Advanced Micro: Recitation 11
Extensive Form Games 1

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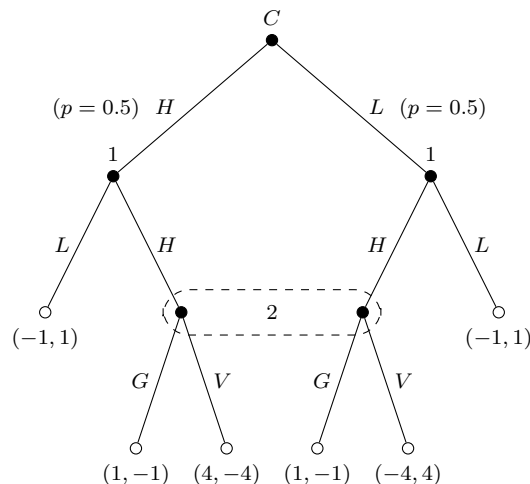
1 Associated Strategic Form

Question

First player 1 receives a card that is either H or L with equal probabilities. Player 2 does not see the card. Player 1 may announce that her card is L, in which case she must pay 1 to player 2, or may claim her card is H, in which case player 2 may choose to give up (G) or to insist on viewing player 1's card (V). If player 2 gives up then he must pay 1 to player 1. If he insists on seeing player 1's card, then player 1 must pay him 4 if her card is L and he must pay her 4 if her card is H. Assume the utility functions are simply $u(x) = x$, i.e. losing 1 yields a utility of -1.

1. Draw the game tree

2. Write the associated strategic form

3. Show that there is no pure strategy NE

4. Find the (mixed strategy) NE

Solution

The game tree looks as follows:

The strategic form can be written in this table - it is effectively a game of incomplete information:

High (probability 0.5)

	G	V
H	$\underline{1}$, $\underline{-1}$	$\underline{4}$, -4
L	-1, $\underline{1}$	-1, $\underline{1}$

Low (probability 0.5)

	G	V
H	$\underline{1}$, -1	-4, $\underline{4}$
L	-1, $\underline{1}$	$\underline{-1}$, $\underline{1}$

So the associated strategic form game is:

- Three players: $1H$, $1L$, 2

- Strategies: $S_{1H} = \{L, H\}$, $S_{1L} = \{L, H\}$, $S_2 = \{G, S\}$

- Payoffs: see matrices above

Could we have a PSNE? For player $1H$, H is a strictly dominant strategy so they must be playing that. In the low game, player 1 best responds to G with H and best responds to V with L. So the only possible PSNE are (H, H, G) and (H, L, V). Let's consider player 2's expected payoff:

$$E[u_2(H, s_{1L}, G)] = 0.5(-1) + 0.5u_2(H, s_{1L}, G; L) = \begin{cases} -1 & \text{if } s_{1L} = H \\ 0 & \text{if } s_{1L} = L \end{cases}$$

$$E[u_2(H, s_{1L}, V)] = 0.5(-4) + 0.5u_2(H, s_{1L}, V; L) = \begin{cases} 0 & \text{if } s_{1L} = H \\ -1.5 & \text{if } s_{1L} = L \end{cases}$$

So (H, H, G) cannot be a NE because player 2 would want to deviate to V (-1 vs 0). Also, (H, L, V) cannot be a NE because player 2 would want to deviate to G (-1.5 vs 0).

Next, we want to find a mixed strategy. As we said, $1H$ has to be playing H. Suppose that $1L$ is using the mixed strategy $m_{1L} = (p, 1-p)$. Then to make player 2 indifferent we would need:

$$E[u_2(H, m_{1L}, G)] = E[u_2(H, m_{1L}, V)]$$
$$p(-1) + (1-p)(0) = p(0) + (1-p)(-1.5)$$
$$-p = 1.5p - 1.5$$
$$2.5p = 1.5$$
$$p = 0.6$$

Suppose that player 2 is mixing with the strategy $m_2 = (q, 1-q)$. To make player $1L$ indifferent, we would need:

$$E[u_1(H, H, m_2)] = E[u_1(H, L, m_2)]$$
$$q(1) + (1-q)(-4) = q(-1) + (1-q)(-1)$$
$$5q - 4 = -1$$

$$q = 0.6$$

So the MSNE is $m_{1H} = (1,0)$, $m_{1L} = (0.6, 0.4)$, and $m_2 = (0.6, 0.4)$.

2 Take It or Leave It

Question

(Jehle and Reny 7.29) The following game, taken from Reny (1992), is called 'Take-it-or-leave-it'. A referee is equipped with N dollars. He places one dollar on the table. Player 1 can either take the dollar or leave it. If he takes it, the game ends. If he leaves it, the referee places a second dollar on the table. Player two is now given the option of taking the two dollars or leaving them. If he takes them, the game ends. Otherwise the referee places a third dollar on the table and it is again player 1's turn to take or leave the three dollars. The game continues in this manner with the players alternately being given the choice to take all the money the referee has so far placed on the table and where the referee adds a dollar to the total whenever a player leaves the money. If the last player to move chooses to leave the N dollars the game ends with neither player receiving any money. Assume that N is public information.

1. Draw the game tree for $N = 6$

2. Calculate the backward induction strategies.

3. Prove that the backward induction strategies form a Nash equilibrium

4. Prove that the outcome that results from the backward induction strategies is the unique outcome in any Nash equilibrium. Is there a unique Nash equilibrium?

Solution

The game tree looks as follows:
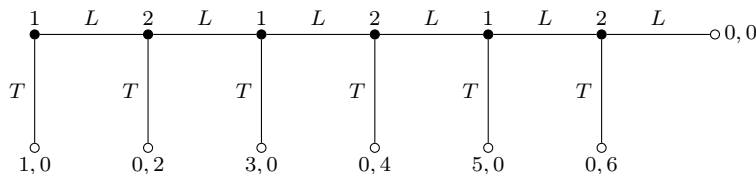
Let $i(x_n)$ denote the player choosing at n^{th} node or round of the game, i.e. $i(x_0) = 1, i(x_1) = 2, i(x_2) = 3, ...,$ where $i(x_{N-1}) = 1$ if N is odd and 2 if N is even. Using backward induction:

- Player at the last node, $i(x_{N-1})$, should take because N is a better payoff than 0

- The player at $i(x_{N-2})$ should take because they know that the next player will take, and $N - 1$ is a better payoff than 0

- This continues all the way until $i(x_0)$, where player 1 should take at the first stage of the game and get 1.

Therefore, the backward induction strategies are $(T, T, T, ..., T)$.

To show that this is a Nash equilibrium, let's look at the associated strategic form game. For simplicity, let's take the $N = 6$ case, but you can easily generalize this. Below is the payoff table based on the set of strategies each player is choosing. Note that strategies are written in the form s_{jn} where j is the player and n is the node, i.e. player one's strategies are (s_{10}, s_{12}, s_{14}) and player two's are (s_{21}, s_{23}, s_{25}). To remove redundancy, I will keep strategies as s_{jn} if the same payoff results for either value of $s_{jn} \in \{L, T\}$.

	T, s_{23}, s_{25}	L, T, s_{25}	L, L, T	L, L, L
T, s_{12}, s_{14}	$\underline{1}, \underline{0}$	$1, \underline{0}$	$1, \underline{0}$	$1, \underline{0}$
L, T, s_{14}	$0, \underline{2}$	$\underline{3}, 0$	$3, 0$	$3, 0$
L, L, T	$0, 2$	$0, \underline{4}$	$\underline{5}, 0$	$\underline{5}, 0$
L, L, L	$0, 2$	$0, 4$	$0, \underline{6}$	$0,0$

Notice that once player 1 chooses $s_{10} = T$, it doesn't matter what else they or player 2 choose - the payoff will always be $(1, 0)$. Similarly, if player 1 chooses $s_{10} = L$ and player 2 plays $s_{21} = T$, then the payoff is always $(0, 2)$. The best responses are also underlined, and we can see that the Nash equilibrium occurs when $s_{10} = T$ and $s_{21} = T$ resulting in the payoff of $(1, 0)$. The backward induction definitely fits this bill and therefore is a NE. However, note that many other strategies also give us this outcome. For example, $s_1 = (T, L, L)$ and $s_2 = (T, T, L)$. As long as each player chooses T as their first move, we get a Nash equilibrium. In fact, there are 4 possible strategies for each player, so there are 16 Nash equilibria! The point is that everything after the first node is off the equilibrium path, so we don't need to put any restriction on them. This is what distinguishes NE from the backward induction method.

3 Rotten Kid

Question

Suppose a parent and child play the following game. First the child takes an action, A, that produces income for the child $I_C(A)$ and income for the parent $I_P(A)$. Second the parent observes the incomes I_C and I_P and chooses a bequest B to leave to the child. The payoff to the child is $u(I_C(A) + B)$ and the payoff to the parent is $v(I_P(A) - B) + \delta u(I_C(A) + B)$ where $\delta > 0$ (so the parent cares about her child's welfare). Let A be any non-negative number. Let the income functions $I_C(A)$ and $I_P(A)$ both be strictly concave in A and maximized at $A_C > 0$ and $A_P > 0$, respectively. The bequest, B, can be any real number. Finally, assume that both u and v are strictly increasing and strictly concave in total income (income net bequest). Show that in the backward induction solution, the child chooses an action A^* that maximizes total family income, $I_C(A) + I_P(A)$, even though only the parent is altruistic.

As an extra exercise (or possibly to make this question clearer), show this result for these specific functions:

$$u(x) = \ln x$$
$$v(x) = \ln x$$

$$I_C(A) = A(10 - A)$$
$$I_P(A) = A(30 - A)$$

Solution

Using backward induction, we should first consider the last action in the game: the parent's choice of the bequest B. Their goal is to maximize their payoff:

$$\max_{B \in \mathbb{R}} \ v(I_P(A) - B) + \delta u(I_C(A) + B)$$

This gives an FOC of:

$$v'(I_P(A) - B)(-1) + \delta u'(I_C(A) + B) = 0$$
$$\therefore v'(I_P(A) - B) = \delta u'(I_C(A) + B)$$

Since u and v are strictly concave, we know that the choice of B will be the global maximizer. As the bequest is chosen after observing the child's action, and the parent is choosing the best response to that action, we know that B must be a function of A. This means that $B(A)$ is the solution to the parent's problem. In other words, the FOC above implicitly defines $B = B(A)$, and it holds true for all A.

Moving back a step, in the first stage, the child will choose their action A to maximize their own payoff - knowing that the parents choose the bequests according to $B(A)$.

$$\max_{A \in \mathbb{R}_+} \ u(I_C(A) + B(A))$$

This gives an FOC of:

$$u'(I_C(A) + B(A)) \left[I_C'(A) + B'(A) \right] = 0$$
$$\implies I_C'(A) + B'(A) = 0$$

Since u is strictly increasing, this means that $u' > 0$. This equation implicitly defines the child's choice of A^*.

We need to then show that this choice of A^* also maximizes total family income. Call A^{**} this value, which satisfies the FOC:

$$I_P'(A^{**}) + I_C'(A^{**}) = 0$$

Using the parent's FOC, let's differentiate with respect to A (remember that this FOC holds for all A):

$$v''(I_P(A) - B(A)) \left[I_P'(A) - B'(A) \right] = \delta u''(I_C(A) + B) \left[I_C'(A) + B'(A) \right]$$

If we let $A = A^*$, then by the child's FOC, the RHS becomes 0. Therefore, we get:

$$v''(I_P(A^*) - B(A^*)) \left[I_P'(A^*) - B'(A^*) \right] = 0$$

Since v is strictly concave, then $v'' < 0$ and so:

$$I'_P(A^*) - B'(A^*) = 0$$
$$\therefore B'(A^*) - I'_P(A^*) = 0$$

Therefore, we have that at A^*:

$$I'_C(A^*) + B'(A^*) = B'(A^*) - I'_P(A^*)$$
$$\implies I'_P(A^*) + I'_C(A^*) = 0$$

Therefore, A^* satisfies the same equation as A^{**}. Since $I_P(A)$ and $I_C(A)$ are both strictly concave, so is their sum, which has a unique global maximizer. Therefore, $A^* = A^{**}$, which is what we wanted to show.

Using the example functions, let's first solve the game. The parent's FOC is:

$$\frac{1}{A(30 - A) - B} = \frac{\delta}{A(10 - A) + B}$$
$$A(10 - A) + B = \delta A(30 - A) - \delta B$$
$$B(1 + \delta) = \delta A(30 - A) - A(10 - A)$$
$$\therefore B(A) = \frac{1}{1 + \delta}\left[A^2(1 - \delta) + A(30\delta - 10)\right]$$

The child's FOC is:

$$\frac{1}{A(10 - A) + B(A)}\left[10 - 2A + B'(A)\right] = 0$$
$$10 - 2A + \frac{2(1 - \delta)}{1 + \delta}A + \frac{30\delta - 10}{1 + \delta} = 0$$
$$(1 + \delta)(10 - 2A) + 2(1 - \delta)A + 30\delta - 10 = 0$$
$$40\delta - 4\delta A = 0$$
$$\therefore A^* = 10$$

Finally, let's check that 10 does indeed maximize family income:

$$10 - 2A + 30 - 2A = 0$$
$$40 = 4A$$
$$\therefore A^{**} = 10$$

As expected, it does!