Intermediate Micro: Recitation 3
Preferences and Utility

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1 Preferences

1.1 Introduction

Let's recall that our goal here is to model how a consumer makes their choice of goods to purchase. Put in another way: say we observe two people making a different purchase - why do they differ in their decisions? In the last recitation, we discussed budget sets. This was a constraint in the consumer's optimization problem, i.e. it would affect the consumer's choice of goods. The budget constraint defined the set of bundles that the consumer was allowed to pick from. This suggests that part of the reason that we see consumers picking different goods is because they have different budget constraints. For example, when the newest iPhone comes out, some people are able to afford it (and do purchase it), and some people do not. But clearly this cannot be the entire story. When the newest iPhone comes out, there are lots of people who can afford it but choose not to purchase it. Here's another thought experiment to motivate this section. Imagine if I gave everyone in class the same budget, say \$10 for lunch, and told everyone to go out and purchase some lunch (without using your own money). We would all probably come back with different things. Why? Clearly, we all like different things, so our *preferences* must also play a role in the consumer's problem.

Preferences are something we all have, but it is often hard to describe what they are. In economics, we're going to think about preferences as a **binary relation**. This just means that preferences takes two objects, compares them, and tells us which one the consumer likes more. Say our two goods are x and y, we would say that a consumer likes x at least as much as y by writing $x \succsim y$. The symbol $\succsim$ is a shorthand for "is liked at least as much as" and is used as a symbol for a preference relation. This doesn't tell me *how much* more I like x more than y - it just tells me that it is liked at least as much. The $\succsim$ is analogous to the greater than symbol $\geq$ we use to compare numbers. It's just like saying that $4 \geq 3$ or $10 \geq 2$ or $5 \geq 5$; I use $\geq$ to compare two numbers but it doesn't say anything about the size of the difference between them (other than it's positive). In math we also use $>$ for strictly greater than and $=$ for equals to. We can define $x > y$ as being whenever $x \geq y$ and not $y \geq x$. Similarly, we can define $x = y$ as being whenever $x \geq y$ and $y \geq x$. Using the same idea to preferences, we have three symbols to use for preferences:

- $\succsim$: weakly preferred ("is liked at least as much as")
- $\succ$: strictly preferred ("is liked more than"), when $x \succsim y$ but not $y \succsim x$
- $\sim$: indifferent ("is liked the same as"), when $x \succsim y$ and $y \succsim x$

1.2 Properties

Preferences can get pretty unwieldy, so we usually like to put some restrictions on them so it makes them easier to work with. Here are some of the properties that we usually assume hold true for any preferences that we'll see in the class: (note that when I say "prefer", I always mean "weakly prefer" unless I explicitly say "strictly prefer")

- **Complete:** If you give me any two objects, I can always say which one I prefer. Formally: for any two bundles x and y, we must have that $x \succsim y$, $y \succsim x$, or both.
  - **Reflexive:** A good is *weakly* preferred to itself. Formally: $x \succsim x$ for any bundle x. This is usually ignored since complete preferences must also be reflexive (let $y = x$ in the above definition)
- **Transitive:** This essentially says that I make predictable choices which allows me to rank choices unambiguously. Formally: if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.
- **Monotonic:** I always *strictly* prefer having more of a good (this rules out having "bad" goods). Formally: for any two bundles $x = (x_1, x_2)$ and $y = (y_1, y_2)$, if $x_1 \geq y_1$ and $x_2 \geq y_2$ and $x_i > y_i$ for either $i = 1$ or 2, then $x \succ y$.
- **Convex:** I generally prefer having a mix of two goods than having only one type of good ("averages are better than extremes"). Formally: if $x \succsim z$ and $y \succsim z$, then $\lambda x + (1 - \lambda)y \succsim z$ for any $\lambda \in [0, 1]$. Graphically, draw a line between any two points on the same indifference curve, and that entire line needs to be on or above the indifference curve.

Completeness and transitivity are the bedrock for a lot of what we do in economics - it's quite hard to make much progress without these two. We say that a consumer has **rational preferences** if they have complete and transitive preferences. If their preferences satisfy all the above properties, we usually say that they have **well-behaved preferences**. These aren't all absolutely essential, but they are often quite reasonable assumptions that makes our lives much easier.

Let's look at some examples to check our understanding:

1. From a choice of $\{A, B, C\}$, I have $A \succ B$, $B \succ C$, $C \succ A$ (and assume reflexivity). Is this complete and transitive?
2. I choose my favorite car based on its top speed. What properties does this satisfy?
3. I like having more sugar in my coffee than less, up to 1 spoon. After that, I like having less and less sugar. Is this monotonic?

4. The symbol P indicates the relation "is a parent of". Is it complete, reflexive, and transitive?

5. A couple A and B make decisions together. They both have complete and transitive preferences $\succsim_A$ and $\succsim_B$, respectively. The relation $\succsim$ indicates that both A and B prefer it, i.e. if $x \succsim_A y$ and $x \succsim_B y$, then $x \succsim y$. Is it complete and transitive?

6. I am indifferent between having n granules of sugar in my coffee and having $n+1$ granules of sugar for $0 \leq n < 100$. However, I strictly prefer having 100 granules of sugar to 0 granules. Is this monotonic? Transitive?

Here are the answers:

1. Yes it is complete (every pair of bundles is compared). It is not transitive! We have a cycle: $A \succ B \succ C \succ A$ which is indicative of intransitive preferences. No

2. It satisfies all of them. Pick any three numbers x, y, z as top speeds of three cars and apply to the above definitions.

3. It is not monotonic - if it was, I should prefer 2 spoons to 1 spoon.

4. It is not complete - take two siblings A and B for example. Neither are each other's parent, so they cannot be compared. It is not even reflexive - you are not your own parent! It is also not transitive. If A was B's parent, and B was $C's$ parent, then A is not C's parent (your grandparents are not your parents).

5. It is not complete (counter-example: $x \succsim_A y$ and $y \succsim_B x$, then we can't say anything about x and y in terms of $\succsim$). It is transitive. First, assume that $x \succsim y$ and $y \succsim z$. This means that $x \succsim_A y$ and $x \succsim_B y$, as well as $y \succsim_A z$ and $y \succsim_B z$. Since each of them have transitive preferences, then we must also have $x \succsim_A z$ and $x \succsim_B z$, which then means that $x \succsim z$. Therefore this preference is transitive.

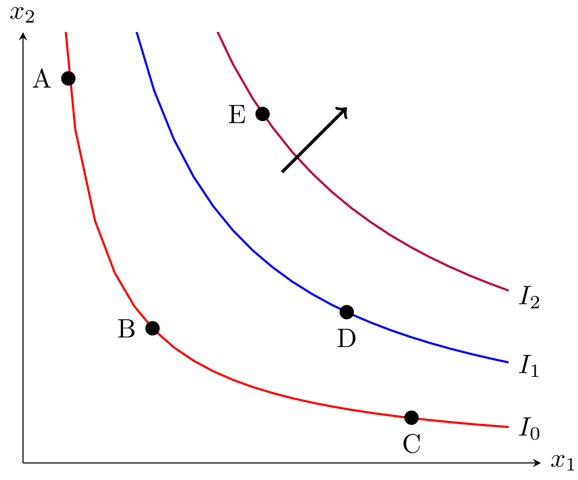
6. It is not monotonic: since I have strictly more sugar, I should also strictly prefer it. It is also not transitive. Let g indicate granules and note that $0g \sim 1g, 1g \sim 2g, ...98g \sim 99g,$ and $99g \sim 100g$. Each indifference implies a weak preference, e.g. $0g \succsim 1g$, so if we use transitivity over and over again, we get $0g \succsim 2g$, $0g \succsim 3g, ...$ etc. Eventually, transitivity would imply that $0g \succsim 100g$. However, we know that $100g \succ 0g$ which means that it *cannot* be the case that $0g \succsim 100g$. So we have a contradiction, and hence transitivity is violated.

1.3 Indifference Curves

We want to have a graphical representation of a person's preferences. We draw them when we are working with the two good case. So our goods are x_1 and x_2 and we are looking at how the consumer prefers different bundles of (x_1, x_2). We represent their preferences through **indifference curves.** Indifference curves indicate the set of bundles that the consumer feels indifferent between. These are usually depicted as in Figure 1.

Any two points on an indifference curve must have the relation $\sim$. In Figure 1, $A \sim B$ and $B \sim C$ and $A \sim C$, since all three of those points are on the indifference curve I_0. Higher indifference curves also indicate

Figure 1: Indifference Curves



a higher preference. How do we know which is the "higher" indifference curve? We usually draw an arrow indicating the direction indicating what is higher. If the arrow isn't drawn, it is assumed to point to the top right, as in Figure 1. In this example, we have I_1 higher than I_0 and I_2 higher than I_1. This means that all points on I_1 are strictly preferred to I_0 and all points on I_2 are strictly preferred to I_1, e.g. $D \succ A$ and $E \succ D$. You can also see transitivity in action too, because I_2 is also higher than I_0 so we also have $E \succ A$ (which is what transitivity would imply). Why are all these strict preferences? Well we know that it should at least be a weak preference (e.g. $E \succsim A$) since it's on a higher indifference curve. But it cannot be the reverse, i.e.$A \succsim E$, because that would imply that the two bundles are indifferent ($E \sim A$). If they were indifferent then they should be on the same indifference curve! So you can think of an indifference curve as a boundary:

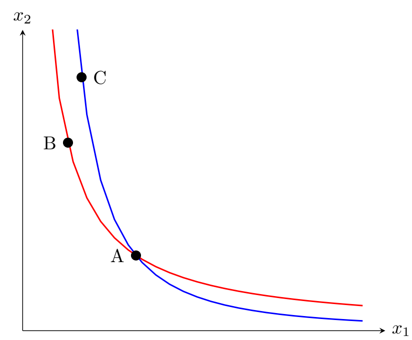
- For any point <u>above</u> the indifference curve, the consumer *strictly prefers* it to any point <u>on</u> the indifference curve
- For any point <u>on</u> the indifference curve, the consumer *is indifferent* between it and any other point <u>on</u> the indifference curve
- For any point <u>below</u> the indifference curve, the consumer *strictly does not prefer* it to any point <u>on</u> the indifference curve

Note that above and below are all relative to the direction of the arrow (we will see examples of this later on). You may encounter some odd looking indifference curves, but for the most part they will generally have the following properties:

1. **Every point has an indifference curve passing through it:** This comes from completeness. We need to be able to compare any two points, so we need to be able to compare which is on the higher indifference curve. This means they need to be on an indifference curve in the first place. Keep in mind that when we draw a diagram, we only draw a few indifference curves. In reality, there are an infinite number of indifference curves taking up the entire space.

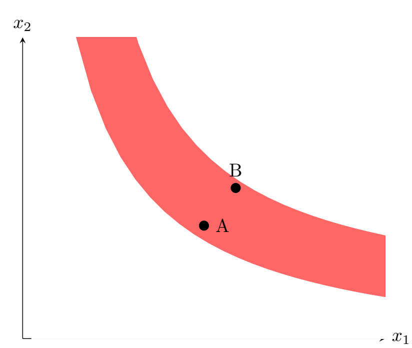
2. **Higher indifference curves are more preferred:** This comes from monotonicity. We often draw an arrow indicating the direction of preference, which is especially useful in weird cases, such as having "bads". But the general idea is that as we increase the quantities of each good, the person prefers the bundles more.

3. **Indifference curves never cross:** This comes from transitivity. Consider the example below where we allow them to cross. Since A and B are on the same red indifference curve, then $A \sim B$. Additionally, both A and C are on the blue indifference curve, so $A \sim C$. By transitivity, we must have that $B \sim C$. But that would mean that B and C should be on the same indifference curve too - and they are clearly not. In fact, C is on a higher indifference curve than B, so it must be that $C \succ B$ (C has more of both goods than B, so it should be strictly preferred by monotonicity). But this means that $B \succsim C$ cannot be true! You can also think of this property as each point only being on one indifference curve.



4. **Downward sloping:** This comes from monotonicity. Say I give you more of x_1 - by monotonicity, this new bundle must be strictly more preferred (you have more of x_1 and the same amount of x_2). But for you to be on the same indifference curve, I need to you to be indifferent to your old bundle, which means I need reduce your happiness. I can do this by taking away some of your x_2 - I take away just enough so that you are just as happy as before. So I increase your x_1 (moving right on the x-axis) and decrease your x_2 (moving down on the y-axis). This is exactly the definition of having a downward slope.

5. **Thin curves:** This comes from monotonicity. Indifference curves are lines - they are not areas, otherwise this would violate transitivity. See the example below where the red shaded area is *one* indifference curve. Both A and B lie on the indifference curve, so we must have that $A \sim B$. However, B has more of both x_1 and x_2 than A. This violates monotonicity.



6. **Convex curves:** This comes from convexity. This gives us indifference curves that are curved towards the origin. It doesn't necessarily have to curve in (we will see straight-line indifference curves), but it cannot curve *outwards*.

1.4 Examples

Let's go through a few examples of indifference curves. Just keep in mind the following rules to interpret an indifference curves:

- Any two points on the same curve must be equally preferred
- The direction of higher indifference curves is usually indicated by an arrow
- The slope indicates the substitution between goods to ensure indifference:
  - A negative slope means that when you get more x_1, I have to take away some x_2 to make sure you are left indifferent to before. This means both goods are "goods" (desirable).
  - A positive slope means that when you get more x_1, I have to give you some x_2 to make sure you are left indifferent to before. This means x_1 is a "bad" (undesirable).
  - A slope of 0 (horizontal line) means that no matter how much more x_1 you get, you will always be left indifferent to before. This means x_1 is a "neutral" (has no impact on happiness).
  - A slope of ∞ (vertical line) means that no matter how much more x_2 you get, you will always be left indifferent to before. This means x_2 is a "neutral".
- The way the slope changes (the second derivative) tells us how much mixing the consumer likes
  - If the slope is constant, then the consumer doesn't care about mixing

The graphs are shown on the next page. I will go over these in more detail in recitation, but some general things to note is as follows:

(a) The ICs slope downwards and are increasing in the usual direction, so monotonicity is satisfied. Even though they are straight lines, they satisfy convexity (but not *strict* convexity). The slope is also constant, since the ICs are linear. This means that the consumer is always willing to substitute the goods at a constant rate - they don't care about mixing at all. This is indicative of **perfect substitutes**. For example, you can think of Left Twix and Right Twix as perfect substitutes. But note that that two objects don't have to be perfect substitutes in a one-to-one proportion. For example, 4 quarts of milk is a perfect substitute for 1 gallon of milk.

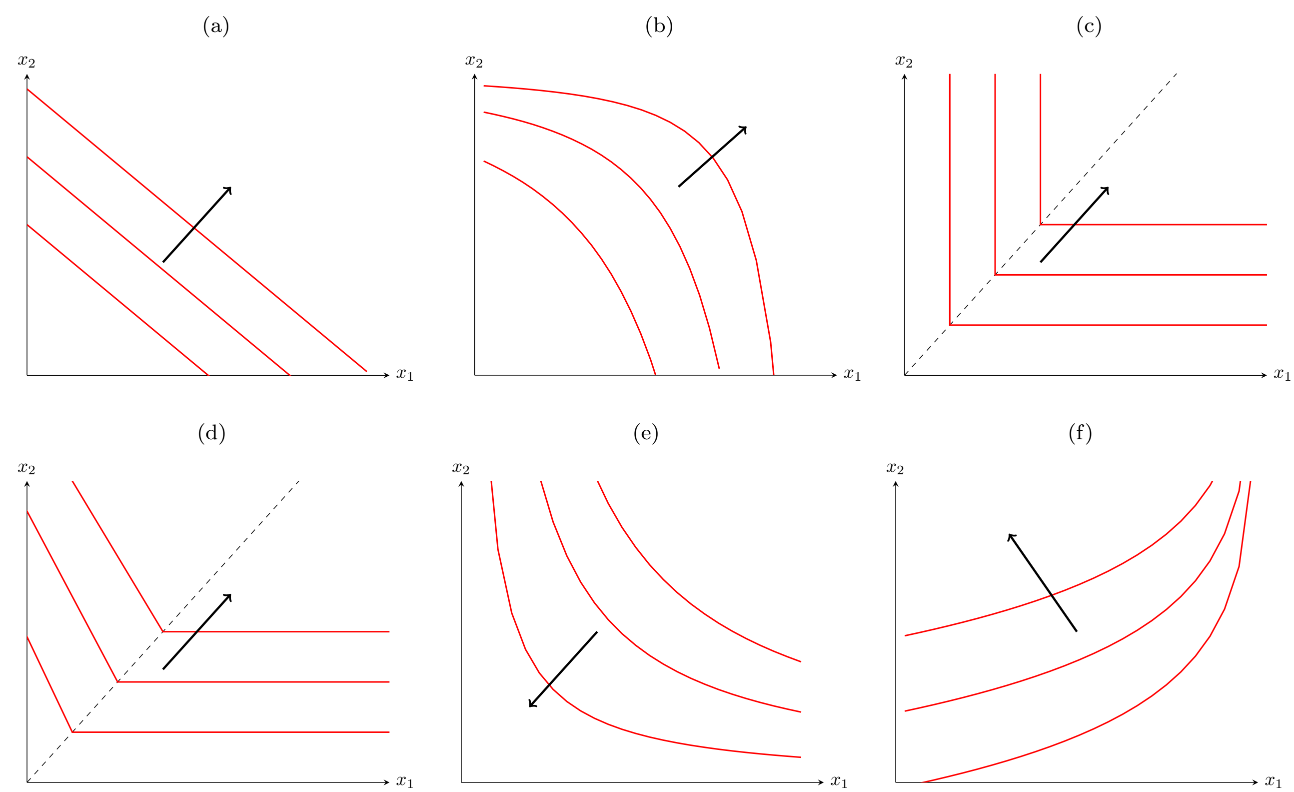
(b) The ICs are clearly not convex, but they are still monotonic. This is the preference of an anti-mixer: I'd much rather have extremes of either good than a mix of both.

(c) These “L-shaped”’ ICs are still convex and monotonic. Take any IC and look at the point it bends (what we call the “kink”). If you move straight upwards from the kink, then it says that adding more x_2 gives you no extra happiness (for the current level of x_1). Similarly, if you move to the right from the kink, then it says that adding more x_1 gives you no extra happiness (for the current level of x_2). This is the ultimate mixer. To get higher utility, it’s not enough to just get more of one good. You have to increase both goods in a particular proportion. These ICs are indicative of **perfect complements**. For example, left shoes and right shoes are perfect complements - it’s hard to make a good use of just one shoe. Again, it doesn’t have to be in a one-to-one proportion. Preferences for having one car and four tires are also an example of perfect complements.

(c) This IC is a mix of (a) and (d). To the left of the kink we have perfect substitutes. But at some point, extra x_1 no longer gives the consumer extra happiness. x_1 goes from being a perfect substitute for x_2 to being a “neutral” good (something that the consumer doesn’t really care about). I have no real world applications of these types of preferences, so let me know if you can come up with one!

(e) These are like standard ICs, but the big difference is the arrow. Now, the arrow points down which means that lower ICs are actually more preferred. So despite their shape, the preference is neither monotonic nor convex.

(f) The slope is upward sloping, which means that one good is actually a bad. In this case, it is x_1. Think about it like this - if I increase your x_1, I also have to increase your x_2 in order to make you just as happy as before and stay on the same IC. This must mean that you really hate having x_1. These preferences are still convex because you still prefer mixing.



2 Utility

2.1 Utility Representation

At this point, we understand preferences and how to express them. But remember that our goal was to model the consumer's problem, and that we wanted to do this using an optimization problem. For this, we need an *objective function* (the thing that the consumer is trying to maximize). We know that the consumer wants to maximize their 'happiness', but we need a mathematical formula for this and preferences are not a function! So the next step is try to think about how we can model a consumer's preferences into a function. Think of this as trying to create a robot that behaves in the same way as a consumer. Suppose I give the consumer a choice between apples and bananas and the consumer chooses apples. If your robot is good at replicating the consumer, then it should also choose the apple (without seeing what the consumer has chosen). Whatever the consumer chooses, your robot would also choose (given the same set of options). Now think about how each of them got to their choice. For the consumer, they used their preferences to decide which thing they preferred. Your robot used some fancy algorithm to figure out the choice - something that is very complicated and definitely does not reflect what the consumer is doing in their head. But, the end result is the same: the robot and consumer make the same choice. The natural question would then be: what is the fancy algorithm that the robot uses? This is where **utility functions** come in.

Utility functions are a mathematical formula that assigns a number to a bundle of goods. We express it as $u(x)$, where x is a bundle and $u(\cdot)$ is the function. This number represents the utility or happiness that the consumer gets from consuming that bundle. So our robot looks at a bundle, plugs it into the utility function, and gets a number for it. When it comes to choosing the most preferred bundle, the robot takes the option that gives the highest utility (the biggest number). Keep in mind that utility functions are just models. Modeling means that we take something we observe in the real world and turn into a (simplified) mathematical formulation. In the case of the budget set, this was easy to model because we can observe and count monetary amounts. I can see how much money you spend (expenditure) versus how much money you have (income) and compare the two to see whether you are meeting your budget constraint. But modeling what makes a person a happy is not as obvious. Just keep in mind the following: **preferences are real, utility functions are models**.

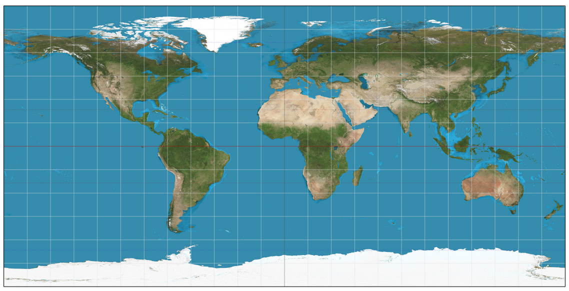
If a function is a good model for a preference, we say that **a utility function $u(\cdot)$ represents a preference relation $\succsim$**. In mathematical terms, this means that whenever the consumer prefers a bundle x to another bundle y (i.e. $x \succsim y$), then it must be the case that the utility from x is higher than y (i.e. $u(x) \geq u(y)$). Additionally, this has to work the other way: if $u(x) \geq u(y)$, then it must be that $x \succsim y$.

What makes utility functions a good model for preferences? Here's an analogy using maps. In Figure 2(a) is a standard map of the world that we usually see. This uses a particular projection to turn a 3D object into a 2D representation. We know this isn't how the Earth looks like, but it's still very useful. It helps us see where countries are relative to one another, for example. This is one model for the Earth. However, it does have some major drawbacks. In particular, it distorts the areas of countries - especially closer to the poles. For example, Greenland looks almost as large as Australia in the map. It looks definitely bigger than, say, Algeria. In reality, Algeria is actually slightly bigger than Greenland (919,600 mi^2 versus 836,300 mi^2).

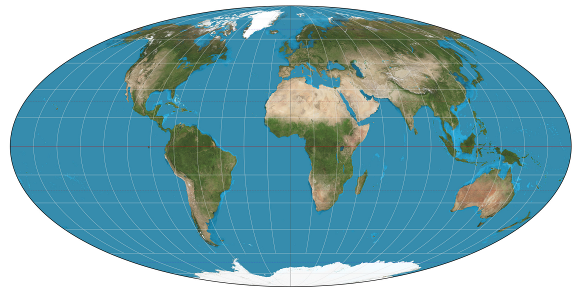
So if we ranked the areas of countries based on the map and compared that to the actual rank[1], the lists would not be the same. This projection - the Equirectangular projection - does not preserve order in terms of country area. We could use another projection, as shown in Figure 2(b), called the Mollweide projection. This projection does preserve area - but again there's a drawback. It distorts the shape of the countries, so it's not a perfect model either. The point here is that a model is an imperfect replica of the real world, but as long as it accurately reflects some *part* of reality, then it can still be useful. You just have to decide what aspect of the real world you want your model to capture.

Figure 2: Maps as Models

(a) Equirectangular Projection



(b) Mollweide Projection



Source: Daniel R. Strebe, 15 August 2011

Now let's bring this back to utility. Remember that preferences only tell us *which* option a consumer prefers, but they don't tell us *how much* they prefer it. Similarly, we want our model to keep the same ranking of choices as the true preferences, but we're ok with it distorting how much they prefer different alternatives. Here's an example highlighting this. Suppose the consumer has to choose between three alternatives: x, y, and z. The consumer's preferences are as follows: $x \succsim y$, $y \succsim z$, and $x \succsim z$. Using this information, we can create a "chain" of preferences: $x \succsim y \succsim z$. This tells us that the consumer prefers x first, y second, and z third. If a utility function represents $\succsim$, then it must be the case that $u(x) \geq u(y) \geq u(z)$. In other words, the utility function has to preserve the order given by the preference relation. If it doesn't preserve the order of preferences, then it doesn't represent the preference. For example, we could have a utility function that has the following values: $u(x) = 10$, $u(y) = 7$, $u(z) = 4$. However, another utility function with values $u(x) = 0.2$, $u(y) = 0.1$, $u(z) = 0.001$ would also work. The fact that that in the first case $u(x) - u(y) = 3$ and in the second case $u(x) - u(y) = 0.1$ means nothing. This is just like prices when we studied budget constraints. The actual number of utilities do not mean anything - what matters is how they compare relative to other values (for the same utility function). A utility function that gave values $u(x) = 1$, $u(y) = 2$, and $u(z) = 3$, would *not* preserve the rank order, and so could not represent the preferences.

There is an important implication of this: **preference relations do not have unique utility representation**. Put another way, two different utility functions can represent the same preferences. Here's another map analogy to help you understand this. Say I was looking at a NYC map to figure out which subway I was closest to. I can look at a large paper map or a map on my phone. Both of these are valid models for the real world (the actual distance), but they use different *scales*. It's the same idea with utility functions - I can always scale a utility function up or down, but as long as the order of preferences remains unchanged,

[1] `https://en.wikipedia.org/wiki/List_of_countries_and_dependencies_by_area`

then I have another valid utility representation. We call this scaling a *monotonic transformation*, which we can represent with a monotonic function $f(\cdot)$. Some examples of such functions are $f(x) = 2x$, $f(x) = x^2$, $f(x) = \log(x)$, $f(x) = x - 10$. The first example, $f(x) = 2x$, means that I can double all the values of the utility function and have another function that still represents the same preferences. Notice that all these functions have positive slopes - it doesn't matter whether the slope is small or large or changing or constant, as long as it's *always positive*, then it's a monotonic transformation.

To summarize, here are the differences between preferences and utility functions:

	Preferences ($\succsim$)	**Utility Functions** ($\boldsymbol{u(\cdot)}$)
Source	Real	Models
Type	Binary relation (comparer)	Function
Form	$x \succsim y$	$u(x)$
Use	To compare between two bundles	To evaluate the utility of one bundle
Input	Two bundles	One bundle
Output	A bundle	A number
Output Interpretation	Preferred bundle	Meaningless (by itself)

And to link them together, we have the idea of representation:

Utility Representation

1. A utility function $u(\cdot)$ represents a preference relation $\succsim$ if:
   (i) $u(x) \geq u(y)$ whenever $x \succsim y$
   (ii) $x \succsim y$ whenever $u(x) \geq u(y)$
2. For any function $u(\cdot)$ that represents a preference relation, another function $v(\cdot)$ also represents the same preferences if it is a monotonic (order-preserving) transformation of $u(\cdot)$
   - $v(x) = f(u(x))$, where $f(\cdot)$ is a monotonic function

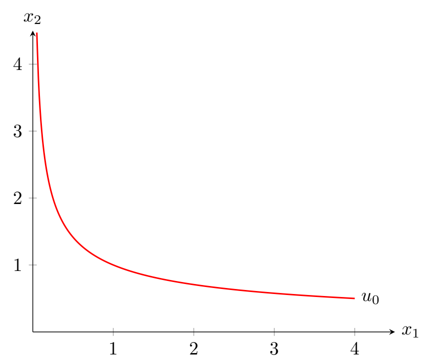
2.2 Indifference Curves and MRS

As always, we will find it useful to have a graphical representation of utility functions. It may not be surprising to find out that the graphs for utility functions look exactly the same as the graphs for indifference curves! You would hope so, because that's exactly what a good model should do. If you have two bundles that the consumer views as indifferent $x \sim y$, then it must be that $u(x) = u(y)$. This means that they must be on the same graph for the utility function.

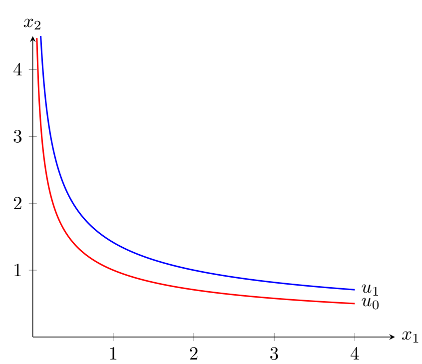
The biggest difference between the two graphs is that we can associate a number (the utility level) with each curve, which is something we didn't do earlier when we had indifference curves. For example, in Figure 1, the indifference curves are just labelled as I_0, I_1, and I_2. Now, each indifference curve will have a utility level, e.g. u_0, u_1, and u_2, respectively, where the higher utility number indicates a more preferred set ($u_2 > u_1 > u_0$). The interpretation will be that all bundles on the same indifference curve will give the consumer the same level utility, e.g. all points on I_0 give a utility of u_0, all points on I_1 give a utility of u_1 and so on. More specifically, we must have that $u(A) = u(B) = u(C) = u_0$ and likewise for points on the other curves.

To plot an indifference curve is quite easy. You will be given a utility function in the form of $u(x_1, x_2)$, i.e. a function of both goods x_1 and x_2. For example, $u(x_1, x_2) = x_1 x_2^2$. Then just follow these steps:

1. Replace $u(x_1, x_2)$ with u in the function. Solve for x_2, i.e. have x_2 by itself on the LHS and have x_1 and u in some function on the RHS
   - From the example function, we would have $u = x_1 x_2^2$. Then solving for x_2 gets us $x_2^2 = \frac{u}{x_1} \implies x_2 = \left(\frac{u}{x_1}\right)^{\frac{1}{2}}$
2. Choose any value for u (let's call this u_0) and plug it into the above formula. Draw this line on an axis with x_2 on the y and x_1 on the x axis. This gives you the indifference curve corresponding to the utility level $u(x_1, x_2) = u_0$
   - From the example, let's say we had $u_0 = 1$. Then plugging that in gives us the line $x_2 = \left(\frac{1}{x_1}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{x_1}}$. Plotting this gives us the red line below, labelled u_0.



3. Repeat step 2 for a different value of u (let's call this u_1). This will give you another indifference curve, corresponding to utility level $u(x_1, x_2) = u_1$
   - Now let's have $u_1 = 4$. Then plugging this in gives us the line $x_2 = \left(\frac{4}{x_1}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{x_1}}$. Plotting this gives us the blue line below, labelled u_1.



4. Keep repeating this for as many values of u as you want, each one giving you a different indifference curve

Recall that we said that a monotonic transformation of the utility function would represent the same preferences. And if it represents the same preferences, then it must have the same indifference curves. Let's see this through the example.

Say the monotonic transformation we want to do is to square the utility function, i.e. $f(x) = x^2$. Our new utility function, called $v(\cdot)$, will be defined as:

$$v(x) = f(u(x)) = \left(x_1 x_2^2\right)^2 = x_1^2 x_2^4$$

Now let's try plotting this function. Following the steps above (and using v instead of u), we can solve for x_2 to get:

$$\begin{aligned} v =& x_1^2 x_2^4 \\ x_2^4 =& \frac{v}{x_1^2} \\ x_2 =& \left(\frac{v}{x_1^2}\right)^{\frac{1}{4}} = \frac{v^{\frac{1}{4}}}{x_1^{\frac{1}{2}}} \end{aligned}$$

Since $v = f(u) = u^2$, this must mean the graph for $v_0 = u_0^2 = 1$ should be the same as u_0, and the graph for $v_1 = u_1^2 = 16$ should be the same as u_1. Let's check this:

$$\begin{aligned} x_2 =& \frac{v_0^{\frac{1}{4}}}{x_1^{\frac{1}{2}}} = \frac{1}{\sqrt{x_1}} \\ x_2 =& \frac{v_1^{\frac{1}{4}}}{x_1^{\frac{1}{2}}} = \frac{16^{\frac{1}{4}}}{\sqrt{x_1}} = \frac{2}{\sqrt{x_1}} \end{aligned}$$

No need to plot them now, we can see by the formulas that graphs will be the same. However, the biggest difference is that each graph is now associated with a different utility number. As before, the red line is associated with a utility of 1 (in terms of $v(x)$), but the blue line is associated with a utility of 16 (again, in terms of $v(x)$). This just highlights that the numbers of the utility don't really mean very much. It may seem that a utility of $u(x) = 4$ is small compared to a utility of $v(x) = 16$, but in fact, they represent exactly the same bundles. The reason they give us the same set of bundles is because regardless of the size of the utility number, both utility functions capture the same substitution a consumer is willing to make to ensure they stay on the same indifference curve. This idea gets us to the very important concept of **marginal rates of substitution**.

The marginal rate of substitution (MRS) captures exactly the idea described above: if I get one more unit of x_1, how many more units of x_2 do you need to give me to make sure I was indifferent to before? Say I had a bundle (x_1', x_2') and I wanted to move to a new bundle $(x_1'', x_2'') = (x_1' + 1, x_2'')$, where $u(x_1', x_2') = u(x_1'', x_2'')$. Surely if the utility is the same (I'm indifferent between the two bundles), but I have more of x_1, then the only way this could happen is if I had *less* x_2 (assuming monotonic preferences, which we usually do). This tells us that the MRS is usually a negative number. To calculate the MRS, we use the following formula:

$$MRS_{12} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = -\frac{MU_1}{MU_2}$$

We divide each of the first order partial derivatives, which we refer to as the **marginal utility**: $MU_i = \partial u(x_1, x_2)/\partial x_i$. The marginal utility of good i captures how much my utility changes when I increase my consumption of good i increases by a little, holding everything else constant (generally this is positive). The graphical interpretation is that the MRS is the slope of the indifference curve. You can see this exactly through the description of the MRS: how much does x_2 (the y variable) change in response to a one unit change in x_1 (the x variable). For those of you who prefer a mathematical explanation, we know that the slope of the indifference curve is $\frac{\Delta y}{\Delta x} = \frac{dx_2}{dx_1}$. Take the total derivative of a utility function for a fixed utility level u and then "divide" through by dx_1:

$$\begin{aligned}
u =& u(x_1, x_2) \\
du =& \frac{\partial u(x_1, x_2)}{\partial x_1} \cdot dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} \cdot dx_2 \\
\frac{du}{dx_1} =& \frac{\partial u(x_1, x_2)}{\partial x_1} \cdot \frac{dx_1}{dx_1} + \frac{\partial u(x_1, x_2)}{\partial x_2} \cdot \frac{dx_2}{dx_1} \\
0 =& MU_1 \cdot 1 + MU_2 \cdot \frac{dx_2}{dx_1} \\
\frac{dx_2}{dx_1} =& -\frac{MU_1}{MU_2} = MRS_{12}
\end{aligned}$$

You may notice that I have written MRS as MRS_{12} with "12" in the subscript. This indicates that it is "the MRS of good 2 with respect to good 1", which means it captures how much good 2 <u>changes</u> for a <u>one unit increase</u> in good 1. This what we usually want because it has the nice interpretation of being the slope of the indifference curve. However, you could also have MRS_{21}, which tells you how much good 1 you will need to take away for a one unit increase in good 2. In this case the formula is $MRS_{21} = -\frac{MU_2}{MU_1} = \frac{1}{MRS_{12}}$. In general, for MRS_{ij}:

- i is the good in the numerator and is the one that changes by one unit
- j is the good in the denominator and is the one that we change to make sure that utility stays at the same level

It's very important that you don't get this order mixed up! This will be important in the next part when we do optimization and this type of interpretation will also be useful when we move onto producer theory later in the course.

Another reason the MRS is useful is because it helps us compare different utility functions. The rule is that the **any two utility functions that have the same MRS represent the same preferences**. Let's go back to our example functions $u(x) = x_1 x_2^2$ and $v(x) = x_1^2 x_2^4$. Now let's calculate the MRS for $u(x)$ (call it

MRS^u) and for $v(x)$ (call it MRS^v):

$$MRS_{12}^u = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = -\frac{x_2^2}{2x_1 x_2} = -\frac{x_2}{2x_1}$$
$$MRS_{12}^v = -\frac{\partial v(x_1, x_2)/\partial x_1}{\partial v(x_1, x_2)/\partial x_2} = -\frac{2x_1 x_2^4}{4x_1^2 x_2^3} = -\frac{x_2}{2x_1}$$

Basically the MRS takes into account the different scales used by the utility functions and *normalizes* them so that they can be comparable. Utilities have no units so we can't compare them, but MRS_{12} is measured in units of x_2. This means that we can compare different MRS's because their unit has a real interpretable meaning. So the lesson here is that if you have two utilities that you think might represent the same preferences, just check the MRS.

A final note about the MRS is that many students often forget that it is a *function*. We could write more formally as $MRS_{12}(x_1, x_2)$ to make this super clear (but we usually don't). This means that the MRS isn't usually some fixed number, it can potentially be different at every bundle. This is because your willingness to substitute between the two goods may of course depend on the bundle you currently have. We can also see this visually: the MRS is the slope of the indifference curve, and the slope of the indifference is often changing.

2.3 Examples

Cobb-Douglas

- General Form: $u(x_1, x_2) = x_1^\alpha x_2^\beta$
- Indifference Curves: Standard vanilla ICs. They must never touch either axis though!
- Preference: Intermediate mixer who likes to have a little bit of everything

Cobb-Douglas utilities are the standard and most commonly used utility functions. At first they may seem intimidating, but you'll soon see that they are often the easiest to work with. The example function I gave above $u(x_1, x_2) = x_1 x_2^2$ is a Cobb-Douglas with $\alpha = 1$ and $\beta = 2$. We also saw an example of a monotonic transformation $v(x_1, x_2) = x_1^2 x_2^4$, which is also Cobb-Douglas with $\alpha = 2$ and $\beta = 4$. Two common monotonic transformations used with Cobb-Douglas are:

1. $f(x) = x^{\frac{1}{\alpha+\beta}}$. This gives the transformed utility function: $v(x_1, x_2) = x_1^{\frac{\alpha}{\alpha+\beta}} x_2^{\frac{\beta}{\alpha+\beta}}$
2. $f(x) = \log(x)$. This gives the transformed utility function: $v(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$

Again, let's check that these represent the same preferences by checking the MRS. Let's use the same function as before $u(x_1, x_2) = x_1 x_2^2$. Using the first transformation, the new utility function would be $v(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$

(note that $\alpha + \beta = 1 + 2 = 3$). The MRS for this function is:

$$
\begin{aligned}
MRS^v_{12} = -\frac{\partial v(x_1,x_2)/\partial x_1}{\partial v(x_1,x_2)/\partial x_2} &= -\frac{\frac{1}{3}x_1^{-\frac{2}{3}}x_2^{\frac{2}{3}}}{\frac{2}{3}x_1^{\frac{1}{3}}x_2^{-\frac{1}{3}}} \\
&= -\frac{1}{2}\cdot x_1^{-\frac{2}{3}-\frac{1}{3}}\cdot x_2^{\frac{2}{3}-(-\frac{1}{3})} \\
&= -\frac{1}{2}\cdot x_1^{-1}\cdot x_2^{1} \\
&= -\frac{x_2}{2x_1}
\end{aligned}
$$

For the second transformation, the new utility function would be $v(x_1, x_2) = \log(x_1) + 2\log(x_2)$. The MRS for this function is:

$$
\begin{aligned}
MRS^v_{12} = -\frac{\partial v(x_1,x_2)/\partial x_1}{\partial v(x_1,x_2)/\partial x_2} &= -\frac{\frac{1}{x_2}}{\frac{2}{x_2}} \\
&= -\frac{1}{x_2}\cdot\frac{x_2}{2} \\
&= -\frac{x_2}{2x_1}
\end{aligned}
$$

So all of these functions represent the same preferences.

Quasi-Linear Utility

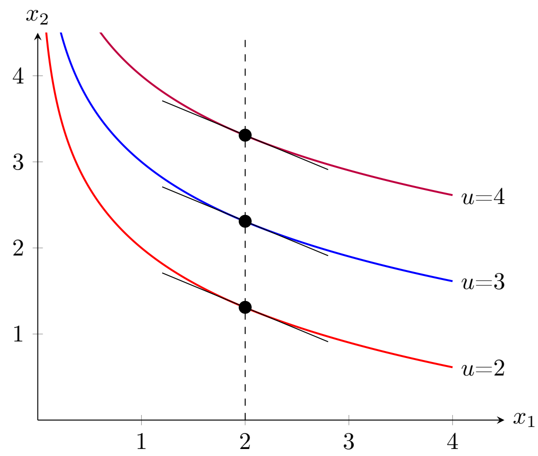
- General Form: $u(x_1, x_2) = f(x_1) + x_2$, where $f(x_1)$ is some (non-linear) function of x_1
- Indifference Curves: Look mostly like standard ICs but often wider. A key property is that each IC is just a parallel shift of other ICs
- Preference: Also a mixer, whose MRS is unaffected by how much x_2 they have

Quasi-linear utilities at first don't look very special, but we will see later on they have a number of useful properties. For now, the biggest thing for you to note is that the ICs are parallel shifts of one another. We can see this by solving for x_2 and getting $x_2 = u - f(x_1)$. Now every time we plug in a different value for u, we are just changing the y intercept, while leaving the slope completely unchanged. Another way to see this is as follows:

$$MRS_{12} = -\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2} = -\frac{f'(x_1)}{1} = -f'(x_1)$$

Notice that the MRS does not depend on x_2. Which means for the same level of x_1, whether we increase or decrease the level of x_2, the slope on the IC we are on will always be the same. This is exactly the definition of parallel lines.

An example of a quasi-linear utility is $u(x_1, x_2) = \ln(x_1) + x_2$. So to plot this we have $x_2 = u - \ln(x_1)$, as shown below for three different levels of utility. Also note that they are parallel. Fixing $x_1 = 2$, we see that the slope of each IC is equal to $-f'(x_1) = -\frac{1}{x_1} = -0.5$ on all three curves.

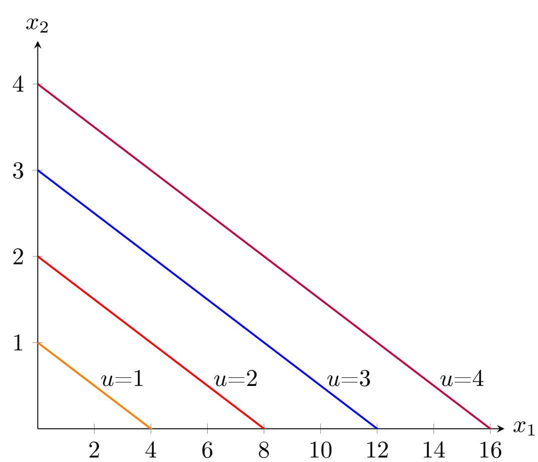


Perfect Substitutes

- General Form: $u(x_1, x_2) = \frac{1}{\alpha}x_1 + \frac{1}{\beta}x_2$
- Indifference Curves: Linear indifference curves
- Preference: A consumer who is equally happy with α units of good 1 as they are with β units of good 2.

This general form takes into account when consumers don't substitute the two goods in one-to-one proportions. For example, let's go back to the milk example I gave before. Say that x_1 is units of quarts of milk and x_2 is units of gallons of milk. That means that 4 units of x_1 is of equal utility to 1 unit of x_2. Using the formula above, $\alpha = 4$ and $\beta = 1$. This means that the formula for the indifference curves is: $u = \frac{1}{4}x_1 + \frac{1}{1}x_2 = \frac{1}{4}x_2 + x_2$.

Let's try plotting this utility function. First we solve for x_2 to give us $x_2 = -\frac{1}{4}x_1 + u$. Notice how this is in our usual "$y = mx + c$" format, so it will be a straight line. Now let's plot for different levels of utility.



Notice how the points $(0, 4)$ and $(1, 0)$ are on the same indifference curves - this is exactly what we should expect given the description of the preferences. Also it's good to double check by plugging into the utility function to make sure we've accurately written out the formula: $u(4,0) = \frac{1}{4}\cdot 4 + 0 = 1$ and $u(0,1) = \frac{1}{4}\cdot 0 + 1 = 1$. If you don't like fractions, we could of course scale up the utility function into something like $v(x_1, x_2) = 4u(x_1, x_2) = x_1 + 4x_2$, i.e. we use the monotonic transformation $f(x) = 4x$. You should check that this still gives us the same MRS and indifference curves (unsurprising spoiler: it does).

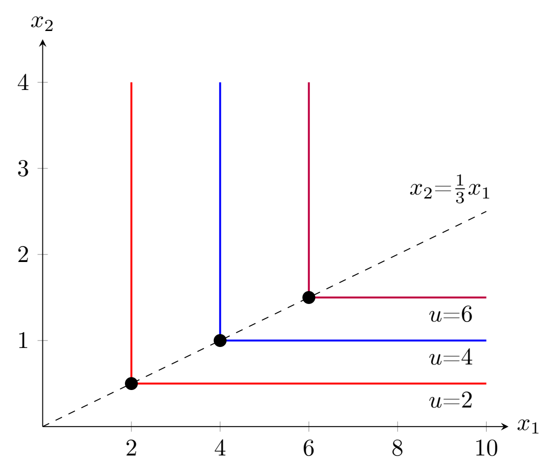
Perfect Complements

- General Form: $u(x_1, x_2) = \min\left\{\frac{1}{\alpha}x_1, \frac{1}{\beta}x_2\right\}$
- Indifference Curves: L-shaped indifference curves (Leontief)
- Preference: A consumer who only prefer to consume bundles in proportions of α units of good 1 with β units of good 2

The general form of this is probably more different that functions you're used to. The min function just chooses the smallest number out of the ones gives to it (see Recitation 1 which covered this in the math review). To plot it you follow these simple two steps:

1. Plot the function $\frac{1}{\alpha}x_1 = \frac{1}{\beta}x_2$ as a dashed line. This represents the bundles where the x_1 and x_2 are in the correct proportions for the consumer. Re-arranging this gives us that $x_2 = \frac{\beta}{\alpha}x_1$, which is just a straight line from the origin.
2. Select a few points along the dashed line. From each of these points, draw a vertical line going upwards (but not below the point) and a horizontal line going rightwards (but not behind the point)

Let's the take example of having 4 tires (x_1) as a perfect complement to 1 car (x_2). Using the formula above, the utility function is $u(x_1, x_2) = \min\left\{\frac{1}{4}x_1, x_2\right\}$ since $\alpha = 4$ and $\beta = 1$. So let's now plot this (as well as the dashed line $x_2 = \frac{\beta}{\alpha}x_1 = \frac{1}{4}x_1$):



A very common mistake people will do is to write the utility function using the wrong proportions. In the case above, many will incorrectly write the utility function as $u(x_1, x_2) = \min\{4x_1, x_2\}$. But let's check this (this is a good habit to get into). We know that if I only have 4 tires, then having more than 1 car will not increase my utility. This means that we should have that $u(4,1) = u(4,2)$, for example. Let's plug this into the incorrect utility function:

$$u(4,1) = \min\{4 \cdot 4, 1\} = \min\{16, 1\} = 1$$
$$u(4,2) = \min\{4 \cdot 4, 2\} = \min\{16, 2\} = 2$$

But this is totally wrong because we now have $u(4,2) > u(4,1)$! Trying the same thing with the correct utility function:

$$u(4,1) = \min\left\{\frac{1}{4} \cdot 4, 1\right\} = \min\{1, 1\} = 1$$
$$u(4,2) = \min\left\{\frac{1}{4} \cdot 4, 2\right\} = \min\{1, 2\} = 1$$

This similar issue also applies to perfect substitutes (try this exercise with the example given). Again, if you don't like dealing with fractions, there's a very simple fix. You can just scale up every component inside the min function. For example, the utility function $v(x_1, x_2) = \min\{x_1, 4x_2\}$ also represents the same preferences. Doing the same check as above:

$$v(4,1) = \min\{4, 4 \cdot 1\} = \min\{4, 4\} = 4$$
$$v(4,2) = \min\{4, 4 \cdot 2\} = \min\{4, 8\} = 4$$

Of course, it's not enough that this works for one point, it has to work for every bundle. Usually to check this we can compare the MRS. Unfortunately, this is hard here because the MRS for Leontief (L-shaped) function is not well-defined. The approach here is to check the dashed line where the kinks occur. Remember that this is the line of "optimal proportions" - at these bundles, there are no goods going to waste (i.e. being consumed but not bringing any extra utility). For $v(\cdot)$, the kinks occur at $x_1 = 4x_2$, which is $x_2 = \frac{1}{4}x_1$, exactly the same as before.