# Advanced Micro: Recitation 12
# Extensive Form Games 2

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## Sequential Equilibrium

Let's establish the definition of sequential equilibrium (SE) that we will use in this recitation.

**Definition.** An assessment $(s, \mu)$ consisting of a strategy profile $s$ and a system of beliefs $\mu$ is a sequential equilibrium if:

1. $s$ is sequentially rational given the beliefs $\mu$. That is, at all information sets $I$ where player $j$ takes an action we have:
$$E[u_j(s_j, s_{-j})|I, \mu] \geq E[u_j(t_j, s_{-j})|I, \mu], \forall t_j$$

2. There exists a sequence of *fully mixed* strategy profiles $\{s^k\} \to s$ such that $\{\mu^k\} \to \mu$, where $\mu^k$ denotes the beliefs derived from the strategy profile $s^k$ using Bayes Rule. That is, for any information set $I$ and any node $x \in I$ we have that:
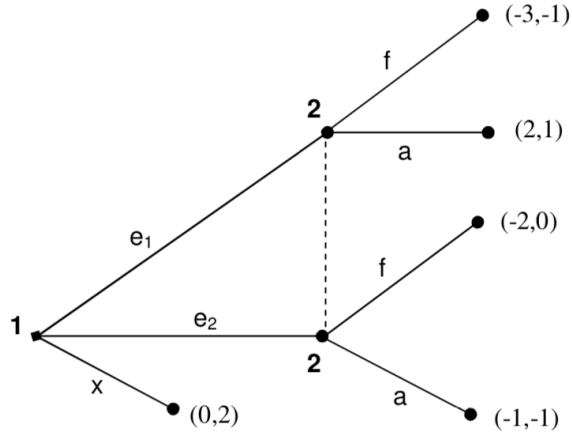$$\mu(x) = \frac{Pr(x|s)}{Pr(I|s)}$$

We have seen refinements of the Nash equilibrium. In extensive form games, the solution concepts can be ordered as follows:
$$\{SE\} \subseteq \{SPNE\} \subseteq \{NE\}$$

In the following exercises, find the NE, SPNE, and SE (for simplicity, just focus on pure strategies)

# Exercise 1

## Game Tree



Let's define the strategies as $(s_1, s_2)$, where $s_1 \in \{e_1, e_2, x\}$, $s_2 \in \{f, a\}$.

## PSNE

The associated strategic form game is:

|       | $f$            | $a$            |
|-------|----------------|----------------|
| $e_1$ | -3, -1         | $\underline{2}, \underline{1}$ |
| $e_2$ | -2, $\underline{0}$ | -1, -1    |
| $x$   | $\underline{0}, \underline{2}$ | $0, \underline{2}$ |

The PSNE are $(e_1, a)$ and $(x, f)$.

## SPNE

Since there are no proper sub-games, the NEs are also SPNE.

## SE

Let $\mu$ be player 2's belief that they are at the node following $e_1$. Then player 2 should play $f$ if and only if:

$$E[u_2(f)|I, \mu] \geq E[u_2(a)|I, \mu]$$
$$\mu(-1) + (1 - \mu)(0) \geq \mu(1) + (1 - \mu)(-1)$$

$$-\mu \geq 2\mu - 1$$
$$3\mu \leq 1$$
$$\mu \leq \frac{1}{3}$$

If player 2 is playing $a$, then player 1 being sequentially rational means they should play $e_1$. The belief consistent with this $\mu = 1$. So this suggests that an SE is $(e_1, a)$ with the belief $\mu = 1$. It is easy to find a fully mixed strategy then that converges to this:

$$Pr(e_1) = 1 - 2\varepsilon_k \qquad\qquad Pr(e_2) = \varepsilon_k \qquad\qquad Pr(x) = \varepsilon_k$$
$$Pr(f) = \delta_k \qquad\qquad\qquad Pr(a) = 1 - \delta_k$$

Where $\varepsilon_k, \delta_k \to 0$. The consistent belief associated with this is:

$$\mu = \frac{Pr(e_1)}{Pr(e_1 \text{ or } e_2)} = \frac{1 - 2\varepsilon_k}{1 - \varepsilon_k} \to 1 = \mu$$

If player 2 is playing $f$, then player 1 being sequentially rational means they should play $x$. $\mu$ becomes off the equilibrium path, but we still need to find a fully mixed strategy which gives a belief consistent with $\mu \leq \frac{1}{3}$. Consider the following:

$$Pr(e_1) = \varepsilon_k \qquad\qquad Pr(e_2) = 2\varepsilon_k \qquad\qquad Pr(x) = 1 - 3\varepsilon_k$$
$$Pr(f) = 1 - \delta_k \qquad\qquad\qquad Pr(a) = \delta_k$$

The consistent belief associated with this is:

$$\mu = \frac{Pr(e_1)}{Pr(e_1 \text{ or } e_2)} = \frac{\varepsilon_k}{3\varepsilon_k} \to \frac{1}{3}$$

Therefore, $(x, f)$ with a belief $\mu = \frac{1}{3}$ is a SE. There are infinitely more, of course, as you can set $\mu < \frac{1}{3}$ and find a fully mixed strategy to make it consistent.

Interestingly, note that even with the refinement of SPNE and SE we are still left with $(x, f)$, which is still not a very credible outcome. Think of it like this: if player 1 was forced to enter, they would choose $e_1$, since that strictly dominates $e_2$. If player 2 is being rational, they should know that player 1 would never play $e_2$, which means *if* they had entered, they must have entered by playing $e_1$. In that case it makes sense for player 2 to play $a$ and not $f$. Knowing that player 2 will think this, player 1 should also deviate from $x$ to $e_1$ and subsequently player 2 deviates to $a$.

# Exercise 2

## Game Tree
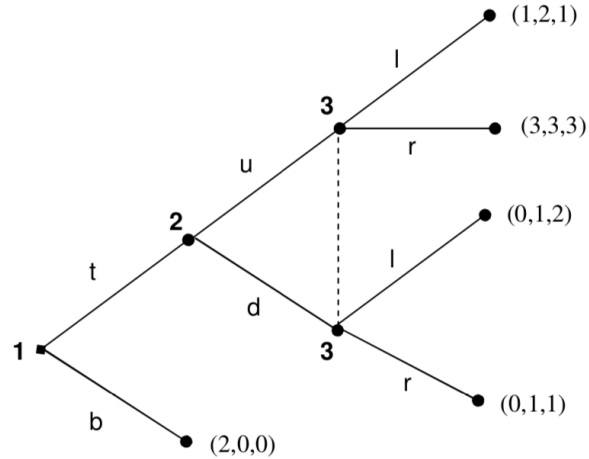


Let's define the strategies as $(s_1, s_2, s_3)$, where $s_1 \in \{t, b\}$, $s_2 \in \{u, d\}$, $s_3 \in \{l, r\}$.

## PSNE

Consider player 1. They get payoff 2 if they choose $b$. The only way to get a higher payoff is if they play $t$, player 2 chooses $u$ and player 3 chooses $r$. Indeed, $(t, u, r)$ is a NE since player 2 is best responding (switching to $d$ will lower payoffs from 3 to 1) and player 3 is best responding too (switching to $l$ will lower payoffs from 3 to 1).

All other PSNEs must be of the form $(b, s_2, s_3)$, where $(s_2, s_3) \neq (u, r)$. In other words, $(b, u, l)$, $(b, d, r)$, and $(b, d, l)$ are all NEs too.

## SPNE

There is one proper sub-game starting at player 2's node. The sub-game's payoff matrix is:

|   | $l$ | $r$ |
|---|---|---|
| $u$ | $\underline{2}$, 1 | $\underline{3}$, $\underline{3}$ |
| $d$ | 1, $\underline{2}$ | 1, 1 |

Therefore, the only SPNE is $(t, u, r)$.

## SE

Since there is only one SPNE, there is only one candidate for SE: $(t, u, r)$. Let's denote the nodes as follows: $x_1, x_2, x_{3u}, x_{3d}$. Let $\mu$ be player 3's belief that they are at the node $x_{3u}$ and their non-singleton information set as $I$. In other words, for a fully mixed strategy $m$, a consistent belief would be:

$$\mu = \frac{Pr(x_{3u}|m)}{Pr(I|m)} = \frac{Pr(x_2|m)Pr(x_{3u}|m, x_2)}{Pr(x_2|m)Pr(I|m, x_2)} = \frac{Pr(x_{3u}|m, x_2)}{Pr(I|m, x_2)} = \frac{m_{2u}}{1}$$

Where $m_{2u}$ is the probability that player 2 assigns to $u$. So any sequence of fully mixed strategies $\{m^k\}$ that converges to $m$ must have $m_{2u}^k \to m_{2u}$ and therefore we must also have that $\mu^k \to m_{2u}$. For player 2 to be sequentially rational, they should be playing $s_2 = u$ (since $u$ strictly dominates $d$); therefore, we must have $m_{2u} = 1$ in a SE. This means that $\mu = 1$. If that is the case, then for player 3 to be sequentially rational, they should be playing $r$. Given all this, player 1 being sequentially rational should be playing $t$. Therefore, the strategies $(t, u, r)$ with the belief $\mu = 1$ is a SE.
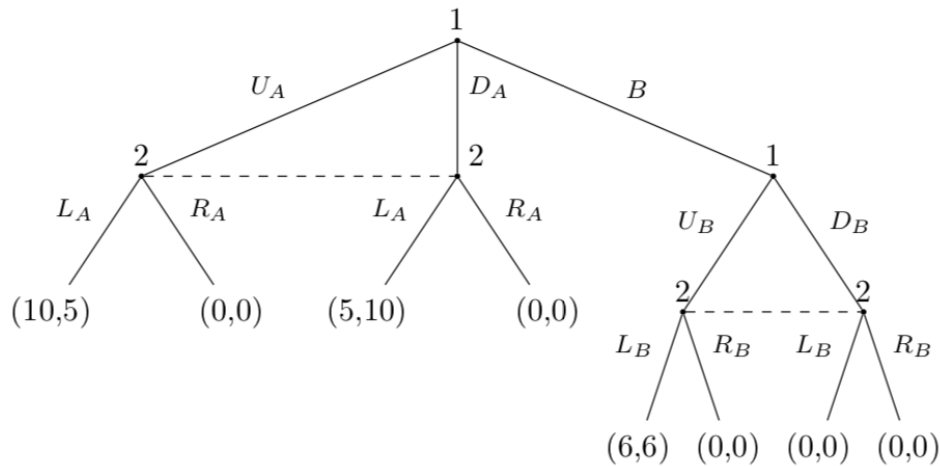
To be precise, we should show an example of a fully mixed strategy that converges to this. The following would work:

$$Pr(t) = 1 - \varepsilon_k \qquad\qquad Pr(b) = \varepsilon_k$$
$$Pr(u) = 1 - \sigma_k \qquad\qquad Pr(d) = \sigma_k$$
$$Pr(l) = \delta_k \qquad\qquad Pr(r) = 1 - \delta_k$$

Where $\varepsilon_k, \sigma_k, \delta_k \to 0$. Clearly, we have $\mu^k = m_{2u} = 1 - \sigma_k \to 1 = \mu$.

# Exercise 3

## Game Tree



Let's define the strategies as $((s_{11}, s_{12}), (s_{21}, s_{22}))$, where $s_{11} \in \{U_A, D_A, B\}$, $s_{12} \in \{U_B, D_B\}$, $s_{21} \in \{L_A, R_A\}$, $s_{22} \in \{L_B, R_B\}$.

## PSNE

Player 1 has 6 possible strategies and player 2 has 4. The associated strategic form game is: (for simplicity, ignore $s_{12}$ if $s_{11} \neq B$ as it will give duplicate rows)

|  | $L_A, L_B$ | $R_A, L_B$ | $L_A, R_B$ | $R_A, R_B$ |
|---|---|---|---|---|
| $U_A, s_{12}$ | $\underline{10}, \underline{5}$ | $0, 0$ | $\underline{10}, \underline{5}$ | $\underline{0}, 0$ |
| $D_A, s_{12}$ | $5, \underline{10}$ | $0, 0$ | $5, \underline{10}$ | $\underline{0}, 0$ |
| $B, U_B$ | $6, \underline{6}$ | $\underline{6}, \underline{6}$ | $0, 0$ | $\underline{0}, 0$ |
| $B, D_B$ | $0, \underline{0}$ | $0, \underline{0}$ | $0, \underline{0}$ | $\underline{0}, \underline{0}$ |

This gives us six PSNE: $((U_A, s_{12}), (L_A, s_{22}))$ where $s_{12} \in \{U_B, D_B\}, s_{22} \in \{L_B, R_B\}$ (4 NEs), $((B, U_B), (R_A, L_B))$, and $((B, D_B), (R_A, R_B))$.

## SPNE

For SPNE, this game has one proper sub-game starting after player 1's second node. The payoff matrix for this sub-game is:

|  | $L_B$ | $R_B$ |
|---|---|---|
| $U_B$ | $\underline{6}, \underline{6}$ | $\underline{0}, 0$ |
| $D_B$ | $0, \underline{0}$ | $\underline{0}, \underline{0}$ |

So there are two PSNEs here. The reduced game's payoff matrix becomes (depending on the strategy played in the sub-game):

**Sub-game NE:** $(U_B, L_B)$

|  | $L_A$ | $R_A$ |
|---|---|---|
| $U_A$ | $\underline{10}, \underline{5}$ | $0, 0$ |
| $D_A$ | $5, \underline{10}$ | $0, 0$ |
| $B$ | $6, \underline{6}$ | $\underline{6}, \underline{6}$ |

**Sub-game NE:** $(D_B, R_B)$

|  | $L_A$ | $R_A$ |
|---|---|---|
| $U_A$ | $\underline{10}, \underline{5}$ | $\underline{0}, 0$ |
| $D_A$ | $5, \underline{10}$ | $\underline{0}, 0$ |
| $B$ | $0, \underline{0}$ | $\underline{0}, \underline{0}$ |

Therefore the four SPNE here are:

$$((U_A, U_B), (L_A, L_B)), ((B, U_B), (R_A, L_B)), ((U_A, D_B), (L_A, R_B)), ((B, D_B), (R_A, R_B))$$

## SE

First, we define the beliefs. There are two information sets (the $A$ information set on the left and the $B$ information set on the right), and therefore we need to define two beliefs. Suppose that $\mu$ is player 2's belief that they are at $U_A$ and $\lambda$ is their belief that they are at $U_B$ (then their belief that they are $D_A$ and $D_B$ are $1 - \mu$ and $1 - \lambda$, respectively).

At the $A$ information set, player 2 has a choice between playing $L_A$ or $R_A$. Given a belief $\mu$, they should play $L_A$ if and only if:

$$E[u_2(L_A)|A, \mu] \geq E[u_2(R_A)|A, \mu]$$
$$\mu(5) + (1 - \mu)(10) \geq \mu(0) + (1 - \mu)(0)$$
$$10 - 5\mu \geq 0$$
$$\mu \leq 2$$

Therefore, $\forall \mu \in [0, 1]$, player 2 strictly prefers $L_A$.

At the $B$ information set, player 2 has a choice between playing $L_B$ or $R_B$. Given a belief $\lambda$, they should play $L_B$ if and only if:

$$E[u_2(L_B)|B, \mu] \geq E[u_2(R_B)|B, \mu]$$
$$\lambda(6) + (1 - \lambda)(0) \geq \lambda(0) + (1 - \lambda)(0)$$
$$6\lambda \geq 0$$
$$\lambda \geq 0$$

Therefore, $\forall \lambda \in (0, 1]$, player 2 strictly prefers $L_B$. At $\lambda = 0$, then player 2 is indifferent between $L_B$ and $R_B$.

Next, we need to think about player 1's action to ensure they are sequentially rational. Player 1 has two information sets, i.e. their two nodes in the game. Given that player 2 will always play $L_A$ at the $A$ information set, then player 1's best response at the first node is to play $U_A$. This gives player 1 a pay-off of 10, which is better than anything else they could do in the $B$ game. So a SE should have $s_{11} = U_A$ and $s_{21} = L_A$. Even though the second node will be the off-path, we should still check sequential rationality. If $s_{22} = L_B$, then player 1's best response is $s_{12} = U_B$. If $s_{22} = R_B$, then player 1 is indifferent.

So possible strategies for SEs are $((U_A, U_B), (L_A, L_B))$, $((U_A, U_B), (L_A, R_B))$, $((U_A, D_B), (L_A, R_B))$. However, notice that $((U_A, U_B), (L_A, R_B))$ is not a SPNE, so this already tells us that we should rule it out as a SE. The reason is that it will not generate a consistent belief. If $s_{22} = R_B$, then it must be because $\lambda = 0$, which means we must have had $s_{12} = D_B$. This leaves us with two real candidates, and all that is left is to check the beliefs. Note that the only way to have $\mu$ be consistent with Bayes Rule given $s_{11} = U_A$ is $\mu = 1$:

$$\mu = \frac{Pr(U_A)}{Pr(U_A \text{ or } D_A)} = \frac{1}{1} = 1$$

To check that the beliefs are consistent, let's consider fully mixed strategies.

Consider a fully mixed strategy $m_k \to ((U_A, U_B), (L_A, L_B))$ with $\mu = 1, \lambda \in [0, 1]$:

$$Pr(U_A) = 1 - 2\varepsilon_k \qquad\qquad Pr(D_A) = \varepsilon_k \qquad\qquad Pr(B) = \varepsilon_k$$
$$Pr(U_B) = 1 - \varepsilon_k \qquad\qquad Pr(D_B) = \varepsilon_k$$
$$Pr(L_A) = 1 - \delta_k \qquad\qquad Pr(R_A) = \delta_k$$
$$Pr(L_B) = 1 - \delta_k \qquad\qquad Pr(R_B) = \delta_k$$

Where $\varepsilon_k \to 0$ and $\delta_k \to 0$. The consistent beliefs with $\sigma_k$ are then:

$$\mu_k = \frac{Pr(U_A)}{Pr(U_A \text{ or } D_A)} = \frac{1 - 2\varepsilon_k}{1 - \varepsilon_k} \to 1$$

$$\lambda_k = \frac{Pr(B \text{ and } U_B)}{Pr(B \text{ and } (U_B \text{ or } D_B))} = \frac{\varepsilon_k (1 - \varepsilon_k)}{\varepsilon_k (1)} = 1 - \varepsilon_k \to 1$$

Therefore, an SE is $((U_A, U_B), (L_A, L_B))$ with $\{\mu = 1, \lambda = 1\}$

Now, let's do another fully mixed strategy but with $m_k \to ((U_A, U_B), (L_A, L_B))$ with $\mu = 1, \lambda = 0$:

$$Pr(U_A) = 1 - 2\varepsilon_k \qquad\qquad Pr(D_A) = \varepsilon_k \qquad\qquad Pr(B) = \varepsilon_k$$
$$Pr(U_B) = \varepsilon_k \qquad\qquad Pr(D_B) = 1 - \varepsilon_k$$
$$Pr(L_A) = 1 - \delta_k \qquad\qquad Pr(R_A) = \delta_k$$
$$Pr(L_B) = \delta_k \qquad\qquad Pr(R_B) = 1 - \delta_k$$

The consistent beliefs with $\sigma_k$ are then:

$$\mu_k = \frac{Pr(U_A)}{Pr(U_A \text{ or } D_A)} = \frac{1 - 2\varepsilon_k}{1 - \varepsilon_k} \to 1$$

$$\lambda_k = \frac{Pr(B \text{ and } U_B)}{Pr(B \text{ and } (U_B \text{ or } D_B))} = \frac{\varepsilon_k (\varepsilon_k)}{\varepsilon_k (1)} = \varepsilon_k \to 0$$

Therefore, another SE is $((U_A, U_B), (L_A, L_B))$ with $\{\mu = 1, \lambda = 0\}$