Intermediate Micro: Recitation 13
Supply and Market Equilibrium

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1 Supply

1.1 Shutdown Decision

We know that the firm faces the profit maximization problem, where it has to choose it's output quantity q to solve the following:

$$\max_{q} \ \pi(q) = pq - C(q)$$

We also saw that the FOC gave us the following condition:

$$p = MC(q)$$

This tells us that, given a price for their output p, the firm should be producing where that price is equal to the marginal cost of production. However, one thing that we left out is that the firm always has the option to shutdown and produce nothing (i.e. $q = 0$). We know this is important because in the last recitation we saw examples where the firm was producing negative profit. If you are losing money, you may be better off just going out of business. So, to fully understand the firm's supply function, we need to understand its shutdown decision. In other words, we want to know for what $q > 0$ do we have $\pi(0) > \pi(q)$?

To understand this, we have to break-down the firm's costs. We know that a firm's costs are composed of two parts: variable costs (which depend on q) and fixed costs (which do not depend on q). If a firm produces $q = 0$, then by definition, we must have that there are no variable costs, i.e. $VC(0) = 0$. The big focus is then about the fixed costs. Usually we say that fixed costs do not depend on q, but this can be slightly misleading. To be precise, fixed costs are costs that do not depend on q, <u>when the firm is producing strictly positive amounts</u> ($q > 0$). We can further break down the fixed costs into two types:

1. **Sunk Costs:** These are fixed costs that you pay even when $q = 0$ (i.e. you cannot recover these costs). I will denote these by SFC, where $SFC(q) = SFC, \forall q \geq 0$.

2. **Non-Sunk or Quasi-Fixed Costs:** These are fixed costs that you do not pay when $q = 0$ (i.e. if you shutdown, you can avoid them). I will denote these by QFC, where $QFC(q) = QFC, \forall q > 0$ and 0 for $q = 0$.

So a firm's total cost can be expressed as:

$$TC(q) = VC(q) + \underbrace{SFC + QFC(q)}_{FC(q)}$$

In particular, if a firm decides to shut down and produce $q = 0$, then it's total costs are just its sunk costs:

$$\begin{aligned} TC(0) &= VC(0) + SFC + QFC(0) \\ &= 0 + SFC + 0 \\ &= SFC \end{aligned}$$

This means its profits when $q = 0$ are:

$$\begin{aligned} \pi(0) &= p \cdot 0 - C(0) \\ &= 0 - SFC \\ &= -SFC \end{aligned}$$

Which means that even if it shuts down, it will still make a loss if it has sunk fixed costs. Therefore, the firm's profit maximization problem can be fully written as:

$$\max_q \ \pi(q) = \begin{cases} pq - VC(q) - SFC - QFC & \text{if } q > 0 \\ -SFC & \text{if } q = 0 \end{cases}$$

But notice that in either case, the company will have to pay the sunk fixed costs. Since it cannot control them, it should only be focusing on trying to maximize the remainder (i.e. the part it has control over). This is why economists often say "sunk costs are sunk". As you cannot recover them no matter what you do, you shouldn't really factor them into your decision making. That's exactly what we do when we take an FOC. Taking the derivative of $\pi(q)$ with respect to q gives us: (assuming $q > 0$)

$$\begin{aligned} \frac{\partial \pi(q)}{\partial q} &= \frac{\partial}{\partial q}\left(pq - VC(q) - SFC - QFC\right) \\ &= p - \frac{\partial VC(q)}{\partial q} - 0 - 0 \\ &= p - MC(q) \end{aligned}$$

In the FOC, the fixed costs drop out. So we are only deciding on the q that maximizes the $pq - VC(q)$ part of the profit function. But this could lead us to making the wrong decision if we are not careful.

Let's take the following example. Suppose the cost function is $C(q) = 2q^2 + 20$, where all of the fixed costs are sunk, and the output price is $p = 8$. Then the FOC says the firm should set q according to

$$p = MC(q)$$

$$\begin{aligned} 8 &= 4q \\ \therefore q &= 2 \end{aligned}$$

Now, you might stop here and say that $q^* = 2$ is the optimal quantity. But we should check how much profit the firm is making:

$$\begin{aligned} \pi(2) &= 8 \times 2 - \left(2 \times 2^2 + 20\right) \\ &= 16 - 8 - 20 \\ &= -12 \end{aligned}$$

They are making negative profits. Since all the fixed costs are sunk, that means if the firm shuts down, then:

$$\pi(0) = -20 < -12 = \pi(2)$$

So shutting down would result in a bigger loss. That means that $q^* = 2$ is indeed the optimal level. Let's instead suppose that the all fixed costs are non-sunk (i.e. $QFC = 20$ and $SFC = 0$). That means if the firm shuts down it would earn $\pi(0) = -SFC = 0$. Now, shutting down is actually better than producing $q^* = 2$ (which means if we stopped after the FOC, we would have the wrong answer).

What if there's a mix? Suppose 25% of the fixed costs are sunk and 75% are non-sunk (i.e. $QFC = 5$ and $SFC = 15$). Now, if the firm shuts down it will earn:

$$\pi(0) = -SFC = -15 < -12 = \pi(2)$$

So shutting down in this case would not be sensible, and q^* is still optimal.

Let's now generalize these results. Suppose from the FOC, we get a candidate value for output, which we'll call $\hat{q}$. The optimal quantity q^* is determined by:

$$q^* = \begin{cases} 0 & \text{if } \pi(\hat{q}) < -SFC \\ \hat{q} & \text{otherwise} \end{cases}$$

So, the firm should shutdown if its profits are less than just paying the sunk fixed costs. We can simplify this shutdown condition:

$$\begin{aligned} \pi(q) &< -SFC \\ pq - VC(q) - SFC - QFC &< -SFC \\ pq - VC(q) - QFC &< 0 \end{aligned}$$

In other words, the firm should shutdown if its revenue less variable and quasi-fixed costs are negative. As you can see here, the sunk costs actually play no role in the firm's shutdown decision. This again shows that sunk costs should play no role in the firm's optimization.

If we define $AQFC$ as average quasi-fixed costs ($\frac{QFC}{q}$), then the general shutdown condition can be written as:

$$p < AVC(q) + AQFC$$

If there are no quasi-fixed costs, then the shutdown condition can simply be expressed as:

$$p < AVC(q)$$

Similarly, if there are no sunk costs, then $AC(q) = AVC(q) + AQFC$, and the shutdown condition is just:

$$p < AC(q)$$

1.2 Supply Function

Now that we can fully characterize the firm's optimization choice, we are ready to derive the firm's supply function. The supply function is the quantity of output the firm produces (q) as a function of price (p). Let's call this $q = S(p)$. It will also be a function of other parameters, in particular, the prices of the inputs p_1 and p_2. We know that if a firm is active (i.e. has $q > 0$), then it produces where:

$$p = MC(q)$$

This expresses price as a function of quantity, which is the opposite of what we want. In fact, this is exactly the **inverse supply function**! So if we just invert it back, we will get the supply function:

$$q = MC^{-1}(p) = S(p)$$

As always, when we plot the supply function, we will want to use the inverse (since we have p on the y-axis and q on the x-axis). Therefore, you can think of the marginal cost curve as just being the supply curve.

Of course, there is a caveat to this, which is when the firm decides to shutdown. We know the condition for shutting down, and we also know that $p = MC(q)$. Putting this together:

$$MC(q) < AVC(q) + AQFC$$

This tells us the the MC curve is the supply curve, except for the part of it that is below the $AVC + AQFC$ curve. For that part, we just have a vertical line (i.e. for those prices, $q = 0$).

Let's see this through an example. Suppose $C(q) = 0.25q^2 + 5q + 9$. At an interior solution, the firm satisfies the following:

$$\begin{aligned} p &= MC(q) \\ &= 0.5q + 5 \end{aligned}$$

This is the inverse supply function. We could also write this as the supply function $q = S(p)$:

$$\begin{aligned} 0.5q &= p - 5 \\ q &= 2p - 10 \end{aligned}$$

Of course, we can't have negative quantities, so we can more accurately write this as:

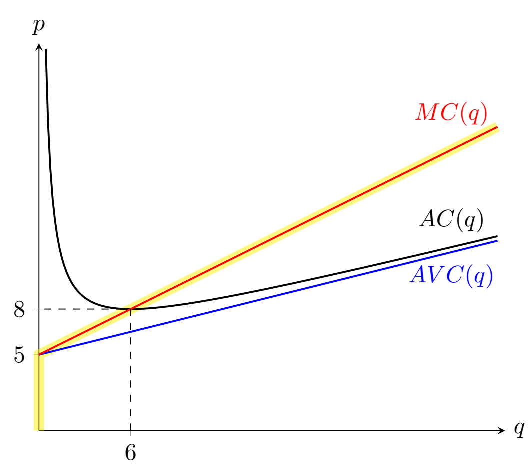
$$q = \begin{cases} 2p - 10 & \text{if } p \geq 5 \\ 0 & \text{if } p < 5 \end{cases}$$

We can see that $VC(q) = 0.25q^2 + 5q$ and $FC(q) = 9$. Let's first suppose that all fixed costs are sunk ($SFC = 9, QFC = 0$). In that case, the firm will only produce when:

$$\begin{aligned} MC(q) &> AVC(q) \\ 0.5q + 5 &> 0.25q + 5 \\ 0.25q &> 0 \\ \therefore q &> 0 \end{aligned}$$

So, at all levels of positive production, we have $MC > AVC$, which means that the firm will never shutdown (except of course when $p < 5$, as we showed above).

We can represent this graphically, by drawing the MC, AVC, and AC graphs.



I've highlighted in yellow where the supply curve occurs. As we can see from the graph, the MC is always above the AVC. And since all fixed costs are sunk, this means that the shutdown condition will never be met. Therefore, the entire MC is part of the supply curve. Notice though that this doesn't mean the firm is always earning positive profits. Whenever $MC < AC$, the profits are negative. So in this case:

$$\begin{aligned} MC(q) &= AC(q) \\ 0.5q + 5 &= 0.25q + 5 + \frac{9}{q} \end{aligned}$$

$$
\begin{aligned}
0.25q &= \frac{9}{q} \\
q &= \frac{36}{q} \\
\therefore q &= \sqrt{36} = 6
\end{aligned}
$$

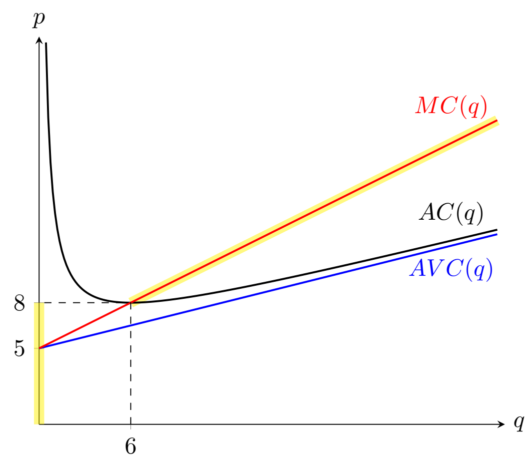
The threshold value is 6. Whenever $q > 6$, we have $\pi > 0$, and whenever $q < 6$, we have $\pi < 0$ (but the firm will still not shutdown).

Next, let's suppose that all fixed costs are non-sunk ($SFC = 0, QFC = 5$). The firm will not shutdown if:

$$
\begin{aligned}
&MC(q) > AC(q) \\
&\implies q > 6
\end{aligned}
$$

This is exactly what we solved for above. This tells us that if whenever $\hat{q} < 6$ (the q we get from the $p = MC$ condition), then the firm would instead set $q^* = 0$ and shutdown. We can use the inverse supply function to make this condition in terms of price:

$$
p > 0.5(6) + 5 = 3 + 5 = 8
$$



Now, the only part of the MC curve that is on the supply curve is when it is above the AC. This happens at $p = 8$. For prices below that, we have the vertical line indicating $q^* = 0$.

So, if all the fixed costs are sunk, the shutdown price is $p = 5$. If all the fixed are non-sunk, the shutdown price is $p = 8$. If we are in some intermediate case, the shutdown price will be somewhere in the middle. For example, if 50% of the fixed costs are sunk, then the shutdown condition is where:

$$
\begin{aligned}
MC(q) &= AVC(q) + AQFC \\
0.5q + 5 &= 0.25q + 5 + \frac{(0.5 \times 9)}{q} \\
0.25q &= \frac{4.5}{q}
\end{aligned}
$$

$$q = \frac{18}{q}$$
$$\therefore q = \sqrt{18} = 3\sqrt{2}$$

This occurs at the price:

$$p = 0.5\left(3\sqrt{2}\right) + 5 = \frac{3}{2}\sqrt{2} + 5 \approx 7.12$$

1.3 Long Run Supply

In the long run, all inputs are adjustable. In particular, there cannot be any fixed sunk costs.[1] Since everything can be adjusted, there shouldn't be a cost that you are permanently stuck with. This means that the firm will shutdown if:

$$p < AC(q)$$

Or equivalently, the long run supply curve is the part of the (long run) MC curve that is above the (long run) AC curve.

Another important aspect of the long run is that firms can enter and exit the market freely. What this means is that if firms in the market are making positive profit, then that will incentivize more firms to enter. As more firms enter, this increases supply and lowers price, leading to lower profits for each firm. Similarly, if firms are making negative profits, then firms will exit the market, thus reducing supply and increases price, leading to higher firm profits. Either way, this process continues until $\pi = 0$. At that point, firms will be indifferent between entering and exiting the market. So, in the long run, firms should be making **zero profit**.

For $\pi = 0$, then it must be the case that $p = AC(q)$. We also know that firms set $p = MC(q)$ by their profit maximization problem. Putting this together, it must be the case that in the long run we have that $MC(q) = AC(q)$. Since we know that the MC crosses the AC at the minimum of the average cost curve, then we must be at the minimum efficient scale (q_{MES}) in the long run.

Let's do an example. Consider a firm with the production function $f(K, L) = K^{1/3}L^{1/3}$. To be in active production, it has to pay a fixed cost of b (think of this like a business license - it is a fixed cost, but it is not sunk in the long run since the firm has long enough to decide whether it will take on this cost or not). In the long run, both inputs are choice variables, so the cost minimization problem is:

$$\min_{K,L} \; rK + wL + b$$
$$\text{s.t. } K^{1/3}L^{1/3} = q$$

The tangency condition is: (note that the fixed cost drops out of the FOCs, so it won't play a role in the tangency condition)

$$\frac{L}{K} = \frac{r}{w}$$

[1] Often, you will hear that "in the long run, there are no fixed costs". That's not technically true. You can still have fixed costs in the long run (there could certainly be costs that are independent of quantity, even in the long run). However, these fixed costs just cannot be sunk.

$$\implies L = \frac{r}{w}K$$

Plugging this into the production function gives:

$$\begin{aligned}
K^{1/3}\left(\frac{r}{w}K\right)^{1/3} &= q \\
K\left(\frac{r}{w}K\right) &= q^3 \\
K^2 &= \frac{wq^3}{r} \\
\therefore K(q) &= \sqrt{\frac{wq^3}{r}} \\
L(q) &= \sqrt{\frac{rq^3}{w}}
\end{aligned}$$

The cost function is then:

$$\begin{aligned}
C(q) &= rK(q) + wL(q) + b \\
&= \sqrt{rwq^3} + \sqrt{wrq^3} + b \\
&= 2\sqrt{rwq^3} + b
\end{aligned}$$

Notice that this has no sunk fixed costs. Moving onto the firm's profit maximization problem, the optimality condition gives us:

$$\begin{aligned}
p &= MC(q) \\
&= 2\sqrt{rw}\left(\frac{3}{2}q^{1/2}\right) \\
&= 3\sqrt{rwq}
\end{aligned}$$

The long run supply function is then:

$$\begin{aligned}
p^2 &= 9rwq \\
q &= \frac{p^2}{9rw}
\end{aligned}$$

In the long run equilibrium, we must have that $MC = AC$ to have zero profits. This occurs at the minimum of the AC:

$$\min_q \; AC(q) = 2\sqrt{rwq} + \frac{b}{q}$$

$$\begin{aligned}
\implies 2\sqrt{rw}\left(\frac{1}{2}q^{-1/2}\right) + b\left(-q^{-2}\right) &= 0 \\
\sqrt{\frac{rw}{q}} + \frac{b}{q^2} &= 0 \\
\sqrt{\frac{rw}{q}} &= \frac{b}{q^2} \\
\frac{rw}{q} &= \frac{b^2}{q^4}
\end{aligned}$$

$$\therefore q = \left(\frac{b^2}{rw}\right)^{1/3}$$

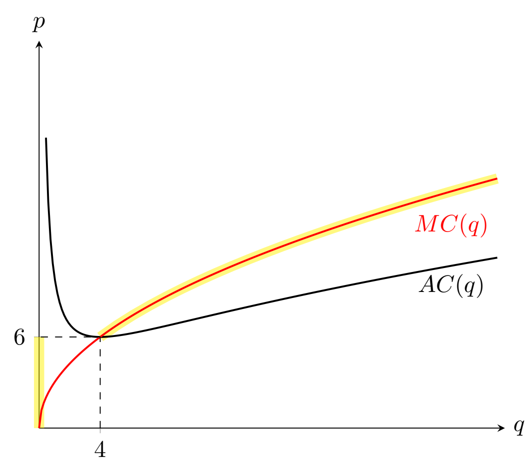
This value is the minimum efficient scale. It also represents the point at which firms will shutdown, i.e. when price is:

$$p(q_{MES}) = 3\sqrt{rw\left(\frac{b^2}{rw}\right)^{1/3}} = 3\sqrt{b^{2/3}\,(rw)^{4/3}} = 3b^{1/3}\,(rw)^{2/3}$$

Let's suppose that $r = 1, w = 1$, and $b = 8$. This gives us the following functions and values:

$$\begin{aligned} K(q) &= q^{3/2} & p(q) = MC(q) &= 3\sqrt{q} \\ L(q) &= q^{3/2} & q(p) &= \frac{p^2}{9} \\ C(q) &= 2q^{3/2} + 8 & q_{MES} &= 4 \\ AC(q) &= 2\sqrt{q} + \frac{8}{q} & p(q_{MES}) &= 6 \end{aligned}$$

From this, we can plot the long run MC, AC, and supply curve (highlighted in yellow):



In the long run equilibrium, we should be where $p = 6$, that way $\pi = 0$ and firms have no incentive to enter or exit.

One property of long run supply curves is that they tend to be more elastic than their short run analogue. For example, let's take the above production function, but change two things in the short run. First, let's assume that capital is fixed at $\bar{K} = 8$ in the short run. Second, let's assume that the fixed cost is sunk. The firm's cost minimization means that labor has to be set such that:

$$\begin{aligned} q &= f(\bar{K}, L) \\ &= 8^{1/3}L^{1/3} \\ &= 2L^{1/3} \\ \therefore L(q) &= \left(\frac{q}{2}\right)^3 = \frac{q^3}{8} \end{aligned}$$

This makes the short run cost function as:

$$\begin{aligned}C(q) &= r\bar{K} + wL(q) + b \\ &= 8r + \frac{wq^3}{8} + b\end{aligned}$$

The firm's profit maximization has the following optimality condition:

$$\begin{aligned}p &= MC(q) \\ &= \frac{3wq^2}{8}\end{aligned}$$

The supply function is then:

$$\begin{aligned}\frac{8p}{3w} &= q^2 \\ \therefore q &= \sqrt{\frac{8p}{3w}}\end{aligned}$$

Since all fixed costs are sunk, the firm's shutdown decision is when $p < AVC(q)$. This means the cutoff is when $MC = AVC$:

$$\begin{aligned}MC(q) &= AVC(q) \\ \frac{3wq^2}{8} &= \frac{wq^2}{8} \\ \frac{2wq^2}{8} &= 0 \\ \therefore q &= 0\end{aligned}$$

Therefore, the firm will always prefer to produce a positive amount rather than shutting down.

We can also see where the minimum of the AC occurs, i.e. where $MC = AC$:

$$\begin{aligned}MC(q) &= AC(q) \\ \frac{3wq^2}{8} &= \frac{wq^2}{8} + \frac{8r+b}{q} \\ \frac{2wq^2}{8} &= \frac{8r+b}{q} \\ \frac{wq^3}{4} &= 8r + b \\ \therefore \bar{q} &= \left(\frac{32r+4b}{w}\right)^{1/3}\end{aligned}$$

Let's suppose that $r = 1, w = 1$, and $b = 8$. This gives us the following functions and values:

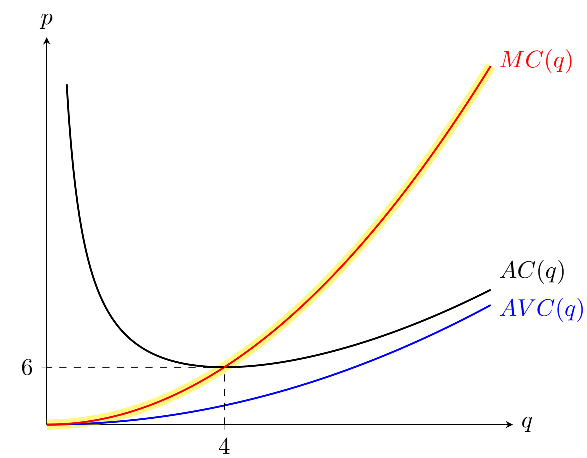
$$L(q) = \frac{q^3}{8} \qquad p(q) = MC(q) = \frac{3q^2}{8}$$

$$C(q) = \frac{q^3}{8} + 16 \qquad q(p) = \sqrt{\frac{8p}{3}}$$

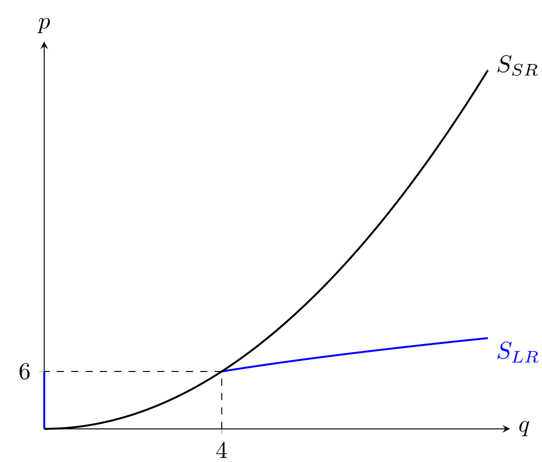
$$AC(q) = \frac{q^2}{8} + \frac{16}{q} \qquad \bar{q} = 4$$

$$AVC(q) = \frac{q^2}{8} \qquad p(\bar{q}) = 6$$

From this, we can plot the short run run MC, AC, and supply curve (highlighted in yellow):



In fact, if we plot the short run and long run supply curves at the same time we get the following:



Notice a few things from this diagram:

- The long run supply curve is a lot flatter (more elastic) than the short run supply curve
- The long run supply is 0 (i.e. vertical) for more prices than the short run supply. This is because in the long run, there are no sunk costs, and so all costs have to be considered when deciding how much to produce (i.e. the firm is not willing to make any negative profits in the LR, but it is willing to take on some losses in the SR)
- The SR and LR curve intersect at $(q, p) = (4, 6)$. This is because at $q = 4$, in the long run we have the optimal level of capital as $K(4) = 4^{3/2} = 8$. This is exactly how much capital is fixed at in the short run case. So the short run looks exactly like the long run at the point where $K_{LR}(q) = \bar{K}$.

2 Market Equilibrium

2.1 Aggregating Demand and Supply

In the first part of the course, we've shown that starting with a consumer's preferences, we can solve the consumer's utility maximization problem (UMP) to get the demand function. Now, on the producer side, once we know the firm's production technology, then we can solve the producer's profit maximization problem (PMP) to get a supply function. This is just for individuals though. To understand the market, we need to have many people. For this, we just need to aggregate the individual demand and supply functions.

We've already done this with demand, and it's exactly the same idea with supply. Let's say there are I consumers in the market, where consumer i has a demand function $q_i(p)$. Then the market demand is:

$$Q_D(p) = \sum_{i=1}^{I} q_i(p)$$

Similarly, there are J producers in the market, where producer j has a supply function $q_j(p)$. Then the market supply is:

$$Q_S(p) = \sum_{j=1}^{J} q_j(p)$$

So to the get the market functions, we just sum up the individual functions. The two things to always be aware of when aggregating are:

1. Make sure to use demand/supply function and *not* the inverse demand/supply, i.e. use $q(p)$ not $p(q)$
2. Watch out for kinks! These occur when there is a "choke price", i.e. a price where some of the individuals are willing to demand/supply a positive quantity while others are not

Example

Suppose we have the following market:

- 20 consumers of Type A with demand $q_A(p) = 20 - 2p$
- 10 consumers of Type B with demand $q_B(p) = 15 - p$
- 5 producers of Type Y with supply $q_Y(p) = 3p$
- 5 producers of Type Z with supply $q_Z(p) = 6p - 3$

To be careful of kinks, we should write out these functions making clear where the choke prices occur: (i.e. just check when $q < 0$ for the above equations)

Demand　　　　　　　　Supply

$$q_A(p) = \begin{cases} 20 - 2p & \text{if } p \leq 10 \\ 0 & \text{if } p > 10 \end{cases} \qquad q_Y(p) = 3p$$

$$q_B(p) = \begin{cases} 15 - p & \text{if } p \leq 15 \\ 0 & \text{if } p > 15 \end{cases} \qquad q_Z(p) = \begin{cases} 6p - 3 & \text{if } p > 2 \\ 0 & \text{if } p \leq 2 \end{cases}$$

Now we are ready to aggregate. The market demand is:

$$\begin{aligned} Q_D(p) &= 20q_A(p) + 10q_B(p) \\ &= \begin{cases} 20\,(20 - 2p) + 10\,(15 - p) & \text{if } p \leq 10 \\ 20\,(0) + 10\,(15 - p) & \text{if } 10 < p \leq 15 \\ 20\,(0) + 10\,(0) & \text{if } p > 15 \end{cases} \\ &= \begin{cases} 550 - 50p & \text{if } p \leq 10 \\ 150 - 10p & \text{if } 10 < p \leq 15 \\ 0 & \text{if } p > 15 \end{cases} \end{aligned}$$

The market supply is:

$$\begin{aligned} Q_S(p) &= 5q_Y(p) + 5q_Z(p) \\ &= \begin{cases} 5\,(3p) + 5\,(6p - 3) & \text{if } p > 2 \\ 5\,(3p) + 5\,(0) & \text{if } p \leq 2 \end{cases} \\ &= \begin{cases} 45p - 15 & \text{if } p > 2 \\ 15p & \text{if } p \leq 2 \end{cases} \end{aligned}$$

To plot these, we should invert the market functions:

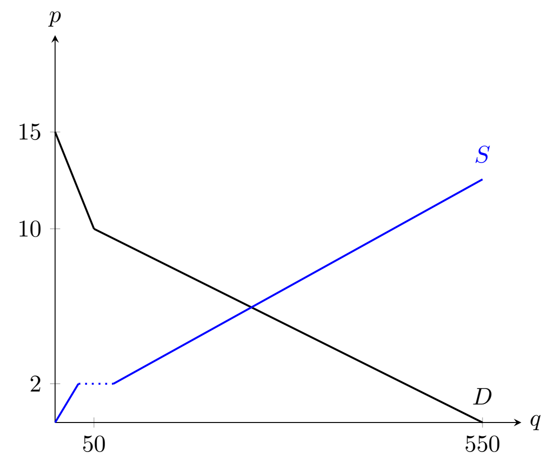
$$P_D(Q) = \begin{cases} 11 - \frac{Q}{50} & \text{if } Q \geq 50 \\ 15 - \frac{Q}{10} & \text{if } Q < 50 \end{cases}$$

$$P_S(Q) = \begin{cases} \frac{Q}{45} + \frac{1}{3} & \text{if } Q > 75 \\ \frac{Q}{15} & \text{if } Q \leq 30 \end{cases}$$

Notice who is participating in the market at each price level:

$$\begin{aligned} p = 0 &: \text{Type Y \& Z producers not in the market (no supply)} \\ p \in (0, 2) &: \text{Type A \& B consumers; Type Y producers in the market} \\ p = 2 &: \text{Type Z producers enter the market (more supply)} \\ p \in (2, 10) &: \text{Type A \& B consumers; Type Y \& Z producers in the market} \\ p = 10 &: \text{Type A consumers exit the market (less demand)} \\ p \in [10, 15) &: \text{Type B consumers; Type Y \& Z producers in the market} \\ p = 15 &: \text{Type B consumers exit the market (no demand)} \end{aligned}$$

With this, we are ready to plot the demand and supply functions:



2.2 Equilibrium

We want to study the market for the firm's output good. So one of the goods that the consumer demands has to be this good. That way we have consumers who are willing to purchase a good and we have producers who are willing to produce that good. So, the next step is to find the equilibrium. Finding the equilibrium is easy - this is something you are used to from principles. All we have to do is equate the market demand and supply curves:

$$Q_D(p) = Q_S(p)$$

From this, we can solve for the equilibrium price p^*, plug it back into either the demand or supply function to get the equilibrium quantity $Q^* = Q_D(p^*) = Q_S(p^*)$. Equivalently, we can equate the inverse functions:

$$P_D(Q) = P_S(Q)$$

This gets us Q^* and the equilibrium price is $p^* = P_D(Q^*) = P_S(Q^*)$.

Example

Let's continue with the example we started above. By the diagram, we can see that the equilibrium occurs when everyone is participating in the market. So we just need worry about those parts of the function (i.e. where $p \in (2, 10)$ and $Q \in (75, 550)$). In particular we want to solve:

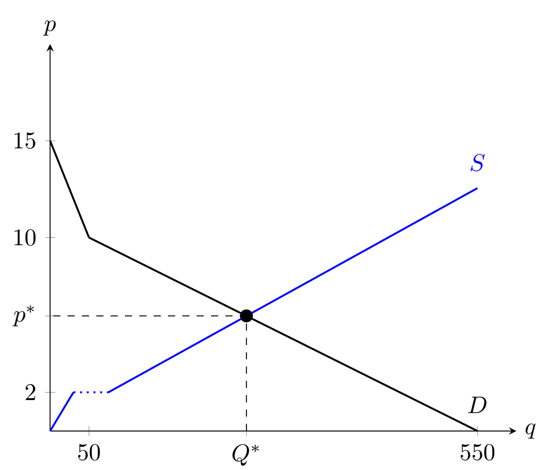
$$\begin{aligned} Q_D(p) &= Q_S(p) \\ 550 - 50p &= 45p - 15 \\ 565 &= 95p \\ \therefore p^* &= \frac{565}{95} = \frac{113}{19} \approx 5.95 \end{aligned}$$

$$
\begin{aligned}
\implies Q^* &= Q_S(p^*) \\
&= 45\left(\frac{113}{19}\right) - 15 \\
&= \frac{4800}{19} \\
&\approx 252.63
\end{aligned}
$$

We could have used the inverse functions too:

$$
\begin{aligned}
P_D(Q) &= P_S(Q) \\
11 - \frac{Q}{50} &= \frac{Q}{45} + \frac{1}{3} \\
11 - \frac{1}{3} &= Q\left(\frac{1}{45} + \frac{1}{50}\right) \\
495 - 15 &= Q\left(1 + \frac{9}{10}\right) \\
480 &= Q\left(\frac{19}{10}\right) \\
\therefore Q^* &= \frac{4800}{19} \\
\implies p^* &= P_D(Q^*) \\
&= 11 - \frac{1}{50}\left(\frac{4800}{19}\right) \\
&= 11 - \frac{96}{19} \\
&= \frac{113}{19}
\end{aligned}
$$

This gives us our equilibrium point where the two curves cross:



2.3 Welfare

When we studied consumers, we learnt about three measures of welfare changes: consumer surplus (CS), EV, and CV. When studying the market, it is convenient to work with the consumer surplus. We also have the firm analog, which is called producer surplus (PS).

These ideas should already be familiar to you, but let's define them in a clear way. For this, let's suppose that consumers pay price of p^c for each unit of the good, and suppliers receive a price of p^s for each unit of good sold. For the most part, we are going to have $p_c = p_s = p^*$, but in the next section we will talk about when these prices could differ. Here is how to think about the surplus measures:

- **(Net) Consumer Surplus:** The area below the inverse demand curve and above the price that consumers pay p_c, from $Q = 0$ to $Q = Q_D(p_c)$
- **(Net) Producer Surplus:** The area above the inverse supply curve and below the price that producers pay p_s, from $Q = 0$ to $Q = Q_S(p_s)$

<u>Example</u>

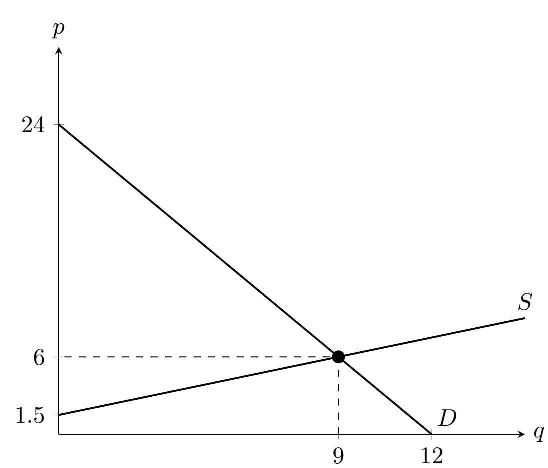
Suppose we have the following:

- Demand: $Q_D(p) = 12 - \frac{1}{2}p$
- Supply: $Q_S(p) = 2p - 3$

Inverting this gives us the following functions:

$$P_D(Q) = 24 - 2Q$$
$$P_S(Q) = \frac{1}{2}Q + \frac{3}{2}$$

We can plot these to give us the following diagram:



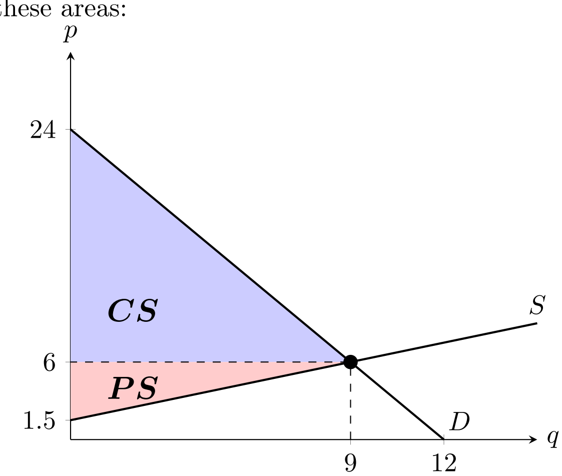
We can find the equilibrium by equating the functions:

$$
\begin{aligned}
Q_D(p) &= Q_S(p) \\
12 - \frac{1}{2}p &= 2p - 3 \\
15 &= \frac{5}{2}p \\
\therefore p^* &= \frac{30}{5} = 6 \\
\implies Q^* &= Q_S(p) \\
&= 2(6) - 3 \\
&= 9
\end{aligned}
$$

So, the surpluses will be calculated as:

- Consumer Surplus: Area below the inverse demand curve and above the price $p_c = 9$, from $Q = 0$ to $Q = 9$
- Producer Surplus: Area below the inverse supply curve and above the price $p_s = 9$, from $Q = 0$ to $Q = 9$

In the diagram, we can shade these areas:



And with some simple geometry, it's easy to calculate them:

$$
\begin{aligned}
CS &= 0.5 \times (24 - 6) \times (9 - 0) = 0.5 \times 18 \times 9 = 81 \\
PS &= 0.5 \times (6 - 1.5) \times (9 - 0) = 0.5 \times 4.5 \times 9 = 20.25
\end{aligned}
$$

2.4 Taxes

Now that we have a market equilibrium, we are often interested in introducing a tax and seeing what effect this has on the market. Suppose the government introduces a tax t.[2] This tax is imposed on either the

[2] If $t > 0$, it is a tax as we usually think about it. If $t < 0$, then it is a subsidy. But you still follow through with the same process

consumers or the producers. In either case, it drives a wedge between the price that consumers pay and the price that producers receive (i.e. we no longer have $p_c = p_s$). Instead the relationship becomes:

$$p_c = p_s + t$$

This just says that part of the price the consumer pays goes to the supplier, and part of it goes to the government as a tax. For example, suppose a good costs the consumer \$10 but \$2 of that is a tax (i.e. the tax is levied on consumers). That means that producers only receive \$8, i.e. $p_c = 10, p_s = 8, t = 2$.

This wedge in the prices has an effect on the equilibrium price, quantity, and welfare. Suppose the tax is imposed on the consumers, then this shifts the demand curve to a new function $Q'_D(p)$:

$$Q'_D(p) = Q_D(p + t)$$

If instead the tax is imposed on the producers, then this shifts the supply curve to a new function $Q'_S(p)$:

$$Q'_S(p) = Q_S(p - t)$$

The equilibrium occurs at the new intersection, which you can solve for by equating the new demand (supply) curve to the old supply (demand) for a tax on consumers (producers):

Consumer Tax:

$$\begin{aligned} Q'_D(p_s) &= Q_S(p_s) \\ \implies Q_D(p_s + t) &= Q_S(p_s) \end{aligned}$$

Producer Tax:

$$\begin{aligned} Q_D(p_c) &= Q'_S(p_c) \\ \implies Q_D(p_c) &= Q_S(p_c + t) \end{aligned}$$

The first way will allow you so solve for the price that producers receive in equilibrium p_s^*. From this, you can get the price that consumers pay $p_c^* = p_s^* + t$. Similarly, the second way will get you solution for p_c^*, and from that you can back out $p_s^* = p_c^* - t$.

<u>Example</u>

Let's continue with the example from before and introduce a tax

- Demand: $Q_D(p) = 12 - \frac{1}{2}p$, i.e. $P_D(Q) = 24 - 2Q$
- Supply: $Q_S(p) = 2p - 3$, i.e. $P_S(Q) = \frac{1}{2}Q + \frac{3}{2}$
- Tax consumers a unit tax of $t = 8$

This tax will shift the demand curve:

$$\begin{aligned} Q'_D(p) &= Q_D(p + t) \\ &= 12 - \frac{1}{2}(p + 8) \\ &= 12 - \frac{1}{2}p - 4 \end{aligned}$$

$$= 8 - \frac{1}{2}p$$
$$\implies P'_D(Q) = 16 - 2Q$$

The new equilibrium will be:

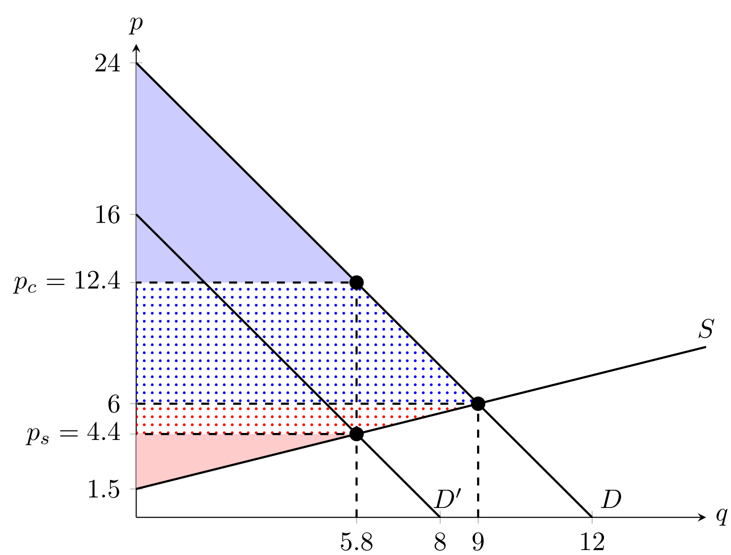
$$\begin{aligned} Q'_D(p_s) &= Q_S(p_s) \\ 8 - \frac{1}{2}p &= 2p - 3 \\ 11 &= 2.5p \\ \therefore p^* &= 4.4 \\ \implies Q^* &= 8 - \frac{1}{2}(4.4) \\ &= 8 - 2.2 \\ &= 5.8 \end{aligned}$$

However, this price is not actually what consumers are paying. You can think of this as the price that sellers receive, i.e. p_s. Consumers also pay an additional tax of $t = 8$:

$$p_c = p_s + t = 4.4 + 8 = 12.4$$

Another way to think about this is that it is the price on the consumers' original demand curve, i.e. $P_D(Q^*) = 24 - 2(5.8) = 12.4$.

Next, we want to evaluate how this changes welfare. For welfare changes, you always want to use the original demand and supply curves. Using the same rules as usual, all of this gives us the following diagram:



The shaded areas are as follows:

- New Consumer Surplus (blue shaded area): This is the area under the *original* demand curve $P_D(Q)$ and the *new* price that *consumers* pay p_c, i.e. $0.5 \times (24 - 12.4) \times 5.8 = 33.64$.
- Lost Consumer Surplus (blue dotted area): This is the change in consumer surplus, i.e. $CS_{old} - CS_{new} = 81 - 33.64 = 47.36$.
- New Producer Surplus (red shaded area): This is the area above the *original* supply curve $P_S(Q)$ and the *new* price that *producer* receive p_s, i.e. $0.5 \times (4.4 - 1.5) \times 5.8 = 8.41$.
- Lost Producer Surplus (red dotted area): This is the change in consumer surplus, i.e. $PS_{old} - PS_{new} = 20.25 - 8.41 = 11.84$.

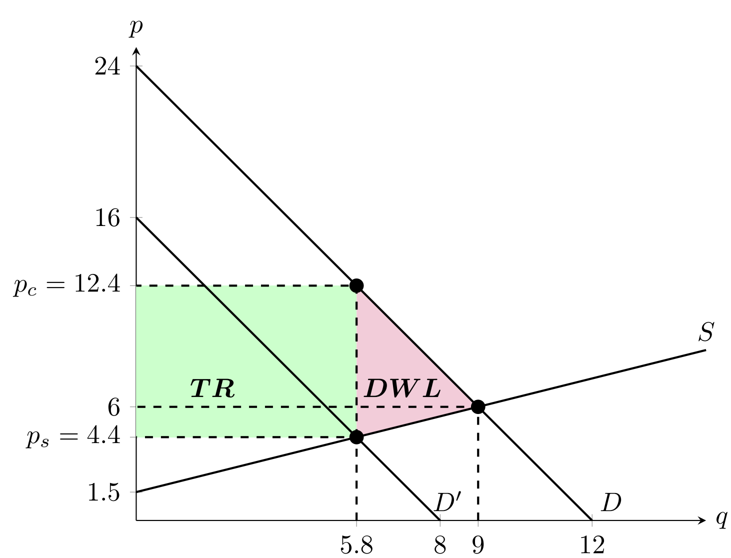
So the tax means that consumers and producers are worse off than before. However, this is not necessarily a bad thing, because the government is also raising tax revenue (TR). The tax revenue is equal to the unit tax rate multiplied by the total number of units sold in the market:

$$TR = t \times Q^* = 8 \times 5.8 = 46.4$$

Graphically, this is the rectangle between p_c and p_s on the y-axis (which must be equal to $t = 8$) and between 0 and $Q^* = 5.8$ on the x-axis. However, the total lost surplus (from consumers and producers) is equal to $47.36 + 11.84 = 59.2$. That means that more surplus is lost than is raised in tax revenue. This difference captures the inefficiency of a tax, which we call the **deadweight loss** (DWL).

$$DWL = 59.2 - 46.4 = 12.8$$

Graphically, this is the triangle that is in the lost surplus, but not in the tax revenue:



So another way to calculate the DWL is to calculate the area of the triangle:

$$\begin{aligned} DWL &= 0.5 \times (12.4 - 4.4) \times (9 - 5.8) \\ &= 0.5 \times 8 \times 3.2 \\ &= 12.8 \end{aligned}$$