Advanced Micro: Recitation 1
Sets and Sequences

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1 Sequences

1.1 Definitions

Sequence A sequence x is a function that maps from the natural numbers $\mathbb{N}$ to X. Instead of using usual function notation, we use subscripts, $x(i) = x_i$, and we can think of a sequence as an infinitely long list, $\{x_1, x_2, ...\}$, where x_i is the i^{th} term of the sequence. We will denote a sequence as $\{x_n\}$

Euclidean Distance The Euclidean distance d is a function that is one way to measure the distance between two vectors x and y in $\mathbb{R}^n$. It is defined as:

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

A distance function (or metric) has three key properties:

1. $d(x,y) = 0$ if and only if $x = y$
2. Symmetry: $d(x,y) = d(y,x)$, $\forall x, y$
3. Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y, z$

Open Ball $B(x,\varepsilon)$ or $B_\varepsilon(x)$ is the open ball centered around $x \in X$ with radius ε is. It is the set of points within ε distance from x (alternatively, "in the ε-neighborhood of x"):

$$B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$$

Convergence A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } x_n \in B(x,\varepsilon), \forall n > N$$

In other words, a sequence converges to x if no matter what ε you choose, I can always find a point in the sequence such that all the remaining terms in the sequence that are within ε distance from x. If a sequence $\{x_n\}$ converges to a point x, we call that the **limit** of the sequence and denote it as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$

Monotone A sequence is *increasing* iff $x_m \leq x_n$, for any $m \leq n$. A sequence is *decreasing* iff $x_m \geq x_n$, for any $m \leq n$. A sequence is *monotone* iff it is increasing or decreasing.

1.2 Useful Results

1. **The limit of a sequence is unique (if it exists)**

   Intuition. For uniqueness proofs, using proof by contradiction is often a good strategy. The intuition here is that if a sequence converges to two points, then it must be in the ε-neighborhood of both points. This seems odd because as ε gets smaller and smaller, this still needs to be true. In particular, if the ε balls do not overlap, then the same point is going to be in two non-overlapping sets. This will be the contradiction that we need to prove our result.

   Proof. Use proof by contradiction, i.e. suppose that the limit is not unique and find a nonsensical result. Suppose that $x_n \to x$ and $x_n \to x'$, where $x \neq x'$. By definition of convergence, we must have that for any $\varepsilon > 0$, $\exists N$ s.t. $x_n \in B(x, \varepsilon), \forall n > N$ and $\exists N'$ s.t. $x_n \in B(x', \varepsilon), \forall n > N'$. Since these two limit points are distinct, we must have that $d(x, x') > 0$. So we can set $\varepsilon = \frac{d(x,x')}{2}$ and find the corresponding N and N' from the convergence definition. Next, set $N^* = \max\{N, N'\}$. This means that we have:

   $$\begin{aligned} &x_n \in B(x, \varepsilon), \forall n > N^*, \text{ and} \\ &x_n \in B(x', \varepsilon), \forall n > N^* \end{aligned}$$

   This is where the contradiction will come: the sequence is in the ε neighborhood of both x and x', even though ε is smaller than the distance between these two points! To formally get the contradiction, take any x_n for $n > N^*$, and start with the triangle inequality property:

   $$\begin{aligned} d(x, x') \leq& d(x, x_n) + d(x_n, x') \\ <& \varepsilon + \varepsilon \\ =& 2\varepsilon \\ =& d(x, x') \end{aligned}$$

   □

2. **A convergent sequence is bounded, i.e. the set $S = \{x_1, x_2, ...\}$ is bounded such that $\exists x \in X$ and $r > 0$ where $S \subset B(x, r)$**

   Intuition. If you have a finite set, it's very easy to bound it (make your r such that you take the biggest element and add 1). The issue with sequence values is that they could be an infinite set. However, looking at the definition for a bounded set, it involves containing elements in an open ball. But we already know that a *convergent* sequence will *eventually* be inside an ε-sized open ball. So that deals with the infiniteness part of the sequence - all we really have to worry about now is how to bound the first part of the sequence before it becomes entirely in inside the ε-ball. But, we know that this is a finite set of values, and as we said, it's easy to bound this.

Proof. Let the sequence be $x_n \to x$. Choose any ε and find the corresponding N from the convergence definition, e.g. let $\varepsilon = 1$ and find the N such that $x_n \in B(x, \varepsilon), \forall n > N$. Now we can divide the sequence into two parts: the first N terms (which have $d(x_n, x) > \varepsilon$) and terms from $N + 1$ onwards (which have $d(x_n, x) < \varepsilon$). The second part is clearly bounded by the open ball $B(x, \varepsilon) = B(x, 1)$, so we just have to worry about bounding the first part. But this possible because it is finite - just find the x_n that is furthest from x. Formally, let $r = \max\{d(x_1, x), d(x_2, x), ..., d(x_N, x)\} + 1$ and then we must have that $S \subset B(x, r)$. □

3. **If there are two convergent sequences in $\mathbb{R}$, $x_n \to x$ and $y_n \to y$, then (i) $x_n + y_n \to x + y$ and (ii) $x_n y_n \to xy$**

   Intuition. Let's prove the first claim, and I'll leave the second as an exercise (see Theorem 12.3 in Simon and Blume). This result is fairly intuitive, but proving it can be tricky if you get lost in the notation. Here, you should make sure you remember what the goal is. We know that $\{x_n\}$ and $\{y_n\}$ will eventually be inside *any* ε-ball, but we want to show that $\{x_n + y_n\}$ will be too. So we need to make the ε for $\{x_n + y_n\}$ arbitrary, but we can strategically choose the ε for $\{x_n\}$ and $\{y_n\}$. In general, if you have two convergent sequences, you usually want to choose an N such that both sequences are inside their respective ε-balls. The only trick comes in setting the ε value.

   Proof. Since we are in $\mathbb{R}$, then $d(x, y) = |x - y|$. Let's prove the first claim directly. We want to show that: $\forall \varepsilon > 0, \exists N$ s.t. $|(x_n + y_n) - (x + y)| < \varepsilon, \forall n > N$. We start by knowing that: $\forall \tilde{\varepsilon} > 0$

$$\exists N_x \text{ s.t. } |x_n - x| < \tilde{\varepsilon}, \forall n > N_x, \text{ and}$$
$$\exists N_y \text{ s.t. } |x_n - x| < \tilde{\varepsilon}, \forall n > N_y$$

   Choose an arbitrary ε and let $\tilde{\varepsilon} = \frac{\varepsilon}{2}$ (i.e. ε is the arbitrary distance we want to use to show that $x_n + y_n$ converges, while $\tilde{\varepsilon}$ is the distance used the convergence definition for x and y). Then we can set $N = \max\{N_x, N_y\}$, so that $|x_n - x| < \tilde{\varepsilon}$ and $|y_n - x| < \tilde{\varepsilon}$, $\forall n > N$. Then we have:

$$\begin{aligned}
|(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| && \text{(re-arrange)} \\
&\leq |x_n - x| + |y_n - y| && \text{(triangle inequality)} \\
&< \tilde{\varepsilon} + \tilde{\varepsilon} && \text{(convergence of } x_n, y_n) \\
&= \varepsilon
\end{aligned}$$

   □

4. **Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence**

   Proof. Use the following two results: (1) Every monotone and bounded sequence in $\mathbb{R}$ is convergent; and (2) Every sequence in $\mathbb{R}$ has a monotone subsequence. You can prove these as an exercise, but here are some hints. For the first one (the monotone convergence theorem), notice that an increasing sequence is bounded from below by its first term, and since it is bounded, then the set have values must be bounded from above and must have a least upper bound. You want to show that sequence converges to the supremum. For the second one, call a term in the sequence x_n a dominant term if $x_n \geq x_m, \forall m \geq n$. What are the two cases of how many dominant terms there are in the sequence? How can you use these to create a monotone sequence? $\square$

2 Sets

2.1 Definitions

Limit Points A limit point of a set S is a point x (not necessarily in S) such that $(B(x, \varepsilon) \backslash \{x\}) \cap S \neq \emptyset$, $\forall \varepsilon > 0$, i.e. any open ball around x (but without x itself) will always contain some points in the set - no matter how small the ball. The set of limit points is denoted as $L(S)$.

Closure The closure of a set S, denoted as $\bar{S}$ or $cl(S)$, is defined as the union of the set and its limit point: $\bar{S} = S \cup L(S)$.

Boundary A point $x \in S$ is in the boundary of a set S if $B(x, \varepsilon) \cap S \neq \emptyset$ and $B(x, \varepsilon) \cap S^c \neq \emptyset$, $\forall \varepsilon > 0$, i.e. any open ball around the point contains both points in S and its complement.

Open A set S is open if for each $x \in S$, $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset S$, i.e. each element in the set contains at least one open ball around it that is entirely in the set

Closed Two ways to define a closed set:

[Topological] A set S is closed if and only if it is equal to its closure: $S = \bar{S}$. Equivalently, the set is closed iff it contains all its limit points: $S' \subset S$.

[Sequential] For any convergent sequence $\{x_n\}$ in S that converges to a point x, we must have $x \in S$

Bounded A set $S \subset X$ is bounded iff $\exists x \in X$ and $r > 0$ such that $S \subset B(x, r)$

Compact A set S is compact if every sequence in S has a convergent subsequence that converges to a point in S

2.2 Simple Results

The following results are fairly obvious if you draw a simple Venn diagram. But they are a good way to practice thinking about exactly what you need to prove before you actually dive into it. Here is a tip for proving claims with sets. For any two sets, A and B, if we want to show that $A \subset B$, then we need to show that $\forall x \in A$, we have $x \in B$. If we want to show that two sets are equivalent, $A = B$, then this is just saying that both $A \subset B$ and $B \subset A$ are true. This means we would need to show that $\forall x \in A, x \in B$ and $\forall x \in B, x \in A$.

1. $\boldsymbol{A \cup B = B}$ ***iff*** $\boldsymbol{A \subset B}$
   - Similarly: $\boldsymbol{A \cap B = A}$ ***iff*** $\boldsymbol{A \subset B}$ (proof left as an exercise)

Proof. To prove this statement, we need to prove the following:

(a) Prove the "if" claim ($\Longleftarrow$): $A \subset B \implies A \cup B = B$

   i. Prove the $\subset$ claim: $A \subset B \implies A \cup B \subset B$, i.e. $\forall x \in A \cup B, x \in B$

   ii. Prove the $\supset$ claim: $A \subset B \implies B \subset A \cup B$, i.e. $\forall x \in B, x \in A \cup B$

(b) Prove the "only if" claim ($\Longrightarrow$): $A \cup B = B \implies A \subset B$

   i. Prove the $\subset$ claim: $A \cup B = B \implies A \subset B$, i.e. $\forall x \in A, x \in B$

First, (a)(i): Take any $x \in A \cup B$. It must be that $x \in A$ or $x \in B$ (or both). Since $A \subset B$, then we have $x \in B$ either way.
Next, (a)(ii): Take any $x \in B$. By definition of the union, we must have $x \in A \cup B$ too.
Finally, (b)(i): Take any $x \in A$. By definition, we must have $x \in A \cup B = B$. Therefore, $x \in B$.

□

2. $\boldsymbol{B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap A_2)}$

   - Similarly: $\boldsymbol{B \cup (A_1 \cap A_2) = (B \cup A_1) \cap (B \cup A_2)}$ (proof left as an exercise)

Proof. To prove this statement, we need to prove the following:

(a) $\subset$: Take any $x \in B \cap (A_1 \cup A_2)$. We must have $x \in B$ and $x \in (A_1 \cup A_2)$. There are two cases then: (not mutually exclusive)

   i. $x \in A_1$. In that case, $x \in B$ and $x \in A_1$, so $x \in (B \cap A_1)$; or

   ii. $x \in A_2$. In that case, $x \in B$ and $x \in A_2$, so $x \in (B \cap A_2)$

   Putting this together, this means that $x \in (B \cap A_1) \cup (B \cap A_2)$

(b) $\supset$: Take any $x \in (B \cap A_1) \cup (B \cap A_2)$. We must have $x \in (B \cap A_1)$ or $x \in (B \cap A_2)$. There are two cases then: (not mutually exclusive)

   i. $x \in (B \cap A_1)$. In that case, $x \in B$ and $x \in A_1$; or

   ii. $x \in (B \cap A_2)$. In that case, $x \in B$ and $x \in A_2$

   In either case, we must have $x \in B$ and $x \in (A_1 \cup A_2)$. Putting all this together, this means that $x \in B \cap (A_1 \cup A_2)$

□

3. **De Morgan's Law:** $\boldsymbol{(A_1 \cup A_2)^c = A_1^c \cap A_2^c}$ **and** $\boldsymbol{(A_1 \cap A_2)^c = A_1^c \cup A_2^c}$

   - Similarly: extend this to an arbitrary number of sets (proof left as an exercise)

Proof. The first statement:

- $\subset$: $x \in (A_1 \cup A_2)^c \implies x \in U \backslash (A_1 \cup A_2) \implies x \notin A_1$ and $x \notin A_2 \implies x \in A_1^c$ and $x \in A_2^c \implies x \in (A_1^c \cap A_2^c)$
- $\supset$: $x \in (A_1^c \cap A_2^c) \implies x \in A_1^c$ and $x \in A_2^c \implies x \notin A_1$ and $x \notin A_2 \implies x \notin (A_1 \cup A_2) \implies x \in (A_1 \cup A_2)^c$

For the second statement, note that $(S^c)^c = S$ for any set S. Use this and the results of the first statement:

$$\begin{aligned}(A_1 \cap A_2)^c &= ((A_1^c)^c \cap (A_2^c)^c)^c && \text{(double complement)}\\ &= ((A_1^c \cup A_2^c)^c)^c && \text{(apply first statement inside brackets)}\\ &= A_1^c \cup A_2^c && \text{(double complement)}\end{aligned}$$

□

2.3 Useful Results

1. **A set S is closed if and only its complement S^c is open.**

   Proof. Homework 1 Question 4. □

2. **The union of open sets is open**

   - Similarly: **The union of closed sets is closed** (proof left as an exercise)

   Intuition. Draw a Venn diagram for intuition. With open sets, you should be able to draw an open ball around any point. If you create a union of sets (even uncountably many!), you aren't "losing" any of the individual sets. This means you should still be able to draw the exact same open ball as before (remember, we only need one!).

   Proof. Start with a set of open sets indexed by $i \in I$: $\{S_i\}_{i \in I}$ (there could potentially be uncountably many sets). Denote their union as $T = \bigcup_{i \in I} S_i$. We want to show that this is open. Take any $x \in T$. By definition, $\exists j \in I$ such that $x \in S_j$. Since S_j is open, then $\exists r > 0$ such that $B(x, r) \subset S_j$. This must mean that $B(x, r) \subset T$ and therefore, we can find an open ball inside T for any x. So the union is open. □

3. **The intersection of a finite set of open sets is open**

   - Similarly: **The intersection of closed sets is closed** (proof left as an exercise)

   Intuition. Draw a Venn diagram for intuition. Unlike the case before, we may not be able to draw the same open ball because we might have lost part of the set in the intersection process. But, the points in the intersection all come from open sets. So for each these points, we can draw a whole set of open balls (one from each of the sets in the intersection) centered at that point. All we need is to take the smallest ball (which we can find since there are finitely many). The smallest open ball has to clearly fit into all the other balls (they all have the same center). But that must mean it is in all the individual sets too, and hence also in the intersection.

Proof. Start with a set of open sets indexed by $i \in \{1, ..., I\}$: $\{S_i\}_{i=1}^I$ (there could potentially be uncountably many sets). Denote their intersection as $T = \bigcap_{i=1}^{I} S_i$. Take any $x \in T$. By definition, it must be the case that $x \in S_i, \forall i$. Moreover, for each i, since S_i is open, then $\exists r_i > 0$ such that $B(x, r_i) \subset S_i$. Let $r = \min\{r_1, r_2, ..., r_I\}$. Then, $B(x, r) \subset B(x, r_i) \subset S_i, \forall i$. By definition then, $B(x, r) \subset \bigcap_{i=1}^{I} S_i = T$. □

4. **In $\mathbb{R}^n$, a set S is compact if and only if it is closed and bounded**

   Proof. First, we prove the "if" direction ($\Longleftarrow$). Take an arbitrary sequence $\{x_n\}$ in S. Since S is bounded, then $\{x_n\}$ must be bounded too. Using the Bolzano-Weierstrass theorem, $\{x_n\}$ must have a convergent subsequence. Since S is closed, then this subsequence must converge in S. Therefore, S is compact. Next, we prove the "only if" direction ($\Longrightarrow$). Suppose S is compact, but not closed. If so, there exists a sequence $\{x_n\}$ that converges to a point $x \notin S$. But then we can trivially find a subsequence that converges to a point outside of S (let the subsequence be the sequence itself). Therefore, S must be closed. Finally, suppose S is compact, but not bounded. Take an arbitrary sequence and construct a subsequence such that $x_n \in S$ and $d(x_n, 0) > n$ (this is possible since S is unbounded). But this sequence cannot have a convergent subsequence, contradicting S being compact. Therefore, S must be bounded. □

2.4 Examples

Open/Closed

For each of the following sets $S \subset X$, determine if they are: open/not open, and closed/not closed

1. The set $S = [0, +\infty)$ in $X = \mathbb{R}$
2. The set $S = [0, +\infty)$ in $X = \mathbb{R}_+$
3. The set $S = \{1/n : n \in \mathbb{N}\}$ in $X = \mathbb{R}$
4. The set $S = \mathbb{Z}$ (set of integers) in $X = \mathbb{R}$

Answers:

1. Not open: take any ball centered around 0 and there will be an element in in $S^c = \mathbb{R} \backslash S$ (i.e. a negative number). However, it is closed. Any sequence $\{x_n\} \in S$ that converges to a non-negative number x will have $x \in S$. If $x < 0$ (i.e. $x \notin S$), then at some point the sequence must have terms $x_n \notin S$, which is a contradiction.

2. This is still closed, but now it is also open as we don't have the same problem as above (since $S^c = \mathbb{R}_+ \backslash S = \emptyset$.

3. Not open: Since $n \in \mathbb{N}$, this means that the set is bounded from above by 1, with $1 \in S$. Therefore, for any $\varepsilon > 0$, $\exists x \in B(1, \varepsilon)$ such that $x > 1$ and therefore $x \notin S$. Not closed: take the sequence $x_n = \frac{1}{n}$. Clearly, we have $x_n \in S, \forall n$ and $x_n \to 0$. But $0 \notin S$.

4. Not open: take any $x \in S$ and any $\varepsilon > 0$. There must be $y \in B(x, \varepsilon)$ with $|x - y| \in (0, 1)$ and therefore $y \notin S$ and so $B(x, \varepsilon) \not\subset S$. It is closed. The interval between any two consecutive integers is open, e.g. $(1, 2)$. The union of all these intervals must therefore also be open. This is exactly the complement set S^c. Since the complement is open, that means S must be closed. Alternatively, $L(S) = \emptyset$ (there are no limit points) and trivially we have $L(S) = \emptyset \subset S$, so S is closed.

Economics Application

Assume that the preference relation $\succsim$ is rational (complete and transitive). Show that the following are all equivalent definitions of continuous preferences:

1. If $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n, \forall n$, then $x \succsim y$
2. $\forall y \in X$, the sets $\{x \in X : x \succsim y\}$ and $\{x \in X : y \succsim x\}$ are closed
3. $\forall y \in X$, the sets $\{x \in X : x \succ y\}$ and $\{x \in X : y \succ x\}$ are open
4. If $x \succ y$, then $\exists \varepsilon > 0$ such that $x' \succ y', \forall x' \in B(x, \varepsilon)$ and $\forall y' \in B(y, \varepsilon)$

Answers

You can do this in a few ways, but essentially you want to create a logic "cycle". I'm going to show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1) to complete the cycle.

[**1** $\implies$ **2**]

- Since we are working with sequences, it's easier to use the sequential definition of closed set. Fix an arbitrary y. Let's consider the set $S = \{z \in X : z \succsim y\}$ (I'm just using z here to avoid confusion with the x we will actually use in the proof). We want to show that this set is closed. By definition of a closed set, this means that if we have a sequence x_n in S such that $x_n \to x$, then $x \in S$.
- Now, let's put this all in $P \implies Q$ notation. We want to show that [1] $\implies$ [2]. But, our premise P (i.e. [1]) and conclusion Q (i.e. [2]) are each themselves a proposition. Let's call them as follows:
  - [$1P$]: $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n, \forall n$
  - [$1Q$]: $x \succsim y$
  - [$2P$]: $x_n \in \{z \in X : z \succsim y\}, \forall n$ and $x_n \to x$
  - [$2Q$]: $x \in \{z \in X : z \succsim y\}$.
- Note that we want to show ([$1P$] $\implies$ [$1Q$]) $\implies$ ([$2P$] $\implies$ [$2Q$]). The logic of this proof is as follows. We want to start in an *arbitrary* place that satisfies [$2P$]. Then, we want to create the conditions to also satisfy [$1P$] (this does not have to be arbitrary). Once we have that, we can then apply the result of [1] to conclude that [$1Q$] is true. With that, we should be to then conclude that [$2Q$] is true too.
- The actual proof:
  - Take a sequence $x_n \in \{z \in X : z \succsim y\}, \forall n$ and $x_n \to x$. In other words, we have $x_n \succsim y, \forall n$ and $x_n \to x$. This is exactly [$2P$].

- Create a sequence $y_n = y$. This implies two things: (i) $y_n \to y$, and (ii) $x_n \succsim y_n, \forall n$.
- With our sequences $\{x_n\}$ and $\{y_n\}$, we have now satisfied [$1P$]. We can now use [1] and conclude [$1Q$] is true and that $x \succsim y$. This means that $x \in \{z \in X : z \succsim y\}$ and hence the set is closed

[**2** $\iff$ **3**]

- You just need to use one of the useful results: The complement of an open set is closed (and vice versa)
- In set notation: $\{x \in X : x \succsim y\}^c = \{x \in X : x \not\succsim y\} = \{x \in X : y \succ x\}$

[**3** $\implies$ **4**]

- Try to find a z such that $x \succ z \succ y$. There are two cases: (1) there exists a z, or (2) no such z exists
- Case 1: ($\exists z$)
  - Find a ε_x and ε_y such that $B(x, \varepsilon_x) \subset \{\alpha \in X : \alpha \succ z\}$ and $B(y, \varepsilon_y) \subset \{\alpha \in X : z \succ \alpha\}$ (we can find these ε-balls because the sets are open)
  - Take $\varepsilon = \min\{\varepsilon_x, \varepsilon_y\}$, and then select any $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$. By transitivity, we have: $x' \succ z \succ y'$
- Case 2: ($\nexists z$)
  - Find a ε_x and ε_y such that $B(x, \varepsilon_x) \subset \{\alpha \in X : \alpha \succ y\}$ and $B(y, \varepsilon_y) \subset \{\alpha \in X : x \succ \alpha\}$ (we can find these ε-balls because the sets are open)
  - $\forall x' \in B(x, \varepsilon_x)$, since $x' \succ y$ and $\nexists z$ such that $x \succ z \succ y$, we cannot have $x \succ x' \implies x' \succsim x$
  - $\forall y' \in B(y, \varepsilon_y)$, since $x \succ y'$ and $\nexists z$ such that $x \succ z \succ y$, we cannot have $y' \succ y \implies y \succsim y'$
  - Take $\varepsilon = \min\{\varepsilon_x, \varepsilon_y\}$, and then select any $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$. By transitivity, we have: $x' \succsim x \succ y \succsim y' \implies x' \succ y'$

[**4** $\implies$ **1**]

- We want to show [4] $\implies$ [1]. As before, let's break this down:
  - [$4P$]: $x \succ y$
  - [$4Q$]: $\exists \varepsilon > 0$ such that $x' \succ y', \forall x' \in B(x, \varepsilon)$ and $\forall y' \in B(y, \varepsilon)$
  - [$1P$]: $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n, \forall n$
  - [$1Q$]: $x \succsim y$
- Let's use proof by contradiction and show that $P \cap Q^c \subset Q$. This means that [4] and NOT [1] $\implies$ [1]. More specifically, we will want to say: ([$4P$] $\implies$ [$4Q$]) and ([$1P$] and NOT [$1Q$]) $\implies$ NOT [$1P$]. We will want to conclude that [$1P$] is not true because NOT [$1Q$] $\implies$ NOT [$1P$] is the contrapositive of [$1P$] $\implies$ [$1Q$] (which is our Q). So we start by assuming that [$1P$] and NOT [$1Q$] are true. We want to then create the conditions that satisfy [$4P$]. Then we will be able to conclude that [$4Q$] is true. With all this, we should be able to conclude that [$1P$] is *not* true, which will be the contradiction.

- The actual proof:
  - Suppose we have sequences $x_n \to x$, $y_n \to y$, and $x_n \succsim y_n \forall n$. This arbitrary setup is exactly $[1P]$.
  - Now also suppose that $y \succ x$. This is NOT $[1Q]$.
  - Since $y \succ x$, we have satisfied the conditions for $[4P]$. We can then conclude that $[4Q]$ is true. This means that we can find a $\varepsilon > 0$ such that $y' \succ x', \forall y' \in B(y, \varepsilon)$ and $\forall x' \in B(x, \varepsilon)$
  - Since the sequences are convergent, for that same ε, $\exists N$ such that $y_n \in B(y, \varepsilon)$ and $x_n \in B(x, \varepsilon), \forall n > N$ (let $N = \max\{N_x, N_y\}$). But since they are both in their respective open balls, then based on $[4Q]$, it must be the case that $y_n \succ x_n$. This contradicts $[1P]$.