Advanced Micro: Recitation 13
Final Exam Review

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Questions

1. Suppose we have a risk-averse agent with the following utility function: $u(x) = x + \gamma x^2$
   (a) What is the sign of γ?
   (b) Show that this utility function cannot be used when x is "too large"? (find the bound for x)
   (c) Suppose X is a random variable with mean $\mu = E[X]$ and variance $\sigma^2 = E[X^2] - E[X]^2$. Show that the expected utility $E[u(X)]$ only depends on the mean and variance. Is it increasing or decreasing with respect to μ and σ^2?
   (d) What is the Arrow-Pratt measure of risk aversion for this utility function? How would you categorize this agent?
2. Prove the following:
   (a) Prove that $CE(g) < E[g]$ for all non-degenerate lotteries $g \in \mathcal{G}$ is a necessary and sufficient condition for risk aversion
   (b) Consider two agents (with preferences over lotteries $\succsim_1$ and $\succsim_2$), where for all lotteries, we have $CE_1(g) \leq CE_2(g)$. For any lottery g and degenerate lottery c, show that $g_1 \succsim_1 c$ implies $g \succsim_2 c$.
   (c) Suppose a consumer's preferences over wealth gambles can be represented by a twice differentiable VNM utility function. Show that the consumer's preferences over gambles are independent of initial wealth if and only if the utility function displays constant absolute risk aversion.
3. Two players are bargaining on how to split one dollar. They each simultaneously announce a share $s_i \in [0, 1]$. If $s_1 + s_2 \leq 1$, then they receive their announced shares. If $s_1 + s_2 > 1$, then they both receive 0. Let $u_i(x) = x$ so that their payoffs are their utilities.
   (a) What are the strictly dominated strategies for each player?
   (b) What are the weakly dominated strategies for each player?
   (c) What are the PSNE of this game?
   (d) Redo parts (a)-(c) but now allow for each $s_i \in [0, \infty)$. How do your answers change?

4. Consider this extension of the first price auction we saw in Homework 9. This is called the common value first price auction. Suppose there are two bidders in a first price auction. Each bidder i observes their private type t_i, that is independently drawn from a uniform distribution over $[0, 1]$. The actual value of the object to each bidder is $t_1 + t_2$.

   (a) Find a symmetric equilibrium where each player uses a strategy of the form: $b_i(t_i) = \alpha + \beta t_i$, where α and β are constants (and the same for the two players)

   (b) Suppose instead that each player's valuation of the good was $\frac{1}{2} + t_i$. Redo part (a).

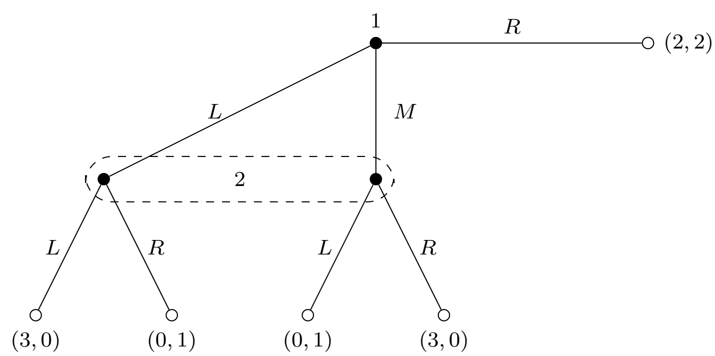
   (c) Show that your answer in part (a) is strictly lower that your answer in part (b), for (almost) all the players.

5. A seller is bargaining to sell a good to a buyer. The good is worth \$0 to the seller and \$1 to the buyer. The bargaining is in two periods. In period 1, one party is chosen with probability $\frac{1}{2}$ to propose a sale price. The other party then accepts or rejects. Acceptance results in sale with the proposed price. Rejection moves the game to the second period. In the second period, one party is again chosen with probability $\frac{1}{2}$ to propose a sale price, and the other party accepts or rejects. Acceptance results in sale at the proposed price. Rejection results in no sale, in which case each player receives zero payoff. Both players discount period 2 payoffs with a discount factor $\delta \in (0, 1)$: if a sale occurs at price p_1 in period 1, then the seller and the buyer collect p_1 and $1 - p_1$, respectively; but if a sale occurs at price p_2 in period 2, the parties collect δp_2 and $\delta(1 - p_2)$, respectively.

   (a) Draw the game tree

   (b) Find all the subgame perfect Nash equilibria

6. Find the sequential equilibria of the following game



7. Prove the following statements:

   (a) Prove that a player can have at most one strictly dominant strategy.

   (b) Prove that a player can have at most one weakly dominant strategy.

   (c) Prove that every finite game has a mixed strategy subgame perfect Nash equilibrium.

   (d) Show that if player j has two weakly dominant strategies, then for every opponent strategy choice s_{-j}, the two strategies yield equal payoffs for player j

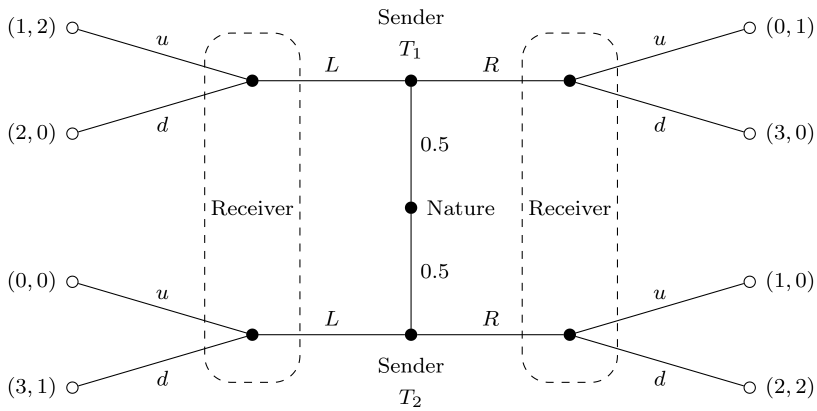
   (e) Consider a 2-player strategic game that is symmetric, i.e. $S_1 = S_2$ and $u_i(s_i, s_{-i}) = u_j(s_j, s_{-j})$ for $s_i = s_j$ and $s_{-i} = s_{-j}$.

i. Suppose that S_i is nonempty, convex, and compact and $u_i(\cdot)$ is continuous and quasi-concave on S_i, for all i. Show that there exists a symmetric pure strategy Nash equilibrium

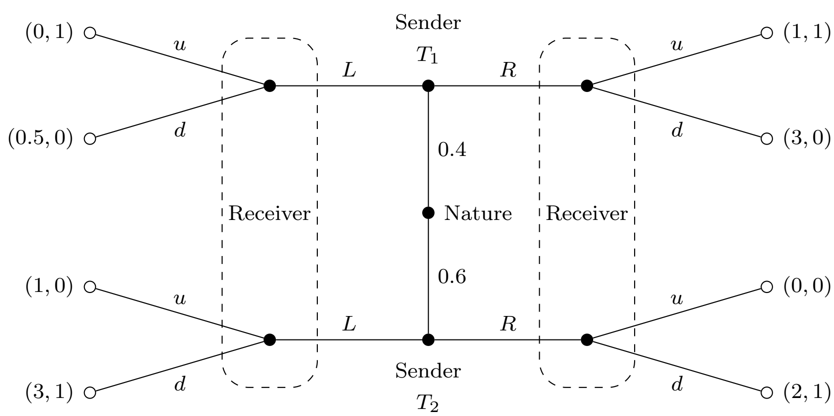
ii. Using a counter-example, show that not every symmetric game has a symmetric PSNE (hint: consider a discrete game, which does not impose the assumptions on S_i as we did in (i))

8. For each game, find the separating and pooling equilibriums:

(a)



(b)



Solutions

Exercise 1

(a) Since the agent is risk-averse, the utility function is concave. To have this we need $u''(x) = 2\gamma < 0$. This happens if and only if $\gamma < 0$.

(b) The first derivative is $\frac{\partial u}{\partial x} = 1 + 2\gamma x$. To ensure $\frac{\partial u}{\partial x} > 0$ (i.e. that utility is increasing in x), we need $x < -\frac{1}{2\gamma}$.

(c) $E[u(X)] = E\left[X + \gamma X^2\right] = E\left[X\right] + \gamma E\left[X^2\right]$. Adding and subtracting $\gamma E\left[X\right]^2$ gives us:

$$\begin{aligned} E[u(X)] &= E[X] + \gamma\left(E\left[X^2\right] - E\left[X\right]^2 + E\left[X\right]^2\right) \\ &= \mu + \gamma\left(\sigma^2 + \mu^2\right) \end{aligned}$$

From this, we can see that $\frac{\partial E[u(X)]}{\partial \mu} = 1 + 2\gamma\mu$, which is positive if $\mu > -\frac{1}{2\gamma} > 0$. Additionally, $\frac{\partial E[u(X)]}{\partial \sigma^2} = \gamma < 0$.

(d) $R_a(x) = -\frac{u''(x)}{u'(x)} = -\frac{2\gamma}{1+2\gamma x}$. Since we have restricted x to $x < -\frac{1}{2\gamma}$ and $\gamma < 0$, then we have $R_a(x) > 0$. In particular, we have $R_a'(x) = \frac{2\gamma}{(1+2\gamma x)^2}(2\gamma) > 0$. Since the Arrow-Pratt measure is increasing in x, the utility function exhibits IARA (increasing absolute risk aversion)

Exercise 2

(a) First, we show that risk aversion implies $CE(g) < E[g], \forall g$. Take an arbitrary $g \in \mathcal{G}$. By definition of risk aversion, $u(E[g]) > u(g)$. By definition of certainty equivalent, $u(g) = u(CE(g))$.By monotonicity, $E[g] > CE(g)$. Second, we show that $CE(g) < E[g], \forall g$ implies risk aversion. Since u is increasing, apply $u(\cdot)$ to both sides of the inequality to get: $u(CE(g)) < u(E[g])$. By the definition of certainty equivalent, $u(g) = u(CE(g))$,so we get $u(g) < u(E[g])$, which is exactly the definition of risk aversion.

(b) By the definition of certainty equivalent, $u_1(CE_1(g)) = u(g)$. By monotonicity, $u_1(CE_1(g)) = u_1(g) \geq u_1(c)$ implies that $CE_1(g) \geq c$. By assumption, $CE_2(g) \geq CE_1(g) \geq c$, so by monotonicity again, we have $u_2(CE_2(g)) \geq u_2(c)$. Using the definition of certainty equivalent again, this means that $g \succsim_2 c$.

(c) Under wealth level w, the consumer's utility from a lottery g is $u(g + w)$. This means that the agent's risk aversion measure is: $R_a(x, w) = -\frac{u''(x+w)}{u'(x+w)} = R_a(x + w)$. CARA means that $R_a(x) = \gamma, \forall x$, for some constant γ. $R_a(x, w) = R_a(x + w) = \gamma, \forall w$ is true iff CARA and iff preferences over lotteries are independent of wealth (i.e. the Arrow-Pratt measure doesn't change with w).

Exercise 3

First, we answer when $S_i = [0, 1]$:

(a) No strategy is strictly dominated. If $s_{-i} = 1$, then $u(s_i, s_{-i}) = 0, \forall s_i \in [0, 1]$. This means that every strategy is a best response to $s_{-i} = 1$, so it is not strictly dominated.

(b) Only $s_i = 0$ is weakly dominated. For example, it is weakly dominated by $s_i = 1$. If $s_{-i} = 0$, then $u(1, s_{-i}) = 1 > 0 = u(0, s_{-i})$. If $s_{-i} > 0$, then $u(1, s_{-i}) = 0 \geq 0 = u(0, s_{-i})$. Since the inequality is weak for all s_{-i} and strict for one, then $s_i = 0$ is weakly dominated. However, no other strategy is weakly dominated. As we saw above, $s_i = 1$ is not weakly dominated. Moreover, for any $s_i \in (0, 1)$, then we can always find some $\varepsilon > 0$ and choose $s_i' = s_i + \varepsilon$ and $s_{-i} = 1 - s_i - \varepsilon$. Then $u(s_i, s_{-i}) = s_i > 0 = u(s_i', s_{-i})$

(c) The best response functions are $BR_i(s_{-i}) = 1 - s_{-i}$ for $s_{-i} \in [0, 1)$. Notice that these lines completely overlap, so any (s_1, s_2) such that $s_1 + s_2 = 1$ are PSNE. However, note that if $s_{-i} = 1$, then any s_i is a best response. That means $(1, 1)$ is another PSNE.

For (d), let's re-do with $S_i = [0, \infty)$:

(a) No strategy is strictly dominated. By the same reasoning as before: if $s_{-i} = 1$, then $u(s_i, s_{-i}) = 0, \forall s_i \in [0, \infty)$. This means that every strategy is a best response to $s_{-i} = 1$, so it is not strictly dominated.

(b) $s_i = 0$ is still weakly dominated, however, so is $s_i \in (1, \infty)$. Call this set $W = \{0\} \cup (1, \infty)$ Again, we can show these are weakly dominated by $s_i = 1$. If $s_{-i} = 0$, then $u(1, s_{-i}) = 1 > 0 = u(s_i, s_{-i})$, $\forall s_i \in W$. If $s_{-i} > 0$, then $u(1, s_{-i}) = 0 \geq 0 = u(0, s_{-i})$, $\forall s_i \in W$. Since the inequality is weak for all s_{-i} and strict for one, then all $s_i \in W$ are weakly dominated. Using the same reasoning as before, $s_i \in (0, 1]$ are not weakly dominated

(c) Now, the best response functions are $BR_i(s_{-i}) = 1 - s_{-i}$ for $s_{-i} \in [0, 1)$ and any strategy for $s_{-i} \in [1, \infty)$. This means that the PSNE are any combinations of (s_1, s_2) where $s_1 + s_2 = 1$ or $\min\{s_1, s_2\} \geq 1$.

Exercise 4

(a) Setting this up just like in the homework, let's look at player 1's expected payoff

$$
\begin{aligned}
u_1\left(b_1, b_2(t_2)\right) &= Pr(\text{Win}) \cdot E[\text{Valuation} \ - \ \text{Bid}|\text{Win}] + Pr(\text{Lose}) \cdot 0 \\
&= Pr\left(b_2(t_2) \leq b_1\right) \cdot E\left[t_1 + t_2 - b_1 |\, b_2(t_2) \leq b\right] \\
&= Pr\left(t_2 \leq \frac{b_1 - \alpha}{\beta}\right) \cdot \left(t_1 - b_1 + E\left[t_2 |\, t_2 \leq \frac{b_1 - \alpha}{\beta}\right]\right) \\
&= \left(\frac{b_1 - \alpha}{\beta}\right) \cdot \left(t_1 - b_1 + \frac{b_1 - \alpha}{2\beta}\right)
\end{aligned}
$$

Notice that we now have an expected utility because player 1 doesn't know their true valuation - however, they can put a bound on it if they know they win.
To find the best response, we take the FOC:

$$
\begin{aligned}
0 &= \frac{\partial u_1}{\partial b_1} \\
&= \frac{1}{\beta} \cdot \left(t_1 - b_1 + \frac{b_1 - \alpha}{2\beta}\right) + \left(\frac{b_1 - \alpha}{\beta}\right) \cdot \left(-1 + \frac{1}{2\beta}\right) \\
&= \frac{1}{2\beta^2}\left(2\beta t_1 + (1 - 2\beta)b_1 - \alpha\right) + \frac{1}{2\beta^2}\left(b_1 - \alpha\right)(1 - 2\beta) \\
\implies 2(1 - 2\beta)b_1 &= 2\left(1 - \beta\right)\alpha - 2\beta t_1
\end{aligned}
$$

$$b_1 = \frac{1-\beta}{1-2\beta}\alpha - \frac{\beta}{(1-2\beta)}t_1$$

Since we must have $b_1 = \alpha + \beta t_1$, then it follows that:

$$\begin{aligned}
\beta &= -\frac{\beta}{1-2\beta} \\
2\beta - 2\beta^2 &= 0 \\
2\beta(1-\beta) &= 0 \\
\therefore \beta &= 1 \\
\implies \alpha &= \frac{1-(1)}{1-2(1)}\alpha \\
\therefore \alpha &= 0
\end{aligned}$$

(Note that we've implicitly assumed that $\beta \neq 0$ throughout these calculations).

Therefore, we have $b_i(t_i) = t_i$.

(b) Let's re-do this:

$$\begin{aligned}
u_1\left(b_1, b_2(t_2)\right) &= Pr(\text{Win}) \cdot E[\text{Valuation} \;-\; \text{Bid}|\text{Win}] + Pr(\text{Lose}) \cdot 0 \\
&= Pr\left(b_2(t_2) \leq b_1\right) \cdot \left(\frac{1}{2} + t_1 - b_1\right) \\
&= Pr\left(t_2 \leq \frac{b_1 - \alpha}{\beta}\right) \cdot \left(\frac{1}{2} + t_1 - b_1\right) \\
&= \left(\frac{b_1 - \alpha}{\beta}\right) \cdot \left(\frac{1}{2} + t_1 - b_1\right)
\end{aligned}$$

Now, there is no uncertainty about your valuation, so we can plug it in straight away. To find the best response, we take the FOC:

$$\begin{aligned}
0 &= \frac{\partial u_1}{\partial b_1} \\
&= \frac{1}{\beta} \cdot \left(\frac{1}{2} + t_1 - b_1\right) + \left(\frac{b_1 - \alpha}{\beta}\right) \cdot (-1) \\
&= \frac{1}{\beta}\left(\frac{1}{2} + t_1 - 2b_1 + \alpha\right) \\
\implies 2b_1 &= \frac{1}{2} + t_1 + \alpha \\
b_1 &= \frac{1}{4} + \frac{\alpha}{2} + \frac{1}{2}t_1
\end{aligned}$$

Since we must have $b_1 = \alpha + \beta t_1$, then it follows that:

$$\begin{aligned}
\beta &= \frac{1}{2} \\
\alpha &= \frac{1}{4} + \frac{\alpha}{2} \\
\frac{\alpha}{2} &= \frac{1}{4}
\end{aligned}$$

$$\therefore \alpha = \frac{1}{2}$$

Therefore, we have $b_i(t_i) = \frac{1}{2} + \frac{1}{2}t_i$.

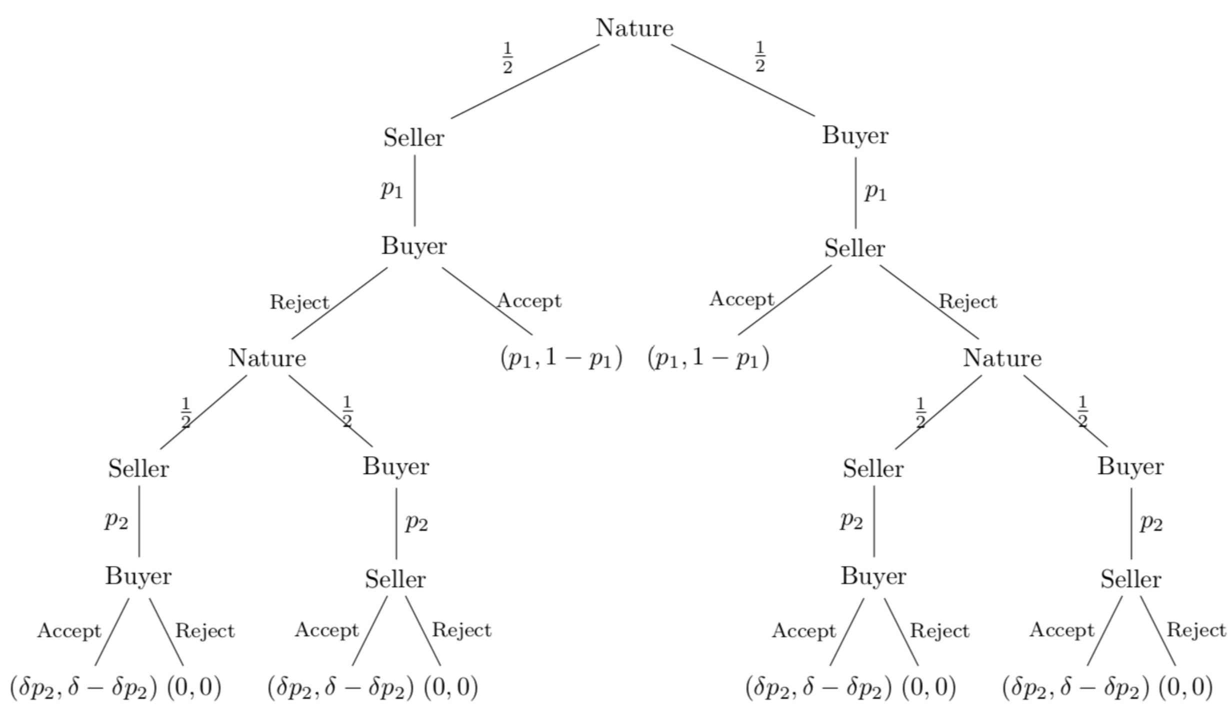
(c) Comparing the bids:

$$\begin{aligned} t_i &\leq \frac{1}{2} + \frac{1}{2}t_i \\ \frac{1}{2}t_i &\leq \frac{1}{2} \\ t_i &\leq 1 \end{aligned}$$

Even though their (unconditional) expected valuations are the same in the two cases, the bid is lower in the first case. The idea here is that in common value auctions, there is the winner's curse. If you win, then you realize that it was because you over-bid. This results in players lowering their bids as compared to the private values.

Exercise 5

The game tree looks as follows: [note, payoffs are in the form (seller, buyer)]



To find the SPNE, let's work backwards:

- At the last nodes, the buyer will accept if and only if $\delta - \delta p_2 \geq 0 \implies p_2 \leq 1$ and the seller will accept if and only if $\delta p_2 \geq 0 \implies p_2 \geq 0$.

- In the nodes preceding them, the seller will offer $p_2 = 1$ if they get to make the offer and the buyer will offer $p_2 = 0$. This guarantees that their offer is accepted by the next player, i.e. the player making the offer gets a payoff of δ (which is better than getting 0). The player not making the offer gets 0.

- In period 1, let's first consider the left node.

  - At the last node in period 1, the buyer compares accepting the offer (and getting $1-p_1$) or rejecting it and getting the expected value of $\frac{1}{2}(0) + \frac{1}{2}(\delta) = \frac{\delta}{2}$. Therefore, the buyer accepts if and only if $1 - p_1 \geq \frac{\delta}{2} \implies p_1 \leq 1 - \frac{\delta}{2}$.
  - Therefore, the seller makes an offer of $p_1 = 1 - \frac{\delta}{2}$, which will get them a payoff of p_1. If they make a lower offer $p_1^L < p_1$, then the buyer accepts and the seller's payoff is lower ($p_1^L < p_1$). If they make a higher offer $p_1^H > p_1$, then the buyer rejects and the seller's expected payoff will be $\frac{\delta}{2}$, which will again be lower ($\delta \in (0,1) \implies p_1 = 1 - \frac{\delta}{2} > \frac{\delta}{2}$).

- The same idea goes on the right node

  - The seller accepts if and only if $p_1 \geq \frac{\delta}{2}$.
  - The buyer proposes $p_1 = \frac{\delta}{2}$

Let's define the strategies as $s_i = (s_{FO}, s_{FA}, s_{LO}, s_{LA}, s_{RO}, s_{RA})$, where the first letter indicates the period (F=first period, L=second period, left branch, R=second period, right branch) and the letter denotes whether the player is making the offer (O) or accepting (A). You should note that this corresponds to the six information sets of each player.

Therefore, the SPNE is $s_S = \left(1 - \frac{\delta}{2}, \text{Accept}, 1, \text{Accept}, 1, \text{Accept}\right)$ and $s_B = \left(\frac{\delta}{2}, \text{Accept}, 0, \text{Accept}, 0, \text{Accept}\right)$

Exercise 6

Strategies are (s_1, s_2), where $s_1 \in \{L, M, R\}$ and $s_2 \in \{L, R\}$

Call the belief μ.

Player 2 plays L if and only if:

$$\begin{aligned}
E[u_2(L)|I,\mu] &\geq E[u_2(R)|I,\mu] \\
\mu(0) + (1-\mu)(1) &\geq \mu(1) + (1-\mu)(0) \\
1 - \mu &\geq \mu \\
\mu &\leq \frac{1}{2}
\end{aligned}$$

So for $\mu \in \left[0, \frac{1}{2}\right)$ we have $s_2 = L$; for $\mu \in \left(\frac{1}{2}, 1\right]$ we have $s_2 = R$; at $\mu = \frac{1}{2}$, player 2 is indifferent between the two strategies.

Next, let's consider player 1's best response. If $s_2 = L$, then player 1 should play L. If $s_2 = R$, then player 1 should play M.

However, both of these generate inconsistent beliefs. If we have (L, L), then $\mu = 1$, so player 2 should deviate to R. If we have (M, R), then $\mu = 0$, so player 2 should deviate to L.

Let's consider a mixed equilibrium then. Suppose player 1's strategy is $m_1 = (p_L, p_M, p_R)$ and player 2's strategy is $(q_L, q_M) = (q, 1-q)$.

To have a mixed strategy, this means we have $\mu = \frac{1}{2}$ and player 2 puts probability q on playing L (and $1-q$ for playing R). To have these beliefs consistent, player 1 could be doing the following:

- Mixing equally on L and M: $m_1 = \left(\frac{p}{2}, \frac{p}{2}, 1-p\right)$
- Only playing R (so μ is off-path): $m_1 = (0,0,1)$ - but notice that this is just a special case of the above, where $p = 0$

Calculating player 1's payoff from each strategy gives us:

$$\begin{aligned} E[u_1(L, m_2)] &= q(3) + (1-q)(0) = 3q \\ E[u_1(M, m_2)] &= q(0) + (1-q)(3) = 3 - 3q \\ E[u_1(R, m_2)] &= 2 \end{aligned}$$

For player 1 to be indifferent between L and M, we need $q = \frac{1}{2}$. But this gives a payoff of 1.5, so player 1 is better off just choosing R. This means that we cannot have player 1 mixing across all three strategies, which means the only possible strategy in SE is if player 1 plays R. In that case, any action by player 2 is a best response. But to ensure that player 1 does not deviate, we need to have $3q < 2$ and $3 - 3q < 2$, which occurs for $q \in \left(\frac{1}{3}, \frac{2}{3}\right)$.

Therefore, the sequential equilibrium are of the form $((0,0,1), (q, 1-q))$ with $\mu = \frac{1}{2}$ and $q \in \left(\frac{1}{3}, \frac{2}{3}\right)$. To confirm this, we need to find a fully mixed strategy sequence. The following would work:

$$\begin{aligned} &Pr(L) = \varepsilon_k && Pr(M) = \varepsilon_k && Pr(R) = 1 - 2\varepsilon_k \\ &Pr(L) = q && Pr(R) = 1 - q \end{aligned}$$

The beliefs consistent with this would be:

$$\mu_k = \frac{Pr(s_1 = L)}{Pr(s_1 = L \text{ or } M)} = \frac{\varepsilon_k}{2\varepsilon_k} = \frac{1}{2} \to \frac{1}{2} = \mu$$

Exercise 7

(a) Suppose not and that player i has two strategies s_i and s_i' that are both strictly dominant (where $s_i \neq s_i'$). Then for any s_{-i}, we must have that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ and $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. Putting this together, we get $u_i(s_i, s_{-i}) > u_i(s_i, s_{-i})$, which is a contradiction.

(b) Suppose not and that player i has two strategies s_i and s_i' that are both weakly dominant (where $s_i \neq s_i'$). Then for any s_{-i}, we must have that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$. Moreover, there must exist some t_{-i} such that $u_i(s_i, t_{-i}) > u_i(s_i', t_{-i})$. But this violates the previous statement, because we must also have $u_i(s_i', t_{-i}) \geq u_i(s_i, t_{-i})$.

(c) Recall that we have seen that every finite game has a mixed strategy Nash equilibrium (MSNE). If the game has no proper subgames, then the set of NE and the set of SPNE are identical, so the claim is proven. So let's only consider the case where the game has proper subgames. In that case, take the all proper subgames that themselves contain no proper subgames. They must exist because the game is finite. Moreover, since the whole game is finite, these subgames must be finite too. Therefore, by there must be a MSNE of the subgame. Using the same algorithm we used in class, we replace the node where the subgame starts with the payoff from one of the MSNE we found. Then we repeat this process by again with every "last" subgame. Since the game is finite, this procedure will end. Taking the MSNEs that we find in each subgame, we can construct a mixed strategy equilibrium that is an SPNE.

(d) Suppose that s_j^1 and $s_j^2 \in S_1$ are two weakly dominant strategies for player j. This means that for $i = 1, 2$, we have: $u_1(s_j^i, s_{-j}) \geq u_1(t_j, s_{-j}), \forall t_j \in S_j$ and $\forall s_{-j} \in S_{-j}$. For $i = 1$ let $t_j = s_j^2$ and for $i = 2$, let $t_j = s_j^1$. This gives us two inequalities that hold true $\forall s_{-j} \in S_{-j}$:

$$u_1(s_j^1, s_{-j}) \geq u_1(s_j^2, s_{-j})$$
$$u_1(s_j^2, s_{-j}) \geq u_1(s_j^1, s_{-j})$$

Therefore, we must have $u_1(s_j^1, s_{-j}) = u_1(s_j^2, s_{-j})$, $\forall s_{-j} \in S_{-j}$

(e) i. For player i, define the function $f_i : S_i \to S_i$ such that $f_i(x) = BR_i(x)$. In other words, it's just the best response function, which we can do since the strategy spaces are symmetric. We know that under the given assumptions, the best response function is convex-valued and has the closed graph property. By Kakutani, this guarantees that it has a fixed point, i.e. $\exists s^*$ such that $f_i(s^*) = s^*$. Therefore, for each player, s^* is a best response to the opponent playing s^*. By definition, this means that $(s_1, s_2) = (s^*, s^*)$ is a Nash equilibrium

ii. You can come up with many examples. Here is one (essentially it is an *anti*-coordination game)

	L	R
L	0, 0	1, 1
R	1, 1	0, 0

The PSNEs here are (R, L) and (L, R), neither of which is symmetric.

Exercise 8

Let's call the receiver's belief that the sender is type T_1 given that they receive the signal L as λ. And the corresponding belief for receiving the signal R is μ. Let's also call the sender's strategies as $m_{s1} = (p, 1-p)$ for T_1 and $m_{s2} = (q, 1-q)$ for T_2, which represent the weights on L and R, respectively. We'll call the receiver's strategy as $m_{rL} = (a, 1-a)$ and $m_{rR} = (b, 1-b)$, corresponding to u and $d$for the L and R information sets, respectively.

Game (a)

For the receiver, their expected payoffs are:

$$E[u|L] = \lambda(2) + (1-\lambda)(0) = 2\lambda$$
$$E[d|L] = \lambda(0) + (1-\lambda)(1) = 1-\lambda$$
$$E[u|R] = \mu(1) + (1-\mu)(0) = \mu$$
$$E[d|R] = \mu(0) + (1-\mu)(2) = 2-2\mu$$

This tells us they play u after L (i.e. $a=1$) iff $2\lambda \geq 1-\lambda \implies \lambda \geq \frac{1}{3}$. And they play u after R (i.e. $b=1$) iff $\mu \geq 2-2\mu \implies \mu \geq \frac{2}{3}$.

For the sender, their expected payoffs are:

$$E[L|T_1] = a(1) + (1-a)(2) = 2-a$$
$$E[R|T_1] = b(0) + (1-b)(3) = 3-3b$$
$$E[L|T_2] = a(0) + (1-a)(3) = 3-3a$$
$$E[R|T_2] = b(1) + (1-b)(2) = 2-b$$

This tells us that T_1 plays L (i.e. $p=1$) iff $2-a \geq 3-3b \implies a \leq 3b-1$ or $b \geq \frac{1}{3}(a+1)$. Also, this tells us that T_2 plays L (i.e. $q=1$) iff $3-3a \geq 2-b \implies a \leq \frac{1}{3}(b+1)$ or $b \geq 3a-1$. Call $f(x) = 3x-1$ and $g(x) = \frac{1}{3}(x+1)$.

To summarize:

$$a = 1 \iff \lambda \geq \frac{1}{3} \qquad\qquad p = 1 \iff a \leq f(b) \iff b \geq g(a)$$
$$b = 1 \iff \mu \geq \frac{2}{3} \qquad\qquad q = 1 \iff a \leq g(b) \iff b \geq f(a)$$

Let's consider the separating equilibriums.

- T_1 plays L, T_2 plays R. This corresponds to $p=1, q=0$. This implies that $\lambda = 1$ and $\mu = 0$. This means that $a=1$ and $b=0$. Since $a \not\leq f(0) = -1$ and $a \not\leq g(0) = \frac{1}{3}$, then $p=0$ and $q=0$. This is a contradiction (T_1 deviates), so this cannot be an equilibrium.
- T_1 plays R, T_2 plays L. This corresponds to $p=0, q=1$. This implies that $\lambda = 0$ and $\mu = 1$. This means that $a=0$ and $b=1$. Since $a \leq f(1) = 2$ and $a \leq g(1) = \frac{2}{3}$, then $p=1$ and $q=1$. This is a contradiction (T_1 deviates), so this cannot be an equilibrium.

Therefore, there are no separating equilibrium. Now let's check for pooling.

- Both types play L. This corresponds to $p=1, q=1$. This implies that $\lambda = \frac{1}{2}$, which means that $a=1$. μ is off-path, but since $p=1$ and $q=1$, we need $b \geq g(1) = \frac{2}{3}$ and $b \geq f(1) = 2$. There are no such b values (T_2 deviates), so this cannot be an equilibrium.
- Both types play R. This corresponds to $p=0, q=0$. This implies that $\mu = \frac{1}{2}$, which means that $b=0$. λ is off-path, but since $p=0$ and $q=0$, we need $a \geq f(0) = -1$ and $a \geq g(0) = \frac{1}{3}$. Therefore, any $a \in \left[\frac{1}{3}, 1\right]$ can generate this pooling equilibrium (this prevents T_2 from deviating).

So the only equilibrium is $m_{s1} = (0, 1)$, $m_{s2} = (0, 1)$, $m_{rL} = (a, 1-a)$ for $a \in \left[\frac{1}{3}, 1\right]$, and $m_{rR} = (0, 1)$, and this is a pooling equilibrium. To properly have a sequential equilibrium, we need to find our strategy sequence to show consistent beliefs. Consider the following for $\varepsilon_k \to 0$:

$$
\begin{aligned}
m_{s1} &= (\varepsilon_k, 1-\varepsilon_k) & \qquad m_{rL} &= (a-\varepsilon_k, 1-a+\varepsilon_k) \\
m_{s2} &= (\varepsilon_k, 1-\varepsilon_k) & \qquad m_{rR} &= (\varepsilon_k, 1-\varepsilon_k)
\end{aligned}
$$

Then we get the following beliefs:

$$
\begin{aligned}
\lambda_k &= \frac{0.5\,(\varepsilon_k)}{0.5\,(\varepsilon_k) + 0.5\,(\varepsilon_k)} = \frac{1}{2} \to \frac{1}{2} = \lambda \\
\mu_k &= \frac{0.5\,(1-\varepsilon_k)}{0.5\,(1-\varepsilon_k) + 0.5\,(1-\varepsilon_k)} = \frac{1}{2} \to \frac{1}{2} = \mu
\end{aligned}
$$

So the equilibrium is the listed above strategies with beliefs $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$.

Game (b)

For the receiver, their expected payoffs are:

$$
\begin{aligned}
E[u|L] &= \lambda(1) + (1-\lambda)(0) = \lambda \\
E[d|L] &= \lambda(0) + (1-\lambda)(1) = 1-\lambda \\
E[u|R] &= \mu(1) + (1-\mu)(0) = \mu \\
E[d|R] &= \mu(0) + (1-\mu)(1) = 1-\mu
\end{aligned}
$$

This tells us they play u after L (i.e. $a = 1$) iff $\lambda \geq 1-\lambda \implies \lambda \geq \frac{1}{2}$. And they play u after R (i.e. $b = 1$) iff $\mu \geq 1-\mu \implies \mu \geq \frac{1}{2}$.

For the sender, their expected payoffs are:

$$
\begin{aligned}
E[L|T_1] &= a(0) + (1-a)\left(\frac{1}{2}\right) = \frac{1}{2}(1-a) \\
E[R|T_1] &= b(1) + (1-b)(3) = 3-2b \\
E[L|T_2] &= a(1) + (1-a)(3) = 3-2a \\
E[R|T_2] &= b(0) + (1-b)(2) = 2-2b
\end{aligned}
$$

This tells us that T_1 plays L (i.e. $p = 1$) iff $\frac{1}{2}(1-a) \geq 3-2b \implies a \leq 4b-5$ or $b \geq \frac{1}{4}(a+5)$. Also, this tells us that T_2 plays L (i.e. $q = 1$) iff $3-2a \geq 2-2b \implies a \leq b+\frac{1}{2}$ or $b \geq a-\frac{1}{2}$. Call $f_1(x) = 4x-5$ and $g_1(x) = \frac{1}{4}(x+5)$. Call $f_2(x) = x+\frac{1}{2}$ and $g_2(x) = x-\frac{1}{2}$.

To summarize:

$$
\begin{aligned}
a = 1 &\iff \lambda \geq \frac{1}{2} & \qquad p = 1 &\iff a \leq f_1(b) \iff b \geq g_1(a) \\
b = 1 &\iff \mu \geq \frac{1}{2} & \qquad q = 1 &\iff a \leq f_2(b) \iff b \geq g_2(a)
\end{aligned}
$$

Let's consider the separating equilibriums.

- T_1 plays L, T_2 plays R. This corresponds to $p = 1, q = 0$. This implies that $\lambda = 1$ and $\mu = 0$. This means that $a = 1$ and $b = 0$. Since $a \nleq f_1(0) = -5$ and $a \nleq f_2(0) = \frac{1}{2}$, then $p = 0$ and $q = 0$. This is a contradiction (T_1 deviates), so this cannot be an equilibrium.

- T_1 plays R, T_2 plays L. This corresponds to $p = 0, q = 1$. This implies that $\lambda = 0$ and $\mu = 1$. This means that $a = 0$ and $b = 1$. Since $a \nleq f_1(1) = -1$ and $a \leq f_2(1) = \frac{3}{2}$, then $p = 0$ and $q = 1$. This is what we started with, so this is an equilibrium.

Now let's check for pooling.

- Both types play L. This corresponds to $p = 1, q = 1$. This implies that $\lambda = 0.4$, which means that $a = 0$. μ is off-path, but since $p = 1$ and $q = 1$, we need $b \geq g_1(0) = \frac{5}{4}$ and $b \geq g_2(0) = -\frac{1}{2}$. There are no such b values (T_1 deviates), so this cannot be an equilibrium.

- Both types play R. This corresponds to $p = 0, q = 0$. This implies that $\mu = 0.4$, which means that $b = 0$. λ is off-path, but since $p = 0$ and $q = 0$, we need $a \geq f_1(0) = -5$ and $a \geq f_2(0) = \frac{1}{2}$. Therefore, any $a \in \left[\frac{1}{2}, 1\right]$ can generate this pooling equilibrium (this prevents T_2 from deviating)

The separating equilibrium is $m_{s1} = (0, 1)$, $m_{s2} = (1, 0)$, $m_{rL} = (0, 1)$, and $m_{rR} = (1, 0)$. Consider the following for $\varepsilon_k \to 0$:

$$
\begin{aligned}
m_{s1} &= (\varepsilon_k, 1 - \varepsilon_k) & \qquad m_{rL} &= (\varepsilon_k, 1 - \varepsilon_k) \\
m_{s2} &= (1 - \varepsilon_k, \varepsilon_k) & \qquad m_{rR} &= (1 - \varepsilon_k, \varepsilon_k)
\end{aligned}
$$

Then we get the following beliefs:

$$
\begin{aligned}
\lambda_k &= \frac{0.4\,(\varepsilon_k)}{0.4\,(\varepsilon_k) + 0.6\,(1 - \varepsilon_k)} = \frac{0.4\varepsilon_k}{0.6 - 0.2\varepsilon_k} \to 0 = \lambda \\
\mu_k &= \frac{0.4\,(1 - \varepsilon_k)}{0.4\,(1 - \varepsilon_k) + 0.6\,(\varepsilon_k)} = \frac{0.4 - 0.4\varepsilon_k}{0.4 + 0.2\varepsilon_k} \to 1 = \mu
\end{aligned}
$$

So the separating equilibrium is the listed above strategies with beliefs $\lambda = 0$ and $\mu = 1$.

The pooling equilibrium is $m_{s1} = (0, 1)$, $m_{s2} = (0, 1)$, $m_{rL} = (a, 1 - a)$ for $a \in \left[\frac{1}{2}, 1\right]$, and $m_{rR} = (0, 1)$. Consider the following for $\varepsilon_k \to 0$:

$$
\begin{aligned}
m_{s1} &= (\varepsilon_k, 1 - \varepsilon_k) & \qquad m_{rL} &= (a - \varepsilon_k, 1 - a + \varepsilon_k) \\
m_{s2} &= (\varepsilon_k, 1 - \varepsilon_k) & \qquad m_{rR} &= (\varepsilon_k, 1 - \varepsilon_k)
\end{aligned}
$$

Then we get the following beliefs:

$$
\begin{aligned}
\lambda_k &= \frac{0.4\,(\varepsilon_k)}{0.4\,(\varepsilon_k) + 0.6\,(\varepsilon_k)} = \frac{0.4\varepsilon_k}{\varepsilon_k} \to 0.4 = \lambda \\
\mu_k &= \frac{0.4\,(1 - \varepsilon_k)}{0.4\,(1 - \varepsilon_k) + 0.6\,(1 - \varepsilon_k)} = \frac{0.4 - 0.4\varepsilon_k}{1 - \varepsilon_k} \to 0.4 = \mu
\end{aligned}
$$

So the pooling equilibrium is the listed above strategies with beliefs $\lambda = 0.4$ and $\mu = 0.4$.