{I} x_{lsi}^* \leq \sum_{i=1}^{I} \omega_{lsi}, \forall l = 1, ..., L \text{ and } \forall s = 1, ..., S$$

The notation can seem very intimidating at first, so let's walk through each of these and interpret them.

First, just as in the Walrasian equilibrium we are looking for both an allocation and prices. But, given the uncertainty, we need to find an allocation quantity and price for every good and every state. That's all the introductory paragraph is saying.

The first condition is the consumer's UMP and it highlights the first difference between this and the

Walrasian equilibrium. The objective function is just expected utility:

$$\underbrace{u_{0i}(x_{0i})}_{\substack{\text{Utility in} \\ \text{period } 0}} + \underbrace{\delta_i}_{\substack{\text{Discount} \\ \text{Factor}}} \underbrace{\sum_{s=1}^{S} \overbrace{\pi_s}^{\substack{\text{prob.} \\ \text{of state } s}} \overbrace{u_{si}(x_{si})}^{\substack{\text{utility} \\ \text{in state } s}}}_{\substack{\text{Expected utility} \\ \text{in period 1 (vNM)}}}$$

There is no expectation or discounting needed for the first term ($t = 0$) because there is no uncertainty or waiting needed. Notice that within each state we evaluate an allocation using that state's Bernoulli utility.

The other part of a UMP is a budget constraint. Since all the allocations are purchased at $t = 0$, this budget constraint needs to be intertemporal. In other words, the total value of your allocations over all possible states should equal the total value of your endowments over all possible states. It also means that even though a state may never end up happening, it can still affect your purchasing power today. The key idea is that because trading is only at done at one point (before the uncertainty is resolved), we must have only one budget constraint.

Conditions (2) and (3) are just feasibility constraints, except now we have them in every state. This highlights the second difference I pointed out, which is one that tends to trip many people up. Under uncertainty, we are often are working in expectations but when it comes to markets clearing, there can be no expectations of whether an allocation is feasible. For example, say I promise my friend I'll deliver 2 apples if state 1 occurs and 10 apples if state 2 occurs (each occur with 50% chance). In both states I will have an endowment of 6 apples (and my friend will have none). If state 1 occurs, then the trade is feasible ($2 \leq 6$) but if state 2 occurs, the trade is not feasible ($10 \nleq 6$). You might say "in expectation, the trade is feasible" (because $0.5 \times 2 + 0.5 \times 10 = 6 \leq 6 = 0.5 \times 6 + 0.5 \times 6$), but that's not how feasibility works. That's as ridiculous as saying "sometimes I float in the air, and sometimes I sink into the ground, but in expectation I obey the laws of gravity". The physical laws of the world (like feasible trades) can't be true sometimes - they have to always be true, no matter the state of the world (and even if the state does not end up occurring).

2.2   Radner Equilibrium

In the above setting, there are a lot of goods being traded and everything is being done before the uncertainty is resolved. This is a little unrealistic, so we want to simplify the number of goods being traded and we want to allow for more trading after the state is known.

To solve the first issue, we are going to introduce another thing that consumers can purchase called an *Arrow-Debreu asset*. Instead of having state-contingent commodities cover *every* good in every state, we just have these assets that cover *one* good in every state. This reduces the number of state-contingent commodity markets from LS to just S. So we fix a good for this asset (say good 1), and an Arrow-Debreu asset s will pay out 1 unit of good 1 in state s and nothing otherwise. We'll denote this by $z_{si} \in \mathbb{R}$, which indicates that the number units of good 1 that consumer i will receive in state s. Note that z_{si} can be positive (if they will get units from someone) or negative (if they must pay units to someone).

To solve the second issue, at $t = 1$ (when the uncertainty is resolved), we will allow L "spot markets" to open up for consumers to trade all L goods with each other. For this, we will use the same notation as before, with x_{si} being the allocation vector for consumer i in state s.

So, now at $t = 0$, a consumer has two things to purchase:

- Commodities to consume today $x_{0i} = (x_{10i}, ..., x_{L0i}) \in \mathbb{R}^L$ at prices $p_0 = (p_{10}, ..., p_{L0}) \in \mathbb{R}^L$, where p_{l0} is the per-unit price of x_{l0i}

- Portfolio of Arrow-Debreu assets $z_i = (z_{1i}, ..., z_{Si}) \in \mathbb{R}^S$ at prices $q = (q_1, ..., q_S) \in \mathbb{R}^S$, where q_s is the per-unit price of z_{si}

At $t = 1$, once we know that state s has occurred, the consumer purchases:

- Commodities to consume today $x_{si} = (x_{1si}, ..., x_{Lsi}) \in \mathbb{R}^L$ at prices $p = (p_{1s}, ..., p_{Ls}) \in \mathbb{R}^L$, where p_{ls} is the per-unit price of x_{lsi}

Let's again highlight the differences between this setup and the one for an Arrow-Debreu equilibrium. Notice now that x_{si} is purchased *after* the state is known (i.e. at $t = 1$) rather than before. This means that the payment and delivery of x_{si} will only happen if state s occurs (previously the payment always happened but the delivery was state-contingent). Even though only one state will occur, we will need to find allocations for all the states, i.e. $x_i \in \mathbb{R}^{LS}$. Moreover, there is now another set of prices q (the asset prices) that are separate to p (the spot prices).

Our choice of good 1 being the payment form of the Arrow-Debreu assets was entirely arbitrary. The important thing to realize is that there really is no difference between z_{si} and x_{1si}: they both represent getting good 1 in state s. So they must have the same price, i.e. the value of z_{si} if state s occurs is $p_{1s}z_{si}$. Moreover, for simplicity, you can also just normalize $p_{1s} = 1, \forall s$ since we won't be able to solve for all prices (just as before). In other words, we can only solve for $L - 1$ prices *within* a state.

> **Market Structure for Radner Equilbrium:**
> Before the uncertainty is resolved, consumers purchase allocations x_{0i} and asset portfolio z_i. After the uncertainty is resolved, L spot markets open up for consumers to trade all L goods with each other. All payments for assets are done in $t = 0$, but the return on assets is realized in $t = 1$

Under this setting, we can now define a *Radner equilibrium*

> **Radner Equilbrium:**
> A Radner equilibrium is
> - An allocation $(x^*, z^*) = ((x_{01}^*, ..., x_{0I}^*), (x_1^*, ..., x_I^*), (z_1^*, ..., z_I^*)) \in \mathbb{R}^{L(S+1)I}$, with spot allocations $x_i^* = (x_{11i}^*, ..., x_{L1i}^*, ..., x_{1Si}^*, ..., x_{LSi}^*)$ and asset portfolios $z_i^* = (z_{1i}^*, ..., z_{Si}^*)$
> - A system of spot prices $(p_0^*, p^*) = (p_0^*, p_1^*, ..., p_S^*) \in \mathbb{R}^{L(S+1)}$, with $p_s^* = (p_{1s}^*, ..., p_{Ls}^*) \in \mathbb{R}^L$
> - A system of asset prices $q^* = (q_1^*, ..., q_S^*) \in \mathbb{R}^S$
>
> that satisfy the following conditions:
>
> *(continued)*

Radner Equilbrium: (continued)

1. Every consumer i maximizes their expected utility subject to a budget constraint in every state under prices (p_0^*, p^*) and q^*

$$\max_{x \in \mathbb{R}^{L(S+1)}} \left[u_{0i}(x_{0i}) + \delta_i \sum_{s=1}^{S} \pi_s u_{si}(x_{si}) \right]$$

$$\text{s.t. } p_0^* \cdot x_{0i} + \sum_{s=1}^{S} q_s^* z_{si} \leq p_0^* \cdot \omega_{0i}$$

$$\text{and } p_s^* \cdot x_{si} \leq p_s^* \cdot \omega_{si} + p_{1s}^* z_{si}, \forall s = 1, ..., S$$

2. Feasibility constraint at $t = 0$

$$\sum_{i=1}^{I} x_{l0i}^* \leq \sum_{i=1}^{I} \omega_{l0i}, \forall l = 1, ..., L$$

3. Feasibility constraint at $t = 1$ (all states)

$$\sum_{i=1}^{I} x_{lsi}^* \leq \sum_{i=1}^{I} \omega_{lsi}, \forall l = 1, ..., L \text{ and } \forall s = 1, ..., S$$

4. Feasibility constraint for Arrow-Debreu assets

$$\sum_{i=1}^{I} z_{si}^* \leq 0, \forall s = 1, ..., S$$

Let's break this definition down step-by-step:

1. The consumer's objective has not changed: it is still about maximizing utility. The key difference is that instead of having one budget constraint, we know have $S + 1$ (one for $t = 0$ and one for each of the S states in $t = 1$). This is because there are purchases being made in every state, so the purchases have to be feasible (regardless of whether the state occurs or not). Also notice that in $t = 0$, assets are on the LHS (because you buy them) and in $t = 1$, assets are on the RHS (because they increase your wealth). However, if z_{si} is negative then this logic is the opposite (it increases your wealth in $t = 0$ and decreases it in state s). Consumer's don't derive utility from their assets, they just use them as a form of insurance to transfer their wealth across different states.

2. This feasibility constraint is the same as before

3. This feasibility constraint is the same as before

4. This ensures that the amount of asset delivered is at least as much as the amount promised (i.e. no scamming). In optimality, we would expect this to bind ($\sum_i z_{si} = 0$) indicating that all the positive z_{si} (someone owes you) exactly cancel out with the negative z_{si} (you owe someone)

2.3   Radner Equilibrium under General Asset Markets

The asset structure we gave above is more realistic but it is still quite restrictive: each asset only gives you payments in one state and the payment is only 1 unit of good 1. We could have a more general structure where there are K assets, that each pay out different returns depending on the state. We can then represent an asset k's state-dependent returns as $(a_{k1}, ..., a_{kS})$, where asset k pays back a_{ks} units of good 1 in state s. Putting this together, we can represent the general asset structure in the following $S \times K$ matrix:

$$
A = \begin{pmatrix} a_{11} & \cdots & a_{K1} \\ \vdots & \ddots & \vdots \\ a_{1S} & \cdots & a_{KS} \end{pmatrix}
$$

The kth column of the matrix A represents the return vector of each asset k. The sth row of A represents the payments the consumer would get in state s from each of the different assets. As before, we will let the asset prices be represented by $q = (q_1, ..., q_K) \in \mathbb{R}^K$, where q_k is the price of asset k. Consumer i's demand for asset k will be z_{ki}, and their asset portfolio is again $z_i = (z_{1i}, ..., z_{Ki}) \in \mathbb{R}^K$.

Notice that this is just a special case of what we saw before. In the usual Radner equilibrium setting, we had $K = S$ and $A = I_S$ (i.e. a $S \times S$ identity matrix). So all we are doing is just generalizing the asset structure, but the idea behind an equilibrium will be just the same as before.

Now we will have a *Radner equilibrium under general asset markets*.

Radner Equilbrium Under General Asset Markets:
A Radner equilibrium under general asset markets is
- An allocation $(x^*, z^*) = ((x^*_{01}, ..., x^*_{0I}), (x^*_1, ..., x^*_I), (z^*_1, ..., z^*_I)) \in \mathbb{R}^{L(S+1)I} \times \mathbb{R}^{KI}$, with spot allocations $x^*_i = (x^*_{11i}, ..., x^*_{L1i}, ..., x^*_{1Si}, ..., x^*_{LSi})$ and asset portfolios $z^*_i = (z^*_{1i}, ..., z^*_{Ki})$
- A system of spot prices $(p^*_0, p^*) = (p^*_0, p^*_1, ..., p^*_S) \in \mathbb{R}^{L(S+1)}$, with $p^*_s = (p^*_{1s}, ..., p^*_{Ls}) \in \mathbb{R}^L$
- A system of asset prices $q^* = (q^*_1, ..., q^*_K) \in \mathbb{R}^K$

that satisfy the following conditions:

1. Every consumer i maximizes their expected utility subject to a budget constraint in every state under prices (p^*_0, p^*) and q^*

$$
\max_{x \in \mathbb{R}^{L(S+1)}} \left[u_{0i}(x_{0i}) + \delta_i \sum_{s=1}^{S} \pi_s u_{si}(x_{si}) \right]
$$

$$
\text{s.t. } p^*_0 \cdot x_{0i} + \sum_{k=1}^{K} q^*_k z_{ki} \leq p^*_0 \cdot \omega_{0i}
$$

$$
\text{and } p^*_s \cdot x_{si} \leq p^*_s \cdot \omega_{si} + p^*_{1s} \sum_{k=1}^{K} z_{ki} a_{ks}, \forall s = 1, ..., S
$$

Radner Equilbrium Under General Asset Markets: (continued)

2. Feasibility constraint at $t = 0$

$$\sum_{i=1}^{I} x_{l0i}^* \leq \sum_{i=1}^{I} \omega_{l0i}, \forall l = 1, ..., L$$

3. Feasibility constraint at $t = 1$ (all states)

$$\sum_{i=1}^{I} x_{lsi}^* \leq \sum_{i=1}^{I} \omega_{lsi}, \forall l = 1, ..., L \text{ and } \forall s = 1, ..., S$$

4. Feasibility constraint for assets

$$\sum_{i=1}^{I} z_{ki}^* \leq 0, \forall k = 1, ..., K$$

Notice that this is basically the same as before, with the biggest difference coming in the $t = 1$ budget constraints for each s.

$$p_s^* \cdot x_{si} \leq p_s^* \cdot \omega_{si} + p_{1s}^* \overbrace{\sum_{k=1}^{K} z_{ki} a_{ks}}^{\substack{\text{Total returns of} \\ \text{good 1 from all assets}}}, \forall s = 1, ..., S$$

$$\underbrace{\phantom{p_{1s}^* \sum_{k=1}^{K} z_{ki} a_{ks}}}_{\substack{\text{Total value of} \\ \text{asset returns}}}$$

Since all K assets could give a return in state s and each return is different, we need the term $\sum_k z_{ki} a_{ks}$. Since Arrow-Debreu assets only give a return in one state, we don't need to sum over assets (because only one is positive and the other are all zeros). Another difference is that sums over assets are now up to K and not S.

3 Evaluating Equilibria

3.1 Pareto Optimality

Preferences are over state-contingent commodities, i.e. $x_i = (x_{1i}, ..., x_{Si})$. This means that $x_i \succsim_i x_i' \iff U_i(x_i) \geq U(x_i')$ where $U_i(\cdot)$ is a vNM utility function. In other words, consumer i prefers x_i over x_i' if and only if

$$U_i(x_i) = \sum_{s=1}^{S} \pi_s u_{si}(x_{si}) \geq \sum_{s=1}^{S} \pi_s u_{si}(x_{si}') = U_i(x_i')$$

Next, we want to think about Pareto optimality. Under uncertainty, we can think about Pareto efficiency in two ways: *ex ante* (before the uncertainty is resolved) and *ex post* (after the uncertainty is resolved). Ex ante efficiency is exactly the same as the standard Pareto optimality definition except using the vNM utility function $U_i(\cdot)$. Ex post efficiency is evaluated using the Bernoulli utility function $u_{si}(\cdot)$.

The key result to remember is that **ex ante efficiency implies ex post efficiency (but not the other way around)**.

The logic is straightforward. Consider an allocation x that is ex ante efficient but suppose that in state s it is not ex post efficient. Then there is some other allocation $(x_{s1}', ..., x_{sI}')$ that Pareto dominates $(x_{s1}, ..., x_{sI})$. But then consider the allocation $x^* = (\{(x_{(-s)i}, x_{si}')\}_{i=1}^I)$, i.e. for each x_i, replace x_{si} with x_{si}'. Then x^* Pareto dominates x, which contradicts the ex ante efficiency of x. To prove that the reverse is not true, notice that ex post efficiency doesn't impose any restrictions on states that do not occur. But it's more that that. Even if we took S ex post efficient allocations $\{x^s\}_{s=1}^S$, that still won't necessarily be ex ante efficient. For ex ante efficiency, we need to also have _efficient risk-sharing._

Without uncertainty, Pareto optimality meant that for any two consumers i and m and two goods l and k, we have that $MRS_{lk}^i = MRS_{lk}^m$. With uncertainty, all we are doing is defining our "goods" to be a good-state combination (i.e. the same good in two different states is treated as two distinct goods). Using the same MRS condition, with good-state pairs l^*, k^*, we get:

$$\frac{u_i'(x_{l^*i})}{u_i'(x_{k^*i})} = \frac{u_m'(x_{l^*m})}{u_m'(x_{k^*m})}, \forall i, m \in \{1, ..., I\}; \forall l^*, k^* \in \{1, ..., L\} \times \{1, ..., S\}$$

Now we just write $l^* = ls$ and $k^* = kr$ where l, k are good indices and s, r are state indices:

$$\frac{u_i'(x_{lsi})}{u_i'(x_{kri})} = \frac{u_m'(x_{lsm})}{u_m'(x_{krm})}, \forall i, m \in \{1, ..., I\}; \forall l, k \in \{1, ..., L\}; \forall s, r \in \{1, ..., S\}$$

Doing some re-arranging, this means that:

$$\frac{u_i'(x_{lsi})}{u_m'(x_{lsm})} \text{ does not depend on the state or good}$$

In a world of no uncertainty, our MRS condition gives us that the ratio of marginal utilities is the same across all goods. Now, we just extend this to say that it is also the same across all states too. This idea that there are no gains from switching is efficient risk sharing. So ex ante efficiency is the more demanding concept: we need ex post efficiency in every state (within-state efficiency) as well as efficient risk sharing (across-state efficiency)

3.2 Comparing Equilibria

The Arrow-Debreu equilibrium setup is actually no different that the standard Walrasian equilibrium. With uncertainty, all we are doing is defining our "goods" to be a good-state combination (i.e. the same good in two different states is treated as two distinct goods). This means that all the previous results will hold too:

- Under local non-satiation, the FWT holds and so a Arrow-Debreu equilibrium is Pareto optimal

- Under convex preferences, the SWT holds and so every Pareto optimal allocation can be supported as a Arrow-Debreu equilibrium with transfers

- Under strict monotonicity, an equilibrium is guaranteed to exist

Another way to see this is to solve for the set of Pareto optimal allocations. We know how to do this: maximize one consumer's utility subject to a minimum utility for all other consumers and feasibility constraints:

$$\max_{\{(x_{0i}, x_i)\}_{i=1}^I} \left[u_{01}(x_{01}) + \delta_1 \sum_{s=1}^S \pi_s u_{s1}(x_{s1}) \right]$$

$$\text{s.t. } u_{0i}(x_{0i}) + \delta_i \sum_{s=1}^S \pi_s u_{si}(x_{si}) \geq \bar{u}_i, \forall i = 2, ..., I$$

$$\sum_{i=1}^I x_{l0i}^* \leq \sum_{i=1}^I \omega_{l0i}, \forall l = 1, ..., L$$

$$\sum_{i=1}^I x_{lsi}^* \leq \sum_{i=1}^I \omega_{lsi}, \forall l = 1, ..., L \text{ and } \forall s = 1, ..., S$$

The Lagrangian for this is:

$$\mathcal{L} = \sum_{i=1}^I \lambda_i \left[u_{0i}(x_{0i}) + \delta_i \sum_{s=1}^S \pi_s u_{si}(x_{si}) - \bar{u}_i \right] + \sum_{s=0}^S \sum_{l=1}^L \mu_{ls} \left[\sum_{i=1}^I \omega_{lsi} - \sum_{i=1}^I x_{lsi} \right]$$

Where $\lambda_1 = 1$ and $\bar{u}_1 = 0$, and $s = 0$ represents $t = 0$ to make the notation more convenient. Consider two consumers (i and j), two states (s and r), and two goods (l and k). Taking FOCs with respect to x_{lsi}, x_{kri}, x_{lsj}, x_{krj} gives us:

$$\lambda_i \delta_i \pi_s \frac{\partial u_{si}(x_{si})}{\partial x_{lsi}} - \mu_{ls} = 0 \qquad\qquad \lambda_j \delta_j \pi_s \frac{\partial u_{sj}(x_{sj})}{\partial x_{lsj}} - \mu_{ls} = 0$$

$$\lambda_i \delta_i \pi_r \frac{\partial u_{ri}(x_{ri})}{\partial x_{kri}} - \mu_{kr} = 0 \qquad\qquad \lambda_j \delta_j \pi_r \frac{\partial u_{rj}(x_{rj})}{\partial x_{krj}} - \mu_{kr} = 0$$

$$\implies \frac{\partial u_{si}(x_{si})/\partial x_{lsi}}{\partial u_{ri}(x_{ri})/\partial x_{kri}} = \frac{\pi_r}{\pi_s} \frac{\mu_{ls}}{\mu_{kr}} = \frac{\partial u_{sj}(x_{sj})/\partial x_{lsj}}{\partial u_{rj}(x_{rj})/\partial x_{krj}}$$

$$\implies \frac{\partial u_{si}(x_{si})/\partial x_{lsi}}{\partial u_{sj}(x_{sj})/\partial x_{lsj}} = \frac{\partial u_{ri}(x_{ri})/\partial x_{kri}}{\partial u_{rj}(x_{rj})/\partial x_{krj}}$$

We have found that the ratio of marginal utility is constant across states and goods, which is exactly the condition we needed for ex ante efficiency. This actually raises an important implication. Remember in this setup we did not allow for trading at $t = 1$. However, since the equilibrium is ex ante efficient, it must also be ex post efficient. This means that even if we opened up spot markets in $t = 1$ (i.e. allowed for trades once the state is known), there would be no trading! If they did trade, then the allocation was not ex post Pareto optimal, which means it couldn't have been ex ante efficient.

Next, we have Radner equilibrium. Here, we've reduced the number of markets for state-contingent commodities (from LS to S), but actually **the set of Arrow-Debreu and Radner equilibria are the same**.

One way to see this is to note that the $t = 1$ budget constraints can be re-arranged (lets assume LNS so that all the constraints are binding):

$$z_{si} = \frac{1}{p_{1s}^*} p_s^* \cdot (x_{si} - \omega_{si})$$

Plugging this back into the $t = 0$ budget constraint:

$$p_0^* \cdot x_{0i} + \sum_{s=1}^{S} q_s^* \left(\frac{1}{p_{1s}^*} p_s^* \cdot (x_{si} - \omega_{si}) \right) = p_0^* \cdot \omega_{0i}$$

$$\implies p_0^* \cdot x_{0i} + \sum_{s=1}^{S} \frac{q_s^*}{p_{1s}^*} p_s^* \cdot x_{si} = p_0^* \cdot \omega_{0i} + \sum_{s=1}^{S} \frac{q_s^*}{p_{1s}^*} p_s^* \cdot \omega_{si}$$

This is basically the same as budget constraint for the Arrow-Debreu equilibrium (we are just normalizing prices in each state with $\frac{q_s^*}{p_{1s}^*}$). Moreover, if we consider z_{si} in optimality and sum over all consumers, we get:

$$\sum_{i=1}^{I} z_{si}^* = \frac{1}{p_{1s}^*} p_s^* \cdot \underbrace{\sum_{i=1}^{I}(x_{si}^* - \omega_{si})}_{\text{Aggregate Excess Demand}}$$

The sum is just the aggregate excess demand, which we know will be equal to 0 in a Walrasian equilibrium (i.e. markets are clearing). So condition 3 in the definition of Arrow-Debreu equilibrium will guarantee condition 4 in the Radner equilibrium. So really there isn't really any difference between the two equilibrium definitions and hence their solutions will be the same too.

Finally, we looked at a Radner equilibrium under general asset structure. Clearly we saw that having $K = S$ and $A = I_S$ was a special case that gives us the Radner equilibrium. But the point is not that A is the identity matrix, it's the fact that we have *complete markets*. This means that consumers can transfer wealth across all states. A simple way is of course to have the Arrow-Debreu assets (one unit payoff in only one state) but more generally, we just need the asset payments to *span* the set of all possible wealth vectors across states ($\mathbb{R}^S$). This is captured by the range or column space of the matrix A:

$$Range(A) = \{w \in \mathbb{R}^S : w = Az \text{ for some } z \in \mathbb{R}^K\}$$

This is capturing the set of wealths we can achieve with all possible portfolio compositions z. The rank of a matrix is the dimension of the column space. In matrix terms, this means an asset structure is complete if:

$$Rank(A) = S$$

Intuitively, this is saying that complete markets means that we need markets for every state (i.e. the asset structure allows agents to perfectly transfer wealth to any state). Another thing to remember is that $Rank(A) \leq \min\{K, S\}$. This means to have complete markets, we at least need $K \geq S$ (but if $K > S$ there will be redundant assets we can get rid of and not lose completeness). If $K < S$ (less assets than states), then markets are necessarily *incomplete* and consumers cannot perfectly transfer wealth across different states.

The point here is that the actual asset structure doesn't really matter, but rather its range. This is because for two asset structures A and A' with $Range(A) = Range(A')$, any wealth transfer done under A can also be done under A' (but with a different asset portfolio). This means that the equilibrium allocation of goods x^* will be the same under A and A'. In particular if $Range(A) = Range(I_S) = \mathbb{R}^S$, then the set of Radner equilibria under general asset structure are the same as the set of Radner equilibria, which in turn is the same as the set of Arrow-Debreu equilibria. This means that **under complete asset markets, the sets of Radner under general asset structure and Arrow-Debreu equilibria are the same**.

With incomplete markets, the equilibrium may no longer be Pareto efficient. In fact, even we add more assets to increase $Rank(A)$ (i.e. allow for trading in more markets), we may not make things better - it could in fact make everyone worse off. Of course opening all markets is optimal, but in general, there is no monotonic relationship between number of markets and efficiency.

The main takeaways from this section are summarized below:

Efficiency of Equilibria:

- *Arrow-Debreu Equilibrium:* This is the uncertainty analogue of a Walrasian equilibrium, and hence is ex ante efficient

- *Radner Equilibrium:* The equilibrium allocation x^* is the same as in Arrow-Debreu

- *Radner Equilibrium under General Asset Structure:* Under complete markets, the equilibrium allocation x^* is the same as in Arrow-Debreu.