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PhD Micro (Part 2)
Intro to General Equilibrium

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1 Framework

1.1 Notation

In this part of the course, we're going to bring together the consumer and producer theory we covered in Part 1 into one economy. The notation can get really tricky here, so let's first start by establishing that. In our economy we will have I consumers, J firms, and L goods.

- Each consumer $i \in \{1, ..., I\}$ has a consumption bundle $x_i = (x_{1i}, ..., x_{Li}) \in \mathbb{R}^L$. Consumer i's preferences $\succsim_i$ of bundles over their consumption set $X_i \subset \mathbb{R}^L$ are represented by the utility function $u_i(.)$
- Each firm $j \in \{1, ..., J\}$ has a production vector $y_j = (y_{1j}, ..., y_{Lj}) \in \mathbb{R}^L$. If $y_{lj} > 0$, good l is an output for firm j, and if $y_{lj} < 0$, then it is an input. Firms have a production set $Y_j \subset \mathbb{R}^L$ and transformation functions $F_j(\cdot)$.[1]
- Each consumer i starts off with an initial endowment of each good l, represented by $\omega_{li} \in \mathbb{R}_+$, i.e. they have an endowment vector $\omega_i = (\omega_{1i}, ..., \omega_{Li}) \in \mathbb{R}^L_+$. We will represent the total *initial* quantity of good l as $\omega^l = \sum_i \omega_{li} \in \mathbb{R}_+$.[2]

Throughout all of this, we will always assume that $\succsim_i$ is rational and continuous. We will also usually have $X_i = \mathbb{R}^L_+$, unless the context suggests otherwise (e.g. one of the goods is leisure hours per day). Though we will basically always have X_i be convex.

1.2 Allocations

The story is that we have two types of agents: consumers and producers. Consumers start off with their initial goods - think of everyone in the world starting off with some various level of wealth. Consumers can trade goods among each other or they could sell goods to the firm, who will then produce new goods to sell back to consumers. There's a lot happening in the economy, and our job is to try to figure out what will happen (and whether that outcome is 'good' or 'bad', to put it in simplistic terms).

[1] If you've forgotten this notation, look back at your producer theory notes from Part 1

[2] The notation for the endowment is especially annoying because it will be convenient to have a shorthand for both the consumer *vector* and the total *sum*. In MWG, they use $\bar{\omega}_l$ to represent what I call ω^l. I'm going to use a superscript so that it is clear that, unlike most of the other variables, this is a sum and not a vector.

An outcome is represented by an *allocation*. Allocations tell us how much of each good the consumers should consume and how much the producers should produce (or use, if it is an input).

> **Allocation:**
> An allocation $(x_1, ..., x_I, y_1, ..., y_J)$ is a consumption vector for each $x_i \in X_i$ and a production vector for each $y_j \in Y_j$

Of course, not every allocation is possible in the economy. It would be great if everyone could have 100 Ferraris all produced at no cost, but that's just not realistic. We are really only interested in *feasible* allocations.

> **Feasibility:**
> An allocation is feasible if:
> $$\sum_{i=1}^{I} x_{li} \leq \omega^l + \sum_{j=1}^{J} y_{lj}, \forall l$$

For convenience, we will denote the set of feasible allocations as A. Notice that the definition for feasibility implies that there are L constraints. So we could write the definition as:

$$A = \left\{ (x, y) \in \mathbb{R}^{(I+J)\times L} : \begin{array}{c} \sum_{i=1}^{I} x_{1i} \leq \omega^1 + \sum_{j=1}^{J} y_{1j} \\ \vdots \\ \sum_{i=1}^{I} x_{Li} \leq \omega^L + \sum_{j=1}^{J} y_{Lj} \end{array} \right\}$$

In consumer theory, we would often say an allocation was "feasible" if it was in the budget set of the consumer. But, notice here that feasibility has nothing to do with prices (in fact, we haven't introduced prices at all yet). We're not talking about what consumers or firms can afford - this is about actual physical constraints.

Here's a simple example. Suppose there are initially only 10 apples in the economy and it takes a firm 3 apples to make an apple pie. Consider the following possible allocations:

- Produce 0 pies using 0 apples. This leaves 10 apples to share among the consumers (feasible)
- Produce 1 pie using 3 apples. This leaves 7 apples and 1 pie to share among the consumers (feasible)
- Produce 2 pies using 6 apples. This leaves 5 apples and 2 pies to share among the consumers (infeasible - feasibility constraint for pies is satisfied but not the one for apples)
- Produce 3 pies using 6 apples. This leaves 4 apples and 3 pies to share among the consumers (infeasible - cannot make 3 pies from 6 apples, i.e. $y_j \notin Y_j$)

The production term can be a little confusing because some goods are inputs and some are outputs. Just think of $\sum_{j=1}^{J} y_{lj}$ as representing the net output of good l in the economy. Say there are J_i firms that use good l as a input and there are J_o firms that use it as an output, where $J_i + J_o = J$. Then we can write the production term as:

$$\sum_{j=1}^{J} y_{lj} = -\underbrace{\sum_{h=1}^{J_i} |y_{lh}|}_{<0} + \underbrace{\sum_{k=1}^{J_o} y_{lk}}_{>0}$$

So if we have $\sum_{h=1}^{J_i} |y_{lh}| > \sum_{k=1}^{J_o} y_{lk}$, then $\sum_{j=1}^{J} y_{lj} < 0$ and overall, we are using more good l than we create, so it is a net input (and vice versa for the output case). So we could equivalently write the feasibility constraint as:

$$\sum_{i=1}^{I} x_{li} + \sum_{h=1}^{J_i} |y_{lh}| \leq \omega^l + \sum_{k=1}^{J_o} y_{lk}, \forall l$$

Notice that now all the sums add up to a positive number, so it is easy to interpret. This tell us that for any feasible allocation, the total quantity of good that is consumed (x_{li}) or used as a production input ($\hat{y}_{lh}$) must come either from the endowment we start off with (ω^l) or what we produce ($\hat{y}_{lk}$). This is like the law of conservation of mass that you learned in high school chemistry.

2 Utility and Allocations

2.1 Pareto Optimality

We've established what allocations are and whether they are feasible. Of course, as we learned in consumer theory, not everything that is feasible is desirable. There, we wanted our consumer to find something within their budget set that maximized their utility. In other words, for any $x \in x(p, w)$, it was the case that $u(x) \geq u(x'), \forall x' \in B(p, w)$, where $x(p, w)$ represents Marshallian demand and $B(p, w)$ represents the budget set. But now we have multiple consumers - and firms too - so what condition should we be looking for?

First, we don't care about firms' payoffs. In fact, since we don't have prices in our economy (yet), we can't really define firm profits. Firms are just production technologies - their purpose is to transform goods that the consumers will consume. We don't think of them as people (contrary to *Citizens United*) and so the social planner doesn't really care too much about finding an allocation that makes them 'happy'.

Second, for consumers we will use the same logic as in consumer theory. There, we essentially said that for any $x \in x(p, w), \nexists x' \in B(p, w)$ where $u(x') > u(x)$ (i.e. we can't find another allocation in the budget set that would give strictly higher payoffs). Here, with multiple consumers, an optimal allocation is one where we cannot find another feasible allocation that would give *at least one* person strictly higher payoffs. We call this concept **Pareto optimality**.

Pareto Optimality:
A feasible allocation $(x_1, ..., x_I, y_1, ..., y_J) \in A$ is Pareto optimal if there is no other feasible allocation $(x'_1, ..., x'_I, y'_1, ..., y'_J) \in A$ such that:
1. $u_i(x'_i) \geq u_i(x_i), \forall i$, and
2. $u_i(x'_i) > u_i(x_i)$ for at least one i

I've emphasized the fact that the allocations have to be feasible because this is something that people often forget. Obviously an optimal allocation should be feasible, but it also must be "better" than any other feasible allocation. Of course, we could make people better off by giving them 100 Ferraris, but that's just not possible (or sensible) and so it's a pointless comparison.

Another common phrasing of Pareto optimality (or Pareto efficiency) is that we cannot make someone else better off without making someone else worse off. That type of language suggests being at some sort of boundary or frontier. Since Pareto optimality is all about finding the 'best' out of a feasible set, this leads us to our next concept.

2.2 Utility Possibility Set

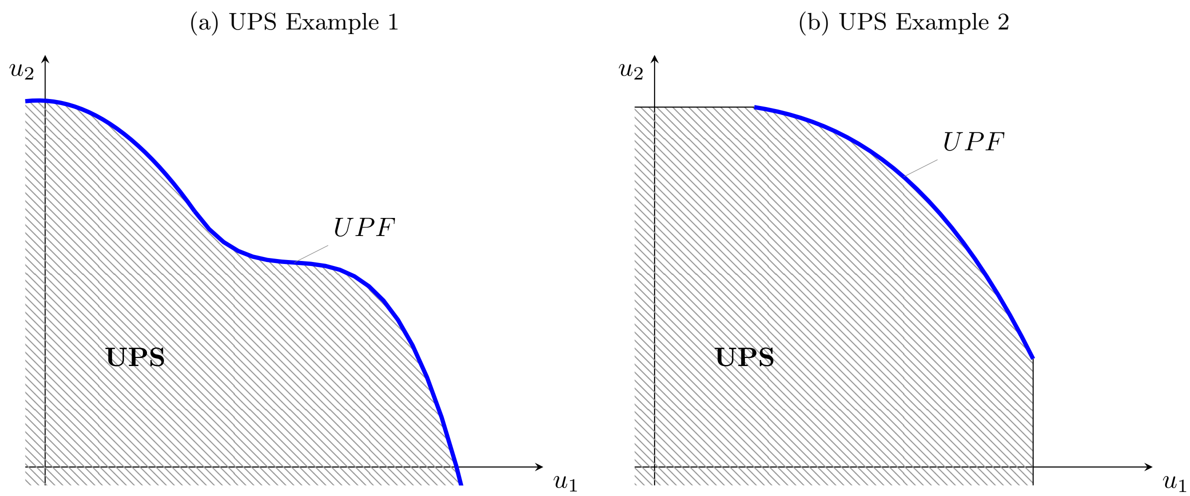
Utility Possibility Set (UPS):
The set of attainable utilities, i.e. $U = \{(u_1, ..., u_I) \in \mathbb{R}^I : \exists (x, y) \in A \text{ where } u_i \leq u_i(x_i), \forall i\}$.

Utility Possibility Frontier (UPF):
A subset of the UPS, $UP \subset U$, defined as the set of $(u_1, ..., u_I) \in U$ such that there is no $(u'_1, ..., u'_i) \in U$ where:

1. $u'_i \geq u_i, \forall i$, and
2. $u'_i > u_i$ for at least one i

To construct the UPS, imagine taking all possible feasible allocations (the set A) and evaluating consumers utilities at each of those allocations. If you plot out all these possible vectors of utilities, you'll get the UPS (the set U). So A gives us the set of feasible *allocations*, but U gives us the set of feasible *utilities.* Of course in higher dimensions this is difficult to conceptualize, so it's easier to think about it in the 2 consumer case. Figure 1 shows two examples of a UPS. The UPS is the shaded area and the UPF is the blue line.

Figure 1: Utility Possibility Set



Note that the axis are labeled u_1 (utility of person 1) and u_2 (utility of person 2). Take any point in the set, say it has coordinates $(\bar{u}_1, \bar{u}_2)$. This tells us that we can find a feasible allocation where person 1 gets utility $\bar{u}_1$ and person 2 gets utility $\bar{u}_2$. The UPS might look quite different depending on the context (e.g. notice that the UPS is not convex in Figure 1a, while it is convex in Figure 1b). However, in both cases (and in general), the UPF is downward sloping; this represents that there is a trade-off in utilities. Increasing utility of person 1 (moving rightwards along the x-axis) means that we will decreasing the utility of person 1 (moving down the y-axis). The *rate* at which this trade-off occurs is captured by the slope of the UPF.

We can even say more than that. Looking at the definition of the UPF, you should see that it is a lot like the one for Pareto optimality, which might make you think that there is a relationship between the two. In fact, they are the same! **An allocation is Pareto efficient if and only if it is part of the UPF.** Here you have to be a little bit careful, because the UPF isn't exactly the *entire* frontier of the UPS. Looking at Figure 1b, you can see that there are horizontal and vertical parts of the UPS boundary. Those points cannot be Pareto optimal. For example, if we move along the horizontal line we make person 1 strictly better off while person 2 is no worse off (they continue to have the same utility). This is why in only the middle downward-sloping part of the boundary is highlighted in blue (i.e. the UPF). So just keep in mind that the entire frontier may not be Pareto optimal!

As mentioned before, the shape of the UPS can depend on a lot of things, including the cardinal representations of the utility functions. However, a special case is that **the UPS is convex if $u_i(\cdot)$ is concave for all i**.

The proof for this is quite straightforward (you can also see it in MWG Ex 16.E.2). For this, we will need to assume that all X_i and Y_j are convex. Also for notational simplicity, let $\mathbf{u} = (u_1, ..., u_I)$ and $\mathbf{u}(x) = (u_1(x_1), ..., u_I(x_I))$ (i.e. these are vectors)

- Take $\mathbf{u}, \mathbf{u'} \in U$. To prove that U is convex, we need to show that $\lambda\mathbf{u} + (1-\lambda)\mathbf{u'} \in U, \forall\lambda \in [0, 1]$
- By the definition of UPS, there exists feasible allocations $(x, y), (x', y') \in A$ such that $\mathbf{u} \leq \mathbf{u}(x)$ and $\mathbf{u'} \leq \mathbf{u}(x')$
- Putting this together, we get:

$$\begin{aligned}\lambda\mathbf{u} + (1-\lambda)\mathbf{u'} \leq & \lambda\mathbf{u}(x) + (1-\lambda)\mathbf{u}(x') \\ & \leq \mathbf{u}(\lambda x + (1-\lambda)x')\end{aligned}$$

  Where the second inequality follows from the fact that all the utility functions are concave (hence $\mathbf{u}(x)$ is concave too). So all that's left to do is show that $\lambda x + (1-\lambda)x'$ is part of some feasible allocation (recall the definition of the UPS)
- Note that since both (x, y) and (x', y') allocations are feasible, then $x \leq \omega + y$ and $x' \leq \omega + y'$. This means that $\lambda x + (1-\lambda)x' \leq \omega + \lambda y + (1-\lambda)y'$. This means that the allocation $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y')$ is also feasible, and so we're done

Another special case to consider is when utilities are quasi-linear. For simplicity, think about the pure exchange economy with utilities $u_i(x_1, x_2) = x_1 + v_i(x_2)$ for $i = 1, 2$. If we just focus on the numeraire, notice that any division of ω^1 would be Pareto optimal. Because all consumers value it in the same way, giving consumer i one more unit of good 1 means having to take away one unit of good 1 from the other consumer i'. Therefore, the only way we can make i better off is by making i' worse off. This logic would be true given any allocation of good 2. So take any Pareto optimal allocation with (x_1^*, x_2^*) that gives utility (u_1^*, u_2^*) (which is on the UPF). Suppose that $x_{1i} \neq 0$ for both i. Now if we did a one unit transfer of good 1, by the logic above, that would get us to another Pareto allocation, so we would still be on the frontier. This change would result in a one unit increase in u_i and a one unit decrease in $u_{i'}$, e.g. we go from (u_1^*, u_2^*) to $(u_1^* + 1, u_2^* - 1)$ but would still remain on the UPF. You can keep doing these transfers until all of the numeraire belongs to one consumer. This intuition tells you that the part of the UPF where we divide up the numeraire is **linear with slope -1**. Of course, once one person has all of the numeraire, then the next step would be to start giving them more of good 2, but at that point we cannot be guaranteed that the frontier will be linear.

2.3 The Core

From the diagrams above, it should be clear that Pareto optimal allocations are not unique - the UPF represents a whole *set* of Pareto optimal allocations. But of course, not all Pareto optimal allocations are "fair" or realistic. For example, take an extreme case where consumer 1 has all the goods and everyone else has nothing. Clearly, consumer 1 can't be made better off but also the only way to make another consumer happier would be give them something. However, this would mean having to take something away from consumer 1 (which would then make consumer 1 strictly worse off). Not only is this not equitable, but it's also an outcome we would never expect to happen. Remember that each of these consumers started off with an endowment; why would they ever give up all their endowments and make themselves worse off than what they started with? This gets to the idea that we should only be looking at Pareto optimal allocations that the agents would actually agree to.

The Core
Let $S \subset \{1, ..., I\}$ be a subset of consumers, called a **coalition**.

A Pareto optimal allocation x is in **the core** if it cannot be **blocked** by any coalition, i.e. there does not exist a feasible allocation $x' \in A$ such that:

1. $u_i(x'_i) \geq u_i(x_i), \forall i \in S$,
2. $u_i(x'_i) > u_i(x_i)$ for at least one $i \in S$, and
3. $\sum_{i \in S} x_{li} \leq \sum_{i \in S} \omega_{li}, \forall l$

For any coalition $S \in \mathcal{P}(\{1, ..., I\}) \backslash \emptyset$ (power set excluding the empty set)

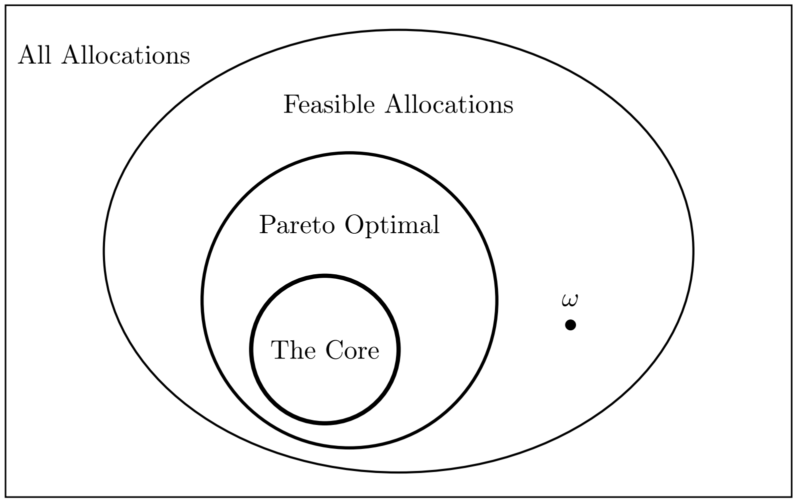
The story here is that a social planner announces a possible allocation x. Everyone in the economy looks at the current state of the economy (the initial endowments) and sees if they can do better than what the social planner is proposing. It could be that a person could be better off with just sticking to their endowment (a coalition of one). It could be that two people could get together and trade amongst themselves (a coalition of two). Or it could even be that everyone in the economy comes together and figures out an allocation better than the social planner (the grand coalition). The coalition pools its resources together and figures out an allocation that Pareto dominates the social planners proposal *for its members*. However, keep in mind that they are restricted to only using the endowments of its members (condition 3 above). The core is only considered about the welfare of its members - its preferred allocation may make other consumers (who are not in the coalition) worse off. If there is one coalition that can do better than the social planner's proposal, then they will *block* the allocation (think of this like a veto power). For an allocation to be in the core, it must make every possible coalition happy.

Since every coalition has to be happy, then no matter the size of I (how many consumers there are), we know that two things must be true in the core:

- No coalition of one blocks a core allocation. This means that no agent gets strictly lower utility in the core than they would have if they stuck with their endowments. If this is true, we say that agents are **individually rational** (i.e. they have an exit option, and they use that option rationally when they are being offered a bad deal)
- The grand coalition does not block a core allocation. But since the grand coalition involves all the members and the full resources of the economy, the no-blocking definition becomes exactly the same as one for Pareto optimality. Hence all allocations in the core must be Pareto optimal.

So the core further shrinks the set of allocations we are looking at. Figure 2 summarizes the possible allocations we have. I have also plotted the endowment point ω but this could really be anywhere within the feasible allocations.

Figure 2: Sets of Allocations

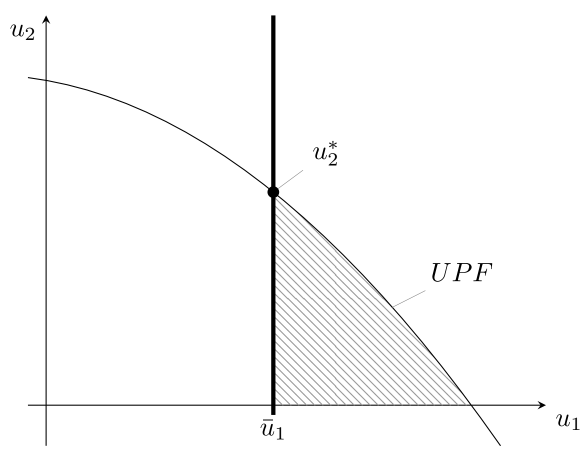


3 Finding Pareto Optimal Allocations

3.1 Utility Maximization

We say that Pareto optimal allocations are where we cannot make someone better off without making someone else worse off. Consider the case of two consumers. That's equivalent to saying that we want to maximize person 2's utility while ensuring that person 1's utility is at least some level, i.e. $u_1 \geq \bar{u}_1$. In optimality, we know that this constraint will hold with equality $u_1 = \bar{u}_1$. So the only way to get a higher objective function (u_2) would be to have $u_1 < \bar{u}_1$, which would obviously make person 1 worse off. We can also see this below in Figure 3:

Figure 3: Finding PO - Method 1



We fix some minimum level of utility for one of the consumers. Then we try to get the highest utility for the other person (i.e. our new feasible set is restricted to the shaded area). This gives us a solution of

u_2^*, which as you can see is on the UPF (i.e. Pareto optimal) and makes the constraint binding (is on the vertical line). This same intuition extends to multiple consumers and gives us the general maximization problem that we solve to find Pareto optimal allocations:

Finding Pareto Optima - Method 1
The Pareto optimal allocations are the solution to the following maximization problem:

$$\begin{aligned} &\max_{(x,y)} u_1(x_1) \\ \text{s.t. } &u_i(x_i) \geq \bar{u}_i, \forall i \neq 1 \\ &F_j(y_j) \leq 0, \forall j \\ &\sum_{i=1}^{I} x_{li} \leq \omega^l + \sum_{j=1}^{J} y_{lj}, \forall l \end{aligned}$$

For some real numbers $(\bar{u}_2, ..., \bar{u}_I) \in \mathbb{R}^{I-1}$

The maximization problem says to find an allocation $(x, y) = (x_1, ..., x_I, y_1, ..., y_J)$ to maximize person 1's utility subject to *three sets* of constraints:

1. Ensure that all other consumers meet their minimum level of utility $\bar{u}_i$ (this is like the utility constraint in the consumer expenditure minimization problem, but now for *other* consumers) [$I - 1$ constraints]
2. Ensure that all firms are producing within their feasible (this is the same constraint in the firm's profit maximization problem) [J constraints]
3. Ensure that the allocation is feasible [L constraints]

Our choice of maximizing consumer 1 is entirely arbitrary - we just need to choose one person and this makes the notation slightly easier. The key part isn't whose utility we are maximizing, but rather whose utilities we aren't maximizing. The choice of the minimum utilities $(\bar{u}_2, ..., \bar{u}_I)$ is important here; as we vary these numbers we will get different solutions to the maximization problem. Since we are looking for a *set* of Pareto optimal allocations, all we have to do is vary the $\bar{u}_i$'s and we can identity the entire set.

Let's try solving this problem (assuming an interior solution - we'll get back to this later). Note that there are $(I - 1) + J + L$ constraints, so we need just as many Lagrange multipliers. When you set up the Lagrangian, you have to be very careful where you put the multipliers (i.e. *inside* the sum). The Lagrangian for the maximization problem is:

$$\mathcal{L} = u_1(x_1) + \sum_{i=2}^{I} \lambda_i \left[u_i(x_i) - \bar{u}_i\right] + \sum_{j=1}^{J} \gamma_j \left[-F_j(y_j)\right] + \sum_{l=1}^{L} \mu_l \left[\omega^l + \sum_{j=1}^{J} y_{lj} - \sum_{i=1}^{I} x_{li}\right]$$

Where the Lagrange multipliers are $\lambda_2, ..., \lambda_I, \gamma_1, ...\gamma_J, \mu_1, ..., \mu_L$. Another way to write this is to let $\lambda_1 = 1$ and $\bar{u}_1 = 0$, so that we have:

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \left[u_i(x_i) - \bar{u}_i\right] + \sum_{j=1}^{J} \gamma_j \left[-F_j(y_j)\right] + \sum_{l=1}^{L} \mu_l \left[\omega^l + \sum_{j=1}^{J} y_{lj} - \sum_{i=1}^{I} x_{li}\right]$$

First, we differentiate with respect to consumer i's allocation of good l, i.e. x_{li}. This gives us:

$$\lambda_i \frac{\partial u_i(x_i)}{\partial x_{li}} = \mu_l \tag{1}$$

Second, we differentiate with respect to firm j's allocation of good l, i.e. y_{lj}. This gives us:

$$\gamma_j \frac{\partial F_j(y_j)}{\partial y_{lj}} = \mu_l \tag{2}$$

Our generic consumer is i, firm is j, and good is l. Let's consider an alternative consumer m, firm n and good k.

Since (1) holds for any two consumers i and m as well as any two goods l and k, then we have:

$$\begin{aligned} \lambda_i \frac{\partial u_i(x_i)}{\partial x_{li}} &= \mu_l & \qquad \lambda_m \frac{\partial u_m(x_m)}{\partial x_{lm}} &= \mu_l \\ \lambda_i \frac{\partial u_i(x_i)}{\partial x_{ki}} &= \mu_k & \qquad \lambda_m \frac{\partial u_m(x_m)}{\partial x_{km}} &= \mu_k \end{aligned}$$

$$\implies \frac{\partial u_i(x_i)/\partial x_{li}}{\partial u_i(x_i)/\partial x_{ki}} = \frac{\partial u_m(x_m)/\partial x_{lm}}{\partial u_m(x_m)/\partial x_{km}} = \frac{\mu_l}{\mu_k} \tag{3}$$

Since (2) holds for any two firms j and n as well as any two goods l and k, then we have:

$$\begin{aligned} \gamma_j \frac{\partial F_j(y_j)}{\partial y_{lj}} &= \mu_l & \qquad \gamma_n \frac{\partial F_n(y_n)}{\partial y_{ln}} &= \mu_l \\ \gamma_j \frac{\partial F_j(y_j)}{\partial y_{kj}} &= \mu_k & \qquad \gamma_n \frac{\partial F_n(y_n)}{\partial y_{kn}} &= \mu_k \end{aligned}$$

$$\implies \frac{\partial F_j(y_j)/\partial y_{lj}}{\partial F_j(y_j)/\partial y_{kj}} = \frac{\partial F_n(y_n)/\partial y_{ln}}{\partial F_n(y_n)/\partial y_{kn}} = \frac{\mu_l}{\mu_k} \tag{4}$$

Putting (3) and (4) together, we also get:

$$\implies \frac{\partial u_i(x_i)/\partial x_{li}}{\partial u_i(x_i)/\partial x_{ki}} = \frac{\partial F_j(y_j)/\partial y_{lj}}{\partial F_j(y_j)/\partial y_{kj}} = \frac{\mu_l}{\mu_k} \tag{5}$$

Put in a different way, since our choice of i, j, l, m, n, k was arbitrary, then we get that:

Finding Pareto Optima - Method 2

In a *smooth and convex* economy, a Pareto optimal allocation must satisfy the following conditions:

- **Distributive Efficiency:** For any $i, m \in \{1, ..., I\}$ and $l, k \in \{1, ..., L\}$

$$MRS^i_{lk} = MRS^m_{lk}$$

- **Productive Efficiency:** For any $j, n \in \{1, ..., J\}$ and $l, k \in \{1, ..., L\}$

$$MRT^j_{lk} = MRT^n_{lk}$$

- **Aggregate Efficiency:** For any $i \in \{1, ..., I\}$, $j \in \{1, ..., J\}$ and $l, k \in \{1, ..., L\}$

$$MRS^i_{lk} = MRT^j_{lk}$$

where MRS^i_{lk} is marginal rate of substitution for consumer i between goods l and k and MRT^j_{lk} is marginal rate of transformation for firm j between goods l and k.

Notice that I qualify these conditions by saying that we require a "smooth and convex" economy. Essentially this is guaranteeing that we get an interior solution for the maximization problem in Method 1. A smooth and convex economy is one where every utility function $u_i(\cdot)$ is **quasi-concave, strictly increasing, and differentiable**.

3.2 Social Welfare Function

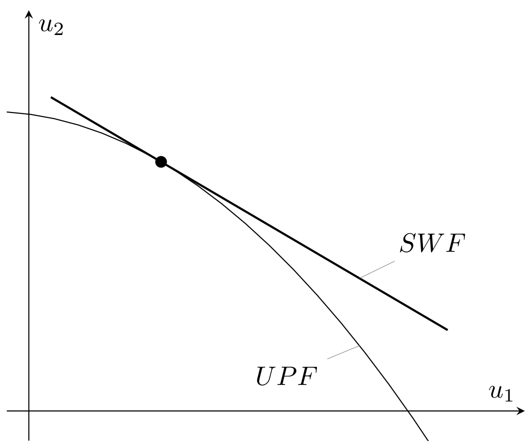
Since we know that the set of Pareto optimal allocations lie on the UPF, we can also try to set up a maximization problem such that we solve for utilities that will get us on the frontier. Recall that the UPS is over the space of utilities (not allocations). But for a given set of utilities, $(u_1, ..., u_I)$, how can we evaluate the value to society from this outcome. For this, we need to define a *social welfare function* (SWF). A SWF is a function $W(\cdot)$, which assigns a social utility value to the various utility vectors in the utility possibility set U. Since we want to capture the fact that higher utility for one individual is better for society (all else equal), $W(u_1, ..., u_I)$ is increasing in all its arguments. The simplest case is the linear SWF where $W(u_1, ..., u_I) = \sum_i \lambda_i u_i$ and λ_i represents the weight for consumer i.

In consumer theory, our utility maximization problem was graphically represented by trying to achieve the highest indifference curve given the budget set. This meant that we were trying to get an indifference curve tangent to the budget set. However, in that case, we are working in the space of allocations. This is a similar idea but now with utilities: our "budget set" is the UPS (set of feasible utilities) and the 'social' indifference curve is the SWF (value of utilities). See Figure 4 for how this works in the 2 consumer case. In Figure 4a, we put greater weight on consumer 2 ($\lambda_2 > \lambda_1$), hence a flatter slope for the SWF and a point that has a relatively high y-coordinate (i.e. higher utility for consumer 2). When we put more weight on consumer 1 ($\lambda_2 > \lambda_1$), as in Figure 4b, we get the opposite result. Notice that since our choice of λ_i determines the slope of the SWF, then means as we change these weights, we get a different tangency point and therefore can capture all the different points along the frontier.

There is one big exception to this idea. This is when the UPS is not convex. If that happens then we get something like in Figure 5. Notice that no matter what weights we choose, we can never achieve the part of the UPF that is in bold. So what condition do we need to avoid this? The answer is that we need all the UPS to be *convex*, but as we saw before, we can guarantee that this will be true if all the utility functions are concave.

Figure 4: Maximizing Social Welfare Function

(a) SWF: More Weight on 2



(b) SWF: More Weight on 1

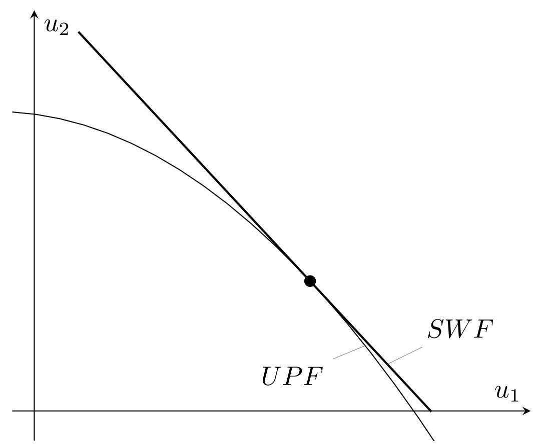
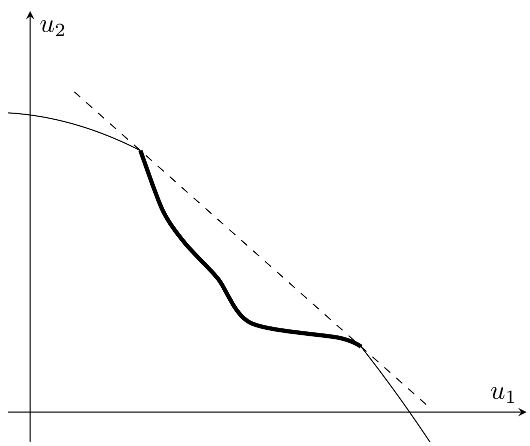


Figure 5: Issues Maximizing SWF



This discussion gives us another way to solve for the set of Pareto optimal allocations.

Finding Pareto Optima - Method 3
The solution to following maximization problem for some vector of weights $\lambda = (\lambda_1, ..., \lambda_I) \gg 0$ is on the UPF:

$$\max_{u \in U} \sum_{i=1}^{I} \lambda_i u_i$$

If all utilities are concave, then for any point u^* on the UPF, we can find weights $\lambda^* \geq 0$ such that the u^* is the solution to the above maximization problem with weights λ^*

Notice here we are not actually finding allocations - we find utilities achieved at a Pareto optimum. However, you can then map this back onto the allocations.

So we can summarize this section as follows. To find Pareto optimal allocations, you can:

1. Method 1: Maximize one person utilities subject to minimum utilities for everyone else + feasibility conditions. This always works.
2. Method 2: Use differentiability conditions. This only works for interior solutions (need strictly increasing, quasiconcave, differentiable utility functions)
3. Method 3: Maximize linear SWF. This only works if the UPS is convex (need concave utilities)

3.3 The Edgeworth Box

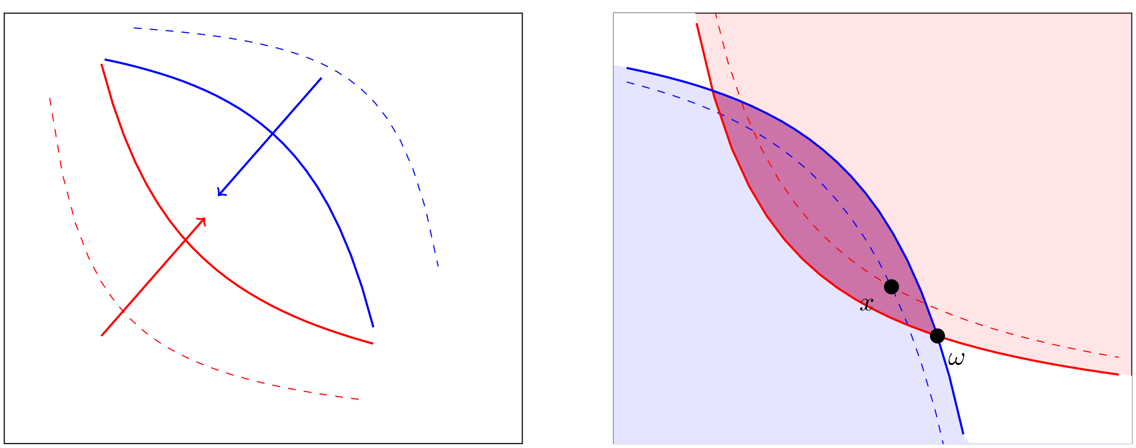
We can see the ideas presented so far through the Edgeworth Box. I won't go into too much detail in this section, but if it is new to you, you should read MWG Chapter 15B.

The Edgeworth Box represents a pure exchange economy, where we have two consumers, two goods, and no producers ($I = 2, J = 0, L = 2$). Recall the usual diagrams we drew for consumer theory with good 1 on the x-axis and good 2 on the y-axis. Now we do this twice: a standard one representing consumer 1's point of view, and another that's been flipped, representing consumer 2's point of view. Putting this together gives us the Edgeworth box, as represented by Figure 6. The box represents feasible allocations where the feasibility constraint holds with equality, i.e. $x_{l1} + x_{l2} = \omega^l, l = 1, 2$. This means allocations are mutually exclusive; if we know the coordinates of a point in terms of the standard axis (i.e. in terms of consumer 1's allocation), then we will also know consumer 2's allocation. For example, a point with coordinates (x_{11}, x_{21}) implies that consumer 2's allocation is $(x_{12}, x_{22}) = (\omega^1 - x_{11}, \omega^2 - x_{21})$

Figure 6: The Edgeworth Box

(a) Indifference Curves

(b) Individual Rationality



For utility, we can draw indifference curves as before. The only trick is to remember that since consumer 2's axis is flipped, their utility is increasing as the indifference curves move down (i.e. towards the southwest corner). For consumer 1, indifference curves are represented as we usually know them. In Figure 6a, consumer 1's indifference curves are in red and consumer 2's are in blue. For both consumers, moving from the dashed to the solid indifference curve indicates an increase in utility.

Now if we take some initial point ω, we know exactly which points each consumer prefers over ω. This is seen in Figure 6b. The red shaded area represents all bundles that consumer 1 prefers over their endowment ω_1. The blue shaded area represents all bundles that consumer 2 prefers over their endowment ω_2. These represent the points where the consumers are being *individually rational* (choosing bundles better than their exit option). The overlap between those two areas is in purple, and is called *the lens*. Since both consumers prefer points in the lens to ω, it must be that ω is not Pareto optimal. Consider a change from ω to x. This would give both consumers higher utility (their new indifference curves are the dashed ones, which is higher from each consumer's perspective). Since x is feasible and makes both consumers better off without making anyone worse off, x Pareto dominates ω. However, x is not Pareto optimal either. Notice that under the new dashed indifference curves, there is still another (yet smaller) lens. Any bundle in the new lens would Pareto dominate x.

This suggests that the presence of the lens indicates that a bundle is not Pareto optimal. In other words, to find Pareto optimality, we need to find bundles where there is no lens. This happens when the **indifference curves are tangent**, as seen in Figure 7a. This is again intuitive because MRS represents the slope of the indifference curves, and we know that Pareto optimality is where the MRS of the consumers are equal (i.e. where the indifference curves have the same slope). Moreover, we know that Pareto optimality is not unique, and we can see this here too. There are many allocations which will give us a tangency point. Figure 7a plots three such examples, but there are infinitely more. If we draw a line connecting all these Pareto optimal allocations, we would get the line in green. This represents the *Pareto set*.

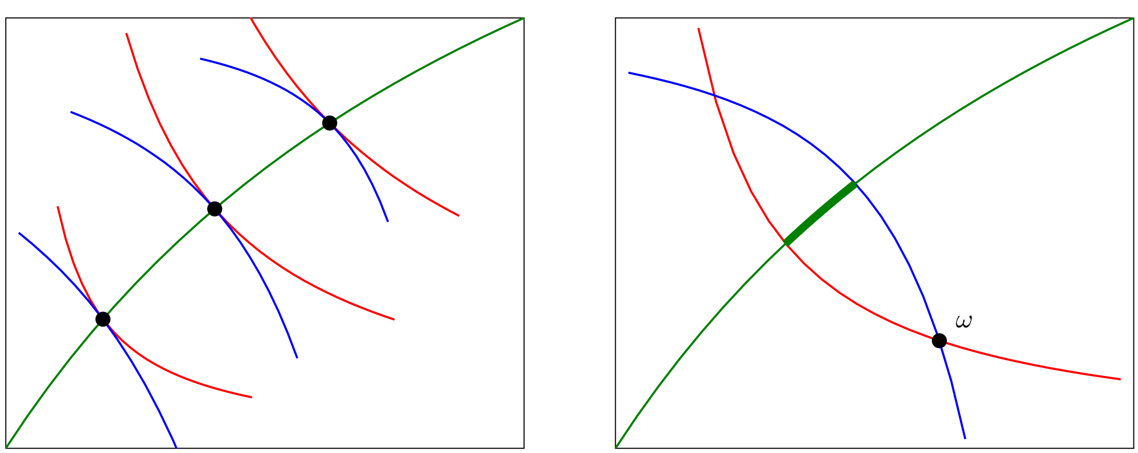
As we also know, not all Pareto optimal allocations are fair. We also want to see where the core is in the Edgeworth box (what won't be blocked by any coalition). Since we have two consumers, we just have to find the area of the Pareto set (green line) that is: individually rational for consumer 1 (red shaded area as per Figure 6b) and individually rational for consumer 2 (blue shaded area). This gives us exactly the lens (purple area). Individual rationality deals with the 1 person coalitions, and the Pareto set deals with the grand coalition. So the core is simply the part of the Pareto set that is inside the lens. This can all be summarized in Figure 7b, where the thick green line represents the core.

You should compare Figure 7a to Figure 2. Everything outside the Edgeworth box is infeasible, and everything inside is feasible. All the points on the green line are Pareto optimal, and the part that is in thick line is the core.

Figure 7: Pareto Optimality in the Edgeworth Box

(a) Pareto optimality is not unique

(b) Summary of Allocations



4 Walrasian Equilibrium

4.1 Definition

At this point, prices haven't played a role in our discussion. Now we're going to bring in prices and think about how agents will react when these are around. To do this, we will be drawing on what we did in Part 1, namely consumer and producer theory. We learned that a consumer, given a choice set and budget set, would try to maximize their utility (the utility maximization problem or UMP). Similarly, a producer facing prices/input costs and a production set would maximize their profits (the profit maximization problem or PMP).

One big difference from what we did before is consumer income. When we did consumer theory, a consumer had a fixed income and the prices affected what was affordable given this fixed income. We also considered a special case of labor supply, where the worker's wage both affected the price of leisure as well as the consumer's income. That same logic will also come up here.

Suppose the prices in the economy are $p = (p_1, ..., p_L)$. Let's consider the consumer's income in a setting, where consumers own all the resources in the economy. This setting is called a *private ownership economy*, i.e. each consumer owns a "piece of the pie". The consumer's source of income comes from two sources, both of which will be endogenous to the price.

Firstly, their wealth is derived from their endowments, but these are in terms of quantities. The only way to get this into dollar amounts is to calculate the value of the endowments, which of course depends on the price of goods. In other words, consumer i's income from their endowments is $p \cdot \omega_i = p_1\omega_{1i} + ... + p_L\omega_{Li}$.

Endowments are one resource in the economy, but we can also think of firms and the profits they generate as another resource. We do this because we think that firms' profit should really go somewhere - it shouldn't just be created for the sake of making profit. Since we can't have profit floating around (remember the law of conservation of mass!), it makes sense that it should go to the consumers. We let $\theta_{ij} \in [0, 1]$ represent the fraction of firm j that consumer i owns, i.e. we have $\sum_i \theta_{ij} = 1$ for all j. For a given price vector p, the firm's profit is $p \cdot y_j$ (recall from producer theory that this is netput notation). So that means consumer i gets an extra $\theta_{ij} p \cdot y_j$ dollars of income from *each firm* j. Therefore, the consumers total income is $p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$. Notice that the value of the income depends on the price, just as we saw in the labor supply case.

The story is that each agent goes to the marketplace. No one is actively setting the prices, but they are changing as agents make trades with one another to try to improve their standing. After the dust settles, we get to an equilibrium where everyone has an allocation and there become some common prices in the market. We know it is an equilibrium because no more trade occurs, i.e. no one can take an action that would make them better off. The great part of the approach we're about to take is that everything is decentralized. Just like the classic demand and supply model you learned in Econ 101, when buyers and sellers come together there is an "invisible hand" such that markets will clear (i.e. quantity demanded = quantity supplied). In this case, every agent acts as both demander and supplier. Consumers will want to buy goods they want to consume and sell off the part of their endowments that they don't need. Firms will buy inputs and sell outputs. Everything is a little more complicated, but the intuition is just same as in the simple Econ 101 case.

Let's note that we're making two key assumptions here:

Assumptions

1. Price Takers: All agents are price takers.
2. Complete Markets: All relevant goods are traded in a market at publicly known prices, and there are no transaction costs.

In this class, we won't really touch the price taking assumption. Obviously, all agents' actions will influence the final prices in the market, but the point is that agents themselves do not internalize that. This may make sense in a large economy, but is admittedly a bit of a dubious assumption in small economies. Note that if these assumptions fail to hold we have *market failures*, e.g. if we have monopolists then the price-taking assumptions fails or if we have externalities or public goods then we violate the complete markets assumption. The second part of this class will look at studying these cases as well as the complete market assumption when we have uncertainty.

For now, all we really need to think about is that consumers are utility maximizers, firms are profit maximizers, and that everyone is a price taker. We then define the equilibrium of the economy, which we call a *Walrasian equilibrium* or *competitive equilibrium*.

Walrasian Equilibrium

A Walrasian Equilibrium is an allocation (x^*, y^*) and a price vector p^* such that:

1. For each consumer i, x_i^* is the solution to the following utility maximization problem (UMP) with price p^*:

$$\max_{x_i \in X_i} u_i(x_i)$$

$$\text{s.t. } p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*$$

2. For each firm j, y_j^* is the solution to the following profit maximization problem (PMP) with price p^*:

$$\max_{y_j \in Y_j} p^* \cdot y_j$$

$$\text{s.t. } F_j(y_j) \leq 0$$

3. The allocation is feasible (markets clear):

$$\sum_{i=1}^{I} x_{li}^* \leq \omega^l + \sum_{j=1}^{J} y_{lj}^*, \forall l$$

There are a few important points to take away from this definition:

- A Walrasian equilibrium consists of an allocation *and* prices. You can't have an equilibrium if you don't know the prices
- Each agent is best responding to the other aspects in the economy. For example, *given* that prices are p^* and production is y^*, *then* consumer i's best response is to have an allocation x^*. This is what gives us an equilibrium - nobody wants to deviate
- It's not enough to just have agents solving their UMPs or PMPs. General equilibrium has to still tie everything together and make sure the allocation can even happen. This comes through the

feasibility constraint in condition (3), which you should note has nothing to do with prices.

The definition of a Walrasian equilibrium is handy because it basically gives us the steps to solving it.

1. First, solve a consumer's UMP. This gives us their demand correspondence, $x(p)$, which is a function of prices and other parameters including endowments. Do this for all consumers.
2. Next, solve a firms' PMP. This gives us their production correspondence, $y(p)$, which is a function of prices and other parameters. Do this for all firms.
3. Plug $x(p)$ and $y(p)$ into the feasibility constraints. Now everything is in terms of prices, so we can solve for *prices* just as a function of the endowments and other parameters (p^*).
4. Plug the prices found back into the demand and production correspondences gives us the *allocations*, i.e. $x^* = x(p^*)$ and $y^* = y(p^*)$.

At this point, we have two main concepts in our toolkit: Pareto optimality and Walrasian equilibrium. Walrasian equilibrium is outcome, i.e. it is an *equilibrium* as we normally think about it. Pareto optimality is an *evaluation* of the allocation. This raises two questions to connect the two concepts:

1. Is a Walrasian equilibrium Pareto optimal?
2. Can any Pareto optimal allocation be a Walrasian equilibrium?

To be precise, the first question is really asking whether the *allocation* of a Walrasian equilibrium is Pareto optimal. An equilibrium consists of allocations and prices, but Pareto optimality says nothing about price, so it doesn't make sense to say that a price is Pareto optimal.

Before we answer these questions, let's work through some examples to build up our intuition of where we're going.

4.2 Intuition through Examples

For this section, we'll develop our intuition mostly using the Edgeworth box. First we have to figure out what a Walrasian equilibrium looks like in the Edgeworth box. In a Walrasian equilibrium, consumers are solving their UMP. In consumer theory, we saw that we could represent a two-good UMP using a budget line and indifference curves. The key part was to find the indifference curve that was tangent to the budget line - this represented the highest utility the consumer could afford. How does this translate to the Edgeworth box?

Since both consumers face the same prices, their budget line is the same. Of course their budget *sets* are different, but the frontier must be the same. We can see this idea in Figure 8, where the black line represents the budget line. Consumer's 1 budget set is shaded in red and consumer 2's budget set is shaded in blue. Notice that the initial endowment point ω has to be on the budget line. Whatever you start off with must obviously be affordable and it always represents one possible allocation to spend *all* of your income (since the value of endowment = value of income).

Moreover, in the Edgeworth box, we have exactly the same idea as before of indifference curves being tangent to the budget line. This captures condition (1) of the Walrasian equilibrium: that consumers are maximizing their utility subject to a budget constraint. To satisfy condition (3), we also need markets to clear, which means that the point of tangency for both indifference curves must be the same. Since the Edgeworth box is a pure exchange economy, we don't need to worry about condition (2). In

Figure 8: Budget Sets in Edgeworth Box

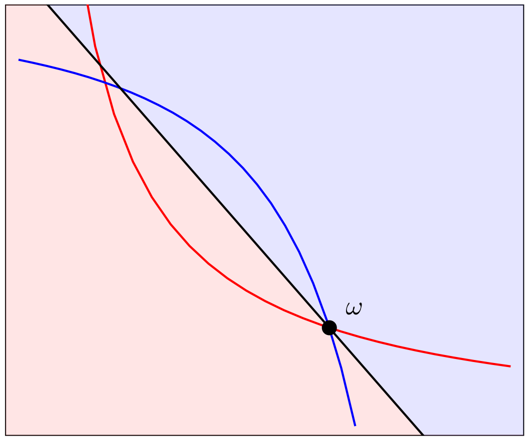
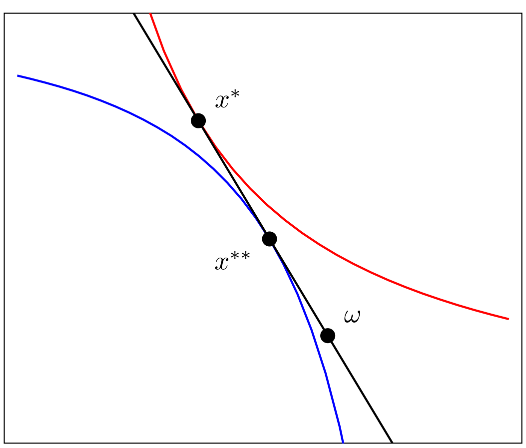


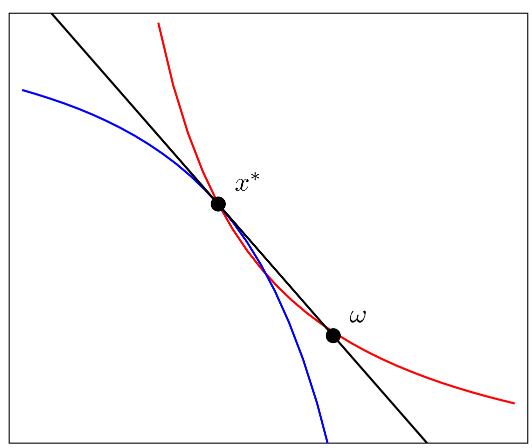
Figure 9, we see how these conditions could fail. In Figure 9a, both agents are solving their UMPs, but their solutions do not align. Since the slope of the budget line is too steep (i.e. p_1 is relatively smaller than p_2), consumer 1 wants to buy extra good 2 ($x^*_{21} - \omega_{21}$) but consumer 2 won't sell that much ($\omega_{22} - x^{**}_{22} = x^{**}_{21} - \omega_{21} < x^*_{21} - \omega_{21}$). In Figure 9b, the market clears at x^* but this can't be an equilibrium because agent 1 is not maximizing utility (indifference curve is not tangent to budget line).

Figure 9: Non-Walrasian Equilibria

(a) Markets not clearing



(b) Incentive to Deviate



In Figure 10a, we see a Walrasian equilibrium. Both agents are solving their UMP (indifference curves are tangent to budget line) and markets clear (tangency point is the same). Notice that x^* (the Walrasian equilibrium allocation) must give the consumers utility that is at least as good as their endowments. In the diagram, the utility of their endowments is represented by the dashed indifference curve. This represents that consumers are being rational - they would only accept a trade that would make them better off than their current state.

We don't just have to restrict ourselves to Walrasian equilibrium in the Edgeworth box. We can also consider the one consumer-one producer case (*the Robinson Crusoe economy*). In Figure 10b, we plot the producer's production set in green, and the consumer's indifference curve in red. Note that the x-axis is the input. You can think of this as labor, and so moving right along the x-axis towards the origin means less labor, which in turn means more leisure (so the indifference curve is still downward sloping). The producer wants to maximize profits, in other words, they want to achieve the highest iso-profit line on their production set. The consumer's problem is the same as before (achieve the highest indifference curve on the budget set).

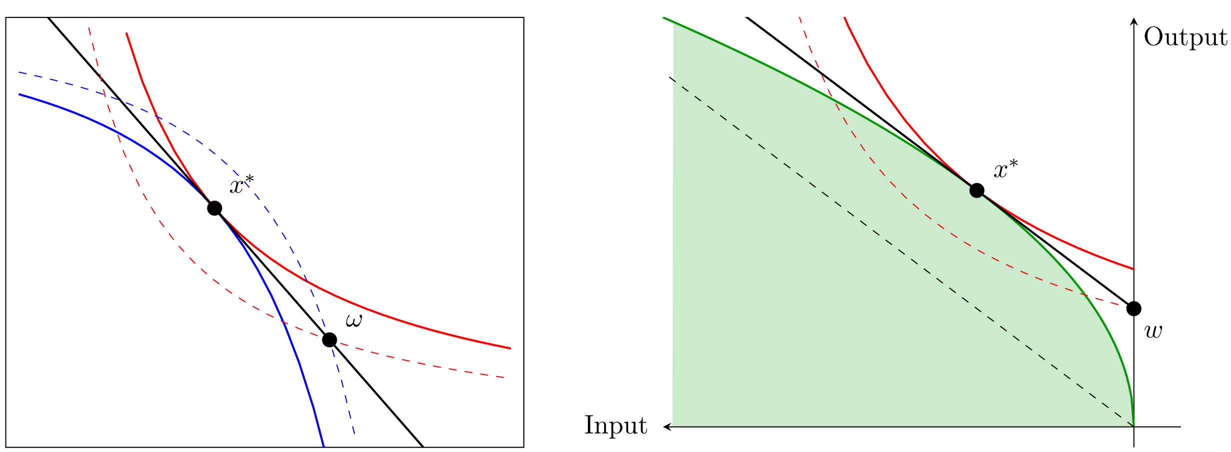
Notice that the firm's iso-profit line is the consumer's budget line; they face the same prices and the consumer owns the firm so all the profit then becomes their wealth. Another way to see this is to consider the point w. Take the firm's isoprofit line through there - it gives the firm some profit level π. The firm's profits are $\pi = p_1 y_1 + p_2 y_2$, where y_2 is the output and y_1 is the input (hence is negative). Re-arranging gives the iso-profit line: $y_2 = \frac{\pi}{p_2} - \frac{p_1}{p_2} y_1$. Hence the point w, where input is 0, must be $w = (0, \frac{\pi}{p_2})$. For the consumer, their income is $p \cdot \omega$ (value of the endowments) plus $\pi = p \cdot y$ (profit of the firm). In the Robinson Crusoe model, we usually assume that the consumer only has endowments for the input good (labor), so their income is $p_1 \omega^1 + \pi$. Hence their budget set is $p_1 x_1 + p_2 x_2 = p_1 \omega^1 + \pi$, which re-arranged would give $x_2 = \frac{\pi}{p_2} + p_1(\omega^1 - x_1)$. By feasibility, $x_1 = \omega^1 + y_1$ and $x_2 = 0 + y_2$, which if you plug into the budget line would give you exactly the same formula as the iso-profit line. At point w, the consumer contributes no input ($x_1 = \omega^1$) and then spends the rest of their income on good 2 ($x_2 = \frac{\pi}{p_2}$). Now of course this is not feasible because it is not in the consumer's budget set - but this explains why the iso-profit and budget lines are the same. Notice if we were on a different isoprofit line, then the consumer's income (and hence budget line) would change because it is a function of π.

The above was a technical point, but the main takeaway from this example is that we have a Walrasian equilibrium: both agents are maximizing their objective and markets are clearing.

Figure 10: Pareto optimal Walrasian Equilibrium

(a) WE in Edgeworth Box

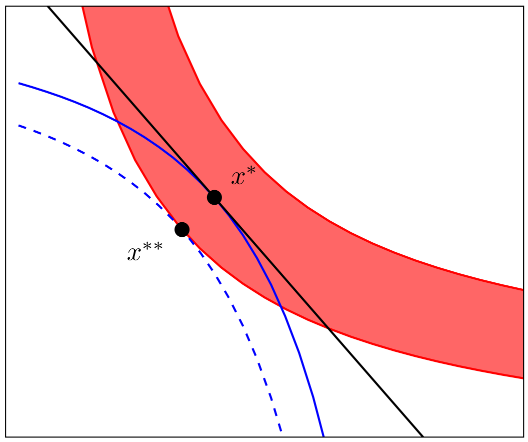
(b) WE in Robinson Crusoe Economy



So now we know what a Walrasian equilibrium looks like, the next step is to try to answer the two questions we laid out. Let's start with the first question: **Is a Walrasian equilibrium Pareto optimal?**

In the pure exchange economy, Figure 10a, we need distribute efficiency (i.e. MRS equal across consumers). Since MRS is simply the slope of the indifference curve, and the indifference curves are tangent to the same line, then both indifference curves have the same slope at x^* (the slope is the price ratio, in fact). Hence this Walrasian equilibrium is also Pareto optimal. In the Robinson Crusoe economy, we need aggregate efficiency (i.e. MRS for the consumer equals MRT for the producer). This is represented by an allocation where the indifference curve and the production function have the same slope. In Figure 10b, this occurs at x^* since the red and green curves are tangent to the same line at that point. So again, the Walrasian equilibrium is Pareto optimal.

Figure 11: Non-Pareto optimal Walrasian Equilibrium



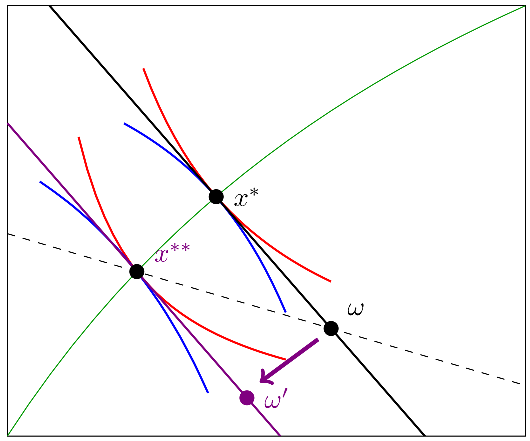
These examples are nice, but before rushing to conclude that all Walrasian equilibria are Pareto optimal, it is best to think of a counter-example. What we need is a situation where the outcome is a Walrasian equilibrium but it is <u>not</u> Pareto optimal. This counter-example is shown in Figure 11, where consumer 1 has a 'thick' indifference curve. This means that every point inside the area shaded in red gives consumer 1 the same level of utility. Consumer 2's utility is standard and the same as before. Consider the point x^*. Is it a Walrasian equilibrium? It definitely fulfills consumer 2's UMP, and clears the market. It also satisfies consumer 1's UMP too because this is the highest utility they can achieve (if you can't see this, try drawing consumer 1's next highest indifference curve, remembering that they should not cross). However, it is not Pareto efficient. Consider the point x^{**}. This gives consumer 2 higher utility, but since it is still inside the 'thick' indifference curve, consumer 1 is indifferent between x^* and x^{**}. Such a move would make consumer 2 better off, without making consumer 1 worse off. To make sure we don't have to worry about these situations, we need to make sure that indifference curves are 'thin'.

At this point, it's good to highlight that the first question did not ask *whether* a Walrasian equilibrium exists. A better way to rephrase that question would be: "*If* a Walrasian equilibrium exists, will it be Pareto optimal?". So in our examples, we didn't have to ensure that a Walrasian equilibrium existed - we just had to draw one and check whether it was Pareto optimal.

Proving existence is a much bigger task, but that's where our second question is going: **Can any Pareto optimal allocations be a Walrasian equilibrium?** As you should recall, Pareto optimal allocations are not unique. We showed that if we had a Walrasian equilibrium (and we have thin indifference curves), we'll end up somewhere in the Pareto set. This question is saying - take any allocation from the Pareto set, can we get to it by letting the market take over? As you can see, this is getting much closer to the idea of the existence of Walrasian equilibrium.

Consider the diagram in Figure 12, where ω and x^* are the same as in Figure 10a. The green line represents the Pareto set. We know with a price vector that defines the budget line in black, we can make the Pareto optimal allocation x^* a Walrasian equilibrium. But what about x^{**}, which is another Pareto optimal allocation? You might suggest putting in a price vector so that the budget line passes

Figure 12: WE with Wealth Transfer



through ω and x^{**} (as in the dotted black line). But recall that if we don't have the 'right' prices, the market will not clear (recall Figure 9a) - this will certainty be the case here since we've made good 1 far too cheap. Notice that the location of x^* is closely tied to the location of ω. In fact, there is just no way to get to x^{**} from ω, because it would not be individually rational for consumer 1. They are strictly worse off with x^{**} than under their initial endowment - so why would they ever give it up? We would instead need to start from an endowment where x^{**} is better for both agents than the endowment. So to get to x^{**}, what if we just moved the endowment point ω? Say we do a wealth transfer so that the endowment moves from ω to ω'. Once the agents have their new endowments, then we let trading start. Now, we can achieve a Walrasian equilibrium: consumers are maximizing utility and markets clear. All we've done is moved from the "black" world to the "purple" world, and then just used the same logic as before. This tells us to achieve any Pareto allocation, all we (might) have to do is find the right wealth transfer.

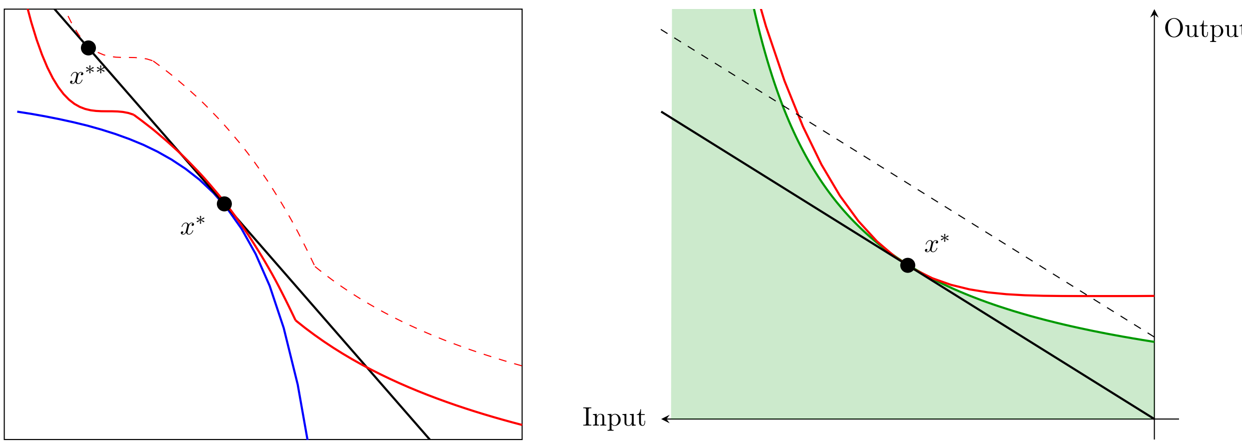
The wealth transfer deals with the fact that the endowment limits which part of the Pareto set we can achieve. However, that's more of a logistical issue and is not the biggest problem. We can still have cases where we have a Pareto optimal allocation but not a Walrasian equilibrium. In other words, we could have an allocation where indifference curves are tangent, but agents are not maximizing their utility.

One issue relates to convexity, which is shown in Figure 13. In the first diagram, Figure 13a, consumer 1 does not have convex preferences. This means that their indifference curves are not convex, so we can have the irregular shaped indifference curve in red. At the point x^*, the slopes of each indifference curve is the same - which means we have MRS between consumers equal and hence we have Pareto optimality (note there is no "lens" to get a better allocation). However, this is not a Walrasian equilibrium because consumer 1 is not maximizing utility. Consider the point x^{**}; this is affordable (it is on the budget line) and it gives consumer 1 a strictly higher utility (as indicated by the higher dotted indifference curve). A similar problem occurs if we do not have convex production sets. This is shown in the Robinson Crusoe economy in Figure 13b, where the green area represents the production set. Again, at x^* we have the slope of the indifference curve (in red) equal to the slope of the production (in green) and hence MRS = MRT, which is Pareto efficient. While the consumer is maximizing utility, the firm is not maximizing profit. For example, they could achieve higher profits on the dotted iso-profit line, which is attainable in their production set. You should compare Figure 13 to Figure 10 to highlight the differences.

Figure 13: Importance of Convexity

(a) Non-Convex Preferences

(b) Non-Convex Production Set

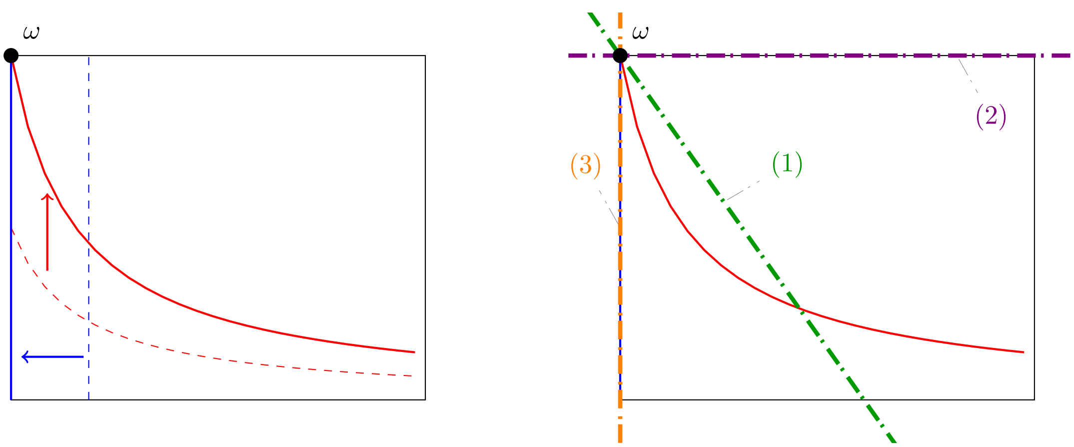


There's one more issue that's even trickier, which is captured by Figure 14a. Here, consumer 2's indifference curves (in blue) shows that they only care about good 1. This means their preference is not strictly monotonic, i.e. $(x_{12}, x_{22} + \Delta) \sim (x_{12}, x_{22})$ for any $\Delta > 0$ when strict monotonicity would imply that this should be $\succ$. The arrows highlight that consumers are better off moving from their respective dashed to solid indifference curves. Suppose that the initial endowment is ω, where all of good 1 is given to consumer 2 and all of good 2 is given to consumer 1. This is a Pareto efficient allocation: the only way to make consumer 1 better off is to give them more of good 1, but that would mean taking it away from consumer 2 which would make them worse off. However, there is no price vector that would make this allocation a Walrasian equilibrium.

Figure 14: Infinite Demand

(a) Indifference Curves

(b) Candidate Budget Sets



Let's consider all the cases, as shown in Figure 14b:

1. $p \gg 0$, i.e. $p_1 > 0$ and $p_2 > 0$. This budget line is green in the figure and labeled with (1). Consumer 1 is not maximizing their utility (their indifference curve is not tangent to the budget

line), so this is not a Walrasian equilibrium (note that consumer 2 is however maximizing their utility).

2. $p_1 = 0$. This budget line is purple in the figure and labeled with (2). In this case, consumer 1's wealth is $p_2 \times \omega^2$, while consumer 2's wealth is $p_1 \times \omega^1 = 0$. Therefore, both consumers have infinite demand for good 1 (try drawing the highest indifference curve they can achieve on the budget line). However, markets cannot clear if demand is infinite, so this is not a Walrasian equilibrium.

3. $p_2 = 0$. This budget line is orange in the figure and labeled with (3). Now, consumer 1's wealth is 0 while consumer 2's wealth is $p_1 \times \omega^1$. Consumer 2 is happy with where they are - they are maximizing utility. However, consumer 1 has infinite demand for good 2 because it costs \$0, which means that they can afford it with their 0 wealth. Again, this cannot be a Walrasian equilibrium.

Let's now summarize what we've seen in this section and then later on we will formalize this intuition:

1. If we have a Walrasian equilibrium, will the allocation be Pareto optimal?
   - *Answer:* Yes, as long as indifference curves are 'thin'
2. Can we make any Pareto optimal allocation a Walrasian equilibrium?
   - *Answer:* Yes, with the right wealth transfer, but there can be some problems...
   - *Problem 1:* If indifference curves are not convex
   - *Problem 2:* If the production set is not convex
   - *Problem 3:* If consumers are demanding infinite amounts of a good

4.3 More Definitions

We're very close to discussing the famous welfare theorems, but before we do that, let's establish some more definitions.

In Part 1, we learned that monotonic preferences gave us 'thin' indifference curves. This works for the most part, but there is actually a weaker notion called **local non-satiation** (LNS). This says that for any $x \in X$ (any bundle in the consumption set), and for all $\varepsilon > 0$, there exists $y \in N_\varepsilon(x)$ (another bundle y in the ε-neighborhood around x) such that $y \succ x$. This still captures the same idea of desirability that monotonicity did, but we're not imposing that you have to have more of a good to strictly prefer it. In fact, y could have less of each component than x. However, it does rule out 'thick' indifference curves (try drawing a ε-ball around x^* in Figure 11 that is entirely inside the indifference curve - now every possible $y \in N_\varepsilon(x)$ will have $y \sim x$, which violates LNS).

As a reminder, here are also definitions for convexity (which we will need to deal with situations like Figure 13):

- *Convex preferences:* If $x \succsim y$, then $\lambda x + (1-\lambda)y \succsim y, \forall \lambda \in [0,1]$. Another definition is that $\forall y \in X$, the set $\{x \in X : x \succsim y\}$ is convex, i.e. the upper contour set of the indifference curve is convex
- *Convex production set:* If $y, z \in Y$, then $\lambda y + (1-\lambda)z \in Y, \forall \lambda \in [0,1]$

The next definitions are different types of equilibria. They're actually very similar to that of the Walrasian equilibrium, so I'll also restate its definition below (in a more compact way).

Walrasian Equilibrium
An allocation (x^*, y^*) and a price vector p^* is a Walrasian equilibrium if it satisfies:

1. Utility Maximization: $x_i^* = \arg\max_{x_i \in X_i} \{u_i(x_i) : p^* \cdot x_i \leq w_i\}, \forall i$
2. Profit Maximization: $y_j^* = \arg\max_{y_j \in Y_j} p^* \cdot y_j, \forall j$
3. Market Clearing: $\sum_{i=1}^I x_{li}^* \leq \omega^l + \sum_{j=1}^J y_{lj}^*, \forall l$

Notice here that we are restricting consumers to only have their initial endowments. But if we were a social planner, we might want to consider different wealth allocations (i.e. if we did a lump-sum transfer of wealth across the consumers as in Figure 12). This gives the idea of a *price equilibrium with transfers*

Price Equilibrium with Transfers
An allocation (x^*, y^*) and a price vector p^* is a price equilibrium with transfers if there is assignment of wealth $(w_1, ..., w_I)$ such that:

1. Utility Maximization: $x_i^* = \arg\max_{x_i \in X_i} \{u_i(x_i) : p^* \cdot x_i \leq w_i\}, \forall i$
2. Profit Maximization: $y_j^* = \arg\max_{y_j \in Y_j} p^* \cdot y_j, \forall j$
3. Market Clearing: $\sum_{i=1}^I x_{li}^* \leq \omega^l + \sum_{j=1}^J y_{lj}^*, \forall l$
4. Feasible Transfer: $\sum_{i=1}^I w_i \leq \sum_{l=1}^L p^* \cdot \omega^l + \sum_{j=1}^J p^* \cdot y_j^*$

Be careful, this notation can be a bit tricky. w (double-u) indicates wealth, while ω (omega) represents endowments. Notice that conditions (1)-(3) is just the same as in Walrasian Equilibrium. So a Walrasian Equilibrium is a special case of a Price Equilibrium with Transfers with wealth assignments:

$$w_i = p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p^* \cdot y_j^*, \forall i$$

The last definition we need has to resolve the issue presented in Figure 14. If you look back, the issue there was that consumers demanded infinite amounts of goods. This comes from the fact that they are trying to find the absolute best bundle in their budget set. Another way we can write utility (or preference) maximization is to say the following:

$$\forall x_i \succ_i x_i^*, p \cdot x_i > w_i \tag{PM}$$

Where x_i^* is the solution to consumer i's UMP, w_i is the consumer's wealth, and $\succ_i$ is i's preference relation. The translation for this is: "For any bundle x_i that i strictly prefers to the optimized bundle x_i^*, it must be the case that x_i was unaffordable for consumer i". Our next definition weakens this, so that consumers are only "weakly" maximizing:

$$\forall x_i \succ_i x^*, p \cdot x_i \geq w_i \tag{WPM}$$

Notice now you can have a bundle that is affordable and makes the consumer strictly better off, and yet the consumer did not choose it. You can think of this as the consumer "making a mistake" - but only when the budget constraint is tight. You still cannot have a strictly preferred bundle that is also strictly cheaper than w, i.e. you can have $p \cdot x \geq w \geq p \cdot x^*$ but you cannot have $p \cdot x < w$. Also notice that $(PM) \implies (WPM)$ but not the other way around.

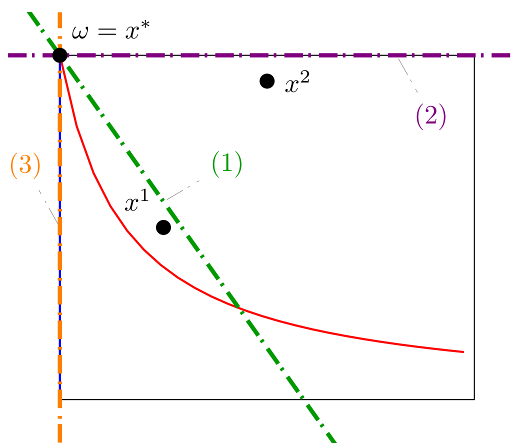
With this we can now define a *price quasi-equilibrium with transfers*.

Price Quasi-Equilibrium with Transfers
An allocation (x^*, y^*) and a price vector p^* is a price quasi-equilibrium with transfers if there is assignment of wealth $(w_1, ..., w_I)$ such that:

1. Weak Utility Maximization: If $x_i \succ_i x_i^*$, then $p \cdot x_i \geq w_i$, $\forall i$
2. Profit Maximization: $y_j^* = \arg\max_{y_j \in Y_j} p^* \cdot y_j, \forall j$
3. Market Clearing: $\sum_{i=1}^{I} x_{li}^* \leq \omega^l + \sum_{j=1}^{J} y_{lj}^*, \forall j$
4. Feasible Transfer: $\sum_{i=1}^{I} w_i \leq \sum_{l=1}^{L} p^* \cdot \omega^l + \sum_{j=1}^{J} p^* \cdot y_j^*, \forall l$

Now compare this with a price equilibrium and you'll see that the only difference is condition (1). Since $(PM) \implies (WPM)$, we have that every price equilibrium with transfers is also a price quasi-equilibrium with transfers.

Figure 15: Price Quasi-Equilibrium



Let's see how this deals with the problem of Figure 14b, which I'll re-produce again in Figure 15. We'll again consider three price vectors to support the allocation $x^* = \omega$ as a price quasi-equilibrium. Note that $p \cdot x^* = w$.

1. $p \gg 0$: x^1 is both strictly preferred to x^* and strictly cheaper than it. This means we have $x^1 \succ x^*$ and $p \cdot x^1 < w$, which violates condition (1).

2. $p_1 = 0$: x^2 is both strictly preferred to x^* and strictly cheaper than it. This means we have $x^2 \succ x^*$ and $p \cdot x^2 < w$, which violates condition (1).

3. $p_2 = 0$: Consider any bundle $x = (x_1, x_2)$ where $x_1 > 0$ and $x_2 \geq x_2^*$. Since this has positive amounts of good 1 (and just as much of good 2), consumer 1 strictly prefers x to x^*. In terms of costs: $p \cdot x^* = (p_1, 0) \cdot x^* = 0$ and $p \cdot x = (p_1, 0) \cdot (x_1, x_2) = p_1 x_1 > 0 = p \cdot x^*$. Therefore, we have satisfied condition (1) and - since consumer 2 is utility maximizing too - this price vector can support x^* as a price quasi-equilibrium.

So here we have $(x^*, (p_1, 0))$ as a price quasi-equilibrium, but we've already shown that it is not a price (Walrasian) equilibrium. As we mentioned before $(PM) \implies (WPM)$, but not the other way round (in general). However, we want to find conditions where $(WPM) \implies (PM)$, i.e. when is "$\forall x \succ x^*, p \cdot x > w$" equivalent to "$\forall x \succ x^*, p \cdot x \geq w$"? Basically all we need is that there exists a cheaper bundle.

Proposition: Cheaper Bundle
Assume that X_i is convex and preferences are continuous. If there exists a $\hat{x}_i \in X_i$ such that $p \cdot \hat{x}_i < w_i$, then $p \cdot x_i > w_i, \forall x_i \succ_i x_i^*$, i.e. $(WPM) \implies (PM)$.

Proof:

- Proof by contradiction - suppose not and there exists a x_i such that $x_i \succ_i x_i^*$ and $p \cdot x_i = w_i$.
- For any $\lambda \in [0,1)$, we must have: $p \cdot (\lambda x_i + (1-\lambda)\hat{x}_i) = \lambda w_i + (1-\lambda) p \cdot \hat{x}_i < w_i$. This holds because X_i is convex, and so $\lambda x_i + (1-\lambda)\hat{x}_i \in X_i$.
- Moreover, since preferences are continuous, the set $S = \{z \in X_i : z \succ x_i^*\}$ is open. So we can find a $\lambda^* \in (0,1)$ such that $\lambda^* x_i + (1-\lambda^*)\hat{x}_i \in S$ (e.g. if λ^* is very close to 1).
- Therefore, we have $\lambda^* x_i + (1-\lambda^*)\hat{x}_i \succ x_i^*$ but $\lambda^* x_i + (1-\lambda^*)\hat{x}_i < w_i$, which is a contradiction.

All we need is to assume that X_i is convex and $\succ_i$ is continuous, which is our standard assumptions. Now we can see why in our example above the price quasi-equilibrium was not also a price equilibrium. Because consumer 1 had wealth of zero, this meant that a cheaper bundle could not have existed. So actually there were two issues going on: the infinite demand and the zero wealth. The above proposition allows us to then conclude:

Proposition: Positive Wealths
Assume that X_i is convex and preferences are continuous. Any price quasi-equilibrium with $w_i > 0, \forall i$ is a price equilibrium with transfers.

4.4 Welfare

Armed with intuition and all the definitions we need, we can now finally state the fundamental welfare theorems.

The first question we asked was: **Is a Walrasian equilibrium Pareto optimal?** The answer is the First Welfare Theorem (FWT):

First Fundamental Theorem of Welfare Economics
Assumptions: (1) preferences are locally nonsatiated.

Theorem: If (x^*, y^*, p) is a price equilibrium with transfers, then the allocation is (x^*, y^*) is a Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

The second question we asked was: **Can any Pareto optimal allocation be a Walrasian equilibrium?** The answer is the Second Welfare Theorem (SWT):

Second Fundamental Theorem of Welfare Economics
Assumptions: (1) preferences are locally nonsatiated, (2) preferences are convex, and (3) Y_j is convex for all j.

Theorem: For every Pareto optimal allocation (x^*, y^*), there exists a price vector $p^* > 0$ such that (x^*, y^*, p^*) is a price quasi-equilibrium with transfers. In particular, if the transfers are such that $w_i > 0, \forall i$, then (x^*, y^*, p^*) is a price equilibrium with transfers.

I won't do the proofs here (you can find them in MWG), but I want to instead highlight a couple of key points:

- The FWT does not rely on as many assumptions as the SWT, in particular you don't need assumptions of convexity.
- The FWT says nothing about the existence of a equilibrium
- You can actually vary the proof for the FWT and make a stronger claim on where the Walrasian equilibrium lies in Figure 2. Not only is it Pareto optimal, but also the **Walrasian equilibrium is in the core**. We actually saw this intuitively in Figure 10a (see the formal proof in MWG Proposition 18.B.1)
- We can make additional assumptions onto the SWT to guarantee that it will be a price equilibrium: (see the discussion at the top of MWG page 556 and Exercise 16.D.3)
  1. $\succsim_i$ is strictly monotonic, $\forall i$
  2. $\omega_i > 0, \forall i$ (initial endowments are positive)
  3. $0 \in Y_j, \forall j$ (firms can shut down)

  These assumptions will guarantee that $p^* \gg 0$ in any price quasi-equilibrium. Then if $x_i^* > 0$, we have $p^* \cdot x_i^* = w_i > 0$ (by LNS, Walras' Law holds so the budget constraint will be tight), which means we have a price equilibrium. If instead $x_i^* = 0$, then $w_i = 0$ but since $p^* \gg 0$, the only affordable bundle is $x_i^* = 0$ (so PM still holds).
- If the economy is "large enough", we can actually weaken the convexity assumptions in the SWT (see the discussion at the end of MWG Section 16.D, starting on pg. 557)
- We are still making other assumptions in the background: complete markets, price takers, no externalities, no uncertainty etc.

4.5 Existence and Uniqueness

For this next part, we want to know under what conditions a Walrasian equilibrium exists and is unique. Through this discussion, we will make the following assumptions:

- Pure exchange economy ($J = 0$)
- $X_i = \mathbb{R}_+^L$
- Preferences are continuous, strictly convex, and strictly monotonic
- Positive endowments: $\sum_i \omega_i \gg 0$

Now, the consumer's UMP becomes to maximize $u_i(x_i)$ subject to the budget constraint: $p \cdot x_i \leq p \cdot \omega_i$. The consumer's Marshallian demand would then be $x_i(p, p \cdot \omega_i)$ since $p \cdot \omega_i$ represents the consumer's income. Using this we can introduce the *excess demand function*

Excess Demand
For a consumer i, $z(p) : \mathbb{R}^L_+ \to \mathbb{R}^L$ is their *excess demand function* defined as:

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$$

For the economy, we have the *aggregate excess demand function*, $z(p) : \mathbb{R}^L_+ \to \mathbb{R}^L$, defined as:

$$z(p) = \sum_{i=1}^{I} z_i(p) = \sum_{i=1}^{I} (x_i(p, p \cdot \omega_i) - \omega_i)$$

In terms of notation, we will use the same logic as we did for endowments. We will denote the l^{th} element $z_i(p)$ as $z_{li}(p) = x_{li}(p, p \cdot \omega_i) - \omega_{li}$. For the aggregate function, since it is a sum, we will use superscripts, i.e. $z^l(p) = \sum_i z_{li}(p)$ be the l^{th} element of $z(p)$.

The excess demand captures how much your demand for each good differs from your endowment. For example, if the l^{th} element is positive, i.e. $z_{li}(p) > 0$, that means you want more of good l than you have in your endowment, i.e. $x_{li}(p, p \cdot \omega_i) > \omega_{li}$. Each of the L elements in $z_i(p)$ could differ in sign. If we add up everyone's excess demand, then we get $z(p)$, the aggregate excess demand. This tells how much total demand differs from total endowment. For example, if $z^l(p) < 0$, that means overall in the economy, people demand less of good l than there is in the economy's endowment, i.e. $\sum_i x_{li}(p, p \cdot \omega_i) < \omega^l$, and there is an over-supply of the good. It doesn't mean that each consumer wants less of good l - we could still have $z_{li}(p) > 0$ for some i - but *overall*, demand is too low for the amount supplied.

As you can probably guess, we would ideally want $z(p) = 0$. That means there is no excess demand (quantity demanded = quantity supplied), and so all markets are clearing. In the two person economy, it would mean that if a consumer i has positive excess demand for a good ($z_{li}(p) > 0$) then the other consumer i' will supply the exact amount to meet this demand ($z_{li'}(p) = -z_{li}(p) < 0$).

To see why this is useful, think about how we would solve for a Walrasian equilibrium (in the pure exchange economy case). First, we would solve each consumer's UMP and get their demand functions: $x(p)$. Then we would plug that into the feasibility constraint, $\sum_i x_{li} \leq \omega^l$, and make sure that this was satisfied for every market. But doing this would give us exactly the aggregate excess demand function! In optimality, since preferences are monotonic, all the constraints will hold with equality. This gives us another way to characterize a Walrasian equilibrium:

Walrasian Equilibrium: Excess Demand Definition
In a pure exchange economy, with continuous, strictly convex, and strictly monotonic preferences, a price p^* is a Walrasian equilibrium if and only if:

$$z(p^*) = 0$$

Equivalently, this can be expressed as:

$$z^l(p^*) = \sum_{i=1}^{I} z_{li}(p^*) = 0, \forall l$$

This is really handy. It says that to find a Walrasian equilibrium, all you have to do is find a p such that $z(p) = 0$. Of course, this is easier said than done. $z(p)$ is a $L \times 1$ vector, so this is in fact a *system of equations*. We're not even sure whether this solution will exist, and if it does, will it be unique? To try and get to those questions, we need to first understand the properties of the function $z(p)$.

From Part 1, we know that the consumer's Marshallian demand is continuous, homogeneous of degree 0, and the budget constraint will be binding in optimality (Walras' Law). Using this, we can also derive conditions on the properties of the excess demand function:

Properties of Excess Demand
In a pure exchange economy, with continuous, strictly convex, and strictly monotonic preferences:

1. $z(p)$ is continuous
2. $z(p)$ is homogeneous of degree 0
3. $p \cdot z(p) = 0, \forall p$ (Walras' Law)
4. There exists $b < 0$ such that $b < z^l(p), \forall l$ and $\forall p$
5. If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then:

$$\max \left\{ z^1(p^n), ..., z^L(p^n) \right\} \to \infty$$

To see (2), just note: $z(\alpha p) = \sum_i x_i(\alpha p, \alpha p \cdot \omega_i) = \sum_i (x_i(p, p \cdot \omega_i) - \omega_i) = z(p)$ (use the homogeneity for the individual demands). To see (3): $p \cdot z(p) = \sum_i (p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i) = \sum_i 0 = 0$. Notice that property (3) applies to *all* prices p. We think of Walras' Law applying in optimality, but this works here because the allocation is already optimized ($x(p, p \cdot \omega)$ is the Marshallian demand).

The last two points are not as immediately clear. Property (4) says that excess demand for each good is bounded from below, which follows from the fact the consumption set is also bounded from below (i.e. the lowest quantity you can demand is 0, so the smallest the excess demand $z^l(p)$ can be is $-\omega^l$). Property (5) roughly says that as the price of one good tends to 0 ($p_l^n \to p_l = 0$), then (at least one) consumer will demand infinite amounts of that good, resulting in an excess demand of infinity.

Properties (2) and (3) are very important to keep in mind when you want to solve for Walrasian equilibrium.

The homogeneity of the aggregate excess demand function shows us that **Walrasian equilibria cannot be pinned down by a unique price vector**. We saw this intuitively in our Edgeworth box examples: prices where defined by the slope of budget line, and the slope is captured by the *price ratio*. Prices (p_1, p_2) and $(\alpha p_1, \alpha p_2)$ for $\alpha > 0$ will give us the same ratio (and budget line) and therefore could support the same allocation. Moreover, in Part 1, we learned that the consumer's Marshallian demand and the producer's production correspondence are both homogenous of degree 0 in price. This means if we scale the prices, we will not change the solutions to the UMP and PMP, i.e. conditions (1) and (2) in the definition of Walrasian equilibrium. Market clearing - condition (3) - has nothing to do with prices, so it also will not be affected. The takeaway here is that if (x^*, y^*, p^*) is a Walrasian equilibrium, then so is $(x^*, y^*, \alpha p^*)$. This tells us that to solve prices, we can just **normalize the price of one good** to 1, and then solve for the other *relative* prices.

Walras' Law is also very helpful to solve for market clearing conditions. Take a price vector $p \gg 0$, and suppose that this price will make the market clear in all but the last market, i.e. $z^1(p) = z^2(p) = ... = z^{L-1}(p) = 0$. By (3), we know that $p \cdot z(p) = \sum_{l=1}^{L} p_l z^l(p) = 0$ (regardless of whether it is an equilibrium price or not). In particular, because $\sum_{l=1}^{L-1} p_l z^l(p) = \sum_{l=1}^{L-1} p_l \times 0 = 0$, this tells us that $p_L z^L(p) = 0$. But by assumption $p \gg 0$, which means that $p_L > 0$ and so it must be the case that $z^L(p) = 0$. Hence the market for the last market clears too. Therefore, in order to check if the market clears in all L markets, **we only need to check if it clears in $L - 1$ markets**.

The next piece of terminology also comes from another property we learned in consumer theory: substitution. We say that two goods are *gross substitutes* if an increase in the price of one good results in a increase in the demand of the other good. If we generalize this to multiple goods, then a price increase in one good l results in an increase in demand for all other goods $k \neq l$. The formal definition is as follows:

Gross Substitute Property
A function $z(p)$ has the gross substitute property if whenever there are two prices p and p' such that $p'_l > p_l$ and $p'_k = p_k, \forall k \neq l$, we have:

$$z^k(p') > z^k(p), \forall k \neq l$$

Notice that in the definition, the two prices only differ on *one* good. Just this one change though results in an increase in demand (and therefore excess demand, since the endowment level will remain unchanged) in the *other* goods. What about the demand for good l? In fact, we can show that demand for it will go down, i.e. $z^l(p') < z^l(p)$ (the proof uses the homogeneity of the excess demand and uses the gross substitute property iteratively - see MWG pg 611). So it means that when p_l increases, I reduce my demand in good l and demand more of all the other goods. Gross substitution is actually a very restrictive assumption, but it does work for Cobb-Douglas utility functions.

Using this terminology, we can now state the following propositions (all without proof):

Existence of Walrasian Equilibrium
Assume: $z(p)$ is defined for $p \gg 0$

If $z(p)$ satisfies the properties of excess demand, i.e. conditions (1)-(5) above, then $z(p) = 0$ has a solution

Uniqueness of Walrasian Equilibrium
Assume: The economy is a pure exchange economy

If $z(p)$ satisfies the gross substitute property, then $z(p) = 0$ has at most one (normalized) solution

Sonnenschein-Mantel-Debreu Theorem
Assume: $z(\cdot)$ is a continuous function defined on a domain bounded away from zero (approximately $\mathbb{R}_{++}$)

If $z(p)$ is homogenous of degree 0 and satisfies Walras' Law ($p \cdot z(p) = 0$), then we can find an economy where $z(p)$ is the aggregate demand function.

For existence, you should look at MWG Figure 17.C.1 to get the intuition. For uniqueness, notice that the gross substitute property was already very restrictive and now we are restricted even more to just the special case of exchange economies. Also note that it says "normalized" solution because we know that scaling the prices will give us another Walrasian equilibrium (i.e. the proposition says that equilibrium is unique in terms of an allocation and relative prices).

Finally, there is the Sonnenschein-Mantel-Debreu Theorem, which is a major negative result for general equilibrium. It tells us that "anything goes". Beyond those few conditions we require, we can't put any more restrictions on the excess demand function. That then means we can't actually say much more about the set of Walrasian equilibria. So we could have a very 'nice' economy with rational agents, but still generate pretty arbitrary excess demand functions (which poses a lot of problems in terms testability, predictions, comparative statics etc). This actually goes back to similar issues that we saw in Part 1 regarding aggregation.