Advanced Micro: Recitation 3
Comparative Statics

Motaz Al-Chanati

February 15, 2019

1 Implicit Function Theorem[1]

1.1 Implicit Functions

In this section, we will consider a problem with n exogenous (independent) variable $x_1, ..., x_n$ and one endogenous (dependent) variable y. We often want an *explicit* function of y as a function of x:

$$y = F(x_1, ..., x_n)$$

Sometimes we can't get this, and instead we have a problem of the form:[2]

$$F(x_1, ..., x_n; y) = 0$$

Where $F(\cdot)$ is too complicated that we can't isolate y. Even still, we are often interested in **comparative statics**: how does a change in an exogenous variable affect the value of the endogenous variable? The purpose of the implicit function theorem (IFT) is help us answer this question (and know when we can ask it!)

As an example, consider the function:

$$F(x; y) = xy^2 - 3y - e^x = 0$$

This implicitly defines y, in other words, if we know the value of x, then we can figure out what the corresponding (unique) value of y will be. In fact, we can just the quadratic formula to solve for an explicit function for y:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 + 4xe^x}}{2x}$$

However, we won't always be so lucky to be able to solve for a function. In fact, sometimes we can't even get a function:

$$F(x; y) = x^2 + y^2 - 1 = 0$$

We know that this plots the unit circle, and in particular, for each value x, we cannot find a unique value for y. If we restrict to $y > 0$, then we can get a function of the upper semi-circle:

[1] This section draws heavily from Simon and Blume Chapter 15 and Chiang and Wainwright Chapter 8.5.

[2] I will always use a semicolon to separate the exogenous from the endogenous variables

$y = +\sqrt{1-x^2}$. What if we restricted it to $x > 0$? Then we still have the same problem as before and cannot define a function for y. This suggests that the neighborhood we are looking at it is going to be very important. In particular, around the points $(x, y) = (1, 0)$ and $(-1, 0)$, we cannot find an open ball where we can define y as a function of x.

1.2 Implicit Function Theorem (1 Endogenous)

Let's generalize the idea we just saw. For a function $F(x; y) = 0$ and point (x^*, y^*) that solves this equation (i.e. $F(x^*, y^*) = 0$), we want to know:

1. Does $F(x; y) = 0$ determine y as a continuous function of x for x near x^* and y near y^*?
2. If so, how do changes in x affect the corresponding y's?

More formally, we can ask the questions as:

1. Given the implicit equation $F(x; y) = 0$ and a point (x^*, y^*) such that $F(x^*, y^*) = 0$, does there exist a continuous function $y = y(x)$ defined on an interval I around x^* such that: (i) $F(x, y(x)) = 0, \forall x \in I$, and (ii) $y(x^*) = y^*$?
2. If $y(x)$ exists and is differentiable, what is $y'(x^*)$?

Let's take $F(x; y) = 0$ and suppose that we can find such a function $y = y(x)$ around the solution point (x^*, y^*). This means that:

$$\begin{aligned}
F(x; y(x)) &= 0 \\
\frac{d}{dx} F(x; y(x)) \bigg|_{x=x^*} &= 0 \\
\frac{\partial}{\partial x} F(x^*; y(x^*)) \cdot \frac{dx}{dx} + \frac{\partial}{\partial y} F(x^*; y(x^*)) \cdot \frac{dy(x)}{dx} \bigg|_{x=x^*} &= 0 \\
\frac{\partial F(x^*; y^*)}{\partial x} + \frac{\partial F(x^*; y^*)}{\partial y} \cdot y'(x^*) &= 0 \\
y'(x^*) &= -\frac{\dfrac{\partial F(x^*; y^*)}{\partial x}}{\dfrac{\partial F(x^*; y^*)}{\partial y}}
\end{aligned}$$

Therefore, if a solution $y(x)$ of the implicit function $F(x; y) = 0$ exists and is differentiable, then a necessary condition is that $\frac{\partial F(x^*;y^*)}{\partial y} \neq 0$. This turns out to be a sufficient condition as well.[3] This result gives us the implicit function theorem: (for the simplest case - one endogenous and one exogenous variable)

[3] Think back to the unit circle example. We could not define a function around (1,0) and using the IFT, it tells us that this is because the tangent at that point is vertical. Why does such a condition imply that we can't find a function to express y in terms of x?

Theorem 1. ***Implicit Function Theorem.*** *Let $F(x; y)$ be a C^1 function on a ball around (x^*, y^*) in $\mathbb{R}^2$. Suppose that:*

$$F(x^*; y^*) = 0, \text{ and}$$
$$\frac{\partial F(x^*; y^*)}{\partial y} \neq 0$$

Then there exists a C^1 function $y = y(x)$ defined on an interval I around the point x^* such that:

1. $F(x; y(x)) = 0, \forall x \in I$
2. $y(x^*) = y^*$
3. $y'(x^*) = -\dfrac{\frac{\partial F(x^*; y^*)}{\partial x}}{\frac{\partial F(x^*; y^*)}{\partial y}}$

Notice a few limitations of the IFT. We don't know what this interval I should be. And, we don't know what $y(x)$ is - but we do know that it does exist! The power of the IFT is that it tells us how $y(x)$ changes without us having to know what it actually is.

Let's see how this works with an example:

Example 2. Consider the equation $F(x; y) = x^2 - 3xy + y^3 - 7 = 0$ around the point $(x^*, y^*) = (4, 3)$. Check that all conditions of the IFT are met. Then, use the IFT to approximate the solution at $x = 4.3$.

Solution. First, we need to check that (x^*, y^*) is indeed a solution point. This is easy to compute: $F(4; 3) = 16 - 36 + 27 - 7 = 0$. Second, we need to check the derivative: (we only need to check the y partial derivative, but lets do both for convenience since we'll need them)

$$\frac{\partial F}{\partial x}(4; 3) = 2x - 3y|_{(4;3)} = 8 - 9 = -1$$
$$\frac{\partial F}{\partial y}(4; 3) = -3x + 3y^2|_{(4;3)} = -12 + 27 = 15 \neq 0$$

This tells us we can use the IFT, i.e. there exists a C^1 function $y = y(x)$ around $(x, y) = (4, 3)$ where:

$$y'(x) = -\frac{\frac{\partial F}{\partial x}(4; 3)}{\frac{\partial F}{\partial y}(4; 3)} = -\frac{(-1)}{(15)} = \frac{1}{15}$$

From this, we can approximate the new value of y:

$$y \approx y^* + y'(x^*)\Delta x = 3 + \left(\frac{1}{15}\right)(0.3) = 3.02$$

The true value is actually $y = 3.01475$, so this a pretty good approximation (and much easier to compute).

Example 3. Consider the utility function $u(x, y)$. Show the derivation of the MRS using the IFT

Solution 4. Consider an indifference curve I at utility level u. This is the locus of points that solve the equation: $F(x; y) = u(x, y) - u = 0$. Suppose we are at a point (x^*, y^*) on I. The MRS tells us the following: "how many units of y am I willing to give up in order to get one more unit of x (and stay indifferent to before)"? This applies directly to the IFT. The IFT tells us for a change in x, how much will y change to still solve the equation: but this exactly means to stay on the same indifference curve I. Applying the IFT to the utility function:

$$y'(x) = -\frac{\frac{\partial F(x;y)}{\partial x}}{\frac{\partial F(x;y)}{\partial y}} = -\frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} = MRS(x, y)$$

We worked with one exogenous variable and one endogenous variable. But it is easy to extend this theorem to having n exogenous variables:

Theorem 5. ***Implicit Function Theorem***. *Let $F(x_1, ..., x_n; y)$ be a C^1 function on a ball around $(x_1^*, ..., x_n^*, y^*)$ in $\mathbb{R}^{n+1}$. Suppose that:*

$$F(x_1^*, ..., x_n^*; y^*) = 0, \text{ and}$$
$$\frac{\partial F}{\partial y}(x_1^*, ..., x_n^*; y^*) \neq 0$$

Then there exists a C^1 function $y = y(x_1, ..., x_n)$ defined on an open ball B around the point $(x_1^*, ..., x_n^*)$ such that:

1. $F(x_1^*, ..., x_n^*; y(x_1, ..., x_n)) = 0, \forall (x_1, ..., x_n) \in B$
2. $y(x_1^*, ..., x_n^*) = y^*$
3. $\frac{\partial y}{\partial x_i}(x_1^*, ..., x_n^*) = -\frac{\frac{\partial F}{\partial x_i}(x_1^*, ..., x_n^*; y^*)}{\frac{\partial F}{\partial y}(x_1^*, ..., x_n^*; y^*)}, \forall i \in \{1, ..., n\}$

1.3 Implicit Function Theorem (Many Endogenous)

The next natural extension is to have many endogenous and exogenous variables. For this consider n exogenous variables $x_1, ..., x_n$ and m endogenous variables $y_1, ..., y_m$. In order to solve this, we need to have as many equations as we have endogenous variables, i.e. m equations. The system of equations is:

$$F^1(x_1, ..., x_n; y_1, ..., y_m) = 0$$
$$\vdots$$
$$F^m(x_1, ..., x_n; y_1, ..., y_m) = 0$$

Where $F^i : \mathbb{R}^{n+m} \to \mathbb{R}$, for $i \in \{1, ..., m\}$. We can collect this set of functions into one multi-dimensional function $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$. We can linearize this system around the point (x^*, y^*) by taking a total derivative and evaluating all the partial derivatives at (x^*, y^*):[4]

$$\frac{\partial F^1(x^*;y^*)}{\partial x_1}dx_1 + \ldots + \frac{\partial F^1(x^*;y^*)}{\partial x_n}dx_n + \frac{\partial F^1(x^*;y^*)}{\partial y_1}dy_1 + \ldots + \frac{\partial F^1(x^*;y)}{\partial y_m}dy_m = 0$$
$$\vdots$$
$$\frac{\partial F^m(x^*;y^*)}{\partial x_1}dx_1 + \ldots + \frac{\partial F^m(x^*;y^*)}{\partial x_n}dx_n + \frac{\partial F^m(x^*;y^*)}{\partial y_1}dy_1 + \ldots + \frac{\partial F^m(x^*;y^*)}{\partial y_m}dy_m = 0$$

In block matrix form, we could write this as:[5]

$$\begin{pmatrix} J_x & J_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = 0$$

Where J_x is the Jacobian of F for the x partials and dx is a vector of dx_i's (and the corresponding definition for y). We can bring the first n terms (with dx_i) to the other side and write the second m terms (with dy_j) as the product of two matrices: a Jacobian matrix for the y partials and a vector of dy_j's.

$$\underbrace{\begin{pmatrix} \frac{\partial F^1(x^*;y^*)}{\partial y_1} & \ldots & \frac{\partial F^1(x^*;y^*)}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial y_1} & \ldots & \frac{\partial F^m(x^*;y^*)}{\partial y_m} \end{pmatrix}}_{m\times m} \underbrace{\begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix}}_{m\times 1} = - \underbrace{\begin{pmatrix} \sum_{i=1}^{n} \frac{\partial F^1(x^*;y^*)}{\partial x_i}dx_i \\ \vdots \\ \sum_{i=1}^{n} \frac{\partial F^m(x^*;y^*)}{\partial x_i}dx_i \end{pmatrix}}_{m\times 1}$$

$$J_y dy = -J_x dx$$

If the Jacobian J_y is invertible, then we can find an expression for the dy_j's. In particular, by the IFT, this tells us that we can express y as a function of x: $y_j(x_1, ..., x_n), \forall j \in \{1, ..., m\}$. Note that then each dy_j can be expressed as:

$$dy_j = \frac{\partial y_j}{\partial x_1}dx_1 + \ldots + \frac{\partial y_j}{\partial x_n}dx_n$$
$$\implies dy = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \ldots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \ldots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = J_x^y dx$$

[4]Often in books you will just see the partial derivatives written as something like $\frac{\partial F^j}{\partial x_i}$, but you have to remember that they all being evaluated at the initial solution point

[5]Instead of $\begin{pmatrix} J_x & J_y \end{pmatrix}$, you will often see this written as $\frac{\partial(F^1,...,F^m)}{\partial(x_1,...,x_n,y_1,...,y_m)}$. But I think this is more concise and really highlights that you should be thinking of this as a matrix. Throughout this, I'll be quite loose and non-standard with my notation, but that's because I find the usual matrix notation for the IFT to be confusing and overwhelming (I hope my approach is less so for you!)

Where J_x^y is the Jacobian of y with respect to x. This allows us to re-write the equation above as:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = - \begin{pmatrix} \frac{\partial F^1(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^1(x^*;y^*)}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F^1(x^*;y)}{\partial x_1} & \cdots & \frac{\partial F^1(x^*;y)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y)}{\partial x_1} & \cdots & \frac{\partial F^m(x^*;y)}{\partial x_m} \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

$$J_x^y dx = -J_y^{-1} J_x dx$$
$$J_x^y = -J_y^{-1} J_x$$

The last line follows since this equation holds for all dx.[6]

Next, we want to know what is the effect of a change in a particular endogenous variable x_h on a particular exogenous variable y_k, i.e. $\partial y_k / \partial x_h$. For this, we consider a one unit change in h and nothing else: $dx_h = 1$ and $dx_i = 0, \forall i \neq h$. This simplifies $dy_j = \partial y_j / \partial x_h, \forall j \in \{1, ..., m\}$ and $\sum_i \frac{\partial F^a}{\partial x_i} dx_i = \partial F^a / \partial x_h, \forall a \in \{1, ..., m\}$. Doing all this simplifies the above equation to:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_h} \\ \vdots \\ \frac{\partial y_m}{\partial x_h} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F^1(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^1(x^*;y^*)}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F^1(x^*;y)}{\partial x_h} \\ \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial x_h} \end{pmatrix}$$
$$\nabla_h y = -J_y^{-1} \nabla_h F$$

Where $\nabla_h y$ is the vector of partial derivatives for y with respect to x_h (and similarly for F).

If we just want the value of one partial, we can use Cramer's rule:[7]

$$\frac{\partial y_k}{\partial x_h} = - \frac{\det \begin{pmatrix} \frac{\partial F^1(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^1(x^*;y^*)}{\partial x_h} & \cdots & \frac{\partial F^1(x^*;y^*)}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial x_h} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial y_m} \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial F^1(x^*;y)}{\partial y_1} & \cdots & \frac{\partial F^1(x^*;y^*)}{\partial y_k} & \cdots & \frac{\partial F^1(x^*;y)}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial F^m(x^*;y^*)}{\partial y_1} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial y_k} & \cdots & \frac{\partial F^m(x^*;y^*)}{\partial y_m} \end{pmatrix}} = - \frac{\det((J_y)_k)}{\det(J_y)}$$

This gives us an IFT that is completely analogous to before:

[6] If $Ax = 0, \forall x$ then A is the zero matrix

[7] For a system of equations $Ax = b$, where A is a $n \times n$ invertible matrix, then Cramer's rule is that $x_i = \frac{\det(A_i)}{\det(A)}$, where A_i is the matrix formed by replacing the i^{th} column of A by the column vector b.

Theorem 6. ***Implicit Function Theorem.*** *Let $\{F^i(x_1,...,x_n;y_1,...,y_m)\}_{i=1}^m$ be C^1 functions on a ball around $(x_1^*,...,x_n^*,y_1^*,...,y_m^*)$ in $\mathbb{R}^{n+m}$. Suppose that:*

$$\begin{pmatrix} J_x^* & J_y^* \end{pmatrix} = \begin{pmatrix} \vdots \\ F^j(x_1^*,...,x_n^*;y_1^*,...,y_m^*) \\ \vdots \end{pmatrix} = 0, \text{ and}$$

$$det(J_y^*) = det \begin{pmatrix} \vdots \\ \frac{\partial F^j}{\partial y}(x_1^*,...,x_n^*;y_1^*,...,y_m^*) \\ \vdots \end{pmatrix} \neq 0$$

Then there exists C^1 functions $\{y = y(x_1,...,x_n)\}_{i=1}^n$ defined on an open ball B around the point $(x_1^*,...,x_n^*)$ such that:

1. $\begin{pmatrix} J_x^* & J_y^* \end{pmatrix} = \begin{pmatrix} \vdots \\ F^i(x_1^*,...,x_n^*;y_1(x^*),..,y_m^*(x^*)) \\ \vdots \end{pmatrix} = 0, \forall (x_1,...,x_n) \in B$

2. $y(x_1^*,...,x_n^*) = y^*$

3. $\frac{\partial y_k}{\partial x_h}(x_1^*,...,x_n^*) \implies$ Invert or use Cramer's rule

1.4 Example

You should go through the IFT example handout we saw in class. As another practice problem, this is Example 15.15 in Simon and Blume (with slightly different notation to make it clearer)

Example 7. Consider the system of equations: $(n = 1, m = 2)$

$$F^1(x;y_1,y_2) = y_1^2 + xy_1y_2 + y_2^2 - 1 = 0$$
$$F^2(x;y_1,y_2) = y_1^2 + y_2^2 - x^2 + 3 = 0$$

Around the point $(x_1, y_1, y_2) = (2, 0, 1)$. If we change x a little to x' near $x = 2$, can we find (y_1', y_2') near $(0, 1)$ so that (x', y_1', y_2') satisfies these two equations?

Solution. First, we should check that this is indeed a solution. $F^1(2; 0, 1) = 0 + 0 + 1 - 1 = 0$. $F^2(2; 0, 1) = 0 + 1 - 4 + 3 = 0$. So it is a solution to the problem.

Second, we need the Jacobian of F with respect to the endogenous variables at $(2, 0, 1)$, i.e. J_y^*:

$$J_y^* = \begin{pmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 2y_1 + xy_2 & xy_1 + 2y_2 \\ 2y_1 & 2y_2 \end{pmatrix}\Bigg|_{(2,0,1)} = \begin{pmatrix} 0 + 2 \cdot 1 & 2 \cdot 0 + 2 \cdot 1 \\ 2 \cdot 0 & 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

We need to make sure this invertible. Checking its determinant:

$$\det(J_y^*) = 2 \cdot 2 - 2 \cdot 0 = 4 \neq 0$$

So we can solve this system for y_1 and y_2 as functions of x near $(2,0,1)$. Let's calculate the effect of the change on x on y_1 by setting up the linearized system of equations and evaluating it at $(x^*, y_1^*, y_2^*) = (2,0,1)$:

$$\begin{pmatrix} J_x & J_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = 0$$

$$\begin{pmatrix} y_1 y_2 & 2y_1 + xy_2 & xy_1 + 2y_2 \\ -2x & 2y_1 & 2y_2 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 2 & 2 \\ -4 & 0 & 2 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = 0$$

Writing this out explicitly gives us:

$$\begin{aligned} 0dx + 2dy_1 + 2dy_2 &= 0 \\ -4dx + 0dy_1 + 2dy_2 &= 0 \end{aligned}$$

And with some substitution, we get:

$$dy_1 = -dy_2 = -2dx$$

Alternatively, you could invert the matrix:

$$\begin{aligned} dy = \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix} dx \\ &= \begin{pmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} dx = \begin{pmatrix} -2dx \\ 2dx \end{pmatrix} \end{aligned}$$

Therefore, an increase in x to $x' = 2.1$ (i.e. $\Delta x = 0.1$), then y_1 will decrease to $y_1' \approx y_1 + \frac{dy_1}{dx} \cdot \Delta x = 0 + (-2)(0.1) = -0.2$.

Let's calculate the effect of the change on x on y_2 around $(2,0,1)$ using Cramer's rule:

$$\begin{aligned} \frac{\partial y_2}{\partial x} = -\frac{\det\left(\begin{pmatrix} F^1_{y_1} & F^1_x \\ F^2_{y_1} & F^2_x \end{pmatrix}\right)}{\det(J_y)} &= -\left.\frac{\det\left(\begin{pmatrix} 2y_1 + xy_2 & y_1 y_2 \\ 2y_1 & -2x \end{pmatrix}\right)}{\det\left(\begin{pmatrix} 2y_1 + xy_2 & xy_1 + 2y_2 \\ 2y_1 & 2y_2 \end{pmatrix}\right)}\right|_{(2,0,1)} \\ &= -\frac{\det\left(\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\right)}{\det\left(\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}\right)} \\ &= -\frac{-8}{4} = 2 > 0 \end{aligned}$$

Notice that this is the same as what we got above when we inverted the matrix (and solved for all the derivatives). So an increase in x to $x' = 2.1$ (i.e. $\Delta x = 0.1$), then y_2 will increase to $y_2' \approx y_2 + \frac{\partial y_2}{\partial x} \cdot \Delta x = 1 + (2)(0.1) = 1.2$.

2 The Envelope Theorem[8]

Note: I recommend you look at pages 19-25 in my PhD Micro notes, which covers the Envelope Theorem, Shephard's Lemma, and Roy's Identity.

3 The Theorem

The Envelope Theorem is a general result about maximization problems. In class, we derived the indirect utility function, $v(p, y)$, which tells us the consumer's utility as a function of prices p and income y. In many situations we want to consider the value some economic agent receives as a function of exogenous variables (e.g. the prices and income in the consumer problem). The Envelope Theorem helps us with comparative statics in these situations, by giving us an expression for how the value function changes as we change the parameters.

Theorem 8. *Consider the program*

$$V(a) = \max_x f(x, a) \;\; s.t. \;\; g(x, a) = 0$$

with associated Lagrangian $\mathcal{L}(x, \lambda, a) = f(x, a) - \lambda g(x, a)$ *and solution* $x(a), \lambda(a)$*. If* V *and* $\mathcal{L}$ *are differentiable at* a*, then*

$$V'(a) = f_a(x(a), a) - \lambda(a) g_a(x(a), a)$$

In this setup, a represents a vector of exogenous variables, and x the choice variables. We won't prove this result formally, but let's think about what it's saying.

Notice $V(a) = \mathcal{L}(x(a), \lambda(a), a)$, since $g(x(a), a) = 0$. Let's differentiate with respect to a:

$$V'(a) = \mathcal{L}_x(x(a), \lambda(a), a)x'(a) + \mathcal{L}_\lambda(x(a), \lambda(a), a)\lambda'(a) + \mathcal{L}_a(x(a), \lambda(a), a)$$

This expression has two parts: the first two terms reflect the fact that when a changes, x and λ change, which changes $\mathcal{L}$. The last term captures the direct effect of a on $\mathcal{L}$. But remember we find $x(a)$ and $\lambda(a)$ is by solving the first-order conditions $\mathcal{L}_x = 0$ and $\mathcal{L}_\lambda = 0$! Thus the expression simplifies to

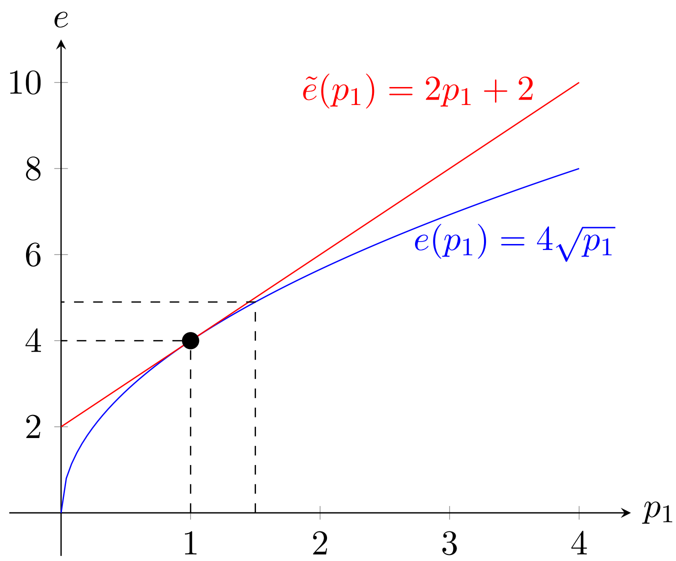
$$\begin{aligned} V'(a) &= \mathcal{L}_a(x(a), \lambda(a), a) \\ &= f_a(x(a), a) - \lambda(a) g_a(x(a), a) \end{aligned}$$

The Envelope Theorem says the following: when we change the parameters of the problem, we can ignore their effect on our optimal actions. Yes, when a changes, $x(a)$ changes as well. However, since we were choosing x optimally, that effect is second order, so the primary effect is the direct effect of changing the parameters.

You can see this intuitively in an example. Consider a consumer with utility function $u(x) = x_1 x_2$. You can solve their EMP to get:

$$h(p, u) = \left(\sqrt{\frac{p_2}{p_1}u}, \sqrt{\frac{p_1}{p_2}u}\right) \qquad\qquad e(p, u) = 2\sqrt{p_1 p_2 u}$$

[8] This section is largely adapted from David Thompson's 2018 recitation notes



Let's suppose we are initially at the point $(p_1, p_2, u) = (1, 1, 4)$. The Hicksian demand tells us that the optimal bundle is $(2, 2)$ and the expenditure for this is \$4. If we plotted the expenditure function on how it varied with p_1 (holding fixed p_2 and u) it would be $e = 2\sqrt{p_1 \cdot 1 \cdot 4} = 4\sqrt{p_1}$ (this is plotted above in blue as $e(p_1)$). But also imagine a "constrained" expenditure function where we are forced to keep $h(p, u)$ the same. This would be $e = p_1 h_1^* + p_2 h_2^* = 2p_1 + 2$ (this is plotted above in red as $\tilde{e}(p_1)$). The black line shows how the expenditure would change if we could resolve optimally. The red line shows us how it would change if we were constrained to keeping the same bundle. These graphs touch at $(1, 4)$, i.e. the initial optimal (p_1, e) point. Moreover, we can see that for a neighborhood around $(1, 4)$, the values are approximately similar. Let's consider 50 cent change in p_1 to 1.5. This gives us $\Delta e = e(1.5) - e(1) \approx 0.899$ and $\Delta\tilde{e} = \tilde{e}(1.5) - \tilde{e}(1) = 1$. This is the point of the Envelope Theorem: this constrained change is a good approximation for the optimal change (for a small change in the parameter)

You can also see this by drawing an indifference curve-budget set diagram. Compare three budget lines: 1) the BL at the old prices going through the old optimal bundle; 2) the BL at the new prices going through the new optimal bundle; 3) the BL at the new prices going through the old optimal bundle. Cases (1) and (2) are both optimal points, but (3) is not. However, you will notice that the the change in expenditure of going from (1) to (2) is largely captured by the change of (1) to (3). You've also probably seen the Envelope Theorem in action in intermediate micro with short-run and long-run average cost curves. If you recall, the LRAC "envelopes" all the SRACs. Here, a similar idea holds. The true expenditure function "envelopes" all the constrained expenditure functions. If you zoom in on a the neighborhood of where they are tangent, then the constrained and fully-flexible value functions will look very similar.

Next, let's look at some concrete examples.

3.1 Unconstrained Optimization: Profit Maximization

Consider the case of unconstrained optimization. In this case, there is no constraint, so $g(x, a) = 0$. In this case, the Envelope Theorem says $V'(a) = f_a(x(a), a)$. Consider the following example:

- A monopolist has constant marginal costs c and faces downward-sloping demand $Q = D(p)$
- The monopolist's goal is to maximize profits. Write her profits as a function of c:

$$\pi(c) = \max_p pQ - cQ = (p - c)D(p)$$

- What is $\pi'(c)$? The Envelope Theorem says: calculate the partial derivative of the objective function with respect to c, and then evaluate it at the optimal prices:

$$\pi'(c) = \underbrace{\frac{\partial \pi}{\partial p}}_{=0} \frac{dp}{dc} + \frac{\partial \pi}{\partial c} = -D(p(c))$$

- How simple was that? We never even had to solve for $p(c)$! The intuition in this example is almost deceivingly simple: if the monopolist's costs increase by \$1 and she sells D units, her profits decrease by \$$D$. Yes, when her costs increase her optimal price changes, which changes the demand she faces, but that effect is negligible relative to the direct cost effect

3.2 Equality Constrained Optimization: The Consumer's Problem

In class we've studied the consumer's problem:

$$v(p, y) = \max_x u(x) \text{ s.t. } px \leq y$$

This fits into the envelope theorem setup with:

- $a = (p, y)$
- $f(x, a) = u(x)$
- $g(x, a) = y - px$

Let's apply the Envelope Theorem: to find v_p and v_y, form the Lagrangian, take derivatives with respect to p and y, and evaluate at the optimal bundle. Since u depends only on x, its partial derivative with respect to p and y is 0.

$$\begin{aligned} v_p(p, y) &= -\lambda(p, y)x(p, y) \\ v_y(p, y) &= \lambda(p, y) \end{aligned}$$

Let's make a few comments about these equations:

- For the second equation: $v_y(p, y) = \lambda(p, y)$. This says the Lagrange multiplier measures additional value the consumer receives from \$1. This intuition holds more generally: Lagrange multipliers measure the value of relaxing constraints by 1 unit

- For the first equation: if the price of a good increases by \$1 and we buy x units, then we have to spend \$$x$ more to afford our bundle. Since we value money at a rate of λ, our total utility loss is λx.

- Lastly, dividing these two equations gives us Roy's Identity:

$$x(p, y) = -\frac{v_p(p, y)}{v_y(p, y)}$$

In a similar fashion, Shephard's Lemma for the expenditure function also follows immediately from the Envelope Theorem.