Intermediate Micro: Recitation 9
Welfare and Elasticity

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1 Welfare

When prices change, we know that consumers re-optimize and choose a new optimal bundle. When we consider an individual consumer, we can say that they are "better off" or "worse off" by just comparing how their utility changes. The issue with this is that it doesn't tell us *how much* the consumer is better or worse off. Remember that utility functions tell us an order of how bundles are ranked, but the actual values themselves are quite meaningless. We want to have an objective way of measuring how much a consumer's welfare has changed.

In this recitation, let's consider the following setup where prices change (note that this is very similar to how I set up the decomposition in Recitation 8). For this, we will go from what I call the "old world" (with the old prices) to a "new world" (with the new prices). We will also have a thought-experiment and imagine a third "alternate world" (to be explained soon). The notation for this setup will be as follows: (note that income stays the same, i.e. $M^0 = M^0$)

	Old World (0)	**New World (1)**	**Alternate World (a)**
$\boldsymbol{p_1}$	p_1^0	p_1^1	p_1^a
$\boldsymbol{p_2}$	p_2^0	p_2^1	p_2^a
$\boldsymbol{M}$	M^0	M^0	M^a
Optimal Bundle $(\boldsymbol{x_1^*, x_2^*})$	$x^0 = (x_1^0, x_2^0)$	$x^1 = (x_1^1, x_2^1)$	$x^a = (x_1^a, x_2^a)$
Budget Line	B^0	B^1	B^a
Indifference Curve	I^0	I^1	I^a

There are three ways we can measure consumer welfare:

1. **Change in Consumer Surplus (ΔCS):** Consumer surplus is how much more you would have been willing to pay above the current price (i.e. the area under the demand curve). The change in the consumer surplus is the new world CS minus the old world CS

2. **Equivalent Variation (EV):** How much money to get you to the same utility as in the new world, while keeping prices the same as the old world

3. **Compensating Variation (CV):** How much money to get you to the same utility as in the old world, while keeping prices the same as the new world

Consumer surplus is an idea that should be quite familiar to you already, so we're going to focus on EV and CV here. For this we do the following thought-experiment using our alternate world:

1. Consider an alternate world where there is no substitution effect and only an income effect *relative to* the old world ($a = e$)
   - No substitution effect means that $p^e = p^0$ (the alternate world prices are the same as the old world prices)
   - The income effect has to be such that $I^e = I^1$ (the alternate world utility is the same as the new world utility)
   - The change in income from the old world to the alternate world is the **equivalent variation**: $EV = M^e - M^0$
   - Intuition: *starting at the old world, how much income would I have to give you so that the welfare effect is equivalent to the price change (i.e. so that you are indifferent to changing to the new world)*

2. Consider an alternate world where there is no substitution effect and only an income effect *relative to* the new world ($a = c$):
   - No substitution effect means that $p^c = p^1$ (the alternate world prices are the same as the new world prices)
   - The income effect has to be such that $I^c = I^0$ (the alternate world utility is the same as the old world utility)
   - The change in income from the new world to the alternate world is the **compensating variation**: $CV = M^c - M^0$
   - Intuition: *starting at the new world, how much income would I have to give you so that I compensate you for the price change (i.e. so that you are indifferent to changing to the old world)*

Intuitively, we want to capture the change in consumer welfare as a change in income. This gives us a dollar amount for their welfare change, which is both objective and has a meaningful interpretation (unlike utility). The difference between the EV and CV is a matter of perspective. For the EV, you start at the old world and consider a change to the new world. For the CV, you start at the new world and consider a change back to the old world. In other words, the starting prices provide a reference point for your alternate world. So, in general, EV and CV are not going to be equal.

We interpret them as follows:

- If $EV < 0$ or $CV > 0$: consumer welfare **falls** after the price change
  - Starting at the old world, to make you indifferent to the new world, I need to *take* money away from you ($EV < 0$). This must mean you were unhappy with the price change since just taking money away from you would have an equivalent effect
  - Starting at the new world, to make you indifferent to the old world, I need to *give* you money ($CV > 0$). This must mean you were unhappy with the price change since I have to give you money to reverse the effects of the price change
- If $EV > 0$ or $CV < 0$: consumer welfare **rises** after the price change
  - Starting at the old world, to make you indifferent to the new world, I need to *give* you money ($EV > 0$). This must mean you were happy with the price change since just giving you money would have an equivalent effect
  - Starting at the new world, to make you indifferent to the old world, I need to *take* money away from you ($CV < 0$). This must mean you were happy with the price change since I have to take money away from you to reverse the effects of the price change

You can summarize the four "worlds" in the following grid:

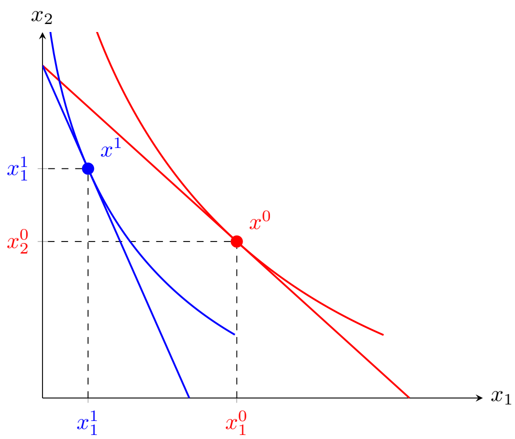
	Old Utility ($\boldsymbol{u^0}$)	**New Utility** ($\boldsymbol{u^1}$)
Old Prices ($\boldsymbol{p^0}$)	Old World	EV World
New Prices ($\boldsymbol{p^1}$)	CV World	New World

Equivalent Variation

For the graphs, suppose we have p_1 increase, i.e. $p_1^1 > p_1^0$, and everything else stays the same. However, the steps are general and apply for any setting.

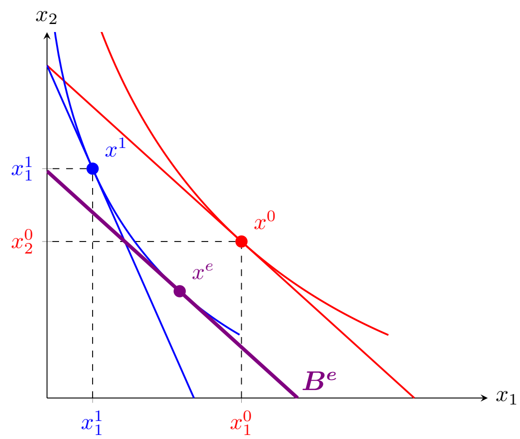
Step 1

- Derive the consumer's demand function $x(p, M)$
- Plug in parameters to get the old bundle x^0 and new bundle x^1 (x^0 is not necessary for calculations)
- Plot the graphs with BL B^0 and IC I^0 for the old world and BL B^1 and IC I^1 for the new world



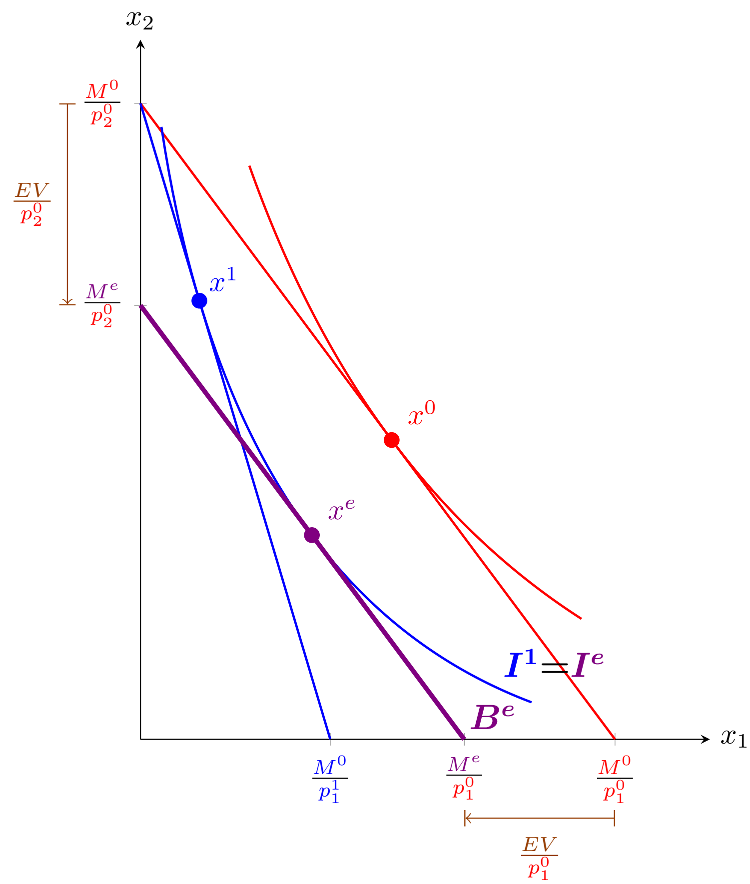
Step 2

- Draw the EV budget line B^e
- This BL has to:
  - have the same slope as B^0, and
  - be tangent to I^1 at x^e



Step 3

- Calculate the EV income M^e
  - Since x^e and x^1 are both on I^1
    $\Longrightarrow u(x_1^e, x_2^e) = u(x_1^1, x_2^1)$
  - From demand function:
    $x_i^e = x_i(p^e, M^e) = x_i(p^0, M^e)$
  - Can calculate new utility: $u(x_1^1, x_2^1) = u^1$
- Plug in new utility and old prices
  $\Longrightarrow u(x_1(p^0, M^e), x_2(p^0, M^e)) = u^1$
  - Only unknown is M^e, which you can solve for



Step 4

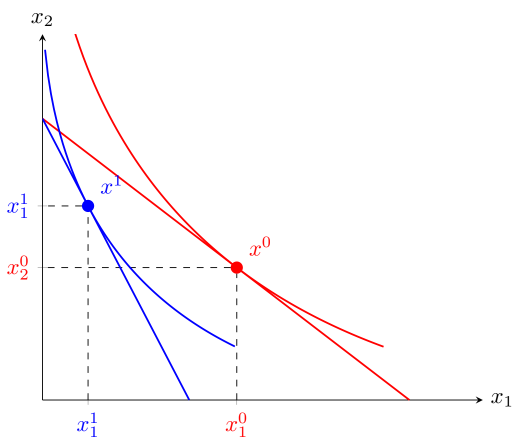
- Calculate EV: $EV = M^e - M^0$

Compensating Variation

For the graphs, suppose we have p_1 increase, i.e. $p_1^1 > p_1^0$, and everything else stays the same. However, the steps are general and apply for any setting.

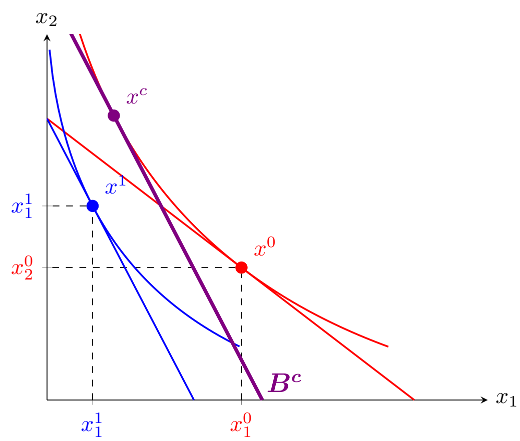
Step 1

- Derive the consumer's demand function $x(p, M)$
- Plug in parameters to get the old bundle x^0 and new bundle x^1 (x^1 is not necessary for calculations)
- Plot the graphs with BL B^0 and IC I^0 for the old world and BL B^1 and IC I^1 for the new world



Step 2

- Draw the CV budget line B^c
- This BL has to:
  - have the same slope as B^1, and
  - be tangent to I^0 at x^c

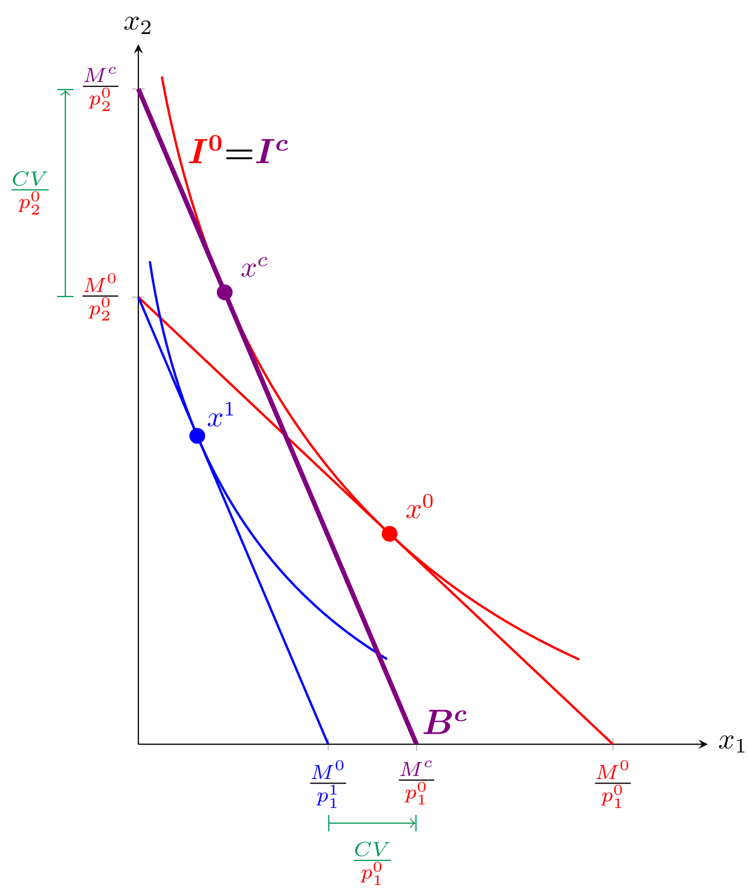


Step 3

- Calculate the CV income M^c
  - Since x^c and x^0 are both on I^0
    $\implies u(x_1^c, x_2^c) = u(x_1^0, x_2^0)$
  - From demand function:
    $x_i^c = x_i(p^c, M^c) = x_i(p^1, M^c)$
  - Can calculate old utility: $u(x_1^0, x_2^0) = u^0$
- Plug in old utility and new prices
  $\implies u(x_1(p^1, M^c), x_2(p^1, M^c)) = u^0$
  - Only unknown is M^c, which you can solve for

Step 4

- Calculate CV: $CV = M^c - M^0$



2 Welfare Examples

2.1 Consumer Surplus

Usually when you get a question on consumer surplus, you get given a simple linear demand curve. This makes finding the area under the curve easier since we can use geometry instead of having to use integration. The only trick is that you have to re-write the demand curve as the inverse demand function. Since we can calculate the CS using algebra, two useful formulas to remember are:

1. Area of a triangle with base of length b and height h is: $\frac{1}{2}bh$
2. Area of a trapezoid/trapezium with base 1 of length a, base 2 of length b, and height h is: $\frac{a+b}{2}h$

Consider the following example:

- Demand is given by: $Q = 400 - 20P$
- Price changes from $P = 15$ to $P = 8$
- Calculate the change in consumer surplus

First, we find the inverse demand function (i.e. solve for P):

$$\begin{aligned} Q &= 400 - 20P \\ 20P &= 400 - Q \\ P &= 20 - \frac{1}{20}Q \end{aligned}$$

Second, we find the old and new equilibrium:

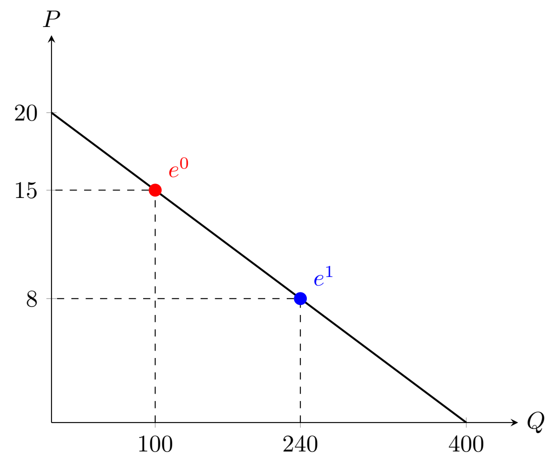
Before:

$$\begin{aligned} P =& 15 \\ Q =& 400 - 20 \cdot 15 \\ =& 400 - 300 \\ =& 100 \end{aligned}$$

After :

$$\begin{aligned} P =& 8 \\ Q =& 400 - 20 \cdot 8 \\ =& 400 - 160 \\ =& 240 \end{aligned}$$

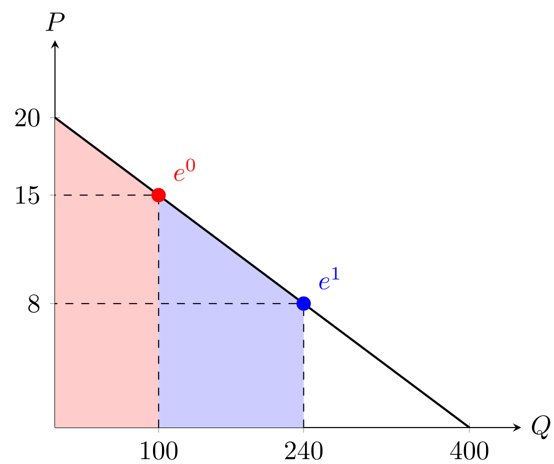
Third, we plot this change on a graph



Fourth, we calculate consumer surplus. There are two ways to measure this: (call the equilibrium point (Q^*, P^*))

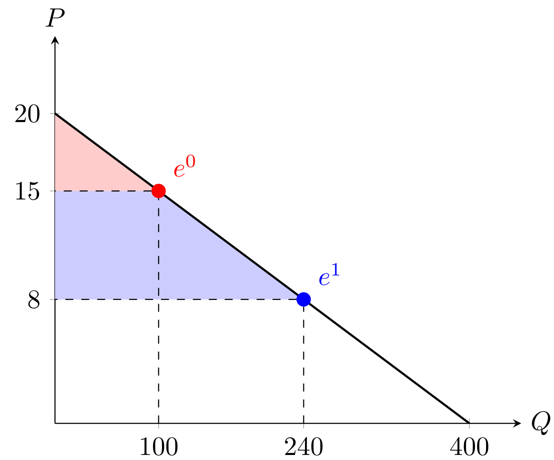
- Gross consumer surplus: the area under the demand curve from $Q = 0$ to Q^*
- Net consumer surplus: the area under the demand curve from $Q = 0$ to Q^* *and* above P^*

Gross surplus is shaded in the diagram below. The red area is the gross CS of the first equilibrium. The red plus blue area is the gross CS of the second equilibrium. The blue area represents the change (increase) in gross CS.



- Gross CS 1: Trapezoid with $a = 15$, $b = 20$, $h = 100$. Therefore area is $\frac{15+20}{2} \times 100 = 17.5 \times 100 = 1750$
- Gross CS 2: Trapezoid with $a = 8$, $b = 20$, $h = 240$. Therefore area is $\frac{8+20}{2} \times 240 = 14 \times 240 = 3360$
- The change in gross CS is: Gross CS 2 $-$ Gross CS 1 $= 3360 - 1750 = 1610$
  - Equivalently: Trapezoid with $a = 8$, $b = 15$, $h = 240 - 100 = 140$. Therefore area is $\frac{8+15}{2} \times 140 = 11.5 \times 140 = 1610$

Net surplus is shaded in the diagram below. The red area is the net CS of the first equilibrium. The red plus blue area is the net CS of the second equilibrium. The blue area represents the change (increase) in net CS.



- Net CS 1: Triangle with $b = 20 - 15 = 5$, $h = 100$. Therefore area is $\frac{1}{2} \times 5 \times 100 = 250$
- Net CS 2: Triangle with $b = 20 - 8 = 12$, $h = 240$. Therefore area is $\frac{1}{2} \times 12 \times 240 = 1440$
- The change in net CS is: Net CS 2 $-$ Net CS 1 $= 1440 - 250 = 1190$
  - Equivalently: Trapezoid with $a = 100$, $b = 240$, $h = 15 - 8 = 7$. Therefore area is $\frac{100+240}{2} \times 7 = 170 \times 7 = 1190$

2.2 Cobb-Douglas

Consider the following setup:

- $u(x_1, x_2) = x_1^3 x_2^2$
- $p_1 = 8, p_2 = 2, M = 120$
- Then p_1 decreases to $p_1 = 4$

We want to calculate the EV and CV of the price change

[Same first step for EV and CV]

<u>**Step 1**</u>

First, derive the demand function:

$$|MRS| = \frac{3x_2}{2x_1} = \frac{p_1}{p_2}$$
$$x_2 = \frac{2p_1}{3p_2} x_1$$

$$\therefore p_1x_1 + p_2\left(\frac{2p_1}{3p_2}x_1\right) = M$$
$$p_1x_1\left(1+\frac{2}{3}\right) = M$$
$$x_1(p,M) = \frac{3M}{5p_1}$$
$$\implies x_2(p,M) = \frac{2M}{5p_2}$$

Then, we calculate the optimal bundles before and after the price change:

Before:

$$x_1(8,2,120) = \frac{3\times 120}{5\times 8} = 9$$
$$x_2(8,2,120) = \frac{2\times 120}{5\times 2} = 24$$

After :

$$x_1(4,2,120) = \frac{3\times 120}{5\times 4} = 18$$
$$x_2(4,2,120) = \frac{2\times 120}{5\times 2} = 24$$

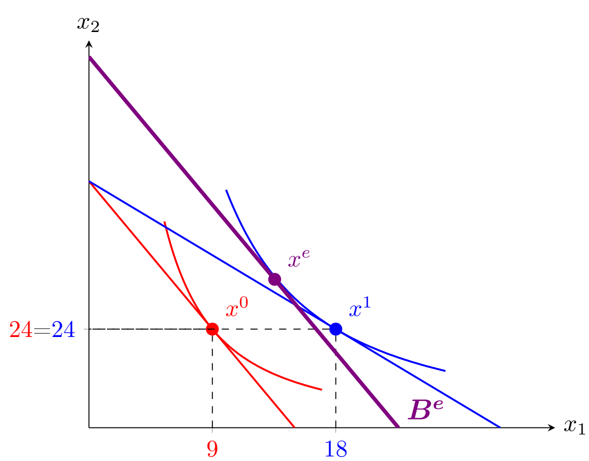
[First, do the EV]

Step 2

Next, we determine the EV budget line. This is the old prices with an income level M^e, which we will need to solve for:

$$8x_1 + 2x_2 = M^e$$

Drawing this in the diagram, we know that this budget line has to be tangent to the blue I^1 indifference curve and have the same slope as the red B^0 budget line.



Step 3

Now, we find the value of the EV income M^e. We also know the prices in the alternate world are the same as the old prices. Therefore, we know that the demanded quantity (i.e. x^e) is:

$$(x_1^e, x_2^e) = \left(\frac{3M^e}{5p_1^0}, \frac{2M^e}{5p_2^0}\right) = \left(\frac{3M^e}{5\times 8}, \frac{2M^e}{5\times 2}\right) = \left(\frac{3M^e}{40}, \frac{M^e}{5}\right)$$

We also know that $u(x_1^e, x_2^e) = u^1$. Let's calculate the value of the new utility:

$$\begin{aligned} u^1 &= \left(x_1^1\right)^3 \left(x_1^1\right)^2 \\ &= 18^3 \times 24^2 \end{aligned}$$

Therefore, we can now solve for the EV income M^e

$$\begin{aligned} u(x_1^e, x_2^e) =& u^1 \\ \left(\frac{3M^e}{40}\right)^3 \left(\frac{M^e}{5}\right)^2 =& 18^3 \times 24^2 \\ (M^e)^5 \left(\frac{3}{40}\right)^3 \left(\frac{1}{5}\right)^2 =& 18^3 \times 24^2 \\ (M^e)^5 =& 18^3 \times 24^2 \times \left(\frac{3}{40}\right)^{-3} \times \left(\frac{1}{5}\right)^{-2} \\ (M^e)^5 =& \left(\frac{18 \times 40}{3}\right)^3 \times (24 \times 5)^2 \\ M^e =& (240)^{3/5} \times (120)^{2/5} \\ =& (120 \times 2)^{3/5} \times (120)^{2/5} \\ =& 120 \times 2^{3/5} \\ \approx& 181.86 \end{aligned}$$

Step 4

Finally, we calculate the EV:

$$\begin{aligned} EV =& M^e - M^0 \\ \approx& 181.86 - 120 \\ \approx& 61.86 \end{aligned}$$

The EV is positive, which means that the price drop has an equivalent effect on consumer welfare as giving the consumer an extra \$61.86 income.

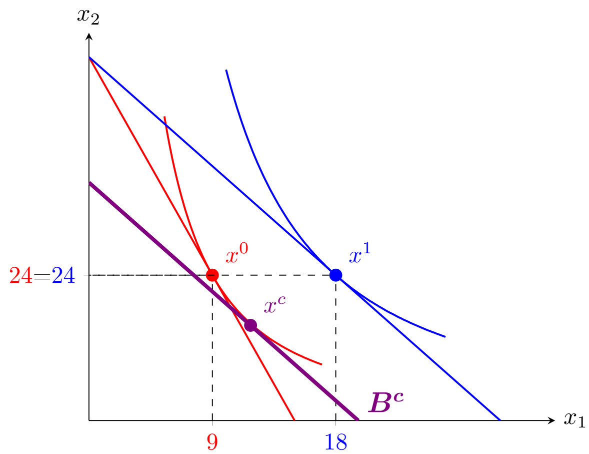
[Second, do the CV]

Step 2

Next, we determine the CV budget line. This is the new prices with an income level M^c, which we will need to solve for:

$$4x_1 + 2x_2 = M^e$$

Drawing this in the diagram, we know that this budget line has to be tangent to the red I^0 indifference curve and have the same slope as the blue B^1 budget line.



Step 3

Now, we find the value of the CV income M^c. We also know the prices in the alternate world are the same as the new prices. Therefore, we know that the demanded quantity (i.e. x^c) is:

$$(x_1^c, x_2^c) = \left(\frac{3M^c}{5p_1^1}, \frac{2M^c}{5p_2^1}\right) = \left(\frac{3M^c}{5\times 4}, \frac{2M^c}{5\times 2}\right) = \left(\frac{3M^c}{20}, \frac{M^c}{5}\right)$$

We also know that $u(x_1^e, x_2^e) = u^0$. Let's calculate the value of the old utility:

$$\begin{aligned} u^0 &= \left(x_1^1\right)^3 \left(x_1^1\right)^2 \\ &= 9^3 \times 24^2 \end{aligned}$$

Therefore, we can now solve for the CV income M^c

$$\begin{aligned} u(x_1^c, x_2^c) =& u^0 \\ \left(\frac{3M^c}{20}\right)^3 \left(\frac{M^c}{5}\right)^2 =& 9^3 \times 24^2 \\ (M^c)^5 \left(\frac{3}{20}\right)^3 \left(\frac{1}{5}\right)^2 =& 9^3 \times 24^2 \\ (M^c)^5 =& 9^3 \times 24^2 \times \left(\frac{3}{20}\right)^{-3} \times \left(\frac{1}{5}\right)^{-2} \\ (M^c)^5 =& \left(\frac{9\times 20}{3}\right)^3 \times (24\times 5)^2 \\ M^c =& (60)^{3/5} \times (120)^{2/5} \\ =& (120\times 0.5)^{3/5} \times (120)^{2/5} \\ =& 120 \times 2^{-3/5} \\ \approx& 79.17 \end{aligned}$$

Step 4

Finally, we calculate the CV:

$$\begin{aligned} CV =& M^c - M^0 \\ \approx& 79.17 - 120 \\ \approx& -40.83 \end{aligned}$$

The CV is negative, which means that to fully compensate the consumer for the price drop, we have take money away from them.

2.3 Quasi-Linear

Consider the following setup:

- $u(x_1, x_2) = 4\sqrt{x_1} + 2x_2$
- $p_1 = 2, p_2 = 2, M = 10$
- Then p_2 increases to $p_2 = 4$

We want to calculate the EV and CV of the price change

[Same first step for EV and CV]

Step 1

First, derive the demand function:

$$\begin{aligned} |MRS| = \frac{4 \cdot \frac{1}{2} x_1^{-1/2}}{2} &= \frac{p_1}{p_2} \\ x_1^{-1/2} &= \frac{p_1}{p_2} \\ x_1 &= \left(\frac{p_2}{p_1}\right)^2 \end{aligned}$$

For an interior solution, the demands are: (let's not worry about corners for this question)

$$\begin{aligned} x_1(p, M) &= \left(\frac{p_2}{p_1}\right)^2 \\ \implies x_2(p, M) &= \frac{1}{p_2}\left(M - p_1 x_1(p, M)\right) \\ &= \frac{M}{p_2} - \frac{p_1}{p_2}\left(\frac{p_2}{p_1}\right)^2 \\ &= \frac{M}{p_2} - \frac{p_2}{p_1} \end{aligned}$$

Then, we calculate the optimal bundles before and after the price change:

Before:

$$x_1(2,2,10) = \left(\frac{2}{2}\right)^2 = 1^2 = 1$$
$$x_2(2,2,10) = \frac{10}{2} - \frac{2}{2} = 4$$

After :

$$x_1(2,4,10) = \left(\frac{4}{2}\right)^2 = 2^2 = 4$$
$$x_2(2,4,10) = \frac{10}{4} - \frac{4}{2} = 0.5$$

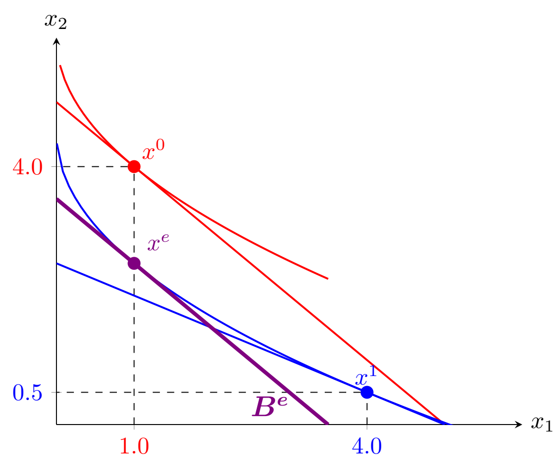
[First, do the EV]

Step 2

Next, we determine the EV budget line. This is the old prices with an income level M^e, which we will need to solve for:

$$2x_1 + 2x_2 = M^e$$

Drawing this in the diagram, we know that this budget line has to be tangent to the blue I^1 indifference curve and have the same slope as the red B^0 budget line.



Step 3

Now, we find the value of the EV income M^e. We also know the prices in the alternate world are the same as the old prices. Therefore, we know that the demanded quantity (i.e. x^e) is:

$$(x_1^e, x_2^e) = \left(\left(\frac{p_2^0}{p_1^0}\right)^2, \frac{M^e}{p_2^0} - \frac{p_2^0}{p_1^0}\right) = \left(\left(\frac{2}{2}\right)^2, \frac{M^e}{2} - \frac{2}{2}\right) = \left(1, \frac{M^e}{2} - 1\right)$$

We also know that $u(x_1^e, x_2^e) = u^1$. Let's calculate the value of the new utility:

$$\begin{aligned} u^1 &= 4\sqrt{x_1^1} + 2x_2^1 \\ &= 4\sqrt{4} + 2 \times 0.5 \\ &= 9 \end{aligned}$$

Therefore, we can now solve for the EV income M^e

$$\begin{aligned} u(x_1^e, x_2^e) =& u^1 \\ 4\sqrt{1} + 2\left(\frac{M^e}{2} - 1\right) =& 9 \\ 4 + M^e - 2 =& 9 \\ M^e =& 7 \end{aligned}$$

Step 4

Finally, we calculate the EV:

$$\begin{aligned} EV =& M^e - M^0 \\ =& 7 - 10 \\ =& -3 \end{aligned}$$

The EV is negative, which means that the price drop has an equivalent effect on consumer welfare as taking \$3 away from the consumer.

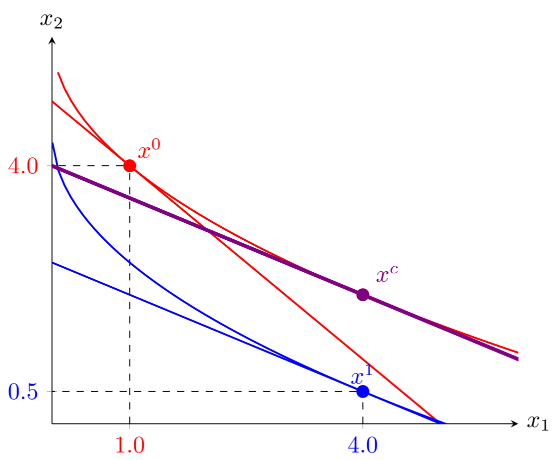
[Second, do the CV]

Step 2

Next, we determine the CV budget line. This is the new prices with an income level M^c, which we will need to solve for:

$$2x_1 + 4x_2 = M^c$$

Drawing this in the diagram, we know that this budget line has to be tangent to the red I^0 indifference curve and have the same slope as the blue B^1 budget line.



Step 3

Now, we find the value of the CV income M^c. We also know the prices in the alternate world are the same as the new prices. Therefore, we know that the demanded quantity (i.e. x^c) is:

$$(x_1^c, x_2^c) = \left(\left(\frac{p_2^1}{p_1^1}\right)^2, \frac{M^c}{p_2^1} - \frac{p_2^1}{p_1^1}\right) = \left(\left(\frac{4}{2}\right)^2, \frac{M^c}{4} - \frac{4}{2}\right) = \left(4, \frac{M^c}{4} - 2\right)$$

We also know that $u(x_1^e, x_2^e) = u^0$. Let's calculate the value of the old utility:

$$\begin{aligned} u^0 &= 4\sqrt{x_1^0} + 2x_2^0 \\ &= 4\sqrt{1} + 2 \times 4 \\ &= 12 \end{aligned}$$

Therefore, we can now solve for the CV income M^c

$$\begin{aligned} u(x_1^c, x_2^c) =& u^0 \\ 4\sqrt{4} + 2\left(\frac{M^c}{4} - 2\right) =& 12 \\ 8 + \frac{M^c}{2} - 4 =& 12 \\ M^c =& 16 \end{aligned}$$

Step 4

Finally, we calculate the CV:

$$\begin{aligned} CV =& M^c - M^0 \\ =& 16 - 10 \\ =& 6 \end{aligned}$$

The CV is positive, which means that the price drop is fully compensated by giving the consumer \$6.

A different price change

This result may be surprising to you because you may have learnt that quasi-linear is an exception where we have $|EV| = |CV| = |\Delta CS_{net}|$. That is true, but only for changes in the price of the non-numeraire good. so consider instead a price change in p_1 of $p_1 = 4$. Using the demand function, we can see that the consumer changes their optimal bundle to:

$$\begin{aligned} &\textbf{After}: \\ x_1(4, 2, 10) &= \left(\frac{2}{4}\right)^2 = \frac{1}{2}^2 = 0.25 \\ x_2(4, 2, 10) &= \frac{10}{2} - \frac{2}{4} = 5 - 0.5 = 4.5 \end{aligned}$$

[EV]

The EV budget line is:

$$2x_1 + 2x_2 = M^e$$

The demanded quantity is:

$$(x_1^e, x_2^e) = \left(\left(\frac{p_2^0}{p_1^0} \right)^2, \frac{M^e}{p_2^0} - \frac{p_2^0}{p_1^0} \right) = \left(\left(\frac{2}{2} \right)^2, \frac{M^e}{2} - \frac{2}{2} \right) = \left(1, \frac{M^e}{2} - 1 \right)$$

Notice that this is the same as before because we are still using the old prices. The difference comes in calculating the new utility:

$$\begin{aligned} u^1 &= 4\sqrt{x_1^1} + 2x_2^1 \\ &= 4\sqrt{\frac{1}{4}} + 2 \times 4.5 \\ &= 4 \times \frac{1}{2} + 9 \\ &= 11 \end{aligned}$$

Therefore, we can now solve for the EV income M^e

$$\begin{aligned} u(x_1^e, x_2^e) =& u^1 \\ 4\sqrt{1} + 2\left(\frac{M^e}{2} - 1 \right) =& 11 \\ 4 + M^e - 2 =& 11 \\ M^e =& 9 \end{aligned}$$

Finally, we calculate the EV:

$$\begin{aligned} EV =& M^e - M^0 \\ =& 9 - 10 \\ =& -1 \end{aligned}$$

[CV]

The CV budget line is:

$$4x_1 + 2x_2 = M^c$$

The demanded quantity is:

$$(x_1^c, x_2^c) = \left(\left(\frac{p_2^1}{p_1^1} \right)^2, \frac{M^c}{p_2^1} - \frac{p_2^1}{p_1^1} \right) = \left(\left(\frac{2}{4} \right)^2, \frac{M^c}{2} - \frac{4}{2} \right) = \left(\frac{1}{4}, \frac{M^c}{2} - \frac{1}{2} \right)$$

The old utility is unchanged at $u^0 = 4\sqrt{1} + 2 \times 4 = 12$. Therefore, we can now solve for the CV income M^c

$$u(x_1^c, x_2^c) = u^0$$

$$
\begin{aligned}
4\sqrt{\frac{1}{4}} + 2\left(\frac{M^c}{2} - \frac{1}{2}\right) =& 12 \\
2 + M^c - 1 =& 12 \\
M^c =& 11
\end{aligned}
$$

Finally, we calculate the CV:

$$
\begin{aligned}
CV =& M^c - M^0 \\
=& 11 - 10 \\
=& 1
\end{aligned}
$$

[CS]

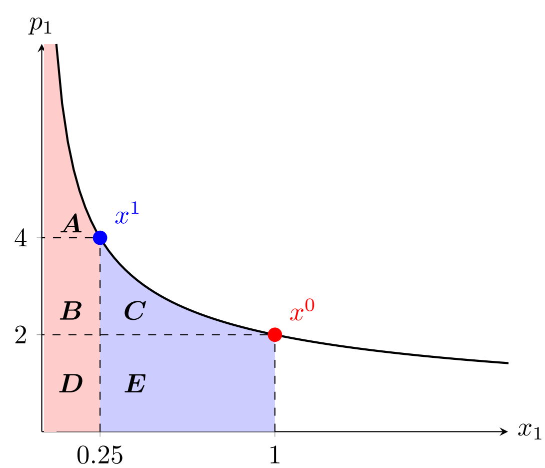
If we want to get very fancy, we can even check the change in consumer surplus. We are considering a change in the price of good 1, which has the following demand function:

$$x_1(p_1; p_2, M) = \left(\frac{p_2}{p_1}\right)^2$$

We have to first solve for p_1 to get the inverse demand function:

$$p_1(x_1; p_2, M) = \frac{p_2}{\sqrt{x_1}}$$

We are going from $p_1 = 2$ to $p_2 = 4$ (holding fixed $p_2 = 2$ and $M = 10$). We saw that means that x_1 changes from 1 to 0.25. Let's draw this in a diagram:



Note that the consumer surplus is as follows:

- Change in Gross CS $= C + E$
  - Old gross CS: $A + B + C + D + E$
  - New gross CS: $A + B + D$

- Change in Net CS $= B + C$
  - Old net CS: $A + B + C$
  - New net CS: A

To find the area under the curve, we can integrate the demand function. The change in gross CS is the integral from $x_1 = 0.25$ to $x_1 = 1$:

$$\begin{aligned}
\int_{0.25}^{1} p_1(x_1; 2, 10) &= \int_{0.25}^{1} \frac{2}{\sqrt{x_1}} \\
&= \int_{0.25}^{1} 2x_1^{-\frac{1}{2}} \\
&= \left[2 \cdot 2x_1^{\frac{1}{2}}\right]_{0.25}^{1} \\
&= 4\sqrt{1} - 4\sqrt{0.25} \\
&= 4 - 4(0.5) \\
&= 2
\end{aligned}$$

The change in net surplus is $B + C$. The change in gross surplus is $C + E$. So all we need to do is add B and subtract E from the value above to get the change in net CS. This is easy to do since that involves calculating the area of a rectangle:

$$\begin{aligned}
B &= (4 - 2) \times (0.25 - 0) = 2 \times 0.25 = 0.5 \\
E &= (2 - 0) \times (1 - 0.25) = 2 \times 0.75 = 1.5
\end{aligned}$$

Therefore, change in net CS is:

$$\begin{aligned}
\Delta CS_{net} &= \Delta CS_{gross} + B - E \\
&= 2 + 0.5 - 1.5 \\
&= 1
\end{aligned}$$

As we can see, we found that $|EV| = |CV| = |\Delta CS_{net}| = 1$ as expected.

2.4 Perfect Complements

Consider the following setup:

- $u(x_1, x_2) = \min\{3x_1, 4x_2\}$
- $p_1 = 1, p_2 = 2, M = 30$
- Then p_1 increases to $p_1 = 3$

We want to calculate just the EV of the price change

Step 1

First, derive the demand function. Since this is perfect complements, we don't use the tangency condition but instead find where the "kinks line" and budget line intersect:

$$
\begin{aligned}
3x_1 =& 4x_2 \\
x_2 =& \frac{3}{4}x_1 \\
\therefore p_1 x_1 + p_2\left(\frac{3}{4}x_1\right) =& M \\
x_1\left(4p_1 + 3p_2\right) =& 4M \\
x_1(p, M) =& \frac{4M}{4p_1 + 3p_2} \\
\implies x_2(p, M) =& \frac{3M}{4p_1 + 3p_2}
\end{aligned}
$$

Then, we calculate the optimal bundles before and after the price change:

Before:

$$
\begin{aligned}
x_1(1,2,30) =& \frac{4 \times 30}{4 \times 1 + 3 \times 2} = 12 \\
x_2(1,2,30) =& \frac{3 \times 30}{4 \times 1 + 3 \times 2} = 9
\end{aligned}
$$

After :

$$
\begin{aligned}
x_1(3,2,30) =& \frac{4 \times 30}{4 \times 3 + 3 \times 2} = \frac{20}{3} \approx 6.67 \\
x_2(3,2,30) =& \frac{3 \times 30}{4 \times 3 + 3 \times 2} = 5
\end{aligned}
$$

Step 2

Next, we determine the EV budget line. This is the new prices with an income level M^e, which we will need to solve for:

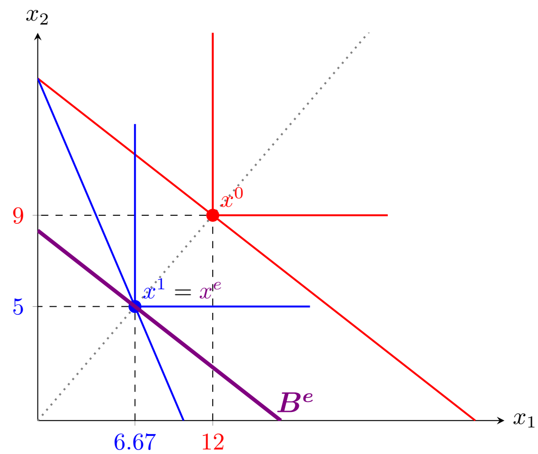
$$x_1 + 2x_2 = M^e$$

Drawing this in the diagram, we know that this budget line has to be tangent to the blue I^1 indifference curve and have the same slope as the red B^0 budget line. Since for perfect complements the "tangency" occurs at the kink, then it must be the case $x^1 = x^e$.

Step 3

Now, we find the value of the EV income M^e. We also know the prices in the alternate world are the same as the old prices. Therefore, we know that the demanded quantity (i.e. x^e) is:

$$
\begin{aligned}
(x_1^e, x_2^e) &= \left(\frac{4M^e}{4p_1^0 + 3p_2^0}, \frac{3M^e}{4p_1^0 + 3p_2^0}\right) \\
&= \left(\frac{4M^e}{4 \times 1 + 3 \times 2}, \frac{3M^e}{4 \times 1 + 3 \times 2}\right) \\
&= \left(\frac{4M^e}{10}, \frac{3M^e}{10}\right)
\end{aligned}
$$



We also know that $u(x_1^e, x_2^e) = u^1$. Let's calculate the value of the new utility:

$$\begin{aligned} u^1 &= \min\left\{3x_1^1, 4x_2^1\right\} \\ &= \min\left\{3 \times \frac{20}{3}, 4 \times 5\right\} \\ &= \min\left\{20, 20\right\} = 20 \end{aligned}$$

Therefore, we can now solve for the EV income M^e

$$\begin{aligned} u(x_1^e, x_2^e) =& u^1 \\ \min\left\{3 \times \frac{4M^e}{10}, 4 \times \frac{3M^e}{10}\right\} =& 20 \\ \min\left\{\frac{12M^e}{10}, \frac{12M^e}{10}\right\} = \frac{12M^e}{10} =& 20 \\ M^e =& \frac{200}{12} \\ =& \frac{100}{6} \approx 16.67 \end{aligned}$$

<u>**Step 4**</u>

Finally, we calculate the EV:

$$\begin{aligned} EV =& M^e - M^0 \\ \approx& 16.67 - 30 \\ \approx& -13.33 \end{aligned}$$

The EV is negative, which means that the price drop has an equivalent effect on consumer welfare as taking $\$13\frac{1}{3}$ away from the consumer.

2.5 Perfect Substitutes

Consider the following setup:

- $u(x_1, x_2) = 2x_1 + x_2$
- $p_1 = 3, p_2 = 5, M = 15$
- First, consider a change of p_2 decreasing to $p_2 = 3$
- Second, consider instead a change of p_2 decreasing to $p_2 = 1$

We want to calculate just the CV of the price change

<u>Step 1</u>

First, derive the demand function. Since this is perfect substitutes, we just need to compare the MRS and price ratio. If we have $|MRS| < \frac{p_1}{p_2}$, then the optimal choice is to only purchase x_2. If we have $|MRS| > \frac{p_1}{p_2}$, then the optimal choice is to only purchase x_1. The MRS is constant and equal to:

$$|MRS| = \frac{2}{1} = 2$$

Therefore, the demand function is:

$$x_1(p, M) = \begin{cases} 0 & \text{if } 2 < \frac{p_1}{p_2} \\ \frac{M}{p_1} & \text{if } 2 > \frac{p_1}{p_2} \\ \in \left[0, \frac{M}{p_1}\right] & \text{if } 2 = \frac{p_1}{p_2} \end{cases} \qquad x_2(p, M) = \begin{cases} \frac{M}{p_2} & \text{if } 2 < \frac{p_1}{p_2} \\ 0 & \text{if } 2 > \frac{p_1}{p_2} \\ \in \left[0, \frac{M}{p_2}\right] & \text{if } 2 = \frac{p_1}{p_2} \end{cases}$$

Then, we calculate the optimal bundles before and after the two price changes.

Before:

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{3}{5} = 0.6 \\ x_1(3, 5, 15) &= \frac{15}{3} = 5 \\ x_2(3, 5, 15) &= 0 \end{aligned}$$

After 1 :

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{3}{3} = 1 \\ x_1(3, 3, 15) &= \frac{15}{3} = 5 \\ x_2(3, 3, 15) &= 0 \end{aligned}$$

After 2 :

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{3}{1} = 3 \\ x_1(3, 1, 15) &= 0 \\ x_2(3, 1, 15) &= \frac{15}{1} = 15 \end{aligned}$$

<u>Step 2</u>

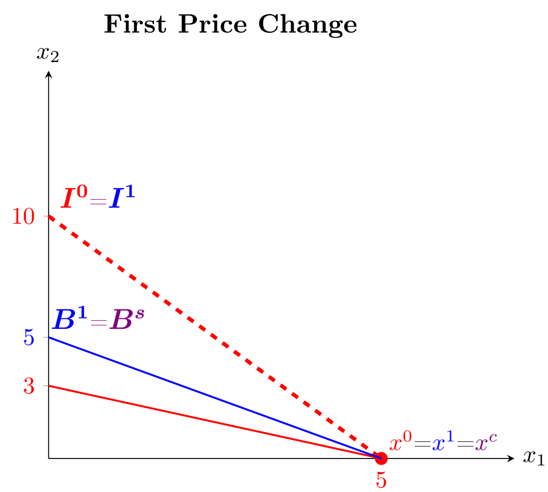
Next, we determine the CV budget line. This is the old prices with an income level M^c, which we will need to solve for. For the first price change, this will be:

$$3x_1 + 3x_2 = M^e$$

For the second price change, this will be:

$$3x_1 + x_2 = M^e$$

Drawing this in the diagram, we know that this budget line has to be tangent to the blue I^1 indifference curve and have the same slope as the red B^0 budget line. I will draw these in separate diagrams, where the dashed line indicates the indifference curve (to distinguish it from the budget line).



In the first price change, note that since the optimal bundle never changed, it must mean we are on the same indifference curve. This is why we have $I^0 = I^1$ and can only see one indifference curve.

Step 3

Now, we find the value of the CV income M^c. We also know the prices in the alternate world are the same as the new prices. Therefore, we know that the demanded quantity (i.e. x^c) for the first price change is:

$$(x_1^c, x_2^c) = \left(\frac{M^c}{p_1^1}, 0\right) = \left(\frac{M^c}{3}, 0\right)$$

For the second price change, it is:

$$(x_1^c, x_2^c) = \left(0, \frac{M^c}{p_2^1}\right) = (0, M^c)$$

We also know that $u(x_1^e, x_2^e) = u^0$. Let's calculate the value of the old utility:

$$\begin{aligned} u^0 &= 2x_1^0 + x_2^0 \\ &= 2 \times 5 + 0 \\ &= 10 \end{aligned}$$

Therefore, we can now solve for the CV income M^c. For the first price change:

$$\begin{aligned} u(x_1^c, x_2^c) =& u^0 \\ 2 \times \frac{M^c}{3} + 0 =& 10 \end{aligned}$$

$$M^c = \frac{30}{2} = 15$$

For the second price change:

$$\begin{aligned} u(x_1^c, x_2^c) =& u^0 \\ 2 \times 0 + M^c =& 10 \\ M^c =& 10 \end{aligned}$$

<u>**Step 4**</u>

Finally, we calculate the CV. For the first price change:

$$\begin{aligned} CV =& M^c - M^0 \\ =& 15 - 15 \\ =& 0 \end{aligned}$$

Unsurprisingly, we didn't change bundles, so the CV is zero (i.e. there is no change to welfare).

For the second price change:

$$\begin{aligned} CV =& M^c - M^0 \\ =& 10 - 15 \\ =& -5 \end{aligned}$$

The CV is negative, which means that to fully compensate the consumer for the price drop, we have take money away from them (i.e. the price drop was welfare improving).

3 Elasticity

Elasticity is a measure of responsiveness. In general, we will ask how demand changes in response to a change in one of the parameters v_1 (i.e. p_1, p_2, or M). The formulas for all of these are quite similar, so it's important to understand the overall intuition. To not worry about units, we want to express elasticity as a number from 0 to infinity (in absolute value), where you can interpret it as the *percentage change in quantity as a response to a percentage change in the variable of interest* v_1. When elasticity is equal to 1, we call this <u>unit elastic</u>. This says that for a 1% increase in v_1, we see a proportional 1% change in demand. A larger number (in absolute value) means it is more <u>elastic</u> (i.e. responds more than proportionately). On the other hand, a smaller number (in absolute value) means demand is more <u>inelastic</u> (i.e. less than proportionate responsiveness).

The most common elasticity is the **(own) price elasticity of demand** (ε_p). This asks: *what is the percentage change in quantity demanded in response to a 1% change in (own) price?* This interpretation tells us that the elasticity formula should look something like this:

$$\varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}$$

Where Q is quantity demanded and P is price. But this price elasticity is a local measure - it changes as the quantity demanded changes. So we actually want to consider very tiny changes. You can think of a derivative as a tiny change! This means that we can just write elasticity as:

$$\varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\frac{dQ}{Q}}{\frac{dP}{P}}$$

Where dX means a tiny change in the variable X. Now, despite what some of your math teachers say, you can pretty much treat derivatives as fractions. With some re-arranging this gives us:

$$\varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\frac{dQ}{Q}}{\frac{dP}{P}} = \frac{dQ}{dP} \times \frac{P}{Q}$$

This is the formula for the price elasticity of demand. Notice I've done this with the notation of P and Q, which isn't something we have used too much so far. So let me translate it into the notation we've been using so far in the course. It isn't too hard though. Just replace P with the appropriate price p_i and replace Q with the quantity demanded.

Consider the demand functions $x_1(p, M)$ and $x_2(p, M)$. Then the own price elasticity of good i is:

$$\begin{aligned} \varepsilon^i_{p_i} &= \frac{\partial x_i(p, M)}{\partial p_i} \times \frac{p_i}{x_i} \\ \implies \varepsilon^i(p_i; p_j, M) &= \frac{\partial x_i(p_i; p_j, M)}{\partial p_i} \times \frac{p_i}{x_i(p_i; p_j, M)} \end{aligned}$$

In the second line I have written it using the notation I used in Recitation 7. You should interpret $x_i(p_i; p_j, M)$ as the demand function. I want to make it clear that the elasticity is a <u>function</u>. In particular, we will write it as a function of all the variables *except* demand. In this case, we cannot have any x_i in our formula.[1]

Using the same idea, we can also come up with two other elasticities:

- The cross-**price elasticity of demand**. This asks: *what is the percentage change in quantity demanded in response to a 1% change in the other good's price?*
- The income **elasticity of demand**. This asks: *what is the percentage change in quantity demanded in response to a 1% change in income?*

The formulas for all elasticities, using both notation styles, are listed below:

[1] You can alternatively write it as a function of demand (i.e. remove p_i or whichever is the variable that you are changing). Both ways are correct, but we'll follow the textbook and have the price elasticity of demand as a function of price

	P-Q Notation	p-x Notation
Own Price Elasticity	$\frac{dQ}{dP} \times \frac{P}{Q}$	$\dfrac{\partial x_i(p_i; p_j, M)}{\partial p_i} \times \dfrac{p_i}{x_i(p_i; p_j, M)}$
$\varepsilon_P = \varepsilon^i(p_i; p_j, M)$		
Cross Price Elasticity	$\frac{dQ_i}{dP_j} \times \frac{P_j}{Q_i}$	$\dfrac{\partial x_i(p_j; p_i, M)}{\partial p_j} \times \dfrac{p_j}{x_i(p_j; p_i, M)}$
$\varepsilon_{P_j} = \varepsilon^i(p_j; p_i, M)$		
Income Elasticity	$\frac{dQ}{dM} \times \frac{M}{Q}$	$\dfrac{\partial x_i(M; p_i, p_j)}{\partial M} \times \dfrac{M}{x_i(M; p_i, p_j)}$
$\varepsilon_M = \varepsilon^i(M; p_i, p_j)$		

3.1 Examples in P-Q Notation

(1) Consider the following demand function: $Q = 10 - 2P$. Calculate the price elasticity:

$$\begin{aligned}\varepsilon_P &= \frac{dQ}{dP} \times \frac{P}{Q} \\ &= (-2) \times \frac{P}{Q}\end{aligned}$$

We're not done! Notice that there is a Q in there. Since this is price elasticity, we cannot have that P in there. To get rid of it, plug in the demand function

$$\begin{aligned}\varepsilon_P &= (-2) \times \frac{P}{(10 - 2P)} \\ &= \frac{-2P}{10 - 2P} \\ &= \frac{1}{1 - \frac{10}{2P}}\end{aligned}$$

This, as you can see, is a function. For example, at $P = 4$, the price elasticity is -4. But at $P = 2.5$, the price elasticity is -1

(2) Consider the following demand function (for good x): $Q_x = 10 - 2P_x + 3P_y$. Calculate the cross-price elasticity:

$$\begin{aligned}\varepsilon_{P_y} &= \frac{dQ_x}{dP_y} \times \frac{P_y}{Q_x} \\ &= (3) \times \frac{P_y}{Q_x}\end{aligned}$$

Again, we're not done as there is a Q_y in there. So plug in the demand function into the above

$$\begin{aligned}
\varepsilon_{P_y} &= (3) \times \frac{P_y}{10 - 2P_x + 3P_y} \\
&= \frac{3P_y}{10 - 2P_x + 3P_y} \\
&= \frac{1}{1 + \frac{10}{P_y} - 2\frac{P_x}{P_y}}
\end{aligned}$$

Notice that the cross-price elasticity depends not just on P_y (as we expect), but also on P_x. For example, if $P_y = 2$ and $P_x = 3$ (and therefore $Q_x = 10$), then:

$$\varepsilon_{P_y} = \frac{3 \times 2}{10 - 2 \times 3 + 3 \times 2} = \frac{6}{10} = 0.6$$

3.2 Examples in p-x_i Notation

(1) Consider the following Cobb-Douglas demand function: $x_1(p, M) = \frac{3M}{5p_1}$ (this is from section 2.2). Calculate the own price elasticity:

$$\begin{aligned}
\varepsilon^i(p_i; p_j, M) &= \frac{\partial x_1(p_1; p_2, M)}{\partial p_1} \times \frac{p_1}{x_1} \\
&= \frac{3M}{5}\left(-p_1^{-2}\right) \times \frac{p_1}{x_1} \\
&= -\frac{3M}{5p_1 x_1}
\end{aligned}$$

Now we plug in the demand function. This gives us:

$$\varepsilon^i(p_i; p_j, M) = -\frac{3M}{5p_1\left(\frac{3M}{5p_1}\right)} = -\frac{3M \times 5p_1}{3M \times 5p_1} = -1$$

So a special property of Cobb-Douglas is that the demand is unit-elastic everywhere. This means that at all price levels, quantity always responds proportionately to prices.

(2) Consider the same Cobb-Douglas demand function as above: $x_1(p, M) = \frac{3M}{5p_1}$ (this is from section 2.2). Calculate the income elasticity:

$$\begin{aligned}
\varepsilon^i(M; p_i, p_j) &= \frac{\partial x_1(M; p_1, p_2)}{\partial M} \times \frac{M}{x_1} \\
&= \frac{3}{5p_1} \times \frac{M}{x_1}
\end{aligned}$$

Now we plug in the Engel curve. This gives us another special property of Cobb-Douglas utility:

$$\begin{aligned}
\varepsilon^i(M; p_i, p_j) &= \frac{3}{5p_1} \times \frac{M}{\left(\frac{3M}{5p_1}\right)} \\
&= \frac{3}{5p_1} \times \frac{5p_1}{3} = 1
\end{aligned}$$