Advanced Micro: Recitation 7
Preferences over Lotteries

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1 Simplex

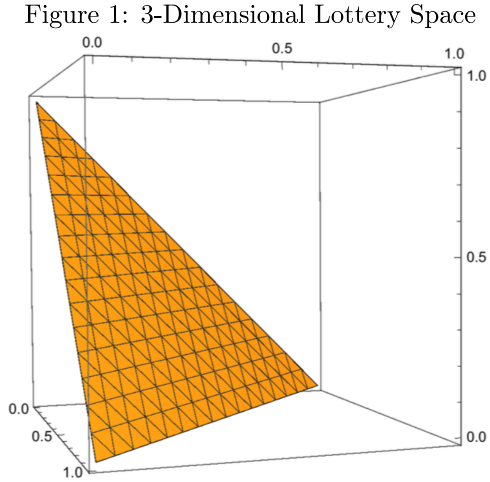
Recall that we saw the simplex in Homework 2:

Definition. The set $S^{n-1} = \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = 1, ; x_i \geq 0, i = 1, ..., n\}$ is called the $(n-1)$-dimensional unit simplex

The idea here is that even though x is a n-dimensional object, given the constraint, we only need to $n-1$ components to fully characterize x. This is very intuitive with probabilities. If there are three possible outcomes, where $p_1 = 0.25$ and $p_2 = 0.3$, then we know that the third outcome must occur with probability $p_3 = 1 - (p_1 + p_2) = 0.45$. Therefore, to represent a 3-dimensional probability, all we need is to plot it in a 2-dimensional space. This is the key idea of the simplex. Since we want to think about preferences over lotteries, the simplex captures all possible lotteries that we could face (i.e. all possible probabilities over the outcomes). In other words, the simplex represents our consumption set!

Next, we want to know how to actually draw the simplex. Let's do this in 3-dimensions (i.e. consider a lottery over three possible outcomes). First, we start by plotting this in 3-dimensions. This gives us the triangle shown in Figure 1, where each vertex represents a lottery were we get an outcome for sure. Spaces on the interior represent a lottery where each outcome occurs with non-zero probability.

Figure 1: 3-Dimensional Lottery Space



To get the simplex, we want to map this 3-dimensional triangle into a 2-dimensional triangle (which is much easier to draw!). Let's consider transforming it an equilateral triangle with sides of length 1. This is intuitive because we can give the vertices of the triangle the following interpretation:

- Bottom left corner: (0,0), which corresponds to the lottery with probabilities (1,0,0).
- Bottom right corner: (1,0), which corresponds to the lottery with probabilities (0,1,0).
- Top corner: $(0.5, \frac{\sqrt{3}}{2})$,which corresponds to the lottery with probabilities (0,0,1).[1]

Note that you can switch up the order, but the point is that the vertex in both the 3D and 2D case represents the degenerate lotteries. So how can we generalize this transformation to points outside of the corners? Well, this is just some linear transformation. Let's call it $F : \mathbb{R}^3 \to \mathbb{R}^2$ and write it as follows:

$$F(p) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So this transformation takes in a 3-dimensional vector p (the probabilities), multiplies it by a 2×3 matrix and gives the x-y coordinates to plot on the simplex. We must then solve for the values of this matrix, but this is easy to do since we know that it satisfies these three equations:

$$F(1,0,0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$F(0,1,0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$F(0,0,1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

Therefore, the mapping is simply:

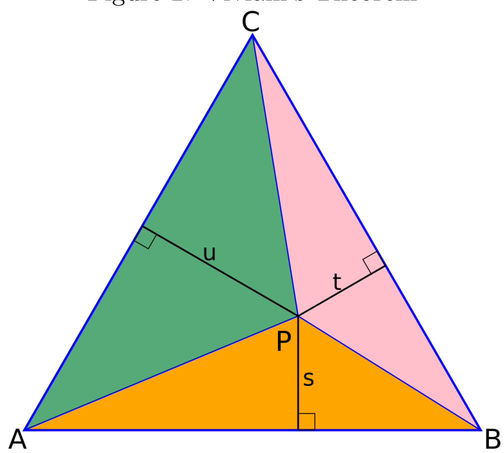
$$F(p) = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} p$$

Another way to think about the simplex is to use something called Viviani's Theorem, which states: "states that the sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's height". This is shown in Figure 2. In this picture, we take a point P and measure the distance between P and each side of the triangle. This gives us the lengths s, t, u. Viviani's Theorem tells us that $s + t + u$ equals the height of the triangle. This can be adapted to our situation in a really nice way. If we make the triangle have height 1, then the distances represent probabilities, which must sum up to 1. Then, the distance between the point P and the side opposite a corner represents the probability assigned to that corner's outcome. For example, in the picture below, P assigns the probability t to outcome A. Or in our notation P is the lottery: $(t \circ A, u \circ B, s \circ C)$. Another way to think about this is that the closer a point is to a corner, then the higher the probability that is assigned to that outcome.

Again, we want to know how to construct this triangle. The triangle we did before had a height of $\frac{\sqrt{3}}{2}$. To have an equilateral triangle with height 1, we will need the triangle to have the following vertices:

[1]For the top corner, we know it occurs at the midpoint of the opposite length, i.e. $x = 0.5$. To get the y coordinate, we just use Pythagoras' theorem: $1^2 = 0.5^2 + y^2 \implies y^2 = \frac{3}{4}$

Figure 2: Viviani's Theorem



- Bottom left corner: (0,0), which corresponds to the lottery with probabilities (1,0,0).
- Bottom right corner: $\left(\frac{2}{\sqrt{3}}, 0\right)$, which corresponds to the lottery with probabilities (0,1,0).
- Top corner: $\left(\frac{1}{\sqrt{3}}, 1\right)$,which corresponds to the lottery with probabilities (0,0,1).

These vertices are quite easy to get. We know the height is 1 and call the length of the sides as x. Therefore we can construct a right-angled triangle with $a = \frac{1}{2}x$, $b = 1$, and $c = x$. By Pythagoras' theorem, we have $\frac{1}{4}x^2 + 1 = x^2$, which means that $\frac{3}{4}x^2 = 1 \implies x = \frac{2}{\sqrt{3}}$. Using the same process as above:

$$F(1,0,0) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$F(0,1,0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix}$$
$$F(0,0,1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}$$

Therefore, the transformation is:

$$F(p) = \begin{pmatrix} 0 & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 \end{pmatrix} p$$

This is useful because now the distance to the sides represent the probability of the corresponding vertex. Let's label the outcomes as A, B, C, as in Figure 2. The sides (and their corresponding vector) also have the following formulas:

$$C: \ y = 0$$
$$B: \ y = \sqrt{3}x$$
$$A: \ y = 2 - \sqrt{3}x$$

Let's consider a point P with coordinates (x_0, y_0). We want to know what probabilities this point correspond to, so we need to calculate the distances. For C, this is easy as the distance is easy to get as it is just the

y value of the point. For the other two outcomes, note that the distance between a point (x_0, y_0) and a line $ax + by + c = 0$ is:

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Therefore the probabilities corresponding to a point (x_0, y_0) are:

$$\begin{aligned}
p_A &= \frac{\left|\sqrt{3}x_0 + y_0 - 2\right|}{\sqrt{3+1}} = \frac{\left|y_0 + \sqrt{3}x_0 - 2\right|}{2} \\
p_B &= \frac{\left|-\sqrt{3}x_0 + y_0 + 0\right|}{\sqrt{3+1}} = \frac{\left|y_0 - \sqrt{3}x_0\right|}{2} \\
p_C &= \frac{|0x_0 + y_0 + 0|}{\sqrt{0+1}} = y_0
\end{aligned}$$

For example, let's take the gamble $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$. To plot this, we find the point using the F transformation:

$$F\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right) = \begin{pmatrix} 0 & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \cdot \frac{2}{\sqrt{3}} + \frac{1}{3} \cdot \frac{1}{\sqrt{3}} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3\sqrt{3}} \\ \frac{1}{3} \end{pmatrix} \approx \begin{pmatrix} 0.384 \\ 0.333 \end{pmatrix}$$

As an example, let's check the distance from the right side. This should give us the probability assigned to A:

$$\begin{aligned}
d &= \frac{\left|\sqrt{3}\left(\frac{2}{3\sqrt{3}}\right) + \left(\frac{1}{3}\right) + (-2)\right|}{\sqrt{(\sqrt{3})^2 + (1)^2}} \\
&= \frac{\left|\frac{2}{3} + \frac{1}{3} - 2\right|}{\sqrt{3+1}} \\
&= \frac{1}{2} = p_A
\end{aligned}$$

2 Gambles

In these exercises, let f, g, and h be gambles over outcomes $a_1, ..., a_n$ (ranked from most to least preferred).

2.1 Exercise 1

Question: Take any two gambles g and h where $g \succsim h$. Given G1-G6, show that $\forall \alpha, \beta \in [0, 1]$:

$$(\alpha \circ g, (1-\alpha) \circ h) \succsim (\beta \circ g, (1-\beta) \circ h) \iff \alpha \geq \beta$$

Note: show this using the axioms, not the utility functions

Note that this differs from G4 (monotonicity) which was in terms of outcomes, but this is in terms of gambles. Let's prove this.

By G3 (continuity), we know that $\exists \lambda_g, \lambda_h \in [0,1]$ s.t.

$$g \sim (\lambda_g \circ a_1, (1-\lambda_g) \circ a_n)$$
$$h \sim (\lambda_h \circ a_1, (1-\lambda_h) \circ a_n)$$

By G2 (transitivity) and G4 (monotonicity):

$$g \succsim h \implies \lambda_g \geq \lambda_h$$

By G6 (reduction to simple gambles):

$$\begin{aligned}(\alpha \circ g, (1-\alpha) \circ h) &\sim ((\alpha\lambda_g + (1-\alpha)\lambda_h) \circ a_1, (\alpha(1-\lambda_g) + (1-\alpha)(1-\lambda_h)) \circ a_n) \\ &\sim ((\alpha\lambda_g + (1-\alpha)\lambda_h) \circ a_1, (1 - (\alpha\lambda_g + (1-\alpha)\lambda_h)) \circ a_n) = L_\alpha\end{aligned}$$

and

$$\begin{aligned}(\beta \circ g, (1-\beta) \circ h) &\sim ((\beta\lambda_g + (1-\beta)\lambda_h) \circ a_1, (\beta(1-\lambda_g) + (1-\beta)(1-\lambda_h)) \circ a_n) \\ &\sim ((\beta\lambda_g + (1-\beta)\lambda_h) \circ a_1, (1 - (\beta\lambda_g + (1-\beta)\lambda_h)) \circ a_n) = L_\beta\end{aligned}$$

By G2 (transitivity) and G4 (monotonicity):

$$\begin{aligned}&(\alpha \circ g, (1-\alpha) \circ h) \succsim (\beta \circ g, (1-\beta) \circ h) \\ \iff & L_\alpha \succsim L_\beta \\ \iff & \alpha\lambda_g + (1-\alpha)\lambda_h \geq \beta\lambda_g + (1-\beta)\lambda_h \\ \iff & (\alpha-\beta)(\lambda_g - \lambda_h) \geq 0 \\ \iff & \alpha - \beta \geq 0\end{aligned}$$

Since we know that $\lambda_g \geq \lambda_h$.

2.2 Exercise 2

One property of preferences over gambles is called the Archimedean property, which states:

- For any three lotteries f, g, h, if $f \succ g \succ h$, then $\exists \alpha, \beta \in (0,1)$ s.t. $(\alpha \circ f, (1-\alpha) \circ h) \succ g \succ (\beta \circ f, (1-\beta) \circ h)$.

Show that G1-G6 imply the Archimedean property.

First, let's express G3 in terms of gambles. This will make everything a lot easier (and you can also use these in future questions!). For G3 (continuity), we know it as follows:

For any lottery g, $\exists \alpha \in [0,1]$ s.t. $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$.

Let's extend it to:

For any three lotteries f, g, h, if $f \succsim g \succsim h$, then $\exists \alpha \in [0,1]$ s.t. $g \sim (\alpha \circ f, (1-\alpha) \circ h)$.

Take any arbitrary three lotteries f, g, h where $f \succsim g \succsim h$. By continuity, we have that $\exists \lambda_f, \lambda_g, \lambda_h \in [0,1]$ such that:

$$\begin{aligned} f &\sim (\lambda_f \circ a_1, (1-\lambda_f) \circ a_n) \\ g &\sim (\lambda_g \circ a_1, (1-\lambda_g) \circ a_n) \\ h &\sim (\lambda_h \circ a_1, (1-\lambda_h) \circ a_n) \end{aligned}$$

Consider a lottery that is a function α, call it $L(\alpha)$, that is defined as $L(\alpha) = (\alpha \circ f, (1-\alpha) \circ h)$. By G6 (reduction to simple gambles):

$$\begin{aligned} L(\alpha) &\sim ((\alpha\lambda_f + (1-\alpha)\lambda_h) \circ a_1, (\alpha(1-\lambda_f) + (1-\alpha)(1-\lambda_h)) \circ a_n) \\ &\sim ((\alpha\lambda_f + (1-\alpha)\lambda_h) \circ a_1, (1-(\alpha\lambda_f + (1-\alpha)\lambda_h)) \circ a_n) \end{aligned}$$

Therefore, if we want $g \sim L(\alpha)$, then by G4 (monotonicity), we need:

$$\begin{aligned} \lambda_g &= \alpha\lambda_f + (1-\alpha)\lambda_h \\ \therefore \alpha &= \frac{\lambda_g - \lambda_h}{\lambda_f - \lambda_h} \end{aligned}$$

Therefore we have found the α and the claim is proved.

Now we can prove the Archimedean property directly. Take an arbitrary three lotteries f, g, h where $f \succ g \succ h$. By the extension of G3 that we proved, $\exists \lambda \in [0,1]$ s.t. $g \sim (\lambda \circ f, (1-\lambda) \circ h)$. Since $f \succ g \succ h$, then it must be the case that $\lambda \in (0,1)$.

Take two numbers $\alpha \in (\lambda, 1)$ and $\beta \in (0, \lambda)$ - note these are non-empty sets because $\lambda \in (0,1)$. From Exercise 1, since $f \succ h$ and $\alpha > \lambda > \beta$, then: (use the above result twice)

$$(\alpha \circ f, (1-\alpha) \circ h) \succ (\lambda \circ f, (1-\lambda) \circ h) \succ (\beta \circ f, (1-\beta) \circ h)$$

By transitivity, since $g \sim (\lambda \circ f, (1-\lambda) \circ h)$, and so we get the result.

2.3 Exercise 3

Show the result in Exercise 1, but *without* G3 (continuity).

To prove this, it is useful to have the "independence" assumption, which is just an extension of G5.

For any three lotteries f, g, h, if $g \succsim h$, then $(\alpha \circ g, (1-\alpha) \circ f) \succsim (\alpha \circ h, (1-\alpha) \circ f), \forall \alpha \in [0,1]$

You will show this is in homework (question 2). This essentially says that to compare lotteries, we should only look at what's different and ignore things that are the same (i.e. the same outcome occurring with the same probability).

Prove "if" direction ($\Longleftarrow$)

If $g \sim h$, then its trivial that any α and β would work, so let's only consider the strict case $\succ$.

If $\alpha = 1$, take any $\beta \in [0, 1)$. By independence:

$$\begin{aligned} g &\succ h \\ \implies (\beta \circ g, (1-\beta) \circ g) &\succ (\beta \circ g, (1-\beta) \circ h) \\ g &\succ (\beta \circ g, (1-\beta) \circ h) \\ (\alpha \circ g, (1-\alpha) \circ h) &\succ (\beta \circ g, (1-\beta) \circ h) \end{aligned}$$

Since $g \sim (\alpha \circ g, (1-\alpha) \circ h)$. You can follow the same idea for $\beta = 0$.

So, let's now consider if we have $0 < \beta < \alpha < 1$. By independence, we have:

$$\begin{aligned} g &\succ h \\ \implies (\alpha \circ g, (1-\alpha) \circ h) &\succ (\alpha \circ h, (1-\alpha) \circ h) \sim h \\ L_\alpha &\succ h \end{aligned}$$

Where $L_\alpha = (\alpha \circ g, (1-\alpha) \circ h)$. Since $\frac{\beta}{\alpha} \in (0, 1)$, then use independence (and transitivity) again:

$$\begin{aligned} L_\alpha &\succ h \\ \implies \left(\frac{\beta}{\alpha} \circ L_\alpha, \left(1 - \frac{\beta}{\alpha}\right) \circ L_\alpha\right) &\succ \left(\frac{\beta}{\alpha} \circ L_a, \left(1 - \frac{\beta}{\alpha}\right) \circ h\right) \\ L_\alpha &\succ \left(\frac{\beta}{\alpha} \circ (\alpha \circ g, (1-\alpha) \circ h), \left(1 - \frac{\beta}{\alpha}\right) \circ h\right) \\ L_\alpha &\succ \left(\beta \circ g, \left(\frac{\beta}{\alpha} - \beta + 1 - \frac{\beta}{\alpha}\right) \circ h\right) \\ (\alpha \circ g, (1-\alpha) \circ h) &\succ (\beta \circ g, (1-\beta) \circ h) \end{aligned}$$

Which is exactly what we set out to prove.

Prove "only if" direction ($\Longrightarrow$)

We want to show that $0 \leq \beta < \alpha \leq 1$. Suppose not, and instead assume that $1 \geq \beta \geq \alpha \geq 0$.

By independence:

$$\begin{aligned} g &\succ h \\ \implies (\beta \circ g, (1-\beta) \circ h) &\succ (\beta \circ g, (1-\beta) \circ h) \sim h \end{aligned}$$

By transitivity and the starting assumption, we have that

$$
\begin{gathered}
(\alpha \circ g, (1-\alpha) \circ h) \succ (\beta \circ g, (1-\beta) \circ h) \succ h \\
L_\alpha \succ L_\beta \succ h
\end{gathered}
$$

Note that $\frac{\alpha}{\beta} \in [0,1]$ by our starting assumption. So by independence (using it twice):

$$
\begin{aligned}
& L_\alpha \succ L_\beta \succ h \\
\implies & \left(\frac{\alpha}{\beta} \circ L_\alpha, \left(1-\frac{\alpha}{\beta}\right) \circ L_\alpha\right) \succ \left(\frac{\alpha}{\beta} \circ L_\beta, \left(1-\frac{\alpha}{\beta}\right) \circ L_\alpha\right) \succ \left(\frac{\alpha}{\beta} \circ L_\beta, \left(1-\frac{\alpha}{\beta}\right) \circ h\right)
\end{aligned}
$$

Note that the LHS is:

$$
\left(\frac{\alpha}{\beta} \circ L_\alpha, \left(1-\frac{\alpha}{\beta}\right) \circ L_\alpha\right) \sim L_\alpha
$$

Note that the RHS is:

$$
\begin{aligned}
\left(\frac{\alpha}{\beta} \circ L_\beta, \left(1-\frac{\alpha}{\beta}\right) \circ h\right) &\sim \left(\frac{\alpha}{\beta} \circ (\beta \circ g, (1-\beta) \circ h), \left(1-\frac{\alpha}{\beta}\right) \circ h\right) \\
&\sim \left(\alpha \circ g, \left(\frac{\alpha}{\beta}(1-\beta) + 1 - \frac{\alpha}{\beta}\right) \circ h\right) \\
&\sim \left(\alpha \circ g, \left(\frac{\alpha}{\beta} - \alpha + 1 - \frac{\alpha}{\beta}\right) \circ h\right) \\
&\sim (\alpha \circ g, (1-\alpha) \circ h) = L_\alpha
\end{aligned}
$$

Therefore, we get that $L_\alpha \succ L_\alpha$, which is a contradiction.