Intermediate Micro: Recitation 11
Producer Theory

Motaz Al-Chanati

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1 Producer's Problem

1.1 Notation

In the first part of the course, we focused on modeling a consumer's decision making process. For this, we had the *utility maximization problem*, where consumers chose a bundle of good that maximized their utility subject to a budget constraint. Now, we move onto the other side of the market: producers. One thing you will quickly realize is that many of the concepts we have already covered come up again when we study producers. For a firm, their problem involves trying to maximize their profit subject to the production technology they have available. Naturally, this is the *profit maximization problem*. So, a consumer wants to maximize utility, while a producer wants to maximize profits.

Let's establish the notation for this section:

- A firm uses two input goods, called x_1 and x_2, to create an output good, called q
- The way a firm transforms the inputs into the output is through the *production function*, denoted by $f(x_1, x_2)$
- The inputs have a cost (price) of p_1 and p_2, respectively and the output good has a price p
- The firm's *profit* is denoted by π

Profit is defined as revenue from selling the output good minus the costs of the inputs it uses. Using the notation above, we can write it as follows:

$$\pi = \underbrace{pq}_{\text{Revenue}} - \underbrace{p_1x_1}_{\text{Cost of Input 1}} - \underbrace{p_2x_2}_{\text{Cost of Input 2}}$$

Moreover, since the output good is created using the production function, we can write the firm's profit maximization problem as:

$$\max_{q,x_1,x_2} pq - p_1x_1 - p_2x_2$$
$$\text{s.t. } q = f(x_1, x_2)$$

So the firm has to choose the quantity of inputs and outputs in order to maximize its profits, subject to the technological constraint of its production function. Of course, the firm doesn't really choose q: once you choose x_1 and x_2, the amount of q is automatically decided by the production function. So we can actually just plug the constraint straight into the profit function, giving us the following problem:

$$\max_{x_1,x_2} pf(x_1, x_2) - p_1x_1 - p_2x_2$$

Notice that unlike the consumer's problem, this is a *unconstrained* problem. We'll talk about how to solve this later.

Commonly, we will have our input goods be capital (K) and labor (L). The price of capital is rent, which we denote as r. The price of labor are wages, which we denote as w. So, in those situations, the profit maximization problem would be:

$$\max_{K,L} pf(K, L) - rK - wL$$

1.2 Production Function

Just like consumer's had utility functions, firms will have production functions. A lot of the same ideas will apply here. For example, we will see three types of production functions:

- **Cobb-Douglas:** $f(x_1, x_2) = x_1^\alpha x_2^\beta$
- **Perfect Compliments:** $f(x_1, x_2) = \alpha x_1 + \beta x_2$
- **Perfect Substitutes:** $f(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$

Of course, these are exactly the same as what we saw with utility functions!

We can also graphically represent production functions, just like we did with utility functions. With consumers, any two bundles that are on the same indifference curve give the consumer the same utility level. For producers, instead of drawing indifference curves, we draw *iso-quants.* "Iso" means same and "quant" means quantity, so "iso-quant" literally means "same quantity". In other words, any two bundles that are on the same iso-quant give the producer the same output quantity amount. When you draw the iso-quant, it is exactly the same as plotting indifference curves.

With utility functions, a useful concept was the marginal rate of substitution (MRS). This told us how many units of x_2 the consumer could give up in order to get one more unit of x_1 and be at the same utility as before. For production functions, we will have the *marginal rate of technical substitution* (MRTS), but the interpretation is similar. The MRTS tells us how many units of x_2 the producer can replace with one more unit of x_1 and be at the same production level as before. Just like the MRS, the MRTS is also the slope of the iso-quant and can be calculated as follows:

$$MRTS = -\frac{MP_1}{MP_2} = -\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}}$$

The MRS was the ratio of the marginal utilities (MU). The MRTS is the ratio of the *marginal products* (MP).

There is one major difference between production and utility, though. With utility functions, we could always do a monotonic transformation and it would not affect the consumer's problem. The reason we could do this is because utility values have no meaningful interpretation (e.g. what does it mean to have a utility of 4?). However, production functions do have a meaningful interpretation: they tell us the *physical quantity* of output produced. For example, doubling a production function literally means that the firm now produces twice as much as before (e.g. it has opened up twice as many factories). Obviously doing such a monotonic transformation completes changes the firm's profits and its entire problem!

1.3 Returns to Scale

Since we care about the quantity of output produced, one important feature of production functions we will focus on are its *returns to scale*. The intuition for this is simple: if we double the amount of inputs, will we get double the amount of output? More than double? Less than double? Mathematically, we start with $q = f(x_1, x_2)$. Then suppose we scale up both inputs by a factor of $\lambda > 1$. We want to compare these two values:

$$f(\lambda x_1, \lambda x_2) \text{ vs. } \lambda f(x_1, x_2)$$

Depending on how they compare, we can categorize the production function:

- **Increasing Returns to Scale (IRTS):** $f(\lambda x_1, \lambda x_2) > \lambda f(x_1, x_2)$
  - If we scale the inputs up by λ, the output amount scales up by *more than* λ
- **Constant Returns to Scale (CRTS):** $f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2)$
  - If we scale the inputs up by λ, the output amount scales up by *exactly* λ
- **Decreasing Returns to Scale (DRTS):** $f(\lambda x_1, \lambda x_2) < \lambda f(x_1, x_2)$
  - If we scale the inputs up by λ, the output amount scales up by *less than* λ

It's really important that when calculating the returns to scales that you scale *all* the inputs.

2 Examples

2.1 Cobb-Douglas

Consider the following production function:

$$f(K, L) = K^{\alpha} L^{\beta}$$

This production function suggests that mixing capital and labor is essential to produce output. Think of this like baking a cake, where the capital is the oven (and the cake ingredients) and the labor is bakers. With only bakers and no oven, there is no way to bake the cake mixture. With only ovens and no bakers, there is no one to put the cake mixture in and out of the oven. Ideally, we want some mix of the two inputs.

What happens if we double the number of bakers and ovens? To answer this, we need to check the returns to scale:[1]

$$\begin{aligned} f(\lambda K, \lambda L) &= (\lambda K)^{\alpha} (\lambda L)^{\beta} \\ &= \lambda^{\alpha} K^{\alpha} \lambda^{\beta} L^{\beta} \\ &= \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} \\ &= \lambda^{\alpha+\beta} f(K, L) \end{aligned}$$

So, the returns to scale depends on the exponents α and β! In particular, for a production function $f(K, L) = K^{\alpha} L^{\beta}$, we have:

- If $\alpha + \beta > 1$, the production function exhibits IRTS
- If $\alpha + \beta = 1$, the production function exhibits CRTS
- If $\alpha + \beta < 1$, the production function exhibits DRTS

Notice, that unlike utility functions, now we really care about the value of the exponents (just looking at the ratio isn't enough).

Let's go through a few examples:

1. $f(K, L) = KL$
   - $MRTS = \frac{L}{K}$
   - Return to scales: IRTS $(1 + 1 = 2)$
2. $f(K, L) = K^2 L$
   - $MRTS = \frac{2L}{K}$

[1] Don't make the following mistake: $f(\lambda K, \lambda L) = \lambda K^{\alpha} \lambda L^{\beta}$! A useful trick is to just replace K with (λK) and L with (λL). It's very important to include the brackets!

- Return to scales: IRTS $(2+1=3)$

3. $f(K,L) = K^{0.25}L^{0.5}$
   - $MRTS = \frac{L}{2K}$
   - Return to scales: DRTS $(0.25+0.5=0.75)$

4. $f(K,L) = K^{0.5}L^{0.5}$
   - $MRTS = \frac{L}{K}$
   - Return to scales: CRTS $(0.5+0.5=1)$

5. $f(K,L) = 3K^{0.5}L^{0.5}$
   - $MRTS = \frac{L}{K}$
   - Return to scales: CRTS $(3(\lambda K)^{0.5}(\lambda L)^{0.5} = \lambda 3K^{0.5}L^{0.5} = \lambda f(K,L))$

2.2 Perfect Complements

Consider the following production function:

$$f(K,L) = \min\{\alpha K, \beta L\}$$

The interpretation here is that β units of x_1 have to be combined with α units of x_2 to produce an extra unit of output. This production function suggests that you need to have capital and labor in an exact proportion. For example, suppose your output is deliveries, the capital input is cars, and the labor input is drivers. This suggests a 1:1 ratio (one driver per car) . Having extra drivers doesn't help you because they don't have a car to drive. Similarly, having extra cars doesn't help because there isn't anyone to drive it.

In terms of returns to scale, let's first start with our initial production function:

$$f(K,L) = \min\{\alpha K, \beta L\} = \begin{cases} \alpha K & \text{if } \alpha K < \beta L \\ \alpha K = \beta K & \text{if } \alpha K = \beta L \\ \beta K & \text{if } \alpha K > \beta L \end{cases}$$

Then, suppose we scale the inputs by λ.

$$f(\lambda K, \lambda L) = \min\{\alpha \lambda K, \beta \lambda L\} = \begin{cases} \lambda \alpha K & \text{if } \alpha K < \beta L \\ \lambda \alpha K = \lambda \beta L & \text{if } \alpha K = \beta L \\ \lambda \beta L & \text{if } \alpha K > \beta L \end{cases}$$

We can see that in all three cases, we have $\lambda f(K,L) = f(\lambda K, \lambda L)$. This means that we have CRTS, regardless of the value of α and β.

2.3 Perfect Substitutes

Consider the following production function:

$$f(K, L) = \alpha K + \beta L$$

The interpretation here is that β units of x_1 are perfectly substitutable with α units of x_2 to produce an extra unit of output. This production function suggests that you can fully substitute between capital and labor. For example, consider an assembly line in a factory. You could have workers physically putting the materials together themselves. Alternatively, you could fully automate the process and have machines do everything.

In terms of returns to scale:

$$\begin{aligned} f(\lambda K, \lambda L) &= \alpha\lambda K + \beta\lambda L \\ &= \lambda\left(\alpha K + \beta L\right) \\ &= \lambda f(K, L) \end{aligned}$$

Again, we will get CRTS regardless of the value of α and β.

2.4 Other Functions

As a final test, let's consider functions outside of the three formats we're used to.

$f(x_1, x_2) = 2x_1\sqrt{x_2}$

- $MRTS = \frac{2\sqrt{x_2}}{2\frac{\frac{1}{2}x_1}{\sqrt{x_2}}} = \frac{2x_2}{x_1}$
- Returns to scale: $f(\lambda x_1, \lambda x_2) = 2\left(\lambda x_1\right)\sqrt{\left(\lambda x_2\right)} = \lambda\sqrt{\lambda}\cdot 2x_1\sqrt{x_2} = \lambda^{1.5}f(x_1, x_2)$. Hence, IRTS (note this is just Cobb-Douglas!)

$f(x, y, z) = xy^2z^3$

- Returns to scale: $f(\lambda x, \lambda y, \lambda z) = \left(\lambda x\right)\left(\lambda y\right)^2\left(\lambda z\right)^3 = \lambda^{1+2+3}xy^2z^3 = \lambda^6 f(x, y, z)$. Hence, IRTS

$f(x_1, x_2) = x_1^2 + x_2^2$

- $MRTS = \frac{2x_1}{2x_2} = \frac{x_1}{x_2}$

- Returns to scale: $f(\lambda x_1, \lambda x_2) = (\lambda x_1)^2 + (\lambda x_2)^2 = \lambda^2 x_1^2 + \lambda^2 x_2^2 = \lambda^2 \left(x_1^2 + x_2^2\right) = \lambda^2 f(x_1, x_2)$. Hence, IRTS

$f(x_1, x_2) = (x_1^3 + x_2^3)^{1/3}$

- $MRTS = \frac{\frac{1}{3}(x_1^3+x_2^3)^{-2/3} 3x_1^2}{\frac{1}{3}(x_1^3+x_2^3)^{-2/3} 3x_2^2} = \frac{x_1^2}{x_2^2}$
- Returns to scale: $f(\lambda x_1, \lambda x_2) = \left((\lambda x_1)^3 + (\lambda x_2)^3\right)^{1/3} = \left(\lambda^3 x_1^3 + \lambda^3 x_2^3\right)^{1/3} = \left(\lambda^3 \left(x_1^3 + x_2^3\right)\right)^{1/3} = \lambda \left(x_1^3 + x_2^3\right)^{1/3} = \lambda f(x_1, x_2)$. Hence, CRTS

$f(x_1, x_2) = \sqrt{x_1} + x_2$

- $MRTS = \frac{\frac{1}{\sqrt{x_1}}}{1} = \sqrt{x_1}$
- Returns to scale: $f(\lambda x_1, \lambda x_2) = \sqrt{\lambda x_1} + \lambda x_2 = \lambda \left(\frac{1}{\sqrt{\lambda}}\sqrt{x_1} + x_2\right) < \lambda \left(\sqrt{x_1} + x_2\right) = \lambda f(x_1, x_2)$. Hence, DRTS.
  - This is assuming $\lambda > 1$ so that $\frac{1}{\sqrt{\lambda}} < 1$

$f(x_1, x_2) = \ln x_1 + 2 \ln x_2$

- $MRTS = \frac{\frac{1}{x_1}}{\frac{2}{x_2}} = \frac{x_2}{2x_1}$
- Returns to scale: $f(\lambda x_1, \lambda x_2) = \ln(\lambda x_1) + 2\ln(\lambda x_2)$. This is hard to simplify, so let's just compare directly to $\lambda f(x_1, x_2)$

$$\begin{aligned} \lambda f(x_1, x_2) &= \lambda \ln x_1 + 2\lambda \ln x_2 \\ &= \ln x_1^\lambda + 2 \ln x_2^\lambda \end{aligned}$$

  For any $\lambda > 1$, we have:

$$\begin{aligned} \lambda x &< x^\lambda \\ \lambda &< x^{\lambda - 1} \\ x &> \lambda^{\frac{1}{\lambda-1}} \end{aligned}$$

  - Therefore, if we have $x > \lambda^{\frac{1}{\lambda-1}}$, then we have $\lambda x < x^\lambda$, and so $f(\lambda x_1, \lambda x_2) < \lambda f(x_1, x_2)$. Hence, DRTS in this range.
  - However, if we have $x < \lambda^{\frac{1}{\lambda-1}}$ then we have $\lambda x > x^\lambda$, and so $f(\lambda x_1, \lambda x_2) > \lambda f(x_1, x_2)$. Hence, IRTS in this range.