Last updated: December 7, 2017

PhD Micro (Part 2)
Market Failures

Motaz Al-Chanati

December 5, 2017

1 Public Goods

1.1 Framework

So far in the course, the market structures we studied were efficient (i.e. they gave us Pareto optimal allocations, as per the First Welfare Theorem). In this last part, we're going to look where this doesn't happen (we call these situations *market failures*). Essentially we would like to know when the First Welfare Theorem fails to hold, or equivalently, when the *assumptions* behind the FWT are broken. We actually already saw one example: the case of incomplete markets, which we saw in the uncertainty section. In this part, we will look at two more types of market failures - public goods and externalities.

We start by studying public goods. Up until now, we've made an implicit assumption that consumption of goods is mutually exclusive; either you consume the good, or I consume it (we can't both consume it). We also assumed that to increase your utility, you have to buy the good yourself. In public goods, these two assumptions no longer hold.

> **Public Good:**
> A (pure) public good is a good that satisfies two conditions:
> 1. *Non-rival:* Each agent's consumption of a good does not reduce the availability of that good for other agents
> 2. *Non-excludable:* Once a good has been made available for one agent, it becomes available to all agents

Some examples of public goods are lighthouses and national defense. These assumptions are hard to apply perfectly in real life. For example, a road is a common example of a public good, but you can easily think of traffic/congestion as violating the non-rivalry assumption and toll roads as violating non-exclusion. We're only going to be thinking about *pure* public goods, where both these assumptions hold perfectly.

In this economy, we will still have I consumers, J firms, and L *private* goods (i.e. the same type of goods that we've always been considering so far). Now we are going to introduce M public goods, where X^m denotes the quantity of public good m that is available in the economy. For a consumer i, they get utility from <u>their allocation</u> of the private goods $(x_{1i}, ..., x_{Li})$ and the <u>total allocation</u> of the public goods $(X^1, ..., X^M)$. Notice that the private goods have an i subscript, but the public goods do not -

this captures the fact that everyone has perfect access to the public goods. Moreover, I index the public goods with a superscript so that you remember it represents a total allocation and not something that belongs to an individual.

We also need to think about the production side for public goods, where we will make three key assumptions:

1. All public goods are provided by one firm (e.g. the government)
2. There are no initial endowments of the public goods (so the only way to get a public good is to produce it)
3. Public goods are produced from private goods

This means we will need just one production function for public goods of the form $G(z, X)$, where $z = (z_1, ..., z_L) < 0$ is the input of private goods and $X = (X^1, ..., X^M) > 0$ is the output of public goods

So to summarize, the economy is characterized by the following:

- I consumers, J firms, L private goods, M public goods
- Consumer i has utility $u_i(x_i, X) = u_i(x_{1i}, ..., x_{Li}, X^1, ..., X^M)$
- Producer j has transformation function $F_j(y_j) \leq 0$
- Public good transformation function is $G(z, X) \leq 0$

Our end goal is to see that markets are inefficient here. To do this, we first have to characterize a Pareto optimal allocation. We then characterize the market allocation, and see what makes the two different.

1.2 Pareto Optimality

To find a Pareto optimal allocation, we just use the same approach as before: maximize one consumer's utility subject to a minimum utility for all other consumers and feasibility constraints. All we have to change in the problem is to include the public good transformation function to capture feasible allocations of the public good.

Consider the maximization problem given by:

$$\begin{aligned}
& & \max u_1(x_1, X) & & \\
\text{s.t.} & & u_i(x_i, X) \geq \bar{u}_i & & \forall i = 2, ..., I \\
& & F_j(y_j) \leq 0 & & \forall j = 1, ..., J \\
& & G(z, X) \leq 0 & & \\
& & \sum_{i=1}^{I} x_{li} \leq \omega^l + z_l + \sum_{j=1}^{J} y_j & & \forall l = 1, ..., L
\end{aligned}$$

The Lagrangian for this problem is: (setting $\lambda_1 = 1$ and $\bar{u}_1 = 0$)

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \left[u_i(x_i, X) - \bar{u}_i\right] + \sum_{j=1}^{J} \gamma_j \left[-F_j(y_j)\right] + \eta \left[-G(z, X)\right] + \sum_{l=1}^{L} \mu_l \left[\omega^l + z_l + \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} x_{li}\right]$$

Notice that there is only one multiplier on the public good constraint (because there is only transformation function). As in the standard case, we need to choose x_{li} and y_{lj}, but we now also need to choose X^m and z_l. Let's do the FOCs for each of these.

The FOC for x_{li} gives us (as before):

$$\lambda_i \frac{\partial u_i(x_i, X)}{\partial x_{li}} = \mu_l \tag{1}$$

And the FOC for y_{lj} will also be the same as what we saw before:

$$\gamma_j \frac{\partial F_j(y_j)}{\partial y_{lj}} = \mu_l \tag{2}$$

So far nothing is different, which means that we can use equations (1) and (2) to get exactly the same conditions as before: $MRS^i_{lk} = MRS^m_{lk}$ (distributive efficiency), $MRT^j_{lk} = MRT^n_{lk}$ (productive efficiency) and $MRS^i_{lk} = MRT^j_{lk}$ (aggregate efficiency) for all consumers i, m, producers j, n and goods l, k.

Let's now do the FOC for X^m. You have to be careful here because it appears in the utility of *every* consumer as well as the production constraint. This gives us:

$$\sum_{i=1}^{I} \lambda_i \frac{\partial u_i(x_i, X)}{\partial X^m} = \eta \frac{\partial G(z, X)}{\partial X^m} \tag{3}$$

Finally, we do the FOC for z_l, which appears in the public good production and the market clearing conditions:

$$\mu_l = \eta \frac{\partial G(z, X)}{\partial z_l} \tag{4}$$

To bring this all together, divide equation (3) by equation (4) to get rid of η:

$$\sum_{i=1}^{I} \lambda_i \frac{\partial u_i(x_i, X)/\partial X^m}{\mu_l} = \frac{\partial G(z, X)/\partial X^m}{\partial G(z, X)/\partial z_l}$$

To get rid of the remaining multipliers λ_i and μ_l, re-arrange equation (1):

$$\frac{\lambda_i}{\mu_l} = \frac{1}{\partial u_i(x_i, X)/\partial x_{li}}$$

And plug this into the above, giving us:

$$\sum_{i=1}^{I} \frac{\partial u_i(x_i, X)/\partial X^m}{\partial u_i(x_i, X)/\partial x_{li}} = \frac{\partial G(z, X)/\partial X^m}{\partial G(z, X)/\partial z_l} \tag{5}$$

So now we have an extra set of conditions that need to be met in Pareto optimality. We call equation (5) the *Bowen-Lindahl-Samuelson* (BLS) conditions, which must hold for any public good m and private good l. Another way to express this is to notice that the terms inside the sum are simply the marginal rates of substitution between (public) good m and (private) good l. Similarly, the RHS is just the marginal rate of transformation between (public) good m and (private) good l. So we can write the BLS condition as:

$$\sum_{i=1}^{I} MRS_{ml}^{i} = MRT_{ml}, \forall m = 1, ..., M; l = 1, ..., L$$

This derivation allows us to characterize the Pareto optimal allocations in an economy with private goods (assuming an interior solution).

Pareto Optimal Allocation with Public Goods
In a *smooth and convex* economy with public goods, a Pareto optimal allocation must satisfy the following conditions:

- **Distributive Efficiency:** For any $i, m \in \{1, ..., I\}$ and $l, k \in \{1, ..., L\}$
$$MRS_{lk}^{i} = MRS_{lk}^{m}$$
- **Productive Efficiency:** For any $j, n \in \{1, ..., J\}$ and $l, k \in \{1, ..., L\}$
$$MRT_{lk}^{j} = MRT_{lk}^{n}$$
- **Aggregate Efficiency:** For any $i \in \{1, ..., I\}$, $j \in \{1, ..., J\}$ and $l, k \in \{1, ..., L\}$
$$MRS_{lk}^{i} = MRT_{lk}^{j}$$
- **Bowen-Lindahl-Samuelson Conditions:** For any $m \in \{1, ..., M\}$ and $l \in \{1, ..., L\}$
$$\sum_{i=1}^{I} MRS_{ml}^{i} = MRT_{ml}$$

where MRS_{lk}^{i} is marginal rate of substitution for consumer i between goods l and k, MRT_{lk}^{j} is marginal rate of transformation for firm j between private goods l and k, and MRT_{ml} is marginal rate of transformation between public good m and private good l.

The BLS conditions have a very nice interpretation. In essence, at an optimality we should see that the marginal benefit of producing an extra unit of public good should be equal to the cost of that extra unit of production. The cost is captured by the RHS: it tells us how much more of the private good l we need to use to produce one more unit of public good m (this is just a standard interpretation of MRT). On the LHS, we are seeing the benefit: how much of the private good l are consumers willing to give up to get an extra unit of public good m (standard interpretation of MRS). However, since every consumer benefits from more public goods, we need to sum up the valuations. So the LHS tells us the total quantity of good l all the consumers are willing to give up to get one more unit of m and this has to be equal to the extra unit of good l needed to produce that unit. In other words: the sum of the marginal benefits equals the marginal cost.

1.3 Walrasian Equilibrium

Now we look at what happens if the market tries to provide the public good. We call this *decentralized provision*. We have to be a little careful with how we define the market structure. I think the best real-life analogy is to think about going to a museum, where the ticket price is however much you're willing to pay (e.g. The Met).[1] You can choose to pay your true MWTP or you could pay as little as \$0.01, which reflects how much you value the public good (though there is no expectation that you have to be honest).

This is not quite how we will do things here, but the general idea is the same. In the museum example, the quantity was fixed (1 all-day entrance to the museum) and consumers were able to choose their prices. We tend to think about this the other way: the price is fixed and consumers choose how much they will consume (or in this case, contribute). In this analogy, imagine if the museum charged a price of \$10 per hour for a visit and you had to tell them how long your visit would be. However, this would not be enforceable. For example, you could say you only wanted a 30 minute visit (and thus pay \$5), but then spend the whole day there.

How should the museum (the firm) respond? Suppose there are three consumers (A,B,C). A demands 1 hour of the museum, B demands 5 hours and C demands 2 hours (or at least that's what they announce to the museum). So the total hours demanded is 8 hours, at a price of \$10/hour. Of course, adding up the hours like this suggests that the visits are mutually exclusive (its almost as if each person was getting a private tour of the museum), but that's not what is happening here. To keep things simple, suppose it costs the firm \$10/hour to keep the museum open. The total amount paid by the consumers is \$80/hour, so it has received enough revenue to keep the museum open for 8 hours. Does it then really matter who comes to the museum and when? As long as the firm gets the money to produce the good, it doesn't really care who ends up consuming it. So from the firm's perspective, its best to still treat it like a private good.

What about the consumers? If B is going to demand 5 hours of museum visits, and A and C know that B will do this, then wouldn't it make more sense for them to say that they demand 0 hours? Once the museum opens up for 5 hours, A and C are free to use it as much as they please. Of course, if B thinks that A and C will announce a total demand of 3 hours, then shouldn't she only announce that she wants the museum open for 2 hours (and then it will open for 5 hours just as she would actually want)? This type of logic keeps going, but in equilibrium, nobody will want to deviate (remember this is the idea of a best response, which you will see a lot of next semester in game theory).

To capture these dynamics, we will introduce the notation $X_i = (X_{1i}, ..., X_{Mi})$, where X_{mi} represents consumer i's contribution of public good m. Notice I am calling this a "contribution" and not consumption. In the end, since the total allocation of the public good is available to all, each consumer will end up getting utility from the full amount of public good provided. Let's also have one producer of public goods (a profit maximizing firm), whose outputs are $(Y_1, ..., Y_M)$. Since there are no initial endowments of public goods, we must have for each public good m:

$$X_{m1} + ... + X_{mI} = Y_m$$

Using our old notation, we can let $X^m = \sum_i X_{mi}$ and so $X^m = Y_m$ (now you can see the superscript notation is useful because it indicates a sum not a vector). For a consumer, their utility is $u_i(x_i, X) = u_i(x_{1i}, ..., x_{Li}, X^1, ..., X^M)$. This means that their utility not only depends on their own X_{mi} but also $X_{mk}, \forall k \neq i$.

In this setting, we will need two key assumptions:

1. The price for public good m is q_m. The price vector for all public goods is $q = (q_1, ..., q_m)$
2. When consumers make their decision of X_{mi}, they take everyone else's demand $(\sum_{k\neq i} X_{mk})$ of the public good as given and fixed.

[1] Of course, a museum is not a pure public good because it is quite easy to exclude people. However, I think the intuition comes out nicely here so just bear with me

Some final notation points. We'll use the notation $X_{-i} = \sum_{k \neq i} X_k$, where the "$-i$" indexes "everybody except i". Finally, since it is not really our focus, let's make consumer i's income from firm profits as $\pi_i(p, q)$ to keep the notation is simple. Now that we have all this, we can define a competitive equilibrium with public goods.

Walrasian Equilibrium with Public Goods

A Walrasian equilibrium with public goods is an allocation (x^*, y^*, z^*) of L private goods, an allocation (X^*, Y^*) of M public goods, and corresponding price vectors p^*, q^* such that:

1. For each consumer i, (x_i^*, X_i^*) is the solution to the following utility maximization problem (UMP) with prices (p^*, q^*):

$$\max_{x_i, X_i} u_i\left(x_i, X_i + X_{-i}^*\right)$$
$$\text{s.t. } p^* \cdot x_i + q^* \cdot X_i \leq p^* \cdot \omega_i + \pi_i(p^*, q^*)$$
$$\text{and } X_{mi} \geq 0, \forall m$$

2. For each firm j, y_j^* is the solution to the following profit maximization problem (PMP) with price p^*:

$$\max_{y_j} p^* \cdot y_j$$
$$\text{s.t. } F_j(y_j) \leq 0$$

3. For the public good producing firm, (Y^*, z^*) is the solution to the following profit maximization problem (PMP) with prices (p^*, q^*):

$$\max_{Y, z} q^* \cdot Y - p^* \cdot z$$
$$\text{s.t. } G(Y, z) \leq 0$$

4. The allocation is feasible (markets clear):

$$\sum_{i=1}^{I} x_{li}^* \leq \omega^l + z_l^* + \sum_{j=1}^{J} y_{lj}^*, \forall l$$
$$\sum_{i=1}^{I} X_{mi}^* \leq Y_m^*, \forall m$$

This is basically our usual definition of a Walrasian equilibrium. The biggest difference comes in the fact that there is now a profit-maximizing firm producing the public good (but other than having two different price vectors for the two types of goods, it is a fairly standard maximization problem). The other major difference is in the consumer's utility function. Notice that they are choosing (x_i, X_i) given everybody else's equilibrium public good demands $X_{-i}^* = \sum_{k \neq i} X_k^*$ (the fact that the X_k's have a $*$ indicates that consumer's are best-responding to each other in equilibrium).

We've seen this setup so many times that we don't need to go through and fully solve it again. But, you do have to be careful with corner solutions here! Normally we assume an interior solution because it usually doesn't make sense for someone to consume 0 of a good. With public goods, however, you can *demand* no units ($X_{mi} = 0$) but *consume* non-zero units ($X^m > 0$), since your consumption relies on total demand. To remind you of this, I've include the non-negativity constraint in the consumers UMP.

Let's look at the consumer's FOC for X_{mi} in their UMP. Remember that we can express their utility as $u_i(x_i, X) = u_i(x_{1i}, ..., u_{Li}, X^1, ..., X^M)$, and since $\frac{\partial X^m}{\partial X_{mi}} = \frac{\partial}{\partial X_{mi}} \sum_{k=1}^{I} X_{mk} = 1$, then:

$$\frac{\partial u_i}{\partial X_{mi}} = \frac{\partial u_i}{\partial X^m} \underbrace{\frac{\partial X^m}{\partial X_{mi}}}_{=1} = \frac{\partial u_i}{\partial X^m}$$

The Lagrangian for the UMP (including non-negativity constraint for public goods) is:

$$\mathcal{L} = u_i\left(x_i, X\right) + \lambda\Big(p \cdot \omega_i + \pi_i(p, q) - p \cdot x_i - q \cdot X_i\Big) + \sum_{m=1}^{M} \mu_m\left(X_{mi}\right)$$

The FOC for X_{mi} is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X_{mi}} &= \frac{\partial u_i}{\partial X_{mi}} + \lambda(-q_m) + \mu_m = 0 \\ \implies \frac{\partial u_i}{\partial X^m} &= \lambda q_m - \mu_m \end{aligned}$$

Recall that $\mu_m \geq 0$ and by complementary slackness if $X_{mi} > 0$ then $\mu_m = 0$ (and if $X_{mi} = 0$ then $\mu_m > 0$). Therefore:

$$\begin{aligned} \frac{\partial u_i}{\partial X^m} &= \lambda q_m \quad \text{if } X_{mi} > 0 \\ \frac{\partial u_i}{\partial X^m} &< \lambda q_m \quad \text{if } X_{mi} = 0 \end{aligned}$$

Let's consider the consumers who contribute to the public good m (i.e. have an interior solution) and call this set of people K_m. Then for any one of these people, we can derive the following condition for the public good m and a private good l: (by plugging in $\partial u_i / \partial x_{li} = \lambda p_l$)

$$MRS^i_{ml} = \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} = \frac{q_m}{p_l}$$

This is not surprising, we know that MRS should equal price ratio (this should hold at an interior regardless of whether the good is public or private). Denote the size of the set of contributers K_m as k_m ($|K_m| = k_m$). This allows us to then say:

$$\begin{aligned} \sum_{i=1}^{I} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} &= \sum_{i \in K_m} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} + \sum_{i \notin K_m} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} \\ &= \sum_{i \in K_m} \frac{q_m}{p_l} + \sum_{i \notin K_m} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} \\ &= k_m \frac{q_m}{p_l} + \underbrace{\sum_{i \notin K_m} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}}}_{\in \left[0, (I - k_m) \frac{q_m}{p_l}\right)} \\ &\geq k_m \frac{q_m}{p_l} \end{aligned}$$

Where the inequalities relies on the fact that $\frac{q_m}{p_l}$ is positive. So if there is more than one person who contributes to the public good, i.e. if $k_m > 1$, then:

$$\sum_{i=1}^{I} MRS_{ml}^{i} = \sum_{i=1}^{I} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial x_{li}} > \frac{q_m}{p_l} \tag{6}$$

Now we consider the firm that produces public goods. Its PMP is quite standard so we should get MRT equal to the price ratio:

$$MRT_{ml} = \frac{\partial G(Y,z)/\partial Y_m}{\partial G(Y,z)/\partial z_l} = \frac{q_m}{p_l} \tag{7}$$

Putting (6) and (7) together, we get that:

$$\sum_{i=1}^{I} MRS_{ml}^{i} > MRT_{ml}$$

But this violates the BLS conditions! In a Pareto optimal allocation, we must have this hold by equality. Since it is $>$, then it indicates that the total marginal benefit outweighs the marginal cost of an additional unit of public good. Therefore when people voluntarily contribute to the public good, we actually have under-provision of the public good. This is the *free-rider problem.* Consumers enjoy the benefits of the public good that others have contributed (at no cost to them), but they don't want to contribute themselves because it is costly. Since consumers fail to internalize that their contribution to the public good would in fact benefit everyone else, we end up with a Pareto inefficient allocation.

1.4 Lindahl Pricing

In the competitive equilibrium setting, every consumer paid the same price for the public good. Under *Lindahl pricing*, each consumer faces a personalized price q_{mi} for public good m. You can think of X_{mi} as a personalized public good. What this essentially does is create markets for each person's consumption of public goods. The consumer's UMP becomes:

$$\max_{x_i, X_i} u_i\left(x_i, X_i\right)$$
$$\text{s.t. } p \cdot x_i + q_i \cdot X_i \leq p \cdot \omega_i + \pi_i(p,q)$$

Where $q_i = (q_{1i}, ..., q_{Mi})$ is the personalized price vector and $X_i = (X_{1i}, ..., X_{Mi})$ is the vector of personalized public goods.

Now, the FOCs with respect to X_{mi} and x_{li} would give us:

$$MRS_{ml}^{i} = \frac{\partial u_i / \partial X^m}{\partial u_i / \partial X_{li}} = \frac{q_{mi}}{p_l}$$

Suppose these prices are set such that every consumer demands a strictly positive amount $X_{mi} > 0$. Then adding the FOCs across all consumers gives:

$$\sum_{i=1}^{I} \frac{\partial u_i / \partial X^m}{\partial u_i / \partial X_{li}} = \frac{\sum_{i=1}^{I} q_{mi}}{p_l}$$

For the firm, they now need to not just make a public good Y_m, but rather produce personalized public goods for every consumer: $Y_{m1}, ..., Y_{mI}, \forall m$. But rather than dealing with $M \times I$ different outputs, we make a simplifying assumption that the firm's production of one unit of a public good creates one unit of a personalized public good for all consumers. You can think of this as a Leontif production function, where the output is $Y_m = \min(Y_{m1}, ..., Y_{mI})$ so that $Y_m = Y_{m1} = ... = Y_{mI}$, or alternatively that the price of one unit of public good Y_m is equal to $\sum_i q_{mi}$. This means that the firm's PMP is:

$$\max_{Y,z} \sum_{m=1}^{M} \left(\left(\sum_{i=1}^{I} q_{mi} \right) Y_m \right) - p^* \cdot z$$
$$\text{s.t. } G(Y, z) \leq 0$$

The FOCs would then give us:

$$MRT_{ml} = \frac{\partial G / \partial Y_m}{\partial G / \partial z_l} = \frac{\sum_{i=1}^{I} q_{mi}}{p_l}$$

Putting this together with the consumer FOCs, gives us exactly the BLS conditions:

$$\sum_{i=1}^{I} MRS_{ml}^{I} = \frac{\sum_{i=1}^{I} q_{mi}}{p_l} = MRT_{ml}$$

Therefore, under Lindahl pricing, we get a *Lindahl equilibrium* where the BLS conditions are satisfied and hence the decentralized outcome is efficient. The point of the Lindahl equilibrium is that we can get an efficient market allocation - we just had to define the right markets!

We need to set the personalized prices equal to the consumer's marginal willingness to pay and then consumers will reveal their true preferred allocation of the public good. Notice that the utility function in the UMP is $u_i(x_i, X_i)$. This means each person's demand $X_{mi}(p, q)$ indicates how much of the personalized public good they want available. But since markets have to clear ($X_{mi} = Y_{mi}$) and production is in fixed-proportions ($Y_{mi} = Y_{mk}, \forall i, k$), then it must the case that all consumers demand the same amount of public good ($X_{m1} = ... = X_{mI}$). The personalized prices will adjust so that everyone ends up agreeing on the quantity of public good demanded.

There are some issues with this setup. The utility function is $u_i(x_i, X_i)$ and not $u_i(x_i, X)$. Each market for good X_{mi} is a market for person i's consumption of public good m. In other words, the only way consumers will get to consume a public good is by purchasing the personalized public good. This means that you have to be able to exclude people who do not purchase it - but this contradicts the idea of a pure public good! If we don't do this, then we just get back to the same problem as before. The pricing system is also quite bizarre. Only consumer i participates in the market for X_{mi} and yet they are treated as price-takers. Moreover, it takes a lot of information to properly set the prices. You have to able to see their true preferences, otherwise a consumer has an incentive to lie and say that they don't care much about the public good (and hence get a low q_i).

2 Externalities

2.1 Framework

Now we move onto another form of market failure - externalities. This occurs when the actions of one agent has an impact on another agent. There are a lot of ways we can think about this, but our framework here will be a simple model of pollution.

In this economy, there are I consumers, $J = 2$ firms, and $L = 2$ goods. Good 2 is a "dirty" good, in that its presence creates pollution, which affects the consumers. The economy is then characterized by:

- Consumer i's utility over both goods and pollution: $u_i(x_{1i}, x_{2i}, P)$, where $\frac{\partial u_i}{\partial P} < 0, \forall i$ (consumer's don't like pollution)
- The firm's produces good 2 out of good 1. To produce y_2 units of good 2, the firm needs to have at least $C(y_2)$ units of good 1. Therefore its production set is captured by: $y_1 + C(y_2) \leq 0$.[2]
- Pollution is created from good 2: $P = ky_2$, with $k \geq 0$
- The total endowment of the economy is $(\omega^1, 0)$, i.e. there is no good 2 initially in the economy

2.2 Pareto Optimality

Let's find the Pareto optimal allocation. We will setup the maximization problem:

$$\begin{aligned}
&\max u_1(x_{11}, x_{21}, P) \\
\text{s.t.} \quad & u_i(x_{1i}, x_{2i}, P) \geq \bar{u}_i \quad \forall i = 2, ..., I \\
& y_1 + C(y_2) \leq 0 \\
& P = ky_2 \\
& \sum_{i=1}^{I} x_{1i} \leq \omega^1 + y_1 \\
& \sum_{i=1}^{I} x_{2i} \leq y_2
\end{aligned}$$

To make this a bit simpler, plug in $y_1 = -C(y_2)$ into the feasibility constraint for good 1 and $P = ky_2$ into the utility functions. The Lagrangian for this problem is then: (setting $\lambda_1 = 1$ and $\bar{u}_1 = 0$)

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \left[u_i(x_{1i}, x_{2i}, ky_2) - \bar{u}_i\right] + \mu_1 \left[\omega^1 - C(y_2) - \sum_{i=1}^{I} x_{1i}\right] + \mu_2 \left[y_2 - \sum_{i=1}^{I} x_{2i}\right]$$

Now all we need to choose is x_{1i}, x_{2i}, y_2. Let's do the FOCs for the consumption goods, which are fairly standard:

$$\begin{aligned}
x_{1i}: \quad \lambda_i \frac{\partial u_i}{\partial x_{1i}} &= \mu_1 \\
x_{2i}: \quad \lambda_i \frac{\partial u_i}{\partial x_{2i}} &= \mu_2
\end{aligned}$$

The interesting one is the FOC for y_2. Notice it appears in many places: the third argument for *all* the utilities, the feasibility constraint for good 1 (through the cost function), and the good 2 feasibility

[2] This cost function approach rather than using a transformation function is something we saw in Problem Set 1

constraint. This gives us: (I'm going to change the index for the consumer sum, it'll be clearer later on)

$$\sum_{n=1}^{I}\left(\lambda_n \frac{\partial u_n}{\partial P}\cdot\frac{\partial[ky_2]}{\partial y_2}\right)-\mu_1\frac{\partial C(y_2)}{\partial y_2}+\mu_2=0$$

Do some re-arranging and eliminate μ_l using the corresponding x_{li} FOC for an arbitrary consumer i:

$$\begin{aligned}
\mu_2 &= \mu_1 C'(y_2)-\sum_{n=1}^{I}\lambda_n\frac{\partial u_n}{\partial P}k\\
\frac{\mu_2}{\mu_1} &= C'(y_2)-k\sum_{n=1}^{I}\frac{\lambda_n}{\mu_1}\frac{\partial u_n}{\partial P}\\
\frac{\lambda_i\partial u_i/\partial x_{2i}}{\lambda_i\partial u_i/\partial x_{1i}} &= C'(y_2)-k\sum_{n=1}^{I}\frac{\lambda_n}{\lambda_n\partial u_n/\partial x_{1n}}\frac{\partial u_n}{\partial P}\\
\frac{\partial u_i/\partial x_{2i}}{\partial u_i/\partial x_{1i}} &= C'(y_2)-k\sum_{n=1}^{I}\frac{\partial u_n/\partial P}{\partial u_n/\partial x_{1n}}
\end{aligned}$$

Notice if we had $k=0$ (no pollution), then this equation would just be $MRS_{21}^i = MRT_{21}$. But now, we can interpret this equation as:

$$\underbrace{\frac{\partial u_i/\partial x_{2i}}{\partial u_i/\partial x_{1i}}}_{\text{Private MB}} = \underbrace{\underbrace{C'(y_2)}_{\text{Private MC}} \underbrace{-k\sum_{n=1}^{I}\frac{\partial u_n/\partial P}{\partial u_n/\partial x_{1n}}}_{\text{Pollution Cost}}}_{\text{Marginal Social Cost}}$$

These are marginal benefits and costs in terms of good 1. Notice that the externality of the pollution makes the social marginal cost *higher* than the private marginal cost (this is because $\frac{\partial u_i}{\partial P}<0$). This gets to the general idea that in a Pareto optimum with externalities, we need Social MB = Social MC. When there are no externalities, there are no differences between private and social marginal benefits/costs.

2.3 Walrasian Equilibrium

In this part we see what a Walrasian equilibrium gives us and how that diverges from the Pareto optimum. The conditions are exactly as before:

1. Consumers chooses x_i^* to solve their UMP, *taking P as given*:

$$\max_{(x_{1i},x_{2i})} u_i(x_{1i},x_{2i},P^*)$$
$$\text{s.t. } p_1^*x_{1i}+p_2^*x_{2i}\le p_1^*\omega_{1i}+\theta_i p^*\cdot y^*$$

2. The firm chooses y^* to solve their PMP:

$$\max_{y} p_1y_1+p_2y_2$$
$$\text{s.t. } y_1+C(y_2)\le 0$$

3. Feasibility constraints are satisfied:

$$\sum_{i=1}^{I} x_{1i}^* \leq \omega^1 + y_1^*$$
$$\sum_{i=1}^{I} x_{2i}^* \leq y_2^*$$

4. Pollution constraint is satisfied:

$$P^* = ky_2^*$$

The key part is that consumers do not realize that they can affect the level of pollution. As they increase their demand of good 2, the firm will supply more of it, which in turn will cause more pollution. However, they don't realize this and instead just treat P as fixed.

Looking at this setup, you can see that nothing major has really changed. For any consumer, we must still have $MRS_{lm}^i = \frac{p_l}{p_m}$. For the firm, we must still have $MRT_{lm} = \frac{p_l}{p_m}$. Therefore, we will have:

$$MRS_{21}^i = \frac{\partial u_i / \partial x_{2i}}{\partial u_i / \partial x_{1i}} = C'(y_2) = MRT_{21}$$

This is just equating private MB to private MC, which then means if we evaluate the condition for Pareto optimality, we get $<$ rather than an equality (remember that private MC $<$ social MC, due to the externality):

$$\frac{\partial u_i / \partial x_{2i}}{\partial u_i / \partial x_{1i}} < C'(y_2) - k \sum_{n=1}^{I} \frac{\partial u_n / \partial P}{\partial u_n / \partial x_{1n}}$$

Clearly you can see that this distortion means that a Walrasian equilibrium is not Pareto optimal (and hence the FWT fails to hold).

2.4 Piguovian Tax

We want to make the firm "internalize the externality" (realize that their production of y_2 is harming other agents). To do this, we will implement a *Pigouvian tax*, which is a tax that the firm pays for its pollution. This is modeled as the firm paying q dollars per unit of pollution. The revenue raised from the tax is redistributed to the consumers, i.e. each consumer gets T_i dollars from the tax revenue such that $\sum_i T_i = qP$.

The consumer's problem doesn't change in this setting (they get a boost of income from the tax revenue, but the FOCs remain the same). That means we still get $MRS_{lm}^i = \frac{p_l}{p_m}$.

For the firm, their PMP now becomes:

$$\max_{y_1, y_2, P} p_1 y_1 + p_2 y_2 - qP$$
$$\text{s.t. } y_1 + C(y_2) \leq 0$$
$$\text{and } P = ky_2$$

Which can equivalently be expressed as:

$$\max_{y_2} p_2 y_2 - p_1 C(y_2) - qky_2$$

This means that the FOC for y_2 is:

$$\begin{aligned}
&p_2 - p_1 C'(y_2) - qk = 0 \\
\implies &\frac{p_2}{p_1} = C'(y_2) - k\frac{-q}{p_1}
\end{aligned}$$

This means that we have:

$$MRS^i_{21} = C'(y_2) - k\frac{-q}{p_1}$$

Which if you compare it to the Pareto optimality condition, tells us we need:

$$\begin{aligned}
\frac{-q}{p_1} &= \sum_{n=1}^{I} \frac{\partial u_n/\partial P}{\partial u_n/\partial x_{1n}} \\
\implies q^* &= -p_1 \sum_{n=1}^{I} \frac{\partial u_n/\partial P}{\partial u_n/\partial x_{1n}}
\end{aligned}$$

Setting the tax in this way will ensure that the competitive equilibrium is Pareto efficient. The interpretation for this is that q is set to be equal to the social cost of the externality. The sum represents the pollution cost in terms of good 1, so multiplying by p_1 converts it into a dollar value (and the minus sign makes it a positive number since $\frac{\partial u_i}{\partial P} < 0$).