

Advanced Micro: Recitation 2

Convexity

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1 Convex Sets

1.1 Definitions

Convex Set A set S is convex iff $\lambda x + (1 - \lambda)y \in S$ for any $\lambda \in [0, 1]$ and $x, y \in S$

Convex Combination A convex combination of n finite vectors x_1, \dots, x_n is the vector $\sum_{i=1}^n \lambda_i x_i$ for scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^n \lambda_i = 1$.

1.2 Alternative Definitions

Convex Set A set S is convex iff any convex combination of $x_1, \dots, x_n \in S$ is also in S

1.3 Useful Results

Proposition 1. *The intersection of convex sets is convex*

Theorem 2. (*Minkowski's Separating Hyperplane*) Let S_1 and S_2 be two disjoint non-empty and convex sets in \mathbb{R}^N . Then there exists a $p \in \mathbb{R}^N \setminus \{0\}$ and $c \in \mathbb{R}$ s.t. we can find a hyperplane $H(p, c) = \{x \in \mathbb{R}^N \mid p \cdot x = c\}$ that weakly separates them, i.e. $p \cdot x \geq c$ for any $x \in S_1$ and $p \cdot x \leq c$ for any $x \in S_2$.

(*Brouwer's Fixed Point Theorem*) Let X be a nonempty, compact, and convex set in \mathbb{R}^N , and consider a continuous function $f : X \rightarrow X$. Then there exists a $x^* \in X$ such that $f(x^*) = x^*$

2 Convex Functions

2.1 Definitions

Convex/Concave Function Consider a function $f : S \rightarrow \mathbb{R}$, where S is a convex subset of \mathbb{R}^N .

1. The function f is **convex** iff

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for any $x, y \in S$ and $\lambda \in [0, 1]$. It is strictly convex if the inequality is strict and with $x \neq y$ and $\lambda \in (0, 1)$.

2. The function f is **concave** iff

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for any $x, y \in S$ and $\lambda \in [0, 1]$. It is strictly concave if the inequality is strict and with $x \neq y$ and $\lambda \in (0, 1)$.

Graph The graph of a function $f : S \rightarrow \mathbb{R}$ is defined as the set: $G(f) = \{(x, y) \in S \times \mathbb{R} \mid y = f(x)\}$. The **epigraph** of f is $G^+(f) = \{(x, y) \in S \times \mathbb{R} \mid y \geq f(x)\}$. The **subgraph** of f is $G^-(f) = \{(x, y) \in S \times \mathbb{R} \mid y \leq f(x)\}$.

C^k Functions A function is C^k (or k^{th} continuously differentiable) iff f is k^{th} differentiable everywhere in the domain of f and its k^{th} order derivative $f^{(k)}$ is a continuous function.

Gradient Consider a real-valued function $f : \mathbb{R}^N \rightarrow \mathbb{R}$. The gradient of the function f at x , denoted as $\nabla f(x)$, is a $1 \times N$ row vector of the partial derivatives evaluated at x :

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_N}(x) \right)$$

Hessian Consider a real-valued function $f : \mathbb{R}^N \rightarrow \mathbb{R}$. The Hessian of the function f at x , denoted as $H_f(x)$ or simply $H(x)$, is a $N \times N$ matrix of second derivatives evaluated at x :

$$H_f(x) = f''(x) = (\nabla f)'(x) = \begin{bmatrix} \left(\nabla \frac{\partial f}{\partial x_1} \right)(x) \\ \left(\nabla \frac{\partial f}{\partial x_2} \right)(x) \\ \vdots \\ \left(\nabla \frac{\partial f}{\partial x_N} \right)(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_N}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_N}(x) & \cdots & \frac{\partial^2 f}{\partial x_N^2}(x) \end{bmatrix}$$

Definiteness Let A be an $n \times n$ real symmetric matrix. The matrix A is said to be:

1. Positive definite iff $x^T Ax > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and iff all its eigenvalues are positive
2. Negative definite iff $x^T Ax < 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and iff all its eigenvalues are negative
3. Positive semi-definite iff $x^T Ax \geq 0, \forall x \in \mathbb{R}^n$ and iff all its eigenvalues are non-negative
4. Negative semi-definite iff $x^T Ax \leq 0, \forall x \in \mathbb{R}^n$ and iff all its eigenvalues are non-positive

2.2 Alternative Definitions

Jensen's Inequality Let f be a function defined on a convex set $S \subset \mathbb{R}^N$.

1. f is convex iff for any $x_1, \dots, x_n \in S$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^n \lambda_i = 1$, we have:

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

2. f is concave iff for any $x_1, \dots, x_n \in S$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^n \lambda_i = 1$, we have:

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \geq \sum_{i=1}^n \lambda_i f(x_i)$$

Graph Definition Let f be a function defined on a convex set $S \subset \mathbb{R}^N$.

1. f is convex iff its epigraph $G^+(f)$ is a convex set.
2. f is concave iff its subgraph $G^-(f)$ is a convex set.

C^1 Functions Let f be a C^1 function on convex and open set $S \subseteq \mathbb{R}^N$.

1. f is convex iff

$$f(y) - f(x) \geq \nabla f(x) \cdot (y - x)$$

for any $x, y \in S$. It is strictly convex if the inequality is strict and $x \neq y$.

2. f is concave iff

$$f(y) - f(x) \leq \nabla f(x) \cdot (y - x)$$

for any $x, y \in S$. It is strictly concave if the inequality is strict and $x \neq y$.

Hessian Definition Let f be a C^2 function on a convex and open set $S \subseteq \mathbb{R}^N$.

1. f is convex iff its Hessian matrix $H(x)$ is positive semi-definite for any $x \in S$. It is strictly convex if $H(x)$ is positive definite (but not only if).
2. f is concave iff its Hessian matrix $H(x)$ is negative semi-definite for any $x \in S$. It is strictly concave if $H(x)$ is negative definite (but not only if).

Note that if $N = 1$, then this just says that f is convex if $f''(x) \geq 0$ and concave if $f''(x) \leq 0$.

2.3 Useful Results

Proposition 3. Consider two real valued functions f and g on a convex subset S of \mathbb{R}^N . If f and g are both convex/concave function, then:

1. $f + g$ is a convex/concave function; and
2. cf is a convex/concave function, for any $c \in \mathbb{R}_+$

Proposition 4. Consider a real valued function f on a convex subset S of \mathbb{R}^N

1. If f is convex and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly increasing and convex, then $\phi \circ f$ is convex

2. If f is concave and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly increasing and concave, then $\phi \circ f$ is concave
3. If f is convex and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly decreasing and concave, then $\phi \circ f$ is concave
4. If f is concave and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly decreasing and convex, then $\phi \circ f$ is convex

Proposition 5. Let f be a C^1 concave (convex) function on a convex subset S of \mathbb{R}^N . If $f'(x) = 0$, then x is a global maximum (minimum) of f .

Proof. This follows immediately from the definition for C^1 functions. If f is concave and $f'(x) = 0$, then for any $y \in S$, $f(y) \leq f(x)$, so x is a global maximum of f . \square

3 Quasi-convex Functions

3.1 Definitions

Quasi-convex/Quasi-concave Function Consider a function $f : S \rightarrow \mathbb{R}$, where S is a convex subset of \mathbb{R}^N .

1. The function f is **quasi-convex** iff

$$f(\lambda x + (1 - \lambda)y) \leq \max \{f(x), f(y)\}$$

for any $x, y \in S$ and $\lambda \in [0, 1]$. It is strictly quasi-convex if the inequality is strict and with $x \neq y$ and $\lambda \in (0, 1)$.

2. The function f is **quasi-concave** iff

$$f(\lambda x + (1 - \lambda)y) \geq \min \{f(x), f(y)\}$$

for any $x, y \in S$ and $\lambda \in [0, 1]$. It is strictly quasi-concave if the inequality is strict and with $x \neq y$ and $\lambda \in (0, 1)$.

Contour Set For a function $f : S \rightarrow \mathbb{R}$, the **upper contour set** of f with cutoff α is $C^+(f, \alpha) = \{x \in S \mid f(x) \geq \alpha\}$, and the **lower contour set** of f with cutoff α is $C^-(f, \alpha) = \{x \in S \mid f(x) \leq \alpha\}$.

3.2 Alternative Definitions

Proposition Form Consider a function $f : S \rightarrow \mathbb{R}$, where S is a convex subset of \mathbb{R}^N .

1. f is quasi-convex iff $f(x) \geq f(y) \Rightarrow f(tx + (1 - t)y) \leq f(x)$, $\forall t \in (0, 1)$ and $\forall x, y \in S$.
2. f is quasi-concave iff $f(x) \geq f(y) \Rightarrow f(tx + (1 - t)y) \geq f(y)$, $\forall t \in (0, 1)$ and $\forall x, y \in S$.

Contour Sets Consider a function $f : S \rightarrow \mathbb{R}$, where S is a convex subset of \mathbb{R}^N .

1. f is quasi-convex iff lower contour set $C^-(f, \alpha)$ is a convex set for all $\alpha \in \mathbb{R}$
2. f is quasi-concave iff upper contour set $C^+(f, \alpha)$ is a convex set for all $\alpha \in \mathbb{R}$

C^1 Functions Consider a C^1 function $f : S \rightarrow \mathbb{R}$, where S is an open convex subset of \mathbb{R}^N .

1. f is quasi-convex iff $f(x) \geq f(y) \Rightarrow \nabla f(x) \cdot (y - x) \leq 0, \forall x, y \in S$.
2. f is quasi-concave iff $f(x) \leq f(y) \Rightarrow \nabla f(x) \cdot (y - x) \geq 0, \forall x, y \in S$.

Hessian Definition There is definition of quasi-convexity using the bordered Hessian - but we will discuss this in a future recitation.

3.3 Useful Results

Proposition 6. Consider a real valued function f on a convex subset S of \mathbb{R}^N

1. If f is quasi-convex and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly increasing, then $\phi \circ f$ is quasi-convex
2. If f is quasi-concave and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly increasing, then $\phi \circ f$ is quasi-concave
3. If f is quasi-convex and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly decreasing, then $\phi \circ f$ is quasi-concave
4. If f is quasi-concave and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is weakly decreasing, then $\phi \circ f$ is quasi-convex

4 Exercises

4.1 Example Functions

Questions: Check whether the following functions are concave/convex and quasi-convex/quasi-concave.

1. $f(x) = x^3 + x$
2. $f(x) = x^3 - x$
3. $f(x) = x^4 + x^2$
4. $f(x_1, x_2) = x_1^2 + x_2^2$

Answers:

1. $f''(x) = 6x$, which can be positive or negative. So neither concave nor convex. Note that $f'(x) = 2x^2 + 1 \geq 0$, so this is an increasing function. Therefore, for any x^* , the set $\{x \in \mathbb{R} : f(x) \geq f(x^*)\}$ is the set $[x^*, \infty)$, which is a convex set. So the upper contour set is convex, and hence the function is quasi-concave. Similarly, the set $\{x \in \mathbb{R} : f(x) \leq f(x^*)\}$ is the set $(-\infty, x^*]$, which is also convex and hence the function is also quasi-convex. So it is both quasi-concave and quasi-convex (which is true of a linear function). This is going to be true of any increasing or decreasing function.
2. As before, neither concave nor convex. It is also not quasi-convex nor quasi-concave. Consider: $x = -1$, $y = 1$, where we have $f(-1) = 0$ and $f(1) = 0$. Take a convex combination, e.g. $\lambda = 0.75$ so $\lambda x + (1 - \lambda)y = -0.5$. Here we have $f(-0.5) = 0.375$, so clearly not quasi-convex. Similarly, with $\lambda = 0.25$, then $f(0.5) = -0.375$, and it is clearly not quasi-concave.

3. $f''(x) = 12x^2 + 2 \geq 0, \forall x$. So this is convex, and therefore quasi-convex too. It is not concave nor quasi-concave (take $x = -1$ and $y = 1$, both give $f(x) = f(y) = 2$, but $f(0) = 0$).

4. f is convex iff $f(y) - f(x) \geq \nabla f(x) \cdot (y - x)$:

$$\begin{aligned} (y_1^2 + y_2^2) - (x_1^2 + x_2^2) &\geq \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix} \\ &= 2x_1(y_1 - x_1) + 2x_2(y_2 - x_2) \\ &= 2x_1y_1 - 2x_1^2 + 2x_2y_2 - 2x_2^2 \end{aligned}$$

Re-arranging gives us:

$$\begin{aligned} y_1^2 + y_2^2 - x_1^2 - x_2^2 - 2x_1y_1 + 2x_1^2 - 2x_2y_2 + 2x_2^2 &\geq 0 \\ y_1^2 + y_2^2 - 2x_1y_1 - 2x_2y_2 + x_1^2 + x_2^2 &\geq 0 \\ (y_1 - x_1)^2 + (y_2 - x_2)^2 &\geq 0 \end{aligned}$$

This is true for all $x, y \in \mathbb{R}^2$. Since it is convex it is also quasi-convex. It is not quasi-concave. For example $x = (1, 1)$ and $y = (-1, -1)$. We have $f(x) = f(y) = 1 + 1 = 2$. With $\lambda = 0.5$, we get $(0, 0)$ which should be in the upper contour set $C^+(f, 2)$ - however: $f(0, 0) = 0$.

4.2 General Ideas

Questions:

1. Prove that the intersection of convex sets is convex. Argue that, in general, a union of convex sets is not convex.
2. Argue that a quasi-concave function $f : \mathbb{R} \rightarrow \mathbb{R}$ is at most a single peak, but that a critical point need not be a maximum
3. Prove that a *concave* monotonic transformation of a concave function is concave. Show using an example that convexity is a cardinal property. Prove that quasi-convexity is preserved under a monotonic transformation. Why is this important for consumer theory (hint: think about the utility function)
4. Let f be a function defined on a convex set $X \subseteq \mathbb{R}^N$. Prove that f is concave iff its subgraph is convex (the proof for convexity is symmetric).
5. Show that a strictly concave function f may not necessarily have a negative definite Hessian

Answers:

1. Call the intersection of the sets $\{S_i\}_{i \in I}$ as $T = \cap_{i \in I} S_i$. Take two arbitrary points $x, y \in T$. For any i , we must have $x \in S_i$ and $y \in S_i$. And since S_i is convex, then $\lambda x + (1 - \lambda)y \in S_i, \forall \lambda \in [0, 1]$. Since our choice of set was arbitrary, this must be true for all sets S_i in the intersection. Therefore, for all $\lambda \in [0, 1]$, we must have $\lambda x + (1 - \lambda)y \in S_i, \forall i \in I$ by definition, $\lambda x + (1 - \lambda)y \in T$. Hence the intersection is convex. To show a union is not convex, take any two disjoint convex sets (draw the

picture). It's easy to see how a convex combination of a point in set 1 and a point in set 2 may not be in either set.

2. You can argue this formally, but intuitively, draw a picture of a function with two peaks (i.e. two local maxima). Shade in the upper contour set for some α above the trough and you will see that the set is the union of two disjoint sets, which is not convex. For the second part, consider the function $f(x) = x^3$. This is an increasing function and therefore quasi-concave, but $f'(x) = 0$ occurs at $x = 0$, which is a saddle point.
3. Take a function f that is concave and g that is concave and increasing. Then:

$$\begin{aligned}
 (g \circ f)(tx + (1-t)y) &= g(f(tx + (1-t)y)) \\
 &\geq g(tf(x) + (1-t)f(y)) && (f \text{ concave and } g \text{ increasing}) \\
 &\geq tg(f(x)) + (1-t)g(f(y)) && (g \text{ concave}) \\
 &= t(g \circ f)(x) + (1-t)(g \circ f)(y)
 \end{aligned}$$

To show that a monotonic transformation is not enough, take for example $f(x) = \frac{-x^2}{2}$. This is a convex function since $f''(x) = -1 < 0$. $\phi(x) = e^x$ is a monotonic function. However, $g(x) = \phi(f(x)) = e^{-\frac{x^2}{2}}$ is not concave: $g''(x) = e^{-\frac{x^2}{2}}(x^2 - 1)$ is not always negative. This is important because in consumer theory we know that utility representations are unique up to affine transformation. So we want to be able to transform a utility function and not lose any important information, which is why we need them to be quasi-concave.

4. Let's only do the \Rightarrow direction since the reverse is similar. Define the epigraph as $G^+(f) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} | y \leq f(x)\}$ and take two elements $g_1, g_2 \in G^+(f)$. We want to show that their convex combination is also in $G^+(f)$. Note that $g_i \in \mathbb{R}^N \times \mathbb{R}$, i.e. $g_i = (x_i, y_i)$. Take any $t \in (0, 1)$. We then see: $tg_1 + (1-t)g_2 = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$. Since f is concave, we have

$$\begin{aligned}
 f(tx_1 + (1-t)x_2) &\geq tf(x_1) + (1-t)f(x_2) \\
 &\geq ty_1 + (1-t)y_2
 \end{aligned}$$

where the second line follows from $y_i \leq f(x_i)$. Therefore $tg_1 + (1-t)g_2 \in G^+(f)$, so $G^+(f)$ is convex.

5. Take the function $f(x) = -x^4$. This is a strictly concave function (draw the picture). Its second derivative (i.e. the Hessian since we are in $N = 1$) is $f''(x) = -12x^2 \leq 0$. Since $f''(0) = 0$ this is not negative definite (i.e. strictly negative in \mathbb{R}).

4.3 Utility Maximization Problem

Question: Consider a consumer with preferences \succsim with price vector p and income y . The consumption space is denoted by X and the budget set is $B(p, y)$. From the consumer's UMP, we get the Marshallian demand $x(p, y)$ and the indirect utility function $v(p, y) = u(x(p, y))$. Show the following:

1. If \succsim is strictly convex, then $x(p, y)$, if non-empty, is a singleton for every $(p, y) \in \mathbb{R}_{++}^K \times \mathbb{R}$
2. $v(p, y)$ is quasi-convex in (p, y)

Answers:

- First, we show that $x(p, y)$ is a convex set. If $x(p, y)$ is empty, then this is trivially true. If it is non-empty, then for any $x^* \in x(p, y)$, we can express the solution set as $x(p, y) = B(p, y) \cap \{x \in X | x \succsim x^*\}$ (i.e. the intersection of the feasible and no-worse-than sets). $B(p, y)$ is convex (you will prove this in a homework) and the NWT set $\{x \in X | x \succsim x^*\}$ is also convex (by the convexity of \succsim). The intersection of two convex sets is convex, so $x(p, y)$ is convex.

Next, we show it is a singleton (if it is non-empty). Suppose not, and take two elements $x, x' \in x(p, y)$ where $x \neq x'$. By strict convexity of \succsim , $\lambda x + (1 - \lambda)y \succ x, \forall \lambda \in (0, 1)$. By convexity of $B(p, y)$, $\lambda x + (1 - \lambda)y \in B(p, y)$. Since it is feasible but strictly preferred to x and x' , this contradicts the optimality of x and x' .

- Another way to express this is to say that the set $\{(p, y) : v(p, y) \leq v\}$ is convex for any v . Fix some v , and suppose that $v(p, y) \leq v$ and $v(p', y') \leq v$. We want to show that $v(\lambda p + (1 - \lambda)p', \lambda y + (1 - \lambda)y') \leq v$.

For any $x \in x(\lambda p + (1 - \lambda)p', \lambda y + (1 - \lambda)y')$, we must have that $[\lambda p + (1 - \lambda)p'] \cdot x \leq [\lambda y + (1 - \lambda)y']$. Re-arranging this gives us: $\lambda[p \cdot x - y] + (1 - \lambda)[p' \cdot x - y'] \leq 0$. Therefore, it must be the case that $p \cdot x \leq y$ or $p' \cdot x \leq y'$. In the first case, this implies x is feasible for the UMP with parameters p and y . Since it is feasible, it must be that $u(x) \leq v(p, y)$. Similarly for the second case: $u(x) \leq v(p', y')$. In either case, $v(\lambda p + (1 - \lambda)p', \lambda y + (1 - \lambda)y') = u(x) \leq v$.

4.4 Simon & Blume 21.12

Question: Suppose that a one-product monopolist faces an inverse demand function $p = F(q)$ and a cost function $C(q)$. What assumptions on F and C yield a concave profit function?

Solution: The profit function is written as $\pi(q) = F(q) \cdot q - C(q)$. For it to be concave, we need:

$$\begin{aligned} \frac{\partial^2 \pi}{\partial q^2} &= \frac{\partial}{\partial q} \left(\frac{\partial F}{\partial q} q + F(q) - \frac{\partial C}{\partial q} \right) \\ &= \frac{\partial^2 F}{\partial q^2} q + \frac{\partial F}{\partial q} + \frac{\partial F}{\partial q} - \frac{\partial^2 C}{\partial q^2} \\ &= \frac{\partial^2 F}{\partial q^2} q + 2 \frac{\partial F}{\partial q} - \frac{\partial^2 C}{\partial q^2} < 0 \end{aligned}$$

To guarantee this, we need $\frac{\partial F}{\partial q} < 0$ and $\frac{\partial^2 F}{\partial q^2} < 0$ and $\frac{\partial^2 C}{\partial q^2} > 0$. This means that we need F to be decreasing and concave, and C to be convex.

4.5 Jehle and Reny 1.23

Question: Prove Theorem 1.3 Namely: Let \succsim be represented by $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$. Then:

- i) $u(x)$ is strictly increasing iff \succsim is strictly monotonic.
- ii) $u(x)$ is quasi-concave if and only if \succsim is convex.

iii) $u(x)$ is strictly quasi-concave if and only if \succsim is strictly convex.

Solution:

Recall the definitions

- \succsim is strictly monotonic if for all $x, y \in \mathbb{R}_+^N$, $x \geq y \implies x \succsim y$ and $x \gg y \implies x \succ y$
- \succsim is convex if $x \succsim y \implies \lambda x + (1 - \lambda)y \succsim y, \forall \lambda \in [0, 1]$
- \succsim is strictly convex if $x \succsim y \implies \lambda x + (1 - \lambda)y \succ y, \forall \lambda \in (0, 1)$ and $x \neq y$

- i) (\Rightarrow) Suppose $u(\cdot)$ is strictly increasing. Let $x, y \in \mathbb{R}_+^N$ with $x \geq y$. Since $u(\cdot)$ is increasing, $u(x) \geq u(y)$, so $x \succsim y$. Moreover, if $x > y$, $u(x) > u(y)$, so $x \succ y$, which implies \succsim is strictly monotonic.
(\Leftarrow) Suppose \succsim is strictly monotonic. Let $x, y \in \mathbb{R}_+^N$ with $x > y$. By strict monotonicity, $x \succ y$, so $u(x) > u(y)$, which implies u is strictly increasing.
- ii) (\Rightarrow) Suppose $u(\cdot)$ is quasi-concave. Let $x, y \in \mathbb{R}_+^N$ with $x \succsim y$. Since quasi-concave functions have convex upper-level sets, the set $S = \{z | u(z) \geq u(y)\}$ is convex. Note $x, y \in S$, so for any $\lambda \in [0, 1]$, $u(\lambda x + (1 - \lambda)y) \geq u(y)$. Thus $\lambda x + (1 - \lambda)y \succsim y$, so \succsim is convex.
(\Leftarrow) Suppose \succsim is convex. Let $x, y \in \mathbb{R}_+^N$. Define $z = \arg \min_{s \in \{x, y\}} u(s)$. That is, z is the least-preferred element between x and y . If $x \sim y$, assign z however you like. Since \succsim is convex, for any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \succsim z$, so $u(\lambda x + (1 - \lambda)y) \geq u(z) = \min\{u(x), u(y)\}$, so $u(\cdot)$ is quasi-concave
- iii) The argument follows the exact same logic as part (b), with weak inequalities (preferences) replaced by strict inequalities (preferences) as needed, and taking $\lambda \in (0, 1)$ instead of $[0, 1]$.