Intermediate Micro: Recitation 7
Demand Functions

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1 Overview and Notation

Let's re-state the consumer's **utility maximization problem** (UMP):

$$\max_{x_1,x_2} u(x_1,x_2)$$
$$\text{s.t. } p_1x_1+p_2x_2 \le M$$

As we know, the solution from this is the demand function (formally, it is called the **Marshallian** or **Walrasian demand function**), which we write as $x(p,M)$.[1] This is the choice variable (quantity demanded) as a function of the parameters (prices and income). This recitation covers properties of the demand function.

Since the demand function is a function of three variables (p_1, p_2, M), we want to see how demand changes with respect to each parameter. This means we will usually change one variable of interest (v_1) while holding the other two variables (v_2, v_3) fixed. To make it super clear, I will write the demand function of good i as $x_i(v_1;v_2,v_3)$, where the semi-colon (;) separates the changing variable to the fixed variables.

As economists, we will also want to have graphical interpretation. In this setup, you can think of x_i as the dependent variable and v_1 as the independent variable. So we want to graph x_i as a function of v_1 (for a given level of v_2 and v_3). However, as economists, we also do a weird thing of plotting the *independent* variable on the y-axis and the *dependent* variable on the x-axis. This means two important things:

1. You will always plot quantity (x_i) on the x-axis and the variable of interest (v_1) on the y-axis. So when you want to plot a curve, you will always want to invert the demand function to get v_1 as a function of x_i, i.e. solve for v_1 to get $v_1(x_i;v_2,v_3)$.
2. If v_1 changes, we move along the curve to get a new value of x_i. If v_2 or v_3 changes, we have a shift of the curve.

[1] Recall that p is shorthand for p_1,p_2, so we can write this more verbosely as $x(p_1,p_2,M)$

Notice that this graph is being plotted on the x_i-v_1 axis. When we draw our usual indifference curves and budget sets, we do this on the x_1-x_2 axis. However there is a very nice mapping between the two. The thought experiment goes as follows:

- Change the value of v_1, while holding fixed v_2 and v_3
- Re-solve the consumer's UMP (i.e. find a new optimal bundle given the new parameter values)
- Observe how your quantity x_i changes as v_1 changes. This gives you a point on the $x_i(v_1; v_2, v_3)$ curve
- If you do this for many v_1 values and connect up your *optimal* (x_1, x_2) values, this gives you an **offer curve**

So the $x_i(v_1; v_2, v_3)$ curve is plotted on the x_i-v_1 axis and tells you directly how x_i changes in response to v_1. The offer curve tells you how the bundle (x_1, x_2) changes as v_1 changes. It is plotted on the x_1-x_2 axis and you can think of it as the locus of points that satisfy $(x_1(v_1; v_2, v_3), x_2(v_1; v_2, v_3))$. To find a formula for the offer curve, you take the expression of x_2 as a function of x_1 that you get after the tangency condition (i.e. $x_2 = h(x_1, p)$). Then you plug in $v_1(x_1; v_2, v_3)$ to replace v_1. This will give you a formula of the form $x_2(x_1; v_2, v_3)$, i.e. you want to ensure that v_1 doesn't appear anywhere in the equation. Plotting this function will give you the offer curve.

2 Function of Own Price

Here, our variable of interest is the good's own price (p_i), so we want to plot $x_i(p_i; p_j, M)$. In other words, we have $v_1 = p_i, v_2 = p_j, v_3 = M$. This is asking: how does quantity of good i change as the price of good i changes (holding everything else fixed)? This is just gives us the usual **demand curve** that you've seen back in Principles! The inverse of this, $p_i(x_i; p_j, M)$, is called (unimaginatively) the **inverse demand** curve**.** The offer curve here is called the **price offer curve**.

2.1 Cobb-Douglas

Let's take our usual Cobb-Douglas demand functions:

$$x_1(p, M) = x_1(p_1; p_2, M) = \frac{\alpha}{\alpha + \beta} \cdot \frac{M}{p_1} \qquad x_2(p, M) = x_2(p_2; p_1, M) = \frac{\beta}{\alpha + \beta} \cdot \frac{M}{p_2}$$

Given that p_i is in the denominator for each $x_i(p, M)$, this tells us that as own price increases, then quantity demanded decreases (this is simply the Law of Demand).

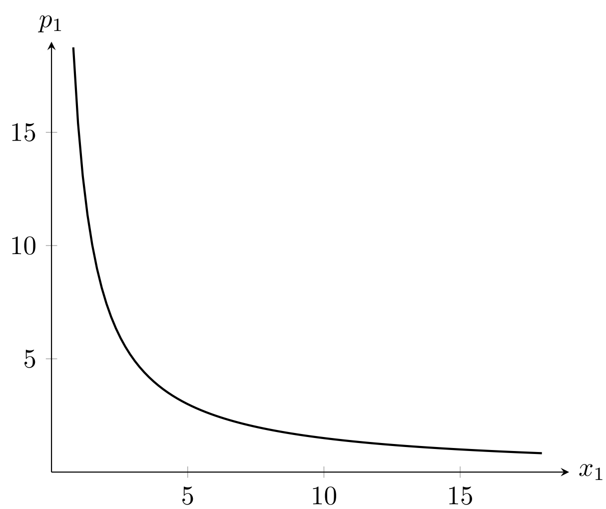
To plot demand as a function of own price, we want to solve for p_i. Re-arranging gives us the inverse demand curve $p_i(x_i; p_j, M)$:

$$p_1(x_1; p_2, M) = \frac{\alpha}{\alpha + \beta} \cdot \frac{M}{x_1} \qquad p_2(x_2; p_1, M) = \frac{\beta}{\alpha + \beta} \cdot \frac{M}{x_2}$$

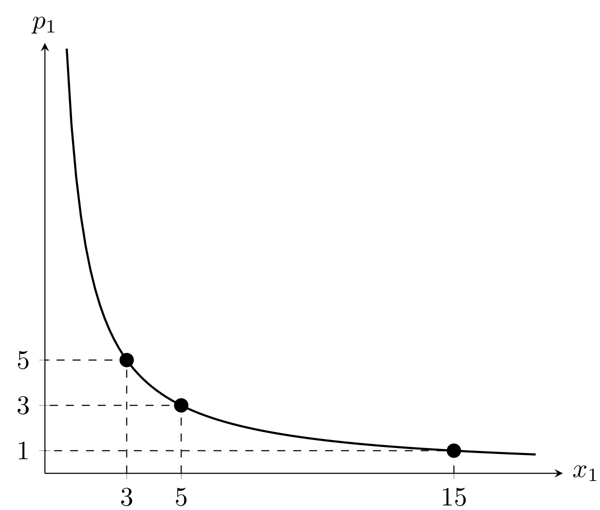
Let's see a concrete example. Suppose $\alpha = 0.75, \beta = 0.25, p_1 = 3, p_2 = 1, M = 20$. This gives us the following demand and inverse demand curves:

$$x_1(p_1; 1, 20) = \frac{3}{4} \cdot \frac{20}{p_1} = \frac{15}{p_1} \qquad x_2(p_2; 3, 20) = \frac{1}{4} \cdot \frac{20}{p_2} = \frac{5}{p_2}$$
$$p_1(x_1; 1, 20) = \frac{15}{x_1} \qquad p_2(x_2; 3, 20) = \frac{5}{x_2}$$

Plotting the demand curve for x_1 gives us the following graph. Notice that x_1 is on the x-axis and p_1 is on the y-axis: (so we say that we are "plotting the demand curve", but actually, we use the inverse demand curve to plot it)



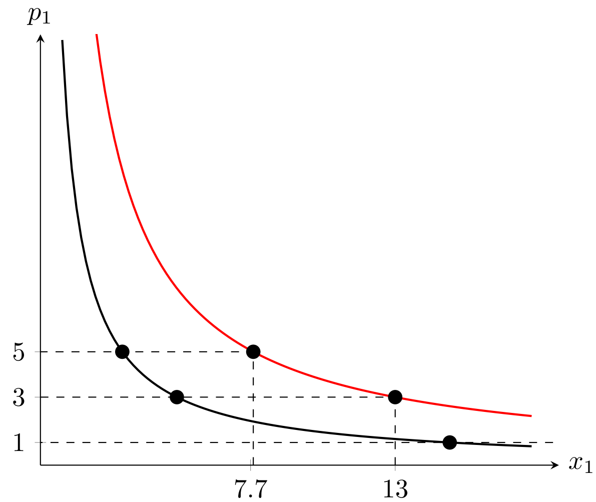
Now, we want to see what happens when prices change. What is the demand of good 1 if $p_1 = 1$? If $p_1 = 3$? If $p_1 = 5$? We can read all of these directly off the graph. Notice that we move along the demand curve.



Next, let's consider a change in income. Suppose M increases to 52. As you should recall from Principles, an increase in income means that quantity demanded increases at each and every price level. This means that the demand curve should shift outwards. We can see this in our example where the demand function for good 1 becomes:

$$x_1(p_1; p_2, M) = \frac{3}{4} \cdot \frac{52}{p_1} = \frac{39}{p_1}$$

And graphically, we can see this in the new red curve.



Notice that for the same price level as before, we get higher amounts of the good demanded.

Now, let's do the price offer curve. Recall for the offer curve is the locus of points that satisfy the conditions:

$$(x_1(p_1; p_2, M), x_2(p_1; p_2, M)) = \left(\frac{\alpha}{\alpha + \beta} \cdot \frac{M}{p_1}, \frac{\beta}{\alpha + \beta} \cdot \frac{M}{p_2} \right)$$

For the same starting parameter values $\alpha = 0.75, \beta = 0.25, p_1 = 3, p_2 = 1, M = 20$, the offer curve will be defined by:

$$(x_1(p_1; 1, 20), x_2(p_1; 1, 20)) = \left(\frac{15}{p_1}, 5 \right)$$

As we know, x_2 is not affected by p_1, so we shouldn't be surprised to see a constant value. This is an easy locus of points to plot (it's just $x_2 = 5$ for any value of x_1), but let's do it the "proper" way for practice. Recall that from the tangency condition we get:

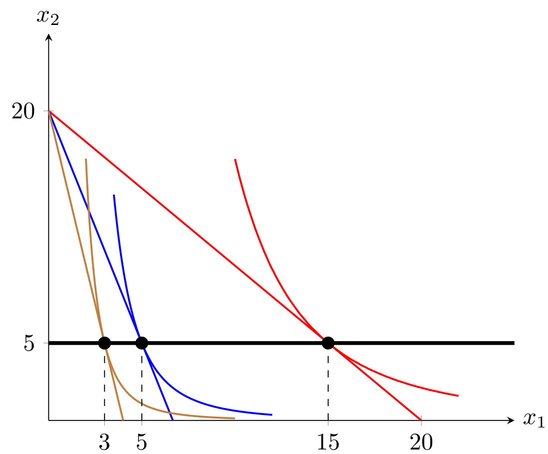
$$x_2 = \frac{\beta}{\alpha} \cdot \frac{p_1}{p_2} x_1$$

Notice there is a p_1 in here, so we want to get rid of it by plugging in $p_1(x_1; p_2, M) = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{x_1}$. This gives us:

$$\begin{aligned} x_2 &= \frac{\beta}{\alpha} \cdot \frac{1}{p_2} \cdot \left(\frac{\alpha}{\alpha + \beta} \cdot \frac{M}{x_1} \right) \cdot x_1 \\ \therefore x_2(x_1; p_2, M) &= \frac{\beta}{\alpha + \beta} \cdot \frac{M}{p_2} \end{aligned}$$

Now we don't have p_1 in the equation, which is what we wanted. But we also don't have x_1 - which tells us (as we expected) - that the optimal x_2 is independent of x_1 as we change p_1. Plugging in the parameter values gets us $x_2(x_1; 1, 20) = \frac{1}{4} \cdot \frac{20}{1} = 5$, which is the formula for the price offer curve.

Let's plot the price offer curve (in black), and the optimal bundles (+ BC + IC) for $p_1 = 1, 3$, and 5. As we should expect, each of the tangency points lies on the price offer curve (since the price offer curve is a locus of all optimal bundles, it wouldn't make sense to have an optimal bundle that *wasn't* on the offer curve).



2.2 Giffen Goods

For almost all goods, we think that the Law of Demand holds (so we just call these **ordinary goods**). However, it could be the case that as price increases, demand actually increases too. We call these types of goods **Giffen goods**. Later on in the class, we'll see exactly why this happens. But let's see an example of this with the following utility function:[2]

$$u(x_1, x_2) = \ln(x_1 - 1) - 4\ln(10 - x_2)$$

Here, to make everything valid (since ln(0) is undefined), we must have $x_1 > 1$ and $0 \leq x_2 < 10$. Basically, we are imposing a minimum on x_1 and a maximum on x_2. Let's solve this the way we usually do:

$$\begin{aligned}
|MRS| = \frac{\frac{1}{x_1-1}}{\frac{-4}{10-x_2}(-1)} = \frac{\frac{1}{x_1-1}}{\frac{4}{10-x_2}} &= \frac{p_1}{p_2} \\
\frac{10 - x_2}{4(x_1 - 1)} &= \frac{p_1}{p_2} \\
x_2 &= 10 - \frac{4p_1}{p_2}(x_1 - 1)
\end{aligned}$$

Plug this into the budget constraint:

$$\begin{aligned}
p_1 x_1 + p_2 \left(10 - \frac{4p_1}{p_2}(x_1 - 1)\right) &= M \\
p_1 x_2 + 10p_2 - 4p_1 x_1 + 4p_1 &= M \\
-3p_1 x_1 &= M - 4p_1 - 10p_2 \\
x_1 &= \frac{-M}{3p_1} + \frac{4}{3} + \frac{10p_2}{3p_1}
\end{aligned}$$

[2]See Haagsma (2012): `https://www.hindawi.com/journals/isrn/2012/608645/`

$$\therefore x_1(p, M) = \frac{4}{3} - \frac{1}{3p_1}(M - 10p_2)$$

Notice that the effect of a change in p_1 depends on the sign of the term in the brackets: $M - 10p_2$. Let's call this value α. If $\alpha > 0$, then as p_1 *increases*, $\frac{\alpha}{3p_1}$ decreases, which means we are subtracting a smaller number, which means that x_1 *increases*. But if $\alpha < 0$, then the opposite is true.

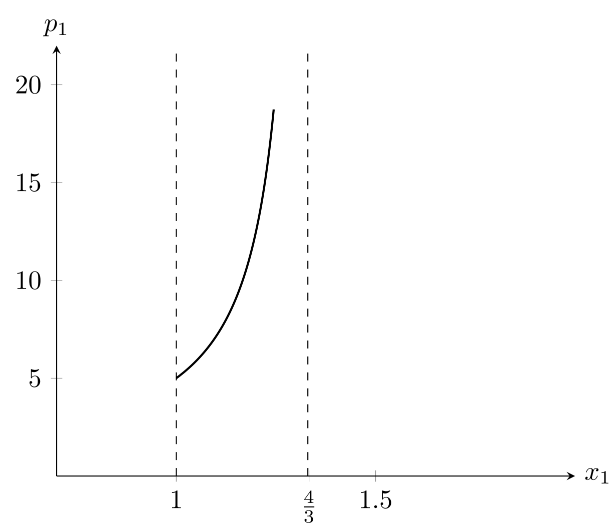
To see this more clearly, let's suppose we have $M = 25$ and $p_2 = 2$. Then our demand function becomes:

$$x_1(p_1; 2, 25) = \frac{4}{3} - \frac{1}{3p_1}(25 - 20) = \frac{4}{3} - \frac{5}{3p_1}$$

Inverting the demand function gets us:

$$\begin{aligned} \frac{5}{3p_1} &= \frac{4 - 3x_1}{3} \\ p_1 &= \frac{5}{3}\left(\frac{3}{4 - 3x_1}\right) \\ p_1(x_1; 2, 25) &= \frac{5}{4 - 3x_1} \end{aligned}$$

Note that we had a restriction that $x_1 > 1$; this means that $p_1 > \frac{5}{4-3\times 1} = 5$. Similarly, we can't have $p_1 < 0$, so we also need $4 - 3x_1 > 0 \implies x_1 < \frac{4}{3}$.[3] So we are actually going to be looking at a very small window of relatively small amount of x_1. However, in this very small window we get the Giffen good behavior we were looking for:



3 Function of Other Price

Here, our variable of interest is the price of the *other* good (p_j), so we want to plot $x_i(p_j; p_i, M)$. In other words, we have $v_1 = p_j, v_2 = p_i, v_3 = M$. This is asking: how does quantity of good i change as the price

[3] To be very technical, we actually only have demand defined over $p_1 \in (5, 18.75)$ or similarly $x_1 \in (1, 1.24)$

of good j changes (holding everything else fixed)? This tells us whether the two goods are complements or substitutes. The offer curve here is still the **price offer curve**.[4]

3.1 Perfect Complements

Recall that if we have a utility function of the form $u(x_1, x_2) = \min\{\beta x_1, \alpha x_2\}$, then the demand functions are:

$$x_1(p, M) = \frac{\alpha M}{\alpha p_1 + \beta p_2} \qquad\qquad x_2(p, M) = \frac{\beta M}{\alpha p_1 + \beta p_2}$$

Let's consider for the following parameter values: $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_1 = 3, p_2 = 2, M = 12$. Let's find demand as a function of the other price for the above parameter values:

$$\begin{aligned} x_1(p_2; 3, 12) &= \frac{\frac{1}{3} \cdot 12}{\frac{1}{3} \cdot 3 + \frac{1}{4} \cdot p_2} = \frac{4}{1 + \frac{1}{4}p_2} = \frac{16}{4 + p_2} \\ x_2(p_1; 2, 12) &= \frac{\frac{1}{4} \cdot 12}{\frac{1}{3} \cdot p_1 + \frac{1}{4} \cdot 2} = \frac{3}{\frac{1}{3}p_1 + \frac{1}{2}} = \frac{18}{2p_1 + 3} \end{aligned}$$

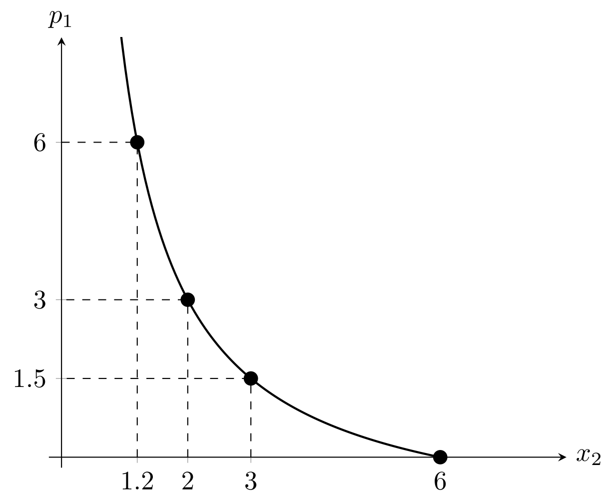
Since the price of the other good is in the denominator, this tells us that as the price of good j increases, the quantity demanded of good i decreases. This tells us that the goods are complements (shockingly, it's also in name of these preferences).

Next, we'll plot the demand function for x_2 as a function of p_1. Again, we should start by first solving for p_1 to get us the inverse function $p_1(x_2; p_2, M)$:

$$\begin{aligned} x_2 &= \frac{\frac{1}{4}M}{\frac{1}{3}p_1 + \frac{1}{4}p_2} = \frac{3M}{4p_1 + 3p_2} \\ \therefore 4p_1 + 3p_2 &= \frac{3M}{x_2} \\ p_1 &= \frac{1}{4}\left(\frac{3M}{x_2} - 3p_2\right) \\ p_1(x_2; p_2, M) &= \frac{3M}{4x_2} - \frac{3}{4}p_2 \\ \therefore p_1(x_2; 2, 12) &= \frac{9}{x_2} - \frac{3}{2} \end{aligned}$$

Let's plot this function and highlight a few price points as well: $p_1 = 1.5, p_1 = 3, p_1 = 6$, and $p_1 = 0$. This gives us the graph below. Note a few important things. First, we have x_2 labeled on the x-axis but p_1 on the y-axis. Second, we don't actually care about $p_1 = 0$ (we always assume prices are strictly positive), but the point is that this curve touches the axis (unlike, the Cobb-Douglas demand curve we drew before), so it's just an important point about always checking the intercepts.

[4] Obviously, "own price" and "other price" are relative terms. One good's own price offer curve is another good's other price offer curve. In a homework/exam question, you will be asked to draw the price offer curve for a particular price (e.g. "the p_1 price offer curve"). If it doesn't say which price, you can assume it is the own price (e.g. "draw the price offer curve for x_1" means draw the p_1 offer curve).



Next, let's change the price of good 2 and have it increase to $p_2 = 4$. Obviously, by the law of demand, we know that this should decrease the demand of x_2. But the question is actually: how does the change in p_2 affect the relationship between x_2 and p_1? We can see from the formula of $p_1(x_2; p_2, M)$ that a change in p_2 only affects the intercept of the graph, but not the slope. Likewise, what if instead we had a decrease in income to $M = 8$. Now, demand will fall but the strength of the relationship between x_2 and p_1 will increase (i.e. there will be flatter slope). To see this in the graph, note that these new parameter values give us the following functions:

$$\begin{aligned} x_2(p_1; 4, 12) =& \frac{3 \cdot 12}{4p_1 + 3 \cdot 4} \\ =& \frac{9}{p_1 + 3} \end{aligned} \qquad \begin{aligned} x_2(p_1; 2, 8) =& \frac{3 \cdot 8}{4p_1 + 3 \cdot 2} \\ =& \frac{12}{2p_1 + 3} \end{aligned}$$

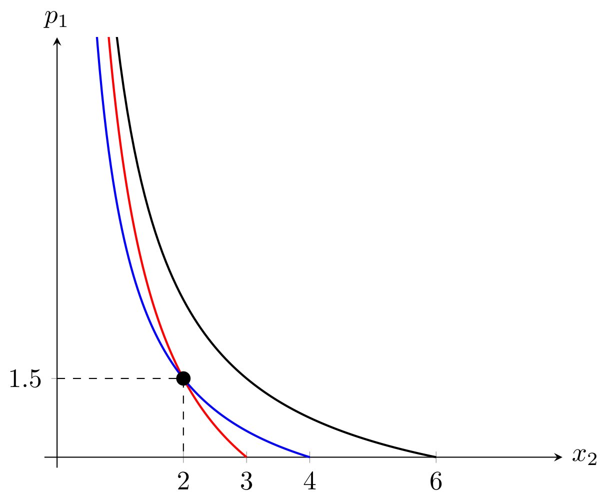
$$\begin{aligned} p_1(x_2; 4, 12) =& \frac{3 \cdot 12}{4x_2} - \frac{3}{4} \cdot 4 \\ =& \frac{9}{x_2} - 3 \end{aligned} \qquad \begin{aligned} p_1(x_2; 2, 8) =& \frac{3 \cdot 8}{4x_2} - \frac{3}{4} \cdot 2 \\ =& \frac{6}{x_2} - \frac{3}{2} \end{aligned}$$

I will plot the change in p_2 graph in red and the change in M graph in blue. This gives us the graph below.

Next, let's do the price offer curve. Here, we will do the p_1 price offer curve. Let's use the same initial parameter values $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, p_1 = 3, p_2 = 2, M = 12$. So our price offer curve is the locus of points satisfying the condition:

$$\begin{aligned} (x_1(p_1; p_2, M), x_2(p_1; p_2, M)) &= \left(\frac{\frac{1}{3}M}{\frac{1}{3}p_1 + \frac{1}{4}p_2}, \frac{\frac{1}{4}M}{\frac{1}{3}p_1 + \frac{1}{4}p_2} \right) = \left(\frac{4M}{4p_1 + 3p_2}, \frac{3M}{4p_1 + 3p_2} \right) \\ \therefore (x_1(p_1; 2, 12), x_2(p_1; 2, 12)) &= \left(\frac{48}{4p_1 + 6}, \frac{36}{4p_1 + 6} \right) = \left(\frac{24}{2p_1 + 3}, \frac{18}{2p_1 + 3} \right) \end{aligned}$$

This is a bit harder to visualize than the Cobb-Douglas locus. So let's calculate the formula for the price offer curve. First, the "tangency condition" of perfect complements is that the optimal bundle must occur at



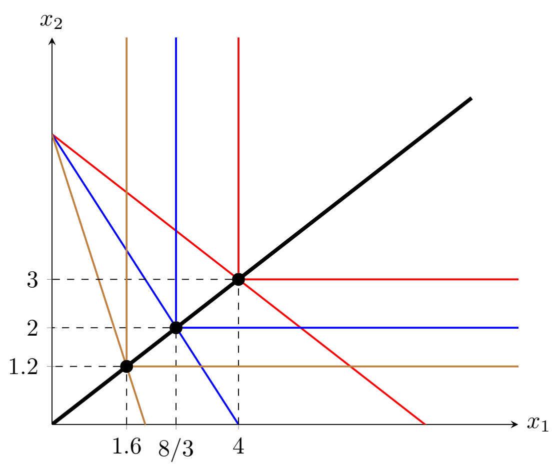
the kink of an indifference curve:

$$x_2 = \frac{\beta}{\alpha}x_1$$

Notice, this doesn't have p_1 in it, so we're done! This is exactly the formula for the price offer curve:

$$x_2(x_1; p_2, M) = \frac{\beta}{\alpha}x_1 = \frac{1/4}{1/3}x_1 = \frac{3}{4}x_1$$

Now we plot the offer curve as well as the optimal bundles for $p_1 = 1.5, p_1 = 3, p_1 = 6$. This gives us the following graph:



3.2 Perfect Substitutes

Let's consider a consumer with perfect substitute preferences: $u(x_1, x_2) = \beta x_1 + \alpha x_2$. We know their demand function looks as follows:

$$(x_1, x_2) = \begin{cases} \left(0, \frac{M}{p_2}\right) & \text{if } \frac{\beta}{\alpha} < \frac{p_1}{p_2} \\ \text{Budget Line} & \text{if } \frac{\beta}{\alpha} = \frac{p_1}{p_2} \\ \left(\frac{M}{p_1}, 0\right) & \text{if } \frac{\beta}{\alpha} > \frac{p_1}{p_2} \end{cases}$$

This means that demand for x_2 as a function of p_1 can be expressed as:

$$x_2(p_1; p_2, M) = \begin{cases} \frac{M}{p_2} & \text{if } \frac{\beta}{\alpha} < \frac{p_1}{p_2} \implies p_1 > \frac{\beta}{\alpha}p_2 \\ \in \left[0, \frac{M}{p_2}\right] & \text{if } \frac{\beta}{\alpha} = \frac{p_1}{p_2} \implies p_1 = \frac{\beta}{\alpha}p_2 \\ 0 & \text{if } \frac{\beta}{\alpha} > \frac{p_1}{p_2} \implies p_1 < \frac{\beta}{\alpha}p_2 \end{cases}$$

Let's set $\alpha = 2$ and $\beta = 1$. We will consider the following parameter values:

Parameters: (plot in blue)
$p_1 = 1, p_2 = 2, M = 10$

Demand: $x_2(p_1; 2, 10) =$

$$\begin{cases} 5 & \text{if } p_1 > 1 \\ \in [0, 5] & \text{if } p_1 = 1 \\ 0 & \text{if } p_1 < 1 \end{cases}$$

Inverse: $p_1(x_2; 2, 10) =$

$$\begin{cases} \in (0, 1] & \text{if } x_2 = 0 \\ 1 & \text{if } x_2 \in (0, 5) \\ \in [1, \infty) & \text{if } x_2 \geq 5 \end{cases}$$

Parameters: (plot in red)
$p_1 = 1, p_2 = 5, M = 10$

Demand: $x_2(p_1; 5, 10) =$

$$\begin{cases} 2 & \text{if } p_1 > 2.5 \\ \in [0, 2] & \text{if } p_1 = 2.5 \\ 0 & \text{if } p_1 < 2.5 \end{cases}$$

Inverse: $p_1(x_2; 5, 10) =$

$$\begin{cases} \in (0, 2.5] & \text{if } x_2 = 0 \\ 2.5 & \text{if } x_2 \in (0, 2) \\ \in [2..5, \infty) & \text{if } x_2 \geq 2 \end{cases}$$

Parameters: (plot in orange)
$p_1 = 1, p_2 = 2, M = 8$

Demand: $x_2(p_1; 2, 8 =$

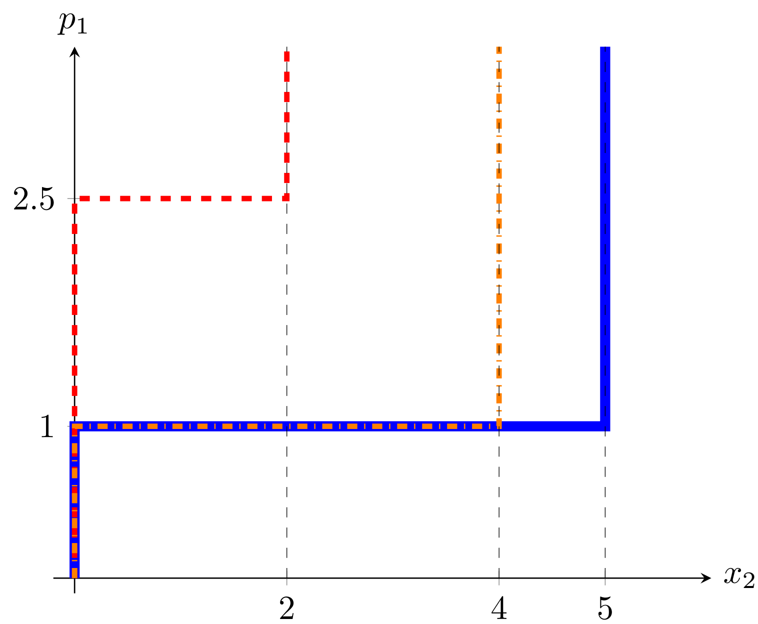
$$\begin{cases} 5 & \text{if } p_1 > 1 \\ \in [0, 5] & \text{if } p_1 = 1 \\ 0 & \text{if } p_1 < 1 \end{cases}$$

Inverse: $p_1(x_2; 2, 8) =$

$$\begin{cases} \in (0, 1] & \text{if } x_2 = 0 \\ 1 & \text{if } x_2 \in (0, 4) \\ \in [1, \infty) & \text{if } x_2 \geq 4 \end{cases}$$

Plotting this gives us the graph below. The colors are a little bit hard to see, so definitely try drawing it yourself. But you should notice that a change in M doesn't affect the cutoff value (where the straight line occurs), only how much x_2 is purchased (i.e. the x-intercept). However, a change in p_2 affects both how much x_2 is purchased, as well as where the cut-off occurs.

This tells us that at the current price of $p_1 = 1$, in the first and third case, the consumer is indifferent between any bundles on the budget line, but in the second case (when $p_2 = 5$, they only purchase x_1 (i.e. $x_2 = 0$). These curves look a bit odd, but overall they are trending upwards. This is true for any two goods that are substitutes: an *increase* in p_j leads to an *increase* in x_i.



4 Function of Income

Here, our variable of interest is the consumer's income (M), so we want to plot $x_i(M; p_i, p_j)$. In other words, we have $v_1 = M, v_2 = p_i, v_3 = p_j$. This is asking: how does quantity of good i change as income changes (holding everything else fixed)? This tells us whether the good is a normal or inferior good. We call this curve the **Engel curve**. The offer curve here is called the **income offer curve** or the **income expansion path**.

4.1 Cobb-Douglas

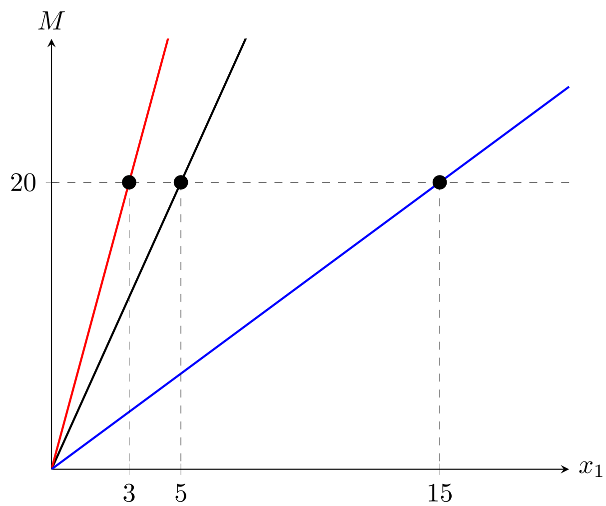
Let's go back to our Cobb-Douglas example, with parameter values $\alpha = 0.75, \beta = 0.25, p_1 = 3, p_2 = 1, M = 20$. Our Engel curves are:

$$x_1(M; p_1, p_2) = \frac{\alpha}{\alpha + \beta} \cdot \frac{M}{p_1} \qquad x_2(M; p_1, p_2) = \frac{\beta}{\alpha + \beta} \cdot \frac{M}{p_2}$$
$$\therefore x_1(M; 3, 1) = \frac{M}{4} \qquad \therefore x_2(M; 3, 1) = \frac{M}{4}$$

As per usual, we will want to invert the Engel curve to make plotting them easier:

$$M(x_1; p_1, p_2) = \frac{\alpha + \beta}{\alpha} \cdot p_1 x_1 \qquad M(x_2; p_1, p_2) = \frac{\alpha + \beta}{\beta} \cdot p_2 x_2$$
$$\therefore M(x_1; 3, 1) = 4x_1 \qquad \therefore x_2(M; 3, 1) = 4x_2$$

Next, we can plot the Engel curve for $x_1(M; 3, 1)$ (in black) as well as $x_1(M; 5, 1)$ (in red) and $x_1(M; 1, 2)$ (in blue), which I'll leave for you to solve for. Notice we have x_1 on the x-axis and M on the y-axis. I'll also plot the optimal bundle for the given income $M = 20$.



As we expect, Cobb-Douglas preferences give us normal goods: as income increases, we demand more of the goods. We also know a property of Cobb-Douglas is that share of income spent on each good stays constant. Since we are holding prices fixed, that also means that an increase in income will result in a proportional increase in quantity demanded. In other words, we get linear Engel curves. This is exactly what we see above. While a change in price affects the slope of the Engel curve, it is always linear.

This isn't just specific to Cobb-Douglas, we also get this for perfect substitutes and perfect complements too. All of these utilities represent preferences that are called **homothetic preferences**. Homothetic preferences are preferences where if you have $x \succsim y$, then you must also have $\lambda x \succsim \lambda y$, for some constant number $\lambda > 0$. In other words, you can scale up/down everything in the bundle, and the preference ordering stays the same. In terms of utility functions, homothetic preferences have a utility representation where $u(\lambda x_1, \lambda x_2) = \lambda u(x_1, x_2)$.[5] Note that I have emphasized "a" in the previous statement. We aren't saying that all utility functions that represent homothetic preferences satisfy this property. We are saying that if you have homothetic preferences, we can find at least one utility function that represents your preferences *and* satisfies this property.

To give an example, let's take the C-D utility function: $u(x_1, x_2) = x_1^2 x_2^2$. You know this means that the preferences are homothetic. Let's check if the utility function satisfies the property: $u(\lambda x_1, \lambda x_2) = (\lambda x_1)^2 (\lambda x_2)^2 = \lambda^2 x_1^2 \lambda^2 x_2^2 = \lambda^4 x_1^2 x_2^2 = \lambda^4 u(x_1, x_2)$. Clearly it doesn't - so what's going on?! Remember that utility representation isn't unique, so the following utility function also represents the same preferences: $\tilde{u}(x_1, x_2) = x_1^{0.5} x_2^{0.5}$. It also satisfies the property:

$$\tilde{u}(\lambda x_1, \lambda x_2) = (\lambda x_1)^{0.5} (\lambda x_2)^{0.5} = \lambda^{0.5} x_1^{0.5} \lambda^{0.5} x_2^{0.5} = \lambda x_1^{0.5} x_2^{0.5} = \lambda \tilde{u}(x_1, x_2)$$

So, we have found one utility that ticks both boxes, hence we're not violating the conditions of homothetic preferences.

Homothetic preferences always have linear Engel curves that pass through the origin. This is really handy. Suppose we have solved a consumer's UMP for an income level M and got the solution (x_1^*, x_2^*). Then, if

[5] You may have seen something similar in a macro or math class. If a function has this property, we say it is "homogenous of degree 1"

we scale the consumer's income to λM, it must be that the solution is $(\lambda x_1^*, \lambda x_2^*)$. You can see this directly from the Engel curve. Since it is linear and passes through the origin, we can write it as follows $M = bx_i$, where b is the slope of the Engel curve for good i (and is a constant number). Therefore, $x_i = M/b$ tells us the optimal x_i for any level of income M. Suppose we have $x_i^* = M^*/b$ for income level M^*. If you plug in λM^*, then you get the optimal quantity as $\lambda M^*/b = \lambda x_i^*$, which is exactly what we expected.

Let's finish off by doing the income offer curve. We know that this is locus of points that satisfy:

$$(x_1(p_1; p_2, M), x_2(p_1; p_2, M)) = \left(\frac{\alpha}{\alpha+\beta} \cdot \frac{M}{p_1}, \frac{\beta}{\alpha+\beta} \cdot \frac{M}{p_2}\right)$$

For the same starting parameter values $\alpha = 0.75, \beta = 0.25, p_1 = 3, p_2 = 1, M = 20$, the offer curve will be defined by:

$$(x_1(M; 3, 1), x_2(M; 3, 1)) = \left(\frac{M}{4}, \frac{M}{4}\right)$$

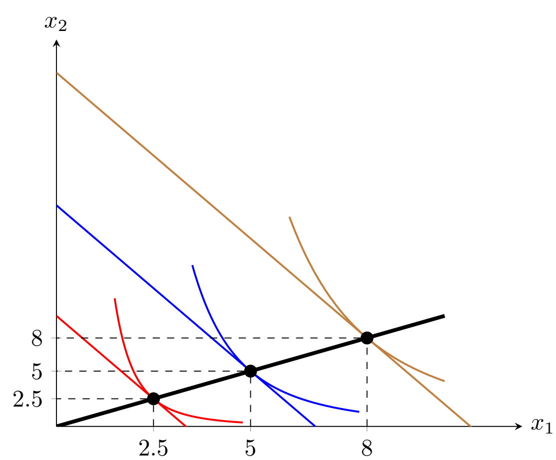
To find the formula, we again start from $x_2 = h(x_1, p)$:

$$x_2 = \frac{\beta}{\alpha} \cdot \frac{p_1}{p_2} x_1$$

Notice that there is no M in here, so we're done:

$$\therefore x_2(x_1; 3, 1) = \frac{1}{3} \cdot \frac{3}{1} \cdot x_1 = x_1$$

Let's plot the income offer curve (in black), and the optimal bundles (+ BC + IC) for $M = 10, 20$, and 32.



4.2 Inferior, Necessity, Luxury

Normal goods are what we are most used to. But, we can also have **inferior goods**. These are goods where quantity demanded decreases as income increases. In other words, we have downward sloping Engel curves. The most famous of these are Giffen goods. However, keep in mind the following (which many don't!!):

All Giffen goods are inferior goods, but not all inferior goods are Giffen goods

For normal goods, we can divide them further into two sub-types:

- **Necessities** are goods where quantity demanded increases as income increases, but the *share* of income spent on the good *decreases* as income increases
- **Luxuries** are goods where quantity demanded increases as income increases, but the *share* of income spent on the good *increases* as income increases

Let's not worry about finding utility functions that satisfy these conditions. Instead, imagine you had two consumers A and B that had the following Engel curves:[6]

$$x_1^A(M; p_1, p_2) = \frac{M^2}{p_1 p_2}$$
$$x_1^B(M; p_1, p_2) = \frac{p_2\sqrt{M}}{p_1}$$

To plot these, we need to invert them:

$$M^A(x_1; p_1, p_2) = \left(\frac{x_1}{p_1 p_2}\right)^{1/2}$$
$$M^B(x_1; p_1, p_2) = \left(\frac{p_1 x_1}{p_2}\right)^2$$

Suppose $p_1 = 1$, $p_2 = 1$ to keep things simple. This means our Engel curves and the inverted functions are:

$$x_1^A(M; 1, 1) = M^2 \qquad\qquad M^A(x_1; 1, 1) = \sqrt{x_1}$$
$$x_1^B(M; 1, 1) = \sqrt{M} \qquad\qquad M^B(x_1; 1, 1) = x_1^2$$

Now, let's check which person considers good 1 a necessity and which person considers it a luxury. For this, we should calculate the share of income spent on good 1. Let's call this share as $s_1 = \frac{p_1 x_1}{M}$ (i.e. expenditure on good 1 divided by income)

A:

$$\begin{aligned} s_1^A &= \frac{p_1 x_1^A(M; p_1, p_2)}{M} \\ &= \frac{p_1 \frac{M^2}{p_1 p_2}}{M} \\ &= \frac{M^2}{p_2 M} \\ &= \frac{M}{p_2} \\ &= M \end{aligned}$$

B:

$$\begin{aligned} s_1^B &= \frac{p_1 x_1^B(M; p_1, p_2)}{M} \\ &= \frac{p_1 \frac{p_2\sqrt{M}}{p_1}}{M} \\ &= \frac{p_2\sqrt{M}}{M} \\ &= \frac{p_2}{\sqrt{M}} \\ &= \frac{1}{\sqrt{M}} \end{aligned}$$

[6]Bonus question: from these functions, can you determine whether the goods are ordinary or Giffen? What about substitutes or complements with x_2?

So what we're interested in is how s_1 changes with income, i.e. $\partial s_1/\partial M$. Calculating this for each person:

$$A: \qquad \frac{\partial s_1^A}{\partial M} = 1 \qquad\qquad B: \qquad \frac{\partial s_1^B}{\partial M} = \frac{-1}{2\sqrt{M^3}}$$

Since $\frac{\partial s_1^A}{\partial M} > 0$, this means that as income increases, A spends a greater share on good 1, making it a luxury. Since $\frac{\partial s_1^B}{\partial M} < 0$, this means that as income increases, B spends a lower share on good 1, making it a necessity.

We can actually see this directly from the Engel curves. In general, since both goods are normal, then we need the first derivative of the Engel curve to be positive, but the second derivative (how fast the slope is changing) is what determines the category of the good.

Necessity:

$$\frac{\partial x_i(M; p_1, p_2)}{\partial M} > 0$$
$$\frac{\partial^2 x_i(M; p_1, p_2)}{\partial M^2} < 0$$

Luxury:

$$\frac{\partial x_i(M; p_1, p_2)}{\partial M} > 0$$
$$\frac{\partial^2 x_i(M; p_1, p_2)}{\partial M^2} > 0$$

This should be intuitive. As income increases, the increase in quantity demanded progressively gets smaller for necessity but progressively gets larger for luxuries. However, things are a little bit more confusing if we are looking at the inverted function:

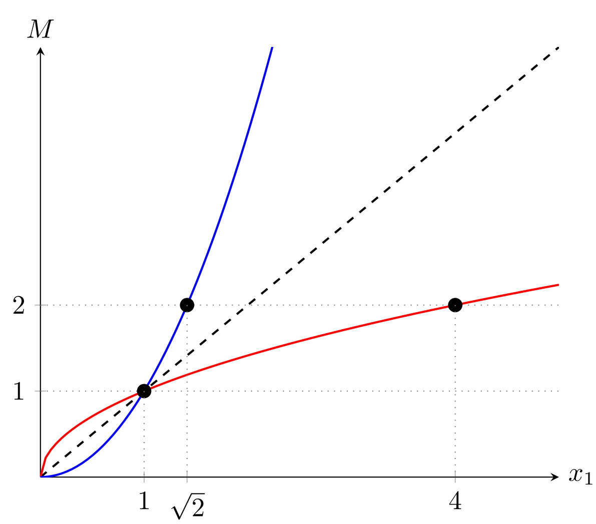
Necessity:

$$\frac{\partial M(x_i; p_1, p_2)}{\partial x_i} > 0$$
$$\frac{\partial^2 M(x_i; p_1, p_2)}{\partial x_i^2} > 0$$

Luxury:

$$\frac{\partial M(x_i; p_1, p_2)}{\partial x_i} > 0$$
$$\frac{\partial^2 M(x_i; p_1, p_2)}{\partial x_i^2} < 0$$

The first derivative is still positive but now the signs of the second derivative are flipped. Don't get this mixed up! At first it might seem very odd, but there is some nice intuition. Let's plot the two Engel curves above on the x_1-M axis, with A in red and B in blue. I'll also plot the 45° degree line.



Notice that A has an Engel curve that has a relatively flat slope, while B has an Engel curve that has a relatively steep slope. In particular, we can see that for A, $\partial^2 M/\partial x_1^2 < 0$ (the slope is decreasing), while for B, $\partial^2 M/\partial x_1^2 > 0$ (the slope is increasing). Now consider a change of income from $M = 1$ to $M = 2$. Since $A's$ Engel curve is flatter, we get a huge increase in x_1 from $x_1 = 1$ to $x_1 = 4$. But for B, it's only a small change of $x_1 = 1$ to $x_1 = \sqrt{2} \approx 1.414$. Remember that prices are constant here, so the change in share is going to be driven by the change in quantity demanded (for the same change in income).

Finally, let's try to summarize this concisely. When you observe a change in income and then observe the a new optimal bundle, this tells you the type of each good (in terms of its income relationship). Suppose you start at an optimal bundle (x_1^*, x_2^*) with income equal to M. Then income increases to M', and you observe a new bundle (x_1', x_2'). We want to see whether quantity demanded increases (i.e. is $x_i' > x_i^*$?) and whether the share of income increases (i.e. is $s_i' = \frac{p_i x_i'}{M'} > \frac{p_i x_i^*}{M} = s_i^*$?). This first tells us whether it is an inferior or normal good, and the second tells us whether it is a necessity or luxury.

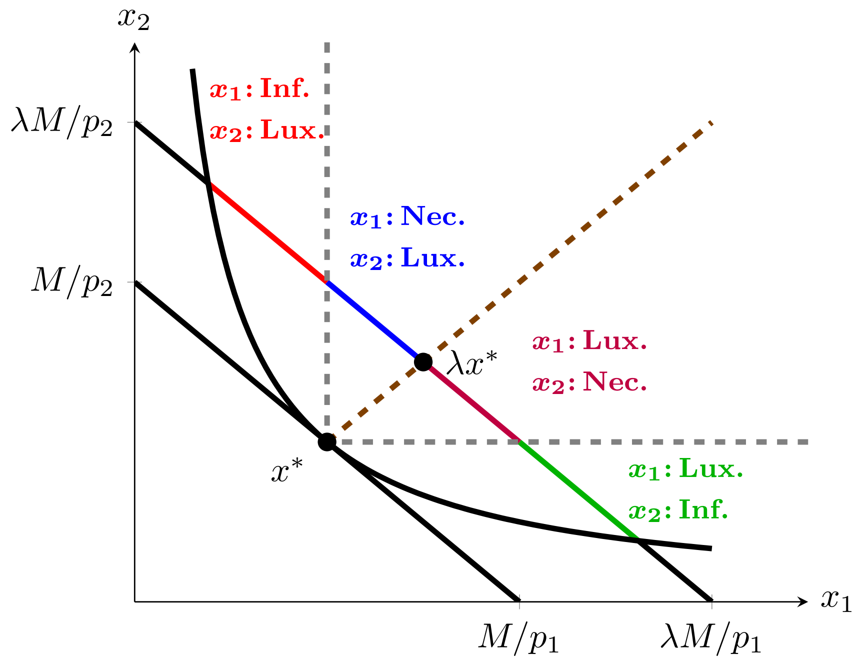
Before we go through this thought experiment, a few things to note.

- First, for any $M' > M$, we can always find some constant $\lambda > 1$ such that $M' = \lambda M$. λ just captures how much income has been scaled up.
- Second, we know that the budget constraint must always be satisfied at the optimal bundle. This means that $p_1 x_1^* + p_2 x_2^* = M$ and $p_1 x_1' + p_2 x_2' = M' = \lambda M$. You can do a little simple algebra to see the following relations.
  - If $x_i' = \lambda x_i^*$, then we must have $x_j' = \lambda x_j^*$. If $x_i' < \lambda x_i^*$, then we must have $x_j' > \lambda x_j^*$. If $x_i' > \lambda x_i^*$, then we must have $x_j' < \lambda x_j^*$.
  - We can *never* be in the case where both goods have $x_i' < \lambda x_i^*$.
  - If $x_i' < x_i^*$, then $x_i' < x_i^* < \lambda x_i^*$, and so it must be the case that $x_j > \lambda x_j^*$
- Third, if $x_i' > \lambda x_i^*$, then the share of income spent on x_i increases (call this s_i). To see this: $s_i' = \frac{p_i x_i'}{M'} > \frac{p_i \lambda x_i^*}{M'} = \frac{p_i \lambda x_i^*}{\lambda M} = \frac{p_i x_i^*}{M} = s_i^*$. Similarly, if $x_i' < \lambda x_i^*$, then $s_i' < s_i^*$ and if $x_i' = \lambda x_i^*$, then $s_i' = s_i^*$.

Now we are ready to consider all the cases:

1. If x_1 and x_2 both increase by a factor of λ ($x_1' = \lambda x_1^* > x_1^*$ and $x_2' = \lambda x_2^* > x_2^*$): then the preferences are homothetic and both goods are normal (but neither luxuries nor necessities)
2. If x_1 increases by more than a factor of λ ($x_1' > \lambda x_1^* > x_1^*$) and x_2 increases by less than a factor of λ ($x_2^* < x_2' < \lambda x_2^*$): this means that x_1 is a normal (luxury) good and x_2 is a normal (necessity) good
3. If x_1 increases by less than a factor of λ ($x_1^* < x_1' < \lambda x_1^*$) and x_2 increases by more than a factor of λ ($x_2' > \lambda x_2^* > x_2^*$): this means that x_1 is a normal (necessity) good and x_2 is a normal (luxury) good
4. If x_1 decreases ($x_1' < x_1^* < \lambda x_1^*$) and x_2 increases ($x_2' > \lambda x_2^* > x_2^*$): this means that x_1 is an inferior good and x_2 is a normal (luxury) good
5. If x_1 increases ($x_1' > \lambda x_1^* > x_1^*$) and x_2 decreases ($x_2' < x_2^* < \lambda x_2^*$): this means that x_1 is a normal (luxury) good and x_2 is an inferior good

Putting this together, we can represent this information in the following diagram:[7]



You interpret this diagram using the information above. Here, we have an increase in income from the black budget line (which had a tangency point at x^*) to the multi-colored budget line. The brown line represents the cut-off on whether a good is a necessity ("Nec.") or a luxury ("Lux."). Above the line is where x_2 is a luxury and below the line is where x_1 is a luxury. The gray line dotted line represents the cut-off of whether a good is normal or inferior ("Inf."). The location of the new optimal bundle x' determines which case we are in. If x' is in the red area, we are in case 4. If x' is in the blue area, we are in case 3. If x' is in the purple area, we are in case 2. If x' is in the green area, we are in case 5. If $x' = \lambda x^*$, i.e. on the brown line, we are in case 1. Notice that we can never be on the black areas of the new budget line. This is defined by where the old indifference curve intersects the budget line. Any points on this part of the line would give you strictly lower utility than x^*, so you would never choose it because you can still afford x^* in the new budget set (which would be a better choice since it gives higher utility).

[7]Credits to Professor Anna Caterina Musatti for this great diagram