

Advanced Micro: Recitation 6

Midterm Review

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1 Calculations Exercise¹

1.1 UMP and EMP

Consider a consumer with the following utility function:

$$u(x_1, x_2) = 2\sqrt{x_1} + 4\sqrt{x_2}$$

Solve the UMP to get the Marshallian demand and indirect utility:

UMP

The problem:

$$\begin{aligned} \max_{x_1, x_2} & 2\sqrt{x_1} + 4\sqrt{x_2} \\ \text{s.t.} & p_1x_1 + p_2x_2 = y \end{aligned}$$

The Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = 2\sqrt{x_1} + 4\sqrt{x_2} + \lambda(y - p_1x_1 - p_2x_2)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{2}{2\sqrt{x_1}} - \lambda p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{4}{2\sqrt{x_2}} - \lambda p_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= y - p_1x_1 - p_2x_2 = 0 \end{aligned}$$

¹Loosely adapted from MWG 3.G.15

Solve for interior solution:

$$\begin{aligned}\frac{\frac{1}{\sqrt{x_1}}}{\frac{2}{\sqrt{x_2}}} &= \frac{\lambda p_1}{\lambda p_2} \\ \frac{\sqrt{x_2}}{2\sqrt{x_1}} &= \frac{p_1}{p_2} \\ x_2 &= \left(\frac{p_1}{p_2}\right)^2 4x_1\end{aligned}$$

$$\begin{aligned}\therefore p_1 x_1 + p_2 \left(\frac{p_1}{p_2}\right)^2 4x_1 &= y \\ \left(\frac{p_1 p_2 + 4p_1^2}{p_2}\right) x_1 &= y \\ \implies x_1(p, y) &= \frac{p_2 y}{4p_1^2 + p_1 p_2} \\ x_2(p, y) &= \left(\frac{p_1}{p_2}\right)^2 4 \left(\frac{p_2}{p_1} \cdot \frac{y}{4p_1 + p_2}\right) \\ &= \frac{4p_1 y}{p_2^2 + 4p_1 p_2}\end{aligned}$$

Get the indirect utility:

$$\begin{aligned}v(p, y) &= 2\sqrt{x_1(p, y)} + 4\sqrt{x_2(p, y)} \\ &= 2\sqrt{\frac{p_2 y}{4p_1^2 + p_1 p_2}} + 4\sqrt{\frac{4p_1 y}{p_2^2 + 4p_1 p_2}} \\ &= 2\sqrt{\frac{p_1^{-1} p_2 y}{4p_1 + p_2}} + 8\sqrt{\frac{p_1 p_2^{-1} y}{4p_1 + p_2}} \\ &= 2\sqrt{\frac{y}{4p_1 + p_2}} \left(\sqrt{p_1^{-1} p_2} + 4\sqrt{p_1 p_2^{-1}} \right) \\ &= 2\sqrt{\frac{y}{4p_1 + p_2}} \left(\frac{\sqrt{p_1 p_2}}{p_1} + 4\frac{\sqrt{p_1 p_2}}{p_2} \right) \\ &= 2\sqrt{\frac{y}{4p_1 + p_2}} \left(\frac{(4p_1 + p_2) \sqrt{p_1 p_2}}{p_1 p_2} \right) \\ &= 2\sqrt{\frac{(4p_1 + p_2) y}{p_1 p_2}} \\ &= 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}}\end{aligned}$$

Should we be worried about corner solutions? The answer is no. To have a corner on the y-axis, we need $|MRS| < \frac{p_1}{p_2}$. To have a corner on the x-axis, we need $|MRS| > \frac{p_1}{p_2}$. However, consider the MRS at a corner. At a y-axis corner, we have $|MRS(0, x_2)| = \infty$, which will never be strictly less than the price ratio. At a x-axis corner, we have $|MRS(x_1, 0)| = 0$, which will never be strictly greater than the price ratio. These Inada conditions guarantee an interior solution.

EMP

The problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & p_1 x_1 + p_2 x_2 \\ \text{s.t.} \quad & 2\sqrt{x_1} + 4\sqrt{x_2} \geq u \end{aligned}$$

The Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda(u - 2\sqrt{x_1} - 4\sqrt{x_2})$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \lambda \frac{2}{2\sqrt{x_1}} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \lambda \frac{4}{2\sqrt{x_2}} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= u - 2\sqrt{x_1} - 4\sqrt{x_2} = 0 \end{aligned}$$

Solve for interior solution:

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{\lambda \frac{1}{\sqrt{x_1}}}{\lambda \frac{2}{\sqrt{x_2}}} \\ \frac{p_1}{p_2} &= \frac{\sqrt{x_2}}{2\sqrt{x_1}} \\ x_2 &= \left(\frac{p_1}{p_2}\right)^2 4x_1 \end{aligned}$$

$$\begin{aligned} \therefore 2\sqrt{x_1} + 4\sqrt{x_2} &= u \\ 2\sqrt{x_1} + 4\sqrt{\left(\frac{p_1}{p_2}\right)^2 4x_1} &= u \\ \left(2 + 8\frac{p_1}{p_2}\right) \sqrt{x_1} &= u \\ \implies x_1^h(p, u) &= \left(\frac{p_2 u}{2(4p_1 + p_2)}\right)^2 \\ x_2^h(p, u) &= \left(\frac{p_1}{p_2}\right)^2 4 \left(\frac{p_2 u}{2(4p_1 + p_2)}\right)^2 \\ &= \left(\frac{p_1 u}{4p_1 + p_2}\right)^2 \end{aligned}$$

Get the expenditure function:

$$e(p, u) = p_1 x_1^h(p, u) + p_2 x_2^h(p, u)$$

$$\begin{aligned}
&= p_1 \left(\frac{p_2 u}{2(4p_1 + p_2)} \right)^2 + p_2 \left(\frac{p_1 u}{4p_1 + p_2} \right)^2 \\
&= \left(\frac{u}{4p_1 + p_2} \right)^2 \left(\frac{1}{4} p_1 p_2^2 + p_1^2 p_2 \right) \\
&= \left(\frac{u}{4p_1 + p_2} \right)^2 \frac{1}{4} p_1 p_2 (p_2 + 4p_1) \\
&= \frac{p_1 p_2 u^2}{4(4p_1 + p_2)}
\end{aligned}$$

1.2 Derivations

1. Derive the Marshallian demand from the indirect utility
2. Derive the Hicksian demand from the expenditure function
3. Derive the expenditure function from the indirect utility
4. Derive the indirect utility from the expenditure function
5. Derive the Slutsky matrix from the Marshallian demand
6. Derive the substitution matrix from the Hicksian demand
7. Derive a utility function from the expenditure function

Answers:

1. The indirect utility is:

$$v(p, y) = 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} = 2\sqrt{\phi(p, y)}$$

To get the Marshallian demands, we use Roy's Identity:

$$x_i(p, y) = - \frac{\frac{\partial v(p, y)}{\partial p_i}}{\frac{\partial v(p, y)}{\partial y}}$$

For good 1:

$$\begin{aligned}
x_1(p, y) &= - \frac{2 \frac{1}{2\sqrt{\phi(p, y)}} \left(-\frac{y}{p_1^2} \right)}{2 \frac{1}{2\sqrt{\phi(p, y)}} \left(\frac{1}{p_1} + \frac{4}{p_2} \right)} \\
&= \frac{\frac{y}{p_1^2}}{\frac{1}{p_1} + \frac{4}{p_2}} \\
&= \frac{y}{p_1^2 \left(\frac{p_2 + 4p_1}{p_1 p_2} \right)} \\
&= \frac{p_2 y}{4p_1^2 + p_1 p_2}
\end{aligned}$$

For good 2:

$$\begin{aligned}
 x_2(p, y) &= -\frac{2\frac{1}{2\sqrt{\phi(p, y)}}\left(-\frac{4y}{p_2}\right)}{2\frac{1}{2\sqrt{\phi(p, y)}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} \\
 &= \frac{\frac{4y}{p_2}}{\frac{1}{p_1} + \frac{4}{p_2}} \\
 &= \frac{4y}{p_2^2\left(\frac{p_2+4p_1}{p_1p_2}\right)} \\
 &= \frac{4p_1y}{p_2^2 + 4p_1p_2}
 \end{aligned}$$

2. The expenditure function is:

$$e(p, u) = \frac{p_1p_2u^2}{4(4p_1 + p_2)}$$

To get the Hicksian demands, we use Shephard's Lemma:

$$x_i^h(p, u) = \frac{\partial e(p, u)}{\partial p_i}$$

For good 1:

$$\begin{aligned}
 x_1^h(p, u) &= \frac{p_2u^24(4p_1 + p_2) - p_1p_2u^24(1)}{(4(4p_1 + p_2))^2} \\
 &= \frac{4p_2u^2(4p_1 + p_2 - 4p_1)}{(4(4p_1 + p_2))^2} \\
 &= \frac{4p_2^2u^2}{(4(4p_1 + p_2))^2} \\
 &= \left(\frac{p_2u}{2(4p_1 + p_2)}\right)^2
 \end{aligned}$$

For good 2:

$$\begin{aligned}
 x_2^h(p, u) &= \frac{p_1u^24(4p_1 + p_2) - p_1p_2u^24(1)}{(4(4p_1 + p_2))^2} \\
 &= \frac{4p_1u^2(4p_1 + p_2 - p_2)}{(4(4p_1 + p_2))^2} \\
 &= \frac{(4p_1)^2u^2}{(4(4p_1 + p_2))^2} \\
 &= \left(\frac{p_1u}{4p_1 + p_2}\right)^2
 \end{aligned}$$

3. The indirect utility is:

$$v(p, y) = 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}}$$

To get the expenditure function, let $v(p, y) = u$ and solve for y (then let $y = e(p, u)$):

$$\begin{aligned} u &= 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \\ \frac{u^2}{4} &= \left(\frac{1}{p_1} + \frac{4}{p_2}\right)y \\ \frac{u^2}{4} &= \left(\frac{4p_1 + p_2}{p_1 p_2}\right)y \\ \therefore e(p, u) &= \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \end{aligned}$$

4. The expenditure function is:

$$e(p, u) = \frac{p_1 p_2 u^2}{4(4p_1 + p_2)}$$

To get the indirect utility, let $e(p, u) = y$ and solve for u (then let $u = v(p, y)$):

$$\begin{aligned} y &= \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\ u &= \sqrt{\frac{4(4p_1 + p_2)y}{p_1 p_2}} \\ &= 2\sqrt{\frac{4y}{p_2} + \frac{y}{p_1}} \\ \therefore v(p, y) &= 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \end{aligned}$$

5. The Slutsky matrix is defined as:

$$\begin{aligned} S(p, y) &= D_p x(p, y) + D_y x(p, y) x(p, y)' \\ &= \begin{pmatrix} \frac{\partial x_1(p, y)}{\partial p_1} + \frac{\partial x_1(p, y)}{\partial y} x_1(p, y) & \frac{\partial x_1(p, y)}{\partial p_2} + \frac{\partial x_1(p, y)}{\partial y} x_2(p, y) \\ \frac{\partial x_2(p, y)}{\partial p_1} + \frac{\partial x_2(p, y)}{\partial y} x_1(p, y) & \frac{\partial x_2(p, y)}{\partial p_2} + \frac{\partial x_2(p, y)}{\partial y} x_2(p, y) \end{pmatrix} \end{aligned}$$

The Marshallian demands are:

$$x_1(p, y) = \frac{p_2 y}{4p_1^2 + p_1 p_2} \quad x_2(p, y) = \frac{4p_1 y}{p_2^2 + 4p_1 p_2}$$

Therefore the Slutsky matrix is:

$$\begin{aligned} S(p, y) &= \begin{pmatrix} -\frac{p_2 y (8p_1 + p_2)}{(4p_1^2 + p_1 p_2)^2} + \frac{p_2}{4p_1^2 + p_1 p_2} \cdot \frac{p_2 y}{4p_1^2 + p_1 p_2} & \frac{y(4p_1^2 + p_1 p_2) - p_2 y(p_1)}{(4p_1^2 + p_1 p_2)^2} + \frac{p_2}{4p_1^2 + p_1 p_2} \cdot \frac{4p_1 y}{p_2^2 + 4p_1 p_2} \\ \frac{4y(p_2^2 + 4p_1 p_2) - 4p_1 y(4p_2)}{(p_2^2 + 4p_1 p_2)^2} + \frac{4p_1}{p_2^2 + 4p_1 p_2} \cdot \frac{p_2 y}{4p_1^2 + p_1 p_2} & -\frac{4p_1 y(2p_2 + 4p_1)}{(p_2^2 + 4p_1 p_2)^2} + \frac{4p_1}{p_2^2 + 4p_1 p_2} \cdot \frac{4p_1 y}{p_2^2 + 4p_1 p_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{p_2^2 y - p_2 y(8p_1 + p_2)}{(4p_1^2 + p_1 p_2)^2} & \frac{4p_1^2 y}{(4p_1^2 + p_1 p_2)^2} + \frac{4p_1 p_2 y}{(4p_1^2 + p_1 p_2)(p_2^2 + 4p_1 p_2)} \\ \frac{4p_2^2 y}{(p_2^2 + 4p_1 p_2)^2} + \frac{4p_1 p_2 y}{(4p_1^2 + p_1 p_2)(p_2^2 + 4p_1 p_2)} & \frac{(4p_1)^2 y - 4p_1 y(2p_2 + 4p_1)}{(p_2^2 + 4p_1 p_2)^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{8p_1 p_2 y}{A^2} & \frac{4p_1^2 y}{A^2} + \frac{4p_1 p_2 y}{AB} \\ \frac{4p_2^2 y}{B^2} + \frac{4p_1 p_2 y}{AB} & -\frac{8p_1 p_2 y}{B^2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{-8p_1p_2y}{A^2} & \frac{(Bp_1+Ap_2)(4p_1y)}{A^2B} \\ \frac{(Bp_1+Ap_2)(4p_2y)}{AB^2} & \frac{-8p_1p_2y}{B^2} \end{pmatrix}$$

Where $A = (4p_1^2 + p_1p_2)$ and $B = (p_2^2 + 4p_1p_2)$. Moreover, note that:

$$\begin{aligned} Bp_1 + Ap_2 &= (p_2^2 + 4p_1p_2)p_1 + (4p_1^2 + p_1p_2)p_2 \\ &= p_1p_2^2 + 4p_1^2p_2 + 4p_1^2p_2 + p_1p_2^2 \\ &= 2p_1p_2(4p_1 + p_2) \end{aligned}$$

$$\begin{aligned} \frac{4p_1}{A} &= \frac{4p_1}{4p_1^2 + p_1p_2} = \frac{4}{4p_1 + p_2} \\ \frac{4p_2}{B} &= \frac{4p_2}{p_2^2 + 4p_1p_2} = \frac{4}{4p_1 + p_2} \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{(Bp_1 + Ap_2)(4p_1y)}{A^2B} &= \frac{2p_1p_2(4p_1 + p_2)}{AB} \cdot \left(\frac{4}{4p_1 + p_2} \right) \cdot y \\ &= \frac{8p_1p_2y}{AB} \\ \frac{(Bp_1 + Ap_2)(4p_2y)}{AB^2} &= \frac{2p_1p_2(4p_1 + p_2)}{AB} \cdot \left(\frac{4}{4p_1 + p_2} \right) \cdot y \\ &= \frac{8p_1p_2y}{AB} \end{aligned}$$

This means that the Slutsky matrix is simply:

$$S(p, y) = \begin{pmatrix} \frac{-8p_1p_2y}{A^2} & \frac{8p_1p_2y}{AB} \\ \frac{8p_1p_2y}{AB} & \frac{-8p_1p_2y}{B^2} \end{pmatrix} = 8p_1p_2y \begin{pmatrix} -\frac{1}{A^2} & \frac{1}{AB} \\ \frac{1}{AB} & -\frac{1}{B^2} \end{pmatrix}$$

You could also write it like this. Let $Q = 4p_1 + p_2$. Then $A = Qp_1$ and $B = Qp_2$ and the Slutsky matrix becomes:

$$S(p, y) = 8p_1p_2y \begin{pmatrix} -\frac{1}{Q^2p_1^2} & \frac{1}{Q^2p_1p_2} \\ \frac{1}{Q^2p_1p_2} & -\frac{1}{Q^2p_2^2} \end{pmatrix} = \frac{8y}{Q^2} \begin{pmatrix} -\frac{p_2}{p_1} & 1 \\ 1 & -\frac{p_1}{p_2} \end{pmatrix}$$

6. The substitution matrix is:

$$\begin{aligned} \sigma(p, u) &= D_p x^h(p, u) \\ &= \begin{pmatrix} \frac{\partial x_1^h(p, u)}{\partial p_1} & \frac{\partial x_1^h(p, u)}{\partial p_2} \\ \frac{\partial x_2^h(p, u)}{\partial p_1} & \frac{\partial x_2^h(p, u)}{\partial p_2} \end{pmatrix} \end{aligned}$$

The Hicksian demands are:

$$x_1^h(p, u) = \left(\frac{p_2 u}{2(4p_1 + p_2)} \right)^2 \quad x_2^h(p, u) = \left(\frac{p_1 u}{4p_1 + p_2} \right)^2$$

Therefore the substitution matrix is:

$$\begin{aligned}
\sigma(p, u) &= \begin{pmatrix} 2 \cdot \frac{p_2 u}{2(4p_1 + p_2)} \cdot \frac{-p_2 u}{2(4p_1 + p_2)^2} \cdot 4 & 2 \cdot \frac{p_2 u}{2(4p_1 + p_2)} \cdot \frac{u \cdot 2(4p_1 + p_2) - p_2 u \cdot 2}{(2(4p_1 + p_2))^2} \\ 2 \cdot \frac{p_1 u}{4p_1 + p_2} \cdot \frac{u \cdot (4p_1 + p_2) - p_1 u \cdot 4}{(4p_1 + p_2)^2} & 2 \cdot \frac{p_1 u}{4p_1 + p_2} \cdot \frac{-p_1 u}{(4p_1 + p_2)^2} \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2p_2^2 u^2}{(4p_1 + p_2)^3} & \frac{\frac{1}{2} p_2 u [u(4p_1 + p_2) - p_2 u]}{(4p_1 + p_2)^3} \\ \frac{2p_1 u [u(4p_1 + p_2) - p_1 u]}{(4p_1 + p_2)^3} & -\frac{2p_1^2 u^2}{(4p_1 + p_2)^3} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2p_2^2 u^2}{(4p_1 + p_2)^3} & \frac{2p_1 p_2 u^2}{(4p_1 + p_2)^3} \\ \frac{2p_1 p_2 u^2}{(4p_1 + p_2)^3} & -\frac{2p_1^2 u^2}{(4p_1 + p_2)^3} \end{pmatrix} \\
&= \frac{2u^2}{(4p_1 + p_2)^3} \begin{pmatrix} -p_2^2 & p_1 p_2 \\ p_1 p_2 & -p_1^2 \end{pmatrix}
\end{aligned}$$

7. We want to find a $u(x) = \max \{u : p \cdot x \geq e(p, u); \forall p \gg 0\}$. This means we need:

$$\begin{aligned}
p_1 x_1 + p_2 x_2 &\geq \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\
\frac{4}{p_1} (4p_1 + p_2) \frac{1}{p_2} (p_1 x_1 + p_2 x_2) &\geq u^2 \\
4 \left(4 + \frac{p_2}{p_1} \right) \left(\frac{p_1 x_1}{p_2} + x_2 \right) &\geq u^2 \\
2 \sqrt{\left(4 + \frac{1}{p} \right) (p x_1 + x_2)} &\geq u
\end{aligned}$$

Where we will let p capture the relative prices (with $p_2 = 1$). We want to minimize the LHS, so taking the FOC with respect to p :

$$\begin{aligned}
2 \frac{1}{2} \left(4 + \frac{1}{p} \right)^{-1} (p x_1 + x_2)^{-1} \left[-\frac{1}{p^2} (p x_1 + x_2) + \left(4 + \frac{1}{p} \right) x_1 \right] &= 0 \\
\frac{x_1}{p x_1 + x_2} - \frac{1}{p^2 \left(4 + \frac{1}{p} \right)} &= 0 \\
x_1 p^2 \left(4 + \frac{1}{p} \right) &= p x_1 + x_2 \\
4p^2 x_1 + p x_1 &= p x_1 + x_2 \\
\therefore p &= \sqrt{\frac{x_2}{4x_1}}
\end{aligned}$$

Plug this into the LHS:

$$\begin{aligned}
u(x) &= 2 \sqrt{\left(4 + \sqrt{\frac{4x_1}{x_2}} \right) \left(\sqrt{\frac{x_2}{4x_1}} x_1 + x_2 \right)} \\
&= 2 \sqrt{\left(\frac{4x_2 + \sqrt{4x_1 x_2}}{x_2} \right) \left(\frac{x_1 (4x_2 + \sqrt{4x_1 x_2})}{4x_1} \right)} \\
&= 2 (4x_2 + \sqrt{4x_1 x_2}) \sqrt{\frac{1}{4x_2}}
\end{aligned}$$

$$= 4\sqrt{x_2} + 2\sqrt{x_1}$$

1.3 Verifications

1. Verify that the Marshallian demands satisfy budget balance
2. Verify that the Marshallian and Hicksian demand are equal at the “right” level of utility
3. Verify that the Marshallian and Hicksian demand are equal at the “right” level of income
4. Verify that the Slutsky equation holds for good 1 and price of good 2
5. Verify that for the Slutsky matrix $S(p, y)p = 0$
6. Verify that the Slutsky matrix is symmetric and negative semidefinite
7. Verify that Engel aggregation holds
8. Verify that Cournot aggregation holds (for good 2)
9. Verify that the indirect utility is h.o.d. 0 in prices and income
10. Verify that the expenditure function is concave in prices

Answers

1. The Marshallian demands are:

$$x_1(p, y) = \frac{p_2 y}{4p_1^2 + p_1 p_2} \qquad x_2(p, y) = \frac{4p_1 y}{p_2^2 + 4p_1 p_2}$$

Check budget balance:

$$\begin{aligned} & p_1 \left(\frac{p_2 y}{4p_1^2 + p_1 p_2} \right) + p_2 \left(\frac{4p_1 y}{p_2^2 + 4p_1 p_2} \right) \\ &= p_1 \left(\frac{1}{p_1} \cdot \frac{p_2 y}{4p_1 + p_2} \right) + p_2 \left(\frac{1}{p_2} \cdot \frac{4p_1 y}{p_2 + 4p_1} \right) \\ &= \frac{4p_1 y + p_2 y}{4p_1 + p_2} \\ &= y \end{aligned}$$

2. The “right” level of utility is when $u = v(p, y)$. Then we should have $x^h(p, v(p, y)) = x(p, y)$.
For good 1:

$$\begin{aligned} x_1^h(p, v(p, y)) &= \left(\frac{p_2}{2(4p_1 + p_2)} \cdot 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \right)^2 \\ &= \left(\frac{p_2}{(4p_1 + p_2)} \sqrt{\frac{(4p_1 + p_2)y}{p_1 p_2}} \right)^2 \\ &= \left(\sqrt{\frac{p_2 y}{p_1(4p_1 + p_2)}} \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{p_2 y}{p_1(4p_1 + p_2)} \\
&= \frac{p_2 y}{4p_1^2 + p_1 p_2} \\
&= x_1(p, y)
\end{aligned}$$

For good 2:

$$\begin{aligned}
x_2^h(p, v(p, y)) &= \left(\frac{p_1}{4p_1 + p_2} \cdot 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \right)^2 \\
&= \left(\frac{p_1}{4p_1 + p_2} \cdot 2\sqrt{\frac{(4p_1 + p_2)y}{p_1 p_2}} \right)^2 \\
&= \left(2\sqrt{\frac{p_1 y}{p_2(4p_1 + p_2)}} \right)^2 \\
&= \frac{4p_1 y}{p_2(4p_1 + p_2)} \\
&= \frac{4p_1 y}{p_2^2 + 4p_1 p_2} \\
&= x_2(p, y)
\end{aligned}$$

3. The “right” level of income is when $y = e(p, u)$. Then we should have $x(p, e(p, u)) = x^h(p, u)$.

For good 1:

$$\begin{aligned}
x_1(p, e(p, u)) &= \frac{p_2}{4p_1^2 + p_1 p_2} \cdot \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\
&= \frac{p_2}{p_1(4p_1 + p_2)} \cdot \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\
&= \frac{p_2^2 u^2}{4(4p_1 + p_2)^2} \\
&= \left(\frac{p_2 u}{2(4p_1 + p_2)} \right)^2 \\
&= x_1^h(p, u)
\end{aligned}$$

For good 2:

$$\begin{aligned}
x_2(p, e(p, u)) &= \frac{4p_1}{p_2^2 + 4p_1 p_2} \cdot \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\
&= \frac{4p_1}{p_2(4p_1 + p_2)} \cdot \frac{p_1 p_2 u^2}{4(4p_1 + p_2)} \\
&= \frac{p_1^2 u^2}{(4p_1 + p_2)^2} \\
&= \left(\frac{p_1 u}{4p_1 + p_2} \right)^2 \\
&= x_2^h(p, u)
\end{aligned}$$

4. The Slutsky equation for good 1 and price of good 2 is:

$$\frac{\partial x_1(p, y)}{\partial p_2} = \frac{\partial x_1^h(p, u)}{\partial p_2} \Big|_{u=v(p, y)} - x_2(p, y) \frac{\partial x_1(p, y)}{\partial y}$$

This just means we need the (1, 2) element in the Slutsky and substitution matrix to be equal:

$$\begin{aligned} \frac{8p_1p_2y}{AB} &= \frac{2p_1p_2u^2}{(4p_1 + p_2)^3} \Big|_{u=v(p, y)} \\ \frac{4y}{(4p_1^2 + p_1p_2)(p_2^2 + 4p_1p_2)} &= \frac{1}{(4p_1 + p_2)^3} \cdot \left(2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \right)^2 \\ \frac{y}{(4p_1^2 + p_1p_2)(p_2^2 + 4p_1p_2)} &= \frac{1}{(4p_1 + p_2)^3} \cdot \left(\frac{y}{p_1} + \frac{4y}{p_2} \right) \\ \frac{y}{p_1p_2(4p_1 + p_2)(4p_1 + p_2)} &= \frac{1}{(4p_1 + p_2)^3} \cdot \left(\frac{y(4p_1 + p_2)}{p_1p_2} \right) \\ \frac{y}{p_1p_2(4p_1 + p_2)^2} &= \frac{y}{(4p_1 + p_2)^2 p_1p_2} \\ 1 &= 1 \end{aligned}$$

Alternatively, if we used the more simplified Slutsky matrix: (where $Q = 4p_1 + p_2$)

$$S(p, y) = \frac{8y}{Q^2} \begin{pmatrix} -\frac{p_2}{p_1} & 1 \\ 1 & -\frac{p_1}{p_2} \end{pmatrix}$$

Then we would need:

$$\begin{aligned} \frac{8y}{Q^2} &= \frac{2p_1p_2u^2}{Q^3} \Big|_{u=v(p, y)} \\ 4y &= \frac{p_1p_2}{Q} \cdot \left(2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} \right)^2 \\ y &= \frac{p_1p_2}{Q} \cdot \left(\frac{yQ}{p_1p_2} \right) \\ 1 &= 1 \end{aligned}$$

5. Simply evaluate:

$$\begin{aligned} S(p, y)p &= \frac{8y}{Q^2} \begin{pmatrix} -\frac{p_2}{p_1} & 1 \\ 1 & -\frac{p_1}{p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &= \frac{8y}{Q^2} \begin{pmatrix} -p_2 + p_2 \\ p_1 - p_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

6. After doing the simplifications, it is clearly a symmetric matrix. To check if it is negative semidefinite, we would need all the principal minors of order k to be $(-1)^k \Delta_k \geq 0$. The order 1 principal minors are just the diagonals, which are all negative. The order 2 principal minor is just the determinant, which

we need to be non-negative:²

$$\begin{aligned}\det(S) &= \left(\frac{8y}{Q^2}\right)^2 \begin{vmatrix} -\frac{p_2}{p_1} & 1 \\ 1 & -\frac{p_1}{p_2} \end{vmatrix} \\ &= \left(\frac{8y}{Q^2}\right)^2 (1 - 1) \\ &= 0\end{aligned}$$

7. Engel aggregation is:

$$\sum_{i=1}^n s_i \eta_i = 1$$

In our case, we have $n = 2$ and:

$$\begin{aligned}s_1 &= \frac{p_1 x_1(p, y)}{y} = \frac{p_1 p_2}{4p_1^2 + p_1 p_2} & s_2 &= \frac{p_2 x_2(p, y)}{y} = \frac{4p_1 p_2}{p_2^2 + 4p_1 p_2} \\ \eta_1 &= \frac{\partial x_1(p, y)}{\partial y} \cdot \frac{y}{x_1(p, y)} & \eta_2 &= \frac{\partial x_2(p, y)}{\partial y} \cdot \frac{y}{x_2(p, y)} \\ &= \left(\frac{p_2}{4p_1^2 + p_1 p_2}\right) \cdot \left(\frac{4p_1^2 + p_1 p_2}{p_2}\right) & &= \left(\frac{4p_1}{p_2^2 + 4p_1 p_2}\right) \cdot \left(\frac{p_2^2 + 4p_1 p_2}{4p_1}\right) \\ &= 1 & &= 1\end{aligned}$$

This means:

$$\begin{aligned}\sum_{i=1}^2 s_i \eta_i &= \left(\frac{p_1 p_2}{4p_1^2 + p_1 p_2}\right) \cdot 1 + \left(\frac{4p_1 p_2}{p_2^2 + 4p_1 p_2}\right) \cdot 1 \\ &= \frac{p_1 p_2}{p_1 (4p_1 + p_2)} + \frac{4p_1 p_2}{p_2 (4p_1 + p_2)} \\ &= \frac{p_2 + 4p_1}{4p_1 + p_2} \\ &= 1\end{aligned}$$

8. Cournot aggregation is:

$$\sum_{i=1}^n s_i \varepsilon_{ij} = -s_j$$

In our case, we have $n = 2$ and $j = 2$:

$$\begin{aligned}\varepsilon_{12} &= \frac{\partial x_1(p, y)}{\partial p_2} \cdot \frac{p_2}{x_1(p, y)} & \varepsilon_{22} &= \frac{\partial x_2(p, y)}{\partial p_2} \cdot \frac{p_2}{x_2(p, y)} \\ &= \left(\frac{y (4p_1^2 + p_1 p_2) - p_2 y (p_1)}{(4p_1^2 + p_1 p_2)^2}\right) \cdot \left(p_2 \cdot \frac{4p_1^2 + p_1 p_2}{p_2 y}\right) & &= \left(\frac{-4p_1 y}{(p_2^2 + 4p_1 p_2)^2} (2p_2 + 4p_1)\right) \cdot \left(p_2 \cdot \frac{p_2^2 + 4p_1 p_2}{4p_1 y}\right) \\ &= \left(\frac{y (4p_1^2 + p_1 p_2) - p_2 y (p_1)}{(4p_1^2 + p_1 p_2)}\right) \left(\frac{1}{y}\right) & &= \frac{-1}{(p_2^2 + 4p_1 p_2)} (2p_2 + 4p_1) (p_2)\end{aligned}$$

²Recall that for a $n \times n$ matrix A and scalar k , $\det(kA) = k^n \det(A)$

$$\begin{aligned}
&= 1 - \frac{p_1 p_2}{(4p_1^2 + p_1 p_2)} &= \frac{-(4p_1 + 2p_2)}{(4p_1 + p_2)} \\
&= 1 - \frac{p_2}{(4p_1 + p_2)} &= -1 - \frac{p_2}{(4p_1 + p_2)}
\end{aligned}$$

Recall that the shares are:

$$\begin{aligned}
s_1 &= \frac{p_1 p_2}{4p_1^2 + p_1 p_2} & s_2 &= \frac{4p_1 p_2}{p_2^2 + 4p_1 p_2} \\
&= \frac{p_2}{(4p_1 + p_2)} & &= \frac{4p_1}{(4p_1 + p_2)}
\end{aligned}$$

This means:

$$\begin{aligned}
\sum_{i=1}^2 s_i \varepsilon_{i2} &= \frac{p_2}{(4p_1 + p_2)} \cdot \left(1 - \frac{p_2}{(4p_1 + p_2)}\right) + \frac{4p_1}{(4p_1 + p_2)} \cdot \left(-1 - \frac{p_2}{(4p_1 + p_2)}\right) \\
&= \left(\frac{p_2}{(4p_1 + p_2)} - \frac{4p_1}{(4p_1 + p_2)}\right) - \frac{p_2}{(4p_1 + p_2)} \left(\frac{p_2}{(4p_1 + p_2)} + \frac{4p_1}{(4p_1 + p_2)}\right) \\
&= \frac{p_2 - 4p_1}{(4p_1 + p_2)} - \frac{p_2}{(4p_1 + p_2)} (1) \\
&= \frac{-4p_1}{(4p_1 + p_2)} \\
&= -s_2
\end{aligned}$$

9. Check directly:

$$v(\lambda p, \lambda y) = 2\sqrt{\frac{\lambda y}{\lambda p_1} + \frac{4\lambda y}{\lambda p_2}} = 2\sqrt{\frac{y}{p_1} + \frac{4y}{p_2}} = v(p, y)$$

10. We need to show that the Hessian of the expenditure function is negative semidefinite

$$e(p, u) = \frac{p_1 p_2 u^2}{4(4p_1 + p_2)}$$

Note that the Hessian is exactly the substitution matrix we derived above:

$$H = \sigma(p, u) = \frac{2u^2}{(4p_1 + p_2)^3} \begin{pmatrix} -p_2^2 & p_1 p_2 \\ p_1 p_2 & -p_1^2 \end{pmatrix}$$

To show it is negative semidefinite, we need the diagonals (order 1 principal minors) to be weakly negative and the determinant (order 2 principal minor) to be weakly positive. Clearly the diagonals are negative. Let's check the determinant:

$$\begin{aligned}
\det(H) &= \left(\frac{2u^2}{(4p_1 + p_2)^3}\right)^2 \begin{vmatrix} -p_2^2 & p_1 p_2 \\ p_1 p_2 & -p_1^2 \end{vmatrix} \\
&= \left(\frac{2u^2}{(4p_1 + p_2)^3}\right)^2 (p_1^2 p_2^2 - p_1^2 p_2^2) \\
&= 0
\end{aligned}$$

We should have also known this was going to be a negative semidefinite matrix (with determinant 0),

because that's exactly what we found for the Slutsky matrix (and these two matrices are the same).

2 Proofs³

2.1 Questions

1. JR 1.23: Prove Theorem 1.3. Namely: Let \succsim be represented by $u : \mathbb{R}_+^N \rightarrow \mathbb{R}$. Then:
 - (a) $u(x)$ is strictly increasing iff \succsim is strictly monotonic.
 - (b) $u(x)$ is quasiconcave iff \succsim is convex.
 - (c) $u(x)$ is strictly quasiconcave iff \succsim is strictly convex.
2. JR 1.24: Let $u(x)$ represent some consumer's monotonic preferences over $x \in \mathbb{R}_+^n$. For each of the functions $f(x)$ that follow, state whether or not f *also* represents the preferences of this consumer. In each case, be sure to justify your answer with either an argument or a counterexample
 - (a) $f(x) = u(x) + (u(x))^3$
 - (b) $f(x) = u(x) - (u(x))^2$
 - (c) $f(x) = u(x) + \sum_{i=1}^n x_i$
3. JR 1.44: In a two-good case, show that if one good is inferior, the other good must be normal.
4. JR 1.47: Suppose that $u(x)$ is a linear homogeneous utility function.
 - (a) Show that the expenditure function is multiplicatively separable in p and u and can be written in the form $e(p, u) = e(p, 1)u$.
 - (b) Show that the marginal utility of income depends on p , but is independent of y .
5. JR 1.48: Suppose that the expenditure function is multiplicatively separable in p and u so that $e(p, u) = k(u)g(p)$, where $k(\cdot)$ is some positive monotonic function of a single variable, and $g : \mathbb{R}_+^n \rightarrow \mathbb{R}$. Show that the income elasticity of (Marshallian) demand for every good is equal to unity.
6. JR 1.49: You are given the following information about the demand functions and expenditure patterns of a consumer who spends all his income on two goods: (1) At current prices, the same amount is spent on both goods; (2) at current prices, the own-price elasticity of demand for good 1 is equal to -3.
 - (a) At current prices, what is the elasticity of demand for good 2 with respect to the price of good 1?
 - (b) Can statements (1) and (2) both hold at all prices? Why or why not?
7. JR 1.51: Consider the utility function, $u(x_1, x_2) = (x_1)^{1/2} + (x_2)^{1/2}$.
 - (a) Compute the demand functions, $x_i(p_1, p_2, y), i = 1, 2$
 - (b) Compute the substitution term in the Slutsky equation for the effects on x_1 of changes in p_2 .
 - (c) Classify x_1 and x_2 as (gross) complements or substitutes.

³All these questions are from Jehle and Reny Chapter 1 (the numbers correspond to the third edition). This section is called "proofs" in the exam, but it's probably more accurate to think of it as "short exercises"

8. JR 1.53: Agents A and B have the following expenditure functions. In each case, state whether or not the observable market behavior of the two agents will be identical. Justify your answers.

- (a) $e^A(p, u)$ and $e^B(p, u) = e^A(p, 2u)$.
 (b) $e^A(p, u) = k(u)g(p)$, where $k'(u) > 0$, and $e^B(p, u) = 2e^A(p, u)$.

9. JR 1.54: The n -good Cobb-Douglas utility function is:

$$u(x) = A \prod_{i=1}^n x_i^{\alpha_i}$$

where $A > 0$ and $\sum_{i=1}^n \alpha_i = 1$.

- (a) Derive the Marshallian demand functions.
 (b) Derive the indirect utility function.
 (c) Compute the expenditure function.
 (d) Compute the Hicksian demands.
10. JR 1.56: What restrictions must the α_i , $f(y)$, $w(p_1, p_2)$, and $z(p_1, p_2)$ satisfy if each of the following is to be a legitimate indirect utility function?

- (a) $v(p_1, p_2, p_3, y) = f(y)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$
 (b) $v(p_1, p_2, y) = w(p_1, p_2) + z(p_1, p_2)/y$

11. JR 1.62: According to Hick's Third Law:

$$\sum_{j=1}^n \frac{\partial x_i^h(p, u)}{\partial p_j} p_j = 0, \forall i$$

or equivalently, in elasticity form,

$$\sum_{j=1}^n \varepsilon_{ij}^h = 0, \forall i$$

Prove this and verify it for a consumer with the n -good Cobb-Douglas utility function in Exercise 1.54.

12. JR 1.63: The substitution matrix of a utility-maximizing consumer's demand system at prices $(8, p)$ is

$$\begin{pmatrix} a & b \\ 2 & -1/2 \end{pmatrix}$$

Find a , b , and p .

13. JR 1.65: Show that the utility function is homothetic if and only if all demand functions are multiplicatively separable in prices and income and of the form $x(p, y) = \phi(y)x(p, 1)$. Note that a real-valued function h is homothetic if it can be written in the form $h(x) = g(f(x))$, where g is strictly increasing and f is homogeneous of degree 1.

14. JR 1.66: A consumer with income y_0 faces prices p_0 and enjoys utility $u_0 = v(p_0, y_0)$. When prices change to p_1 , the cost of living is affected. To gauge the impact of these price changes, we may define a cost of living index as the ratio:

$$I(p_0, p_1, u_0) \equiv \frac{e(p_1, u_0)}{e(p_0, u_0)}$$

- (a) Show that $I(p_0, p_1, u_0)$ is greater than (less than) unity as the outlay necessary to maintain base utility, u_0 , rises (falls).
- (b) Suppose consumer income also changes from y_0 to y_1 . Show that the consumer will be better off (worse off) in the final period whenever y_1/y_0 is greater (less) than $I(p_0, p_1, u_0)$.

2.2 Answers

1. (JR 1.23)

- (a) (\Rightarrow) Suppose $u(\cdot)$ is strictly increasing. Let $x, y \in \mathbb{R}_+^N$ with $x \geq y$. Since $u(\cdot)$ is increasing, $u(x) \geq u(y)$, so $x \succsim y$. Moreover, if $x > y$, $u(x) > u(y)$, so $x \succ y$, which implies \succsim is strictly monotonic.
- (\Leftarrow) Suppose \succsim is strictly monotonic. Let $x, y \in \mathbb{R}_+^N$ with $x > y$. By strict monotonicity, $x \succ y$, so $u(x) > u(y)$, which implies u is strictly increasing.
- (b) (\Rightarrow) Suppose $u(\cdot)$ is quasiconcave. Let $x, y \in \mathbb{R}_+^N$ with $x \succsim y$. Since quasiconcave functions have convex upper-level sets, the set $S = \{z | u(z) \geq u(y)\}$ is convex. Note $x, y \in S$, so for any $t \in [0, 1]$, $u(tx + (1-t)y) \geq u(y)$. Thus $tx + (1-t)y \succsim y$, so \succsim is convex.
- (\Leftarrow) Suppose \succsim is convex. Let $x, y \in \mathbb{R}_+^N$. Define $z = \arg \min_{s \in \{x, y\}} u(s)$.⁴ Since \succsim is convex, for any $t \in [0, 1]$, $tx + (1-t)y \succsim z$, so $u(tx + (1-t)y) \geq u(z) = \min\{u(x), u(y)\}$, so $u(\cdot)$ is quasiconcave (I am using the characterization of quasiconcavity from definition A1.24 here).
- (c) The argument follows the exact same logic as part (b), with weak inequalities (preferences) replaced by strict inequalities (preferences) as needed, and taking $t \in (0, 1)$ instead of $[0, 1]$.

2. (JR 1.24)

- (a) Yes. The function $g(x) = x + x^3$ is monotonic: $g'(x) = 1 + 3x^2 > 0$ as long as $x > 0$. We can also check the MRS

$$MRS_{ij}^f = \frac{f_i}{f_j} = \frac{u_i + 3(u(x))^2 u_i}{u_j + 3(u(x))^2 u_j} = \frac{u_i (1 + 3(u(x))^2)}{u_j (1 + 3(u(x))^2)} = \frac{u_i}{u_j} = MRS_{ij}^u$$

Since the MRS is the same, it represents the same preferences.

- (b) No. The function $g(x) = x - x^2$ is not monotonic: $g'(x) = 1 - 2x < 0$ if $x > 0.5$. As an example, consider $u(x) = 1$, $u(y) = 0.5$, which means that $x \succ y$. However, this means that $f(x) = 1 - 1^2 = 0$ and $f(y) = 0.5 - 0.5^2 = 0.25$, which means that $y \succ x$.
- (c) No. Consider $u(x) = x_1 x_2$. Then $u(1, 1) = 1$ and $u(2, 0.5) = 1$, which means that $(1, 1) \sim (2, 0.5)$. But $f(1, 1) = 1 + 2 = 3$ and $f(2, 0.5) = 1 + 2.5 = 3.5$, which suggests that $(1, 1) \prec (2, 0.5)$. We

⁴That is, z is the least-preferred element between x and y . If $x \sim y$, assign z however you like.

can also see this by looking at the MRS:

$$MRS_{ij}^f = \frac{f_i}{f_j} = \frac{u_i + 1}{u_j + 1} \neq \frac{u_i}{u_j} = MRS_{ij}^u$$

3. (JR 1.44) Use Engel aggregation: $s_1\eta_1 + s_2\eta_2 = 1$. WLOG, suppose that good 1 is inferior, which means that $\eta_1 < 0$. This means that:

$$\eta_1 = \frac{1}{s_1} (1 - s_2\eta_2) < 0$$

Since $s_i \in [0, 1], \forall i$, then the only way the inequality can hold is if $\eta_2 > 0$ (in particular, we need $\eta_2 > \frac{1}{s_2}$).

4. (JR 1.47) Note that linearly homogenous is just h.o.d. 1., i.e. $u(\lambda x) = \lambda u(x)$.

(a) Just use the definition of the expenditure and the homogeneity to simplify the constraint:

$$\begin{aligned} e(p, u_0) &= \min_x p \cdot x \text{ s.t. } u(x) \geq u_0 \\ &= \min_x p \cdot x \text{ s.t. } \frac{1}{u_0} u(x) \geq 1 \\ &= \min_x p \cdot x \text{ s.t. } u\left(\frac{x}{u_0}\right) \geq 1 \\ &= \min_x u_0 \left(p \cdot \frac{x}{u_0}\right) \text{ s.t. } u\left(\frac{x}{u_0}\right) \geq 1 \\ &= u_0 \left(\min_z p \cdot z \text{ s.t. } u(z) \geq 1\right) \\ &= u_0 \cdot e(p, 1) \end{aligned}$$

- (b) By duality, we know that: $y = e(p, v(p, y))$. By (a), we know that $e(p, v(p, y)) = v(p, y)e(p, 1)$. Putting this together:

$$\begin{aligned} y &= v(p, y)e(p, 1) \\ v(p, y) &= \frac{y}{e(p, 1)} \end{aligned}$$

The marginal utility of income is then:

$$\frac{\partial v(p, y)}{\partial y} = \frac{\partial}{\partial y} \left(\frac{y}{e(p, 1)} \right) = \frac{1}{e(p, 1)}$$

Which clearly depends on p but not on y .

5. (JR 1.48) Start from the expenditure function, use Shephard's Lemma to get the Hicksian demand.

$$\begin{aligned} e(p, u) &= k(u)g(p) \\ \implies x_i^h(p, u) &= \frac{\partial e(p, u)}{\partial p_i} \\ &= k(u) \frac{\partial g(p)}{\partial p_i} \end{aligned}$$

By duality, we can express the Marshallian demand as:

$$x_i(p, e(p, u)) = x_i^h(p, u)$$

Take the partial derivative of both sides with respect to u :

$$\begin{aligned}\frac{\partial x_i(p, e(p, u))}{\partial u} &= \frac{\partial x_i^h(p, u)}{\partial u} \\ \frac{\partial x_i(p, y)}{\partial y} \frac{\partial e(p, u)}{\partial u} &= k'(u) \frac{\partial g(p)}{\partial p_i} \\ \frac{\partial x_i(p, y)}{\partial y} k'(u) g(p) &= k'(u) \frac{\partial g(p)}{\partial p_i} \\ \frac{\partial x_i(p, y)}{\partial y} &= \frac{1}{g(p)} \cdot \frac{\partial g(p)}{\partial p_i}\end{aligned}$$

Use this to calculate the income elasticity: (evaluated at $y = e(p, u)$)

$$\begin{aligned}\eta_i &= \frac{\partial x_i(p, y)}{\partial y} \cdot \frac{y}{x_i(p, y)} \\ &= \left(\frac{1}{g(p)} \cdot \frac{\partial g(p)}{\partial p_i} \right) \cdot \frac{e(p, u)}{x_i(p, e(p, u))} \\ &= \frac{1}{g(p)} \cdot \frac{\partial g(p)}{\partial p_i} \cdot \frac{k(u)g(p)}{k(u) \frac{\partial g(p)}{\partial p_i}} \\ &= 1\end{aligned}$$

6. (JR 1.49)

(a) According to Cournot aggregation:

$$s_1 \epsilon_{11} + s_2 \epsilon_{21} = -s_1$$

We're given $s_1 = s_2 = \frac{1}{2}$ and $\epsilon_{11} = -3$, so $\epsilon_{21} = -1 + 3 = 2$.

(b) No. Suppose statement (1) holds for all prices. In that case $p_2 x_2 = \frac{y}{2}$, so $x_2 = \frac{y}{2p_2}$. In this case, x_2 does not depend on p_1 at all, so $\epsilon_{21} = 0$, which implies (from Cournot aggregation) that $\epsilon_{11} = -1$, not -3 .

7. (JR 1.51)

(a) The Lagrangian is:

$$x_1^{1/2} + x_2^{1/2} + \lambda(y - p_1 x_1 - p_2 x_2)$$

The FOCs are

$$\begin{aligned}\frac{1}{2x_1^{1/2}} &= \lambda p_1 \\ \frac{1}{2x_2^{1/2}} &= \lambda p_2\end{aligned}$$

Taking the ratio of these equations and squaring gives:

$$\frac{x_2}{x_1} = \frac{p_1^2}{p_2^2}$$

Finally, substituting this relationship into the budget constraint lets us solve for the demands:

$$\begin{aligned} x_1 &= \frac{p_2 y}{p_1(p_2 + p_1)} \\ x_2 &= \frac{p_1 y}{p_2(p_2 + p_1)} \end{aligned}$$

(b) Recall the Slutsky equation is:

$$\underbrace{\frac{\partial x_i(p, y)}{\partial p_j}}_{\text{Total Effect}} = \underbrace{\frac{\partial x_i^h(p, u^*)}{\partial p_j}}_{\text{Substitution Effect}} + \underbrace{-x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}}_{\text{Income Effect}}$$

Thus the substitution effect is:

$$\frac{\partial x_i^h(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, y)}{\partial p_j} + x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}$$

We see

$$\partial x_1 / \partial p_2 = \frac{y}{(p_1 + p_2)^2}$$

and

$$\partial x_1 / \partial y = \frac{p_2}{p_1(p_1 + p_2)}$$

Combining gives:

$$\begin{aligned} \frac{\partial x_i^h(p, u^*)}{\partial p_j} &= \frac{\partial x_1(p, y)}{p_2} + x_2(p, y) \frac{\partial x_1(p, y)}{y} \\ &= \left[\frac{y}{(p_2 + p_1)^2} \right] + \left[\frac{p_1 y}{p_2(p_2 + p_1)} \right] \left[\frac{p_2}{p_1(p_2 + p_1)} \right] \\ &= \frac{y}{(p_2 + p_1)^2} + \frac{y}{(p_2 + p_1)^2} \\ &= \frac{2y}{(p_2 + p_1)^2} \end{aligned}$$

(c) Since $\partial x_1 / \partial p_2 > 0$, these goods are gross substitutes: when good 2 becomes more expensive, I substitute more towards good 1.

8. (JR 1.53)

(a) The observable behavior is the same. Using the fact that $v(p, e(p, u)) = u$, we first note $v^A(p, e^A(p, 2u)) = 2u = 2v^B(p, e^A(p, 2u))$. Thus for any p, y , we have:

$$v^A(p, y) = 2v^B(p, y)$$

Since v^A is a monotonic transformation of v^B , they represent the same preferences (we proved this Homework 3, Question 1).

(b) The behavior is again identical. We see

$$\begin{aligned} y &= e^A(p, v^A(p, y)) \\ &= k(v^A(p, y))g(p) \end{aligned}$$

Since k is strictly increasing, we can invert to solve for v^A :

$$v^A(p, y) = k^{-1} \left(\frac{y}{g(p)} \right)$$

Similarly, we find $v^B(p, y) = k^{-1} \left(\frac{y}{2g(p)} \right)$. Thus:

$$v^A(p, y) = k^{-1} [2k \{v^B(p, y)\}]$$

This is an increasing transformation of v^B , and so by the result referenced above, these agents have the same underlying preferences.

9. (JR 1.54)

(a) To simplify the algebra, I will work with the log-transformed utility $u(x) = \log(A) + \sum_{i=1}^n \alpha_i \log x_i$. The Lagrangian is:

$$\mathcal{L}(x, \lambda) = \log(A) + \sum_{i=1}^n \alpha_i \log(x_i) + \lambda(y - px)$$

The FOC⁵ wrt x_i is:

$$\frac{\alpha_i}{x_i} = \lambda p_i$$

Rearranging gives $\alpha_i = \lambda p_i x_i$. Summing this over i gives $1 = \lambda y$, which implies $\lambda = 1/y$. Therefore the Marshallian demand is:

$$x_i = \frac{\alpha_i y}{p_i}$$

(b) Since $v(p, y) = u(x(p, y))$ we see:

$$v(p, y) = A \prod_{i=1}^n \left(\frac{\alpha_i y}{p_i} \right)^{\alpha_i} = Ay \prod_{i=1}^n \alpha_i^{\alpha_i} p_i^{-\alpha_i}$$

(c) This is rather simple in this case since the indirect utility function is linear in y . Note if $v(p, y) = u^0$, then:

$$y = \frac{u^0}{A} \prod_{i=1}^n \alpha_i^{-\alpha_i} p_i^{\alpha_i} = e(p, u^0)$$

(d) Using Sheppard's lemma, we have:

$$x_j^h(p, u) = \frac{\partial e(p, u)}{\partial p_j} = \frac{u}{A} \frac{\alpha_j}{p_j} \prod_{i=1}^n \alpha_i^{-\alpha_i} p_i^{\alpha_i}$$

10. (JR 1.56)

⁵The FOC is sufficient here since the Cobb-Douglas function is concave.

(a) Run through the properties:

- i. Continuity: we need $f(y)$ to be continuous
- ii. Homogeneity in (p, y) : we need $f(\lambda y) = \lambda^{-(\alpha_1 + \alpha_2 + \alpha_3)} f(y)$.
- iii. Strictly increasing in y : we need $f(y)$ to be strictly increasing in y , i.e. $f'(y) > 0$.
- iv. Decreasing in p : we need $\frac{\partial v}{\partial p_i} = \alpha_i f(y) p_i^{\alpha_i - 1} \prod_{j \neq i} p_j^{\alpha_j} \leq 0$. Since prices are positive, we require that $\alpha_i f(y) \leq 0, \forall i$.
- v. Quasiconvex in (p, y) : Do the usual bordered Hessian condition

(b) Run through the properties:

- i. Continuity: we need $w(\cdot)$ and $z(\cdot)$ to be continuous
- ii. Homogeneity in (p, y) : we need $w(\lambda p) + \frac{z(\lambda p)}{\lambda y} = w(p) + \frac{z(p)}{y}$. For example, having w being h.o.d. 0 and z h.o.d. 1 is sufficient.
- iii. Strictly increasing in y : we need $\frac{\partial v}{\partial y} = -\frac{z(p)}{y^2} > 0$, which means we need $z(p) < 0$
- iv. Decreasing in p : we need $\frac{\partial v}{\partial p_i} = \frac{\partial w}{\partial p_i} + \frac{1}{y} \frac{\partial z}{\partial p_i} \leq 0$, which means $\nabla w \leq 0$ and $\nabla z \leq 0$
- v. Quasiconvex in (p, y) : Do the usual bordered Hessian condition

(c) $v(p_1, p_2, p_3) = w(p_1, p_2) + z(p_1, p_2)/y$

11. (JR 1.62) Since $x_i^h(p, u)$ is h.o.d. 0 in p , we can use Euler's Theorem:

$$\begin{aligned} \sum_{j=1}^n \frac{\partial x_i^h(p, u)}{\partial p_j} p_j &= 0 \cdot x_i^h(p, u) \\ \sum_{j=1}^n \frac{\partial x_i^h(p, u)}{\partial p_j} \cdot \frac{p_j}{x_i^h(p, u)} &= 0 \\ \sum_{j=1}^n \varepsilon_{ij}^h &= 0 \end{aligned}$$

The Hicksian demand from Exercise 1.54 is:

$$x_i^h(p, u) = \frac{u}{A} \frac{\alpha_i}{p_i} \prod_{k=1}^n \alpha_k^{-\alpha_k} p_k^{\alpha_k}$$

Let $Q_j = \prod_{k \neq j} \alpha_k^{-\alpha_k} p_k^{\alpha_k}$. The partial derivative for $j \neq i$ is:

$$\begin{aligned} \frac{\partial x_i^h(p, u)}{\partial p_j} &= \frac{\partial}{\partial p_j} \left(\frac{u \alpha_i Q_j}{A p_i} \alpha_j^{-\alpha_j} p_j^{\alpha_j} \right) \\ &= \frac{u \alpha_i Q_j}{A p_i} \alpha_j^{-\alpha_j} \alpha_j p_j^{\alpha_j - 1} \\ &= \frac{u \alpha_i Q_j}{A p_i} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j - 1} \end{aligned}$$

The partial derivative for $j = i$ is:

$$\begin{aligned} \frac{\partial x_i^h(p, u)}{\partial p_i} &= \frac{\partial}{\partial p_i} \left(\frac{u \alpha_i Q_i}{A p_i} \alpha_i^{-\alpha_i} p_i^{\alpha_i} \right) \\ &= \frac{\partial}{\partial p_i} \left(\frac{u Q_i}{A} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i - 1} \right) \end{aligned}$$

$$= \frac{uQ_i(\alpha_i - 1)}{A\alpha_i} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i - 2}$$

Finally, let's put all this together. Let $Q = \prod_{k=1}^n \alpha_k^{-\alpha_k} p_k^{\alpha_k}$. Note that $Q = Q_j \alpha_j^{-\alpha_j} p_j^{\alpha_j}, \forall j$. Recall that $\sum_j \alpha_j = 1$.

$$\begin{aligned} \sum_{j=1}^n \frac{\partial x_i^h(p, u)}{\partial p_j} p_j &= \frac{uQ_i(\alpha_i - 1)}{A\alpha_i} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i - 2} p_i + \sum_{j \neq i} \frac{u\alpha_i Q_j}{Ap_i} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j - 1} p_j \\ &= \frac{u(\alpha_i - 1)}{A\alpha_i} \underbrace{Q_i \alpha_i^{-\alpha_i} p_i^{\alpha_i}}_{=Q} \left(\frac{\alpha_i}{p_i} \right)^2 p_i + \sum_{j \neq i} \frac{u\alpha_i \alpha_j}{Ap_i} \underbrace{Q_j \alpha_j^{-\alpha_j} p_j^{\alpha_j}}_{=Q} \\ &= \frac{uQ\alpha_i(\alpha_i - 1)}{Ap_i} + \sum_{j \neq i} \frac{uQ\alpha_i \alpha_j}{Ap_i} \\ &= \frac{uQ\alpha_i}{Ap_i} \left[\alpha_i - 1 + \sum_{j \neq i} \alpha_j \right] \\ &= \frac{uQ\alpha_i}{Ap_i} \left[-1 + \sum_{j=1}^n \alpha_j \right] \\ &= \frac{uQ\alpha_i}{Ap_i^2} [0] = 0 \end{aligned}$$

12. (JR 1.63) The general substitution matrix is:

$$\sigma = \begin{pmatrix} \frac{\partial x_1^h}{\partial p_1} & \frac{\partial x_1^h}{\partial p_2} \\ \frac{\partial x_2^h}{\partial p_1} & \frac{\partial x_2^h}{\partial p_2} \end{pmatrix} = \begin{pmatrix} a & b \\ 2 & -1/2 \end{pmatrix}$$

Since we know that $\frac{\partial x_i^h}{\partial p_j} = \frac{\partial x_j^h}{\partial p_i}$, that means that $b = 2$. Moreover, we also know that $\sigma \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0$.

This means that:

$$\begin{pmatrix} a & b \\ 2 & -1/2 \end{pmatrix} \begin{pmatrix} 8 \\ p \end{pmatrix} = \begin{pmatrix} 8a + bp \\ 16 - \frac{1}{2}p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies p = 32$$

$$a = -\frac{bp}{8} = -\frac{(2)(32)}{8} = -8$$

Therefore, we have $a = -8, b = 2, p = 32$

13. (JR 1.65) The Marshallian demand is:

$$\begin{aligned} x(p, y) &= \operatorname{argmax}_x u(x) \text{ s.t. } p \cdot x \leq y && \text{(definition of Marshallian demand)} \\ &= \operatorname{argmax}_x g(f(x)) \text{ s.t. } p \cdot x \leq y && \text{(since } u(x) \text{ is homothetic)} \\ &= \operatorname{argmax}_x f(x) \text{ s.t. } p \cdot x \leq y && \text{(since } g(\cdot) \text{ is monotonic)} \end{aligned}$$

$$\begin{aligned}
&= \operatorname{argmax}_x y f\left(\frac{x}{y}\right) \text{ s.t. } p \cdot \frac{x}{y} \leq 1 && \text{(since } f(\cdot) \text{ is h.o.d 1)} \\
&= y \left(\operatorname{argmax}_z f(z) \text{ s.t. } p \cdot z \leq 1 \right) && \text{(let } z = x/y \text{ and pull out } y) \\
&= yx(p, 1) && \text{(definition of Marshallian demand)}
\end{aligned}$$

Therefore, we have it in the multiplicative form where $\phi(y) = 1$.

14. (JR 1.66)

- (a) Note $e(p_0, u_0) = y_0$. Thus $I > 1$ if we need to spend more money to attain u_0 , and $I < 1$ if we can spend less money to attain u_0 .
- (b) Again, note $y_0 = e(p_0, u_0)$. Therefore $y_1/y_0 > I(p_0, p_1, u_0)$ iff $y_1 > e(p_1, u_0)$. If $y_1 > e(p_1, u_0)$, then $V(p_1, y_1) > V(p_1, e(p_1, u_0)) = u_0$, so the consumer is better off. A similar analysis shows the worse off case.

3 Long Exercise⁶

3.1 Question

In an economy with L commodities, a consumer has the following utility function:

$$u(x_1, \dots, x_L) = x_1 + \alpha_2 \ln x_2 + \dots + \alpha_L \ln x_L$$

With $\sum_{i=2}^L \alpha_i = 1$. He has an income y and faces a price vector $p = (1, p_2, \dots, p_L)$, where $p_1 = 1$.

1. Let z denote the amount spent on commodities 2 to L :

$$z = \sum_{l=2}^L p_l x_l(p, y)$$

where $x_l(p, y)$ denotes the consumer's Marshallian demand for commodity l . What does the optimal (x_2, \dots, x_L) maximize? Write x_l (for $l \geq 2$) as a function of (p, z) .

2. Compute the agent's utility and budget constraint as a function of x_1 and z
3. Compute the optimal values of x_1 and z
4. We first assume $y > 1$. How does the demand for commodity l ($l = 1, \dots, L$) depend on income?
5. Compute the indirect utility and expenditure function.
6. Consider a reform that changes the price vector from $p = (1, p_2, \dots, p_L)$ to $p' = (1, p'_2, \dots, p'_L)$.
 - (a) Compute the compensating variation
 - (b) Compute the equivalent variation

⁶This question was the Fall 2017 PhD Micro Midterm

- (c) Compare the two results. Can you explain this finding?
 - (d) When does the reform increase the consumer's utility?
7. We now assume $y < 1$.
- (a) Compute the compensating variation
 - (b) Compute the equivalent variation
 - (c) Are the latter equal?
 - (d) When does the reform increase the consumer's utility?

3.2 Answers

1. We are looking for $x(p, z) = (x_2(p, z), \dots, x_L(p, z))$, which is the optimal (utility maximizing) amount spent on goods 2- L , *given* that we are spending z on goods 2- L . This problem is expressed as:

$$x(p, z) = \underset{x}{\operatorname{argmax}} \sum_{l=2}^L \alpha_l \ln x_l \text{ s.t. } \sum_{l=2}^L p_l x_l \leq z$$

But notice that this is just a UMP for a Cobb-Douglas utility function. Therefore, the “conditional” Marshallian demand is: (conditional on spending z on goods 2- L)

$$x_l(p, z) = \frac{\alpha_l z}{p_l}$$

2. Plug in from Q1 into the utility function:

$$\begin{aligned} u(x_1, z) &= x_1 + \sum_{l=2}^L \alpha_l \ln \left(\frac{\alpha_l z}{p_l} \right) \\ &= x_1 + \ln z + \sum_{l=2}^L \alpha_l (\ln \alpha_l - \ln p_l) \\ &= x_1 + \ln z + A(p) \end{aligned}$$

Where $A(p) = \sum_{l=2}^L \alpha_l (\ln \alpha_l - \ln p_l)$. The UMP is then just:

$$\begin{aligned} \max_x \quad & x_1 + \ln z + A(p) \\ \text{s.t.} \quad & x_1 + z \leq y \end{aligned}$$

3. We solve the above UMP. At an interior solution, we get that:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{1} \\ \therefore z &= 1 \end{aligned}$$

Plugging this into the budget constraint, we get $x_1 = y - z = y - 1$.

However, we're not done. What if $y < 1$? Then we would get $x_1 < 0$. In that case, we would instead need to worry about corner solutions (this utility is quasi-linear, so we should always be worried about corners). In fact, all we need to is make the good consumed in negative quantities as zero and put all income into the other good, i.e. $x_1 = 0$ and $z = y$. Therefore, the Marshallian demands are:

$$x_1 = \begin{cases} y - 1 & \text{if } y \geq 1 \\ 0 & \text{if } y < 1 \end{cases} \quad z = \begin{cases} 1 & \text{if } y \geq 1 \\ y & \text{if } y < 1 \end{cases}$$

4. We have that:

$$\frac{\partial x_1}{\partial y} = 1 \quad \frac{\partial x_l}{\partial y} = 0, \forall l \geq 2$$

Which is what we expect for a quasi-linear utility function.

5. Plugging into the utility function:

$$\begin{aligned} v(p, y) &= (y - 1) + \ln(1) + A(p) \\ &= y + A(p) - 1 \end{aligned}$$

Invert this to get the expenditure function:

$$\begin{aligned} u &= y + A(p) - 1 \\ \therefore e(p, u) &= u - A(p) + 1 \end{aligned}$$

6. Let's recall the following:⁷

$$\begin{aligned} EV &= e(p, v(p', y)) - e(p, v(p, y)) & CV &= e(p', v(p', y)) - e(p', v(p, y)) \\ &= e(p, v(p', y)) - y & &= y - e(p', v(p, y)) \end{aligned}$$

If you forget this, just remember the following table:

	Old Utility [$v(p^0, y)$]	New Utility [$v(p^1, y)$]
Old Prices [p^0]	Old World	EV World
New Prices [p^1]	CV World	New World

To make things clearer, let $A(p) = B - F(p)$, where $B = \sum_{l=2}^L \alpha_l \ln \alpha_l$ and $F(p) = \sum_{l=2}^L \alpha_l \ln p_l$. We need to do this because we are going to be changing p , and A is a function of p , but we don't care as

⁷The way these are written CV and EV have the same sign, where if they are positive, then the price change improved consumer welfare. In previous classes, you may have seen the CV expressed as $e(p', v(p, y)) - y$, which represents the change in income required to achieve the old utility under the new prices. In that case, the EV and CV have opposite signs, where a positive EV and negative CV indicate an increase in consumer welfare.

much about the B part (which is not a function of p). So our expenditure and indirect utility functions are:

$$\begin{aligned} e(p, u) &= u - B + F(p) + 1 \\ v(p, y) &= y + B - F(p) - 1 \end{aligned}$$

(a) The CV is

$$\begin{aligned} CV &= y - (v(p, y) - B + F(p') + 1) \\ &= y - (y + B - F(p) - 1 - B + F(p') + 1) \\ &= F(p) - F(p') \\ &= \sum_{l=2}^L \alpha_l (\ln p_l - \ln p'_l) \\ &= \sum_{l=2}^L \alpha_l \ln \left(\frac{p_l}{p'_l} \right) \end{aligned}$$

(b) The EV is:

$$\begin{aligned} EV &= (v(p', y) - B + F(p) + 1) - y \\ &= (y + B - F(p') - 1 - B + F(p) + 1) - y \\ &= F(p) - F(p') \\ &= \sum_{l=2}^L \alpha_l \ln \left(\frac{p_l}{p'_l} \right) \end{aligned}$$

(c) The CV and the EV are equal. This is a property of quasi-linear utility functions. To see this, think about the shift of the budget line we do to get the CV and EV . Since the indifference curves for quasi-linear utility are parallel, these amounts must necessarily be the same.

(d) The reform increases welfare if and only if $F(p) - F(p') \geq 0$

7. If $y < 1$, then $x_1 = 0$ and $z = y$. Then:

$$\begin{aligned} v(p, y) &= \ln y + A(p) \\ e(p, u) &= \exp(u - A(p)) \end{aligned}$$

(a) The CV is:

$$\begin{aligned} CV &= y - \exp(v(p, y) - B + F(p')) \\ &= y - \exp(\ln y + B - F(p) - B + F(p')) \\ &= y - y \exp(F(p') - F(p)) \\ &= y(1 - \exp(F(p') - F(p))) \\ &= y \left(1 - \exp \left(- \sum_{l=1}^L \alpha_l \ln \left(\frac{p_l}{p'_l} \right) \right) \right) \end{aligned}$$

(b) Compute the equivalent variation

$$\begin{aligned}
EV &= \exp(v(p', y) - B + F(p)) - y \\
&= \exp(\ln y + B - F(p') - B + F(p)) - y \\
&= y \exp(F(p) - F(p')) - y \\
&= y (\exp(F(p) - F(p')) - 1) \\
&= y \left(\exp \left(\sum_{l=1}^L \alpha_l \ln \left(\frac{p_l}{p'_l} \right) \right) - 1 \right)
\end{aligned}$$

(c) Let $K = \exp(F(p) - F(p')) = \exp \sum_{l=1}^L \alpha_l \ln \left(\frac{p_l}{p'_l} \right)$. Then $K^{-1} = \exp(F(p) - F(p'))^{-1} = \exp(F(p') - F(p))$, and:

$$\begin{aligned}
CV &= y \left(1 - \frac{1}{K} \right) \\
EV &= y(K - 1)
\end{aligned}$$

Clearly, we can see that unless $K = 1$, we have $CV \neq EV$.

(d) Since $y > 0$, CV is positive when $(1 - K^{-1}) \geq 0 \implies 1 \geq K^{-1} \implies K \geq 1$. Similarly, EV is positive when $(K - 1) \geq 0 \implies K \geq 1$. So the condition is the same for both, and this occurs when:

$$\begin{aligned}
K &\geq 1 \\
\exp(F(p) - F(p')) &\geq 1 \\
F(p) - F(p') &\geq 0
\end{aligned}$$

Which is the same condition we found in 6(d).