Last updated: September 19, 2017

# PhD Micro (Part 1)
# Introduction to Consumer Theory

Motaz Al-Chanati

September 12, 2017

## 1 Preferences and Utility

### 1.1 Notation

Before we dive into this, let's make sure we are all comfortable with the setting and notation. For the first part of the course, we are going to be mostly thinking about a consumer making a decision over what commodities to consume.

In this world, there are $K$ goods to choose from and the consumer has to decide how much of each good they will consume.[1] We will define the quantity of good $k$ that the consumer chooses as $x_k$, where $k$ is an integer between 1 and $K$ representing each good. We can take all of these $x_k$'s and put them into a *commodity vector* (or *bundle*), which we'll call $x$. You'll often see $x$, which might look like a scalar, but don't let the notation mislead you - this is a vector! Sometimes you'll also see it written like so: $x = (x_1, ..., x_K)$, but keep in mind that it is actually a $1 \times K$ column vector. Since each $x_k$ is a quantity, we think that this should probably be some real number, i.e. $x_k \in \mathbb{R}$. Since there are $K$ goods, we can then say that the commodity vector lives in the *commodity space* in $K$-dimensional Euclidean space, i.e. $x \in \mathbb{R}^K$.

Next we want to think about what values of $x_k$ we will allow the consumer to choose from. This is called the *consumption set*, denoted by $X$. This is a subset of the commodity space, $X \subset \mathbb{R}^K$. You can think of some natural restrictions here that may depend on the context. Usually, we think it is unusual to have negative quantities, so we often restrict to positive numbers, i.e. $\mathbb{R}^K_+$. If we are dealing with time and $x_k$ represents hours per day, then we have a physical limit of $x_k \leq 24$. For the most part, we will let consumers choose any non-negative quantity for each good, i.e. $X = \mathbb{R}^K_+ = \left\{x \in \mathbb{R}^K : x_k \geq 0, \forall k = 1, ..., K\right\}$.

So to summarize, $x_k$ is the $k^{\text{th}}$ element of the vector $x$. The vector $x$ is an element of the set $X$, which in turn is a subset of $\mathbb{R}^K$ ($x \in X \subset \mathbb{R}^K$).

Finally, I will use the following convention: for two vectors $x, y \in \mathbb{R}^K$:

- $x \geq y$ if $x_k \geq y_k$, for all $k = 1, ..., K$
- $x > y$ if $x_k \geq y_k$, for all $k = 1, ..., K$ and $x_k > y_k$ for at least one $k$ (i.e. $x \neq y$)
- $x \gg y$ if $x_k > y_k$, for all $k = 1, ..., K$

[1] To be pedantic, they are commodities not goods but I will use these terms interchangeably

## 1.2 Preferences

Preferences are described by a binary relation $\succsim$ on $X$. Recall (from math camp) that a binary relation is just a subset of ordered pairs, $B \subset X \times X$, such that if $(x, y) \in B$, then $x \succsim y$. Our interpretation of this is that $x \succsim y$ means that "$x$ is at least as good as $y$" or that "$x$ is weakly preferred to $y$". This ordering actually induces two more relations:

**Induced Relations:**

- *Strict preference:* If $x \succsim y$ but not $y \succsim x$, then $x$ is strictly preferred to $y$. We denote this as $x \succ y$.
- *Indifference:* If $x \succsim y$ and $y \succsim x$, then the consumer is indifferent between $x$ and $y$ (equally happy with either). We denote this as $x \sim y$.

If it is helpful, think of these induced relations in the following way. Say that the agent is repeatedly choosing between $x$ and $y$. If $x \succ y$, then they should always select $x$ over $y$. If $x \sim y$, then we should see them picking both (sometimes they pick $x$, other times they pick $y$).

We say that preferences are *rational* if they satisfy two quite intuitive assumptions:

**Rational Preferences:** A preference relation $\succsim$ is rational if it satisfies:

1. *Completeness:* For any $x, y \in X$, we either have $x \succsim y$ or $y \succsim x$ (or both)
2. *Transitivity:* If $x \succsim y$ and $y \succsim z$, then $x \succsim z$

The math notation probably makes this look more confusing than it actually is, but it will become clear when you apply it to an example.

Completeness just says that the agent will always be able to compare two bundles. Say I ask you: "do you prefer apples or bananas?". If you didn't have complete preferences, you might not be able to give me an answer. Note that this is not the same as indifference; it's not that you don't care which one I give you, it's that you are incapable of even being able to make the comparison. This might seem unlikely when it comes to food, but maybe you could think of other contexts where that assumption may not hold. Maybe it is due to information constraints or it could be a very difficult moral dilemma. For example, what is your answer to the famous trolley problem?[2] If someone said that they couldn't possibly choose either option, would you say that they are not 'rational'?

Transitivity allows us to infer preferences based on what we have already observed. So if I prefer apples to bananas, and bananas to cherries, then transitivity would say I prefer apples to cherries. Be really careful here: if I (weakly) preferred cherries to apples that does not violate transitivity. Maybe I'm indifferent between all three! Transitivity only rules out that I *strictly* prefer cherries to apples.

Now, we're going to add some more assumptions. Let's go through the definitions and the intuition before we see why they're useful.

[2] https://en.wikipedia.org/wiki/Trolley_problem

**Continuity:** The following statements are all equivalent definitions of a continuous preference:

1. $\forall y \in X$, the sets $\{x \in X : x \succsim y\}$ and $\{x \in X : y \succsim x\}$ are closed
2. $\forall y \in X$, the sets $\{x \in X : x \succ y\}$ and $\{x \in X : y \succ x\}$ are open
3. If $x^n \to x$ and $y^n \to y$ such that $x^n \succsim y^n, \forall n$, then $x \succsim y$

*Proof:*

- (1) $\Leftrightarrow$ (2) : The complement of an open set is closed (and vice versa). In set notation, this means: $\{x \succsim y\}' = \{x \not\succsim y\} = \{y \succ x\}$
- (1) $\Rightarrow$ (3) : See MWG Exercise 3.C.3
- (3) $\Rightarrow$ (1) : Let $x^n \in \{z \in X : z \succsim y\}$ and $x^n \to x$. Let $y^n = y$, so $y^n \to y$. By this definition, we have $x \succsim y$, which means that $x \in \{z \in X : z \succsim y\}$ and hence the set is closed.

The idea behind continuity is that preferences don't "jump". We saw the first two definitions in class, but the third one really gets to the intuition more clearly. For example, say I prefer $1/8^{\text{th}}$ of a pizza to $1/8^{\text{th}}$ of a cake. And I prefer $1/7^{\text{th}}$ of a pizza to $1/7^{\text{th}}$ of a cake; $1/6^{\text{th}}$ of a pizza to $1/6^{\text{th}}$ of a cake and so on (i.e. the sequence is $1-1/n$). Now you offer me the whole pizza and the whole cake. If my preferences were not continuous, it *could* be the case that I *strictly* prefer the cake over the pizza even though the sequence of preferences would say I preferred 99.99% of the pizza to the cake. Continuous preferences rules out these 'preference reversals'. Continuous preferences do not, however, rule out indifference in the limit. Say the sequence was instead $1/n$, i.e. I preferred a whole pizza to a whole cake, $1/2$ a pizza to $1/2$ a cake and so on. These sequences converge to no pizza and no cake, which I'm probably indifferent between, but I could then say that I do *weakly* prefer no cake over no pizza.

To finish this section, let's bring in the last two assumptions.

**Additional Assumptions:** In addition to rationality and continuity, we will also assume that preferences have the following properties:

1. *Monotonicity:* If $x \gg y$ $(x_k > y_k, \forall k)$, then $x \succ y$
   - Strict Monotonicity: If $x > y$ $(x_k \geq y_k, \forall k$ and $x \neq y)$, then $x \succ y$.
2. *Convexity:* If $x \succsim y$, then $\lambda x + (1-\lambda)y \succsim y, \forall \lambda \in [0,1]$.
   - Equivalently: $\forall y \in X$, the set $\{x \in X : x \succsim y\}$ is convex
   - Strict Convexity: If $x \succsim y$ and $x \neq y$, then $\lambda x + (1-\lambda)y \succ y, \forall \lambda \in (0,1)$.

Monotonicity captures the idea of desirability (that more of a good thing is generally better than having less of it). This gives us "thin" indifference curves. To see why, try drawing a thick indifference curve where you would have a bundle that was larger than another but both be 'inside' the curve. This would mean that the consumer is indifferent between them which violates monotonicity. Convexity gives us convex indifference curves, which captures that we enjoy variety, i.e. that we need larger amounts of one good to make up for successive losses of another good.

## 1.3 Utility Representation

We now move from preferences to utility. Let's keep in mind the five assumptions that we've made:
1. Completeness
2. Transitivity
3. Continuity
4. Monotonicity
5. Convexity

You should also be comfortable with the following terms:

**Mathematical Terms:** A function $f(\cdot)$ is
- *Continuous:* $x_n \to x \implies f(x_n) \to f(x)$
- *Weakly increasing:* $x \geq y \implies f(x) \geq f(y)$.
- *Strictly increasing:* $x > y \implies f(x) > f(y)$
- *Quasi-concave:* $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}, \forall x, y \in X$ and $\forall \lambda \in [0,1]$
- *Strictly quasi-concave:* $f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}, \forall x, y \in X,\ x \neq y$, and $\forall \lambda \in (0,1)$

Our goal is utility representation, both its existence and its properties. First, let's define what we mean by representation:

**Utility Representation:** A function $u : X \to \mathbb{R}$ represents a preference relation $\succsim$ if:

$$x \succsim y \iff u(x) \geq u(y)$$

This also has implications on the induced relations: $x \succ y \iff u(x) > u(y)$ and $x \sim y \iff u(x) = u(y)$. So a utility function maps from the consumption set to real numbers. This is very useful because maximizing a real-valued function is relatively easy to do mathematically. One thing you might be able to see from the definition is that utility representation is not unique. If a function $u(\cdot)$ represents $\succsim$ and $f(\cdot)$ is a strictly increasing function, then $v(x) = f(u(x))$ is also going to represent $\succsim$:

$$x \succsim y \overset{\substack{\text{utility}\\ \text{rep.}}}{\iff} u(x) \geq u(y) \overset{\substack{\text{strictly}\\ \text{incr.}}}{\iff} v(x) \geq v(y)$$
$$\therefore x \succsim y \iff v(x) \geq v(y)$$

Note that this has to be *strictly* increasing function. Let's say $f(\cdot)$ was just weakly increasing and we have $x \succ y$, which implies $u(x) > u(y)$, which in turn implies $v(x) \geq v(y)$. So we could have a situation with $x \succ y$ and $v(x) = v(y)$, which violates representation.

This captures the idea of cardinal versus ordinal properties of utility functions. Compare the following values: $u(x) = 2, u(y) = 1, u(z) = 3$ against $v(x) = 33, v(y) = 31, v(z) = 100$. While the actual values of the utility functions differ (this is cardinal), the order does not change (ordinal). In both cases, we have the preference ranking of the three bundles being $z \succ x \succ y$. This ranking is much more informative than the levels. The strictly increasing transformation may change the cardinal values, but the utilities will remain ordinally equivalent, which is the key idea.

Now let's get into more properties of the utility functions:

**Properties of Utility Representation:**

- *Existence:* $\succsim$ is complete, transitive, and continuous $\iff$ there exists a continuous utility function $u(\cdot)$ that represents $\succsim$
- *Monotonicity:* $\succsim$ is (strictly) monotonic $\iff$ $u(\cdot)$ is (strictly) monotonic
- *Convexity:* $\succsim$ is (strictly) convexity $\iff$ $u(\cdot)$ is (strictly) quasi-concave

The first theorem tell us that a preference being rational is not enough for existence of utility representation, you also need to add in continuity. If you would like to see an example of why rationality is not enough, try Example 3.C.1 in MWG (lexicographic preferences). In fact, continuity (of preferences) is a sufficient condition for utility representation: not only does it guarantee the existence of a utility function, it also ensures that there exists a *continuous* utility function that represents the preference.[3] Remember that utility representation is not unique. For a preference $\succsim$ there are infinitely many valid utility representations, but this theorem tells us that we can be sure that at least one of the functions is continuous.

The next two properties are quite straightforward but there's a very important nuance here. The first theorem just said that there is *at least one* continuous representation. The second and third apply to *all* utility functions. For example, whenever you have a monotonic preference, you can be sure that any utility function that represents it will be weakly increasing.

So now you see that our *assumptions* about preferences have *implications* on the properties of utility representation. Next, we will move on to the utility maximization problem and see why those properties are extremely useful to find a solution.

# 2 Utility Maximization Problem

## 2.1 Notation

For this section, we will introduce some new notation. Prices will have a similar notation structure to the consumption bundle, where $p_k$ is the price of good $k$ and the $k^{\text{th}}$ element of the price vector $p = (p_1, ..., p_K)$. We will also put in a restriction of all prices being strictly above zero, i.e. $p \gg 0$ or $p_k > 0, \forall k = 1, ..., K$. Note that this is a $K \times 1$ column vector.

A consumer has wealth $w \in \mathbb{R}_+$. In class, you will see this written as $y$. I'm going to follow MWG's notation and use $w$ to avoid confusion when I want to use $y$ to represent an an alternative bundle to our usual bundle $x$.

## 2.2 The Problem

Before, we essentially got two bundles and just asked the consumer which bundle they preferred. Now, the consumer has to *purchase* a bundle that will make them the happiest (maximize utility). Of course, we would all like to have infinite quantities of each good but we are restricted by two things: how much things cost (prices) and how much money we have (wealth). We're going to assume that the consumer takes prices and their wealth as given.

[3]Here, we use the term "continuous" for both preferences and functions - make sure you notice the difference depending on the context

This is the basic motivation of the *utility maximization problem* (UMP), which we will write as follows:

$$\max_{x \in X} u(x)$$
$$\text{s.t. } p \cdot x \leq w$$

Let's pick this apart and get comfortable with the terminology:

- $\max_{x \in X}$: This tells us to choose the bundle $x$ in the set $X$ that will maximize whatever comes to the right of it (while satisfying the constraint below it)
- $u(x)$: This is the consumer's utility function. In optimization language, we call this the *objective function* (the thing we are trying to maximize or minimize)
- $p \cdot x$: This is the total cost of the bundle chosen. Remember that this is a dot product (we are working with vectors) so it actually means $p \cdot x = p_1 x_1 + ... + p_K x_K$.[4]
- s.t. $p \cdot x \leq w$: This is the constraint which defines the *feasible set* of solutions. We call the set $B(p, w) = \{x \in X : p \cdot x \leq w\}$ the *budget set.* It says that the total cost of the bundle chosen must be less than or equal to the consumer's income (which makes sense - you can't buy something if you don't have the money for it!)

With an optimization problem, and especially in this section, you have to be able to distinguish between a *feasible* bundle and an *optimal* bundle. Everything inside the budget set is feasible. For example, if you had $10, you could buy ten $1 items. However, you could also buy one $4 item. Feasible doesn't mean that you have to spend all your money or that it will even make you happy. An optimal bundle, on the other hand, will maximize your utility *out of all your feasible options.*

Now that we understand the UMP, let's define two important concepts:

**UMP Correspondences:**

- *Marshallian/Walrasian Demand Correspondence:* This is the set of solutions to the UMP (optimal bundles), which we will denote as $x(p, w)$
- *Indirect Utility Function:* This is the optimal value of the UMP (utility achieved with an optimal bundle), which we will denote as $v(p, w) = u(x), \forall x \in x(p, w)$

If you are having a hard time wrapping your head around what these are, just look closely at the notation. A UMP for a given consumer/preferences and consumption set can be characterized by just two things: price and wealth. Look back at the UMP and you can see three variables $(x, p, w)$, but $p$ and $w$ are given (we call these parameters or state variables). So all that's left is for the consumer to choose $x$ (that's why it's under the "max").

The Marshallian demand $x(p, w)$ maps price and wealth (the parameters) to the optimal choice variable $x$. In other words, it says "give me the prices of goods and the consumer's wealth, and I will tell you what bundles are optimal for the consumer". $x(p, w)$ maps back to the consumption set (so keep in mind that it's vector). The indirect utility $v(p, w)$ is slightly different - it also takes in prices and wealth but it tells you what the consumer's utility will be in optimality (i.e. it maps to real numbers). In other words, it says "give me the prices of goods and the consumer's wealth, and I will tell you what their maximized utility will be".

[4] In class, you will see the notation $p^T x$ or $p'x$, but this is just another way of writing the dot product of two column vectors

I've called them correspondences to highlight the fact that we are mapping from the UMP's parameters to some outcome (bundle or utility). Keep in mind that $x(p, w)$ is a general correspondence - there could be many optimal bundles. However, $v(p, w)$ is a single-valued correspondence, i.e. a function, because every optimal bundle must give you the same utility. If not, then you could have $x, y \in x(p, w)$ but $u(x) > u(y)$; clearly $y$ is not optimal, which is a contradiction.

### 2.3 Example: Cobb-Douglas

Since these concepts are really important but sometimes hard to grasp when we're talking about them abstractly, we're going to work through a simple example so you can see exactly what these things even look like.

**Example:** Suppose the consumer's utility function is $u(x) = x_1 x_2$ (so $K = 2$).[5] Then the Lagrangian from the UMP would be:

$$\mathcal{L} = x_1 x_2 + \lambda(w - p_1 x_1 - p_2 x_2)$$

We take the FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= x_2 - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= x_1 - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w - p_1 x_1 - p_2 x_2 = 0
\end{aligned}$$

Simplifying gives us the following equations:

$$\begin{aligned}
x_2 &= \lambda p_1 \\
x_1 &= \lambda p_2 \\
w &= p_1 x_2 + p_2 x_2
\end{aligned}$$

Let's get rid of the pesky $\lambda$ by dividing the first equation by the second equation:

$$\begin{aligned}
\frac{x_2}{x_1} &= \frac{\lambda p_1}{\lambda p_2} \\
\implies \frac{x_2}{x_1} &= \frac{p_1}{p_2}
\end{aligned}$$

This is probably a familiar relationship to you: marginal rate of substitution is equal to the price ratio. Great, we have an equation with demand and prices. Are we done? Definitely not! Let's find the Marshallian demand of good 1. Recall that the notation for this would be $x_1(p, w)$. This tells us that we should be expressing $x_1$ only as a function of $p = (p_1, p_2)$ and $w$. That means we need to get rid of that $x_2$ term (and bring in $w$). This is when we bring in the binding budget constraint, i.e. the third FOC: $w = p_1 x_1 + p_2 x_2$.

First, let's re-arrange the equation we found so that we express $x_2$ (the variable we want to get rid of) in terms of $p_1, p_2$ and $x_1$ (the variables we want). Re-arranging gives:

$$x_2 = \frac{p_1}{p_2} x_1$$

[5]In this case , we can clearly see that the utility function is differentiable. In general, we will just assume that it is differentiable unless explicitly told, e.g. Leontief preferences

Plug this into the budget constraint and simplify:

$$\begin{aligned}
w &= p_1 x_1 + p_2 \frac{p_1}{p_2} x_1 \\
&= 2p_1 x_1 \\
\implies x_1(p,w) &= \frac{w}{2p_1}
\end{aligned}$$

Now we've got $x_1$ just as a function of wealth and prices, so we're done with this one. This is a special case where $p_2$ does not affect the demand of $x_1$, but this is not true in general. You can plug this back into the budget constraint to get the demand for $x_2$

$$x_2(p,w) = \frac{w}{2p_2}$$

Therefore the Marshallian demand is:

$$x(p,w) = \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right)$$

Now let's find the indirect utility function. We know that it should be the utility that you get from an optimal bundle so we just plug the Marshallian demand into the utility function:

$$v(p,w) = u\left(x(p,w)\right) = u\left(\left(\frac{w}{2p_1}, \frac{w}{2p_2}\right)\right) = \frac{w}{2p_1} \cdot \frac{w}{2p_2} = \frac{w^2}{4p_1p_2}$$

To make this more concrete, suppose that good 1 costs \$1, good 2 costs \$2, and wealth is \$10. The demand for good 1 would be $\frac{10}{2} = 5$ and for good 2 it would be $\frac{10}{4} = 2.5$. The utility of such a bundle would be $\frac{100}{8} = 12.5$, which is the highest utility the consumer could achieve given these prices and their wealth.

## 2.4 Properties

Ok, we understand the problem and the terminology, so now we can start asking some questions that we should always be thinking about in an optimization problem:

1. When does a solution exist?
2. When is the solution unique?
3. When is the constraint binding?
4. What are the properties of the solution set (Marshallian demand)?
5. What are the properties of the optimal objective value (indirect utility function)?

Before we get into answering each question, I'm going to throw out one more question that is specific to the UMP. We already established that utility representation is not unique - so does the utility function we choose in the UMP affect the solution? Suppose the consumer preference relation is $\succsim$ and can be represented by two utility functions $u(\cdot)$ and $\hat{u}(\cdot)$. Then we have two UMPs:

| UMP($u$) | UMP($\hat{u}$) |
|---|---|
| $\max_{x \in X} u(x)$ | $\max_{x \in X} \hat{u}(x)$ |
| s.t. $p \cdot x \leq w$ | s.t. $p \cdot x \leq w$ |

If UMP($u$) has a solution $x^*$, this means that $x^* \in \arg\max_{B(p,w)} u(x)$. By definition, $x^* \succsim x, \forall x \in B(p,w)$ (it gives the highest utility out of all feasible options). Since $\hat{u}$ is another utility representation for $\succsim$, then $x^* \succsim x \implies u(x^*) \geq u(x), \forall x \in B(p,w)$. By definition, this means that $x^* \in \arg\max_{B(p,w)} \hat{u}(x)$. This works the other way and so the two UMPs have the same set of solutions. This is a very handy property that you should use in practice. You can sometimes simplify your problem by applying some strictly increasing transformation to the utility function presented and still get the same solution as the original problem.

Now let's answer the five questions above:[6]

**Properties of UMP**

**(1) Existence:** If $\succsim$ is continuous, then a solution to the UMP exists for every $(p,w)$

**(2) Uniqueness:** If $\succsim$ is strictly convex, then a unique solution to the UMP exists for every $(p,w)$

**(3) Binding Constraint:** [Walras' Law] If $\succsim$ is monotonic, then the budget constraint is binding, i.e. $p \cdot x = w, \forall x \in x(p,w)$, for every $(p,w)$

**(4) Marshallian Demand Properties:**
1. If $u(\cdot)$ is continuous and $\succsim$ strictly convex, $x(p,w)$ is continuous (and single-valued)
2. $x(p,w)$ is homogeneous of degree zero in $(p,w)$, i.e. $x(\lambda p, \lambda w) = x(p,w), \forall \lambda > 0$
3. If $\succsim$ is convex, then $x(p,w)$ is convex

**(5) Indirect Utility Properties:**
1. If $u(\cdot)$ is continuous, then $v(p,w)$ is continuous
2. If $\succsim$ is monotonic, then $v(p,w)$ is weakly decreasing in $p$ and strictly increasing in $w$
3. $v(p,w)$ is homogeneous of degree zero in $(p,w)$, i.e. $v(\lambda p, \lambda w) = v(p,w), \forall \lambda > 0$
4. $v(p,w)$ is quasi-convex in $(p,w)$, i.e. the set $\{(p,w) : v(p,w) \leq v\}$ is convex for any $v$

Let's do the proofs for these. You shouldn't try to memorize these proofs because there's little value in that. Instead work through them so that you can understand what these statements are saying and what role our assumptions are playing.

(1) Existence:

- For any $(p,w)$, the budget set is closed and bounded, and hence compact (since in $\mathbb{R}^K$) [Heine-Borel Theorem]
  - Closed: Take any sequence $x^n \to x$, such that $p \cdot x^n \leq y, \forall n$. By the property of limits, we must have $p \cdot x \leq y$ and so $x \in B(p,w)$
  - Bounded: Clearly each $x_k$ is bounded from below by 0. Additionally, the most we can buy of each good $k$ is $\frac{w}{p_k}$ (spend all income on it). So $0 \leq x_k \leq \frac{w}{p_k}, \forall k$ and this is true for all $x \in B(p,w)$
- Since $\succsim$ is continuous, it has a utility representation $\hat{u}$ that is continuous ($\hat{u}$ may or may not be the same as $u(\cdot)$ in the UMP)

[6] In the following statements, I'm going to be making claims related to $\succsim$ being monotonic. You can actually weaken this to have $\succsim$ be locally non-satiated, which itself implies monotonicity. However, I will follow how it was presented in class (but now you know why MWG does it differently)

- By Weierstrass Theorem, a continuous function ($\hat{u}$) is maximized over a compact space ($B(p, w)$), so there is a solution for UMP($\hat{u}$)
- The UMP solution is invariant to utility representation, so therefore there is a solution for UMP($u$)

(2) Uniqueness

- For any $(p, w)$, $x(p, w)$ is a convex set. If empty, then done.
  - If $x(p, w)$ is non-empty, then for any $x^* \in x(p, w)$ we can express the solution set as $x(p, w) = B(p, w) \cap \{x \in \mathbb{R}_+^K | x \succsim x^*\}$ (intersection of feasible set and the set of bundles no worse than the optimal bundle)
  - $B(p, w)$ is convex: For any $x, y \in B(p, w)$ and $\lambda \in [0, 1]$, $p \cdot [\lambda x + (1 - \lambda)y] = \lambda p \cdot x + (1 - \lambda) p \cdot y \leq \lambda w + (1 - \lambda) w = w$, since $p \cdot x \leq w$ and $p \cdot y \leq w$ (they are both feasible).
  - The set $\{x \in \mathbb{R}_+^K | x \succsim x^*\}$ is convex because $\succsim$ is convex
  - The intersection of two convex sets in convex, so $x(p, w)$ is convex
- If non-empty, $x(p, w)$ is a singleton.
  - Suppose not and take two elements $x, y \in x(p, w)$, where $x \neq y$
  - By strict convexity of $\succsim$, $\alpha x + (1 - \alpha)y \succ x, \forall \alpha \in (0, 1)$
  - By convexity of $B(p, w)$, $\alpha x + (1 - \alpha)y \in B(p, w)$
  - Since it is feasible but strictly preferred to $x$ and $y$, this contradicts the optimality of $x$ and $y$
- Alternative method: Maximization theorem tells us that when maximizing a function over a convex set, the set of maximizers of a (strictly) quasi-concave function is a convex set (singleton). We maximize $u(x)$, which is quasi-concavey by the convexity of $\succsim$, over the convex set $B(p, w)$, so the theorem applies here.

(3) Binding Constraint

- Proof by contradiction: Suppose there exists a $x \in x(p, w)$ and $(p, w)$ such that $p \cdot x < w$
- There exists $\varepsilon \in \mathbb{R}_{++}^K$ such that $p \cdot y < w$, where $y = x + \varepsilon$
- Since $y \gg x$, by monotonicity, $u(y) > u(x)$
- Since $y$ is feasible but strictly preferred to $x$, this contradicts the optimality of $x$

(4) Properties of Marshallian Demand

*[1] Continuity*

- $B(p, w)$ is a continuous and compact valued correspondence ($f(x) = p \cdot x$ is a continuous function and we have already proved compactness in the existence proof)
- $u(x)$ is continuous (by assumption)
- By Berge's Maximum Theorem, the two claims imply that $x(p, w)$ is upper hemicontinuous

- If $\succsim$ is strictly convex, then $x(p, w)$ is a single-valued correspondence (uniqueness). A correspondence that is single-valued and uhc is simply a continuous function.[7]

*[2] Homogeneous of degree 0*

- Scaling both prices and wealth by the same amount does not change the budget set

*[3] Convex*

- See the first part of the uniqueness proof

<u>(5) Properties of Indirect Utility</u>

*[1] Continuous*

- $B(p, w)$ is a continuous and compact valued correspondence
- $u(x)$ is continuous (by assumption)
- By Berge's Maximum Theorem, the two claims imply that $v(p, w)$ is continuous

*[2] Weakly decreasing in $p$ and strictly increasing in $w$*

- If $p \geq p'$ then $B(p, w) \subset B(p', w)$. Since the feasible set does not increase (but may shrink), either the optimal bundle is still affordable (corner solution) or you have to now choose a bundle that previously was feasible but not optimal. This means that $v(p, w)$ is weakly decreasing in $p$
- If $w < w'$ then $B(p, w) \subsetneq B(p, w')$ (feasible set must expand if income increases). If $x \in x(p, w)$, then $p \cdot x < w'$. By monotonicity, we can find a $x^*$ that is feasible but $x^* \succ x$. This means that $v(p, w)$ is strictly increasing in $w$

*[3] Homogeneous of degree 0*

- Since $x(p, w)$ is h.o.d. 0, then the utility level in optimality will not change either (you choose the same bundle)

*[4] Quasi-convex*

- Suppose $v(p, w) \leq v$ and $v(p', w') \leq v$. We want to show that $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq v$
- Take any $x \in x(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w')$. This is an optimal bundle, so it must be feasible too, i.e. $[\lambda p + (1-\lambda)p'] \cdot x \leq [\lambda w + (1-\lambda)w']$. This can be re-written as $\lambda(p \cdot x - w) + (1-\lambda)(p' \cdot x - w') \leq 0$
- For the inequality to hold, it must be that $p \cdot x \leq w$ or $p' \cdot x \leq w'$ (or both)
- In the first case, this implies $x$ is feasible for the UMP with price vector $p$ and wealth $w$. Since it is feasible but may or may not be optimal, we must have $u(x) \leq v(p, w)$. Similarly, for the second case $u(x) \leq v(p', w')$
- In either case, since $x$ is optimal for the UMP with price vector $\lambda p + (1-\lambda)p'$ and wealth $\lambda w + (1-\lambda)w'$, then we must have: $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') = u(x) \leq v$

[7] Technical note: Since we want to apply Berge's Theorem, we need to directly assume that $u(x)$ is continuous

## 2.5 Duality: Inverse Demand Function

Normally we think of demand in the following way: "you tell me how much things cost, and I'll tell you how much I'm willing to buy". This is exactly what the Marshallian demand, $x(p, w)$, is saying. The inverse demand is, well, the inverse! It says: "you tell me how much you want me to buy, and I'll tell you at what price I'm willing to take it for". So we want something that will map from the commodity set to real numbers (the opposite of Marshallian demand).

Notice here we are talking about prices but we haven't mentioned income at all. Since the inverse demand function will map into one-dimensional space $\mathbb{R}$, there is no way to get two separate values for price and income. Instead, we are going to be working with *relative* prices, i.e. we're going to normalize income to 1. You might be worried that this is going to affect the solution, but if you look above, we've already proved that $x(p, w)$ and $v(p, w)$ are h.o.d. 0. So we can just set our $\lambda = 1/w$ and therefore conclude:

$$x(p, w) = x\left(\tfrac{1}{w}p, \tfrac{1}{w}w\right) = x\left(\tfrac{p}{w}, 1\right) \qquad\qquad v(p, w) = v\left(\tfrac{1}{w}p, \tfrac{1}{w}w\right) = v\left(\tfrac{p}{w}, 1\right)$$

So our goal is to find an inverse demand function, but our UMP doesn't quite give us that. Let's compare what we have in the UMP with what we would like in this new problem.

In our UMP, we took prices and wealth as parameters and chose consumption bundles $x$. We maximized an objective function $u(x)$ that depended on our choice variable (bundles) and the optimal value was indirect utility function $v(p, w)$, which depends only on the parameters $p$ and $w$. What if we wanted to go the other way? Could we setup a problem which had $v(p, w)$ as the objective function and $u(x)$ as the optimal solution? Let's use similar logic - since the 'outcome' is now $u(x)$, which only depends on $x$, it must be the case that $x$ is now a parameter. This means that $p$ and $w$ are our choice variables in this problem. To simplify the complexity though we are going to work with relative prices and only think about $p/w$. So in this new problem, we have to choose relative prices taking the bundles as given.

Ok, so now we know our objective function ($v(p, w)$), we know that we have to choose relative prices and take bundles as given, so what's left? Well, we have to figure out if we are trying minimize or maximize our objective function! Here's how to think about this. We know that for any bundle in the feasible set, we must have that $u(x) \leq v(p, w)$. Our goal is to find the combination of variables $(x^*, p^*, w^*)$ so that we have $u(x^*) = v(p^*, w^*)$. To do that, we can either 'increase' the LHS or 'decrease' the RHS. In the UMP, we are choosing $x$ to 'increase' the LHS so that the equality holds, i.e. we need to *maximize* $u(x)$. In this 'inverse' problem, we are choosing $p/w$ to 'decrease' the RHS so that the equality holds, i.e. we need to *minimize* $v(p, w)$.

Great, we have all the components for our new optimization problem so now we can write it out:

$$\begin{aligned} &\min_{p \in \mathbb{R}^K_{++}} \; v(p, 1) \\ &\text{s.t. } \; p \cdot x \leq 1 \end{aligned}$$

And our intuition has also told us the optimal solution to this "dual problem" is exactly $u(x)$. But we haven't discussed the solution set though - what would be the analogue of the Marshallian demand $x(p, w)$? If the UMP's solution set gave us the demand for our choice variable $x$, then in this new problem our solution set must surely be the demand for our choice variable $p$. And the demand for prices is exactly what the inverse demand function represents, so we've found exactly what we're looking for. We will express this as $p(x)$.

As a final note, in optimization we can easily change between maximizing and minimizing. If I want to find the $x$ that makes $f(x)$ as big as possible, that's the same thing as saying I want to find the $x$ that makes $-f(x)$ as small as possible. Clearly if $f(x^*) \geq f(x), \forall x$, then $-f(x^*) \leq -f(x), \forall x$. So another way we can write our dual problem so that it is still a maximization problem is:

$$\max_{p \in \mathbb{R}^K_{++}} -v(p, 1)$$
$$\text{s.t. } p \cdot x \leq 1$$

But now be careful here because this will give us an optimal objective function equal to $-u(x)$. So while the solution set is exactly the same, the objective function has definitely changed.

I've focused here on the intuition behind this duality, but if you do want to see this done in a more 'math-y' way, then you should refer to Jehle and Reny 2.1.3, but your priority here should be to have a good grasp of the intuition.

# 3 Expenditure Minimization Problem

## 3.1 The Problem

In the UMP, our story was that the consumer was trying to maximize their utility subject to a budget constraint. What if we flipped that story so that utility was the constraint while the money spent was changing. The story would then be that the consumer is trying to minimize their expenditure while still achieving some baseline level of utility. This is exactly the *expenditure minimization problem* (EMP). In both the UMP and EMP we are still choosing bundles of goods (that part hasn't changed) but the objective function and constraint have switched positions. The EMP is written as:

$$\min_{x \in X} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

Where $u > u(0)$ represents the lowest acceptable utility such that $u(0)$ is what the consumer would get from consuming nothing. Let's compare this to the UMP. In that problem, while we would all like to buy infinitely many things, we obviously are limited by how much we can afford. In the EMP, you really want to avoid spending your money but you know that you still have to get something to satisfy some minimal amount of happiness. This might seem quite weird to you, so let me give you a concrete example. Imagine being a poor and hungry grad student looking for some lunch (this may not require too much imagination). You go to the store and check the price of each item to try to find the cheapest item that will fill you up. While you wish you could spend nothing, you at least recognize that you have to eat *something*. So, conditional on finding something filling, you want to spend the least amount possible (e.g. no guac at Chipotle because that burrito is plenty filling already).

Hopefully at this point I've convinced you that the EMP really isn't too different from the UMP. So now we're going to go through the same analysis as before and while the results will differ you'll see that you'll use basically the same intuition throughout. If you don't feel comfortable with the UMP at this point, I highly recommend going back and working through it again until it clicks. Once you do, you'll be able to breeze through the EMP with relative ease. Ok, let's dive into this!

Let's first analyze the problem:

- $\min_{x \in X}$: Like before, we are still choosing a bundle $x$, except now we are minimizing
- $p \cdot x$: This is our new objective function and, like in the UMP's constraint, it represents the total amount spent by the consumer, i.e. $p \cdot x = p_1 x_1 + ... + p_K x_K$.
- s.t. $u(x) \geq u$: This is the constraint which defines the *feasible set* of solutions. There are two important things to notice here. First, the feasible set is defined just by the utility parameter $u$. Secondly, this set is actually unbounded, but we'll deal with this later.

Now we'll define the EMP's analogues of the Marshallian demand and the indirect utility function:

**EMP Correspondences:**

- *Hicksian Demand Correspondence:* This is the set of solutions to the EMP (optimal bundles), which we will denote as $h(p, u)$
- *Expenditure Function:* This is the optimal value of the EMP (money spent under an optimal bundle), which we will denote as $e(p, u) = p \cdot x, \forall x \in h(p, u)$

In class, we are often working with functions so we can use the other term for $h(p, u)$ which is the *compensated demand function*, but at least you now know where the $h$ comes from. This isn't too different from what we saw before. The Marshallian demand told us the optimal bundles for the UMP and the Hicksian demand is similarly the optimal bundles for the EMP. The objective function has a nuance. While the optimal amount is still a real number, before it represented the optimal *utility*. Now it is the optimal *expenditure*, so they're not as directly comparable.

Just to be safe, we'll just go over this in another way. The Hicksian demand $h(p, u)$ maps price and utility (the parameters) to the optimal choice variable $x$. In other words, it says "give me the prices of goods and the minimum utility level you're okay with, and I will tell you what bundles are optimal for the consumer". The expenditure function $e(p, u)$ also takes in prices and utility but it tells you what the consumer's expenditure will be in optimality (remember - it maps to real numbers). In other words, it says "give me the prices of goods and the minimum utility level you're okay with, and I will tell you what their minimized expenditure will be". Notice that wealth plays absolutely no role here. Money is not the constraint (the consumer just wants to minimize how much they spend), the constraint comes from the fact that utility cannot go below the threshold set by $u$.

## 3.2 Example: Cobb-Douglas

To show the duality of the problems, we're going to use the same example as in the UMP but express it as EMP:

**Example:** Suppose the consumer's utility function is $u(x) = x_1 x_2$ (so $K = 2$). Then the Lagrangian from the EMP would be:

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda(u - x_1 x_2)$$

We take the FOCs:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \lambda x_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \lambda x_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= u - x_1 x_2 = 0\end{aligned}$$

Simplifying gives us the following equations:

$$\begin{aligned}p_1 &= \lambda x_2 \\ p_2 &= \lambda x_1 \\ u &= x_1 x_2\end{aligned}$$

Dividing the first equation by the second equation to eliminate $\lambda$:

$$\begin{aligned}\frac{p_1}{p_2} &= \frac{\lambda x_2}{\lambda x_1} \\ \implies \frac{p_1}{p_2} &= \frac{x_2}{x_1}\end{aligned}$$

Look back to what we get at this point in the UMP example - it's exactly the same! Unsurprisingly, MRS is still equal to the price ratio in the EMP. At this point in the UMP we brought in the budget constraint, but now we have this 'utility constraint'. That shouldn't cause any problems, so let's continue as normal and start by finding the Hicksian demand of good 1.

First, let's re-arrange the equation we found so that we express $x_2$ (the variable we want to get rid of) in terms of $p_1, p_2$ and $x_1$ (the variables we want). Re-arranging gives:

$$x_2 = \frac{p_1}{p_2} x_1$$

Plug this into the utility constraint and simplify:

$$\begin{aligned}u &= x_1 x_2 \\ &= x_1 \frac{p_1}{p_2} x_1 \\ \frac{p_2}{p_1} u &= x_1^2 \\ \implies h_1(p, u) &= \sqrt{\frac{p_2}{p_1} u}\end{aligned}$$

Now we've got $x_1$ just as a function of prices and the utility threshold, so we're done with this one. You can plug this back into the utility constraint to get the demand for $x_2$

$$\begin{aligned}u &= \sqrt{\frac{p_2}{p_1} u} x_2 \\ \therefore h_2(p, u) &= \sqrt{\frac{p_1}{p_2} u}\end{aligned}$$

Therefore the Hicksian demand is:

$$h(p,u) = \left( \sqrt{\frac{p_2}{p_1} u}, \sqrt{\frac{p_1}{p_2} u} \right)$$

Now let's find the expenditure function. We know that it should be the total cost of an optimal bundle so we just plug the Hicksian demand into the cost function $p \cdot x$:

$$\begin{aligned} e(p,u) &= \begin{pmatrix} p_1 & p_2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{p_2}{p_1} u} & \sqrt{\frac{p_1}{p_2} u} \end{pmatrix}^T \\ &= p_1 \sqrt{\frac{p_2}{p_1} u} + p_2 \sqrt{\frac{p_1}{p_2} u} \\ &= 2\sqrt{p_1 p_2 u} \end{aligned}$$

Maybe you're surprised that what we found in the EMP looks so different than the UMP example. I mean, it's basically the same problem and $x(p,w)$ and $h(p,u)$ both represent demand - why do they look so different? As is usually helpful, look at their inputs. The Marshallian demand works in the space of prices and income, while the Hicksian demand is working with prices and utility. These are in completely different spaces, it's like comparing apples and oranges! Ok, so they shouldn't be exactly the same but surely there should be some connection between them, right? Absolutely! But we'll come to that later.

## 3.3 Properties

Now, we're going to ask and answer the same five questions again:

1. When does a solution exist?
2. When is the solution unique?
3. When is the constraint binding?
4. What are the properties of the solution set (Hicksian demand)?
5. What are the properties of the optimal objective value (expenditure function)?

At this point in the UMP section, I asked whether the solution would change if we changed the utility function. We showed that the answer was no because the feasible set didn't change and the order of preferences within this set also didn't change. But let's say that we try to change the utility function $u(x)$ to $\hat{u}(x) = f(u(x))$ for some strictly increasing $f(\cdot)$, which we know will represent the same preferences. Does that affect the EMP? Let's compare:

| $\underline{\text{EMP}(u)}$ | $\underline{\text{EMP}(\hat{u})}$ |
|---|---|
| $\min_{x \in X} p \cdot x$ | $\min_{x \in X} p \cdot x$ |
| s.t. $u(x) \geq u$ | s.t. $\hat{u}(x) \geq u$ |

Obviously the objective function is the same but look at the constraint. It used to be $u(x) \geq u$ but after the change it becomes $\hat{u}(x) \geq u$. Has the feasible set changed? Definitely! The fact that the lower bound on utility is still the same real number $u$ means that the cardinal values of the utility function are determining the feasible set. Suppose that $u(a) = 1, u(b) = 3, u(c) = 5$ with $u = 2$. In this case, the feasible set is $\{b, c\}$. Now let's transform the utilities with $f(x) = 2x$ so that $\hat{u}(a) = 2, \hat{u}(b) = 6, \hat{u}(c) = 10$. Now the feasible set is $\{a, b, c\}$ so the problem has changed completely. How can we fix this? All we have to do is also transform $u$ so that the constraint becomes $f(u(x)) \geq f(u)$.

Now let's answer the five questions above:

**Properties of EMP**

**(1) Existence:** If $u(\cdot)$ is continuous, then a solution to the EMP exists for every $(p, u)$

**(2) Uniqueness:** If $\succsim$ is strictly convex, then a unique solution to the EMP exists for every $(p, u)$

**(3) Binding Constraint:** If $u(\cdot)$ is continuous, the utility constraint is binding, i.e. $u(x) = u, \forall x \in h(p, u)$, for every $(p, u)$

**(4) Hicksian Demand Properties:**
1. If $u(\cdot)$ is continuous and $\succsim$ strictly convex, $h(p, u)$ is continuous (and single-valued)
2. $h(p, u)$ is homogeneous of degree zero in $p$, i.e. $h(\lambda p, u) = h(p, u), \forall \lambda > 0$
3. If $\succsim$ is convex, then $h(p, u)$ is convex
4. [Compensated Law of Demand] If $p, p' \geq 0$, $x \in h(p, u)$, and $x' \in h(p', u)$, then $(p' - p) \cdot (x' - x) \leq 0$

**(5) Expenditure Function Properties:**
1. If $u(\cdot)$ is continuous, then $e(p, u)$ is continuous
2. $e(p, u)$ is weakly increasing in $p$ and strictly increasing in $u$
3. $e(p, u)$ is homogeneous of degree one in $p$, i.e. $e(\lambda p, u) = \lambda e(p, u), \forall \lambda > 0$
4. $e(p, u)$ is concave in $p$

Let's move onto the proofs for these claims.

(1) Existence:

- The constraint set is unbounded since the only conditions are $u(x) \geq u$ and $x \in \mathbb{R}^K_+$, which means there is no upper bound for $x$
- Let $\bar{v} = \sup_{x \in X} u(x)$ (i.e. the upper bound of utilities that can be achieved on the consumption set). For any $u \in [u(0), \bar{v})$, there exists a $\hat{x} \in \mathbb{R}^K_+$ such that $u(\hat{x}) > u$
- Since $\hat{x}$ is feasible, then any optimal solution $x^*$ has to also be at least as cheap as $\hat{x}$, i.e. $p \cdot x^* \leq p \cdot \hat{x}$. Therefore, adding the constraint $p \cdot x \leq p \cdot \hat{x}$ to the EMP bounds the constraint set from above but does not change $h(p, u)$.
- Since $u(\cdot)$ is continuous and the constraint set is now compact, by Weierstrass, the minimum exists.
- Technical note: Unlike the UMP we have to explicitly have $u(\cdot)$ be continuous rather than just have $\succsim$ be continuous. If $u(\cdot)$ is not continuous then the feasible set could be open and so existence would not be guaranteed.

(2) Uniqueness

- For any $(p, u)$, $h(p, u)$ is a convex set. If empty, then done.
  - For any $\hat{x} \in h(p, u)$, we can express the solution set as $h(p, u) = \{x \in X : p \cdot x \leq p \cdot \hat{x}\} \cap \{x \in X : u(x) \geq u\}$

- The first set is essentially like a budget set, which we showed in the UMP section is convex. The second set is convex by the convexity of $\succsim$. Since both of these sets are convex, their intersection $h(p, u)$ is convex

- If non-empty, $h(p, u)$ is a singleton.
  - Proof by contradiction: Suppose there are $x, x' \in h(p, u)$ where $x \neq x'$
  - By above, $h(p, u)$ is convex, so $\lambda x + (1 - \lambda)x' \in h(p, u)$ for $\lambda \in (0, 1)$
  - By strict convexity of $\succsim$, $u(\lambda x + (1 - \lambda)x') > \min\{u(x), u(x')\}$
  - Since the constraint is binding for optimal solutions, then $u(x) = u$ and $u(x') = u$
  - Therefore, $u(\lambda x + (1 - \lambda)x') > u$, which is contradiction since it should be binding

(3) Binding Constraint

- Proof by contradiction: Suppose that $x \in h(p, u)$ but $u(x) > u$ (and by assumption $u > u(0)$)
- Since $u(\cdot)$ is continuous, $\exists \alpha \in (0, 1)$ s.t. $u(\alpha x) > u$ (intermediate value theorem between $u(0)$ and $u(x)$)
- Since $\alpha < 1$, then $p \cdot \alpha x < p \cdot x$, contradicting the optimality of $x$

(4) Properties of Hicksian Demand

*[1] Continuous*

- As in the proof for existence, we add an extra constraint of $p \cdot x \leq p \cdot \hat{x}$ to make the feasible set compact
- Now, we can apply Berge's Maximum Theorem, which tells us that $h(p, u)$ is a compact-valued and upper hemicontinuous correspondence.
- If $\succsim$ is strictly convex, then by uniqueness $h(p, u)$ is single-valued, and hence must be a continuous function.

*[2] Homogeneous of degree 0 in p*

- Scaling the prices does not change the feasible set

*[3] Convex*

- See the first part of the uniqueness proof

*[4] Compensated Law of Demand*

- $x$ is feasible under $h(p', u)$ and $x'$ is feasible under $h(p, u)$, since the constraint set does not change with $p$. However they may not be optimal.
- Since $x' \in h(p', u)$, then $p' \cdot x' \leq p' \cdot x$. Rearranging: $p' \cdot (x' - x) \leq 0$
- Since $x \in h(p, u)$, then $p \cdot x \leq p \cdot x'$. Rearranging: $p \cdot (x - x') \leq 0$
- Combine results: $p' \cdot (x' - x) + p \cdot (x - x') = p' \cdot (x' - x) - p \cdot (x' - x) = (p' - p) \cdot (x' - x) \leq 0$

(5) Properties of Indirect Utility

*[1] Continuous*

- Follow the steps in the continuity proof for the Hicksian demand. Applying Berge's Maximum Theorem, tells us that $e(p,u)$ is continuous in $(p,u)$

*[2] Weakly increasing in $p$ and strictly increasing in $u$*

- Weakly increasing in $p$
  - Let $p \le p'$, $x \in h(p,u)$, and $x' \in h(p',u)$.
  - Since $u$ is the same in both, then $x'$ is feasible for the EMP under $p$.
  - However $x'$ is feasible but not necessarily optimal under $p$, so $p \cdot x \le p \cdot x'$. Since $p \le p'$, then we also have $p \cdot x' \le p' \cdot x'$.
  - Putting this together gives $e(p,u) = p \cdot x \le p \cdot x' \le p' \cdot x' = e(p',u)$. This means that $e(p,u)$ is weakly increasing in $p$
- Strictly increasing in $u$
  - Let $u < u'$, $x \in h(p,u)$, and $x' \in h(p,u')$.
  - Since the constraint is binding for optimal choices, then $u(x') = u' \implies u(x') > u \implies x' \notin h(p,u)$.
  - Since $x'$ is feasible but not optimal for $h(p,u)$, then $e(p,u') = p \cdot x' > p \cdot x = e(p,u)$. This means that $e(p,u)$ is strictly increasing in $u$

*[3] Homogeneous of degree 1 in $p$*

- If prices are scaled by $\lambda$, we showed that this doesn't affect the solution set. So the same bundle is being chosen but it's cost has been scaled up by $\lambda$, which clearly scales up the optimal objective function too

*[4] Concavity*

- Let $x \in h(\lambda p + (1-\lambda)p', u)$. Since the constraint set only depends on $u$, then $x$ is feasible for the EMP with parameters $(p,u)$ and for the EMP with parameters $(p',u)$
- Since $x$ is feasible but may not be optimal in both cases, this implies that $p \cdot x \ge e(p,u)$ and $p' \cdot x \ge e(p',u)$
- $e(\lambda p + (1-\lambda)p', u) = [\lambda p + (1-\lambda)p'] \cdot x = \lambda p \cdot x + (1-\lambda)p' \cdot x \ge \lambda e(p,u) + (1-\lambda)e(p',u)$

# 4 Important Relationships

## 4.1 Math You Should Know

In this section, we're going to be assuming that everything is differentiable so that we can use calculus to derive some key relationships. We'll also be working with matrices, which means being comfortable with their terminology and operations. This first section will go over the concepts we'll need, all of which you covered in math camp. I will be going through them again but without getting too technical (refer back to your math camp notes for a more precise presentation of the material).

**Envelope Theorem**
Suppose we have a problem of the following form: $\max_{x \in X} f(x,\alpha)$ *s.t.* $g(x,\alpha) \le c$. Note that we have

to choose $x$ given a vector of parameters $\alpha = (\alpha_1, ..., \alpha_N)$. The Lagrangian of the problem is:

$$\mathcal{L}(x, \alpha, \lambda) = f(x, \alpha) + \lambda \left( c - g(x, \alpha) \right)$$

For notation, any time you see $\square^*$, I am referring to the optimized value of $\square$. Once everything is optimized, it should only be a function of the parameter $\alpha$. For example, $f^*(\alpha)$ is the optimized objective function, i.e. the value function, and $\mathcal{L}^*(\alpha)$ is the optimized Lagrangian which has the multiplier $\lambda^*$, i.e. $\mathcal{L}^*(\alpha) = f^*\left(x^*(\alpha), \alpha\right) + \lambda^*\left(c - g\left(x^*(\alpha), \alpha\right)\right)$.

The Envelope Theorem says that:

$$\frac{\partial f^*(\alpha)}{\partial \alpha_n} = \frac{\partial \mathcal{L}^*(\alpha)}{\partial \alpha_n} = \frac{\partial f(x^*(\alpha), \alpha)}{\partial \alpha_n} + \lambda^* \frac{\partial g(x^*(\alpha), \alpha)}{\partial \alpha_n}$$

In practice, the Envelope Theorem just says that if you want to know how the value function changes in response to a change in parameter, then just take a partial derivative of the Lagrangian with respect to that parameter (holding fixed the choice variable) and then evaluate it at the optimal. A change in the parameter $\alpha$ will directly affect the value of the objective function but it will also have an indirect effect by changing the solution set $x^*$. The Envelope Theorem says we can ignore this indirect effect *at the optimum* because the first order conditions will always ensure that the objective $f$ is maximized with respect to $x$.

**Derivatives**

You should know the following terms and notation:

- For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the *Jacobian matrix* denoted by $Df$ is a $m \times n$ matrix where the $i$-$j^{\text{th}}$ element is the partial derivative of the $i^{\text{th}}$ component of $f(x)$ with respect to the $j^{\text{th}}$ component of $x$

$$Df = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

- For a function $f : \mathbb{R}^n \to \mathbb{R}$, the special case of the Jacobian matrix is the *gradient* denoted by $\nabla f = Df^T$. We have the transpose to ensure that it is a column vector where the $i^{\text{th}}$ element is the partial derivative of $f(x)$ with respect to the $i^{\text{th}}$ component of $x$

$$\nabla f = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

- For a function $f : \mathbb{R}^n \to \mathbb{R}$, the *Hessian matrix* denoted by $\nabla^2 f$ is a $n \times n$ matrix of second derivatives, where the $i$-$j^{\text{th}}$ element is the cross partial of $f(x)$ with respect to the $i^{\text{th}}$ and $j^{\text{th}}$ component of $x$. In fact, $\nabla^2 f = D[\nabla f]$

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \, \partial x_1} & \frac{\partial^2 f}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

- For a function $f(t,x)$, the *total derivative* denoted by $\frac{df}{dt}$ is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$$

So to simplify, the Jacobian is a general form of a matrix of first derivatives. A special case for scalar-valued functions is the gradient. The Hessian is also for scalar-valued functions, but it is a matrix of second derivatives. Note that if the functions have many inputs, we can specify which one the derivative is with respect to by using a subscript. For example, take $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, and $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$. Then $D_x f(x,y)$ is the Jacobian of $f$ with respect to $x$, and this is a $m \times n$ matrix.

**Matrix**

You can read up on linear algebra from your math camp notes but for this section make sure you know these two terms:

- *Symmetric:* A square matrix $A$ is symmetric iff it is equal to its transpose $A' = A$
- *Negative semi-definite:* A $n \times n$ real symmetric matrix $A$ is negative semi-definite iff $x^T A x \leq 0$ for any $x \in \mathbb{R}^n$

Tying the last two sections together is the following theorem:

A function $f$ is concave iff its Hessian matrix is negative semi-definite

## 4.2 EMP

Let's start with analyzing the EMP relationships, which is a bit easier to do. Recall that the EMP gives us the Hicksian demand correspondence $h(p,u)$ (the solution set) and the expenditure function $e(p,u)$ (the optimal value of the EMP). By definition, we must have that $e(p,u) = p \cdot h, \forall h \in h(p,u)$. In other words, take any element from the solution set and multiply the quantity of each good by its corresponding price and that will give you the optimal expenditure amount. Now, we want to go the other way, i.e. we want to get the demand as a function of the expenditure. The answer it turns out is the following, called Shephard's Lemma:

**Shephard's Lemma**

$$h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}, \forall k = 1, ..., K$$

Equivalently, in vector notation this can be written as:

$$\begin{aligned} h(p,u) =& \nabla_p e(p,u) \\ \begin{pmatrix} h_1(p,u) \\ \vdots \\ h_K(p,u) \end{pmatrix} =& \begin{pmatrix} \frac{\partial e(p,u)}{\partial p_1} \\ \vdots \\ \frac{\partial e(p,u)}{\partial p_K} \end{pmatrix} \end{aligned}$$

Before we prove this, what is this even saying? It says that the Hicksian demand is the gradient (with respect to price) of the expenditure. Looking more closely at the right hand side, it is just the change in the value of the optimal objective function because of a change in the price of a good (a parameter). That phrasing should make you think of our good friend, the Envelope Theorem!

So, we know what to do: differentiate the Lagrangian with respect to the parameter ($p_k$) and then evaluate at the optimal (i.e. where $x = h(p,u), \lambda = \lambda^*$)[8]

$$
\begin{aligned}
\frac{\partial e(p,u)}{\partial p_k} &= \left.\frac{\partial \mathcal{L}(p,u)}{\partial p_k}\right|_{(x,\lambda)=(h(p,u),\lambda^*)} \\
&= \left.\frac{\partial}{\partial p_k}\left[p \cdot x + \lambda(u - u(x))\right]\right|_{(x,\lambda)=(h(p,u),\lambda^*)} \\
&= \left.\left(\frac{\partial p}{\partial p_k} \cdot x - \lambda \frac{\partial u(x)}{\partial p_k}\right)\right|_{(x,\lambda)=(h(p,u),\lambda^*)} \\
&= e_k^T \cdot x\big|_{(x,\lambda)=(h(p,u),\lambda^*)} \\
&= x_k\big|_{(x,\lambda)=(h(p,u),\lambda^*)} \\
&= h_k(p,u)
\end{aligned}
$$

The key part in the derivations above is that even though we know $p$ has an effect on $h(p,u)$, we have to treat $x$ as if it's a constant. If you're not quite convinced of this, you can see this by calculating $\partial \mathcal{L}^*(p,u)/\partial p_k$ directly (i.e. we first evaluate at the optimal and then we differentiate). This makes everything a function of $p$ so we can see how a change in $p_k$ affects the Lagrangian at optimality.

So, using our notation from above, where $\alpha = (p,u) = (p_1, ..., p_K, u)$, $f^*(\alpha) = e(p,u)$, $x^*(\alpha) = h(p,u)$, $f(x,a) = p \cdot x$, and $g(x,\alpha) = u(x)$, this gives us:

$$
\begin{aligned}
\frac{\partial \mathcal{L}^*(p,u)}{\partial p_k} &= \frac{\partial\left[p \cdot h(p,u)\right]}{\partial p_k} - \lambda^* \frac{\partial u(h(p,u))}{\partial p_k} && \text{[plug in from general formula]} \\
&= p \cdot \frac{\partial h(p,u)}{\partial p_k} + \frac{\partial p}{\partial p_k} h(p,u) - \lambda^* \nabla u(h(p,u)) \cdot \frac{\partial h(p,u)}{\partial p_k} && \text{[product and chain rule]} \\
&= \left[p - \lambda^* \nabla u(h(p,u))\right] \cdot \frac{\partial h(p,u)}{\partial p_k} + \frac{\partial p}{\partial p_k} h(p,u) && \text{[simplify]} \\
&= 0 \cdot \frac{\partial h(p,u)}{\partial p_k} + e_k^T \cdot h(p,u) && \text{[plug in FOC; use } \partial p/\partial p_k = e_k\text{]} \\
&= h_k(p,u)
\end{aligned}
$$

In the last step, the FOC is just taking the derivative of the Lagrangian with respect to $x$ and setting it equal to 0. So the derivative is $\mathcal{L}_x = p - \lambda \nabla u(x)$, and in optimality this is $p - \lambda^* \nabla u(h(p,u)) = 0$. The reason we can ignore the 'indirect' effect of the parameter change is *not* because $\frac{\partial h(p,u)}{\partial p_k}$ is equal to zero, but because it is always going to be multiplied by zero (due to the FOC). Since the parameter does not affect the constraints at all, we just need to look at its direct effect on the objective function.

Here's how to think about Shephard's Lemma. Imagine starting off at $h(p,u)$ and now the price of good $k$ goes up by 1. So you re-optimize and might choose a new mix of goods in your bundle. How much should your expenditure change? It wouldn't make sense for your expenditure to change by *more than* $h_k(p,u)$, because instead of choosing a new bundle you could have been better off by sticking with your old bundle (where we know the expenditure change will be exactly $(p+e_k-p) \cdot h(p,u) = h_k(p,u)$). It also wouldn't make sense for the change to be *less than* $h_k(p,u)$. Since the change in price does not affect the feasible set (which is only determined by utility), and no other prices change, then this new mix

[8] Note that in what follows I will use the notation $e_k$ which represents a $1 \times K$ vector with zeros in all rows and a 1 in the $k^{\text{th}}$ row

must have been available and cheaper than $h(p, u)$ before the price change. But this would contradict the optimality of $h(p, u)$, and so we conclude the change must be weakly greater than $h_k(p, u)$. Those two arguments together imply that the change is exactly equal to $h_k(p, u)$.

Remember how another name for the Hicksian demand is compensated demand? This actually shows why it gets that name. Imagine you've optimized so that the indifference curve is tangent to the budget line, but now the price of a good changes increases (let's say good 2, and so the budget line tilts down to a flatter slope). Obviously you choose a different bundle; but why? There are two reasons for this:

1. The relative prices of goods has changed (good 2 is now relatively more expensive than before)
2. The consumer's real income has fallen (all goods are now more expensive so your income simply buys less of all goods)

However we can't really disentangle these two effects by just looking at the new bundle. So, here's our thought experiment: how much do I have to increase your income so that you're back to being indifferent to your original bundle? Notice you just have to be indifferent, so I need shift the budget line *out* so that it is just tangent to the first indifference curve. This will probably mean it won't be the same bundle as before, but that's ok because it's on the same indifference curve. This will mean you've been compensated for the drop in your real income and any other changes must purely be due to the relative price effects.

Why is this relevant when we're looking at $\partial e(p, u)/\partial p_k$? Here, $u$ is being held fixed, which means staying on the same indifference curve, while $p$ is changing. So it's exactly the same story we just outlined. Therefore a change in the expenditure function due to price must only be through the relative price channel. This is why our intuition for Shephard's Lemma works, because $u$ is being held fixed, any changes to $h(p, u)$ must be because of the relative prices of goods and not because some bundle becomes unaffordable.

## 4.3 UMP

Now we're going to follow very similar steps but with the UMP.

Recall that the UMP gives us the Marshallian demand correspondence $x(p, w)$ (the solution set) and the indirect utility function $v(p, w)$ (the optimal value of the UMP). By definition, we must have that $v(p, w) = u(x), \forall x \in x(p, w)$. In other words, take any element from the solution set and plug that into the utility function and that will give you the optimal utility amount. Now, we want to go the other way, i.e. we want to get the demand as a function of the utility. The answer it turns out is the following, called Roy's Identity:

**Roy's Identity**

$$x_k(p, w) = -\frac{\partial v(p, w)/\partial p_k}{\partial v(p, w)/\partial w}, \forall k = 1, ..., K$$

Equivalently, in vector notation this can be written as:

$$x(p, w) = -\frac{\nabla_p v(p, w)}{\partial v(p, w)/\partial w}$$

$$\begin{pmatrix} x_1(p, w) \\ \vdots \\ x_K(p, w) \end{pmatrix} = \begin{pmatrix} -\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w} \\ \vdots \\ -\frac{\partial v(p,w)/\partial p_K}{\partial v(p,w)/\partial w} \end{pmatrix}$$

Again, let's first make sure we understand what this saying. Like before, the Marshallian demand is essentially the gradient (with respect to price) of the indirect utility function - we can see that in the numerator. But what about the denominator? Before, we had the expenditure function, where everything is expressed in a common amount (dollars). With utilities, however, we don't have that same standard unit and so the cardinal values themselves have very little meaning. So the denominator normalizes the gradient depending on the particular utility function we use.

To prove this, we will use the Envelope Theorem again. We will set up the Lagrangian, differentiate with respect to $p_k$ (holding $x$ constant) and then evaluate at the optimum (i.e. where $x^* = x(p, w), \lambda = \lambda^*$).

$$\begin{aligned}
\frac{\partial v(p, w)}{\partial p_k} &= \frac{\partial \mathcal{L}(p, w)}{\partial p_k}\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \frac{\partial}{\partial p_k}\left[u(x) + \lambda(w - p \cdot x)\right]\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \left(\frac{\partial u(x)}{\partial p_k} - \lambda\frac{\partial p}{\partial p_k} \cdot x\right)\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= -\lambda e_k^T \cdot x\big|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= -\lambda x_k\big|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= -\lambda^* x_k(p, w)
\end{aligned}$$

The key part to realize is that unlike the EMP, the parameter $p_k$ doesn't appear in objective function, but in the constraint. This means we now have to deal with the Lagrange multiplier $\lambda^*$. The trick here is to repeat the same steps but now take the derivative with respect to income $w$:

$$\begin{aligned}
\frac{\partial v(p, w)}{\partial w} &= \frac{\partial \mathcal{L}(p, w)}{\partial w}\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \frac{\partial}{\partial w}\left[u(x) + \lambda(w - p \cdot x)\right]\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \left(\frac{\partial u(x)}{\partial w} + \lambda\right)\bigg|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \lambda\big|_{(x,\lambda)=(x(p,w),\lambda^*)} \\
&= \lambda^*
\end{aligned}$$

Recall that we said that price changes affected bundle choices through two channels: relative price changes and real income effects. With Shephard's Lemma we considered how price changes affected expenditure and showed that this was coming entirely through the relative price channel (Hicksian demand). With Roy's Identity, we want to see how price changes affect utility. The idea of relative price changes is to make sure that the consumer is still on the same indifference curve - but that means by definition relative price changes have zero effect on utility! So $\partial v/\partial p_k$ must entirely be composed of the real income effects. If the price of good $k$ goes up by $\Delta$ then you will 'feel' poorer by $\Delta \times x_k(p, w)$ dollars (you have to 'give up' these units to satisfy your budget constraints). But this is in dollars while we were initially looking for utilities. Now we bring in the marginal utility of wealth $\partial v/\partial w$ which essentially 'converts' units from dollars to utility because it represents how much happier one more dollar of wealth will make you.

## 4.4 UMP-EMP Link

Throughout this you've seen a constant comparison between the UMP and EMP, and at this point you could probably deduce that there is some connection between them. There are two ways we can connect the two problems: by definitions and by derivatives.

The first way parallels the simpler "plug-in" relationships we found in the UMP and EMP based on the definitions of our terms. These are following:

1. *EMP to UMP:* $h\left(p, v(p, w)\right) = x(p, w)$ and $e\left(p, v(p, w)\right) = w$
2. *UMP to EMP:* $x\left(p, e(p, u)\right) = h(p, u)$ and $v\left(p, e(p, u)\right) = u$

Let's prove these statements, which is a good way to get you comfortable with the concepts, especially the distinction between feasible and optimal. Remember that the first part of each statement relates to sets so if we want to prove $A = B$ where $A$ and $B$ are sets, we need to prove $A \subset B$ and $A \supset B$.

*[Proof of (1)]*

- Show $h\left(p, v(p, w)\right) \subset x(p, w)$
  - Proof by contradiction: Suppose that $x \in h\left(p, v(p, w)\right)$ but $x \notin x(p, w)$
  - Since EMP constraint is binding, then $u(x) = v(p, w)$. So the only way that $x \notin x(p, w)$ is if $p \cdot x > w$
  - For $x' \in x(p, w)$, we have $u(x') = v(p, w)$ (so $x'$ feasible for EMP) and, by Walras' Law, $p \cdot x' = w$
  - Putting this together we get $p \cdot x' = w < p \cdot x = e\left(p, v(p, w)\right)$, which contradicts the optimality of $x$ for the EMP
- Show $e\left(p, v(p, w)\right) = w$ and $h\left(p, v(p, w)\right) \supset x(p, w)$
  - Let $x \in x(p, w)$. This means $v(p, w) = u(x)$ ($x$ is optimal) and $p \cdot x \leq w$ ($x$ is feasible).
  - Then $e\left(p, v(p, w)\right) = e\left(p, u(x)\right) \leq w$ (the minimum expenditure to achieve $u(x)$ cannot be more than $w$ since $x$ is feasible for the EMP and $p \cdot x \leq w$)
  - By Walras' Law, all $x'$ such that $p \cdot x' < w$ cannot be optimal for UMP (but still feasible), so $u(x') < v(p, w) = u(x)$. This means that all such $x'$ cannot be feasible for the EMP.
  - This rules out $e\left(p, v(p, w)\right) < w$, and so it must be that $e\left(p, v(p, w)\right) = w$
  - Since $e\left(p, v(p, w)\right) = w = p \cdot x$ and $x$ is feasible for the EMP, then we must have $x \in h\left(p, v(p, w)\right)$

*[Proof of (2)]*

- Show $h(p,u) \subset x(p, e(p,u))$
  - Proof by contradiction: Suppose $x \in h(p,u)$ but $x \notin x(p, e(p,u))$
  - $x \in h(p,u)$ implies that $u(x) = u$ (EMP constraint is binding) and $p \cdot x = e(p,u)$ ($x$ is optimal)
  - For all $x' \in x(p, e(p,u))$, $p \cdot x' = e(p,u)$ (Walras' Law) and $u(x') > u(x) = u$ (since $x$ is feasible but not optimal for UMP)
  - By continuity of $u(\cdot)$, then $\exists \alpha \in (0,1)$ s.t. $u(\alpha x') = u$
  - Then: $p \cdot \alpha x' < p \cdot x' = e(p,u) = p \cdot x$, but $\alpha x'$ is feasible for the EMP, which contradicts the optimality of $x$. Therefore, $h(p,u) \subset x(p, e(p,u))$
- Show $h(p,u) \supset x(p, e(p,u))$ and $v(p, e(p,u)) = u$
  - Proof by contradiction: Suppose $x \in x(p, e(p,u))$ but $x \notin h(p,u)$
  - $x \in x(p, e(p,u))$ means that $p \cdot x = e(p,u)$ (Walras' Law) and that $u(x) = v(p, e(p,u))$
  - Since $x$ achieves the optimal objective for EMP but is not in the solution set ($x \notin h(p,u)$), then it must be because $x$ is not feasible: $u(x) < u$
  - This means that $v(p, e(p,u)) < u$ (i.e. the highest utility achievable with income $e(p,u)$ is strictly less than $u$).
  - For any $x' \in h(p,u)$, then $p \cdot x' = e(p,u)$ (optimal) and $u(x') = u$ (constraint binding). But then $x'$ is feasible for the UMP and $u(x') > v(p, e(p,u))$, which contradicts the optimality of $x$. Therefore, $x \in h(p,u)$.
  - For any $x \in x(p, e(p,u))$, since $h(p,u) = x(p, e(p,u))$, then $x = h(p,u)$, and so $u(x) = u$. Therefore, $v(p, e(p,u)) = u$

While this is very long-winded, the point here is that we didn't do anything overly complicated. It was just about applying the definitions and results we saw before over and over again.

Another way we can connect them, like before, is by using calculus. The result is called the Slutsky Equation and it is as follows:

**Slutsky Equation**

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w), \forall k, l = 1, ..., K$$

Equivalently, in matrix notation this can be written as:

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

$$\underbrace{\begin{pmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \cdots & \frac{\partial h_1(p,u)}{\partial p_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_K(p,u)}{\partial p_1} & \cdots & \frac{\partial h_K(p,u)}{\partial p_K} \end{pmatrix}}_{K \times K} = \underbrace{\begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_K(p,w)}{\partial p_1} & \cdots & \frac{\partial x_K(p,w)}{\partial p_K} \end{pmatrix}}_{K \times K} + \underbrace{\begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_K(p,w)}{\partial w} \end{pmatrix}}_{K \times 1} \underbrace{\begin{pmatrix} x_1(p,w) & \cdots & x_K(p,w) \end{pmatrix}}_{1 \times K}$$

The proof for this is as follows: For a fixed $(p,w)$, set $u = v(p,w)$. From before, we know that $e(p,u) = e(p, v(p,w)) = w$ and $h(p,u) = x(p, e(p,u))$. Putting this together, we can say that: $h(p,u) = x(p,w)$.

Next we differentiate this with respect to $p_k$ for an arbitrary $l^{\text{th}}$ element in the vectors:

$$\begin{aligned}
\frac{\partial h_l(p,u)}{\partial p_k} &= \frac{dx_l(p,e(p,u))}{dp_k} \\
&= \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}\frac{\partial e(p,u)}{\partial p_k} && \text{[total derivative]} \\
&= \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}h_k(p,u) && \text{[Shephard's Lemma]} \\
&= \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}x_k(p,w) && \text{[by starting equality]}
\end{aligned}$$

Given our interpretation's for Shephard's Lemma and Roy's Identity, this becomes quite easy to interpret. First, a little re-arranging:

$$\frac{\partial x_l(p,w)}{\partial p_k} = \frac{\partial h_l(p,u)}{\partial p_k} - \frac{\partial x_l(p,w)}{\partial w}x_k(p,w)$$

The LHS asks: what is the change in the bundle chosen due to a change in price? We argued before that this change is composed of two parts: relative price changes and real income effects. This is exactly what we see here!

The first term captures the *substitution effect* due to relative prices changing. It tells us how much the compensated demand for good $l$ will change when the price of good $k$ increases. If this term is positive, then goods $k$ and $l$ are *substitutes*. If it is negative, then they are *complements*. If $l = k$, then it be negative due to the compensated law of demand (we'll talk about this a little more later).

The second term captures the *income effect* due to real income changes. $\frac{\partial x_l(p,w)}{\partial w}$ tells us how much the Marshallian demand for good $l$ will change when income increases. If this term is positive, then good $l$ is a *normal good*. If it is negative, then it is an *inferior good*. We multiply this term by $x_k(p,w)$ because (as we saw before) this captures the magnitude of real income change due to change in $p_k$. So multiplying the amount of income change due to price change ($x_k(p,w)$) by the rate at which demand for good $l$ changes due to an income change ($\frac{\partial x_l(p,w)}{\partial w}$) will give us exactly the income effect on the demand for good $l$ due to the price change.

What does all of this mean for the LHS? It might be tempting to conclude that demand and price are inversely related and this means that $\frac{\partial x_l(p,w)}{\partial p_k} < 0$, but this would be wrong. If the good is strongly inferior ($\frac{\partial x_l(p,w)}{\partial w}$ is very negative), the income effect may outweigh the substitution effect and make the total value positive. We call these types of goods *Giffen goods*.

## 4.5 Matrix Results

Imagine we take the RHS of the Slutsky Equation for every $k$-$l$ combination and put it all into the matrix. We call this the Slutsky Matrix, denoted by $S(p,w)$, a $K \times K$ matrix where the element in the $l^{\text{th}}$ row and $k^{\text{th}}$ column is the RHS of the Slutsky equation, as written in the section above:

$$S(p,w) = \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{w}x_1(p,w) & \cdots & \frac{\partial x_1(p,w)}{\partial p_K} + \frac{\partial x_1(p,w)}{w}x_K(p,w) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_K(p,w)}{\partial p_1} + \frac{\partial x_K(p,w)}{w}x_1(p,w) & \cdots & \frac{\partial x_K(p,w)}{\partial p_K} + \frac{\partial x_K(p,w)}{w}x_K(p,w) \end{pmatrix}$$

Note that we can also write this in matrix notation:

$$\begin{aligned} S(p,w) =& D_p x(p,w) + D_w x(p,w) x(p,w)^T \\ =& \underbrace{\begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_K(p,w)}{\partial p_1} & \cdots & \frac{\partial x_K(p,w)}{\partial p_K} \end{pmatrix}}_{K\times K} + \underbrace{\begin{pmatrix} \frac{\partial x_1(p,w)}{w} \\ \vdots \\ \frac{\partial x_K(p,w)}{w} \end{pmatrix}}_{K\times 1} \underbrace{\begin{pmatrix} x_1(p,w) & \cdots & x_K(p,w) \end{pmatrix}}_{1\times K} \end{aligned}$$

By the Slutsky equation, we then also know:

$$S(p,w) = D_p h(p,u)$$

This tells us that to understand the properties of the Slutsky matrix, we can equivalently look at the properties of the Jacobian matrix of the Hicksian demand. Let's state them first and then provide proofs:

**Hicksian Demand Jacobian Properties:**
1. $D_p h(p,u) = \nabla_p^2 e(p,u)$
2. $D_p h(p,u)$ is a negative semidefinite matrix
3. $D_p h(p,u)$ is a symmetric matrix
4. $D_p h(p,u) p = 0$

*Proofs*

1. By Shephard's Lemma, $h(p,u) = \nabla_p e(p,u)$. We differentiate with respect to price to get the result that the Hicksian demand Jacobian matrix is equal to the Hessian matrix of the expenditure function
2. Since $e$ is concave in $p$, its Hessian (with respect to $p$) must be negative semidefinite. By (1), this must mean $D_p h(p,u)$ must be too
3. Since $e$ is twice continuously differentiable, the Hessian matrix is symmetric (cross partials are identical)
4. Since $h(p,u)$ is homogeneous of degree 0 in $p$, then $h(\alpha p, u) = h(p,u), \forall \alpha$. We differentiate this with respect to $\alpha$, giving $D_p h(p,u)p$ on the LHS (by chain rule) and 0 on the RHS (since $\alpha$ does not appear). Putting this together gives: $D_p h(p,u)p = 0$

Note that the negative semidefiniteness implies that the diagonal entries $\partial h_k(p,u)/\partial p_k$ are negative for all $k$. We mentioned this in the previous result referring to the (compensated) law of demand, which says that demand and price move in opposite directions when there is compensation for the real income effect. So if we went from price $p$ to $p'$, where only $p_k$ changed to $p'_k$ (and all other prices stayed the same), then the compensated law of demand would tell us that for $x \in h(p,u)$, and $x' \in h(p',u)$, we have:

$$(p'-p)\cdot(x'-x) = \left(\begin{pmatrix} p_1 \\ \vdots \\ p'_k \\ \vdots \\ p_K \end{pmatrix} - \begin{pmatrix} p_1 \\ \vdots \\ p_k \\ \vdots \\ p_K \end{pmatrix}\right)^T \cdot \left(\begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ \vdots \\ x'_K \end{pmatrix} - \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_K \end{pmatrix}\right)$$

$$
\begin{aligned}
&= \begin{pmatrix} 0 & \cdots & p'_k - p_k & \cdots & 0 \end{pmatrix} \begin{pmatrix} x'_1 - x_1 \\ \vdots \\ x'_k - x_k \\ \vdots \\ x'_K - x_K \end{pmatrix} \\
&= (p'_k - p_k)(x'_k - x_k) \leq 0
\end{aligned}
$$

The negative semidefiniteness in fact represents the compensated law of demand. We can write the differences instead as derivatives, e.g. $p' - p$ has a differential analogue of $dp$, and from the law of demand we get exactly the result of negative semidefiniteness:

$$
\begin{aligned}
\text{Compensated Law of Demand: } & (p' - p) \cdot (h(p,u)' - h(p,u)) \leq 0 \\
\text{Differential Analogue: } & dp \cdot dh(p,u) \leq 0 \\
\text{Chain Rule: } & dp \cdot D_p h(p,u) \cdot dp \leq 0
\end{aligned}
$$

Note, again, that this is specific to the Hicksian demand. It is because income is being compensated so all else equal price would lower demand. In the Marshallian demand, it captures the total effect of a price change so the negative result may not hold.

Now, as we approach the end, you might be wondering why we did all of this! A good theory would be one that has testable implications. The problem with consumer theory is that we cannot observe preferences or utilities. It also means we can't observe Hicksian demand (which has utility as one of its inputs). But we do observe prices and wealth, and so we could observe the Marshallian demand. And since the Slutsky matrix can be derived entirely using just $x(p, w)$, then we could observe that too. The question now is that if we did have some demand function, how can we know if it could be derived from a utility function? Our results give us three tests for this:

**Testable Implications of Consumer Theorey:** A demand function can be derived from a utility function if:

1. $x(p, w)$ is homogeneous of degree zero
2. Walras' Law, $p \cdot x = w$, is satisfied
3. The Slutsky matrix is symmetric and negative semidefinite

## 4.6 Example: Cobb-Douglas

To finish off and as a final test of our understanding, let's bring in the simple Cobb-Douglas example we've worked on throughout. We want to check if all these relationships that we found do hold true.

**Example:** A consumer with utility $u(x) = x_1 x_2$ has the following results from their UMP and EMP:

$$\text{UMP:} \qquad x(p,w) = \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right) \qquad v(p,w) = \frac{w^2}{4p_1p_2}$$

$$\text{EMP:} \qquad h(p,u) = \left(\sqrt{\frac{p_2}{p_1}u}, \sqrt{\frac{p_1}{p_2}u}\right) \qquad e(p,u) = 2\sqrt{p_1p_2u}$$

Let's now verify each of the relationships we found

*EMP*

- $\boldsymbol{e(p,u) = p \cdot h, \forall h \in h(p,u)}$

$$\begin{aligned} p \cdot h(p,u) &= \begin{pmatrix} p_1 & p_2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\frac{p_2}{p_1}u} & \sqrt{\frac{p_1}{p_2}u} \end{pmatrix}^T \\ &= p_1\sqrt{\frac{p_2}{p_1}u} + p_2\sqrt{\frac{p_1}{p_2}u} \\ &= \sqrt{p_1p_2}u + \sqrt{p_1p_2}u \\ &= 2\sqrt{p_1p_2u} \\ &= e(p,u)\checkmark \end{aligned}$$

- Shephard's Lemma: $\boldsymbol{h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}}$

$$\begin{aligned} \frac{\partial e(p,u)}{\partial p_1} &= \frac{\partial[2\sqrt{p_1p_2u}]}{\partial p_1} \\ &= 2\frac{1}{2}(p_1p_2u)^{-\frac{1}{2}}p_2u \\ &= \frac{\sqrt{p_2u}}{\sqrt{p_1}} \\ &= h_1(p,u)\checkmark \ (p_2 \text{ holds by symmetry}) \end{aligned}$$

*UMP*

- $\boldsymbol{v(p,w) = u\,(x)\,, \forall x \in x(p,w)}$

$$\begin{aligned} u\,(x(p,w)) &= \left(\frac{w}{2p_1}\right)\left(\frac{w}{2p_2}\right) \\ &= \frac{w^2}{4p_1p_2} \\ &= v(p,w)\checkmark \end{aligned}$$

- Roy's Identity:
$\boldsymbol{x_k(p,w) = -\frac{\partial v(p,w)/\partial p_k}{\partial v(p,w)/\partial w}}$

$$\begin{aligned} \frac{\partial v(p,w)}{\partial p_1} &= \frac{\partial}{\partial p_1}\left[\frac{w^2}{4p_1p_2}\right] \\ &= -\frac{w^2}{4p_1^2p_2} \\ \frac{\partial v(p,w)}{\partial w} &= \frac{\partial}{\partial w}\left[\frac{w^2}{4p_1p_2}\right] \\ &= \frac{2w}{4p_1p_2} \\ \therefore -\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w} &= -\frac{-\frac{w^2}{4p_1^2p_2}}{\frac{2w}{4p_1p_2}} \\ &= \frac{w}{2p_1} \\ &= x_1(p,w)\checkmark \ (p_2 \text{ holds by symmetry}) \end{aligned}$$

*EMP to UMP*

- $\boldsymbol{h\left(p, v(p,w)\right) = x(p,w)}$

$$\begin{aligned} h\left(p, v(p,w)\right) &= \left(\sqrt{\frac{p_2}{p_1}u}, \sqrt{\frac{p_1}{p_2}u}\right)\Bigg|_{u=v(p,w)} \\ &= \left(\sqrt{\frac{p_2}{p_1}\frac{w^2}{4p_1p_2}}, \sqrt{\frac{p_1}{p_2}\frac{w^2}{4p_1p_2}}\right) \\ &= \left(\sqrt{\frac{w^2}{4p_1^2}}, \sqrt{\frac{w^2}{4p_2^2}}\right) \\ &= \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right) \\ &= x(p,w)\checkmark \end{aligned}$$

- $\boldsymbol{e\left(p, v(p,w)\right) = w}$

$$\begin{aligned} e\left(p, v(p,w)\right) &= 2\sqrt{p_1p_2\frac{w^2}{4p_1p_2}} \\ &= 2\sqrt{\frac{w^2}{4}} \\ &= w\checkmark \end{aligned}$$

*UMP to EMP*

- $\boldsymbol{x\left(p, e(p,u)\right) = h(p,u)}$

$$\begin{aligned} x\left(p, e(p,u)\right) &= \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right)\Bigg|_{w=e(p,u)} \\ &= \left(\frac{2\sqrt{p_1p_2u}}{2p_1}, \frac{2\sqrt{p_1p_2u}}{2p_2}\right) \\ &= \left(\frac{\sqrt{p_2u}}{\sqrt{p_1}}, \frac{\sqrt{p_1u}}{\sqrt{p_2}}\right) \\ &= h(p,u)\checkmark \end{aligned}$$

- $\boldsymbol{v\left(p, e(p,u)\right) = u}$

$$\begin{aligned} v\left(p, e(p,u)\right) &= \frac{\left(2\sqrt{p_1p_2u}\right)^2}{4p_1p_2} \\ &= \frac{4p_1p_2u}{4p_1p_2} \\ &= u\checkmark \end{aligned}$$

*Slutsky*

- $\boldsymbol{\frac{\partial h_l(p,u)}{\partial p_k}}$

$$\begin{aligned} \frac{\partial h_1(p,u)}{\partial p_1} &= \frac{\partial}{\partial p_1}\left[\sqrt{\frac{p_2u}{p_1}}\right] \\ &= -\frac{1}{2}\sqrt{\frac{p_2u}{p_1^3}} \end{aligned}$$

$$\begin{aligned} \frac{\partial h_1(p,u)}{\partial p_2} &= \frac{\partial}{\partial p_2}\left[\sqrt{\frac{p_2u}{p_1}}\right] \\ &= \frac{1}{2}\sqrt{\frac{u}{p_1p_2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial h_2(p,u)}{\partial p_1} &= \frac{\partial}{\partial p_1}\left[\sqrt{\frac{p_1u}{p_2}}\right] \\ &= \frac{1}{2}\sqrt{\frac{u}{p_1p_2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial h_2(p,u)}{\partial p_2} &= \frac{\partial}{\partial p_2}\left[\sqrt{\frac{p_1u}{p_2}}\right] \\ &= -\frac{1}{2}\sqrt{\frac{p_1u}{p_2^3}} \end{aligned}$$

- $\boldsymbol{\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}$

$$\begin{aligned}
&\frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \\
&\qquad = \frac{\partial}{\partial p_1}\left[\frac{w}{2p_1}\right] + \frac{\partial}{\partial w}\left[\frac{w}{2p_1}\right]\left(\frac{w}{2p_1}\right) \\
&\qquad = -\frac{w}{2p_1^2} + \frac{1}{2p_1}\frac{w}{2p_1} = -\frac{w}{4p_1^2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial x_2(p,w)}{\partial p_1} + \frac{\partial x_2(p,w)}{\partial w} x_1(p,w) \\
&\qquad = \frac{\partial}{\partial p_1}\left[\frac{w}{2p_2}\right] + \frac{\partial}{\partial w}\left[\frac{w}{2p_2}\right]\left(\frac{w}{2p_1}\right) \\
&\qquad = 0 + \frac{1}{2p_2}\frac{w}{2p_1} = \frac{w}{4p_1p_2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial x_1(p,w)}{\partial p_2} + \frac{\partial x_1(p,w)}{\partial w} x_2(p,w) \\
&\qquad = \frac{\partial}{\partial p_2}\left[\frac{w}{2p_1}\right] + \frac{\partial}{\partial w}\left[\frac{w}{2p_1}\right]\left(\frac{w}{2p_2}\right) \\
&\qquad = 0 + \frac{1}{2p_1}\frac{w}{2p_2} = -\frac{w}{4p_1p_2}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial x_2(p,w)}{\partial p_2} + \frac{\partial x_2(p,w)}{\partial w} x_2(p,w) \\
&\qquad = \frac{\partial}{\partial p_2}\left[\frac{w}{2p_2}\right] + \frac{\partial}{\partial w}\left[\frac{w}{2p_2}\right]\left(\frac{w}{2p_2}\right) \\
&\qquad = -\frac{w}{2p_2^2} + \frac{1}{2p_2}\frac{w}{2p_2} = -\frac{w}{4p_2^2}
\end{aligned}$$

- $\boldsymbol{D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T}$ (let $u = v(p,w)$)

$$\begin{aligned}
D_p h(p,u)\big|_{u=v(p,w)} &= \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{p_2 u}{p_1^3}} & \frac{1}{2}\sqrt{\frac{u}{p_1p_2}} \\ \frac{1}{2}\sqrt{\frac{u}{p_1p_2}} & -\frac{1}{2}\sqrt{\frac{p_1 u}{p_2^3}} \end{pmatrix}\Bigg|_{u=v(p,w)=\frac{w^2}{4p_1p_2}} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{w^2}{4p_1^4}} & \frac{1}{2}\sqrt{\frac{w^2}{4p_1^2p_2^2}} \\ \frac{1}{2}\sqrt{\frac{w^2}{4p_1^2p_2^2}} & -\frac{1}{2}\sqrt{\frac{w^2}{4p_2^4}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{w}{4p_1^2} & \frac{w}{4p_1p_2} \\ \frac{w}{4p_1p_2} & -\frac{w}{4p_2^2} \end{pmatrix} = D_p x(p,w) + D_w x(p,w) x(p,w)^T \checkmark
\end{aligned}$$

## 4.7 Summary Table

| | **Utility Maximization Problem (UMP)** | **Expenditure Minimization Problem (EMP)** |
|---|---|---|
| *Problem Setup* | $\max u(x)$<br>s.t. $p \cdot x \leq w$<br>$x \in \mathbb{R}^K_+$<br><br>$p \in \mathbb{R}^K_{++}$; $w \in \mathbb{R}_+$ | $\min p \cdot x$<br>s.t. $u(x) \geq u$<br>$x \in \mathbb{R}^K_+$<br><br>$p \in \mathbb{R}^K_{++}$; $u \in \left[u(0), \sup_{x \in X} u(x)\right)$ |
| *Optimal Objective* | $v(p,w)$<br>Indirect utility function | $e(p,u)$<br>Expenditure function |
| *Solution Set* | $x(p,w)$<br>Marshallian demand correspondence | $h(p,u)$<br>Hicksian demand correspondence |
| *Feasible Set* | $B(p,w) := \{x \mid p \cdot w \leq x\}$ | $\{x \mid p \cdot x \leq p \cdot \hat{x}\} \cap \{x \mid u(x) \geq u\}$<br>**for some $\boldsymbol{\hat{x}}$ s.t. $\boldsymbol{u(\hat{x}) > u}$** |
| *Existence of Solution* | $\succsim$ is continuous | $u(\cdot)$ is continuous |
| *Uniqueness of Solution* | $\succsim$ is strictly convex $\iff$ $u(\cdot)$ is strictly quasiconcave | $\succsim$ is strictly convex $\iff$ $u(\cdot)$ is strictly quasiconcave |
| *Binding Constraint* | **If $\boldsymbol{\succsim}$ is monotonic:**<br>[Walras' Law] $p \cdot x = w, \forall x \in x(p,w)$ | $u(x) = u, \forall x \in h(p,u)$ |
| *Homogeneity* | $x(\lambda p, \lambda w) = x(p,w)$ [degree 0]<br>$v(\lambda p, \lambda w) = v(p,w)$ [degree 0] | $h(\lambda p, u) = h(p,u)$ [degree 0 in $p$]<br>$e(\lambda p, u) = \lambda e(p,u)$ [degree 1 in $p$] |
| *Monotonicity* | **If $\boldsymbol{\succsim}$ is monotonic:**<br>$v(p,w)$: weakly decreasing in $p$,<br>and strictly increasing in $w$ | $e(p,u)$: weakly increasing in $p$,<br>and strictly increasing in $u$ |
| *Convexity* | $v(p,w)$ is quasi-convex in $(p,w)$<br><br>**If $\boldsymbol{\succsim}$ is convex:** $x(p,w)$ is convex | $e(p,u)$ is concave in $p$<br><br>**If $\boldsymbol{\succsim}$ is convex:** $h(p,w)$ is convex |
| *Continuity* | **If $\boldsymbol{u(.)}$ is continuous:**<br>$v(p,w)$ is continuous<br><br>**If $\boldsymbol{\succsim}$ is strictly convex:**<br>$x(p,w)$ is single-valued and continuous | **If $\boldsymbol{u(.)}$ is continuous:**<br>$e(p,u)$ is continuous<br><br>**If $\boldsymbol{\succsim}$ is strictly convex:**<br>$h(p,u)$ is single-valued and continuous |
| *Law of Demand* | $\frac{\partial x_k(p,w)}{\partial p_k}$ ambiguous sign | $\frac{\partial h_k(p,u)}{\partial p_k} \leq 0$<br><br>[Compensated Law of Demand]<br>$(p' - p) \cdot (h(p',u) - h(p,u)) \leq 0$ |
| *Relationship between Value and Solution Set* | $v(p,w) = u(x), \forall x \in x(p,w)$<br><br>[Roy's Identity]<br>$x_k(p,w) = -\dfrac{\partial v(p,w)/\partial p_k}{\partial v(p,w)/\partial w}$ | $e(p,u) = p \cdot h, \forall h \in h(p,u)$<br><br>[Shephard's Lemma]<br>$h_k(p,u) = \dfrac{\partial e(p,u)}{\partial p_k}$ |
| *Relationship to Dual Problem* | $x(p, e(p,u)) = h(p,u)$<br>$v(p, e(p,u)) = u$ | $h(p, v(p,w)) = x(p,w)$<br>$e(p, v(p,w)) = w$ |
| *Slutsky* | [Equation] $\dfrac{h_l(p,u)}{\partial p_k} = \dfrac{\partial x_l(p,w)}{\partial p_k} + \dfrac{\partial x_l(p,w)}{\partial w} x_k(p,w)$<br><br>[Matrix] $S(p,u) = D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$<br><br>[Properties] Symmetric, negative semidefinite, $S \cdot p = 0$ | |