Intermediate Micro: Recitation 4
Optimal Choice

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1 Consumer's Optimization Problem

1.1 Overview

So far in the course we have been working towards modeling the consumer's problem (what to buy and how much?). In particular, we said that we want to set this up as an optimization problem. Such a problem has the following parts:

1. *Objective Function:* What function do we want to maximize?
2. *Constraint:* What restrictions are placed on our problem?
3. *Choice Variables:* What variables do we want to choose to achieve the optimal objective function?
4. *Parameters:* What variables affect the problem but we are not able to choose?

Now, we finally have all the ingredients to create our optimization problem:

1. *Objective Function:* The consumer's utility function $u(x_1, x_2)$
2. *Constraint:* The budget constraint $(p_1x_1 + p_2x_2 \leq M)$
3. *Choice Variables:* The quantity of each good (x_1 and x_2)
4. *Parameters:* The prices of each good (p_1 and p_2) and the consumer's income (M)

We write the problem as follows:

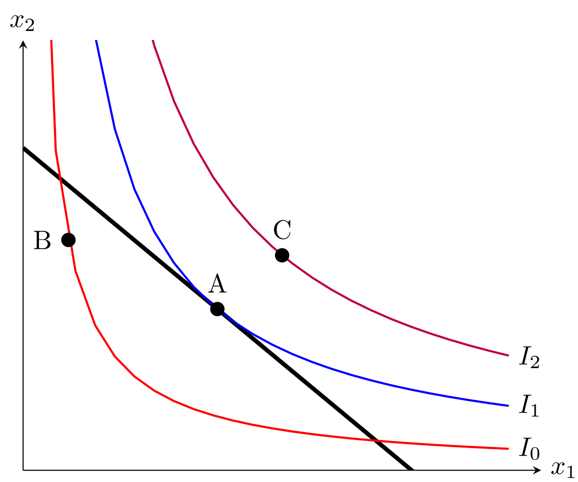
$$\max_{x_1,x_2} u(x_1, x_2)$$
$$\text{s.t. } p_1x_1 + p_2x_2 \leq M$$

We can interpret this as follows: the consumer's problem is to choose an affordable bundle of goods so that they maximize their utility. Note the key words - the consumer wants to make themselves as happy as possible, but they can't just get whatever they want, otherwise the problem would be totally un-interesting. The solution to this problem is called the **optimal bundle**, often denoted as $\boldsymbol{x^* = (x_1^*, x_2^*)}$. This is the bundle that gives the consumer the highest level of utility out of all the **feasible bundles** (options that are within the budget constraint). From this, you should be able to see these important results:

1. The optimal bundle must be feasible.
2. If a bundle is feasible, but was not chosen by the consumer, then it must give less utility than the optimal bundle.
3. If a bundle gives higher utility than the optimal bundle, but it was not chosen by the consumer, then it must not have been feasible.
4. There cannot be a feasible option that the consumer would strictly prefer over the optimal bundle (i.e. it gives strictly more utility)

1.2 Graphical Solution

Recall that we've drawn two types of graphs so far: budget constraints and indifference curves. Budget constraints gave us a *set* while indifference curves are essentially *contour plots*. Given the description of the optimal bundle, we can see how this naturally maps onto the graph. **The optimal bundle is the point in the budget constraint that lies on the highest indifference curve**.



Consider the diagram above, where the black line is the budget line. Point A is within the budget set and it is on the highest indifference curves out of all feasible options. Try taking any other point and draw the indifference curve going through it - you won't find a higher indifference curve! Let's look at such a point - for example, point B. This is also feasible, but the consumer gets lower utility on I_0 than on I_1. Therefore, B cannot be optimal - there's a better option in A. In fact, all points between the red indifference curve I_0 and the black budget line are all better options than B (they give a higher utility while still being feasible). On the flip side, consider point C. This gives higher utility than point A, but yet the consumer would not choose this point. Why? Because C is outside of the budget constraint, so this point is unaffordable for the consumer (even though they would prefer C over A).

Of course, we do not want to try every point until we find the one on the highest indifference curves. Notice that an indifference curve usually curves towards the origin (due to the concavity of preferences). This means that the "lowest"' part of an IC is the curved part (looking from the origin). In other words, this is the cheapest way to achieve a given level utility, because it requires having the least amount of goods. So, to try to get to the highest indifference curve, we should try to be on this curved part. In terms of the budget set, it doesn't make sense to be below the budget line (in the example above, this would be something like point B). If we don't spend all our money, then we surely cannot be at an optimal bundle: if we spent a bit more money and bought a little more of each good, then that must increase our utility (by monotonicity of preferences). This intuition tells us two things about the optimal point: it should be on the "bottom curve" of an IC and it should be on the budget line. Formally, this just says that **the optimal bundle is where an indifference curve is tangent to the budget line**.

1.3 Algebraic Solution

To solve an optimization problem, we set up the Lagrangian (if this is new to you, please read the math review in Recitation 1). The Lagrangian for our problem would be:

$$\mathcal{L} = u(x_1, x_2) + \lambda (M - p_1 x_1 - p_2 x_2)$$

Next, we take the FOCs (first order derivatives with respect to the two choice variables)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u(x_1, x_2)}{\partial x_1} + \lambda(-p_1) = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u(x_1, x_2)}{\partial x_2} + \lambda(-p_2) = 0\end{aligned}$$

We can re-arrange the equations and divide the first by the second:

$$\begin{aligned}\frac{\partial u(x_1, x_2)}{\partial x_1} &= \lambda p_1 \\ \frac{\partial u(x_1, x_2)}{\partial x_2} &= \lambda p_2 \\ \implies \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} &= \frac{p_1}{p_2}\end{aligned}$$

Notice that the partial derivatives are just the marginal utilities so the LHS can be written as MU_1/MU_2. But this is just the definition of the MRS (in absolute value). So the above equation can be written as:

$$|MRS_{12}| = \frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

We call this the **optimality condition**. Notice that the good indices have to match - if you have the marginal utility of good 1 on the LHS numerator then you need the price of good 1 on the RHS numerator (similarly for good 2 in the denominator). The Lagrangian we wrote was incredibly general - we didn't specify a utility function (other than assuming it is differentiable) or say anything specific about the prices and incomes (other than they are constant and exogenous). So any time you see an optimization problem that fits into the general one above, you can (usually) skip the Lagrangian and go straight to the optimality condition. There are some exceptions, and we'll see how to solve these later.

Now let's tie this back to the graphs. You should know that the tangent of a curve must have the same slope as the curve at that point. Therefore, the tangency condition can be expressed as "the slope of the indifference curve equals the slope of the budget constraint". From previous recitations, we know that the MRS is the slope of the indifference curve and the (negative) ratio of prices is the slope of the budget constraint. So the tangency condition is just:

$$\text{Slope of the IC} = \text{Slope of the BC}$$
$$MRS_{12} = -\frac{MU_1}{MU_2} = -\frac{p_1}{p_2}$$
$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

This is exactly the optimality condition! This further proves that this condition characterizes where our optimal bundle should be. We now have an algebraic and graphical justification for the condition. There's also economic intuition behind this too. The MRS tells us how much the consumer is willing to substitute between goods, while the slope of the budget constraint tells us the opportunity cost. Suppose that $|MRS_{12}| > \frac{p_1}{p_2}$, e.g. if $|MRS_{12}| = 4$ and $\frac{p_1}{p_2} = 3$. This says that the consumer, at their current bundle, is willing to give up 4 units of good 2 to get one more unit of good 1, and be just as indifferent as before. However, the opportunity cost of getting one more unit of good 1 is only 3 units of good 2. That means the consumer could give up those 3 units of good 2, get 1 more unit of good 1, and be *happier* than before (remember if they lost 4 units of good 2 they would have been indifferent, so only losing 3 units must be more preferred). Clearly this new option is still feasible and makes the consumer happier - which means that the original starting point could not have been optimal. You can make the same argument if $|MRS_{12}| < \frac{p_1}{p_2}$. This means that the only way you can be at an optimal point is when $|MRS_{12}| = \frac{p_1}{p_2}$. This is when the amount the consumer is willing to substitute is exactly equal to the opportunity cost they face; at this point, there is no trade the consumer can make that would make them strictly better off.

Of course, we are not done. We still need to actually find the optimal bundle itself. The end goal is to get the choice variables only as a function of the parameters. To finish solving the problem, we take the following steps:

1. MRS_{12} is a function of x_1 and x_2. This means the optimality condition is one equation with two unknowns. We solve it for one of the variables (say, x_2). This gives us x_2 as a function of x_1 and the prices: $x_2(x_1, p)$
2. Plug in the above into the budget constraint: $p_1x_2 + p_2x_2(x_1, p) = M$. This gives us one equation with one unknown. We can then solve for x_1 as a function of the parameters (prices and income). This gives us the **demand function** $x_1(p, M)$.
3. Plug in the demand function from (2) either into the budget constraint or into the function found in (1). This gives us the demand function for good 2: $x_2(p, M)$.

4. Plug in the specific prices and incomes into the demand functions (let's call them p^*, M^*). This gives us the optimal bundle: $x_i^* = x(p^*, M^*)$ for $i = 1, 2$.

2 Cobb-Douglas Utility

2.1 General Results

These are the most common types of utility functions you will see. Recall what we know about Cobb-Douglas utilities:

- General Form: $u(x_1, x_2) = x_1^\alpha x_2^\beta$
- Indifference Curves: Standard vanilla ICs. They must never touch either axis though!
- Preference: Intermediate mixer who likes to have a little bit of everything
- Utility Parameters: α and β, which indicate how much the consumer (relatively) cares about each good

To solve an optimization problem with Cobb-Douglas utility, let's first calculate the MRS for a general Cobb-Douglas utility:

$$|MRS_{12}| = \frac{MU_1}{MU_2} = \frac{\alpha x_1^{\alpha-1} x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

Therefore, the optimality condition gives us $x_2(x_1, p)$:

$$\begin{aligned}\frac{\alpha x_2}{\beta x_1} =& \frac{p_1}{p_2} \\ x_2 =& \frac{p_1}{p_2} \cdot \frac{\beta}{\alpha} \cdot x_1\end{aligned}$$

Plugging this into the budget constraint we can get the demand function $x_1(p, M)$:

$$\begin{aligned}p_1 x_1 + p_2 \left(\frac{p_1}{p_2} \cdot \frac{\beta}{\alpha} \cdot x_1\right) =& M \\ p_1 x_1 + p_1 \cdot \frac{\beta}{\alpha} \cdot x_1 =& M \\ p_1 x_1 \left(1 + \frac{\beta}{\alpha}\right) =& M \\ p_1 x_1 \left(\frac{\alpha + \beta}{\alpha}\right) =& M \\ p_1 x_1 =& \frac{\alpha}{\alpha + \beta} \cdot M \\ \therefore x_1^* =& \frac{\alpha}{\alpha + \beta} \cdot \frac{M}{p_1}\end{aligned}$$

This is the optimal quantity of x_1. To get the optimal quantity x_2^* we plug this into $x_2(x_1, p)$:

$$\begin{aligned} x_2 =& \frac{p_1}{p_2} \cdot \frac{\beta}{\alpha} \cdot \left(\frac{\alpha}{\alpha+\beta} \cdot \frac{M}{p_1} \right) \\ \therefore x_2^* =& \frac{\beta}{\alpha+\beta} \cdot \frac{M}{p_2} \end{aligned}$$

The results have a very nice interpretation. The parameters $\alpha$and β tell us how much the consumers cares about the good. To make this into a "relative" amount, we normalize the values by dividing by the sum of $\alpha$and β. This gives us a number between 0 and 1. The interpretation of this is *what share of the consumer's income they would optimally spend on each good*. So the general Cobb-Douglas result is:

For an optimal bundle with Cobb-Douglas utility $u(x_1, x_2) = x_1^\alpha x_2^\beta$, it must be that:

$$\begin{aligned} \text{Expenditure on good 1} &= \frac{\alpha}{\alpha+\beta} \text{ share of income} \\ \text{Expenditure on good 2} &= \frac{\beta}{\alpha+\beta} \text{ share of income} \end{aligned}$$

Keep in mind this is expenditure and not quantity, i.e. it is $p_i x_i$. From the equations, you should be able to see a number of important results about Cobb-Douglas utility:

1. The consumer will spend a constant share of their income on each good. If income increases, the consumer will buy more of both goods, but the *share* of income will remain unchanged
2. If the price of a good i increases, the consumer will decrease the optimal quantity of good i (but the total expenditure on good i is unchanged)
3. If the price of good i increases, the consumer will not change their quantity of good j (the other good)
4. The consumer will always choose non-zero amounts of each good
5. If α increases, the consumer will buy more x_1 and spend a higher share of their income on it. Similarly for β and x_2.

2.2 Example

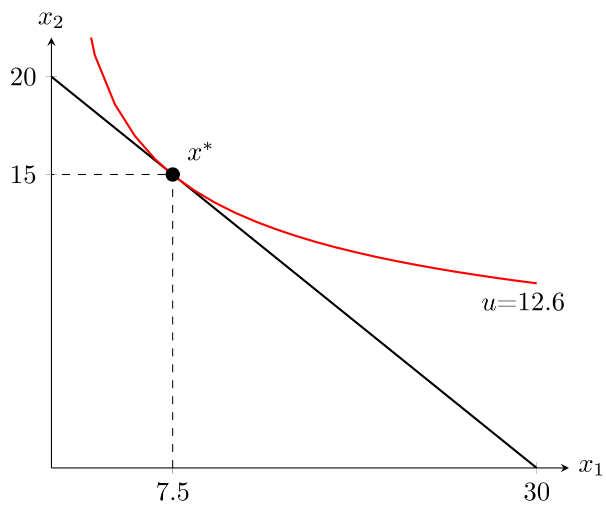
Suppose our question was as follows:

- Utility function: $u(x_1, x_2) = x_1^{0.25} x_2^{0.75}$
- Parameters: $p_1 = 2$, $p_2 = 3$, $M = 60$

Starting from the Lagrangian would be correct but would take a long time. You could also start at the optimality condition, but the Cobb-Douglas results are so important that we expect you to have them memorized (and understood!). So you can just jump straight to the demand functions:

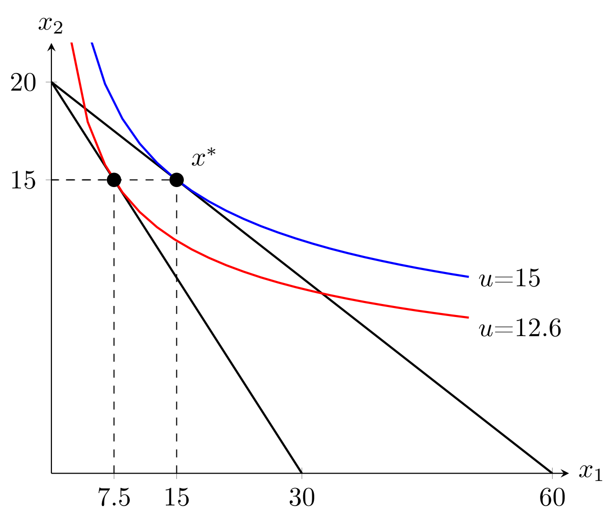
$$x_1^* = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{p_1} = \frac{0.25}{0.25+0.75} \cdot \frac{60}{2} = 0.25 \times 30 = 7.5$$
$$x_2^* = \frac{\beta}{\alpha+\beta} \cdot \frac{M}{p_2} = \frac{0.75}{0.25+0.75} \cdot \frac{60}{3} = 0.75 \times 20 = 15$$

Note that this optimal bundle is feasible: $p_1 x_1^* + p_2 x_2^* = 2 \times 7.5 + 3 \times 15 = 15 + 45 = 60 = M$. The utility that the consumer gets from this optimal bundle is $u(x_1^*, x_2^*) = 7.5^{0.25} \times 15^{0.75} \approx 12.6$. Plotting this in a graph gives us:



Now suppose the price of good 1 fell to \$1. How would the optimal bundle change? In the above equations, we only need to replace p_1 with 1 and keep everything else the same. This means that the shares do not change, and that the amount x_2^* does not change. The new optimal quantity x_1^* is:

$$x_1^* = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{p_1} = 0.25 \times \frac{60}{1} = 15$$

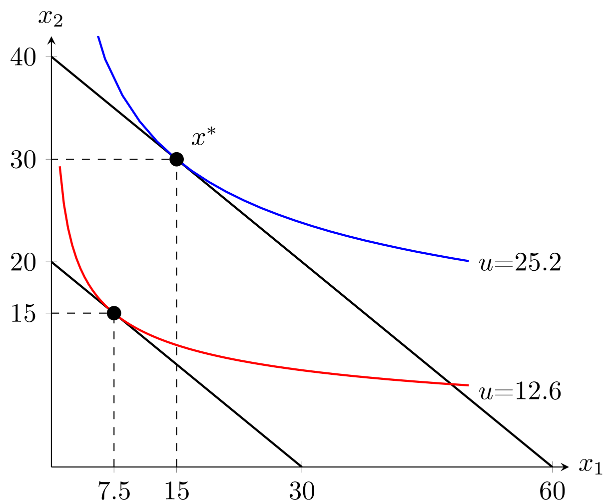


The consumer's utility has now increased to $15^{0.25} \times 15^{0.75} = 15$. Note that this new bundle is still feasible:

$p_1x_1^* + p_2x_2^* = 15 + 45 = 60 = M$. In fact, the expenditure on good 1 remains unchanged! It is \$15 in both cases, which is exactly what we expected. This is because the consumer cares "25%" about x_1 and so they spend 25% of their income on it: $25\% \times 60 = 15$, since their income is \$60.

Now suppose instead the consumer's income increases to \$120 (and p_1 is still 2). Both quantities will change (but again, the shares do not):

$$x_1^* = \frac{\alpha}{\alpha+\beta} \cdot \frac{M}{p_1} = 0.25 \times \frac{120}{2} = 0.25 \times 60 = 15$$
$$x_2^* = \frac{\beta}{\alpha+\beta} \cdot \frac{M}{p_2} = 0.75 \times \frac{120}{3} = 0.75 \times 40 = 30$$



Again, this is all still feasible: $2 \times 15 + 3 \times 30 = 30 + 90 = 120 = M$. Unsurprisingly, our consumer got richer, so now they buy more of both goods. But you should note that this was a proportional increase. The consumer's income doubled (from 60 to 120) and so the consumer now spends double the amount on each good. While the share of income spent on each good has stayed the same, it is now a share of a larger income. This means that the consumer spends double on x_1 (from 15 to 30) and double on x_2 (from 45 to 90). But since the prices have stayed the same, then this can only come from doubling the quantity purchased.

Final test of our knowledge. What if the utility function was instead $v(x_1, x_2) = x_1x_2^3$? Would this change the optimal bundle? The answer is no. Note this is still Cobb-Douglas, with $\frac{\alpha}{\alpha+\beta} = \frac{1}{1+3} = \frac{1}{4}$ and $\frac{\beta}{\alpha+\beta} = \frac{3}{1+3} = \frac{3}{4}$. So the shares spent on income would still be the same, and everything would go through as before. In fact, this utility function is just a monotonic transformation of our old utility function: $v(x_1, x_2) = f(u(x_1, x_2))$, where $f(x) = x^4$. This means that the preferences are still the same, which means that the solution is still the same. The only difference is that the value of the objective function will be different at the optimal bundle, i.e. $v(x_1^*, x_2^*) \neq u(x_1^*, x_2^*)$.

What if the utility function was $v(x_1, x_2) = 3\log x_1 + 9\log x_2$? This is also a monotonic transformation, where $f(x) = 12\log x$. If this is hard to see, another trick you can use is to check the MRS of each function.

If $u(x_1, x_2)$ and $v(x_1, x_2)$ have the same MRS, then they represent the same preferences.

$$|MRS^u| = \frac{\alpha x_2}{\beta x_2} = \frac{x_2}{3x_1}$$
$$|MRS^v| = \frac{MU_1}{MU_2} = \frac{\frac{3}{x_1}}{\frac{9}{x_2}} = \frac{3x_2}{9x_1} = \frac{x_2}{3x_1}$$

So the MRS's are the same, which again means that the optimal bundle would not change under this utility function.

2.3 Taxes

Let's suppose we had the following problem:

> A consumer has utility $u(x_1, x_2) = x_1^2 x_2^2$. The price of each good is 1 and the consumer's income is 100.
>
> (a) What is the consumer's optimal bundle and utility? What is the budget line? Draw the graph representing this problem.
>
> (b) The government puts a tax on good 1 of \$3 per unit. What is the consumer's new optimal bundle and utility? How much tax revenue does the government get? What is the new budget line? Draw these changes on the graph.
>
> (c) Instead of the tax in (b), the government decides to tax the consumer's income instead. They want to ensure they get the same amount of tax revenue from our consumer as the tax in (b) raised. What should the income tax be? Under this income tax system, what would the consumer's optimal bundle and utility be? What is their new budget line? Draw these changes on the graph.

Question (a)

From the question, we have the following values for our parameters: $p_1 = 1$, $p_2 = 1$, $M = 100$, $\alpha = 2$, $\beta = 2$. Using the formulas for Cobb-Douglas demand, we get:

$$x_1^* = \frac{2}{4} \cdot \frac{100}{1} = 50$$
$$x_2^* = \frac{2}{4} \cdot \frac{100}{1} = 50$$

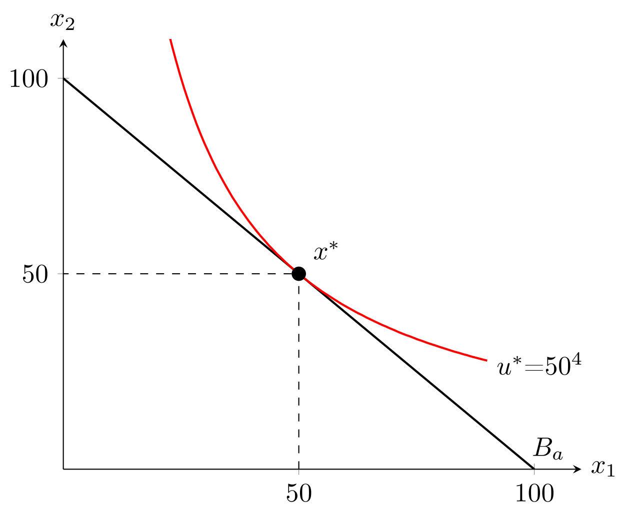
In other words, the consumer spends 50% of their income on good 1 and 50% on good 2, which means \$50 on each. Since both goods have a price of 1, they buy 50 units of each. Their utility from this optimal bundle is then:

$$u^* = u(x_1^*, x_2^*) = 50^2 50^2 = 50^4$$

The budget line, which we will call B_a, is then:

$$p_1x_1 + p_2x_2 = M$$
$$x_1 + x_2 = 100$$

Our graph looks like the following:



Question (b)

The tax increases the price of good 1 from 1 to 4, while everything else remains unchanged. This means that the only x_1^* should change (while x_2^* stays at 50). Let's call this new point $x^{**} = (x_1^{**}, x_2^{**})$:

$$x_1^{**} = \frac{1}{2} \cdot \frac{100}{4} = 12.5$$

As we said, we must have $x_2^{**} = x_2^* = 50$. The new utility, u^{**}, becomes:

$$u^{**} = 12.5^2 \times 50^2 = \left(\frac{25}{2}\right)^2 (25 \times 2)^2 = \frac{25^2 \cdot 25^2 \cdot \cancel{2^2}}{\cancel{2^2}} = 25^4$$

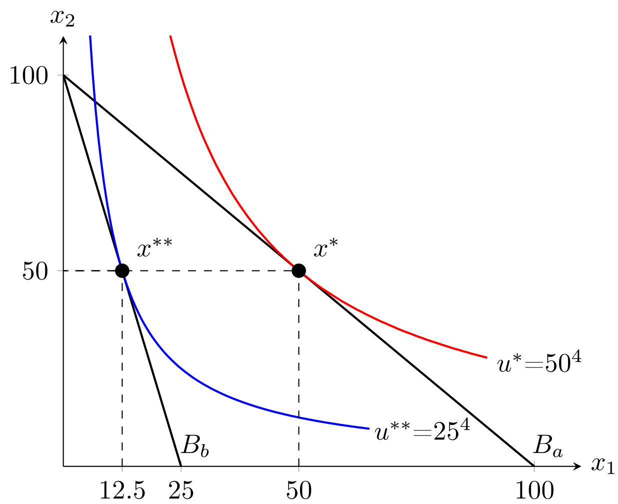
So our utility has fallen. Let's call the prices and income under this tax system as p_1', p_2', and M'. Obviously we have that $p_2' = p_2$ and $M' = M$, using the notation from before. But now $p_1' = p_1 + t$, where t is the unit tax. The budget line, which we will call B_b, is given by the equation:

$$p_1'x_1 + p_2'x_2 = M'$$
$$4x_1 + x_2 = 100$$

Since the government gets $t = 3$ for each unit of good 1 purchased, their tax revenue is simply:

$$R = t \times x_1^{**} = 3 \times 12.5 = 37.5$$

We can represent this change in a graph:



Question (c)

To get the same tax revenue as above, the government could simply tax the income for \$37.5. Equivalently, they could set the income tax rate at 37.5%. This reduces the consumer's income to $100 - 37.5 = 62.5$. The lower income affects the optimal quantity of both goods. Let's call the new bundle $x^{***} = (x_1^{***}, x_2^{***})$:

$$x_1^{***} = \frac{1}{2} \cdot \frac{62.5}{1} = 31.25$$
$$x_2^{***} = \frac{1}{2} \cdot \frac{62.5}{1} = 31.25$$

Now the newest utility u^{***} is:

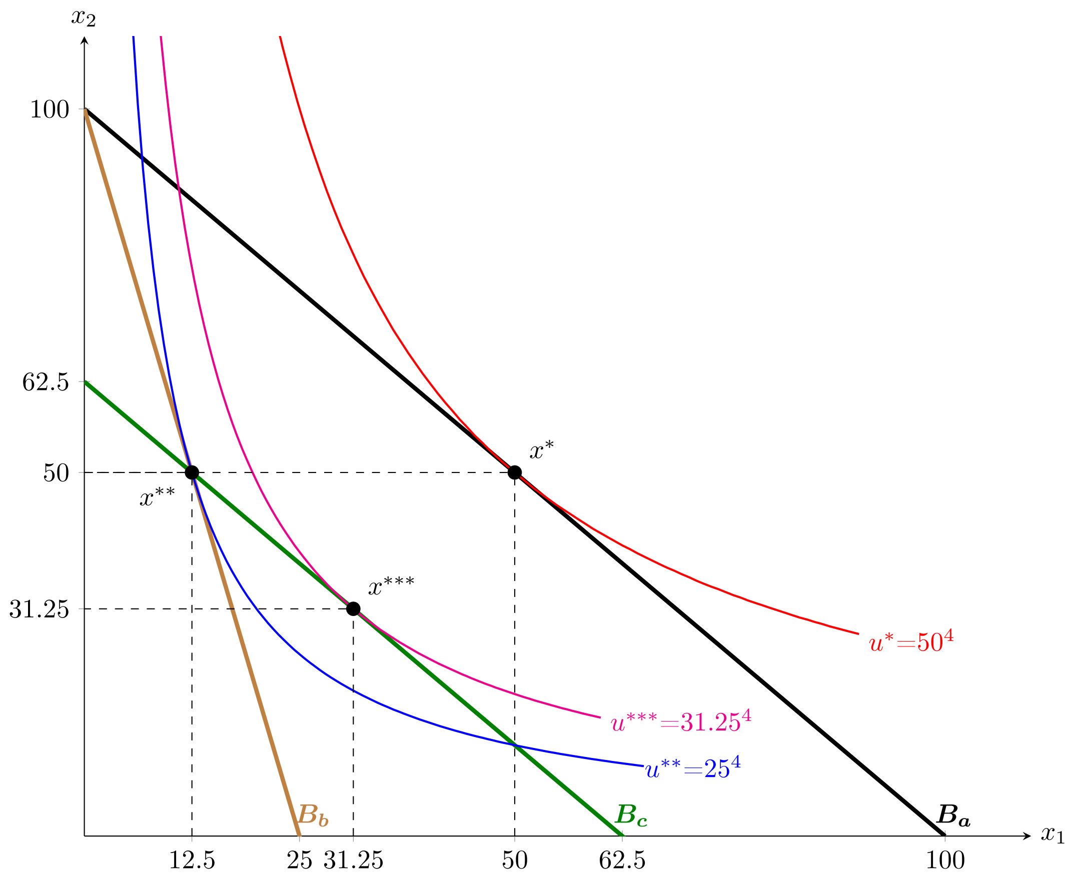
$$u^{***} = 31.25^2 \times 31.25^2 = 31.25^4$$

This is higher than in (b), but still lower than in (a), i.e. $u^{**} < u^{***} < u^*$. The graph on the next page, with a new budget line B_c, shows this change.

In fact, looking at the graph below, we can see that x^{**} is on the budget line B_c. This means that in fact it is affordable given this income tax system, but the consumer has chosen another bundle (which we can infer gives them higher utility). This is not a co-incidence - this occurs because of the way we defined the income tax.

We set the income tax to be equal to $R = tx_1^{**}$, where t was the unit tax. Let's call the prices and income under the income tax system as p_1'', p_2'', and M''. Reminding ourself of the notation so far, note that:

$p_1'' = p_1$	$p_2'' = p_2' = p_2$	$M'' = M - R$
$p_1' = p_1 + t$		$M' = M$
		$R = tx_1^{**}$



This means that the budget line B_c can be written as:

$$\begin{aligned}
p_1''x_1 + p_2''x_2 &= M'' \\
p_1x_1 + p_2x_2 &= M - R \\
p_1x_1 + p_2x_2 &= M - tx_1^{**} \\
p_1x_1 + tx_1^{**} + p_2x_2 &= M
\end{aligned}$$

Now we have to check whether $x^{**} = (x_1^{**}, x_2^{**})$ satisfies this equation and therefore sits on the budget line B_c. Plugging in this point into the LHS and using the relationships described above gets us:

$$\begin{aligned}
p_1x_1^{**} + tx_1^{**} + p_2x_2^{**} &= M \\
(p_1 + t)x_1^{**} + p_2x_2^{**} &= M \\
p_1'x_1^{**} + p_2'x_2^{**} &= M \\
M' &= M
\end{aligned}$$

Note that in the last step, we are simply using the equation of the B_b budget line. Since $M' = M$, this means that the equation must hold true. Therefore, x_1^{**} will always be on budget line B_c, as long as the income tax is set equal to the revenue from the income tax. So, if the government only cares about tax revenue, clearly they would be indifferent over which system to use. But the consumer has a clear preference: they prefer the income tax over the unit tax since it allows them to achieve a higher utility. We will see these ideas much more in the next part of course and will understand why this result holds true.

2.4 Harder Example

Here's a more challenging example. It is unlikely that you'll see a problem like this in an exam, but this is a good exercise to test your intuition.

Suppose a consumer faces the following prices: The price of good 1 is \$6 for the first 5 units and \$12 for any additional units above 5. The price of good 2 is constant at \$10. The consumer has an income of \$90. What is the optimal bundle if the consumer has the following utility functions:

1. $u(x_1, x_2) = x_1^2 x_2$
2. $u(x_1, x_2) = x_1 x_2$
3. $u(x_1, x_2) = x_1 x_2^2$

Send me an email or come to my office hours if you want to discuss your solution. Here are some hints if you get stuck:

- This setup will give us a kinked budget line. See the Recitation 2 notes for how to draw a kinked budget line
- You can think of a kinked budget line as two individual budget lines that have been shortened and combined into one. The point where they meet is the "kink". Imagine if there was no kink. What would the solution be if we only had the first budget line over the entire range of x_1? What if it was only the second budget line? Are those hypothetical optimal bundles achievable in the kinked budget line?
- How does the utility of the consumer change *along* the budget line? Can you express this in a function as $u = f(x_1)$, i.e. utility as a function of good 1? Plot this function for the first and second budget line. What happens at the kink point? What happens at the hypothetical optimal bundles from the hint above? What does this function look like for the kinked budget line? From this graph, are you able to identify the optimal bundle?