Intermediate Micro: Recitation 12
Producer Optimization

Motaz Al-Chanati

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1 Profit Maximization Problem

1.1 Setup

In the last recitation, we said that a firm's objective is to maximize profits, which we expressed through the *profit maximization problem*:

$$\max_{q,x_1,x_2} \; pq - p_1x_1 - p_2x_2$$
$$\text{s.t. } q = f(x_1, x_2)$$

Recall the four components of an optimization problem:

1. *Objective Function:* The producer's profit function $\pi = pq - p_1x_1 - p_2x_2$
2. *Constraint:* The production function ($q = f(x_1, x_2)$)
3. *Choice Variables:* The quantity of each input good (x_1 and x_2) and the quantity of the output good q
4. *Parameters:* The prices of each input good (p_1 and p_2) and the price of the output good (p)

We also discussed, how the output good is automatically determined by the constraint, so we can just re-write it as the following unconstraint maximization problem:

$$\max_{x_1,x_2} \; pf(x_1, x_2) - p_1x_1 - p_2x_2$$

This has the following components:

1. *Objective Function:* The producer's profit function $\pi = pf(x_1, x_2) - p_1x_1 - p_2x_2$
2. *Constraint:* None

3. *Choice Variables:* The quantity of each input good (x_1 and x_2)

4. *Parameters:* The prices of each input good (p_1 and p_2) and the price of the output good (p)

Now let's discuss how to solve this.

1.2 Single-Variable Optimization

Let's consider the following simpler problem, where there is only one input good which has a price of c:

$$\max_{x} \ pf(x) - cx$$

If you are asked to solve this problem, you would take the usual step of taking an FOC:

$$\begin{aligned} \frac{d\pi}{dx} = \frac{d\left[pf(x) - cx\right]}{dx} &= pf'(x) - c = 0 \\ \therefore f'(x) &= \frac{c}{p} \end{aligned}$$

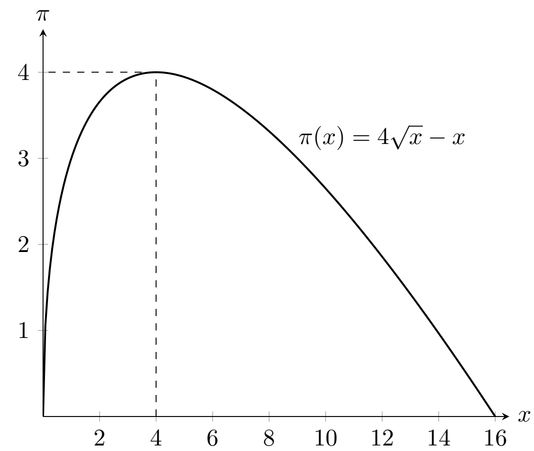
From this, we could find an optimal value of x^*. For example, if $f(x) = \sqrt{x}$, then our problem and solution would be

$$\max_{x} \ p\sqrt{x} - cx$$

$$\begin{aligned} \implies f'(x) = \frac{1}{2\sqrt{x}} &= \frac{c}{p} \\ 2\sqrt{x} &= \frac{p}{c} \\ x &= \left(\frac{p}{2c}\right)^2 \\ \therefore x^* &= \frac{p^2}{4c^2} \end{aligned}$$

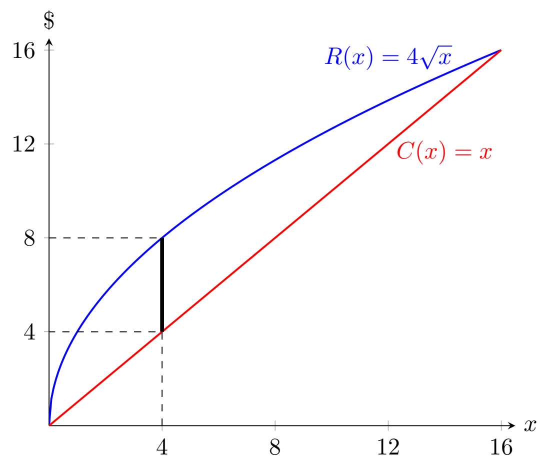
Graphically, we can represent this problem in three ways. Let's consider the case of $f(x) = \sqrt{x}$, with parameters $p = 4$ and $c = 1$. This gives us the profit function $\pi(x) = 4\sqrt{x} - x$. The solution to this problem is $x^* = \frac{4^2}{4 \cdot 1^2} = 4$, which gives a profit of $\pi(4) = 4\sqrt{4} - 4 = 4$. With this input, we produce $q = \sqrt{4} = 2$ units of output.

The first approach is to plot the profit function as a function of x, which is shown below:



As we can see, the solution to the profit maximization is the point where (shockingly) the profit function reaches its highest point.

Another way to show this problem is to plot two graphs. One for the revenue, $R(x) = pf(x)$, and one for the cost, $C(x) = cx$. Profit is $R(x) - C(x)$, so profit will be maximized where the gap between these lines is the biggest. Plotting this for our example gives us:



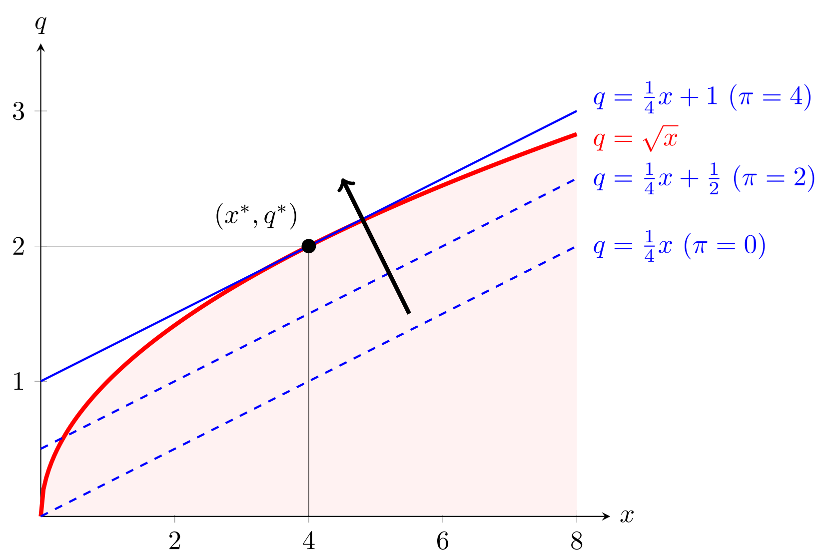
The gap between the two exactly represents the firm's profits (i.e. the size of the gap can be represented by the first graph we plotted).

Finally, we can also graph this problem using something similar to our usual setup of budget constraints and indifference curves. Consider bundles of (x, q), where x is the input and q is the output. So, for example, the bundle $(2, 5)$ means 2 units of the input are used and 5 units of the output are produced. Of course, not all bundles are technologically feasible. Only the ones where $q \leq f(x)$ are feasible. This defines our feasible set (just like how the budget constraint defined the feasible set in the consumer problem). The objective function is to maximize profits, written as $\pi = pq - cx$. This defines an *iso-profit* line. The iso-profit line represents the set of bundles that give the same profit to the firm. For example, $(1, 1)$ and $(2, 1 + \frac{c}{p})$ both give a profit of $\pi = p - c$. From the profit function, for a fixed level of profit π, we can then solve for q to get the formula for the iso-profit line:

$$\pi = pq - cx$$
$$pq = cx + \pi$$
$$q = \frac{c}{p}x + \frac{\pi}{p}$$

Notice that this is completely analogous to when we solve for x_2 from the utility function in order to find the formula for the indifference curve. So the firm's problem can be equivalently expressed as trying to find the highest iso-profit line in the feasible region (as defined by the production function).

Let's take the example of $f(x) = \sqrt{x}$ with $p = 4, c = 1$ again and plot it:



Notice a few key things:

- The axis here has the input x on the x-axis and the output q on the y-axis
- The feasible set is defined by the production function $q = f(x)$ (in red). All areas including and *below* the line (in the shaded area) are all feasible bundles. However, we know that an optimal should be at the border of the feasible region, i.e. where $q = f(x)$ and not $q < f(x)$.
- The iso-profit lines (in blue) are increasing as we move to the top-left corner (unlike indifference curves which generally increase as we move to the top-right corner). You can see this using the formula for the iso-profit line: for the same x, a higher q must necessarily mean a higher π. Or intuitively: more quantity produced with the same amounts of input means higher profits.
- The iso-profit lines are always upward sloping. Again, you can see this in the formula (the coefficient on x is $\frac{c}{p}$, which is always positive). It also makes sense because increasing both the inputs and outputs should 'cancel out' (at the right ratio) meaning you are left with the same profit as before

- As before, the optimal point occurs at a tangency. This is where we can find the highest iso-profit line in the feasible region. For the consumer's problem, we typically have the budget line as a linear graph that is fixed and move the curved indifference curves until we find the tangency point. Here, it's the opposite. We have a fixed curve (the production function) and move the linear iso-profit lines until we get to a tangency.
- The tangency point is $(x^*, q^*) = (4, 2)$, which occurs on the iso-profit line where $\pi = 4$. This is exactly the same solution we got before using the other graphs.

This approach suggests that instead of working with the input good, we could instead be working with the output good. Note that we can (usually) invert the production function:

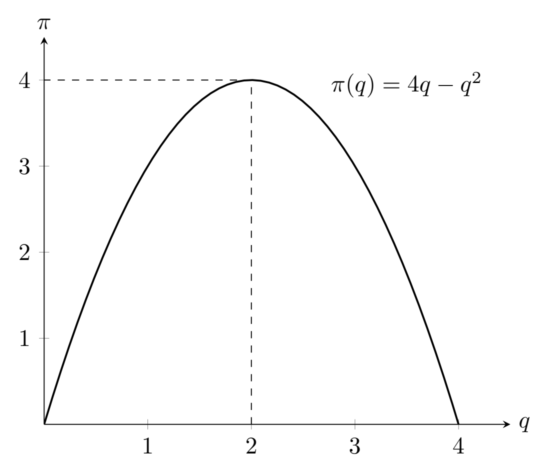
$$\begin{aligned} q &= f(x) \\ \implies x &= f^{-1}(q) \end{aligned}$$

This inverse production has the following interpretation: tell me how much output you want to produce (q), and I'll tell you how much input good you will need to produce it (x). For example, in the case of $f(x) = \sqrt{x}$, we have that $f^{-1}(q) = q^2$. This suggests we can write the profit maximization problem as follows:

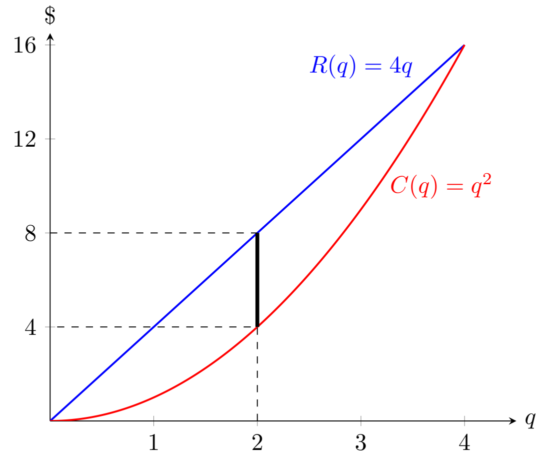
$$\max_q \; pq - cf^{-1}(q)$$

As you can see, this is still a single-variable optimization problem, but now in terms of q instead of x. We can still have the revenue function $R(q) = pq$ and the cost function $C(q) = cf^{-1}(q)$, which are now both functions of q.

Graphically, we can also have similar graphs as before, but note now that q is on the x-axis. Taking the same example as before, we have a profit function of $\pi(q) = 4q - q^2$, which gives us the following plot:



Similarly, we could plot the revenue function $R(q) = 4q$ and the cost function $C(q) = q^2$ to find the biggest gap between the two:



1.3 Multi-Variable Optimization

When we have multiple input goods, our problem is this:

$$\max_{x_1,x_2} \; pf(x_1,x_2) - p_1x_1 - p_2x_2$$

The trick to solving this is to **make it into a single-variable problem**. Then, we can just do the same steps as above (take an FOC and solve for the optimal input level). Right now we have two choice variables. So what we need to do is express one choice variable as a function of the other one.

To start, let's take FOCs with respect to each choice variable (i.e. x_1 and x_2).

FOC x_1:

$$p\frac{\partial f(x_1,x_2)}{\partial x_1} - p_1 = 0$$
$$\implies p\frac{\partial f(x_1,x_2)}{\partial x_1} = p_1$$

FOC x_2:

$$p\frac{\partial f(x_1,x_2)}{\partial x_2} - p_2 = 0$$
$$p\frac{\partial f(x_1,x_2)}{\partial x_2} = p_2$$

Dividing these by one another gives us:

$$\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_1}} = \frac{p_1}{p_2}$$

But notice that the left hand side is exactly the MRTS! So we basically get our familiar tangency condition:

$$MRTS = \frac{p_1}{p_2}$$

We know that from the tangency condition, we can express one choice variable as a function of the other one (e.g. usually x_2 as a function of x_1). This is exactly what we were looking for! So we will get some function $x_2 = h(x_1)$. Then, we plug this back into the objective function and we will have a single-variable problem:

$$\max_{x_1} \; pf\left(x_1, h(x_1)\right) - p_1 x_1 - p_2 h(x_2)$$

Let's do an example so that this concept is clear. Suppose we have a Cobb-Douglas production function $f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$. The tangency condition is:[1]

$$MRTS = \frac{\frac{1}{2}x_2}{\frac{1}{4}x_1} = \frac{2x_2}{x_1} = \frac{p_1}{p_2}$$

Re-arranging this gives us:

$$x_2 = \frac{p_1}{2p_2} x_1$$

Now, we plug this back into the profit function to have a single-variable problem:

$$\begin{aligned}
\pi(x_1) &= p x_1^{1/2} \left(\frac{p_1}{2p_2} x_1\right)^{1/4} - p_1 x_1 - p_2 \left(\frac{p_1}{2p_2} x_1\right) \\
&= p \left(\frac{p_1}{2p_2}\right)^{1/4} x_1^{3/4} - p_1 x_1 - \frac{p_1}{2} x_1 \\
&= p \left(\frac{p_1}{2p_2}\right)^{1/4} x_1^{3/4} - \frac{3p_1}{2} x_1
\end{aligned}$$

To keep the math simple, let's consider this when $p = 8, p_1 = 4, p_2 = 1$:

$$\begin{aligned}
\pi(x_1) &= 8\,(2)^{1/4} x_1^{3/4} - 6x_1 \\
&= 8 \cdot 2^{1/4} \cdot x_1^{3/4} - 6x_1
\end{aligned}$$

To find the maximum profit, we just take an FOC with respect to the only input variable, i.e. x_1:

$$\begin{aligned}
\frac{d\pi(x_1)}{dx_1} &= 8 \cdot 2^{1/4} \cdot \frac{3}{4} x_1^{-1/4} - 6 = 0 \\
\implies 8 \cdot 2^{1/4} \cdot \frac{3}{4} x_1^{-1/4} &= 6 \\
8 \cdot 2^{1/4} \cdot x_1^{-1/4} &= 8 \\
8^4 \cdot 2 \cdot x_1^{-1} &= 8^4 \\
\frac{x_1}{2} &= 1 \\
\therefore x_1^* &= 2
\end{aligned}$$

Plugging this back into the tangency condition gets us the optimal x_2:

$$x_2^* = \frac{p_1}{2p_2} x_1^* = \frac{4}{2 \cdot 1} \cdot 2 = 4$$

[1] You can use the same shortcuts as you use for Cobb-Douglas utility. In particular, if $f(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, then $MRTS = \frac{\alpha x_2}{\beta x_1}$

Therefore, the optimal input values are $(x_1^*, x_2^*) = (2, 4)$, which yields an output level of $q^* = (2)^{1/2}(4)^{(1/4)} = \sqrt{2}\sqrt{2} = 2$, and profit of $\pi^* = 8 \cdot \left(2^{1/2}\right) \cdot \left(4^{1/4}\right) - 4(2) - 1(4) = 16 - 8 - 4 = 4$.

As you can see, once we have everything in terms of one choice variable, it just becomes a standard single-variable optimization problem (which is easy to solve). Remember that we showed that you could write a single-variable problem either in terms of the input good or in terms of the output good. We would like to do the same thing here - especially when we have many input goods, it makes the interpretation much easier if we think about things in terms of the one output good q.

Now, recall that we can actually write the profit maximization problem as:

$$\max_{q, x_1, x_2} \; pq - p_1 x_1 - p_2 x_2$$
$$\text{s.t. } \; q = f(x_1, x_2)$$

Here, q is a choice variable. So using the same logic, if we express the choice variables x_1 and x_2 as functions of q, then we could get a single-variable problem in terms of q:

$$\max_{q} \; pq - p_1 x_1(q) - p_2 x_2(q)$$

So we would have a revenue function $R(q) = pq$, which is the same as before, and a cost function $C(q) = p_1 x_1(q) + p_2 x_2(q)$. This cost function is what we are most interested in. In particular: how do we derive it and what are its properties? To derive it, we need to figure out a way to find the optimal input levels for any output quantity. For this, we will need the *cost minimization problem* (section 2). Once we've derived it, we'll study properties of the cost function in section 3.

2 Cost Minimization Problem

2.1 Problem

The cost minimization problem aims to find the cheapest way to produce any level of output quantity q. You can think of this as the following story. Imagine you are the manager of a factory that produces an output good for a company. You get a call from headquarters telling you that you need to produce q units of output. Your job then is to figure out how to produce the required number of units as cheaply as possible. Formally, the cost minimization problem can be expressed as:

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2$$
$$\text{s.t. } \; f(x_1, x_2) = q$$

Let's break this down:

1. *Objective Function:* The cost function $C(x_1, x_2) = p_1 x_1 + p_2 x_2$

2. *Constraint:* The production function ($f(x_1, x_2) = q$)
3. *Choice Variables:* The quantity of each input good (x_1 and x_2)
4. *Parameters:* The prices of each input good (p_1 and p_2) and the quantity of the output good (q)

Notice that q, which was a choice variable in the profit maximization problem, is now a parameter in the cost minimization problem.[2] It is not your job to determine whether that q is optimal! Headquarters tells you to produce a certain amount, and you just say "Yes, boss!" and try choose your inputs so that it is produced at minimum cost. In other words, you take q as given and treat it as a parameter. When we solve this, we will get the choice variables as a function of all the parameters, i.e. for the input i, we will have it as a function of the input prices and the required quantity:

$$x_i(p_1, p_2, q)$$

This function is called the **conditional input demand**. The word "conditional" comes in because it is *conditional* on the quantity you are being asked to produce. So you can interpret it as follows: if you want me to produce q units of output (with input prices p_1, p_2), then I will need $x_i(p_1, p_2, q)$ units of input i to produce it *in the cheapest way possible.* For simplicity, let me just write it as $x_i(q)$. Plugging the conditional input demands back into the objective function gives us the cost function we are looking for:

$$C(q) = p_1 x_1(q) + p_2 x_2(q)$$

This object here is our goal. We want to solve the cost minimization problem to get the conditional input demands. With them, we get the cost function $C(q)$, i.e. as a function of q. Then, we plug this into the profit maximization problem, which we can then solve because it is then only a function of one variable (q):

$$\max_q \; pq - C(q)$$

2.2 Solution

To solve the cost minimization problem, we can start by setting up our usual Lagrangian:[3]

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\left(q - f(x_1, x_2)\right)$$

Taking FOCs with respect to the two choice variables x_1 and x_2:

FOC x_1:

$$p_1 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} = 0$$
$$\implies p_1 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_1}$$

[2] Also, note that the price of output good plays no role here either

[3] Even though it is a minimization instead of a maximization problem, we still take the same steps as before. Technically, we should be checking the second order conditions, but we won't be giving you any weird functions. So you just can ignore this and safely assume that you answer is indeed a minimum rather than a maximum.

FOC x_2:

$$p_2 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} = 0$$
$$p_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2}$$

Dividing these by one another gives us:

$$\frac{p_1}{p_2} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_1}}$$

Again, this is just our tangency condition! So we can just skip the Lagrangian and go straight to this condition:

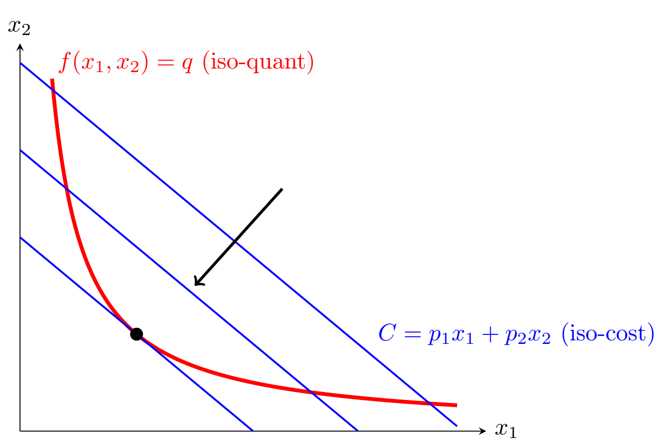
$$MRTS = \frac{p_1}{p_2}$$

As we know, from this, we can typically get x_2 as a function of x_1, i.e. $x_2 = h(x_1)$. In the consumer problem, we would take this and plug it into the optimization constraint (i.e. the budget line). Now, our constraint is the production, so this is where we plug it in:

$$q = f(x_1, x_2) = f\left(x_1, h(x_1)\right)$$

From this, we can solve for x_1 as a function of q, which will give us exactly the conditional input demand we wanted. Plugging this back into $h(x_1)$ will give us the x_2 conditional input demand. Armed with these, we can then go ahead and solve the profit maximization problem.

The cost minimization problem has a very familiar graphical interpretation using the standard x_1-x_2 axis. Our objective function can be represented by *iso-cost* lines: any two bundles on the same iso-cost line cost the same. This is of course just like the budget constraint. Instead of fixing a total cost M, we will just leave it as a variable C, i.e. $p_1x_1 + p_2x_2 = C$. As we change the value of C, we will get different iso-costs. The feasible region here is determined by the *iso-quant* curve, defined by $q = f(x_1, x_2)$. Any two points on the same iso-quant produce the same level of output quantity. These are analogous to indifference curves. We have one iso-quant representing the quantity level of our parameter q. Therefore, the cost minimization problem is simply to find the point on the iso-quant that is on the lowest iso-cost line.



As the diagram shows, we just end up back at the usual tangency condition $MRTS = \frac{p_1}{p_2}$ (i.e. the slope of the iso-quant equals the slope of the iso-cost). This is just like the consumer's utility maximization problem, except now it is like holding one indifference curve fixed and trying to find the lowest budget line on that indifference curve.[4]

2.3 Examples

Cobb-Douglas

Using the same example as before, let's have the production function as $f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$. The tangency condition is:

$$\begin{aligned} MRTS = \frac{\frac{1}{2}x_2}{\frac{1}{4}x_1} &= \frac{p_1}{p_2} \\ \frac{2x_2}{x_1} &= \frac{p_1}{p_2} \\ \therefore x_2 &= \frac{p_1}{2p_2}x_1 \end{aligned}$$

Plugging this into the production gives us:

$$\begin{aligned} f\left(x_1, \frac{p_1}{2p_2}x_1\right) &= q \\ x_1^{1/2}\left(\frac{p_1}{2p_2}x_1\right)^{1/4} &= q \\ \left(\frac{p_1}{2p_2}\right)^{1/4} x_1^{3/4} &= q \\ \left(\frac{p_1}{2p_2}\right) x_1^3 &= q^4 \\ \therefore x_1 &= \left(\frac{2p_2q^4}{p_1}\right)^{1/3} \end{aligned}$$

If the input prices are $p_1 = 4$, $p_2 = 1$, then the conditional input demands are:

$$\begin{aligned} x_1(q) &= \left(\frac{2 \cdot 1 \cdot q^4}{4}\right)^{1/3} \\ &= \left(\frac{q^4}{2}\right)^{1/3} \\ \therefore x_2(q) &= \frac{4}{2 \cdot 1}\left(\frac{q^4}{2}\right)^{1/3} \\ &= \left(2^3 \frac{q^4}{2}\right)^{1/3} \\ &= \left(4q^4\right)^{1/3} \end{aligned}$$

[4] In class, you briefly saw the expenditure minimization problem. This is the consumer analog of the cost minimization problem, and the diagram works in exactly this way.

This means that the cost function is:

$$\begin{aligned}
C(q) &= p_1 x_1(q) + p_2 x_2(q) \\
&= 4\left(\frac{q^4}{2}\right)^{1/3} + \left(4q^4\right)^{1/3} \\
&= \left(2^{5/3} + 2^{2/3}\right) q^{4/3} \\
&= 3 \times 2^{2/3} q^{4/3}
\end{aligned}$$

With an output price of $p = 8$, the profit maximization problem is:

$$\begin{aligned}
\max_q \ \pi(q) =& pq - C(q) \\
=& 8q - 3 \times 2^{2/3} q^{4/3}
\end{aligned}$$

Taking the FOC:

$$\begin{aligned}
8 - 3 \times 2^{2/3}\left(\frac{4}{3} q^{1/3}\right) &= 0 \\
2^{2/3} 4 q^{1/3} &= 8 \\
2^2 4^3 q &= 8^3 \\
q &= \frac{8^3}{4^4} = \frac{2^9}{2^8} = 2
\end{aligned}$$

Therefore, the profit maximizing quantity is $q^* = 2$. At this output level, we have:

$$\begin{aligned}
x_1(2) &= \left(\frac{2^4}{2}\right)^{1/3} = \left(2^3\right)^{1/3} = 2 \\
x_2(2) &= \left(4 \cdot 2^4\right)^{1/3} = \left(2^6\right)^{1/3} = 2^2 = 4 \\
\pi(2) &= 8(2) - 3 \times 2^{2/3} 2^{4/3} = 16 - 3(2^2) = 16 - 12 = 4
\end{aligned}$$

Notice that these values are exactly what we found in doing the profit maximization with respect to x_1. So both methods are totally equivalent.

Perfect Complements

Suppose we had the following problem:

- $f(x_1, x_2) = \min\{x_1, 2x_2\}$
- $p_1 = 1, p_2 = 2, p = 5$

First, let's find the conditional input demands. Since this is perfect complements, we start by finding the equation for where the kinks occur:

$$
\begin{aligned}
x_1 &= 2x_2 \\
\therefore x_2 &= \frac{1}{2}x_1
\end{aligned}
$$

This gives us x_2 as a function of x_1, which we can plug into the production function:

$$
\begin{aligned}
q &= f\left(x_1, \frac{1}{2}x_1\right) \\
&= \min\left\{x_1, 2 \cdot \frac{1}{2}x_1\right\} \\
&= \min\{x_1, x_1\} \\
&= x_1
\end{aligned}
$$

Therefore, the conditional input demands are $x_1(q) = q$ and $x_2(q) = \frac{1}{2}q$. Notice that they don't depend on price. This makes sense: if you must produce exactly q and you know it is optimal to be where the kink in the iso-quant occurs, then you must be exactly where $q = x_1 = 2x_2$, regardless of what the price is. So our cost function becomes:

$$
\begin{aligned}
C(q) &= p_1 x_1(q) + p_2 x_2(q) \\
&= q + 2 \cdot \frac{1}{2}q \\
&= 2q
\end{aligned}
$$

The profit maximization problem is then:

$$
\begin{aligned}
\max_q \ \pi(q) =& pq - C(q) \\
=& 5q - 2q \\
=& 3q
\end{aligned}
$$

Now, we have a problem. In particular, the FOC is just 3, and obviously $3 = 0$ is a nonsensical condition. Notice that the profit function is just a straight line with a slope of 3. This means increasing the output quantity by 1 unit will always increase profits by an extra \$3. Therefore, the optimal quantity is to produce $q = \infty$ and get infinite profits! Notice that the revenue function is linear with slope of 5 and the cost function is also linear with slope of 2. Therefore, the gap between them starts at 0 when $q = 0$, and then increasingly gets bigger and bigger, which means profits are always getting bigger. Why does this happen? It's because the production function exhibits constant returns to scale (CRTS). Increasing the input x_1 by one unit (and consequently x_2 by half a unit as well) only increases costs by \$2. But increasing the input by one unit means that q increases by one unit too, and this additional unit can be sold for \$5. Therefore, for any extra unit produced, the *marginal revenue* for the firm is \$3. This never diminishes, so we should always keep producing more and more q to increase profits. This means that we do not have a finite solution for q.

Perfect Substitutes

Suppose we had the following problem:

- $f(x_1, x_2) = x_1 + 2x_2$
- $p_1 = 4, p_2 = 2, p = 0.5$

For perfect substitutes, we know that we should compare the MRTS to the price ratio:

$$MRTS = \frac{1}{2}$$

Therefore, there are three cases:

$$\begin{aligned} MRTS < \frac{p_1}{p_2} &\implies \text{only } x_2 \\ MRTS > \frac{p_1}{p_2} &\implies \text{only } x_1 \\ MRTS = \frac{p_1}{p_2} &\implies \text{any } x_1, x_2 \text{ where } q = x_1 + 2x_2 \end{aligned}$$

Since we have $\frac{p_1}{p_2} = 2$, then we are in the first case where $MRTS = \frac{1}{2} < 2 = \frac{p_1}{p_2}$. This gives us the following conditional input demands:

$$\begin{aligned} x_1(q) &= 0 \\ x_2(q) &= \frac{q}{2} \end{aligned}$$

Note, that we get the x_2 demand from $q = f(0, x_2) = 0 + 2x_2 = 2x_2$. This means that the cost function is:

$$\begin{aligned} C(q) &= 4 \cdot 0 + 2 \cdot \frac{q}{2} \\ &= q \end{aligned}$$

And the profit function is:

$$\begin{aligned} \pi(q) &= pq - C(q) \\ &= \frac{1}{2}q - q \\ &= -\frac{1}{2}q \end{aligned}$$

Now to maximize profits, we should set $q = 0$. This is similar to the previous example. Each additional unit of output costs an extra \$1 to produce, but we only get \$0.50 as revenue. Therefore, the marginal revenue is $-\$0.50$, i.e. additional output means that we make less money than before. Again, this problem comes up because perfect substitutes production functions exhibit CRTS.

2.4 Returns to Scale and Profit Maximization

As we saw in the examples above, the fact that we had production functions with CRTS lead to "boundary" solutions, i.e. $q = 0$ or $q = \infty$. We would have the same problem with IRTS production functions too. In fact, the only way to get a unique, finite, strictly positive solution is to have DRTS. This comes about because we have linear pricing for our goods. To see this, suppose we have chosen the optimal input levels (x_1^*, x_2^*) that gives an output level $q^* = f(x_1^*, x_2^*)$. The cost for this production is $C^* = C(q^*) = p_1x_1^* + p_2x_1^*$ and the profit $\pi^* = pq^* - C(q^*)$. Suppose we double the amount of inputs to $(x_1^{**}, x_2^{**}) = (2x_1^*, 2x_2^*)$. This leads to double the costs $C^{**} = p_1(2x_1^*) + p_2(2x_2^*) = 2\,(p_1x_1^* + p_2x_1^*) = 2C^*$. Now let's think about whether this could be optimal based on the returns to scale of the production function:

- **IRTS:** Since the production function has IRTS, the new level of output (q^{**}) is *more than* double the old level: $q^{**} > 2q^*$. This means getting more than double the old revenue: $pq^{**} > 2pq^*$. Therefore, the new profit π^{**} is:

$$\pi^{**} = pq^{**} - C^{**} = pq^{**} - 2C^* > 2pq^* - 2C^* = 2\pi^*$$

  So by doubling the inputs, you could in fact make more than double the old profit. Therefore, the initial choice couldn't have been optimal.

- **CRTS:** Since the production function has CRTS, the new level of output (q^{**}) is *exactly* double the old level: $q^{**} = 2q^*$. This means getting exactly double the old revenue: $pq^{**} = 2pq^*$. Therefore, the new profit π^{**} is:

$$\pi^{**} = pq^{**} - C^{**} = pq^{**} - 2C^* = 2pq^* - 2C^* = 2\pi^*$$

  So by doubling the inputs, you could in fact make exactly double the old profit. Therefore, the initial choice couldn't have been optimal.

- **DRTS:** Since the production function has DRTS, the new level of output (q^{**}) is *less than* double the old level: $q^{**} < 2q^*$. This means getting less than double the old revenue: $pq^{**} < 2pq^*$. Therefore, the new profit π^{**} is:

$$\pi^{**} = pq^{**} - C^{**} = pq^{**} - 2C^* < 2pq^* - 2C^* = 2\pi^*$$

  So by doubling the inputs, you would make less than double the old profit. However, that doesn't tell us whether the initial choice was optimal or not. So we would need more information, but it is possible that q^* was indeed optimal.

For simplicity, I imagined doubling the inputs, but this logic will still be true for any scaling $\lambda > 1$. In general, if we scale up the inputs to $(x_1^{**}, x_2^{**}) = (\lambda x_1^*, \lambda x_2^*)$, we will scale up the costs in a similar way: $C^{**} = \lambda C^*$ (since pricing is linear). But, what we need to know is whether $\pi^{**} > \pi^*$. While we know that $q^{**} > q^*$ (more inputs must necessarily leads to more output), to have $\pi^{**} > \pi^*$, we would need:

$$\begin{aligned} pq^{**} - C^{**} = pq^{**} - \lambda C^* &> pq^* - C^* \\ \implies p(q^{**} - q^*) &> (\lambda - 1)C^* \end{aligned}$$

In other words, we should only scale up production if the extra revenue is larger than the extra cost. For IRTS and CRTS, we have $q^{**} - q^* \geq \lambda q^* - q^* = (\lambda - 1)q^*$, which will satisfy the above condition as long as $\pi^* > 0$:

$$\begin{aligned} \pi^* &> 0 \\ pq^* &> C^* \\ p(\lambda-1)q^* &> (\lambda-1)C^* \\ p(q^{**} - q^*) \geq p(\lambda-1)q^* &> (\lambda-1)C^* \end{aligned}$$

This means that if the initial profit is positive, we should keep scaling up q if we have IRTS or CRTS. That means we should aim to have $q = \infty$! On the other hand, if the initial profit is negative ($\pi^* < 0$), then the opposite result holds. We should keep reducing quantity as that way we will make smaller losses than before. We would keep doing this until we have $q = 0$. If $\pi^* = 0$, then any quantity level will also give zero profits, so the firm would be indifferent between any output level. So the only way to get a unique, interior solution (i.e. not $q = 0$ or $q = \infty$) is to have DRTS.

2.5 Short Run Costs

So we want to have production that exhibits decreasing returns to scales, but we also want some realistic motivation for it. The way we have set up the firm's problem, we allow them full flexibility: they can choose any input values they want to produce their quantity. In reality, firms face a lot of frictions that prevent this from happening. In particular, we typically think that **in the short run, some of the firm's inputs are fixed**.

To see this, let's have our two inputs as capital (K) and labor (L) in this section. It seems somewhat plausible that a firm can relatively easily hire and fire workers as needed, so labor is probably quite flexible. But in the short run, we typically think of capital as being fixed. For example, if you have signed a one-year lease for office space, then you are largely stuck with this property for a year. Or if you need more machines in your factory, it can take a long time to order and build them. In the long run, we think of all variables as being flexible, but often we will look at the short-run where the results are more interesting. This is because in the short-run, production functions tend to exhibit DRTS (and hence we get a finite solution for the profit maximization problem). Intuitively this makes sense. Say your output good is cake and you have bakers (L) and one oven (K). Your one oven is fixed in the short run (you can't just run to the store and buy an extra oven). Suppose you double the number of bakers you have - will you also double your output? Probably not, because only a few cakes can be put in the oven at a time. This fixed capital in the short-run tends to be a binding constraint, and thus reducing the returns to scale.

Let's express these ideas mathematically. We can change our problem now so that capital is fixed at some level $\bar{K}$. Then the firm's cost minimization problem becomes:

$$\begin{aligned} &\min_{L} \; wL + r\bar{K} \\ &\text{s.t. } q = f(\bar{K}, L) = f(L) \end{aligned}$$

Notice now that capital is fixed, our only choice variable is labor L. This means we can just simplify the production function as $f(L)$. When that happens, there isn't really an optimization problem to do. Since the firm has to produce q, then L is automatically determined by the constraint:

$$\begin{aligned} q &= f(L) \\ \implies L &= f^{-1}(q) \end{aligned}$$

This represents the conditional input demand. This means that the cost function is $C(q) = wf^{-1}(q) + r\bar{K}$, and the profit maximization problem is:

$$\max_q \; pq - wf^{-1}(q) - r\bar{K}$$

But notice that this is now just a single-variable problem. So we can take an FOC and solve it:

$$\begin{aligned} p - w\frac{\partial f^{-1}(q)}{\partial q} &= 0 \\ \frac{\partial f^{-1}(q)}{\partial q} &= \frac{p}{w} \end{aligned}$$

From this equation, we can solve for the optimal q.

An alternative way to express this problem is to a single-variable profit maximization problem using labor (the flexible input) as the choice variable:

$$\max_L \pi(L) = pf(\bar{K}, L) - wL - r\bar{K}$$

This gives us the FOC:

$$\begin{aligned} p\frac{\partial f(\bar{K}, L)}{\partial L} - w &= 0 \\ \frac{\partial f(\bar{K}, L)}{\partial L} &= \frac{w}{p} \end{aligned}$$

The left-hand side of the optimality condition is known as the *marginal product of labor* (MPL). The derivative with respect to capital $\frac{\partial f(K,L)}{\partial K}$ is known as the *marginal product of capital* (MPK). From this, we can find the optimal level of L^*, and consequently the optimal q^*.

Example: Cobb-Douglas

Consider the following setup:

- $f(K, L) = K\sqrt{L}$
- $r = 3$, $w = 2$, $p = 8$
- In the short run, capital is fixed at 4

Given that capital is fixed, if q units need to be produced, then we would need the following amount of labor:

$$\begin{aligned} q &= K\sqrt{L} \\ q &= 4\sqrt{L} \\ \therefore L(q) &= \left(\frac{q}{4}\right)^2 = \frac{q^2}{16} \end{aligned}$$

With the conditional input demand for labor, we have our cost function:

$$\begin{aligned} C(q) &= wL + r\bar{K} \\ &= 2 \cdot \frac{q^2}{16} + 3 \cdot 4 \\ &= \frac{q^2}{8} + 12 \end{aligned}$$

So the profit maximization problem is:

$$\begin{aligned} \max_q \ \pi(q) &= pq - C(q) \\ &= 8q - \frac{q^2}{8} - 12 \end{aligned}$$

Taking the FOC with respect to q gives us:

$$\begin{aligned} 8 - \frac{q}{4} &= 0 \\ \frac{q}{4} &= 8 \\ \therefore q^* &= 32 \end{aligned}$$

For practice, let's try doing this directly with the profit maximization problem using L as our choice variable:

$$\begin{aligned} \max_L \pi(L) &= pf(\bar{K}, L) - wL - r\bar{K} \\ &= 8\left(4\sqrt{L}\right) - 2L - 3(4) \\ &= 32\sqrt{L} - 2L - 12 \end{aligned}$$

The FOC is then:

$$\begin{aligned} MPL = 32 \cdot \frac{1}{2\sqrt{L}} &= 2 \\ \frac{16}{\sqrt{L}} &= 2 \\ \sqrt{L} &= 8 \\ L^* &= 64 \end{aligned}$$

This means that $q^* = f(\bar{K}, L^*) = 4\sqrt{64} = 4 \times 8 = 32$, which is exactly what we found above.

Notice that the production function is $f(K, L) = K\sqrt{L} = KL^{0.5}$. Since $1 + 0.5 = 1.5 > 1$, we know that this exhibits IRTS. However, when capital is fixed, the production function effectively becomes $f(L) = \bar{K}\sqrt{L} = 4\sqrt{L}$. For this function, we can check the returns to scale:

$$f(\lambda L) = 4\sqrt{\lambda L} = \sqrt{\lambda}\left(4\sqrt{L}\right) = \sqrt{\lambda} f(L) < \lambda f(L)$$

Since we can only scale up labor, but not capital, in the short-run, this function exhibits DRTS. Hence, we can get an interior solution.

3 Cost Functions

3.1 Key Concepts

Now that we know how to derive the cost function $C(q)$, we will spend a bit more time analyzing it. The following are important concepts:

- **Total Cost $TC(q)$:** another name for $C(q)$. This represents all costs required to produce q units of output
- **Variable Cost $VC(q)$:** the component of total cost that varies by quantity (i.e. the cost of the variable input)
- **Fixed Cost FC:** the component of the total cost that does not vary by quantity (i.e. the cost of the fixed input)
- **Average Cost $AC(q)$:** the cost per unit of output, defined as $\frac{TC(q)}{q}$. Also known as the average total cost $ATC(q)$
- **Average Variable Cost $AVC(q)$:** the variable cost per unit of output, defined as $\frac{VC(q)}{q}$
- **Average Fixed Cost $AFC(q)$:** the fixed cost per unit of output, defined as $\frac{FC(q)}{q}$
- **Marginal Cost $MC(q)$:** the cost of an extra unit of output, defined as $\frac{\partial TC(q)}{\partial q}$.

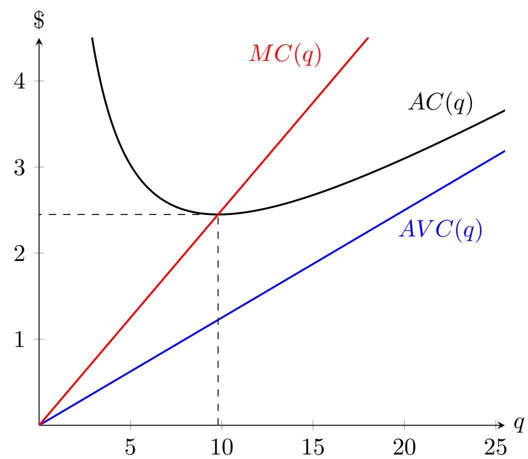
For example, let's take the cost function we derived in the last example:

$$C(q) = \frac{q^2}{8} + 12$$

- Total Cost: $TC(q) = \frac{q^2}{8} + 12$
- Variable Cost: $VC(q) = \frac{q^2}{8}$
- Fixed Cost: $FC = 12$
- Average Cost: $AC(q) = \frac{q^2/8+12}{q} = \frac{q}{8} + \frac{12}{q}$

- Average Variable Cost: $AVC(q) = \frac{q^2/8}{q} = \frac{q}{8}$
- Average Fixed Cost: $AFC(q) = \frac{12}{q}$
- Marginal Cost: $MC(q) = \frac{\partial\left(q^2/8+12\right)}{\partial q} = \frac{2q}{8} = \frac{q}{4}$

Let's plot the AC, AVC, and MC functions on the same graph:



3.2 Properties

Here are some important properties and features of the cost functions:

Marginal and Average Costs

- The average cost follows the marginal cost. In other words, if $MC < AC$, then the AC is downwards sloping; if $MC > AC$, then the AC is upwards sloping.
  - Here's an analogy for intuition: imagine if a group of people in the room, where the average age is 20. If a person aged 30 walks into the room, then the average age goes up (the "marginal" or additional person is aged 30, which is larger than the average). If a person aged 15 walks into the room, then the average age in the room goes down (now, the "marginal" person is aged 15, which is smaller than the average).
  - Mathematically: the slope of the average cost curve is $\frac{\partial}{\partial q}AC(q) = \frac{\partial}{\partial q}\left(\frac{C(q)}{q}\right) = \frac{\frac{\partial C(q)}{\partial q}q - C(q)\frac{\partial q}{\partial q}}{q^2} = \frac{MC(q)q - C(q)}{q^2} = \frac{1}{q}\left(MC(q) - AC(q)\right)$. This uses the quotient rule. Notice that the slope is positive if $MC > AC$ and negative otherwise.
- The marginal cost curve has a positive slope (usually, in the short run)
  - This captures the fact that producing an additional unit of output generally gets increasingly more expensive

- The average cost curve is often U-shaped (usually, in the short run)
  - $AC(q) = AVC(q) + AFC(q)$. At low levels of q, $AFC(q)$ is relatively high. But as q increases, the average fixed cost falls (the same cost is being divided by a larger number). This results in the AC falling. At some point though, average variable costs AVC start becoming relatively high since the costs usually increase at a faster rate than the quantity (due to the diminishing marginal products of the inputs). This results in the average costs increasing. These two dynamics give us the U-shape.
- The marginal cost curve crosses the average cost curve at its minimum. In other words, $MC(q) = AC(q)$ at the $\bar{q}$ where $AC(\bar{q}) = \min_q AC(q)$
  - Consider a U-shaped average cost curve. Let's call the quantity where the minimum occurs as $\bar{q}$. To the left of $\bar{q}$, we have AC decreasing. To the right of $\bar{q}$, we have $MC > AC$.
  - Since AC is falling to the left of $\bar{q}$, then it must be the case that $MC < AC$ to the left of $\bar{q}$. Since AC is increasing to the right of $\bar{q}$, then it must be the case that $MC > AC$ to the right of $\bar{q}$. Therefore, at exactly $\bar{q}$, we must have that $MC = AC$.
  - In the long run, the point $\bar{q}$ is called the "minimum efficient scale" and denoted by q_{MES}

Fixed and Variable Costs

- The marginal cost can be expressed as $MC(q) = \frac{\partial VC(q)}{\partial q}$
  - The fixed costs fall out of the derivative: $MC(q) = \frac{\partial}{\partial q} TC(q) = \frac{\partial}{\partial q} [VC(q) + FC] = \frac{\partial}{\partial q} VC(q) + \frac{\partial}{\partial q} FC = \frac{\partial}{\partial q} VC(q)$.
- The marginal cost crosses the average variable cost curve twice: at its minimum and at the first unit produced ($q \approx 0$)
  - For the minimum: since $MC(q) = \frac{\partial VC(q)}{\partial q}$, you can use the exact same logic as we used with $AC(q)$
  - For $q \approx 0$: technically, the AVC is not defined at $q = 0$, since we can't divide by zero, so we want to imagine the first unit as having a infinitely small amount of quantity. For simplicity, think of this as a discrete change (i.e. going from $q = 0$ to $q = 1$). The average variable cost is just the variable cost $AVC(1) = \frac{VC(1)}{1} = VC(1)$. The marginal cost is $MC(1) = \Delta TC = \Delta VC = VC(1) - VC(0) = VC(1)$ (by definition, you must have that $VC(0) = 0$).
- The average fixed costs tend to zero as quantity increases
  - $AFC(q) = \frac{FC}{q}$. As $q \to \infty$, we have $AFC(q) \to 0$, since FC is a fixed number
- The average costs become similar to the average variable costs as quantity increases
  - $AC(q) = AVC(q) + AFC(q)$. As $q \to \infty$, we have $AFC(q) \to 0$, and so $AC(q) \to AVC(q)$
- In the long run, there are no fixed costs
  - In the short run, some inputs are fixed. This gives rise to the fixed costs. In the long run, however, all inputs are adjustable, which means there should be no fixed costs.

Profit

- For a price p, the firm's optimal choice of output q^* is where $p = MC(q^*)$
  - The profit maximization problem is $\max_q \; pq - C(q)$. The FOC for this is: $p - \frac{\partial C(q)}{\partial q} = 0 \implies p - MC(q) = 0 \implies p = MC(q)$.
  - Intuitively: p represents the marginal revenue of an extra unit of production (you get an extra \$$p$ of revenue from selling one more unit of output). You should only produce that extra unit if you can make up for the extra cost of producing it, i.e. if marginal revenue is greater than marginal cost.
- For a price p and given the firm's choice of output q^*, the firm makes positive profit if $p > AC(q^*)$ and negative profit if $p < AC(q^*)$
  - Profit is $\pi(q) = pq - TC(q)$. Average profit per unit is $\frac{\pi(q)}{q} = \frac{pq}{q} - \frac{TC(q)}{q} = p - AC(q)$.
  - Since quantity is always positive ($q \geq 0$), profit can only be positive if average profit is positive (and similarly for negative).
  - Therefore: $\pi(q) > 0$ if $p - AC(q) > 0 \implies p > AC(q)$, and similarly for negative profits
- For a price p and given the firm's choice of output q^*, the firm's profit is equal to the area of the rectangle formed between p and $AC(q)$ on the y-axis, and 0 and q^* on the x-axis
  - Average profit per unit is $p - AC(q^*)$. This is the height of the rectangle on the y-axis
  - Quantity of units produced is $q^* = q^* - 0$. This is the width of the rectangle on the x-axis
  - Profit is therefore (Average profit per unit) $\times$ (Number of units) $= (p - AC(q^*)) \times q^*$. This is exactly the area of the rectangle.

Returns to Scale

- If the production function exhibits DRTS, then the cost function $TC(q)$ is convex (increasing marginal cost)
- If the production function exhibits IRTS, then the cost function $TC(q)$ is concave (decreasing marginal cost)
- If the production function exhibits CRTS, then the cost function $TC(q)$ is linear (constant marginal cost)
  - Easy to think about this in the one-input case $f(x)$. By definition, a concave function is where $f(\lambda x) < \lambda f(x)$ ($>$ for convex, $=$ for linear). Therefore, if $f(x)$ exhibits DRTS, it is a concave function (compare the definitions). For IRTS, it is convex, and for CRTS, it is linear. Since the cost function is basically the inverse of the the production function, then it has the opposite convexity (try drawing the graph).
  - Intuition: Suppose we double the output, then we want to argue if the production has DRTS, then $C(2q) > 2C(q)$ (definition of convex function). But, since it has DRTS, then to get double the output, we need to more than the double the inputs. This means that costs also have to increase by more than double (since input costs are linear). You can use a similar argument for IRTS and CRTS.

3.3 Examples

Example 1

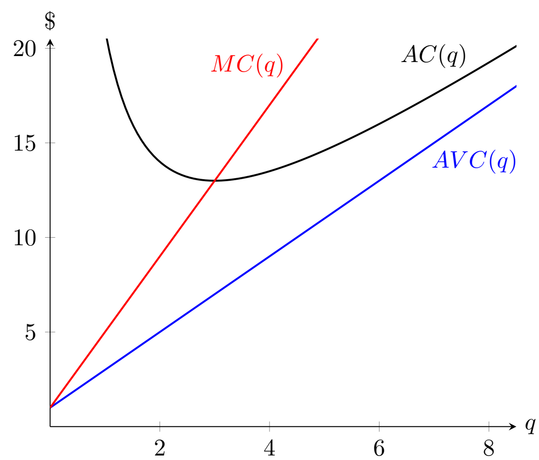
Consider a firm with the following (total) cost function:

$$C(q) = 2q^2 + q + 18$$

Let's calculate the related cost functions:

$$\begin{aligned}
VC(q) &= 2q^2 + q \\
FC(q) &= 18 \\
AC(q) &= 2q + 1 + \frac{18}{q} \\
AVC(q) &= 2q + 1 \\
AFC(q) &= \frac{18}{q} \\
MC(q) &= 4q + 1
\end{aligned}$$

Plotting the AC, AVC, and MC gives us:



We could also want to know where the minimum of the AC occurs. This occurs when $MC(q) = AC(q)$:

$$\begin{aligned}
4q + 1 &= 2q + 1 + \frac{18}{q} \\
2q &= \frac{18}{q} \\
q^2 &= 9 \\
\therefore \bar{q} &= 3
\end{aligned}$$

Suppose the price of the output good was \$17. How much does the firm profit? We can set up the profit maximization problem:

$$\begin{aligned}\max_q \pi(q) &= pq - C(q)\\ &= 17q - 2q^2 - q - 18\end{aligned}$$

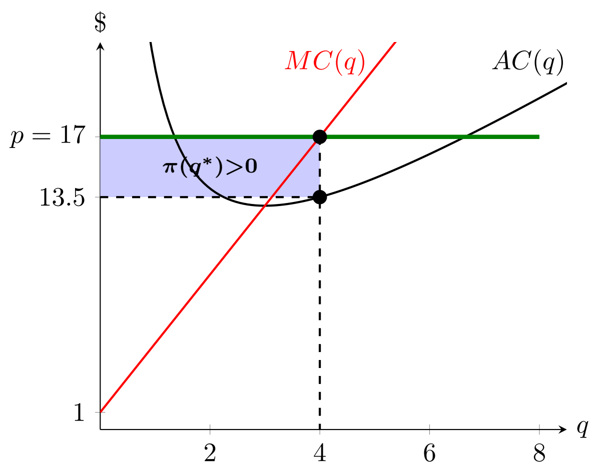
The FOC is:

$$\begin{aligned}17 - 4q - 1 &= 0\\ 16 &= 4q\\ \therefore q^* &= 4\end{aligned}$$

Alternatively, we can just use the condition $p = MC(q)$:

$$\begin{aligned}17 &= 4q + 1\\ 16 &= 4q\\ \therefore q^* &= 4\end{aligned}$$

So the optimal choice for the firm is $q = 4$. Are they making positive profit? Let's look at the graph (I've removed the AVC graph to keep this simple)



Notice that at $q = 4$, we have an average cost of $AC(4) = 8 + 1 + 4.5 = 13.5$. Since $p > AC(q^*)$, we know the firm makes positive profit. This is represented by the blue rectangle shaded in the diagram. The area of this rectangle is the amount of profits made by the firm:

$$\begin{aligned}\pi &= (17 - 13.5) \times (4 - 0)\\ &= 3.5 \times 4\\ &= 14\end{aligned}$$

Example 2

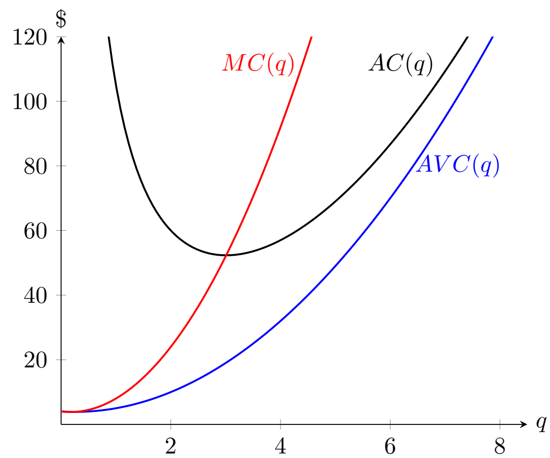
Consider a firm with the following (total) cost function:

$$C(q) = 100 + 4q - q^2 + 2q^3$$

Let's calculate the related cost functions:

$$\begin{aligned}
VC(q) &= 4q - q^2 + 2q^3 \\
FC(q) &= 100 \\
AC(q) &= \frac{100}{q} + 4 - q + 2q^2 \\
AVC(q) &= 4 - q + 2q^2 \\
AFC(q) &= \frac{100}{q} \\
MC(q) &= 4 - 2q + 6q^2
\end{aligned}$$

Plotting the AC, AVC, and MC curves gives us:



Suppose that $p = 24$. Using the optimality condition, we get:

$$\begin{aligned}
p &= MC(q) \\
24 &= 4 - 2q + 6q^2 \\
0 &= 3q^2 - q - 10 \\
0 &= (3q + 5)(q - 2)
\end{aligned}$$

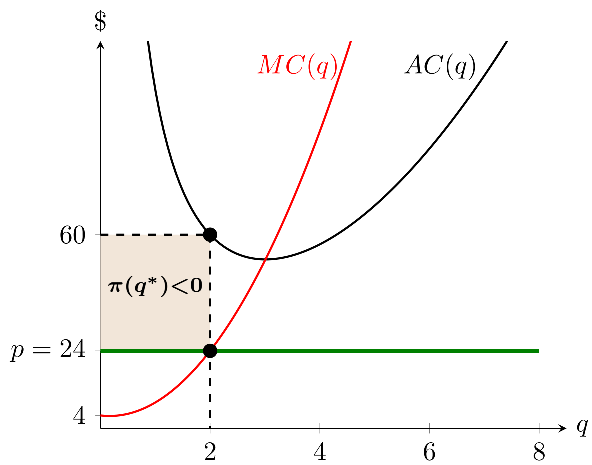
This gives us two possible solutions: $q = -\frac{5}{3}$ or $q = 2$. Obviously, we can't have negative quantities, so the chosen quantity is $q^* = 2$. At this level of quantity though, we have:

$$\begin{aligned} AC(2) &= \frac{100}{2} + 4 - 2 + 2(2)^2 \\ &= 50 + 2 + 8 \\ &= 60 \end{aligned}$$

Therefore, since $p = 24 < 60 = AC(q^*)$, the firm is making negative profits. In fact, their profits are:

$$\begin{aligned} \pi(2) &= (p - AC(2)) \times q^* \\ &= (24 - 60) \times 2 \\ &= -72 \end{aligned}$$

This is shown in the diagram below:



You might feel that it doesn't really make sense for a firm to be making negative profits. What if they just produced nothing instead (i.e. $q = 0$)? In that case, their profit would be:

$$\begin{aligned} \pi(0) &= p0 - C(0) \\ &= 24 \cdot 0 - \left(100 + 4 \cdot 0 - 0^2 + 2 \cdot 0^3\right) \\ &= -100 \end{aligned}$$

So even though they are making a loss of \$72, they would be making an even bigger lose if they produced nothing. Clearly it's better to lose a smaller amount, so the optimal choice is indeed $q = 2$. In the next recitation, we'll talk more about when the firm should shut down or not.