Intermediate Micro: Recitation 2
Budget Constraint

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1 Overview

1.1 Optimization Problem

In the first part of the course, we will be focusing on modeling how a consumer makes decisions. In economics, when we model an agent making a decision, we often think of this as an optimization problem (as discussed in the last recitation). Intuitively, it is easy to think about the consumer's problem because we all face it every single day. Imagine you're going to go buy some food for lunch. You may have budgeted some amount to spend (say, \$10), either explicitly or implicitly. You can spend that \$10 in lots of ways (e.g. a sandwich from Subsconcious or dumplings from a food truck), each one giving you a different level of satisfaction. You end up choosing the option that you like the most (i.e. makes you the happiest), while also costing no more than \$10. That doesn't mean that there other options out there that would make you happier: there might be some very fancy sushi or steak that would you enjoy a lot more than a simple sub. However, this fancy option is outside your budget, so you won't buy it. This story is just an example of the consumer's optimization problem. All we have to do now is turn this intuition into something more formal (i.e. math).

Let's take our story and think about the different 'variables'. In our story, there is a huge number of lunch options you could buy. To keep things simple, we'll only focus on having two goods, called good 1 and good 2. Once we learn how to do things using two goods, then we can easily generalize and apply the same ideas to 3 or more goods (if we only had one good, then there would be no choice for the consumer, so it wouldn't be a very interesting problem). Each good has a price, which we'll denote as p_1 for each unit of good 1, and p_2 for each unit of good 1. These are unit prices, where the unit is going to depend on the type of good (e.g. if good 1 is milk, then p_1 is price per gallon). A consumer is going to choose how much of each good to buy, which we'll denote as x_1 for the quantity of good 1, and x_2 for the quantity of good 2. Finally, in our story there was a limit on how much to spend. Let's call this variable income and denote it as M (for money). So, to sum up, our five variables are: x_1, x_2, p_1, p_2 and M.

Recall that there are four things we have in our optimization problem:

1. *Objective Function:* What function do we want to maximize?
2. *Constraint:* What restrictions are placed on our problem?

3. *Choice Variables:* What variables do we want to choose to achieve the optimal objective function?

4. *Parameters:* What variables affect the problem but we are not able to choose?

For this recitation, let's put aside the objective function. We'll soon see that this will be the consumer's utility, but for now you can think of it is 'happiness'. Let's now consider the next three components:

Constraint

The constraint in the consumer's problem is the budget constraint. This simply says that consumers are not allowed to spend money that they don't have. Our model is super simple; for example, we don't allow the consumer to borrow money (e.g. credit cards). In a mix of words and math, the budget constraint should essentially say:

$$\text{Total Expenditure } \leq \text{ Income}$$

The right-hand side (RHS) of the inequality is easy: income is just M. The left-hand side (LHS) is also not too hard. Total expenditure is just the sum of your expenditure on each good. How much do you spend on good i? (where i is 1 or 2). It's equal to how much of good i you purchase (x_i) multiplied by the price of good i (p_i). So expenditure on good 1 is $p_1 x_1$ and on good 2 it is $p_2 x_2$. This means that total expenditure is $p_1 x_1 + p_2 x_2$. So our budget constraint is going to be:

$$p_1 x_1 + p_2 x_2 \leq M$$

What if we had more than 2 goods? Say there were N goods we could purchase, denoted as $x_1, x_2, ...x_N$. We would just use the same logic as before: the expenditure on a good i is $p_i x_i$ and total expenditure is the sum of all expenditures. This means the budget constraint is:

$$p_1 x_1 + \cdots + p_N x_N = \sum_{i=1}^{N} p_i x_i \leq M$$

Choice Variables

Our choice variables are things that we can choose. When you are buying lunch, you choose which option to purchase (and how much of it). So, here, our choice variables are the quantities: x_1 and x_2.

Parameters

The parameters are the variables that we don't get too choose. By process of elimination, you should see that means our parameters are p_1, p_2, and M. This makes sense - when you go to the store, you don't get to choose the prices that you pay. If you did, you would just set prices equal to 0 and buy whatever you want. Similarly, you also don't get to choose your income, otherwise you would just choose a huge income so that you can afford whatever you want. Of course, you may want to argue that we have some choice in our income based on what job we take, but that's beyond the scope of this class. We have to start with the simplest problem possible and work our way up, so here we will just think of income and prices as being *exogenous.*

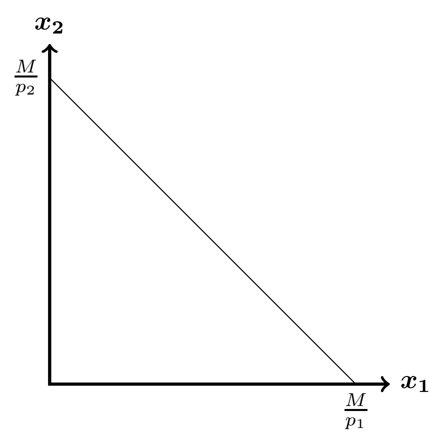
In other words, they are determined outside the problem and the agent (the consumer) has no choice in the matter.

1.2 Plotting

Drawing budget constraints is an important part of this course. Recall from the math review how to plot a linear function (straight line). For now, forget about the inequality and consider the function as $p_1x_1+p_2x_2 = M$. Let's plot this on an axis with x_1 on the horizontal x-axis and x_2 on the vertical y-axis. First we want this in form of $y = mx + c$. Re-arranging the budget equation so that we have x_2 (the y variable) on the LHS gets us:

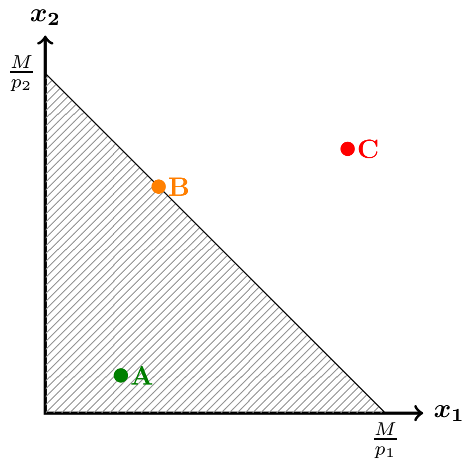
$$\begin{aligned} p_2x_2 &= -p_1x_1 + M \\ x_2 &= -\frac{p_1}{p_2}x_1 + \frac{M}{p_2} \end{aligned}$$

From this we can see that the slope is $-\frac{p_1}{p_2}$ and the y-intercept is $\frac{M}{p_2}$. You should also check and see that the x-intercept is $\frac{M}{p_1}$. With this information, we can now plot the *budget line*.



There's a nice interpretation for each of these. The x-intercept represents how much of good 1 you could buy if you spent all your money on good 1. Similarly, the y-intercept represents how much of good 2 you could buy if you spent all your money on good 2. Moving along the line represents some intermediate case where you buy a mix of goods 1 and 2 (while spending all of your income M). Recall that the slope represents how much the y variable changes along the line if we increase the x variable by 1 unit. This means that the slope represents the *opportunity cost*. If we move along the budget line from the left to the right, increasing the x_1 quantity by 1 means that the x_2 quantity decreases by $\frac{p_1}{p_2}$ (since it is a negative slope). Since we spend all our money on either good 1 or good 2 (and nothing else), then increasing expenditure on good 1 must necessarily mean that we've reduced expenditure on good 2. To buy one extra unit of good 1, we have to pay p_1 dollars. Since good 2's price is p_2, this means that the p_1 dollars could have purchased $\frac{p_1}{p_2}$ units of good 2. So for an extra unit of good 1, we have to give up $\frac{p_1}{p_2}$ units of good 2 - this is exactly the definition of opportunity cost (and the slope).

Notice that the constraint is actually an inequality. This means it represents not just a line, but a whole area on the graph. Below is the graph of the budget constraint, where the shaded area represents $p_1x_1+p_2x_2 \leq M$:



The three points plotted represent the three possible cases we can be in:

- At point **A**, the consumer is spending strictly less than their income: $p_1x_1 + p_2x_2 < M$
- At point **B**, the consumer is spending exactly all their income: $p_1x_1 + p_2x_2 = M$
- At point **C**, the consumer is spending strictly more than their income: $p_1x_1 + p_2x_2 > M$

In an optimization problem, only **A** and **B** are possible solutions. They are both in the feasible set of options (ones that the consumer can afford). **C** is unaffordable, given the current prices and income, and so cannot be a solution.

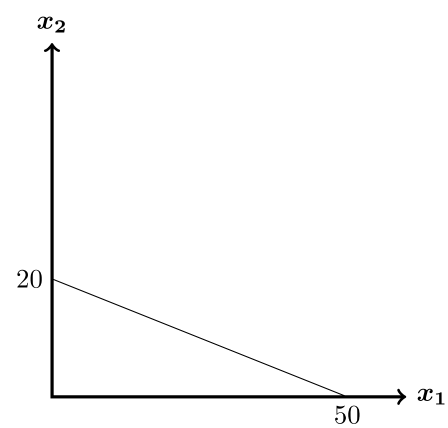
2 Examples

2.1 Constant Prices

Let's practice plotting budget constraints. These will be in the most common and most straightforward form, where prices are constant at all units.

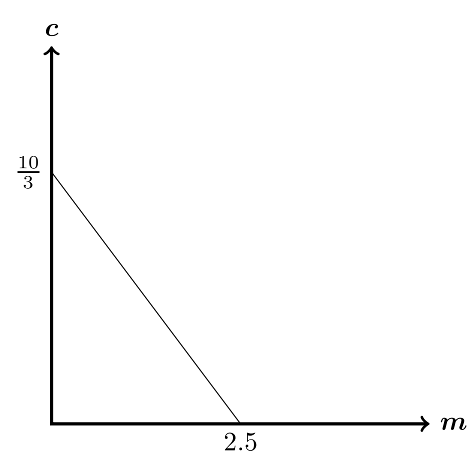
1. **Good 1 costs \$2 per unit, good 2 costs \$5 per unit. The consumer's income is \$100.**

   The x-intercept is $100/2 = 50$. The y-intercept is $100/5 = 20$. The slope is $-2/5 = -0.4$.



2. **A consumer has \$10 to spend on cereal (c) and milk (m). The price of cereal is \$3 per box. The price of milk is \$4 per gallon.**

   Let's put c on the y-axis and m on the x-axis. The x-intercept is $10/4 = 2.5$. The y-intercept is $10/3 \approx 3.33$. The slope is $-4/3 \approx -1.33$.



3. **A consumer spends all their money on apples (a) and bananas (b). Two possible bundles they can purchase are $(a, b) = (4, 14)$ and $(a, b) = (6, 4)$. Draw the budget line.**

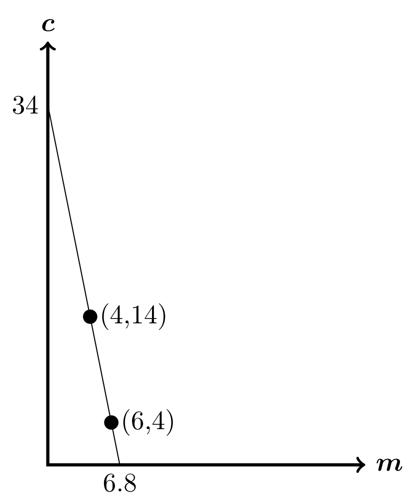
   Recall that we can calculate the formula for a straight line using just two points. Let's call these two points (x_0, y_0) and (x_1, y_1). The first step is to find the slope m:

$$
\begin{aligned}
m =& \frac{y_1 - y_0}{x_1 - x_0} \\
=& \frac{4 - 14}{6 - 4} \\
=& \frac{-10}{2} \\
=& -5
\end{aligned}
$$

The second step is find the intercept c. We plug in the value for m and one of the given points (e.g. $(6, 4)$) into the standard linear line formula to solve for it:

$$
\begin{aligned}
y =& mx + c \\
(4) =& (-5)(6) + c \\
c =& 4 + 30 = 34
\end{aligned}
$$

This means the equation for the budget line is $y = -5x + 34 \implies b = -5x + 34$. This has a x-intercept of $34/5 = 6.8$. The y-intercept is 34 and the slope is -5.



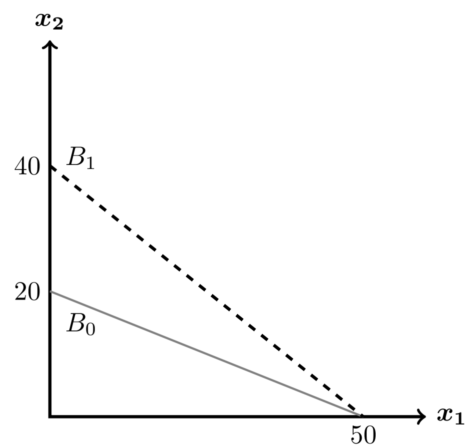
2.2 Change in Parameters

A common question you will see in problem sets and exams is to see how a problem changes when one of the parameter changes. In this part, we want to focus on how the budget set changes when there is a change in a price of a good or income. In the examples below, I'm going to label the old budget line as a grey solid line labelled B_0 and the new budget line as a black dashed line labelled B_1.

1. **Good 1 costs \$2 per unit, good 2 costs \$5 per unit. The consumer's income is \$100. The price of good 2 decreases to \$2.50 per unit.**

B_0 is the same as 2.1 Example 1. For B_1, the y-intercept is $100/2.5 = 40$. The x-intercept remains unchanged. The slope changes from -0.4 to $-2/2.5 = -0.8$.

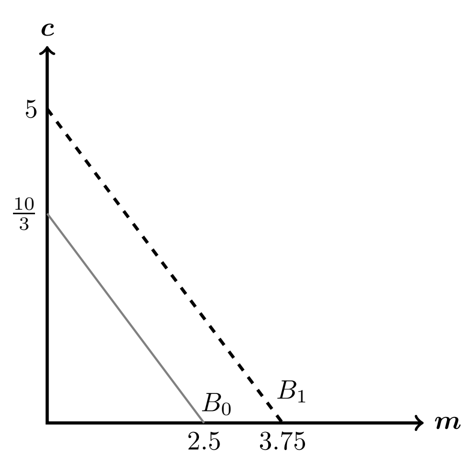
Note that the feasible set (the area under the budget line) has increased. Intuitively this makes sense: goods in this economy have gotten cheaper, so the consumer can now afford more bundles than before. Since p_2 has gotten lower, now $\frac{p_1}{p_2}$ is larger, which means that the opportunity cost of 1 more unit of good 1 has gotten larger (because we can buy more of good 2).



2. **A consumer has \$10 to spend on cereal (c) and milk (m). The price of cereal is \$3 per box. The price of milk is \$4 per gallon. The consumer's income increases to \$15.**

   B_0 is the same as 2.1 Example 2. For B_1, the x-intercept is $15/4 = 3.75$. The y-intercept is $15/3 = 5$. The slope stays the same (-1.33).

   Note that the feasible set (the area under the budget line) has increased. Intuitively this makes sense: the consumer is richer, so they can now afford more bundles than before. However even though the consumer is richer, the opportunity cost between the two goods is still the same (because the slope is unchanged).



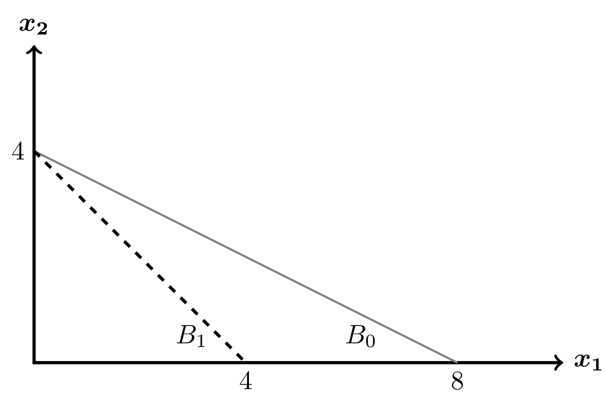
3. **Good 1 costs \$1 and good 2 costs \$2. The consumer's income is \$8. The government introduces a unit tax on good 1 of \$1. Draw the pre- and post-tax budget lines.**

   - Note that a *unit tax* (also known as *quantity tax*) is paid on each unit of a good. So this means that the expenditure on good i becomes $p_i x_i + t x_i$, where t is the unit tax. The first part is the usual expenditure we are used to. The second part is the tax that the consumer pays for buying x_i units of good i. We can re-arrange this into $(p_i + t)x_i$. This shows that a unit tax is just the same as increasing the price from p_i to $p_i + t$.

   For B_0: The x-intercept is $8/1 = 8$. The y-intercept is $8/2 = 4$. The slope is $-1/2 = -0.5$.

   For B_1: The post-tax price of good 1 is $1 + 1 = 2$. The x-intercept is $8/2 = 4$. The y-intercept stays the same. The slope is $-2/2 = -1$.

   Now the feasible set has gotten smaller. The tax has increased the price of goods, and so the consumer cannot afford some bundles that they used to be able to purchase. The tax has also increased the opportunity cost $\frac{p_1}{p_2}$ because we now have to give up more of good 2 in order to buy an additional unit of good 1 (since that one unit is more expensive now).

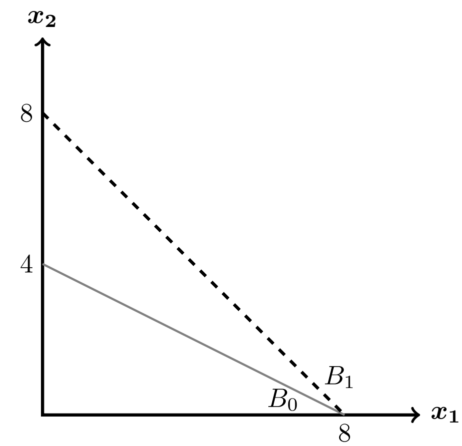


4. **Good 1 costs \$1 and good 2 costs \$2. The consumer's income is \$8. The government introduces a unit subsidy on good 2 of \$1. Draw the pre- and post-subsidy budget lines.**

   - Using the notation above, a subsidy is whenever $t < 0$. This means that $p_i < p_i + t$, so it is effectively the same as a price decrease.

   B_0 is the same as Example 3 above. For B_1, the post-tax price of good 2 is $2 + (-1) = 1$. The y-intercept is $8/1 = 8$. The x-intercept stays the same. The slope is $-1/1 = -1$.

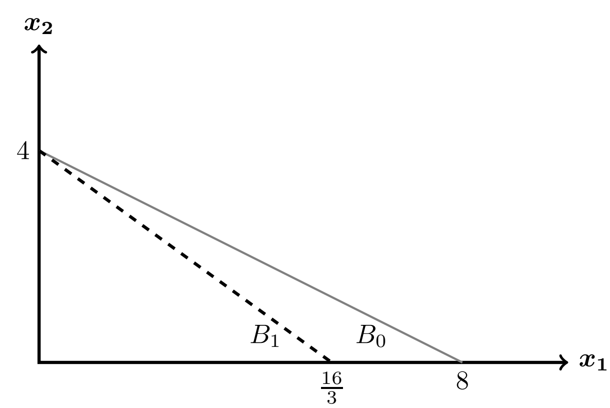
   A subsidy makes goods cheaper, which allows the consumer to purchase more goods. Hence the feasible set increases, as we should expect. It also reduces the price of good 2, which increases the opportunity cost of good 1 (the slope).



5. **Good 1 costs \$1 and good 2 costs \$2. The consumer's income is \$8. The government introduces a value tax on good 1 of 50%. Draw the pre- and post-tax budget lines.**

   - Note that a *value tax* (also known as *ad valorem tax*) is paid on the total value of a good purchased. So this means that the expenditure on good i becomes $p_i x_i + tp_i x_i$, where t is the tax rate. The first part is the usual expenditure we are used to. The second part is the tax that the consumer pays for buying x_i units of good i, which is valued at $p_i x_i$. We can re-arrange this into $(1+t)p_i x_i$. This shows that a unit tax is just the same as increasing the price from p_i to $(1+t)p_i$.

   B_0 is the same as Example 3 above. For B_1, the post-tax price of good 1 is $(1+0.5)\cdot 1 = 1.5$. The x-intercept is $8/1.5 = 5.53$. The y-intercept stays the same. The slope is $-1.5/2 = -0.75$.

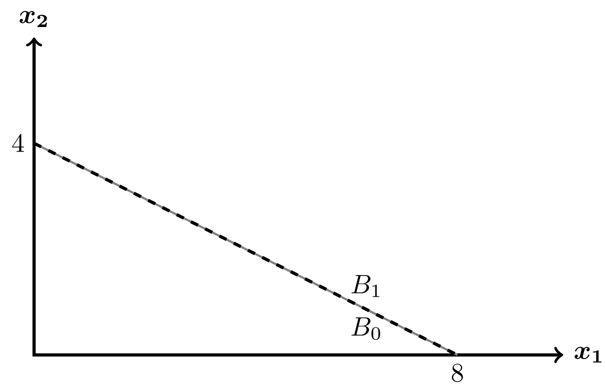


6. **Good 1 costs \$1 and good 2 costs \$2. The consumer's income is \$8. The government decides to double the prices of all goods and also double the consumer's income.**

   B_0 is the same as Example 3 above. For B_1, the new prices are $p_1 = 2$, $p_2 = 4$, $M = 16$. This means that the x-intercept is $16/2 = 8$. The y-intercept is $16/4 = 4$. The slope is $-2/4 = -0.5$. Notice that this is exactly the same as B_0!

   There are two ways for you to see this:

   - Intuitively, this is just saying that <u>only relative prices matter</u>. The actual number of a price has no meaning unless you can compare it to something else. Imagine if your friend told you that Tokyo



was more expensive place than New York because the same good in Tokyo is 1000 yen versus 10 dollars in NYC and the number 1000 is larger than the number 10. You would surely correct your friend and explain how currency exchange rates work, and that all prices in Japan are nominally larger than in the US. So, if your weekly income is 100,000 yen in Japan and 1000 dollars in the US, then that good would make up the same fraction of your income in either country (1%). This is surely a better measure of a good's expensiveness than looking at the number alone. This is exactly what's going on here.

- You can also see this mathematically. Let's denote the new prices and income with a prime symbol: $p_1' = kp_1$, $p_2' = kp_2$, and $M' = kM$, for some constant k. This means that the new budget line B_1 has an x-intercept of:

$$x\text{-intercept of } B_1 = \frac{M'}{p_1'} = \frac{kM}{kp_1} = \frac{M}{p_1} = x\text{-intercept of } B_0$$

  You will see the same thing for the y-intercept and the slope. Given that the intercepts and the slope do not change, that means the lines must be exactly the same.

From these exercises, we can summarize how a change (increase ↑ or decrease ↓) in one of the parameters affects the budget line:[1]

Parameter	**Change**	**x-intercept** $\left(\frac{M}{p_1}\right)$	**y-intercept** $\left(\frac{M}{p_2}\right)$	**Slope (Magnitude)** $\left(\frac{p_1}{p_2}\right)$	**Overall Budget Line Change**
p_1	↑	↓	–	↑	Pivots down (y-int is pivot point)
	↓	↑	–	↓	Pivots up (y-int is pivot point)
p_2	↑	–	↓	↓	Pivots down (x-int is pivot point)
	↓	–	↑	↑	Pivots up (x-int is pivot point)
M	↑	↑	↑	–	Parallel shift out
	↓	↓	↓	–	Parallel shift in

[1] Note for slope, I am using the absolute value, since it will always be negative. So, an increase in the size of the (absolute) slope means that the slope has become more negative (i.e. a smaller number) and the line becomes steeper.

2.3 Non-Linear Prices

Most budget constraints that you see will be covered in everything listed above. The most odd cases are when prices change with the quantity purchased. There are lots of ways this can happen, but you will use the same principles as we've used so far. I'll present just one case here, and we can cover other types later on if they come up.

Under this system, the price changes as you pay for each additional good. For example, consider this pricing system for good 1: $p_1 = 2$ for the first 10 units and $p_1 = 3$ for any additional units beyond 10. So 9 units would cost $\$2 \times 9 = \$18$, 10 units would cost $\$2 \times 10 = \$20$, 11 units would cost $\$2 \times 10 + \$3 \times 1 = \$23$, and 12 units would cost $\$2 \times 10 + \$3 \times 2 = \$23$. Notice that the price change only kicks in for extra units above 10 (not for the entire purchase). This system will give us a kinked budget line.

As an example, let's plot the above price for p_1, and set $p_2 = 4$ and $M = 100$.

If there was no price change (and p_1 stayed constant at 2), then the budget line would be simple. It would have the x-intercept at $100/2 = 50$, the y-intercept at $100/4 = 25$, and a slope of $-2/4 = -0.5$. Let's call this budget line B_0.

After the price change, the prices are $p_1 = 3$ and $p_2 = 4$ and income is still $M = 100$. However, the consumer must have purchased at least 10 units of good 1, which means they must have spent at least $\$2 \times 10 = \$20$. That \$20 is already locked into their expenditure. And whatever their total purchase of good 1 they end up taking (which is x_1), only $x_1 - 10$ of it will be charged at the new price of \$3. This means that the budget equation is:

$$\begin{aligned} 3(x_1 - 10) + 2 \cdot (10) + 4x_2 &= 100 \\ 3x_1 - 30 + 20 + 4x_2 &= 100 \\ 3x_1 + 4x_2 &= 110 \end{aligned}$$

Let's call this budget line B_1. So, in general, if we have a price change from p_1 to p_1' that occurs at a point $\bar{x}_1$, then the budget line is written as:

$$p_1'(x_1 - \bar{x}_1) + p_1 \cdot \bar{x}_1 + p_2 x_2 = M$$

However, we know that the budget lines B_0 and B_1 don't exist for the entire range. There's a cut-off point that occurs when we switch from B_0 to B_1. Once the consumer buys more than 10 units of good 1, the price changes. So this cut-off point occurs when $x_1 = 10$. Since all money is spent on either x_1 or x_2, then we can calculate how much x_2 is purchased.

$$\begin{aligned} p_1 x_1 + p_2 x_2 &= M \\ 2 \cdot 10 + 4 \cdot x_2 &= 100 \\ \therefore x_2 &= 20 \end{aligned}$$

This means that the budget line B_0 only exists from the y-intercept $(0, 50)$ to the point $(10, 20)$. After that, we switch to the budget line B_1 from $(10, 20)$ to $(\frac{110}{3}, \frac{55}{2}) \approx (36.67, 27.5)$.

Here's another way to think about how to get the equation for B_1. We know the cut-off occurs at $(10, 20)$ and with the price change, the slope becomes $-3/4 = -0.75$. Re-call that the general formula for a line is $y = mx + c$. We know that $m = -0.75$ and we know one point on this line $(x, y) = (10, 20)$. Plugging this in allows us to solve for c:

$$
\begin{aligned}
y =& mx + c \\
20 =& -\frac{3}{4} \cdot 10 + c \\
\therefore c =& 27.5
\end{aligned}
$$

So re-arranging and multiplying the whole equation by $p_2 = 4$, gives us the budget line $3x_1 + 4x_2 = 110$. This is exactly what we found before. Now we can plot the budget line. Note that the dashed lines are not part of it, this just continues the lines of B_0 and B_1 to give you a guide. The budget line is the solid red and blue lines and has a kink at $(10, 20)$.