Advanced Micro: Recitation 9
Strategic Form Games 1

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1 Second Price Auction

1.1 Setup

- N bidders are bidding for an item at auction
- Each bidder has a valuation $v_i \in \mathbb{R}$ that is common knowledge (question: how would our answer change if the valuations were private knowledge)
- Bidders simultaneously (and privately) submit a bid $b_i \in \mathbb{R}_+$. The highest bid wins the item, but only pays the second-highest bid price (hence second price auction). In the event of a tie, the item is allocated randomly among the highest bidders (let k denote the number of winning bidders)
- If a bidder wins the item, his payoff is $v_i - p$, where p is the second-highest bid; if a bidder does not win, his payoff is 0

From the setup, we see the strategy sets are $S_i = \mathbb{R}_+$.

1.2 Solving the game

We will show that bidding your valuation is a weakly dominant strategy in this game. That is, for any opponent profile b_{-i} and any possible bid b_i,

$$u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$$

For convenience, define b to be the maximum bid of all of i's opponents: $b = \max b_{-i} = \max \{b_j\}_{j \neq i}$. Therefore, we can express the player's payoff as:

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b & \text{if } b_i > b \\ \frac{1}{k}(v_i - b) & \text{if } b_i = b \\ 0 & \text{if } b_i < b \end{cases}$$

Suppose that the player chooses a bid b_i different to their valuation v_i. This gives us six possible cases:

	$b < b_i$	$b = b_i$	$b > b_i$
$b_i < v_i$	(1)	(2)	(3)
$b_i > v_i$	(4)	(5)	(6)

Let's run through the cases and compare the payoff from playing b_i versus v_i:

1. If $b < b_i < v_i$: player i wins with either strategy and has payoff $v_i - b$ in either case, so $u(v_i, b_{-i}) = u(b_i, b_{-i})$.

2. If $b = b_i < v_i$: player i only wins sometimes when playing b_i, but wins for sure when playing v_i, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

3. If $b_i < v_i$ and $b_i < b$. Three sub-cases:

    (a) $b_i < b < v_i$: player i always wins with v_i and loses for sure with b_i, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

    (b) $b_i < v_i < b$: player i never wins with either strategy, so the payoff is always 0, so $u(v_i, b_{-i}) = u(b_i, b_{-i})$

    (c) $b_i < v_i = b$: player i wins sometimes when playing v_i, but loses for sure when playing b_i, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

4. If $b_i > v_i$ and $b_i > b$. Three sub-cases:

    (a) $b_i > b > v_i$: player i wins with b_i and gets payoff $v_i - b < 0$, while they would lose with v_i and get payoff 0, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

    (b) $b_i > v_i > b$: player i wins with either strategy, so the payoff is always $v_i - b$, so $u(v_i, b_{-i}) = u(b_i, b_{-i})$

    (c) $b_i > v_i = b$: player i sometimes wins with v_i, but their payoff will be 0 whether they win or lose; however, they win for sure with b_i, but get payoff $v_i - b < 0$, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

5. If $b = b_i > v_i$: player i loses for sure with v_i and gets payoff zero, or they sometimes win with b_i and either get zero or negative payoff, so $u(v_i, b_{-i}) > u(b_i, b_{-i})$

6. If $b > b_i > v_i$: player i never wins with either strategy, so the payoff is always 0, so $u(v_i, b_{-i}) = u(b_i, b_{-i})$

Regardless of the opponents' strategies, we see $u(v_i, b_{-i}) \geq u(b_i, b_{-i})$ for any b_i and v_i. This shows that v_i is a weakly dominant strategy.

2 Cournot Competition

2.1 Setup

- Two firms simultaneously choose quantities Q_1, Q_2 to produce
- The market's inverse demand curve is $P = 12 - Q_1 - Q_2$
- Firm's want to maximize profits. For simplicity, assume each firm has marginal cost 0

2.2 Discrete Case

To start off, let's make it easy and suppose firms can only produce two possible quantities: $Q_i \in \{3, 4\}$. The payoff matrix is: (you should verify this)

	3	**4**
3	(18,18)	(15,20)
4	(20,15)	(16,16)

This is just like the Prisoner's Dilemma.

- For firm 1: If firm 2 chooses 3, they should choose 4. If firm 2 chooses 4, they should choose 4.
- For firm 2: If firm 1 chooses 3, they should choose 4. If firm 1 chooses 4, they should choose 4.

Therefore, (4,4) is the (pure strategy) Nash equilibrium. However, this is not Pareto efficient. If the firms colluded and agreed to keep quantities at 3 each, they would both have higher payoff (but this wouldn't be an equilibrium because both firms have an incentive to deviate).

Now, let's add even more quantities:

	3	**4**	**5**	**6**
3	(18,18)	(15,$\underline{20}$)	($\underline{12}$,$\underline{20}$)	($\underline{9}$,18)
4	($\underline{20}$,15)	($\underline{16}$,$\underline{16}$)	($\underline{12}$,15)	(8,12)
5	($\underline{20}$,$\underline{12}$)	(15,$\underline{12}$)	(10,10)	(5,6)
6	(18,$\underline{9}$)	(12,8)	(6,5)	(0,0)

Above, I have underlined the best response for each player and each opponent strategy. No strategy is strictly dominant. However, we can see that 6 is never a best response - therefore, it is a strictly dominated strategy. We can use iterated deletion of strictly dominated strategies (IDSDS) to eliminate it. Now, $Q_i = 4$ is a weakly dominant strategy for both players. We can use iterated deletion of *weakly* dominated strategies to get to it, and therefore show that $(4, 4)$ is indeed still a Nash equilibrium.

Notice now that we actually have 3 Nash equilibria. $(3, 5)$ and $(5, 3)$ are also NE. This is because even though the best response is not unique, the players cannot do strictly better and so do not have an incentive to deviate. However, also note that we cannot find these equilibria using IDWDS. In general, if a solution is found by IDWDS, then it is a Nash equilibria (but the converse is not true).

2.3 Solving the Full Game: Best Response Functions

Now we will let firms choose any quantity: $Q_i \in \mathbb{R}_+$. One way to analyze this game is to consider each player's best response function: $r_i(Q_{-i})$. We'll look at player 1. They take player 2's quantity as given and choose the quantity that maximizes profit (i.e. the best response). For a given Q_2, player 1's problem:

$$\max_{Q_1} \ (12 - Q_1 - Q_2)Q_1$$

Using a FOC, we see player 1's best response function is:

$$12 - 2Q_1 - Q_2 = 0$$

$$\therefore r_1(Q_2) = \frac{12 - Q_2}{2}$$

Likewise, for player 2 we have

$$r_2(Q_1) = \frac{12 - Q_1}{2}$$

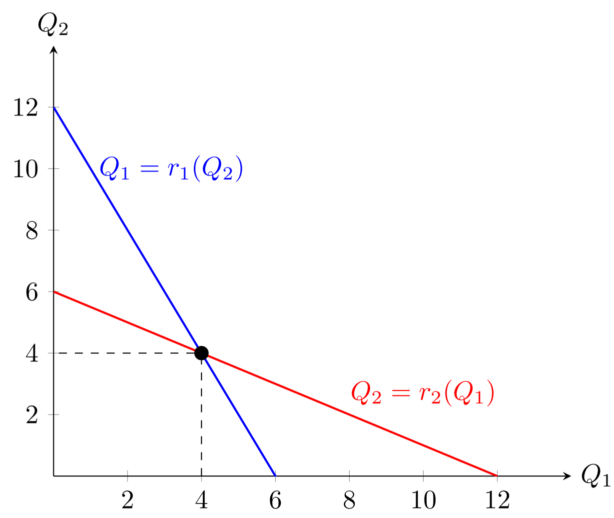
At a Nash Equilibrium, both players should be best responding to each other (if not, one of them would have an incentive to change the strategy and increase their profits). So given the best response functions:

$$Q_1 = r_1(Q_2)$$
$$Q_2 = r_2(Q_1)$$

The Nash equilibrium is where:

$$Q_1 = r_1(r_2(Q_1))$$

This says that player 1 is best responding to player 2, who in turn is also best responding to player 1, and so on. Graphically, this occurs where the best response functions intersect.



There is a unique point satisfying both of these equations: $(Q_1, Q_2) = (4, 4)$. We can see this algebraically:

$$\begin{aligned} Q_1 &= 6 - \frac{1}{2}\left(6 - \frac{1}{2}Q_1\right) \\ Q_1 &= 3 + \frac{1}{4}Q_1 \\ \frac{3}{4}Q_1 &= 3 \end{aligned}$$

$$\therefore Q_1^* = 4$$
$$Q_2 = 6 - \frac{1}{2}Q_1^*$$
$$= 4$$

2.4 Solving the Game: Dominance Reasoning

We can come to the same solution using iterated deletion of weakly dominated strategies. We'll first show that setting $Q_1 > 6$ is weakly dominated by $Q_1 = 6$. First, if $Q_2 \geq 6$, for any $Q_1 > 6$ we have $u_1(6, Q_2) = 0 = u_1(Q_1, Q_2)$.

If $Q_2 < 6$, then for $Q_1 > 6$

$$\begin{aligned} u_1(6, Q_2) - u_1(Q_1, Q_2) &= 36 - 6Q_2 - (12Q1 - Q_1^2 - Q_1Q_2) \\ &= 36 + Q_2 \underbrace{(Q_1 - 6)}_{>0} + Q_1^2 - 12Q_1 \\ &> 36 + Q_1^2 - 12Q_1 \\ &= (Q_1 - 6)^2 \\ &\geq 0 \end{aligned}$$

Thus for any Q_2 and any $Q_1 > 6$, we have $u_1(6, Q_2) \geq u_1(Q_1, Q_2)$, so bidding higher than 6 is weakly dominated. The same argument shows that any $Q_2 > 6$ is weakly dominated for player 2.

After deleting all strategies greater than 6, we can argue that all strategies below 3 are weakly dominated (you will do a version of this on your problem set). Proceeding in this fashion, the only strategy profile that survives this procedure is 4.

If you're curious how to show that the result of this iterative process is 4, here's one approach. The set of strategies the survive IDWDS at each round is: $[0, 12], [0, 6], [3, 6], [3, 4.5], [3.75, 4.5], \ldots$. At each step, we cut the interval in half, and alternate between lowering the right endpoint and raising the left endpoint. Thus the left endpoint increases every 2 steps, increasing one-fourth as much as it did previously. Therefore the limit of the left endpoint is:

$$3 + \frac{3}{4} + \frac{3}{16} + \ldots = \sum_{k=0}^{\infty} \frac{3}{4^k} = 3\frac{1}{1 - \frac{1}{4}} = 4$$

Since the width of the interval is clearly converging to zero, the only strategy that survives IDWDS is 4.