Intermediate Micro: Recitation 1
Math Review

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1 Plotting Functions

1.1 Single-Variable Functions

We will be drawing a lot of graphs in this course, so this will be a useful refresher. Recall that the general formula for a linear (straight line) graph is $y = mx + c$, where m is the slope and c is the y-intercept. You should be quick at solving for the slope and intercepts.

Examples

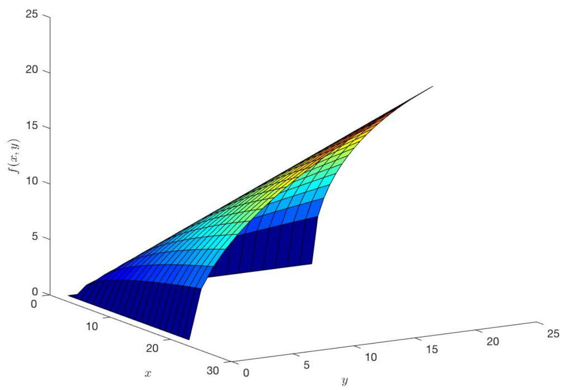
Function	**Slope**	**y-intercept**	**x-intercept**
$y = 2x$	2	0	0
$y = -2x + 4$	-2	4	2
$x + y = 5$	-1	5	5
$3x + 4y = 10$	$-3/4$	10/4	10/3

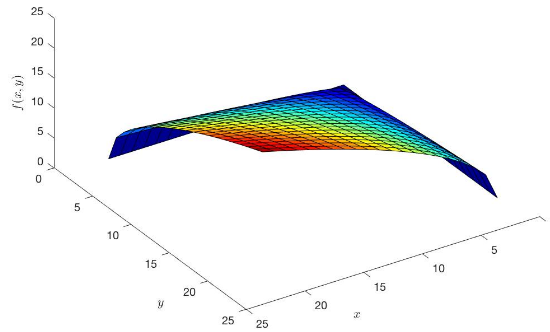
You should also know how to plot non-linear functions, such as:

- $y = x^2$
- $y = \sqrt{x}$
- $y = \log(x)$

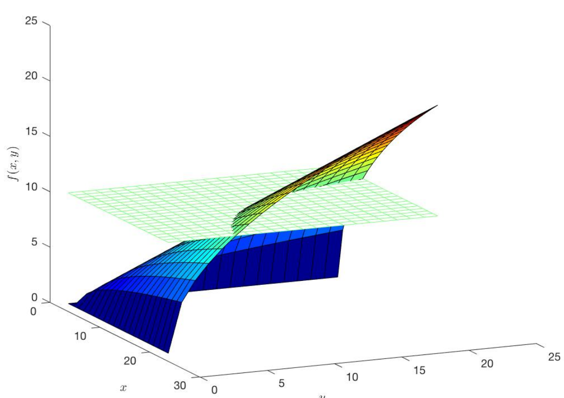
1.2 Multi-Variable Functions

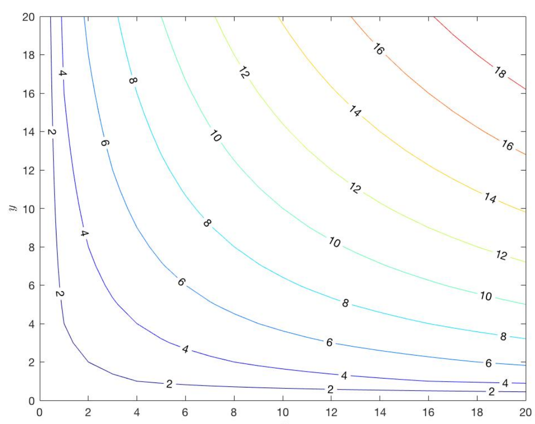
All these functions are relatively simple though because they are single-variable functions, i.e. $y = f(x)$. In this class, we will also be mostly working with multi-variable functions in the form of $z = f(x, y)$. To plot this, we would actually need a 3-dimensional graph. For example, consider the function $z = \sqrt{xy} = x^{0.5}y^{0.5}$. Plotting this gives us (I present two angles so that it is clear that this is a 3D plot):





Of course, drawing 3D graphs is very difficult. Instead we can represent this on a standard axis using contours. This is exactly how maps represent topology. For this imagine taking a flat plane and slicing through the graph at a specific point z^*. Below I show this slice occurring at $z^* = 10$. The intersection of these two graphs will give us all combinations of (x, y) coordinates that achieve $f(x, y) = z^*$. This then gives us a 2D line that we can plot in our standard x-$y$axis. We can do this for all levels of z^* and plot each intersection. This gives us the contour map that is shown below on the right, where the numbers on the line represent the given z^*.





2 Derivatives

2.1 Partial Derivatives

Derivatives are important because they capture the slope of a function: how does the function f change as we change its input variables? This is called a **marginal change**. For single-variable functions $f(x)$, this is usually quite easy to calculate. For multi-variable functions, you will need to be able to take partial derivatives. At this point, you should already be fairly comfortable with partial derivatives. This is not a math class though, so you will not need to take derivatives of overly complicated or trigonometric functions. However, since they will be relatively simple, it's important that you are able to do them quickly and error-free.

General Rules:

1. For $f(x) = x^{\alpha}$, where α is a constant, then $f'(x) = \alpha x^{\alpha-1}$
2. For $f(x) = \log(x)$, then $f'(x) = \frac{1}{x}$
3. For $f(x) = c$, where c is a constant, then $f'(x) = 0$
4. For a constant c, then $\frac{d[c \cdot f(x)]}{dx} = c \cdot f'(x)$
5. For $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$
6. [Product Rule] For $f(x) = u(x) \cdot v(x)$, then $f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$
7. [Quotient Rule] For $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$
8. [Chain Rule] For $f(x) = u(v(x))$, then $f'(x) = \frac{du(v)}{dv} \cdot \frac{dv(x)}{dx} = u'(v(x)) \cdot v'(x)$

Examples

Function	**Derivative (x)**	**Derivative (y)**
$f(x) = x^2$	$f'(x) = 2x$	
$f(x) = x$	$f'(x) = 1$	
$f(x, y) = xy$	$\frac{\partial f}{\partial x} = y$	$\frac{\partial f}{\partial y} = x$
$f(x, y) = x^{1/3}y^{2/3}$	$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3}y^{2/3}$	$\frac{\partial f}{\partial y} = \frac{2}{3}x^{1/3}y^{-1/3}$
$f(x, y) = \log(x) + y$	$\frac{\partial f}{\partial x} = \frac{1}{x}$	$\frac{\partial f}{\partial y} = 1$
$f(x, y) = \log(x^{1/3}y^{2/3})$	$\frac{\partial f}{\partial x} = \frac{1}{3x}$	$\frac{\partial f}{\partial y} = \frac{2}{3y}$

For the last one, you can use the *chain rule*, and set $u(x, y) = (x^{1/3}y^{2/3})$ and then do $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x^{1/3}y^{2/3}} \cdot \frac{1}{3}x^{-2/3}y^{2/3} = \frac{1}{3x}$. A quicker trick would be to use the property of the log function and simplify the function as: $f(x, y) = \log(x^{1/3}) + \log(y^{2/3}) = \frac{1}{3}\log(x) + \frac{2}{3}\log(y)$.

2.2 Non-Differentiable Functions

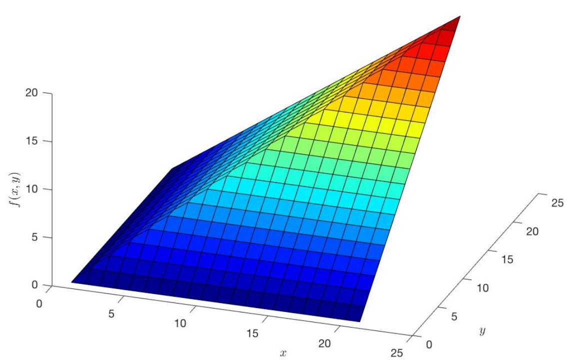
Most functions you will see will be differentiable, but you should be aware that you may not always be able to take a derivative. Consider the function $f(x, y) = \min\{x, y\}$. The min function takes two (or more) numbers and gives you the smallest one. For example: $f(1, 2) = 1$, $f(5, 0) = 0$, $f(3, 3) = 3$. Alternatively, we can write the function as follows:

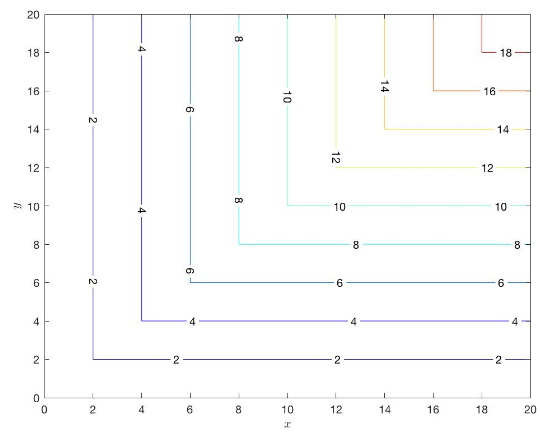
$$f(x,y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

This means the derivative can be written as:

$$\frac{\partial f}{\partial x} = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x > y \end{cases} \text{ and } \frac{\partial f}{\partial y} = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x > y \end{cases}$$

However, whenever $y = x$, this function is not differentiable. The idea here is that for a function to be differentiable it needs to change "smoothly". We cannot differentiate a point in a function where there is a jump (a break) or a kink (a bend). You can see a plot of this function and its contour map below; you'll see that it is not smooth at $y = x$. This example may seem to be a bizarre one, but you'll soon see that this is a very useful function in economics.





3 Optimization

3.1 Unconstrained Optimization

Optimization involves maximizing or minimizing a function while satisfying a constraint. The simplest case is where there is no constraint. In the unconstrained problem, all we have to do is take a *first order condition (FOC)*. This just means taking the first derivative and setting it equal to zero. After that we solve for the variable to give us the optimal value. In all the following cases, we are going to focus on maximizing. Technically, you should also check the second order condition to make sure the value you get is in fact a maximum, but we'll always give you economically-sensible models, so this is something you can ignore.

Examples

1. $f(x) = 10 + 5x - x^2$ for $x \in \mathbb{R}$.
   FOC: $f'(x) = 5 - 2x = 0$

Solve: $\therefore 5 = 2x \implies x^* = 2.5$

2. $f(x) = \log(x)$ for $x > 0$
   FOC: $f'(x) = \frac{1}{x} = 0$
   Solve: $x^* \to \infty$

As Example 2 shows, sometimes with unconstrained optimization, we may get infinity as an answer. This is because some functions do not have a global maximum - they just keep on increasing as you move in one direction.

We can also do unconstrained optimization with multiple variables. This just involves taking FOCs for each variable. This will give us a system of equations, which we will need to solve to find the optimal value for each variable (e.g. use substitution, divide equations, set up a matrix etc).

Example

Choose x and y to maximize the function $f(x, y) = 4x^{1/2}y^{1/4} - x - y$ for $x \in \mathbb{R}_+, y \in \mathbb{R}_+$.

First, we take the FOCs for each variable:
FOC for x: $\partial f/\partial x = 2x^{-1/2}y^{1/4} - 1 = 0$
FOC for y: $\partial f/\partial y = x^{1/2}y^{-3/4} - 1 = 0$

Then, we re-arrange the equations and setup the system of equations:

$$2x^{-1/2}y^{1/4} = 1$$
$$x^{1/2}y^{-3/4} = 1$$

Next, we solve the system. In this case, one option is to divide the equations:

$$\frac{2x^{-1/2}y^{1/4}}{x^{1/2}y^{-3/4}} = \frac{1}{1}$$
$$\frac{2y}{x} = 1$$
$$y = \frac{1}{2}x$$

Finally, we plug this back into one of the FOCs (i.e. back into the system of equations):

$$
\begin{aligned}
x^{1/2}\left(\frac{1}{2}x\right)^{-3/4} &= 1 \\
x^{-1/4}\left(\frac{1}{2}\right)^{-3/4} &= 1 \\
\left(\frac{1}{8}x\right)^{-1/4} &= 1 \\
\frac{1}{8}x &= 1^{-4} = 1 \\
\therefore x^* &= 8 \\
\therefore y^* &= 4
\end{aligned}
$$

Alternatively, we could have substituted one equation into another. For example, re-arranging the FOC for y and plugging it into the FOC for x gives us:

$$
\begin{aligned}
x^{1/2}y^{-3/4} &= 1 \\
\implies x^{-1/2} &= y^{-3/4} \\
\therefore 2\left(y^{-3/4}\right)y^{1/4} &= 1 \\
2y^{-1/2} &= 1 \\
y^{1/2} &= 2 \\
\therefore y^* &= 4 \\
x^* &= \left((y^*)^{-3/4}\right)^{-2} \\
&= 4^{3/2} = 8
\end{aligned}
$$

3.2 Constrained Optimization

Usually we do want to include a constraint in an optimization problem. In the real world, we always face constraints. When you plan out your day, you are constrained by the fact that you have 24 hours in a day. A government may want to implement a popular program but is constrained by how much tax revenue it has raised. So, in general, for an optimization, we will need to identify the following four things:

1. *Objective Function:* What function do we want to maximize?
2. *Constraint:* What restrictions are placed on our problem?
3. *Choice Variables:* What variables do we want to choose to achieve the optimal objective function?
4. *Parameters:* What variables affect the problem but we are <u>not</u> able to choose?

We write an optimization problem in the following way:

$$
\begin{aligned}
&\max_{x,y} f(x,y) \\
&\text{s.t. } b(x,y) \geq 0
\end{aligned}
$$

The max tells us what we are doing (here we are maximizing, if we wanted to minimize, we would write min). So you can read this as “maximize $f(x, y)$”. This tells us that $f(x, y)$ is the objective function. The s.t. stands for “subject to”, so we have to maximize the objective subject to a restriction. This tells us that $b(x, y) \geq 0$ is the constraint. As convention, we write the choice variables under the max (or min), so in this case, x, y are the choice variables. In this setup, we don’t exactly see the parameters, but suppose we had $f(x, y) = \alpha x + \beta y$. Then α and β are parameters - they will affect our solution but we have no control over them.

For the constraint, notice that it is an inequality. This is because it usually represents a set of values that we are allowed to use. The constraint is often also referred to as the *feasible set.* For our purposes though, we will almost always have the constraint be “binding” in optimality. In other words, we can replace the inequality with an equals sign: $b(x, y) = 0$. Once we apply this to economic problems, you’ll see why this is intuitively true.

To solve this, there are generally two ways to do it:

1) Substitution Method

1. Re-arrange the constraint to solve for one variable (y) on the left hand side (LHS) as a function of the other variable (x). This will be in the form $y = y(x)$
2. Plug this into the objective function $f(x, y) = f(x, y(x))$. Now the objective function is only a function of one variable (x)
3. Take the FOC of the objective function (with respect to x)
4. Re-arrange the FOC and solve for the optimal x^*
5. Plug this back into the constraint from (1) to get the optimal $y^* = y(x^*)$

2) Lagrangian Method

1. Set up the Lagrangian in the form: $\mathcal{L} = \textit{Objective} + \lambda[\textit{Constraint}] = f(x, y) + \lambda\,[b(x, y)]$.
   - Make sure the constraint is in the form $= 0$ (i.e. put everything to the LHS and have only 0 on the RHS)
   - λ is called the Lagrange multiplier
2. Take the FOCs of the Lagrangian with respect to x and y: $\frac{\partial \mathcal{L}}{\partial x} = 0$ and $\frac{\partial \mathcal{L}}{\partial y} = 0$
   - This will be in the form: $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial b}{\partial x} = 0$ (and similarly for 0)
3. Re-arrange each FOC to put the term with Lagrange multiplier on the RHS: $\frac{\partial f}{\partial x} = -\lambda \frac{\partial b}{\partial x}$ and $\frac{\partial f}{\partial y} = -\lambda \frac{\partial b}{\partial y}$
4. Divide the two FOCs so that the Lagrange multiplier cancels out: $\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{\partial b/\partial x}{\partial b/\partial y}$
5. Solve for one variable (y) as a function of another variable (x). This will be in the form $y = y(x)$
6. Plug this into the (binding) constraint: $b(x, y) = b(x, y(x)) = 0$. Now the constraint is only a function of one variable (x)

7. Re-arrange the constraint to solve for the optimal x^*
8. Plug this back into your answer for (5) to get the optimal $y^* = y(x^*)$

The Lagrangian Method might seem like it will take longer, but it is more general and can be used in most settings. It is also useful for when you have multiple constraints (we'll discuss this when/if it comes up).

Example

Consider the following problem:

$$\max_{x,y} x^2 y$$
$$\text{s.t. } c - x - 2y = 0$$

In this case, $f(x, y) = x^2 y$ is the objective function, $b(x, y) = c - x - 2y$ is the constraint, the choice variables are x, y, and c is a parameter.

First, let's try solving with the substitution method:

1. Re-arrange the constraint: $y(x) = \frac{1}{2}c - \frac{1}{2}x$
2. Plug into objective: $f(x, y(x)) = x^2 \left(\frac{1}{2}c - \frac{1}{2}x\right) = \frac{1}{2}cx^2 - \frac{1}{2}x^3$
3. FOC: $cx - \frac{3}{2}x^2 = 0$
4. Solve: $x\left(c - \frac{3}{2}x\right) = 0$. This implies that $x^* = 0$ or $x^* = \frac{2c}{3}$. However, note that $x^* = 0$ cannot be optimal (it gives an objective function with a value of zero, which is clearly not a maximum)
5. Plug back in: $y^* = y(x^*) = \frac{1}{2}c - \frac{1}{3}c = \frac{1}{6}c$

Now, let's try solving with the Lagrangian method:

1. Set up the Lagrangian: $\mathcal{L} = x^2 y + \lambda [c - x - 2y]$
2. FOCs:

$$\frac{\partial \mathcal{L}}{\partial x} = 2xy - \lambda = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = x^2 - 2\lambda = 0$$

3. Re-arrange:

$$2xy = \lambda \qquad x^2 = 2\lambda$$

4. Divide:

$$\begin{aligned} \frac{2xy}{x^2} &= \frac{\lambda}{2\lambda} \\ \frac{2y}{x} &= \frac{1}{2} \end{aligned}$$

5. Solve: $y(x) = \frac{1}{4}x$
6. Plug into constraint: $b(x, y(x)) = c - x - 2\left(\frac{1}{4}x\right) = c - x - \frac{1}{2}x$. Therefore: $c - \frac{3}{2}x = 0$
7. Solve: $x^* = \frac{2}{3}c$
8. Plug back in: $y^* = \frac{1}{4} \cdot \frac{2}{3}c = \frac{1}{6}c$