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PhD Micro (Part 1)
Producer Theory

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1 Production Function

For the first part of the course we focused entirely on consumers. Now we're going to focus on producers (firms) and the decisions they need to make. Throughout this section I will be comparing it back to what we saw in consumer theory. You'll soon see that if you understand consumer theory, you'll be able to breeze through producer theory.

1.1 Notation

As always, let's establish notation. We will still have K goods but now we will call the bundle of goods that the producer chooses as $z = (z_1, ..., z_K)$. A big difference is that $z \in \mathbb{R}^K$ (can have negative elements). In consumer theory, we assumed that there were no negative quantities so $x \in \mathbb{R}^K_+$. In producer theory, we will allow negative quantities but give them a meaningful interpretation. If $z_k < 0$, then it is an *input* for the production process and if $z_k > 0$, then it is an *output*. If $z_k = 0$ then it is not used in the production process (or, more specifically, *on net* the firm doesn't use or produce the good). You can think of z as a 'recipe' for the production process. So if $z = (-1, 4, -2)$ then this translates to: "put in 1 unit of good 1 and 2 units of good 3, and you will get out 4 units of good 2". Clearly there can be multiple inputs and outputs. The *production vector* z must be chosen from a set of feasible production plans. We call this feasible set the *production set*, denoted by $Y \subset \mathbb{R}^K$.

I called z a 'recipe' but for that what we need to know how to put together these inputs to get the output. We call this *transformation function* $F(\cdot)$, which basically translates as net output (i.e. the output bundle less the transformed input bundle). Say for a given level of inputs, you can produce 10 units of the output good. Then what are you possible outputs quantities? It could be anything from 0 to 10 (you could just discard what you don't sell) but it definitely can't be more than 10. So the feasible bundles are where $F(z) \leq 0$, i.e. we can define the production set as $Y = \{z \in \mathbb{R}^K : F(z) \leq 0\}$.

Here's another way to describe this process. Let's divide z into two groups: $q = (q_1, ..., q_M) \in \mathbb{R}^L_+$ are the inputs and $y = (y_1, ..., y_L) \in \mathbb{R}^M_+$ are the outputs, where $L + M = K$. In other words, y are the positive elements of z and q are the negative elements of z (but in absolute value). Let's suppose that y is a scalar. Now we can re-write the transformation function as $F(z) = y - f(q)$, where $f(\cdot)$ is the *production function*. All goods are now written in non-negative quantities and we can clearly see which goods are inputs and which are outputs. The z notation is sometimes called netput (net output) notation.

Finally, we will also need prices. We will denote the prices of outputs as $p = (p_1, ..., p_L)$ and the prices (costs) of inputs as $w = (w_1, ..., w_M)$. If we want to work with the single production vector z, then we can combine them into one price vector p^z such that p^z_k is the price of good z_k.

Notation in producer theory can often be really confusing. Here's how my notation compares to MWG's and to what we saw in class (see MWG pg. 137, footnote 6 for clarification on their double use of p):

Variable	**Notes**	**MWG**	**Class**
Output	y	q	y
Input	q	z	q
Netput	z	y	(y, q)
Output Price	p	p	p
Input Price	w	w	c
Netput Price	p^z	p	(p, c)
Transformation Function	$F(q)$	$F(y)$	-
Production Function	$f(q)$	$f(z)$	$F(q)$

1.2 Properties of the Production Set

Let's go through some of the assumptions that we usually make about the production set. Not all of them are interesting or hard to understand, so I'm going to categorize the assumptions.

Basic Assumptions

1. ***Y is non-empty:*** There has to be something feasible that the firm can do
2. ***Y is closed:*** The limit of a sequence of feasible production plans is also feasible
3. ***Possibility of inaction:*** The firm can always choose to do nothing, i.e. $0 \in Y$
4. ***No free lunch:*** You have to use some inputs to produce goods. The production plan can't just be all outputs, i.e. $Y \cap \mathbb{R}^K_+ \subset \{0\}$
5. ***Free disposal:*** You can always get rid of unused goods. If $z \in Y$ and $z' \leq z$, then $z' \in Y$, i.e. either have the same inputs and less output or have the same output and more inputs (remember that inputs are negative in z)

Proportionality Assumptions

6. ***Nondecreasing returns to scale:*** If $z \in Y$, then $\alpha z \in Y, \forall \alpha \geq 1$. We can always scale production *up*.
7. ***Nonincreasing returns to scale:*** If $z \in Y$, then $\alpha z \in Y, \forall \alpha \in [0, 1]$. We can always scale production *down*.
8. ***Constant returns to scale:*** If $z \in Y$, then $\alpha z \in Y, \forall \alpha \geq 0$. We can always scale production *in any direction*.

Convexity Assumptions

9. ***Y is a convex set:*** If $z, z' \in Y$, then $\lambda z + (1-\lambda)z' \in Y$, for all $\lambda \in [0,1]$. Y is strictly convex if $z + (1-\lambda)z'$ is in the interior of Y. This means that we can always scale down and combine different production plans and still get a feasible plan (this is a very strong assumption!)

Notice that convexity of Y and the possibility of inaction ($0 \in Y$) imply decreasing returns to scale. In the convexity definition, we can set $z' = 0$ and therefore $\alpha z \in Y, \forall \alpha \in [0,1]$, which is exactly the decreasing returns to scale definition.

1.3 Properties of the Production Function

Now we want to focus on the production function $y = f(q)$. The firm's production function is somewhat analogous to the consumer's utility function. One difference is that y could be a vector, unlike in utility where our 'output' is one-dimensional. However, for the most part we will work with just one output good, so $f(\cdot) : \mathbb{R}_+^{K-1} \to \mathbb{R}_+$. Let's assume that's the case from now on and set $M = K - 1$.

Standard Assumptions
Our standard assumptions about the production function:

- $f(q)$ is continuous
- $f(q)$ is strictly increasing
- $f(q)$ is strictly quasi-concave
- $f(0) = 0$

Unsurprisingly, these assumptions are basically the same ones we made about utility. There are also similar implications. Continuity of the utility function gave us continuous indifference curves. Monotonicity gave us thin, downward sloping, never-crossing indifference curves. Finally, quasi-concave utility gave us convex indifference curves. What is the producer analogue? Just like the utility function defines indifference curves (bundles of goods which give the same utility), the production function defines *isoquant* (or isoproduct) curves (bundles of inputs which give the same output). Therefore our isoquant curves are going to be continuous, thin, downward sloping, never-crossing, and convex, exactly as before.

A key difference is that, unlike utility functions, the cardinal values of the production function (and therefore the isoquant curves) matter a lot. This is because it has a very clear interpretation as quantity of outputs, so if we double the production function, we've doubled the firm's output for the same level of inputs. That is like working with a completely different firm, which will clearly have an effect on our solutions.

Returns to Scale
Before, we made definitions regarding the proportionality of the production set Y. Now let's define it in terms of the production function. Notice that this is not quite the same as before (except in the case of constant returns):

- *Increasing returns to scale:* A production function exhibits IRTS if $f(\alpha q) > \alpha f(q)$, for all $\alpha > 1$. In other words, if $z \in Y$, then for all $\alpha > 1$, $\exists z' > \alpha z$ such that $z' \in Y$. For example, if we double the inputs we get *more* than double the amount of output.
- *Decreasing returns to scale:* A production function exhibits DRTS if $f(\alpha q) < \alpha f(q)$, for all $\alpha > 1$. In other words, if $z \in Y$, then for all $\alpha \in (0,1)$, $\exists z' < \alpha z$ such that $z' \in Y$. For example, if we double the inputs we get *less* than double the amount of output.

- *Constant returns to scale:* A production function exhibits CRTS if $f(\alpha q) = \alpha f(q)$, for all $\alpha > 0$. This is equivalent to the CRTS scale definition for the set Y. Whichever way we scale the inputs, we will get the same scaling in the outputs.

Derivatives

In consumer theory, we know that $\partial u(x)/\partial x_k$ is the marginal utility of good k. And the ratio of marginal utilities gives us the marginal rate of substitution. The producer analogue is the marginal product, $\partial f(q)/\partial q_k$, and the *marginal rate of technical substitution*:

$$MRTS_{lk}(q^*) = \frac{\partial f(q^*)/\partial q_l}{\partial f(q^*)/\partial q_k}$$

This is the MRTS of input l for input k, at the point q^*. The interpretation is just the same as in the consumer case. It represents how much we have to increase input k to make up for a one unit loss in input l in order to have production at the same level $y = f(q^*)$. Graphically, this is the (absolute) slope of the isoquant curve at point q^*.

We also can define the *elasticity of substitution*, denoted by σ_{lk}:

$$\sigma_{lk} = \frac{d\log(q_k/q_l)}{d\log(MRTS_{lk})}$$

This basically captures the curvature of the isoquant. It asks: how much does the ratio of input k to input l vary as their MRTS (i.e. isoquant slope) varies?[1] Since this is a ratio of logs, then $\sigma_{lk} \geq 0$. $\sigma \to 0$ indicates that it is impossible to substitute the inputs (Leontif isoquant), while $\sigma \to \infty$ indicates that there is perfect substitutability between the inputs (linear isoquant).

Finally, if we want to see how demand for an input q_m changes when its cost w_m changes (i.e. own cost elasticity), then we use:

$$\sigma_{q_m} = \frac{d\log q_m^D(w,\theta)}{d\log w_m}$$

Where $q_m^D(w,\theta)$ is demand for q_m, which will be a function of input costs w and other parameters θ. We used this in the labor demand example in class (elasticity of labor demand is $\frac{d\log L}{d\log w}$, where w is wage), and this is something you will see quite a lot in your macro classes too.

Why do we use logs for elasticities? Let's work with some arbitrary function $f(x)$. The elasticity is basically the *percentage* change of $f(x)$ due to a small *percentage* change in x. A small percentage change $\%\Delta x = \frac{x^{New} - x^{Old}}{x^{Old}}$ can also be expressed as $\frac{dx}{x}$. Since $\frac{d\log x}{dx} = \frac{1}{x}$, then we think of $d\log x = \frac{dx}{x}$. Therefore, elasticity is just: $\frac{\%\Delta f(x)}{\%\Delta x} = \frac{d\log f(x)}{d\log x}$.

1.4 Examples

Let's look at two examples that we saw in class to test what we've learned.

Example 1: Cobb-Douglas

$$f(q) = \prod_{m=1}^{M} q_m^{\alpha_m}$$

[1] Be really careful! It's q_k/q_l, not q_l/q_k

Note that if we scaled production by λ then we get: $f(\lambda q) = \prod_m (\lambda q_m)^{\alpha_m} = \lambda^{\sum \alpha_m} f(q)$. Therefore the value of $\sum_m \alpha_m$ determines the scalability of the production.

$$\sum_m \alpha_m \begin{cases} > 1 & \text{Increasing returns to scale} \\ = 1 & \text{Constant returns to scale} \\ < 1 & \text{Decreasing returns to scale} \end{cases}$$

Since this is Cobb-Douglas, we can already guess the form of MRTS since it will be like the MRS for Cobb-Douglas utility:

$$MRTS_{lk} = \frac{\alpha_l q_k}{\alpha_k q_l}$$

Next, we calculate the elasticity of substitution:

$$\begin{aligned} \log(MRTS_{lk}) &= \log\left(\frac{\alpha_l}{\alpha_k}\right) + \log\left(\frac{q_k}{q_l}\right) \\ \log\left(\frac{q_k}{q_l}\right) &= \log(MRTS_{lk}) - \log\left(\frac{\alpha_l}{\alpha_k}\right) \\ \implies \frac{d\log(q_k/q_l)}{d\log(MRTS_{lk})} &= 1 = \sigma_{lk} \end{aligned}$$

Example 2: Constant Elasticity of Substitution (CES)

$$f(q) = \left(\sum_{m=1}^{M} \alpha_m q_m^\rho\right)^{\frac{\gamma}{\rho}}$$

Where $\rho \leq 1$, $\rho \neq 0$, and $\gamma > 0$. In class, we had $\rho = -\varepsilon$. Usually you will see $\gamma = 1$, but lets keep it general to see its role. Note that: $f(\lambda q) = \left(\sum_m \alpha_m (\lambda q_m)^\rho\right)^{\gamma/\rho} = \left(\lambda^\rho \sum_m \alpha_m q_m^\rho\right)^{\gamma/\rho} = \lambda^\gamma f(q)$. Therefore the value of γ determines the scalability of the production.

$$\gamma \begin{cases} > 1 & \text{Increasing returns to scale} \\ = 1 & \text{Constant returns to scale} \\ \in (0,1) & \text{Decreasing returns to scale} \end{cases}$$

Let's calculate the MRTS. To make notation easier, define $Q = \sum_m \alpha_m q_m^\rho$, i.e. $f(q) = Q^{\gamma/\rho}$.

$$\begin{aligned} MRTS_{lk} &= \frac{\partial Q^{\frac{\gamma}{\rho}}/\partial q_l}{\partial Q^{\frac{\gamma}{\rho}}/\partial q_k} \\ &= \frac{\partial Q^{\frac{\gamma}{\rho}}/\partial Q \cdot \partial Q/\partial q_l}{\partial Q^{\frac{\gamma}{\rho}}/\partial Q \cdot \partial Q/\partial q_k} \\ &= \frac{\frac{\gamma}{\rho} Q^{\frac{\gamma}{\rho}-1} \cdot \alpha_l \rho q_l^{\rho-1}}{\frac{\gamma}{\rho} Q^{\frac{\gamma}{\rho}-1} \cdot \alpha_k \rho q_k^{\rho-1}} \\ &= \frac{\alpha_l}{\alpha_k} \cdot \left(\frac{q_k}{q_l}\right)^{1-\rho} \end{aligned}$$

Next, we calculate the elasticity of substitution:

$$
\begin{aligned}
\log(MRTS_{lk}) &= \log\left(\frac{\alpha_l}{\alpha_k}\right) + (1-\rho)\log\left(\frac{q_k}{q_l}\right) \\
\log\left(\frac{q_k}{q_l}\right) &= \frac{1}{1-\rho}\left[\log(MRTS_{lk}) - \log\left(\frac{\alpha_l}{\alpha_k}\right)\right] \\
\implies \frac{d\log(q_k/q_l)}{d\log(MRTS_{lk})} &= \frac{1}{1-\rho} = \sigma_{lk}
\end{aligned}
$$

A few things to remark here. Remarkably, we found a constant elasticity of substitution, just as the name promised (CES). Note that if $\rho \to 0$, then $\sigma_{lk} \to 1$, which gives us the Cobb-Douglas production function. On the other hand if $\rho \to -\infty$, then $\sigma_{lk} \to 0$, which gives the Leontif production function.

2 Profit Maximization Problem

2.1 The Problem

In consumer theory, agents wanted to maximize their utility, which we saw through the UMP. Firms don't have utility, instead they want to maximize profit. This gives us the *profit maximization problem* (PMP). Their profit is given by: (*Price of Output* $\times$ *Output Quantity*) - (*Price of Input* $\times$ *Input Quantity*). A key assumption here is that firms are price takers, so they take the prices of output and inputs as given, which means that all there is left is to choose quantities. We express the PMP as:

$$
\begin{aligned}
&\max_{y,q} py - w \cdot q \\
&\text{s.t. } y \leq f(q)
\end{aligned}
$$

Similarly, if we want to use netput notation, we can write it as:

$$
\begin{aligned}
&\max_{z} p^z \cdot z \\
&\text{s.t. } F(z) \leq 0
\end{aligned}
$$

Graphically, in the one input/one output case, think of the production set plotted on the z_1-z_2 axis (MWG Figure 5.C.1). The production frontier should be in the second quadrant (where z_1 is negative because it is an input and z_2 is positive because it is an output). On this plane, we can plot *isoprofit* curves (i.e. lines where bundles of goods give the same profit). Since everything enters linearly, the isoprofit curves will also be linear (i.e. $p_1^z z_1 + p_2^z z_2 = \pi^*$ means plotting the line $z_2 = \frac{1}{p_2^z}(\pi^* - p_1^z z_1)$ for all possible π^*). We are trying to find a point that it at or below the production frontier, but also at the highest isoprofit line. This will then be at the point where an isoprofit line is tangent to the production frontier.

Just like in the UMP, we have a solution set and an optimal objective function. Keep in mind that our choice variables are (y, q) and the parameters are (p, w) (i.e. z and p^z, respectively):

- $z(p^z)$ is the *production (or net supply) correspondence*, the solution set of the PMP (analogous to the Marshallian demand). Think of this as consisting of $y^* = y(p, w)$, the optimal supply of output, and $q^* = q(p, w)$, the optimal demand of input.

- $\pi(p^z) = \pi(p, w)$ is the *profit function*, the optimal objective function of the PMP (analogous to the indirect utility function)

Let's look at the FOCs of this problem:[2]

$$\mathcal{L} = py - w \cdot q + \lambda(f(q) - y)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial y} &= p - \lambda = 0 \\
&\implies p = \lambda \\
\frac{\partial \mathcal{L}}{\partial q_m} &= -w_m + \lambda \frac{\partial f(q^*)}{\partial q_m} = 0 \\
&\implies \frac{\partial f(q^*)}{\partial q_m} = \frac{w_m}{p}
\end{aligned}$$

Moreover, if we take any two inputs q_m and q_l and divide the FOCs, we get:

$$\frac{\partial f(q^*)/\partial q_m}{\partial f(q^*)/\partial q_l} = MRTS_{ml} = \frac{w_m}{w_l}$$

In consumer theory we saw that the MRS equaled the price ratio. Here we're seeing exactly the same relation, MRTS is equal to the price (input cost) ratio.

2.2 Properties

Let's go through the properties of the PMP just as we saw in the UMP.

Existence

- The production function needs to exhibit DRTS for a (finite) solution set to exist
  - *Proof:* Suppose not, and that there is an production correspondence q^*, with profit $\pi^* = pf(q^*) - w \cdot z^*$. Then we could increase the inputs by $\alpha > 1$, where we know $f(\alpha q^*) \geq \alpha f(q^*)$. But then $pf(\alpha q^*) - w \cdot \alpha q^* \geq \alpha p f(q^*) - w \cdot \alpha q^* = \alpha \pi^*$. So whatever profit we were making before, we could be making more profit by just scaling up production, contradicting the optimality of q^* (or giving us infinitely many solutions)

Uniqueness

- If Y is strictly convex, then the solution will be unique
  - *Proof:* Suppose not, and that there are two solutions $z, z' \in z(p^z)$. By the strict convexity of Y, then for an arbitrary $\lambda \in [0, 1]$, we have $\lambda z + (1 - \lambda)z'$ is in the interior of Y. But since it is in the interior, then we can scale it up by some $\alpha > 1$ and still be in Y, i.e. $\alpha\left(\lambda z + (1 - \lambda)z'\right) \in Y$. Since both z and z' are optimal, then we must have that $p^z \cdot z = p^z \cdot z'$.

[2] Technical note: For simplicity, we're going to always assume that $\forall z^* \in z(p, w), z^* \gg 0$ so that all our FOCs hold with equality (this will almost always be the situation in your problem sets/exams). Look back on your math camp notes about the Kuhn-Tucker conditions if this part is unclear.

Therefore:

$$\begin{aligned} p^z \cdot \alpha\left(\lambda z + (1-\lambda) z'\right) &= \alpha\left(\lambda p^z \cdot z + (1-\lambda) p^z \cdot z'\right) \\ &= \alpha\left(\lambda p^z \cdot z + (1-\lambda) p^z \cdot z\right) \\ &= \alpha p^z \cdot z > p^z \cdot z \end{aligned}$$

So we have found a feasible production plan that yields strictly larger profits, which is a contradiction.

Binding Constraint

- In optimality, we will have $y^* = f(q^*)$, i.e. $F(z) = 0$
  - *Proof:* If not, then $y < f(q)$. Thus, there is output that has already been paid for and produced but is being 'thrown away'. So the firm can be strictly better off by selling it

Production Correspondence

- $z(p^z)$ homogeneous of degree zero
  - *Proof:* Note that maximizing $\lambda p^z \cdot z$ yields the same solution as maximizing $p^z \cdot z$. Scaling all prices (output and inputs) has no real effect on the optimal choice of quantities
- If Y is convex, then $z(\cdot)$ is convex. If Y is strictly convex, then $z(\cdot)$ is a singleton
  - *Proof:* We proved the singleton claim already for uniqueness. For the convexity, note that $z(p^z) = Y \cap \{z \in \mathbb{R}^K : p^z \cdot z \geq \pi(z^*)\}$, which is the intersection of two convex sets (and therefore must be convex itself)

Profit Function Properties

- $\pi(p^z)$ is continuous
  - *Proof:* This follows from Berge's Theorem of the Maximum
- $\pi(p^z)$ is homogeneous of degree 1
  - *Proof:* $\pi(\lambda p^z) = \max_z \lambda p^z \cdot z = \lambda \max_z p^z \cdot z = \lambda \pi(p^z)$
- $\pi(p^z)$ is convex
  - *Proof:* Take two price vectors p^z and $\tilde{p}^z$ and a solution to the PMP with the parameter being the convex combination of the two prices, i.e. $z \in z(\lambda p^z + (1-\lambda)\tilde{p}^z)$. Note that z is a feasible (but not necessarily optimal) solution for the PMP with price p^z and the PMP with price $\tilde{p}^z$. Therefore, $\pi(\lambda p^z + (1-\lambda)\tilde{p}^z) = (\lambda p^z + (1-\lambda)\tilde{p}^z) \cdot z = \lambda p^z \cdot z + (1-\lambda)\tilde{p}^z \cdot z \leq \lambda \pi(p^z) + (1-\lambda)\pi(\tilde{p}^z)$.
- $\pi(p, w)$ is weakly increasing in p (price of outputs) and weakly decreasing in w (cost of inputs)
  - *Proof (outputs):* Take some $p > p'$, with corresponding production correspondences (y, q) and (y', q'). Then, by definition, $py - w \cdot q \geq py' - w \cdot q'$ because (y, q) is optimal and (y', q') is feasible. Moreover, since $p > p'$, then it must be that $py' - w \cdot q' \geq p'y' - w \cdot q'$. Putting this together, gives $\pi(p, w) = py - w \cdot q \geq p'y' - w' \cdot q' = \pi(p', w)$.
  - *Proof (inputs):* Take some $w > w'$, with corresponding production correspondences (y, q) and (y', q'). Then, by definition, $py' - w' \cdot q' \geq py - w' \cdot q$ because (y', q') is optimal and (y, q) is feasible. Moreover, since $w > w'$, then it must be that $py - w' \cdot q \geq py - w \cdot q$. Putting this together, gives $\pi(p, w') = py' - w' \cdot q' \geq py - w \cdot q = \pi(p, w)$.
  - *Proof (simpler):* Use Hotelling's Lemma (derived next).

Differentiability Results

- *Hotelling's Lemma:* $\frac{\partial \pi(p^z)}{\partial p_k^z} = z_k(p^z)$ for any good k, or equivalently, $\frac{\partial \pi}{\partial p} = y$ and $\frac{\partial \pi}{\partial w_m} = -q_m$. In vector notation, $\nabla \pi(p^z) = z(p^z)$
  - *Proof:* Use the Envelope Theorem. $\partial_{p_k^z} \pi(p^z) = \partial_{p_k^z} \mathcal{L}|_{z=z(p^z)} = \partial_{p_k^z}(p^z \cdot z - \lambda F(z))|_{z=z(p^z)} = z_k|_{z=z(p^z)} = z_k(p^z)$
- $Dz(p^z) = D^2\pi(p^z)$ is symmetric and positive semidefinite matrix with $Dz(p^z)p^z = 0$
  - *Proof (identity):* Start with Hotelling's Lemma and take derivatives on both sides with respect to price
  - *Proof (properties):* This is a Hessian (matrix of second derivatives), so it must be symmetric. Moreover, the Hessian of a convex function is positive semidefinite and π is convex in p^z (as we proved above).
  - *Proof (orthogonality):* Since $z(p^z)$ is homogeneous of degree 0, then $z(\alpha p^z) = z(p^z)$. Take derivatives on both sides with respect to α.

Note that the differentiability results are incredibly analogous to consumer theory. Hotelling's Lemma is essentially the same as Roy's Identity (we don't need to do any normalizing since we're working entirely in dollars though). However be careful because in the UMP, the Slutsky was *negative* semidefinite due to the *concavity* of the expenditure function. The producer analogue is *positive* semidefinite due to the *convexity* of the profit function. In fact, the negative semidefiniteness of the Slutsky matrix captured the law of demand (that, all else equal, prices and quantity demanded move in opposite directions). The positive semidefiniteness here captures the law of supply (that prices and net quantity supplied move in the same direction).

In fact, let's write $Dz(p^z) = D^2\pi(p^z)$ out in full matrix form but using the y, q notation. Remember that $z_k = y_k$ if good k is an output and $-q_k$ if it is an input. For simplicity, we will write $z = (y, q)$ but remember that any ordering of y and q_m's is valid:

$$\begin{pmatrix} \frac{\partial y}{\partial p} & \frac{\partial y}{\partial w_1} & \cdots & \frac{\partial y}{\partial w_M} \\ -\frac{\partial q_1}{\partial p} & -\frac{\partial q_1}{\partial w_1} & \cdots & -\frac{\partial q_1}{\partial w_M} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial q_M}{\partial p} & -\frac{\partial q_M}{\partial w_1} & \cdots & -\frac{\partial q_M}{\partial w_M} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \pi}{\partial^2 p} & \frac{\partial^2 \pi}{\partial w_1 \partial p} & \cdots & \frac{\partial^2 \pi}{\partial w_M \partial p} \\ \frac{\partial^2 \pi}{\partial p \partial w_1} & \frac{\partial^2 \pi}{\partial^2 w_1} & \cdots & \frac{\partial^2 \pi}{\partial w_M \partial w_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial p \partial w_M} & \frac{\partial^2 \pi}{\partial w_1 \partial w_M} & \cdots & \frac{\partial^2 \pi}{\partial^2 w_M} \end{pmatrix}$$

Since we know π is convex, then we know the diagonal elements in its Hessian are non-negative. Therefore $\frac{\partial y}{\partial p} \geq 0$ and $\frac{\partial q_m}{\partial w_m} \leq 0, \forall m$. However, we can't conclude anything about the sign of $\frac{\partial y}{\partial w_m}$ and $\frac{\partial q_m}{\partial p}$ because they are off-diagonal. Due to the symmetry of the Hessian though, we do know that $\frac{\partial y}{\partial w_m} = -\frac{\partial q_m}{\partial p}$ and $\frac{\partial q_m}{\partial w_k} = \frac{\partial q_k}{\partial w_m}$.

3 Cost Minimization Problem

3.1 The Problem

Just as the consumer has an expenditure minimization problem, the producer has a *cost minimization problem* (CMP). For a consumer, expenditure was easily defined as $p \cdot x$ (multiply the quantity of each good by its corresponding price and sum up). For a producer, they only pay the costs of inputs, not outputs. Since a firm pays cost w_k for each input k, their cost function is $w \cdot q = \sum_{m=1}^{M} w_m q_m$. That's not all we need though! In the EMP we also needed to define a minimum utility threshold u, and likewise here we need to define a minimum *output* threshold y. Therefore, the parameters $(w, y) = (w_1, ..., w_M, y)$ define the CMP, where we need to choose inputs $q = (q_1, ..., q_M)$:

$$\min_{q \in \mathbb{R}^M_+} w \cdot q =$$
$$\text{s.t. } f(q) \geq y$$

Be aware of a very big difference between the PMP and CMP. In the PMP, output was variable - it was being endogenously chosen. Here, output is a parameter. So whatever the prices are, we are forced to produce at least y, but we are trying to do so as cheaply as possible.[3] Graphically, in the two input case, think of the isoquant curve for $f(q) = y$ (plotted on q_1-q_2 axis). We can also plot *isocost* lines, which are lines that indicate bundles of goods that *cost* the same (think of this like a budget line). Since costs are linear, then these lines will also be linear (i.e. $w_1 q_1 + w_2 q_2 = c^*$ means plotting the line $q_2 = \frac{1}{w_2}(c^* - w_1 q_1)$ for all possible c^*). We are trying to produce at a point on or above the isoquant curve, but at the lowest isocost line. Obviously this will be at the point where an isocost line is tangent to the isoquant.

Just like in the EMP, we have a solution set and optimal objective function. Keep in mind that our choice variable is q and the parameters are (w, y):

- $q(w, y)$ is the *conditional input demand correspondence*, the solution set of the CMP (analogous to the Hicksian demand). This is demand of inputs, conditional on producing at least y
- $c(w, y)$ is the *cost function*, the optimal objective function of the CMP (analogous to the expenditure function)

Let's look at the FOCs of this problem:

$$\mathcal{L} = w \cdot q + \lambda(y - f(q))$$

$$\frac{\partial \mathcal{L}}{\partial q_m} = w_m - \lambda \frac{\partial f(q^*)}{\partial q_m} = 0$$
$$\implies w_m = \lambda \frac{\partial f(q^*)}{\partial q_m}$$

Moreover, if we take any two inputs q_m and q_l and divide the FOCs, we get:

$$\frac{\partial f(q^*)/\partial q_m}{\partial f(q^*)/\partial q_l} = MRTS_{ml} = \frac{w_m}{w_l}$$

Which is exactly what we saw in the PMP. In fact, if we use the Envelope theorem:

$$\frac{\partial c(w, y)}{\partial y} = \left.\frac{\partial \mathcal{L}}{\partial y}\right|_{q = q(w,y)} = \lambda$$

So λ represents the marginal cost of production, i.e. how much more we will have to pay if we have to increase the minimum threshold y, which we can also see from the way we've written the Lagrangian.

[3] The netput z notation is little bit easier to work with in the PMP, but because we are working only with inputs in the CMP, it's useful to have the y-q notation here

Therefore, we get that $w_m = \frac{\partial c(w,y)}{\partial y} \cdot \frac{\partial f(q^*)}{\partial q_m}$. This is very intuitive: that the cost of input m is equal to how much production changes when q_m changes multiplied by how much optimal costs change when production changes.

Since the constraint is always binding, we can also interpret $\frac{\partial c(w,y)}{\partial y}$ as the cost of increasing production by one unit, which was how we usually think of marginal cost. Let's call marginal cost C_m, which is the cost of an *extra* unit of output. We also have average cost, $C_a = \frac{c(w,y)}{y}$, which is the cost *per* unit of output. These are related because:

$$\frac{\partial C_a}{\partial y} = \frac{\partial(c(w,y)/y)}{\partial y} = \frac{\partial c(w,y)}{\partial y}\frac{1}{y} + c(w,y)\frac{y^{-1}}{\partial y} = \frac{C_m}{y} - \frac{c(w,y)}{y^2} = \frac{C_m - C_a}{y}$$

So if $C_m > C_a$, then $\frac{\partial C_a}{\partial y} > 0$ and average costs are increasing. And if $C_m < C_a$, then $\frac{\partial C_a}{\partial y} < 0$ and average costs are decreasing. In other words, the marginal cost "pulls" the average. You can draw this graphically (or note that if $C_m = C_a$, then $\frac{\partial C_a}{\partial y} = 0$) and see that the marginal cost curve crosses the average cost curve at the minimum of the average cost.

3.2 Properties

Let's go through the properties of the CMP just as we saw in the EMP.

Existence

- A solution always exists for the CMP
  - *Proof:* Let's use Weierstrass Theorem. The objective function $w \cdot q$ is clearly continuous, so we just need to prove that the feasible set is compact (closed and bounded). We have assumed that Y is closed, but it is not bounded. However, note that we can take any $\hat{q}$ such that $f(\hat{q}) > y$. Then (just like in the EMP), we can add the constraint: $w \cdot q \leq w \cdot \hat{q}$, without changing the solution of the problem. This is because if an optimal solution is more expensive than $\hat{q}$, then clearly it is not optimal because $\hat{q}$ is a cheaper alternative that also satisfies the output threshold. So now our constraint set is bounded, and we can apply Weierstrass.

Uniqueness

- If $f(q)$ is strictly quasi-concave (i.e. the set $\{q \geq 0 : f(q) \geq y\}$ is strictly convex for any y), then the solution will be unique
  - *Proof:* Suppose not. Take any $q', q'' \in q(w,y)$. Since they are both feasible, then $q', q'' \in \{q \geq 0 : f(q) \geq y\}$. Since the set is strictly convex, then $\lambda q' + (1-\lambda)q''$ is in the interior of the set for any $\lambda \in (0,1)$, i.e. $f(\lambda q' + (1-\lambda)q'') > y$. Note that $w \cdot q = w \cdot q' = c(p,y)$ (since they are both optimal). But then: $w \cdot [\lambda q' + (1-\lambda)q''] = \lambda w \cdot q' + (1-\lambda) w \cdot q'' = c(p,y)$. But then we have found an input bundle that achieves the optimal expenditure and is strictly above the threshold. As we will show next, in optimality, the constraint should be binding so this is a contradiction (we will be able to find something that is feasible and cheaper).

Binding Constraint

- In optimality, we will have $f(q^*) = y$
  - *Proof:* If not, then $f(q^*) > y$. But since we usually assume DRTS (Y is convex and there is the possibility of inaction), then we can find some $\alpha \in [0,1)$ such that $f(\alpha q^*)$ is feasible and $f(\alpha q^*) \geq y$. But αq^* is clearly cheaper than q^*, contradicting the optimality of q^*.

Production Correspondence

- $q(w, y)$ is homogeneous of degree zero in w. If $f(q)$ is homogeneous of degree 1 (i.e. CRTS), then $q(w, y)$ is also homogeneous of degree zero in y
  - *Proof:* Increasing the costs of all inputs just scales the objective function but has no real effect on the optimal mix of inputs. If $f(q)$ exhibits CRTS, then $q(w, \alpha y)$ needs to satisfy: $f(q(w, \alpha y)) = \alpha y = \alpha f(q(w, y))$ (constraint has to be binding at optimum). By CRTS, $\alpha f(q(w, y)) = f(\alpha q(w, y))$. Putting this together, we get $f(q(w, \alpha y)) = f(\alpha q(w, y))$ and since $f(\cdot)$ is strictly increasing, then $q(w, \alpha y) = \alpha q(w, y)$
- If $f(q)$ is quasi-concave, then $q(w, y)$ is convex. If $f(q)$ is strictly quasi-concave, then $q(w, y)$ is a singleton
  - *Proof:* If $f(q)$ is quasi-concave, then the set $\{q \geq 0 : f(q) \geq y\}$ is convex for any y. Take any two $q', q'' \in q(w, y)$. Then $\lambda q' + (1-\lambda)q'' \in \{q \geq 0 : f(q) \geq y\}, \forall \lambda \in [0, 1]$. Therefore it is feasible for PMP(w, y). Moreover, $w \cdot [\lambda q' + (1-\lambda)q''] = \lambda c(w, y) + (1-\lambda)c(w, y) = c(w, y)$. Hence it is also optimal and therefore $\lambda q' + (1-\lambda)q'' \in q(w, y)$. We proved the singleton property in the uniqueness proof.

Profit Function Properties

- $c(w, y)$ is continuous in w
  - *Proof:* This follows from Berge's Theorem of the Maximum
- $c(w, y)$ is homogeneous of degree one in w. If $f(q)$ is homogeneous of degree 1 (i.e. CRTS), then $c(w, y)$ is also homogeneous of degree one in y
  - *Proof:* Since $q(w, y)$ is homogeneous of degree zero in w, then: $c(\alpha w, y) = \alpha w \cdot q(\alpha w, y) = \alpha w \cdot q(w, y) = \alpha c(w, y)$. If $f(q)$ exhibits CRTS, then $q(w, y)$ is homogeneous of degree 1, and so: $c(w, \alpha y) = w \cdot q(w, \alpha y) = w \cdot \alpha q(w, y) = \alpha c(w, y)$
- $c(w, y)$ is concave in w. If $f(q)$ is concave, then $c(w, y)$ is convex in y (i.e. marginal costs are increasing in y)
  - *Proof (concave in costs):* Take $q \in q(\lambda w + (1-\lambda)w', y)$. Since $f(q) = y$ then it is feasible for the CMP with prices w and w'. Therefore, $w \cdot q \geq c(w, y)$ and $w' \cdot q \geq c(w, y)$. Putting this together, we get that: $c(\lambda w + (1-\lambda)w', y) = [\lambda w + (1-\lambda)w'] \cdot q = \lambda w \cdot q + (1-\lambda)w' \cdot q \geq \lambda c(w, y) + (1-\lambda)c(w', y)$.
  - *Proof (convex in output):* Since $f(q)$ is concave, then $f(\lambda q + (1-\lambda)q') \geq \lambda f(q) + (1-\lambda)f(q') = \lambda y + (1-\lambda)y'$, where $q \in q(w, y)$ and $q' \in q(w, y')$. Therefore, $\lambda q + (1-\lambda)q'$ is feasible (but not necessarily optimal) for the PMP with parameters $(w, \lambda y + (1-\lambda)y')$. Therefore $c(w, \lambda y + (1-\lambda)y') \leq w \cdot (\lambda q + (1-\lambda)q') = \lambda w \cdot q + (1-\lambda)w \cdot q' = \lambda c(w, y) + (1-\lambda)c(w, y')$.
- $c(w, y)$ is weakly increasing in w and strictly increasing in y
  - *Proof (costs):* Take $w \leq w'$ with $q \in q(w, y)$ and $q' \in q(w', y)$. The change in w doesn't change the feasible set. Therefore, $c(w, y) = w \cdot q \leq w \cdot q' \leq w' \cdot q' = c(w', y)$ (the first inequality comes from the fact that q' is feasible but may not be optimal for CMP(w, y) and the second inequality follows from $w \leq w'$).
  - *Proof (output):* Take $y \leq y'$ with $q \in q(w, y)$ and $q' \in q(w, y')$. Since the constraint is binding in optimum and $f(\cdot)$ is strictly increasing, then $f(q') > f(q) = y$. So $q' \notin q(w, y)$, and then: $c(w, y) = w \cdot q < w \cdot q' = c(w, y')$

Differentiability Results

- *Shephard's Lemma:* $\frac{\partial c(w,y)}{\partial w_m} = q_m(w,y)$ for any input m. In vector notation, $\nabla_w c(w,y) = q(w,y)$
  - *Proof:* Use the Envelope Theorem. $\partial_{w_m} c(w,y) = \partial_{w_m} \mathcal{L}|_{q=q(w,y)} = \partial_{w_m}(w \cdot q + \lambda(y - f(q))|_{q=q(w,y)} = q_m|_{q=q(w,y)} = q_m(w,y)$
- $D_w q(w,y) = D_w^2 c(w,y)$ is symmetric and negative semidefinite matrix with $D_w q(w,y) w = 0$
  - *Proof:* Start with Shephard's Lemma and take derivatives on both sides with respect to price. As usual, a Hessian is symmetric and since $c(w,y)$ is concave in w, it is also negative semidefinite. For the orthogonality, use the same trick: $q(\alpha w, y) = q(w,y)$ and take derivatives on both sides with respect to α.
  - Note that this tells us that $\frac{\partial q_m(w,y)}{\partial w_m} \leq 0, \forall m$ and that substitution terms are symmetric, i.e. $\frac{\partial q_m(w,y)}{\partial w_k} = \frac{\partial q_k(w,y)}{\partial w_m}$