

# Intermediate Micro: Recitation 5

## More Optimal Choice

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## 1 Quasi-Linear Utility

### 1.1 General Results

Recall what we know about quasi-linear utility:

- General Form:  $u(x_1, x_2) = f(x_1) + x_2$ , where  $f(x_1)$  is some (non-linear) function of  $x_1$
- Indifference Curves: Look mostly like standard ICs but often wider. A key property is that each IC is just a parallel shift of other ICs
- Preference: Generally a mixer, whose MRS is unaffected by how much  $x_2$  they have

Quasi-linear is special in that the MRS will only be a function of  $x_1$  (since  $x_2$  is the numeraire - the variable that enters linearly). Let the derivative of  $f(x)$  be  $f'(x)$ . Then we can write the optimality condition as:

$$|MRS_{12}| = \frac{MU_1}{MU_2} = \frac{f'(x_1)}{1} = \frac{p_1}{p_2}$$

Normally the optimality condition is one equation with two unknowns. But here, it's one equation with *one* unknown. That means that we can just solve directly for  $x_1$ :

$$\begin{aligned} f'(x_1) &= \frac{p_1}{p_2} \\ \therefore x_1 &= f'^{-1}\left(\frac{p_1}{p_2}\right) \end{aligned}$$

This is a little bit of an abuse of notation. By  $f'^{-1}(x)$ , I mean the inverse of the derivative of  $f(x)$ . This gives us the demand function  $x_1(p, M)$  that we set out for! Notice that it is  $x_1$  as a function of the parameters - but only the prices, it is not a function of income. This means that the optimal choice of  $x_1$  is independent

of income. So if we have  $x_1^*$ , how do we get  $x_2^*$ ? Just like before, we plug what we know into the budget constraint. So plugging in the  $x_1(p, M)$  we solved for gives us  $x_2(p, M)$ :

$$\begin{aligned} p_1 x_1(p, M) + p_2 x_2 &= M \\ p_2 x_2 &= M - p_1 x_1(p, M) \\ \therefore x_2^* &= \frac{1}{p_2} \left( M - p_1 f'^{-1} \left( \frac{p_1}{p_2} \right) \right) \end{aligned}$$

This basically says to us: find your optimal  $x_1^*$  from the optimality condition, and whatever you don't spend on  $x_1$ , put it all towards  $x_2$ . Since  $x_1^*$  does not depend on income, as your income increases, you spend a higher proportion of your income on  $x_2$ .

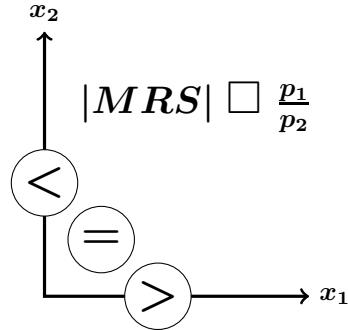
However, you may notice there's a small flaw in our logic. Clearly  $x_1^*$  has to depend at least somewhat on income. Suppose you calculate  $x_1^* = 2$ , and you also have  $p_1 = 5$  and  $M = 5$ . Then you would want to spend  $p_1 x_1^* = 2 \times 5 = 10$  dollars on  $x_1$  - but you don't have enough income for that! So we need to take account of the case where your optimal choice of  $x_1$  isn't affordable (remember, an optimal choice must be in your budget set). The reason this didn't come up is because we just started with the optimality condition  $MRS = \frac{p_1}{p_2}$ . However, the optimality condition only holds true at an **interior solution**. An interior solution is where we have non-zero amounts of both goods:  $x_1^* > 0$  and  $x_2^* > 0$  (i.e. in the "interior" of the graph). The opposite of this is a **corner solution**: either  $x_1^* = 0$  or  $x_2^* = 0$  (i.e. on the axis of the graph).

Corner solutions at first seem tricky, but the nice thing about them is that there are only ever two possibilities, so you can just directly compare them. The two possibilities are:

1. Spend all your money on  $x_1$  and nothing on  $x_2$ . This means that  $(x_1^*, x_2^*) = \left( \frac{M}{p_1}, 0 \right)$
2. Spend all your money on  $x_2$  and nothing on  $x_1$ . This means that  $(x_1^*, x_2^*) = \left( 0, \frac{M}{p_2} \right)$

So if  $u \left( \frac{M}{p_1}, 0 \right) > u \left( 0, \frac{M}{p_2} \right)$ , then the solution is  $(x_1^*, x_2^*) = \left( \frac{M}{p_1}, 0 \right)$ . If it's the other way, then the solution is  $(x_1^*, x_2^*) = \left( 0, \frac{M}{p_2} \right)$ .

Another way to find corner solutions is to use the MRS and a slight variation of the optimality condition. If the corner solution is on the  $y$ -axis (i.e.  $x_1^* = 0$ ) then we have  $|MRS| < \frac{p_1}{p_2}$ . If the corner solution is on the  $x$ -axis (i.e.  $x_2^* = 0$ ) then we have  $|MRS| > \frac{p_1}{p_2}$ . I use this little diagram to remember this relation:



So, you can also find a corner solution by simply checking whether the MRS is *strictly* greater or less than the price ratio. One important thing to note is that at an optimal bundle it cannot be that  $|MRS| = 0$  or  $|MRS| = \infty$ . Otherwise, there would always be a small trade that you could make which would make you strictly better off.<sup>1</sup> Typically, for quasi-linear utility, we will have  $f'(0) \rightarrow \infty$  (i.e.  $|MRS| = \infty$ ), so we usually need to only check for a corner solution of  $x_2 = 0$ .<sup>2</sup>

To summarize, here are the important properties of quasi-linear utility functions of the form  $u(x_1, x_2) = f(x_1) + x_2$ :

1. If at an interior solution, the optimal amount of  $x_1$  is independent of income but a function of both prices
2. Corner solutions are possible (always on the  $x$ -corner, sometimes on the  $y$ -corner too)
3. The results are the opposite if  $x_1$  is the numeraire

## 1.2 Example

Suppose our question was as follows:

- Utility function:  $u(x_1, x_2) = \ln x_1 + x_2$
- Parameters:  $p_1 = 2, p_2 = 4, M = 10$

Here, we have  $f(x) = \ln x$ . We start with the optimality condition and solving for  $x_1(p, M)$ :

$$\begin{aligned} |MRS_{12}| &= \frac{\frac{1}{x_1}}{1} = \frac{p_1}{p_2} \\ \frac{1}{x_1} &= \frac{p_1}{p_2} \\ \implies x_1(p, M) &= \frac{p_2}{p_1} \end{aligned}$$

Next, we plug this into the budget constraint to get  $x_2(p, M)$ :

$$\begin{aligned} p_1 x_1(p, M) + p_2 x_2 &= M \\ p_1 \left( \frac{p_2}{p_1} \right) + p_2 x_2 &= M \\ p_2 + p_2 x_2 &= M \\ \implies x_2(p, M) &= \frac{M - p_2}{p_2} = \frac{M}{p_2} - 1 \end{aligned}$$

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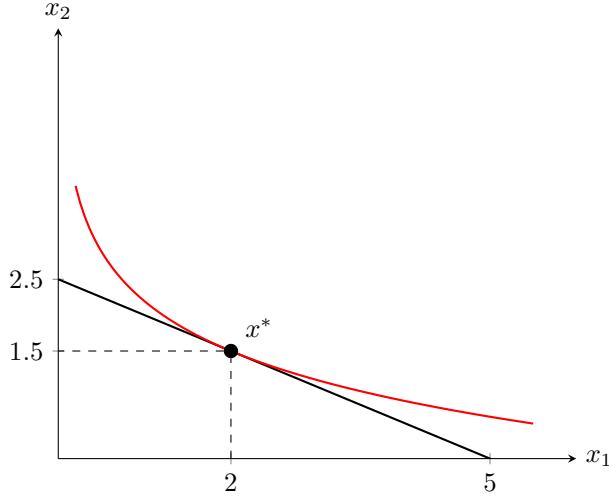
<sup>1</sup>If  $|MRS| = \infty$ , then you would want to buy a little more of  $x_1$  and lose some  $x_2$  (you would have been indifferent if you lost infinite  $x_2$ , so losing only a finite amount is clearly better). If  $|MRS| = 0$ , then you would want to buy a little more of  $x_2$  and lose some  $x_1$  (you would have been indifferent if you gained zero  $x_2$ , so gaining a non-zero amount is clearly better).

<sup>2</sup>For those of you who have done macro, this is one of the Inada conditions. However, we can have quasi-linear utility functions that do not satisfy the Inada conditions.

Now we have our demand functions. Let's plug in value of the parameters to get our optimal bundle:

$$(x_1^*, x_2^*) = \left( \frac{p_2}{p_1}, \frac{M}{p_2} - 1 \right) = \left( \frac{4}{2}, \frac{10}{4} - 1 \right) = (2, 1.5)$$

We can plot this on a graph:



Suppose that income increases to  $M = 20$ . What is the new optimal bundle? Since this is quasi-linear, we should expect no change to  $x_1^*$  and all extra money should be spent on  $x_2$ . Plugging into the demand functions we solved above shows that this is exactly what happens:

$$(x_1^*, x_2^*) = \left( 2, \frac{20}{4} - 1 \right) = (2, 4)$$

Now suppose that income is back to 10 but  $p_2$  has increased dramatically to 20. Let's try plugging into the demand functions:

$$(x_1^*, x_2^*) = \left( \frac{20}{2}, \frac{10}{20} - 1 \right) = (10, -0.5)$$

Clearly this doesn't make sense. We can't have negative quantities. Notice that we are spending  $p_1 x_1^* = 2 \times 10 = 20$  dollars on  $x_1$ , even though we only have \$10. This tells us that the interior solution is not possible, so we need to consider the corner solutions. Let's consider the two possible corner solutions:

$$\begin{aligned} x\text{-corner: } (x_1^*, 0) &= \left( \frac{M}{p_1}, 0 \right) = \left( \frac{10}{2}, 0 \right) = (5, 0) \\ y\text{-corner: } (0, x_2^*) &= \left( 0, \frac{M}{p_2} \right) = \left( 0, \frac{10}{20} \right) = (0, 0.5) \end{aligned}$$

Now let's evaluate their utilities:

$$x\text{-corner: } u(5, 0) = \ln 5 + 0 = \ln 5$$

$$y\text{-corner: } u(0, 0.5) = \ln 0 + 0.5 \rightarrow -\infty$$

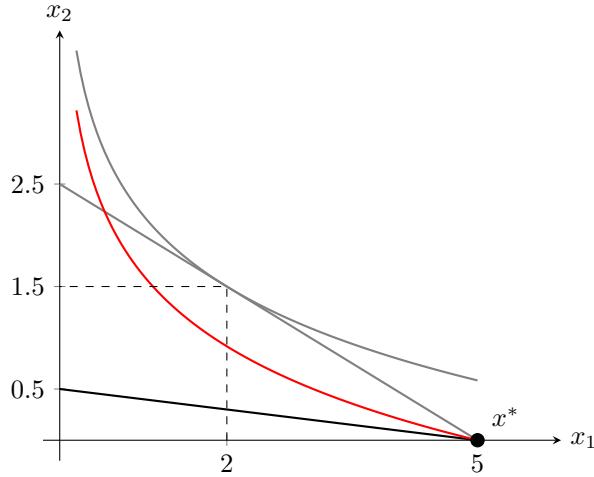
Clearly anything is better than having negative infinite utility. This tells us that the  $x$ -corner is preferred, so

this is the optimal corner solution. In fact, whenever you have the log of a good, you would never consume zero quantity of that good.

Let's also check the MRS to be sure. Remember the the MRS here is  $\frac{1}{x_1}$ .

$$\begin{aligned} x\text{-corner: } |MRS(5, 0)| &= \frac{1}{5} \\ y\text{-corner: } |MRS(0, 0.5)| &= \frac{1}{0} \rightarrow \infty \end{aligned}$$

Again, further proof that the  $y$ -corner cannot be optimal (as we argued before  $|MRS| = \infty$  can never be optimal). Moreover, at the  $x$ -corner, we have  $|MRS| > \frac{p_1}{p_2}$ , since  $\frac{1}{5} > \frac{2}{20} = \frac{1}{10}$ . This is precisely what we should have at an optimal  $x$ -corner. The graph for this is below:



In general, we should look out for corners when one of the interior optimal quantities becomes negative. In this case,  $x_1^*$  will never be negative because  $\frac{p_2}{p_1}$  can never be negative (we assume that prices are always strictly positive). However,  $x_2^*$  can become negative. This happens when:

$$\begin{aligned} \frac{M}{p_2} - 1 &< 0 \\ \frac{M}{p_2} &< 1 \\ M &< p_2 \\ \therefore p_2 &> M \end{aligned}$$

Therefore, corner solutions occur if  $p_2 > M$  (i.e.  $x_2 < 0$ ) and interior solutions occur if  $p_2 \leq M$  (i.e.  $x_2 \geq 0$ ). So, we can summarize the demand functions for this utility function as follows:

$$x_1(p, M) = \begin{cases} \frac{p_2}{p_1} & \text{if } p_2 \leq M \\ \frac{M}{p_1} & \text{if } p_2 > M \end{cases} \quad x_2(p, M) = \begin{cases} \frac{M}{p_2} - 1 & \text{if } p_2 \leq M \\ 0 & \text{if } p_2 > M \end{cases}$$

### 1.3 Harder Example

Let's change the example from before slightly:

- Utility function:  $u(x_1, x_2) = \ln(x_1 + 1) + x_2$
- Parameters:  $p_1 = 2, p_2 = 4, M = 10$

Now,  $f(x) = \ln(x + 1)$ , a seemingly small difference. First, we start with the optimality condition to get  $x_1(p, M)$ :

$$\begin{aligned} |MRS_{12}| &= \frac{\frac{1}{x_1+1}}{1} = \frac{p_1}{p_2} \\ \frac{1}{x_1+1} &= \frac{p_1}{p_2} \\ x_1+1 &= \frac{p_2}{p_1} \\ \implies x_1(p, M) &= \frac{p_2}{p_1} - 1 \end{aligned}$$

Then, we find the demand function for  $x_2$ :

$$\begin{aligned} p_1 x_1(p, M) + p_2 x_2 &= M \\ p_1 \left( \frac{p_2}{p_1} - 1 \right) + p_2 x_2 &= M \\ p_2 - p_1 + p_2 x_2 &= M \\ p_2 x_2 &= M + p_1 - p_2 \\ \implies x_2(p, M) &= \frac{M + p_1}{p_2} - 1 \end{aligned}$$

The optimal bundle for the given parameters is:

$$(x_1^*, x_2^*) = \left( \frac{p_2}{p_1} - 1, \frac{M + p_1}{p_2} - 1 \right) = \left( \frac{4}{2} - 1, \frac{10 + 2}{4} - 1 \right) = (1, 2)$$

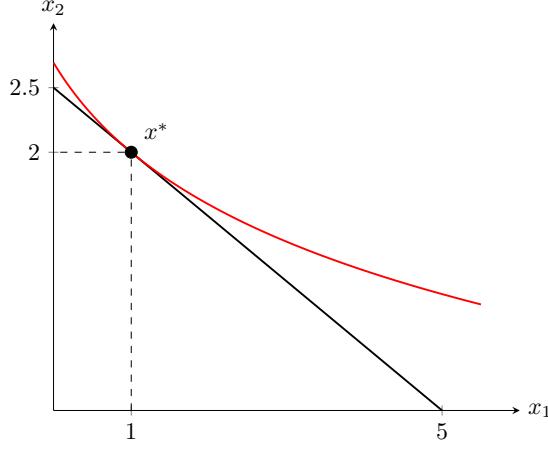
The graph for this is shown below.

Now suppose that  $p_2 = 16$ :

$$(x_1^*, x_2^*) = \left( \frac{16}{2} - 1, \frac{10 + 2}{16} - 1 \right) = \left( 7, \frac{3}{4} - 1 \right) = (7, -0.25)$$

Obviously, this can't happen so let's consider the corner solutions:

$$\begin{aligned} x\text{-corner: } (x_1^*, 0) &= \left( \frac{M}{p_1}, 0 \right) = \left( \frac{10}{2}, 0 \right) = (5, 0) \\ y\text{-corner: } (0, x_2^*) &= \left( 0, \frac{M}{p_2} \right) = \left( 0, \frac{10}{16} \right) = \left( 0, \frac{5}{8} \right) \end{aligned}$$



Let's compare the utilities:

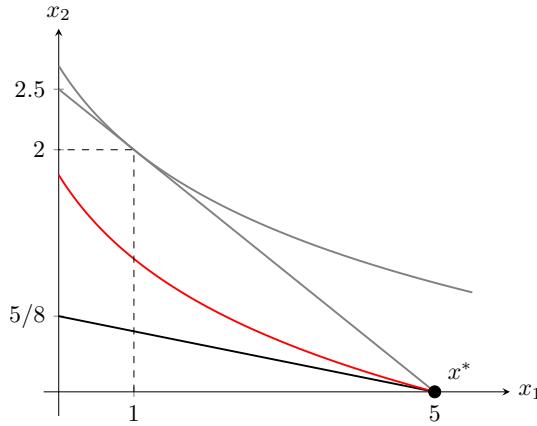
$$x\text{-corner: } u(5, 0) = \ln(5 + 1) + 0 = \ln 6 \approx 1.79$$

$$y\text{-corner: } u(0, 5/8) = \ln(0 + 1) + 5/8 = 0 + 5/8 = 0.625$$

So, this tells us the corner solution should be the  $x$ -corner. But we could also have just looked at  $|MRS| = \frac{1}{x_1+1}$  and compared it to the price ratio:

$$\begin{aligned} \text{Price Ratio: } & \frac{p_1}{p_2} = \frac{2}{16} = \frac{1}{8} \\ x\text{-corner: } & |MRS(5, 0)| = \frac{1}{5+1} = \frac{1}{6} > \frac{1}{8} \\ y\text{-corner: } & |MRS(0, 5/8)| = \frac{1}{0+1} = 1 \not< \frac{1}{8} \end{aligned}$$

We see here that the  $x$ -corner satisfies the “corner optimality condition”, while the  $y$ -corner does not. This tells us that the  $x$ -corner is indeed the solution. The graph representing the change would be:



Ok, so far so good. Now, let's have the following parameter values:  $p_1 = 2$ ,  $p_2 = 1$ ,  $M = 10$  (i.e. the price of

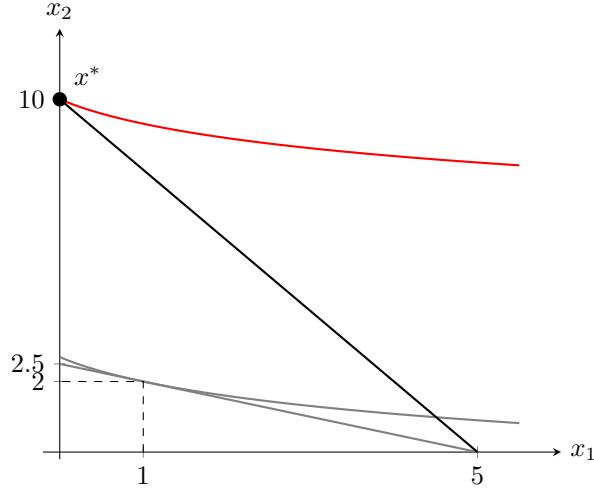
good 2 has dropped). Let's check what the interior solution would be:

$$(x_1^*, x_2^*) = \left( \frac{p_2}{p_1} - 1, \frac{M + p_1}{p_2} - 1 \right) = \left( \frac{1}{2} - 1, \frac{10 + 2}{1} - 1 \right) = (-0.5, 11)$$

We now have the opposite problem: a negative quantity of  $x_1$ ! That still means we need to check for corners. The possible corners are  $(5, 0)$  and  $(0, 10)$ . I'll use the MRS condition to find the optimal bundle and leave it as exercise to check the utilities:

$$\begin{aligned} \text{Price Ratio: } & \frac{p_1}{p_2} = \frac{2}{1} = 2 \\ x\text{-corner: } & |MRS(5, 0)| = \frac{1}{5+1} = \frac{1}{6} \not> 2 \\ y\text{-corner: } & |MRS(0, 10)| = \frac{1}{0+1} = 1 < 2 \end{aligned}$$

In this case, the solution is actually the  $y$ -corner. Even though with quasi-linear utility it's usually an  $x$ -corner, this is showing you that it might not always be the case. The graph in this situation would be:



The more challenging part is figuring out a general demand function. In general, note that we have the  $x$ -corner  $(M/p_1, 0)$  as the optimal corner solution when:

$$\begin{aligned} |MRS(M/p_1, 0)| &= \frac{1}{\frac{M}{p_1} + 1} > \frac{p_1}{p_2} \\ \frac{p_1}{M + p_1} &> \frac{p_1}{p_2} \\ p_2 &> M + p_1 \\ \therefore p_2 - p_1 &> M \end{aligned}$$

Similarly, we have the  $y$ -corner  $(0, M/p_2)$  as the optimal corner solution when:

$$\begin{aligned}|MRS(0, M/p_2)| &= \frac{1}{0+1} < \frac{p_1}{p_2} \\ 1 &< \frac{p_1}{p_2} \\ \therefore p_2 &< p_1\end{aligned}$$

Notice that this lines up with the conditions for negative interior solutions. For example, the interior  $x_2$  solution is negative when:

$$\begin{aligned}\frac{M+p_1}{p_2} - 1 &< 0 \\ M + p_1 &< p_2 \\ p_2 - p_1 &> M\end{aligned}$$

And you may also be unsurprised to see that the interior  $x_1$  solution is negative when:

$$\begin{aligned}\frac{p_2}{p_1} - 1 &< 0 \\ p_2 &< p_1\end{aligned}$$

So when the interior  $x_1$  solution is negative, the optimal bundle is the corner solution where  $x_1^* = 0$  (i.e. the  $y$ -corner), and similarly for  $x_2$  and the  $x$ -corner. Intuitively, the corner solution is realistically the closest we can get to the “true” optimal solution where  $|MRS| = \frac{p_1}{p_2}$ .

Putting all of this together, the demand functions for this utility function can be summarized as:

$$x_1(p, M) = \begin{cases} \frac{M}{p_1} & \text{if } p_1 < p_2 - M \\ 0 & \text{if } p_1 > p_2 \\ \frac{p_2}{p_1} - 1 & \text{otherwise} \end{cases} \quad x_2(p, M) = \begin{cases} 0 & \text{if } p_2 > p_1 + M \\ \frac{M}{p_2} & \text{if } p_2 < p_1 \\ \frac{M+p_1}{p_2} - 1 & \text{otherwise} \end{cases}$$

## 2 Perfect Substitutes

### 2.1 General Results

Recall what we know about perfect substitutes:

- General Form:  $u(x_1, x_2) = \frac{1}{\alpha}x_1 + \frac{1}{\beta}x_2$  or  $u(x_1, x_2) = \beta x_1 + \alpha x_2$
- Indifference Curves: Linear indifference curves
- Preference: A consumer who is equally happy with  $\alpha$  units of good 1 as they are with  $\beta$  units of good 2.

Let's solve the optimization problem, starting with calculating the MRS:

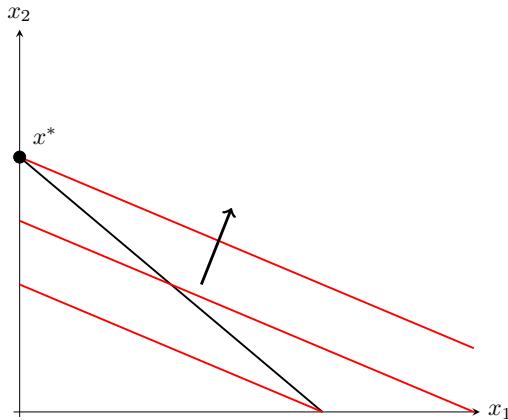
$$|MRS_{12}| = \frac{MU_1}{MU_2} = \frac{\frac{1}{\alpha}}{\frac{1}{\beta}} = \frac{\beta}{\alpha}$$

Notice that the MRS is constant - this is a key property of perfect substitutes because the indifference curves are linear. The price ratio is also constant, which means there's nothing left to solve. In fact, it's not guaranteed that these two values are going to be equal. It is important to remember that the MRS is the slope of the indifference curve while the price ratio is the slope of the budget constraint. So how can we get a tangency condition with two straight lines? Let's think back to the discussion about corners. There are three possible cases.

The first possibility is:

$$|MRS_{12}| = \frac{\beta}{\alpha} < \frac{p_1}{p_2}$$

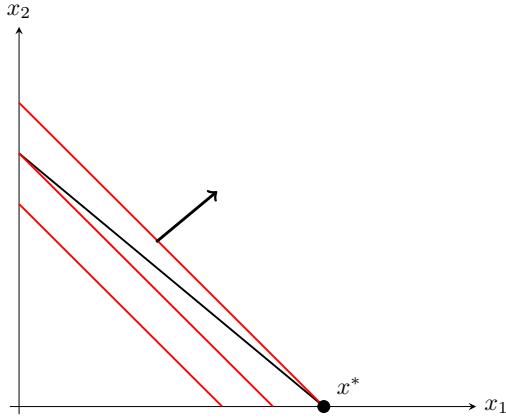
This means that we should have a  $y$ -corner, where  $x_1^* = 0$  and  $x_2^* = \frac{M}{p_2}$ . If this occurs, we have that the indifference curve are flatter than the budget line. For example, as this graph below shows (black line is the budget line, red lines are ICs). Notice that the  $y$ -corner is in fact optimal, because the  $x$ -corner is on a lower indifference curve.



The second possibility is:

$$|MRS_{12}| = \frac{\beta}{\alpha} > \frac{p_1}{p_2}$$

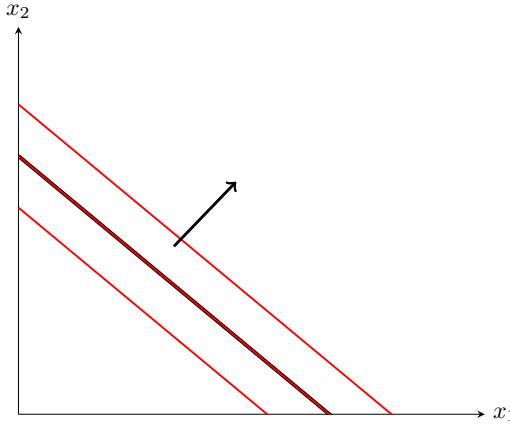
This means that we should have a  $x$ -corner, where  $x_1^* = \frac{M}{p_1}$  and  $x_2^* = 0$ . If this occurs, we have that the indifference curve are steeper than the budget line. We can see this in the graph. Again, notice that now the  $x$ -corner is optimal, because the  $y$ -corner is on a lower indifference curve.



The final possibility is:

$$|MRS_{12}| = \frac{\beta}{\alpha} = \frac{p_1}{p_2}$$

This says that the indifference curve and budget lines have exactly the same slope. This is when we get lucky and the optimality condition is met exactly. At first this is trickier to interpret, so let's look at the graph:



Notice that there is an indifference curve that lies entirely on top of the budget line. In this case, what should the optimal bundle be? The answer is that any bundle on the budget line! This is quite easy to see - any bundle on the budget line gives the consumer the same level of utility, all of which is the highest utility the consumer can achieve in their feasible set. This means that the consumer is indifferent between either the  $x$ -corner and the  $y$ -corner and any mix between the two ("convex combination"). So here, we have both corner and interior solutions.

The properties of perfect substitutes are summarized below:

1. If  $|MRS| < \frac{p_1}{p_2}$ , only purchase  $x_2$
2. If  $|MRS| > \frac{p_1}{p_2}$ , only purchase  $x_1$
3. If  $|MRS| = \frac{p_1}{p_2}$ , any bundle on the budget line is optimal
4. Income does not effect which mix of goods the consumer prefers

## 2.2 Example

Since perfect substitutes are quite easy to solve, let's do multiple variations at the same time. Suppose there are four people with different utility functions, who are each facing four possible sets of parameters:

Utilities	Parameters
(A) $u_A(x_1, x_2) = x_1 + x_2$	(1) $p_1 = 2, p_2 = 3, M = 10$
(B) $u_B(x_1, x_2) = x_1 + 3x_2$	(2) $p_1 = 6, p_2 = 4, M = 10$
(C) $u_C(x_1, x_2) = 4x_1 + 2x_2$	(3) $p_1 = 1, p_2 = 1.5, M = 20$
(D) $u_D(x_1, x_2) = \frac{1}{2}x_1 + \frac{1}{3}x_2$	(4) $p_1 = 1, p_2 = 0.5, M = 5$

As a check, you should try interpret the preference from each of the utility functions. In (A), the consumer is indifferent between 1 unit of  $x_1$  and 1 unit of  $x_2$ . In (B), the consumer is indifferent between 3 units of  $x_1$  and 1 unit of  $x_2$ . In (C), the consumer is indifferent between 1 unit of  $x_1$  and 2 units of  $x_2$ . In (D), the consumer is indifferent between 2 units of  $x_1$  and 3 units of  $x_2$ .

Let's calculate the MRS and the price ratios for each case:

	MRS	Price Ratio
(A) $ MRS  = \frac{1}{1} = 1$	(1) $\frac{p_1}{p_2} = \frac{2}{3}$	
(B) $ MRS  = \frac{1}{3} = \frac{1}{3}$	(2) $\frac{p_1}{p_2} = \frac{6}{4} = \frac{3}{2}$	
(C) $ MRS  = \frac{4}{2} = 2$	(3) $\frac{p_1}{p_2} = \frac{1}{1.5} = \frac{2}{3}$	
(D) $ MRS  = \frac{1/2}{1/3} = \frac{3}{2}$	(4) $\frac{p_1}{p_2} = \frac{1}{0.5} = 2$	

With this, it should be easy to see the solution for each permutation. I'll leave it as an exercise for you to check these are true (note that "BL" means any  $(x_1, x_2)$  on the budget line)

$(x_1^*, x_2^*)$	Parameters			
Utility Functions	(1)	(2)	(3)	(4)
(A)	(5, 0)	(0, $\frac{10}{4}$ )	(40, 0)	(0, 10)
(B)	(0, $\frac{10}{3}$ )	(0, $\frac{10}{3}$ )	(0, $\frac{10}{3}$ )	(0, 10)
(C)	(5, 0)	( $\frac{10}{6}$ , 0)	(40, 0)	BL
(D)	(5, 0)	BL	(40, 0)	(0, 10)

## 3 Perfect Complements

### 3.1 General Results

Recall what we know about perfect complements:

- General Form:  $u(x_1, x_2) = \min \left\{ \frac{1}{\alpha}x_1, \frac{1}{\beta}x_2 \right\}$  or  $u(x_1, x_2) = \min \{\beta x_1, \alpha x_2\}$
- Indifference Curves: L-shaped indifference curves (Leontief)
- Preference: A consumer who only prefers to consume bundles in proportions of  $\alpha$  units of good 1 with  $\beta$  units of good 2

For perfect complements, we are not able to use the optimality condition. This is because the MRS for a L-shaped indifference curve is quite odd: first the line is vertical so it has a slope of  $-\infty$ , then there is a kink (what's the slope of a kink?!), and then the line is flat, so it has a slope of 0. This rapid change of slope means that the MRS is not defined at the kink (i.e. the function is not differentiable at the kink).

This is where we have to use our economic intuition. As previously mentioned, it can never be optimal to be at a point with  $|MRS| = \infty$  (vertical line) or  $|MRS| = 0$  (horizontal line). If we are at such a point, then we are essentially buying units of a good that give us no extra utility. This is clearly a waste of money. The only way to not let our money go to waste is to be right at the kink. So we just need to know where the kinks occur. But that's pretty straightforward: the kinks occur when the two components in the min function equal each other! That is:

$$\begin{aligned}\frac{1}{\alpha}x_1 &= \frac{1}{\beta}x_2 \\ \beta x_1 &= \alpha x_2 \\ \therefore x_2 &= \frac{\beta}{\alpha}x_1\end{aligned}$$

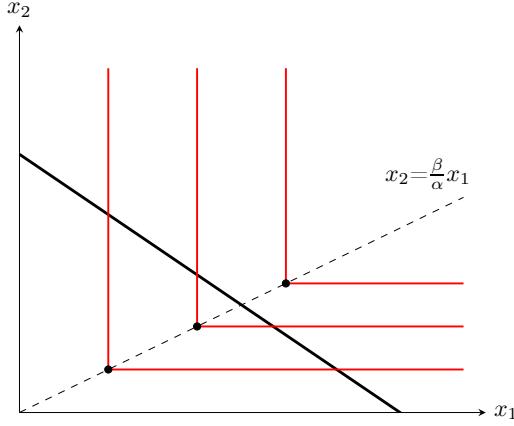
This gives us a line that we can plot.<sup>3</sup> As we can see in the graph below, all the kinks occur along this line (the black line in the graph is the budget line).

So we know that the an optimal solution must be on: (a) the budget line, and (b) the kinks line. This means that the optimal bundle must be where these two lines intersect! We can just plug this directly into the budget line:

$$\begin{aligned}p_1x_1 + p_2 \left( \frac{\beta}{\alpha}x_1 \right) &= M \\ x_1 \left( \frac{\alpha p_1 + \beta p_2}{\alpha} \right) &= M \\ \implies x_1(p, M) &= \frac{\alpha M}{\alpha p_1 + \beta p_2}\end{aligned}$$

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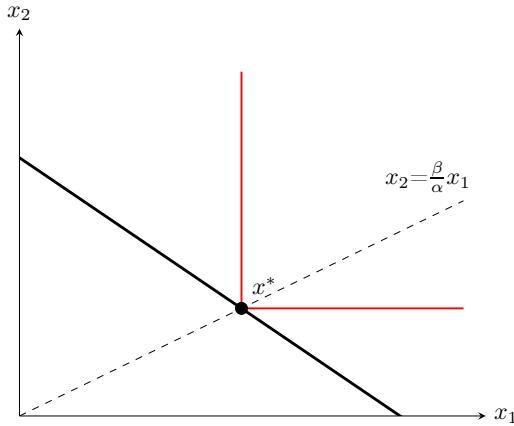
<sup>3</sup>I'm not sure if there is a technical term for this line. I'm just going to call it the "kinks line" or "the line of kinks".



Plugging this back into the equation for the kinks line, gets us the  $x_2$  demand function:

$$\begin{aligned} x_2 &= \frac{\beta}{\alpha}x_1(p, M) \\ \implies x_2(p, M) &= \frac{\beta M}{\alpha p_1 + \beta p_2} \end{aligned}$$

So the optimal bundle is still where the indifference curve is “tangent” to the budget line. It’s not exactly tangent because the slope isn’t defined at the kink (technically any line is tangent at the kink). But it should be clear from the graph that this point is where we can achieve the highest utility on the budget set.



Now with the demand functions we can summarize the properties of perfect complements:

1. The optimal solution must be at a kink, in particular, where the kinks line intersects the budget line
2. Demand for both goods depends on prices of both goods (as well as income)

### 3.2 Example

Suppose our question was as follows:

- Utility function:  $u(x_1, x_2) = \min\{x_1, 3x_2\}$
- Parameters:  $p_1 = 2, p_2 = 4, M = 10$

This says that the consumer needs to consume 1 unit of  $x_1$  with  $\frac{1}{3}$  units of  $x_2$  (or equivalently, 3 units of  $x_1$  for every 1 unit of  $x_2$ ). The line of kinks is given by:

$$\begin{aligned} x_1 &= 3x_2 \\ \implies x_2 &= \frac{1}{3}x_1 \end{aligned}$$

Plugging this into the budget constraint gets us  $x_1(p, M)$ :

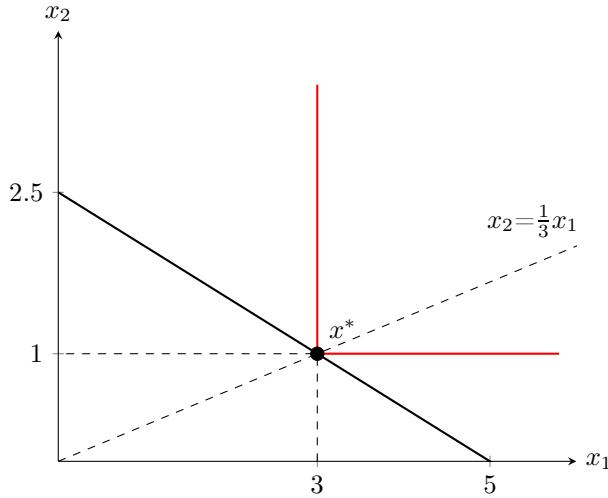
$$\begin{aligned} p_1x_1 + p_2\left(\frac{1}{3}x_1\right) &= M \\ \frac{3p_1 + p_2}{3}x_1 &= M \\ \therefore x_1(p, M) &= \frac{3M}{3p_1 + p_2} \end{aligned}$$

Plug the demand function back into the kinks line formula to get the demand function  $x_2(p, M)$ :

$$\begin{aligned} x_2(p, M) &= \frac{1}{3}x_1(p, M) \\ &= \frac{M}{3p_1 + p_2} \end{aligned}$$

To get the optimal bundle, let's plug in the value of the parameters:

$$\begin{aligned} x_1^* &= \frac{3 \times 10}{3 \times 2 + 4} = \frac{30}{10} = 3 \\ x_2^* &= \frac{10}{3 \times 2 + 4} = \frac{10}{10} = 1 \end{aligned}$$



As expected, we are exactly on the kink that touches the budget line. The utility from this optimal bundle

is  $u^* = \min\{3, 3 \times 1\} = \min\{3, 3\} = 3$ . Notice that there is no wastage in the sense that both the values in the min function equal each other.

Now let's suppose that the price of good 2 falls so that  $p_2 = 2$ . Let's calculate the new optimal bundle by plugging in the new parameters into the demand functions:

$$x_1^* = \frac{3 \times 10}{3 \times 2 + 2} = \frac{30}{8} = 3.75$$

$$x_2^* = \frac{10}{3 \times 2 + 2} = \frac{10}{8} = 1.25$$

The utility from this new bundle is  $u^* = \min\{3.75, 3 \times 1.25\} = \min\{3.75, 3.75\} = 3.75$ . Notice that the change in the quantities ( $\Delta x_1 = 0.75, \Delta x_2 = 0.25$ ) is also in the same proportion as the consumer's preference:  $0.75 : 0.25 \implies 3 : 1$ . The change is shown in the graph below:

