Advanced Micro: Recitation 8
Expected Utility and Risk

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1 Note on Notation

This section's notation can get very daunting, and the textbook's notation makes it even more confusing. Let me show you a different notation that you may want to use. For these notes, I'll stick to the textbook way since that's what we follow in class. But for your own notes or homework, feel free to use whatever notation makes it easier for you.

For this notation, let's use the following conventions:

- Lotteries are always written with upper-case letters. Outcomes are always written with lower-case letters. For example, the lottery G has possible outcomes $a_1, ..., a_n$ with probabilities $p_1, ..., p_n$ and so we can write it as $G = (p_1 \circ a_1, ..., p_n \circ a_n)$
- The space of possible outcomes is denoted as X (i.e. $a_1, ..., a_n \in X$). The space of possible lotteries is denoted as ΔX (the simplex over X). For example, we can compare two lotteries $F, G \in \Delta X$. This highlights that we have to be comparing two lotteries over the same set of outcomes.
- A degenerate lottery (where you get one outcome for sure) is written as δ_x, where x is the amount received for sure. In other words, $\delta_{a_i} = (0 \circ a_1, ..., 1 \circ a_i, ..., 0 \circ a_n)$
- Utility over *lotteries* is expressed with upper-case utility functions (the vNM utilities). Utility over *outcomes* is expressed with lower-case utility functions (Bernoulli utility). For example, $U(G) = \sum_i p_i u(a_i)$
  - The utility for a degenerate lottery can therefore be expressed in two ways: $U(\delta_x) = u(x)$
- A compound lottery is expressed as the convex combination of lotteries, e.g. $L = p_1 F + p_2 G + p_3 H$
  - Any lottery can be expressed as the combination of degenerate lotteries: $G = p_1 \delta_{a_1} + ... + p_n \delta_{a_n}$

Here are some examples of using this notation:

- Risk-aversion is where $\delta_{E[G]} \geq U(G)$, $\forall G \in \Delta X$.
- The certainty equivalent is defined as $u(CE(G)) = U(\delta_{CE(G)}) = U(G)$
- The independence assumption tells us that if $G \geq H$, then $U(\lambda F + (1 - \lambda)G) = U(\lambda F + (1 - \lambda)H)$, $\forall F \in \Delta X$.

2 Risk Aversion Proofs

In class, we saw three equivalent ways of saying that "$\succsim_1$ is more risk averse than $\succsim_2$"

1. If $g \succsim_1 c$, then $g \succsim_2 c$, for all lotteries g and outcomes c
2. $CE_1(g) \leq CE_2(g)$, for all lotteries g
3. There exists a strictly increasing and concave function $\phi(\cdot)$ such that $u_1(x) = \phi(u_2(x))$

We proved this chain by showing that (3) $\implies$ (2) $\implies$ (1) $\implies$ (3). Let's do the other directions and prove: (1) $\implies$ (2) $\implies$ (3) $\implies$ (1)

[1 $\implies$ 2]

- Let $c = CE_1(g)$. Therefore, $g \sim_1 c$ (by definition of the certainty equivalent), which means that $g \succsim_1 c$. By (1), this means that $g \succsim_2 c$.
- By the same logic, $CE_2(g) \sim_2 g$. Therefore, by transitivity, $CE_2(g) \sim_2 g \succsim_2 CE_1(g)$
- Since $CE_2(g) \succsim_2 CE_1(g)$, then $u_2(CE_2(g)) \succsim_2 u_2(CE_1(g))$, which means that $CE_2(g) \geq CE_1(g)$, since utility is increasing

[2 $\implies$ 3]

- We can always find a strictly increasing function ϕ
  - By assumption, u_2 is strictly increasing, so there exists an inverse u_2^{-1} that is strictly increasing
  - Define $\phi(x) = u_1(u_2^{-1}(x))$. Since u_1 and u_2^{-1} are both strictly increasing, then ϕ must be strictly increasing too
- ϕ is also concave. Proof by contradiction: suppose ϕ is strictly increasing but not concave
  - Since ϕ not concave, there must exist $v, v' \in \mathbb{R}$ and some $\lambda \in (0,1)$ such that

$$\phi(\lambda v + (1-\lambda)v') < \lambda\phi(v) + (1-\lambda)\phi(v')$$

  - Find prizes $x, y \in \mathbb{R}$ such that $u_2(x) = v$ and $u_2(y) = v'$ (u_2 is continuous so this is always possible)
  - Define a lottery $g = (\lambda \circ x, (1-\lambda) \circ y)$ (you win x with probability λ and y with probability $1-\lambda$)
  - The expected utility of this lottery under $\succsim_2$ is $u_2(g) = \lambda u_2(x) + (1-\lambda)u_2(y)$ (similarly for $u_1(g)$). Apply the $\phi(\cdot)$ transformation on $u_2(g)$ and use the inequality above:

$$\begin{aligned}
\phi(u_2(g)) =& \phi(\lambda u_2(x) + (1-\lambda)u_2(y)) \\
=& \phi(\lambda v + (1-\lambda)v') \\
<& \lambda\phi(v) + (1-\lambda)\phi(v') \\
&= \lambda\phi(u_2(x)) + (1-\lambda)\phi(u_2(y))
\end{aligned}$$

$$= \lambda u_1(x) + (1-\lambda)u_1(y)$$
$$= u_1(g)$$

where the inequality follows from our assumption that ϕ is not concave at that point

- Since $u_2(g) = u_2(CE_2(g))$ and $u_1(g) = u_1(CE_1(g))$, we can write the inequality as:

$$\phi(u_2(g)) < u_1(g)$$
$$\phi(u_2(CE_2(g))) < u_1(CE_1(g))$$
$$u_1(CE_2(g)) < u_1(CE_1(g))$$

- Since $u_1(\cdot)$ is strictly increasing, then $CE_2(g) < CE_1(g)$. But this violates our starting assumption in (2)

[3 $\implies$ 1]

- $g \succsim_1 c$ means that $\sum_{i=1}^n p_i u_1(a_i) \geq u(c)$ (you can generalize this to the continuous case)
- Since ϕ is strictly increasing, it's inverse ϕ^{-1} must exist and is strictly increasing and convex
  - ϕ^{-1} is strictly increasing: $\sum_{i=1}^n p_i u_1(a_i) \geq u_1(c) \implies \phi^{-1}\left(\sum_{i=1}^n p_i u_1(a_i)\right) \geq \phi^{-1}(u_1(c)) = u_2(c)$
  - ϕ^{-1} is convex: $\phi^{-1}\left(\sum_{i=1}^n p_i u_1(a_i)\right) \leq \sum_{i=1}^n p_i \phi^{-1}(u_1(a_i)) = \sum_{i=1}^n p_i u_2(a_i) = u_2(g)$ (by Jensen's inequality)
- Putting this together: $u_2(c) \leq \phi^{-1}\left(\sum_{i=1}^n p_i u_1(a_i)\right) \leq u_2(g) \implies g \succsim_2 c$

3 Applications

3.1 Demand for Insurance

Adapted from MWG 6.C.1

Consider a strictly risk-averse decision maker who has an initial wealth of w but who runs the risk of a loss of D dollars. The probability of being fine is π and the probability of experiencing a loss is $1-\pi$. It is possible, however, for the decision maker to buy insurance. One unit of insurance costs q dollars and pays 1 dollar if the loss occurs. The decision maker chooses how many units of insurance to buy (call this amount $\alpha \in \mathbb{R}_+$)

1. What is the decision maker's expected utility (as a function of α)? Show that it is concave in α.
2. The decision maker wants to maximize their expected utility. Write the FOC of their problem
3. An *actuarially fair* insurance is one where the cost of the insurance is equal to the expected payout of insurance, i.e. $q = 1-\pi$. If the insurance is actuarially fair, what is the optimal choice of α?
4. Show that if the insurance is not actuarially fair (so that $q > 1-\pi$), then the individual will not insure completely. Does the answer depend on the consumers degree of risk aversion (Arrow-Pratt measure)?

1. Given that α units of insurance are bought, the wealth of the individual will be $w - \alpha q$ if there is no loss and $w - \alpha q - D + \alpha$ if these is a loss (and the insurance is paid out). Therefore, their expected utility (call this $V(\alpha)$) is:

$$V(\alpha) = \pi u\left(w - \alpha q\right) + (1-\pi) u\left(w - \alpha q - D + \alpha\right)$$

   To show that it is concave, we need to show that $V(\lambda a + (1-\lambda) b) \geq \lambda V(a) + (1-\lambda) V(b)$ for any $a, b \in \mathbb{R}_+$ and $\lambda \in [0,1]$.

$$\begin{aligned}
V(\lambda a + (1-\lambda) b) =& \pi u\left(w - \left(\lambda a + (1-\lambda) b\right) q\right) \\
& + (1-\pi) u\left(w - \left(\lambda a + (1-\lambda) b\right) q - D + \left(\lambda a + (1-\lambda) b\right)\right) \\
=& \pi u\left(\lambda\left(w - aq\right) + (1-\lambda)\left(w - bq\right)\right) \\
& + (1-\pi) u\left(\lambda\left(w - aq - D + a\right) + (1-\lambda)\left(w - bq - D + b\right)\right) \\
\geq& \pi\left[\lambda u\left(w - aq\right) + (1-\lambda) u\left(w - bq\right)\right] \\
& + (1-\pi)\left[\lambda u\left(w - aq - D + a\right) + (1-\lambda) u\left(w - bq - D + b\right)\right] \\
& = \lambda\left[\pi u\left(w - aq\right) + (1-\pi) u\left(w - aq - D + a\right)\right] \\
& \quad + (1-\lambda)\left[\pi u\left(w - bq\right) + (1-\pi) u\left(w - bq - D + b\right)\right] \\
& = \lambda V(a) + (1-\lambda) V(b)
\end{aligned}$$

   Where the inequality holds since $u(\cdot)$ is concave (which follows from the agent being risk averse).

2. The consumer's problem is:

$$\max_{\alpha \geq 0} \; V(\alpha) = \pi u\left(w - \alpha q\right) + (1-\pi) u\left(w - \alpha q - D + \alpha\right)$$

   The FOC is:

$$\begin{aligned}
\frac{\partial V(\alpha)}{\partial \alpha} &= -q\pi u'(w - \alpha q) + (1-q)(1-\pi) u'(w - \alpha q - D + \alpha) = 0 \\
& \implies \frac{u'(w - \alpha q)}{u'(w - \alpha q - D + \alpha)} = \frac{(1-\pi)(1-q)}{\pi q}
\end{aligned}$$

   This is essentially our usual tangency condition. The LHS is the MRS (between utility in the no-loss state and utility in the loss-state). The marginal cost of increasing utility in the no-loss state involves taking away one unit of insurance in the loss state, i.e. missing out on $1-q$ (the net insurance payment) with probability $1-\pi$. Similarly, the marginal cost of increasing utility in the loss state involves (unnecessarily) paying a cost of q with probability π.

3. If $q = 1 - \pi$, then the FOC becomes:

$$\begin{aligned}
\frac{u'(w - \alpha(1-\pi))}{u'(w - \alpha(1-\pi) - D + \alpha)} &= \frac{(1-\pi)\pi}{\pi(1-\pi)} \\
\frac{u'(w - \alpha(1-\pi))}{u'(w + \pi\alpha - D)} &= 1 \\
u'(w - \alpha(1-\pi)) &= u'(w + \pi\alpha - D)
\end{aligned}$$

Since u' is strictly decreasing, we can construct an inverse function for it. This implies that:

$$\begin{aligned} w - \alpha(1-\pi) &= w + \pi\alpha - D \\ w + \alpha\pi - \alpha &= w + \pi\alpha - D \\ \therefore \alpha^* &= D \end{aligned}$$

So if the insurance is actuarially fair, then the decision chooses complete coverage.
To be precise, let's also consider the possibility of a corner solution ($\alpha^* = 0$). Note this our solution is still optimal because:

$$\begin{aligned} V(D) &= \pi u(w - D) + (1-\pi)u(w - Dq) \\ &= u(w - (1-\pi)D) \\ &= u(\pi w + (1-\pi)(w - D)) \\ &\geq \pi u(w) + (1-\pi)u(w - D) = V(0) \end{aligned}$$

4. If $q > 1 - \pi$, then $1 - q < \pi$ and so $\frac{1-q}{q} < \frac{\pi}{1-\pi}$. Therefore, the "price ratio" becomes:

$$\frac{(1-\pi)(1-q)}{\pi q} < \frac{(1-\pi)\pi}{\pi(1-\pi)} = 1$$

Therefore, the FOC implies that:

$$u'(w - \alpha q) < u'(w - \alpha q - D + \alpha)$$

Again, since u' is strictly decreasing, this means that:

$$\begin{aligned} w - \alpha q &> w - \alpha q - D + \alpha \\ \therefore \alpha &< D \end{aligned}$$

This tells us if the insurance is actuarially unfair, then the consumer will not fully insure. Notice that this relies on the consumer being risk-averse, but it does not depend on the degree of risk aversion.

3.2 Sale and Buy Price of a Gamble

Consider a gamble g as a random variable with a probability distribution f, i.e. $Pr(g = x) = f(x)$. Suppose we have a consumer with utility over outcomes $u(\cdot)$ and whose initial wealth we normalize to zero. Denote the expected utility from a gamble g as $EU(g) = E[u(g)]$.
We will call the s the consumer's sale price of the gamble, which is implicitly defined as:

$$EU(g) = u(s)$$

Similarly, the buy price b is defined as:

$$EU(g - b) = u(0)$$

The sale price is interpreted as the minimum that the consumer must be paid in order to give up the gamble. The buy price, however, is the maximum that the consumer is willing to pay for the gamble.

1. Suppose that $u(x) = -e^{-\lambda x}$ for some $\lambda > 0$.
   (a) Show that this utility function exhibits CARA
   (b) Show that for someone with this utility function, $s = b$
2. Denote $u_w(x) = u(x + w)$ (you can think of this as utility at wealth level w) and the certainty equivalent of a gamble g at wealth level w as $CE_w(g)$, where

$$u_w(CE_w(g)) = EU(g + w) = E[u(g + w)]$$

   Notice that the LHS is equivalent to $u(CE_w(g) + w)$ and the RHS can be written as $E[u_w(g)]$. Express the prices, s and b, as certainty equivalents (i.e. find the right w to make them each as certainty equivalents of g)
3. Show that a utility function u exhibits DARA if and only if $CE_w(g)$ is increasing in w
4. Suppose that u exhibits DARA. Show that:

$$s \geq b \iff E[u(g)] \geq u(0)$$
$$\text{and}$$
$$s \leq b \iff E[u(g)] \leq u(0)$$

1.
   (a) Calculate the Arrow-Pratt measure:

$$R_a(x) = -\frac{u''(x)}{u'(x)} = -\frac{-e^{-\lambda x}(-\lambda)(-\lambda)}{-e^{-\lambda x}(-\lambda)} = \lambda$$

   Which is constant, and so we have CARA
   (b) Plug in the utility function. The sell price condition gives us:

$$EU(g) = E[u(g)] = u(s)$$

$$E\left[-e^{-\lambda g}\right] = -e^{-\lambda s}$$

The buy price condition gives us:

$$\begin{aligned} EU(g-b) = E[u(g-b)] &= u(0) \\ E\left[-e^{-\lambda(g-b)}\right] &= -e^{-\lambda 0} \\ E\left[-e^{-\lambda g}\right] e^{\lambda b} &= -1 \\ E\left[-e^{-\lambda g}\right] &= -e^{-\lambda b} \end{aligned}$$

Where note that we can pull out $e^{\lambda b}$ out of the expectation since it is a constant (b is not a random variable!). Therefore:

$$\begin{aligned} -e^{-\lambda s} = E\left[-e^{-\lambda g}\right] &= -e^{-\lambda b} \\ \iff s &= b \end{aligned}$$

2. For s, we have $u(s) = EU(g)$. Letting $w = 0$, this implies that $s = CE_0(g)$. For b, we have $u(0) = EU(g-b)$. Moreover, note that $u(0) = u_{-b}(b)$. Therefore $u_{-b}(b) = EU(g-b)$, which implies that $b = CE_{-b}(g)$ (i.e. letting $w = -b$)

3. DARA means us that $R_a(x) = -\frac{u''(x)}{u'(x)}$ is decreasing in x. We want to show that $CE_w(g)$ is increasing in w, i.e. $CE_w(g) \leq CE_v(g)$ for any $w \leq v$. We know that this occurs iff there exists a strictly increasing and concave function $\phi(\cdot)$ such that $u_w(x) = \phi\left(u_v(x)\right)$.[1] Differentiating both sides gives us:

$$\begin{aligned} u_w'(x) &= \phi'\left(u_v(x)\right) u_v'(x) \\ u_w''(x) &= \phi''\left(u_v(x)\right)\left(u_v'(x)\right)^2 + \phi'\left(u_v(x)\right) u_v''(x) \end{aligned}$$

Dividing the second term by the first:

$$\begin{aligned} \frac{u_w''(x)}{u_w'(x)} &= \frac{\phi'\left(u_v(x)\right) u_v'(x)}{\phi''\left(u_v(x)\right)} + \frac{u_v''(x)}{u_v'(x)} \\ -\frac{u''(x+w)}{u'(x+w)} &= -\frac{u''(x+v)}{u'(x+v)} - \frac{\phi'\left(u_v(x)\right)}{\phi''\left(u_v(x)\right)} u_v'(x) \\ R_a(x+w) &= R_a(x+v) - \frac{\phi'\left(u_v(x)\right)}{\phi''\left(u_v(x)\right)} u_v'(x) \end{aligned}$$

Notice that since ϕ is increasing and concave, we have $\phi' > 0, \phi'' < 0$ and, as always, we have $u_v'(x) = u'(x+v) > 0$. This means that the last term positive. Therefore, we showed $R_a(x+w) \geq R_a(x+v)$ for any $w \leq v$ (which is equivalent to saying the $R_a(x)$ is decreasing in x).

Alternatively, you can use the result we saw in class: $CE_w(g) \leq CE_v(g)$ for any $w \leq v$ means that u_w is more risk averse than u_v, and more risk averse means that $R_a^w(x) \geq R_a^v(x) \implies R_a(x+w) \geq R_a(x+v)$, for any $w \leq v$.[2] Therefore, $R_a(x)$ is a decreasing function.

[1] This is just the theorem we saw in class and in Section 2, letting $u_1 = u_w$ and $u_2 = u_v$

[2] Where $R_a^w(x) = -\frac{u_w''(x)}{u_w'(x)} = -\frac{u''(x+w)}{u'(x+w)} = R_a(x+w)$

4. In (2), we showed that $s = CE_0(g)$ and $b = CE_{-b}(g)$. In (3), we showed that $CE_w(g)$ is increasing in w. Therefore:

   - $s \geq b$ iff $CE_0(g) \geq CE_{-b}(g)$, and so $0 \geq -b$, i.e. $b \geq 0$
   - $s \leq b$ iff $CE_0(g) \leq CE_{-b}(g)$, and so $0 \leq -b$, i.e. $b \leq 0$

   Moreover, since $EU(g - b) = E[u(g - b)] = u(0)$ and u is increasing, this tells us that:

   - $b \geq 0$ iff $EU(g) \geq u(0)$
   - $b \leq 0$ iff $EU(g) \leq u(0)$

   Putting all this together gives us:

$$s \geq b \iff b \geq 0 \iff EU(g) \geq u(0)$$
$$s \leq b \iff b \leq 0 \iff EU(g) \leq u(0)$$