Advanced Micro: Recitation 10
Strategic Form Games 2

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April 19, 2019

1 Coordination Game: Stag Hunt

Question

(Jehle and Reny 7.11) Two hunters are on a stag hunt. They split up in the forest and each have two strategies: hunt for a stage (S), or give up the stag hunt and instead hunt for rabbit (R). If they both hunt for a stag, they will succeed and each earn a payoff of 9. If one hunts for stag and the other gives up and hunts for rabbit, the stag hunter receives 0 and the rabbit hunter 8. If both hunt for rabbit then each receives 7. Compute all Nash equilibria for this game, called 'The Stag Hunt', depicted below. Which of these equilibria do you think is most likely to be played? Why?

	S	R
S	9,9	0,8
R	8,0	7,7

Solution

Let's start with the pure strategy Nash equilibria. Underlining the best responses gives us:

	S	R
S	$\underline{9}$,$\underline{9}$	0,8
R	8,0	$\underline{7}$,$\underline{7}$

So, unlike the prisoner's dilemma, we have two PSNE.

We should also check for mixed strategy Nash equilibria. Let's consider player 2 using the mixed strategy $m_2 = (p, 1-p)$, i.e. play S with probability p and R with probability $1-p$. Now let's think about player 1's expected payoff in response to m_2:

- Play S: $E[u_1(S, m_2)] = 9p + 0(1-p) = 9p$

- Play R: $E[u_1(R, m_2)] = 8p + 7(1-p) = 7 + p$

In an equilibrium, these must have the same expected payoff (you are indifferent between the strategies you are mixing over). Therefore:

$$\begin{aligned} 9p &= 7 + p \\ 8p &= 7 \\ p &= \frac{7}{8} \end{aligned}$$

We can do the same for player 2, but it will be symmetrical. Therefore, the MSNE is:

$$m_1 = m_2 = \left(\frac{7}{8}, \frac{1}{8}\right)$$

2 Bertrand Game

2.1 Symmetric

Question

Two firms are engaged in Bertrand competition: they simultaneously choose their (non-negative) prices, with all sales going to the firm with the lower price. They split the market equally if they have the same price. Suppose that both firms have constant marginal costs of c and the market demand curve is $Q(p)$. Assume that $p^* = \text{argmax}_{p \in \mathbb{R}_+}(p - c)Q(p)$ is finite and unique.

1. Write the game as a strategic form game
2. Find the unique PSNE
3. Show that PSNE is the only Nash equilibrium

Solution

The firms strategies are $s_i = p_i$, where $S_i = \mathbb{R}_+$. The payoff for firm i is:

$$u_i(s_i, s_{-i}) = \begin{cases} (s_i - c)\, Q(s_i) & \text{if } s_i < s_{-i} \\ \frac{1}{2}\,(s_i - c)\, Q(s_i) & \text{if } s_i = s_{-i} \\ 0 & \text{if } s_i > s_{-i} \end{cases}$$

We will argue that the unique PSNE is $s_1 = s_2 = c$. This is an equilibrium since neither firm has an incentive to deviate (if you raise the price slightly, you get zero profits; if you lower your price slightly, you get the entire market but get negative profits). We can show it is unique by arguing that there are no other PSNE. Suppose there is a PSNE where $s_1 \geq s_2$ (WLOG) and $s_i \neq c$ for at least one i. This gives us three cases:

1. $s_1 \geq s_2 > c$: firm 1 could deviate to some $s_1 \in (c, s_2)$ and be strictly better off

2. $s_1 > s_2 = c$: firm 2 could deviate to some $s_2 \in (c, s_1)$ and be strictly better off - in fact, deviating to $s_2 = s_1$ would also make them strictly better off too (they would get positive profits instead of zero)

3. $c > s_2$: firm 2 is making losses and can always do better by deviating to $s_2 = c$ (if $s_1 \geq c$, then firm 2 will make positive profits; if $c > s_1 \geq s_2$, then firm 2 will make zero profits instead of a loss)

Finally, to show that it is the unique Nash equilibrium, we need to now also consider mixed strategies. Assume we have a Nash equilibrium with $m = (m_1, m_2)$ that is not $s_1 = s_2 = c$. Define $\underline{m}_i$ as the infimum of the prices that firm i mixes over and $\bar{m}_i$ as the supremum of the prices that firm i mixes over and. WLOG, suppose that $\underline{m}_1 \leq \underline{m}_2$. Let's establish a few things:

- We cannot have $\underline{m}_1 < c$. This means that firm 1 is mixing over prices that give negative profits, even though they are strictly dominated by $s_1 = c$. Therefore, we know that $c \leq \underline{m}_1 \leq \underline{m}_2$.

- We must have that $\bar{m}_i > c$ for at least one i. This is because if $\bar{m}_1 = \bar{m}_2 = c$, then we have $c = \underline{m}_1 = \underline{m}_2 = \bar{m}_1 = \bar{m}_2$, which means we get the PSNE that we found earlier. But we are trying to find another Nash equilibrium, so this cannot be the case. Moreover, we must have $\bar{m}_i > c$ for both i, because a firm that is playing c could increase profits by changing to some $\bar{m}_i \in (c, \bar{m}_{-i})$.

- We must have $p^* \geq \bar{m}_1 = \bar{m}_2 > c$. First, the supremums must be equal. If we had $\bar{m}_i > \bar{m}_{-i}$, then firm i should reduce the weight assigned to prices in the neighborhood of $\bar{m}_i$ and increase the weight to prices at $c + \varepsilon$ (this will raise profits because those high prices lose for sure). Second, they must be smaller than the optimal price p^*. By the same logic, if you mix over prices above p^*, you can be made better off by instead playing p^*. Denote $\bar{m}_1 = \bar{m}_2 = \bar{p}$.

This tells us that the firms are assigning positive probability to the price $\bar{p}$, however, the probability of winning with $\bar{p}$ (i.e. having the strictly lowest price) is a probability zero event. Since you never win with $\bar{p}$, you should instead shift to a lower price. You could split the market if your opponent also plays $\bar{p}$, but in that case, you could do better by playing $\bar{p} - \varepsilon$ for some $\varepsilon > 0$ and winning the whole market outright. You can keep doing this as long as $\bar{p} > c$. Therefore, we get a contradiction and so the only NE is $s_1 = s_2 = c$.

2.2 Asymmetric

Now imagine that the firms have asymmetric costs (but still linear). Call firm i's marginal cost c_i. WLOG, suppose that $c_1 < c_2$. Show that there is no PSNE

<u>Solution</u>

To show that there is no PSNE, consider two cases:

1. Suppose that there is a PSNE where $s_1 \geq s_2$. We must have that $s_2 \geq c_2$, otherwise firm 2 would make negative profits and would instead play $s_2 = c_2$. But this means that $s_1 \geq c_2 > c$, and firm 1 could do better by deviating to some $s_1 \in (c, s_2)$ and make strictly positive profits (instead of zero)

2. Suppose that there is a PSNE where $s_1 < s_2$. Firm 1 can be strictly better off by deviating to $s_2 - \varepsilon$.

3 Incomplete Information

3.1 Modified Prisoner's Dilemma

Two people are playing the prisoner's dilemma, but this game is modified in two ways. First, player 1 is the police chief's son. This means they are able to get out without consequences if they are not implicated in a crime (i.e. 0 payoff if both players don't confess). Second, there is uncertainty about player 2's type. They could either be a squealer or a tightlips, where the tightlips doesn't like ratting out the other person.[1] The probability of being a squealer is $\alpha \in (0, 1)$. The payoffs for this game are as follows:

Squealer (probability α)

	D	C
D	$0, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Tightlips (probability $1 - \alpha$)

	D	C
D	$0, -2$	$-10, -7$
C	$-1, -10$	$-5, -11$

Solution

Call the payoffs $u_i(s_1, s_2; t_2)$. There is no type for player 1, but you could it add into the payoffs if you wanted to be precise. To solve the game, let's consider each game separately:

- Squealer game: $2S$ plays C since $u_2(s_1, C; S) > u_2(s_1, D; S)$ for any $s_1 \in \{D, C\}$. Therefore, 1 should play C to best respond.
- Tightlips game: $2T$ plays D since $u_2(s_1, D; T) > u_2(s_1, C; T)$ for any $s_1 \in \{D, C\}$. Therefore, 1 should play D to best respond.

However, since player 1 doesn't know player 2's type, what will be their best response? In expectation, player 1's payoffs are:

$$\begin{aligned} E[u_1(C, s_2; t_2)] &= \alpha u_1(C, C; S) + (1 - \alpha) u_1(C, D; T) \\ &= \alpha(-5) + (1 - \alpha)(-1) \\ &= -1 - 4\alpha \end{aligned}$$

$$\begin{aligned} E[u_1(D, s_2; t_2)] &= \alpha u_1(D, C; S) + (1 - \alpha) u_1(D, D; T) \\ &= \alpha(-10) + (1 - \alpha)(0) \\ &= -10\alpha \end{aligned}$$

Therefore, player 1 chooses C if:

$$\begin{aligned} -1 - 4\alpha &> -10\alpha \\ 6\alpha &> 1 \end{aligned}$$

[1] https://youtu.be/_ti-KRlZuZI?t=28

$$\therefore \alpha > \frac{1}{6}$$

Otherwise, they choose D.

3.2 Lover or Cheater

Two people are looking to get married. However, player 1 has uncertainty about player 2's type. Is player 2 a lover or a cheater? Player 2 knows their type, and the ex ante distribution is that player 2 is a lover with probability $\mu \in (0, 1)$. Each player can agree to get married or not. If both players agree to not get married, they both get zero payoff. If a player proposes, but the other rejects, then the embarrassment of a failed proposal gives that player a payoff of -1 (while the rejector gets zero). The exception is that a cheater feels no embarrassment from a failed proposal. Finally, if they get married, the payoffs also depend on the type. If player 2 is a lover, they both get 5. If player 2 is a cheater, then player 1 gets -3 and player 2 gets 3.

1. Setup the game, including the payoff matrices
2. Find the pure strategy Bayesian Nash equilibria
3. Redo questions (1) and (2), but where there is no embarrassment of having a failed proposal for any type of player

Solution

The payoff matrices (with the best responses underlined) look as follows:

Lover (probability μ)

	M	N
M	$\underline{5}, \underline{5}$	$-1, 0$
N	$0, -1$	$\underline{0}, \underline{0}$

Cheater (probability $1 - \mu$)

	M	N
M	$-3, \underline{3}$	$-1, 0$
N	$\underline{0}, \underline{0}$	$\underline{0}, \underline{0}$

Three types of players: $1L$, $2L$, $2C$ (player 1 is always a lover). So the strategies are $\{s_{1L}, s_{2L}, s_{2H}\} \in \{M, N\}^3$. The payoffs will be written as follows: $u_i\left(s_1, s_{2L}, s_{2C}; t_1, t_2\right)$.

In the lover game, the Nash equilibria are (M, M) and (N, N). In the cheater game, the Nash equilibria are (N, M) and (N, N). Therefore, this tells us that (N, N, N) and (N, N, M) are all pure strategy BNE (each player 2 type is best responding to player 1, and player 1 is simultaneously best responding to each player 2 type).

Moreover, notice that if player 1 plays M, then both the lover's and cheater's best response is M. This means that (M, M, M) could be a possible BNE. Which profiles are not possible BNE? If player 1 plays M, then player 2 of either type cannot be playing N (it is not a best-response). This eliminates (M, N, M), (M, M, N), and (M, N, N). Similarly, if player 1 plays N, then the lover cannot play M. This eliminates (N, M, M) and (N, M, N).

So, the only possible one left is (M, M, M). Let's consider player 1's expected payoffs in this situation to see when they deviate:

$$E\left[u_1\left(M, M, M; L, t_2\right)\right] = \mu(5) + (1-\mu)(-3) = 8\mu - 3$$
$$E\left[u_1\left(N, M, M; L, t_2\right)\right] = \mu(0) + (1-\mu)(0) = 0$$

This tells us that player 1 chooses M if $8\mu - 3 \geq 0 \implies \mu \geq \frac{3}{8}$. This gives us the BNE (M, M, M).

To summarize, the $2^3 = 8$ possible strategies are:

(M, M, M)	BNE if $\mu \geq \frac{3}{8}$
(M, M, N)	Not BNE. $2C$ deviates to M
(M, N, M)	Not BNE. $2L$ deviates to M
(M, N, N)	Not BNE. $2L$, $2C$ deviates to M
(N, M, M)	Not BNE. $2L$ deviates to N
(N, M, N)	Not BNE. $2L$ deviates to N
(N, N, M)	BNE
(N, N, N)	BNE

Without the embarrassment, the payoff matrices now look as follows:

Lover (probability μ)

	M	N
M	$\underline{5}, \underline{5}$	$\underline{0}, 0$
N	$0, \underline{0}$	$\underline{0}, \underline{0}$

Cheater (probability $1-\mu$)

	M	N
M	$-3, \underline{3}$	$\underline{0}, 0$
N	$\underline{0}, \underline{0}$	$\underline{0}, \underline{0}$

The Nash equilibria in each game stay the same, which means that (N, N, N) and (N, N, M) are all still pure strategy BNE. The difference now is that more strategies are best responses for the player 2s, so we can eliminate less. Consider player 1's expected payoffs:

- If they play M, then $2L$ plays M and $2C$ plays M. Then player 1's expected profit would be:

$$E\left[u_1\left(M, M, M; L, t_2\right)\right] = \mu(5) + (1-\mu)(-3) = 8\mu - 3$$

  - Note that this eliminates (M, N, M), (M, M, N), and (M, N, N) to prevent the player 2s from deviating

- If player 1 plays N, then $2L$ and $2C$ could play M or N. In any case, player 1's expected profit would be 0:

$$E\left[u_1\left(N, s_{2L}, s_{2C}; L, t_2\right)\right] = \mu(0) + (1-\mu)(0) = 0$$

- Note that this doesn't eliminate anything since there is no restriction on what the player 2s are doing. So (N, M, M), (N, M, N), (N, N, M), (N, N, N) are all possible solutions

This tells us that player 1 chooses M if $8\mu - 3 \geq 0 \implies \mu \geq \frac{3}{8}$. This gives us the BNE (M, M, M). Similarly, if $\mu \leq \frac{3}{8}$, then player 1 plays N as a response to M, M. This gives us that (N, M, M) is another BNE.

We're not quite done though. There are other combinations that the player 2s could be playing in response to $s_1 = N$. We have already established that (N, N, N) and (N, N, M) are BNE, so that leaves us with only one more possibility to consider: (N, M, N). But would this be an equilibrium? If $(s_{2L}, s_{2C}) = (M, N)$, then if 1 deviates to M, their expected payoff would be:

$$E\left[u_1\left(M, M, N; L, t_2\right)\right] = \mu(5) + (1 - \mu)(0) = 5\mu$$

So, player 1 deviates if $5\mu \geq 0 \implies \mu \geq 0$. But note that (M, M, N) would not a BNE since $2C$ would deviate. Therefore (N, M, N) is only an equilibrium if $\mu = 0$, but note that we've ruled this out by setting $\mu \in (0, 1)$ (otherwise there's no uncertainty about types, and that's not very interesting).

To summarize, the $2^3 = 8$ possible strategies are:

(M, M, M)	BNE if $\mu \geq \frac{3}{8}$
(M, M, N)	Not BNE. $2C$ deviates to M
(M, N, M)	Not BNE. $2L$ deviates to M
(M, N, N)	Not BNE. $2L$, $2C$ deviate to M
(N, M, M)	BNE if $\mu \leq \frac{3}{8}$
(N, M, N)	Not BNE. 1 deviates to M
(N, N, M)	BNE
(N, N, N)	BNE