Intermediate Micro: Final Exam Review

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1 Key Concepts

Basics

- Firm has a production function $q = f(x_1, x_2)$

- Notation: output good q has price p, input goods x_1 and x_2 have prices p_1 and p_2

- The production function will have some returns-to-scale

  - Compare $f(\lambda x_1, \lambda x_2)$ to $\lambda f(x_1, x_2)$. DRTS is $<$, CRTS is $=$, IRTS is $>$

- The firm's goal is to maximize profits: $\pi(q) = $ Revenue - Costs $= TR(q) - C(q) = pq - C(q)$

- For this, we need the (total) cost function $C(q)$. To derive the cost function, we need to do the cost minimization problem

Cost Minimization Problem

- **Problem:** For a given q, what is my choice of inputs x_1 and x_2 in order to produce it as cheaply as possible

  - The objective function is cost: $p_1 x_1 + p_2 x_2$. The constraint is production: $f(x_1, x_2) = q$

  - Choice variables: x_1 and x_2. Parameters: p_1, p_2, and q (p, the price of q, is irrelevant).

- **Long Run:** (all inputs are flexible)

  - Similar to consumer's utility maximization problem

  - Optimality condition: $TRS = \frac{p_1}{p_2}$, where $TRS = \frac{MP_1}{MP_2} = \frac{\partial f(x_1, x_2)/\partial x_1}{\partial f(x_1, x_2)/\partial x_2}$

  - From the optimality condition, solve for x_2 as a function of x_1. Plug into the production constraint $f(x_1, x_2) = q$

  - Solve for $x_1(q)$ and then $x_2(q)$. These are the conditional input demands (i.e. how much input I demand, given that I have to produce q units of output)

    – Cost function: $C(q) = p_1 x_1(q) + p_2 x_2(q)$

- **Short Run:** (some inputs are fixed)

    – Suppose x_2 is fixed at $\bar{x}_2$. The production function becomes only a function of x_1, i.e. $f(x_1, \bar{x}_2) = f(x_1)$

    – No FOCs needed. Must produce q units so just take the inverse of the production function, i.e. $f(x_1) = q \implies x_1(q) = f^{-1}(q)$

    – Cost function: $C(q) = p_1 x_1(q) + p_2 \bar{x}_2$

- Once you have derived the cost function, you can move onto the profit maximization problem and work entirely in terms of q

- Note: cost function $C(q)$ is effectively determined by the production function

    – IRTS $\implies C(q)$ is concave (decreasing MC). CRTS $\implies C(q)$ is linear (constant MC). DRTS $\implies C(q)$ is convex (increasing MC)

Profit Maximization Problem

Perfect Competition

- **Problem:** For a given market price p, what is the q that will give the highest profit?

    – The objective function is profit: $pq - C(q)$. There is no constraint (the technology constraint is captured by the cost function)

    – Choice variables: q. Parameters: p.

    – Optimality Condition: $p = MC$ (technically $MR = MC$, but for perfect competition, we have $MR = p$)

- **Short Run:**

    – Shut-down rule: shut-down if $p < AVC$ (assuming all fixed costs are sunk)

    – Supply curve: MC curve above AVC

- **Long Run:**

    – Zero-profit condition: $p = AC$ (since $p - AC$ is the profit per unit of output)

    – Profit maximizing condition ($p = MC$) + Zero-profit condition ($p = AC$) means that $MC = AC$. This occurs at the minimum of the AC curve. Therefore, we have that $p = \min AC$ in the long-run

    – Shut-down rule: shut-down if $p < AC$ (assuming no fixed costs, so $AVC = AC$)

    – Supply curve: MC curve above AC

- **Market:**

    – Market supply: $Q_s(p) = \sum q_s(p)$, where $q_s(p)$ is the individual firm's supply function, i.e. solve for q in the inverse supply function $p = MC(q)$

    – If there are N homogenous firms, we have $Q_s(p) = N q_s(p)$. Similarly, if we know the market supply and how much each of the firm makes, we have that the number of firms is $N = Q_s(p)/q_s(p)$

Monopolist

- **Problem:** For a given market demand function $Q_D(p)$, choose price p and quantity q to maximize profits (note that $Q_S = q$ since there is only one firm)

- Optimality condition: $MR(q) = MC(q)$, where $MR(q) = \frac{dTR(q)}{dq}$ and $TR(q) = p(q) \cdot q$ and $p(q)$ is the inverse demand function

- From optimality condition, get optimal quantity q^*. Plug this into the inverse demand function to get the optimal price $p^* = p(q^*)$

- Marginal revenue has two parts: $MR(q) = \frac{dp(q)}{dq} \cdot q + p(q)$ (using chain rule). First part is the price effect (holding quantity supplied fixed, increase in quantity demanded lowers price in the market and so every unit of output gives you lower revenue). The second is the quantity effect (holding price fixed, an increase in quantity gives you an extra \$$p$ of revenue).

2 Exercises

Q1: Worker Skills

A company uses three types of inputs in its production of widgets. It uses capital (K), and two types of workers: low-skill (L_0) and high-skill (L_1) workers. Its production function is:

$$f(K, L_0, L_1) = K^\alpha L_0^\beta L_1^{1-\beta}$$

Where $0 < \alpha < 1$ and $0 < \beta < 1$. The costs for each input are $r = 5$, $w_0 = 2$, and $w_1 = 5$ for K, L_0, and L_1, respectively. The company can sell its output for \$4 on the market.

1. For what values of β does the production function exhibit increasing/decreasing/constant returns to scale?

2. For the rest of the question, set $\alpha = \frac{1}{2}$, and $\beta = \frac{1}{2}$. Capital is fixed at 10 units in the short run. Derive the firm's cost function in terms of units of output q.

3. There is a shortage of high-skill workers in the market.

   (a) Suppose the company can only hire a maximum of 4 high-skill workers. Argue that it will hire up to the maximum (e.g. it will not hire 2 or 3 high-skill workers)

   (b) True or False: Suppose that has currently hired 10 low-skill and 4 high-skill workers. At those input levels, it would need to hire 2.5 low-skill workers to replace one high-skill worker and maintain output at the same level.

   (c) What is its cost function?

4. Given the cost function you derived in (3), what is the firm's profit-maximizing output? How much profit does it earn?

5. The company could hire an instructor to give 10 low-skill workers more training so that they become high-skill workers (and therefore pay them the higher wage w_1). This instructor charges a flat fee, regardless of how many workers are in the training course. What is the maximum the company is willing to pay for the instructor?

Q2: Lemonade Stand

There is a market for lemonade by a beach. There are initially 100 lemonade stands. With so many lemonade stands, this functions as a perfectly competitive market. For any one firm, the cost of production q units of lemonade is $C(q) = 3q + q^2 + 25$. The demand for lemonade is given by $Q_D(p) = 810 - 10P$.

1. Draw a firm's marginal cost, average cost, and average variable cost curves all in one diagram

2. Calculate the market supply of lemonade. What is the equilibrium price and quantity? How much profit do firms make?

3. Describe the long run equilibrium. In particular, what happens to the price, quantity, number and profit of firms? (as compared to the question above)

4. Suppose a large company purchases all the lemonade stands and becomes a monopolist in the market. It re-organizes the factors of production such that its cost function becomes $C(q) = q^2 + \frac{3}{5}q + 50$

   (a) At what quantities does the monopolist have lower costs than an individual lemonade stand?

   (b) Where does the monopolist set price and quantity?

5. True or False: a monopolist who faces a linear inverse demand function of $P = a - bQ$ will have a marginal revenue curve of $MR(Q) = a - 2bQ$

6. True or False: in the long-run, firms will necessarily make zero profits regardless of the market structure (perfect competition or monopoly)

7. How much consumer surplus is lost when we move from the (short-run) perfectly competitive lemonade market to the monopolist?

Q3: Market for Quilts

In the market for quilts (q), there are 10 firms, which all have production technology $q = K\sqrt{L}$. The price of capital (K) is 3 and the price of labor (L) is 2. There are 50 identical consumers with utility functions $u(q, c) = q^{0.4}c^{0.6}$. The price of c (all other consumption) is $p_c = 1$, and consumers have an income of $M = 10$.

1. Does the firm's production have diminishing returns to capital?

2. In the short run, firms have fixed levels of capital at $\bar{K}$ and its cost is sunk. Derive their cost functions as a function of q and $\bar{K}$

3. Conditional on $\bar{K}$, how many quilts does each firm produce at a market price of p?

4. Suppose half the firms have $\bar{K} = 2$ and half have $\bar{K} = 4$. What is the market supply for quilts?

5. What is the market demand for quilts?

6. Find the equilibrium price and quantity for quilts.

7. How much profit does each firm make? Which firms have more workers per capital?

8. At what price level would each firms make zero profits? If the price was below this level, would you recommend to these firms that they shut down?

9. In the long run, capital can be adjusted. What is the shape of the a firm's cost function in the long run? (convex, concave or linear)

Q4:   Subsidy

Demand for a good is $Q_D(p) = 1000 - 20p$. Firms are able to produce this good with a cost of $C(q) = 50 + 20q + \frac{1}{2}q^2$ (this is both their short and long-run cost function).

1. The market is initially in a long-run equilibrium. Find the equilibrium price, quantity, and number of firms in the market.

2. Suppose the government introduces a subsidy of \$20 to producers.

   (a) Find the new short-run equilibrium price and quantity

   (b) How much profit is each firm making?

3. Show in a diagram and calculate the following: change in consumer surplus, change in producer surplus, deadweight loss

4. Find the new long run equilibrium price, quantity and number of firms.

5. Does the subsidy cost the government more in the short-run equilibrium (Q2a) or in the new long-run equilibrium (Q4)?

Q5:   Cobb-Douglas Production

A firm has the following production function:

$$f(x_1, x_2) = \frac{1}{10} x_1^{1/3} x_2^{2/3}$$

1. True or False: this production function exhibits CRTS

2. What is the marginal product of good 2? How does it change as x_2 increases (and what is the economic interpretation of this)?

3. True or False: this is a homothetic production function

4. Suppose $p_1 = 27$ and $p_2 = 2$.

   (a) How much does the firm demand of each input if it wants to produce 18 units of output?

   (b) What does this input combination cost the firm?

5. Verify that this firm's cost function is $C(q) = 90q$. Explain why it is linear. Would you interpret this as a long-run or short-run cost function?

6. Suppose x_2 was fixed at 125 units. How does that change the cost-function?

Q6: Perfect [?] Production

In this question we will compare two firms. Firm 1 has production function:

$$f(x_1, x_2) = \min\{3x_1, 2x_2\}$$

Firm 2 has the following production function:

$$f(x_1, x_2) = 3x_1 + 2x_2$$

Both inputs have a price of 1.

1. True or False: both functions have the same returns-to-scales

2. True or False: these are both homothetic production functions

3. Suppose both firms had to produce $q = 6$ units of output. Would they demand the same bundle of inputs?

4. If you had to hire a firm to produce a given quantity of output q in the cheapest way possible, which firm would you hire? Does your answer depend on q?

5. Can you find an output quantity q and input prices p_1 and p_2 where the firms demand the same bundle of inputs?

6. Suppose that each firm has x_1 fixed at 4 in the short-run. Derive the firms' total cost, average cost, and marginal cost functions.

7. Now imagine you own both firms but x_1 is fixed for firm 1 at 4, while x_1 is fixed for firm 2 at 1. For a given level of q, how would you distribute production across the two firms?

Q7: Monopoly

Consider a market with one firm that has a cost function $C(Q) = 100 + Q^2$. Demand in this market is $Q_D(P) = 45 - \frac{1}{2}P$.

1. True or False: the firm's average revenue curve is equal to the market inverse demand curve.

2. Express the firm's marginal revenue curve (as a function of Q). Decompose this into the price and quantity effect.

3. Find the firm's choice of price and quantity.

4. The firm's mark-up is defined as the amount that price is above the firm's marginal cost. Recall that the price elasticity of demand is defined as $\varepsilon = \frac{dQ_D}{dP} \cdot \frac{P}{Q_D}$. Show that the firm's optimal choice is to set mark-up equal to $-\frac{P}{\varepsilon}$. Note that this is true for <u>any</u> monopolist, not just this specific example. Here are two hints to help you:

- Calculate $MR(Q)/P$. Can you make this into a function of ε?

- In optimality, what does the firm set MC equal to?

5. Using your answer from (3), calculate the price elasticity of demand at the equilibrium and show that the mark-up is indeed equal to $-\frac{P}{\varepsilon}$

6. Suppose that the firm could pay \$50 to purchase new technology that lowers their variable costs to $\frac{1}{4}Q^2$.

   (a) How would this affect the firm's choice of price and quantity?

   (b) What is the maximum the firm is willing to pay for this technology?

3 Solutions

Q1: Worker Skills

1. Scale up all inputs by λ:

$$f(\lambda K, \lambda L_0, \lambda L_1) = (\lambda K)^\alpha (\lambda L_0)^\beta (\lambda L_1)^{1-\beta}$$
$$= \lambda^{\alpha+\beta+1-\beta} K^\alpha L_0^\beta L_1^{1-\beta}$$
$$= \lambda^{\alpha+1} f(K, L_0, L_1)$$

Since $\alpha + 1 > 1$ for all values of α, we have IRTS, regardless of the value of α or β

2. The short-run production function is $f(10, L_0, L_1) = 10^\alpha L_0^\beta L_1^{1-\beta}$. The cost minimization problem is:

$$\min_{L_0, L_1} \quad r\bar{K} + w_0 L_0 + w_1 L_1 = 50 + 2L_0 + 5L_1$$
$$\text{s.t. } q = f(10, L_0, L_1) = 10^\alpha L_0^\beta L_1^{1-\beta}$$

The tangency condition is:

$$\frac{\beta L_1}{(1-\beta)L_0} = \frac{w_0}{w_1}$$
$$\therefore L_1 = \frac{(1-\beta)w_0}{\beta w_1} L_0$$

Plugging this back into the production function gives us:

$$q = 10^\alpha L_0^\beta \left(\frac{(1-\beta)w_0}{\beta w_1} L_0 \right)^{1-\beta}$$
$$= 10^\alpha \left(\frac{(1-\beta)w_0}{\beta w_1} \right)^{1-\beta} L_0$$
$$L_0 = \left(\frac{\beta w_1}{(1-\beta)w_0} \right)^{1-\beta} \frac{q}{10^\alpha}$$

Plugging in the parameter values:

$$L_0(q) = \left(\frac{\frac{1}{2}}{\frac{1}{2}} \cdot \frac{5}{2} \right)^{1/2} \frac{q}{\sqrt{10}}$$
$$= \sqrt{\frac{5}{2}} \cdot \frac{q}{\sqrt{10}}$$
$$= \frac{q}{\sqrt{4}}$$
$$= \frac{q}{2}$$

$$L_1(q) = \frac{\frac{1}{2} \times 2}{\frac{1}{2} \times 5} L_0$$
$$= \frac{2}{5} \cdot \frac{q}{2}$$
$$= \frac{q}{5}$$

This means the cost function is:

$$C(q) = 50 + 2\left(\frac{q}{2}\right) + 5\left(\frac{q}{5}\right)$$
$$= 50 + 2q$$

3.

(a) A firm would set $p = MC$, but here we have:

$$4 > 2$$
$$\implies p > MC(q)$$

Which means that the firm will always want to be producing more output. This happens because, now that capital is fixed, the production function exhibits CRTS (and so the cost curve is linear). Therefore, the firm will want to produce $q \to \infty$ to maximize its profits. Since L_1 demand increases proportionately with q, that means the firm would rather higher 4 units of L_1 over anything less than 4. So it will hire 4 L_1 workers and then since it can't hire any more of them, will try to substitute for more of them with low skill L_0 workers

(b) True. This is exactly asking for the technical rate of substitution. The TRS between L_1 and L_0 is:
$$TRS = \frac{MP_1}{MP_0} = \frac{(1-\beta)L_0}{\beta L_1} = \frac{L_0}{L_1}$$

Plugging in $(L_0, L_1) = (10, 4)$ into the TRS formula:

$$MRTS(10, 4) = \frac{10}{4} = 2.5$$

(c) The production function is then: $f(10, L_0, 4) = \sqrt{10}\sqrt{L_0}\sqrt{4} = \sqrt{40L_0}$. This means the input demand is simply:

$$q = \sqrt{40L_0}$$
$$q^2 = 40L_0$$
$$L_0(q) = \frac{q^2}{40}$$

And the cost function is:

$$C(q) = 5(10) + 2\left(\frac{q^2}{40}\right) + 5(4)$$
$$= 70 + \frac{q^2}{20}$$

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4. With $p = 4$, the company sets $p = MC$

$$p = MC(q)$$
$$4 = \frac{2q}{20}$$
$$q = \frac{80}{2} = 40$$

So the company produces 40. Its profit is:

$$\pi = 4(40) - \left(70 + \frac{(40)^2}{20}\right)$$
$$= 160 - 70 - (40 \times 2)$$
$$= 160 - 150$$
$$= 10$$

5. With 10 additional high-skill workers, the firm would have $L_1 = 4 + 10 = 14$. This makes their input demand for L_0 as:

$$q = \sqrt{140L_0}$$
$$q^2 = 140L_0$$
$$L_0(q) = \frac{q^2}{140}$$

The cost function is then:

$$C(q) = 5(10) + 2\left(\frac{q^2}{140}\right) + 5(14)$$
$$= 120 + \frac{q^2}{70}$$

The profit-maximizing output is:

$$p = MC(q)$$
$$4 = \frac{2q}{70}$$
$$q = \frac{280}{2} = 140$$

The profit would then be:

$$\pi = 4(140) - \left(120 + \frac{(140)^2}{70}\right)$$
$$= 160$$

This gives them $150 more profit than before, which represents how much the firm is willing to pay for the instructor.
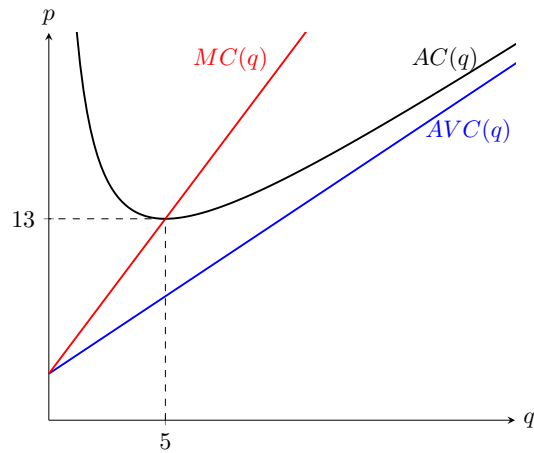
Q2: Lemonade Stand

1. For a cost function of $C(q) = 3q + q^2 + 25$, this gives the following curves:

$$AC(q) = 3 + q + \frac{25}{q}$$

$$AVC(q) = 3 + q$$

$$MC(q) = 3 + 2q$$

2. Each firm's supply is their marginal cost curve, i.e. $p = 3 + 2q \implies q_s = \frac{1}{2}(p - 3)$. With 100 firms, the market supply is:

$$Q_S(p) = 100 \cdot q_s(p) = 50(p - 3) = 50p - 150$$

This means that the equilibrium price is:

$$Q_D(p) = Q_S(p)$$
$$810 - 10p = 50p - 150$$
$$810 + 150 = (50 + 10)p$$
$$p^* = \frac{960}{60} = 16$$

And the equilibrium quantity is:

$$Q^* = Q_S(p^*) = 50 \cdot 16 - 150 = 800 - 150 = 650$$

Since there are 100 firms, each one is producing 6.5 units. Firms' profits are therefore:

$$\pi = pq - C(q)$$
$$= 16(6.5) - 3(6.5) - (6.5)^2 - 25$$
$$= 104 - 19.5 - 42.25 - 25$$
$$= 17.25$$

3. In the long run, we should be at the minimum of the long run AC curve. This occurs where $MC = AC$:

$$MC(q) = AC(q)$$
$$3 + 2q = 3 + q + \frac{25}{q}$$
$$q = \frac{25}{q}$$
$$q^2 = 25$$
$$\therefore q = 5$$

Firms must also be setting $p = MC$ (profit maximizing condition), this means that the price in the market must be:

$$p^* = MC(q^*) = 3 + 10 = 13$$

With a price of 13, the quantity demanded is $Q_D(13) = 810 - 130 = 680$. This must also be equal to the market quantity supplied Q_S. Therefore the number of firms in the market (N) must be:

$$N = \frac{Q_S}{q} = \frac{680}{5} = 136$$

As expected, since firms were making positive profits, more firms have now entered. We could check it, but we should expect that firms are now making zero profits (because $p = AC$).

4.

(a) We need to see where $C_M(q) = q^2 + \frac{3}{5}q + 50$ is less than $C_{PC}(q) = 3q + q^2 + 25$:

$$q^2 + \frac{3}{5}q + 50 < 3q + q^2 + 25$$
$$25 < \left(3 - \frac{3}{5}\right)q$$
$$\frac{12}{5}q > 25$$
$$q > \frac{125}{12} \approx 10.4$$

(b) A monopolist tries to profit maximize. Therefore they set where $MR = MC$. To get the marginal revenue, we first need to get the revenue $TR(Q) = P(Q) \cdot Q$. For this, we will need the inverse

demand function:

$$Q = 810 - 10P$$
$$10P = 810 - Q$$
$$\therefore P(Q) = 81 - \frac{1}{10}Q$$

This means that the marginal revenue is:

$$MR(Q) = \frac{dTR(Q)}{dQ}$$
$$= \frac{d}{dQ}\left(81Q - \frac{1}{10}Q^2\right)$$
$$= 81 - \frac{1}{5}Q$$

The marginal cost function is:

$$MC(Q) = \frac{dC_M(Q)}{dQ}$$
$$= 2Q + \frac{3}{5}$$

Therefore, the monopolist sets quantity at:

$$81 - \frac{1}{5}Q = 2Q + \frac{3}{5}$$
$$81 - \frac{3}{5} = \left(2 + \frac{1}{5}\right)Q$$
$$\frac{11}{5}Q = \frac{402}{5}$$
$$Q = \frac{402}{11}$$
$$\approx 36.55$$

The price they set is then: (using the inverse demand curve)

$$P(36.55) \approx 81 - \frac{1}{10}(36.55)$$
$$\approx 81 - 3.655$$
$$\approx 77.35$$

5. True. Total revenue with this inverse demand curve is:

$$TR(Q) = P(Q) \cdot Q$$
$$= (a - bQ)Q$$
$$= aQ - bQ^2$$

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Therefore, marginal revenue is:

$$MR(Q) = \frac{dTR(Q)}{dQ}$$

$$= a - 2bQ$$

So for a linear (inverse) demand curve, the marginal revenue curve always has the same y-intercept and twice the slope (i.e. half the x-intercept). You can verify this with the demand curve in this example.

6. False. In perfect competition, this is true. Because there are no barriers to entry, if firms are making positive profits, then more firms will enter until we are at an equilibrium of zero profits. Similarly, if firms are making negative profits, then firms will exit until they are making zero profits. Zero profits means that firms are indifferent between entering and exiting the market, which is the long run equilibrium. However, this is not necessarily true for a monopolist. Because there are significant barriers to entry, this means that the monopolist can be making positive profits in the long run as other firms will not be able to enter and take some of the market share.

7. Consumer surplus under perfect competition was:

$$CS_{PC} = \frac{1}{2}(650)(81 - 16) = 21,125$$

However, under the monopolist, price increases and quantity decreases, so the consumer surplus is:

$$CS_M \approx \frac{1}{2}(36.55)(81 - 77.35) \approx 66.70$$

So consumer surplus has fallen by approximately $21,058.30.

Q3: Market for Quilts

1. Marginal product of capital is $MPK = \sqrt{L}$. So the returns to capital depend on L, but this is always a concave function. This means there are diminishing returns to capital.

2. Given $\bar{K}$, the choice of labor is:

$$q = \bar{K}\sqrt{L}$$

$$q^2 = \bar{K}^2 L$$

$$\therefore L(q) = \frac{q^2}{\bar{K}^2}$$

This means that the cost function is:

$$C(q) = wL + rK$$

$$= \frac{wq^2}{\bar{K}^2} + r\bar{K}$$

$$= \frac{2q^2}{\bar{K}^2} + 3\bar{K}$$

3. This is just asking for the supply function. The profit maximizing choice is to have $p = MC$:

$$p = MC(q)$$
$$p = \frac{2wq}{\bar{K}^2} = \frac{4q}{\bar{K}^2}$$
$$q = \frac{\bar{K}^2 p}{2w} = \frac{\bar{K}^2 p}{4}$$

Since fixed costs are sunk, we should only produce where $p > AVC$. Since $p = MC$, we need to check for where $MC > AVC$:

$$MC(q) > AVC(q)$$
$$\frac{4q}{\bar{K}^2} > \frac{2q}{\bar{K}^2}$$
$$2 > 0$$

Which is always true, so the firm will not shut down. Therefore, the supply function is just the MC curve:

$$q = \frac{\bar{K}^2 p}{4}$$

4. The market supply is:

$$Q_S = 5\left(\frac{(2)^2 p}{4}\right) + 5\left(\frac{(4)^2 p}{4}\right)$$
$$= 5(p) + 5(4p)$$
$$= 5p + 20p$$
$$= 25p$$

5. An individual consumer's demand for q is: (using the formula for Cobb-Douglas demand)

$$q = \frac{0.4}{0.4 + 0.6} \cdot \frac{M}{p} = \frac{0.4(10)}{p} = \frac{4}{p}$$

There are 50 consumers, so the market demand is:

$$Q_D = 50\left(\frac{4}{p}\right) = \frac{200}{p}$$

6. The equilibrium occurs at the intersection:

$$Q_D = Q_S$$
$$\frac{200}{p} = 25p$$
$$p^2 = \frac{200}{25} = 8$$
$$\therefore p = \sqrt{8}$$
$$\therefore Q = 25\sqrt{8}$$

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7. For firms with $\bar{K} = 2$, they produce $q = p$, so their profit is:

$$\pi = pq - \frac{2q^2}{\bar{K}^2} - 3\bar{K}$$
$$= \sqrt{8}\sqrt{8} - \frac{(\sqrt{8})^2}{2} - 6$$
$$= 8 - 4 - 6$$
$$= -2$$

They hire $L = \frac{(\sqrt{8})^2}{2^2} = \frac{8}{4} = 2$. So their worker/capital ratio is $2/2 = 1$.

For firms with $\bar{K} = 4$, they produce $q = 4p$, so their profit is:

$$\pi = pq - \frac{2q^2}{\bar{K}^2} - 3\bar{K}$$
$$= \sqrt{8}\left(4\sqrt{8}\right) - \frac{2(4\sqrt{8})^2}{4^2} - 12$$
$$= 32 - 16 - 12$$
$$= 4$$

They hire $L = \frac{(4\sqrt{8})^2}{4^2} = 8$. So their worker/capital ratio is $8/4 = 2$. So these firms have a higher ratio of workers per units of capital.

8. The zero-profit occurs where $MC = AC$:

$$MC(q) = AVC(q)$$
$$\frac{4q}{\bar{K}^2} = \frac{2q}{\bar{K}^2} + \frac{3\bar{K}}{q}$$
$$4q^2 = 2q^2 + 3\bar{K}^3$$
$$q^2 = \frac{3}{2}\bar{K}^3$$
$$\therefore q = \sqrt{\frac{3}{2}\bar{K}^3}$$
$$\implies p = \frac{4}{\bar{K}^2}\sqrt{\frac{3}{2}\bar{K}^3}$$

So for firms with $\bar{K} = 2$, the zero-profit price is:

$$p = \frac{4}{4}\sqrt{\frac{3}{2} \cdot 8} = \sqrt{12}$$

And for firm with $\bar{K} = 4$:

$$p = \frac{4}{16}\sqrt{\frac{3}{2} \cdot 4 \times 16} = \frac{1}{4}\sqrt{6 \times 16} = \sqrt{6}$$

They should not shutdown below this price because even though they would make negative profits, their fixed costs of capital are sunk. So producing below these prices would minimize losses.

9. Since the production function has IRTS (the sum of the exponents is 1.5), the long-run cost curve is

concave

Q4:   Subsidy

1. In the long run, firms produce at the minimum of the average cost curve.

$$\min_{q} \ AC(q) = \frac{50}{q} + 30 + \frac{1}{2}q$$

$$\implies 0 = -\frac{50}{q^2} + \frac{1}{2}$$

$$\frac{50}{q^2} = \frac{1}{2}$$

$$q^2 = 100$$

$$\therefore q = 10$$

Equivalently, this occurs where $MC = AC$:

$$30 + q = \frac{50}{q} + 30 + \frac{1}{2}q$$

$$\frac{1}{2}q^2 = 50$$

$$q = \sqrt{100}$$

$$= 10$$

The long run equilibrium price is then:

$$p = MC(10)$$

$$= 30 + 10$$

$$= 40$$

At a price of 40, the demand is:

$$Q_D(40) = 1000 - 20(40)$$

$$= 1000 - 800$$

$$= 200$$

In equilibrium, we must have $Q_D = Q_S$. So the total market supply is 200, and since each firm produces 10 units, then there must be 20 firms in the long run equilibrium.

2.

(a) Firms produce where $p = MC$:

$$p = 30 + q$$

$$\therefore q = p - 30$$

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This is the individual supply curve. The market supply curve is: (since this is the short run, we still have 20 firms as firms can't enter/exit)

$$Q_S = 20(p - 30)$$
$$= 20p - 600$$

With a subsidy, the supply curve becomes:

$$Q_S = 20(p + 20) - 600$$
$$= 20p - 200$$
$$\implies P_S = \frac{Q}{20} + 10$$

Another way to see this is to first invert the original supply function:

$$P_S = \frac{Q}{20} + 30$$

The subsidy shifts the supply curve *down* at each price level by 20:

$$P_S = \frac{Q}{20} + 30 - 20$$
$$= \frac{Q}{20} + 10$$

This means new equilibrium is where:

$$Q_D = Q_S$$
$$1000 - 20p = 20p - 200$$
$$1200p = 40p$$
$$\therefore p^* = 30$$
$$\implies Q^* = 1000 - 20\,(30)$$
$$= 400$$

(b) Here, consumer pay a price of $p_c = 30$, but suppliers actually receive a price of $p_s = 30 + 20 = 50$. There are 20 firms, so each firm produces $q = \frac{400}{20} = 20$. This means each firm's profit is:

$$\pi = 50 \times 20 - \left(50 + 20(20) + \frac{1}{2}(20)^2\right)$$
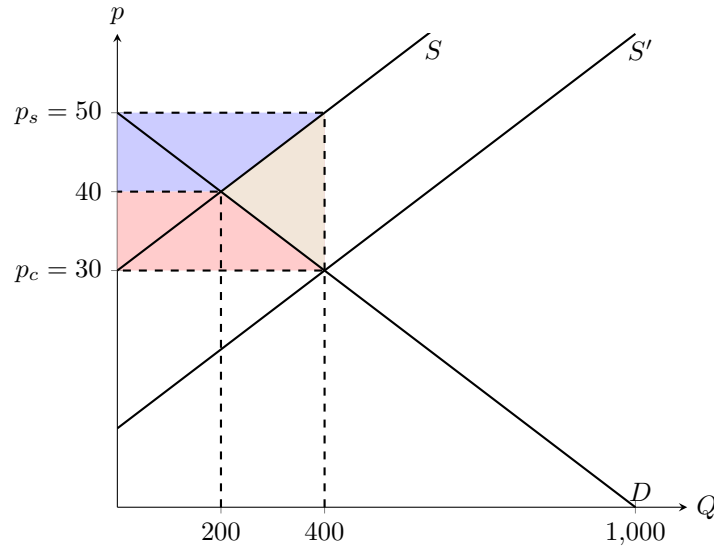$$= 1000 - 50 - 400 - 200$$
$$= 350$$

3. Remember to plot the inverse functions!

$$\text{Demand: } P_D = 50 - \frac{Q}{20}$$

$$\text{Supply: } P_S = \frac{Q}{20} + 30$$

$$\text{Supply w/ Subsidy: } P_S' = \frac{Q}{20} + 10$$

The market diagram is as follows:

The areas shaded are as follows:

- Change in Producer Surplus (blue):

$$\Delta PS = \frac{400 + 200}{2} \times (50 - 40) = 300 \times 10 = 3000$$

- Change in Consumer Surplus (red):

$$\Delta CS = \frac{200 + 400}{2} \times (40 - 30) = 300 \times 10 = 3000$$

- Deadweight Loss (brown):

$$DWL = \frac{1}{2} \times (400 - 200) \times (50 - 30) = \frac{1}{2} \times 200 \times 20 = 2000$$

4. In the long run, we need firms to be back at zero profits. This still occurs at the minimum of the AC (which is unchanged). Which means in the long-run, we still need the price that producers receive to be $p_s = 40$ and each firm producing $q = 10$. However, the price that consumers will actually get is $p_c = p_s - 20 = 20$. This means that demand would be:

$$Q_D(20) = 1000 - 20(20) = 600$$

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So in the long run, the equilibrium quantity is 600 and price is $20 (but producers receive $40 with the $20 subsidy). Since the quantity produced is 600 and each firm's output is 10, that means there are 60 firms. Therefore, the positive profits has incentivized more to firms the market.

5. In the short-run, the there are 400 units sold, each with a $20 subsidy. So the total subsidy cost is $8000. In the long-run, there are 600 units sold, each with a $20 subsidy, so the total cost is $12000. In the long-run, the subsidy will become more expensive for the government.

Q5:   Cobb-Douglas Production

1. True. This is Cobb-Douglas, so just add the exponents: $\frac{1}{3} + \frac{2}{3} = 1$. Since they add to 1, it is CRTS. The coefficient $\frac{1}{10}$ doesn't affect the returns to scale.

2. Calculate the marginal product:

$$
\begin{aligned}
MP_2 &= \frac{\partial f(x_1, x_2)}{\partial x_2} \\
&= \frac{1}{10} x_1^{1/3} \frac{2}{3} x_2^{-1/3} \\
&= \frac{1}{15} \left(\frac{x_1}{x_2} \right)^{1/3}
\end{aligned}
$$

Since x_2 is in the denominator, we see that MP_2 is decreasing in x_2. This captures marginal diminishing returns - for every extra unit of x_2 (holding fixed the amount of x_1), we get less extra unit of output (but we will still get more output).

3. True. All Cobb-Douglas functions are homothetic. Alternatively, look at the TRS:

$$
TRS = \frac{\frac{1}{3} x_2}{\frac{2}{3} x_1} = \frac{x_2}{2x_1}
$$

Scaling up the inputs results in exactly the same TRS as before, i.e. $TRS(\lambda x_1, \lambda x_2) = \frac{(\lambda x_2)}{2(\lambda x_1)} = \frac{x_2}{2x_1} = TRS(x_1, x_2)$. This is the definition of a homothetic function.

4. The firm's cost minimization problem is:

$$
\min_{x_1, x_2} \ 27x_1 + 2x_2
$$
$$
\text{s.t.} \ \frac{1}{10} x_1^{1/3} x_2^{2/3} = 18
$$

The tangency condition is:

$$
TRS = \frac{p_1}{p_2}
$$
$$
\frac{x_2}{2x_1} = \frac{27}{2}
$$
$$
\therefore x_2 = 27x_1
$$

Plug this into the feasibility condition (i.e. the production constraint):

$$\frac{1}{10}x_1^{1/3}(27x_1)^{2/3} = 18$$

$$3^2 x_1 = 180$$

$$\therefore x_1^* = \frac{180}{9} = 20$$

$$x_2^* = 27 \cdot 20 = 540$$

This costs the firm: $27(20) + 2(540) = 540 + 1080 = 1620$.

5. Using the same steps as above but replacing 18 with q, we get:

$$9x_1 = 10q$$

$$\therefore x_1^* = \frac{10q}{9}$$

$$x_2^* = 27 \cdot \frac{10q}{9} = 30q$$

This means that the cost function is:

$$C(q) = 27\left(\frac{10q}{9}\right) + 2(30q)$$

$$= 30q + 60q$$

$$= 90q$$

It is linear because the production function exhibits CRTS. Since under this setup all inputs are flexible, we should interpret this as the long run cost curve.

6. If $x_2 = 125$, then for a given q, the firm solves the following to get its demand of x_1:

$$\frac{1}{10}x_1^{1/3}(125)^{2/3} = q$$

$$25x_1^{1/3} = 10q$$

$$x_1 = \left(\frac{10}{25}q\right)^3 = \frac{8q^3}{125}$$

This makes the cost function:

$$C(q) = 27\left(\frac{8q^3}{125}\right) + 2(125)$$

$$= \frac{216q^3}{125} + 250$$

Now we have a convex (short-run) cost function because the short-run production function exhibits DRTS.

Q6: Perfect [?] Production

1. True. For firm 1: $f(\lambda x_1, \lambda x_2) = \min\{3\lambda x_1, 2\lambda x_2\} = \lambda \min\{3x_1, 2x_2\} = \lambda f(x_1, x_2)$. For firm 2: $f(\lambda x_1, \lambda x_2) = 3\lambda x_1 + 2\lambda x_2 = \lambda(3x_1 + 2x_2) = \lambda f(x_1, x_2)$. So they both have CRTS.

2. True. Perfect complements and perfect substitutes are homothetic. You can also check their TRS and see that they stay the same as you scale the inputs.

3. Firm 1 always wants to set $3x_1 = 2x_2 = q$, i.e.

$$x_1 = \frac{q}{3}$$
$$x_2 = \frac{q}{2}$$

This means that for firm 1, we have $x_1 = 2$ and $x_2 = 3$.
Firm 2 will compare the TRS to the inputs price ratio:

$$\frac{3}{2} > \frac{1}{1}$$

This means that they will only use x_1 and not use any x_2. Therefore, for firm 2, we have $x_1 = \frac{q}{3} = 2$ and $x_2 = 0$. Therefore, they will not demand the same bundle of goods.

4. At the current input prices, the cost functions are:

$$C_1(q) = \frac{q}{3} + \frac{q}{2} = \frac{5q}{6}$$
$$C_2(q) = \frac{q}{3}$$

So no matter the value of q, firm 2 will always be cheaper. This makes sense because firm 2 only has to buy x_1, but firm 2 has to buy the same amount of x_1 in addition to units of x_2 to produce the same level of output.

5. No. For any q, firm 1 will always demand $(\frac{q}{3}, \frac{q}{2})$ regardless of the input prices. If firm 2 demanded these quantities, then they would producing $3(\frac{q}{3}) + 2(\frac{q}{2}) = 2q$ units of output. They only need to be producing q units, so this is unnecessarily increasing their costs. Therefore, this can never happen.

6. With $x_1 = 4$, the firm 1 production function becomes:

$$f(x_1) = \min\{12, 2x_2\} = \begin{cases} 2x_2 & \text{if } x_2 < 6 \\ 12 & \text{if } x_2 \geq 6 \end{cases}$$

This means that *most* output firm 1 is able to make is $q = 12$. To produce an output $q \in [0, 12]$, the firm will demand $x_2 = \frac{q}{2}$. Therefore, its cost function is:

$$C(q) = p_1 x_1 + p_2 x_2 = (1)(4) + (1)\left(\frac{q}{2}\right) = 4 + \frac{q}{2}, \text{ for } q \leq 12$$

Alternatively, you can express it as follows:

$$C(q) = \begin{cases} 4 + \frac{q}{2} & \text{if } q \le 12 \\ \infty & \text{if } q > 12 \end{cases}$$

This captures the fact that it is impossible for the firm to produce $q > 12$ (i.e. it is infinitely costly). This means that we can write the average and marginal cost as:

$$AC(q) = \begin{cases} \frac{4}{q} + \frac{1}{2} & \text{if } q \le 12 \\ \infty & \text{if } q > 12 \end{cases} \qquad MC(q) = \begin{cases} \frac{1}{2} & \text{if } q \le 12 \\ \infty & \text{if } q > 12 \end{cases}$$

For firm 2, the production function becomes:

$$f(x_1) = 3(4) + 2x_2 = 12 + 2x_2$$

This means that the *least* output firm 1 is able to make is $q = 12$. So, to produce an output q, the firm will demand:

$$x_2 = \begin{cases} 0 & \text{if } q \le 12 \\ \frac{q-12}{2} & \text{if } q > 12 \end{cases}$$

Therefore, its cost function is:

$$C(q) = p_1 x_1 + p_2 x_2 = \begin{cases} (1)(4) + (1)(0) = 4 & \text{if } q \le 12 \\ (1)(4) + (1)\left(\frac{q-12}{2}\right) = \frac{q}{2} - 2 & \text{if } q > 12 \end{cases}$$

This means that we can write the average and marginal cost as:

$$AC(q) = \begin{cases} \frac{4}{q} & \text{if } q \le 12 \\ \frac{1}{2} - \frac{2}{q} & \text{if } q > 12 \end{cases} \qquad MC(q) = \begin{cases} 0 & \text{if } q \le 12 \\ \frac{1}{2} & \text{if } q > 12 \end{cases}$$

7. Obviously for firm 1 nothing as changed. But for firm 2, we have the production function as:

$$f(x_1) = 3(1) + 2x_2 = 3 + 2x_2$$

The input demand is then:

$$x_2 = \begin{cases} 0 & \text{if } q \le 3 \\ \frac{q-3}{2} & \text{if } q > 3 \end{cases}$$

Therefore, its cost function is:

$$C(q) = p_1 x_1 + p_2 x_2 = \begin{cases} (1)(1) + (1)(0) = 1 & \text{if } q \le 3 \\ (1)(1) + (1)\left(\frac{q-3}{2}\right) = \frac{q-1}{2} & \text{if } q > 3 \end{cases}$$

In terms of allocating quantity, for each additional unit of output, you should assign it to the firm with the lower marginal cost. Firm 1's marginal cost is:

$$MC(q) = \begin{cases} \frac{1}{2} & \text{if } q \leq 12 \\ \infty & \text{if } q > 12 \end{cases}$$

While for firm 2 is it:

$$MC_2(q) = \begin{cases} 0 & \text{if } q \leq 3 \\ \frac{1}{2} & \text{if } q > 3 \end{cases}$$

This means that for the first 3 units, we have $MC_1 < MC_2$, so everything should be assigned to firm 1. After that, we have $MC_1 = MC_2$, so allocating to either firm makes sense. However, once firm 2 reaches 12 units of output, then everything should go to firm 1. Let's call q_1 and q_2 the output of each firm, with $q = q_1 + q_2$. The allocation is as follows:

$$(q_1, q_2) = \begin{cases} (q, 0) & \text{if } q \leq 3 \\ (q - q_2, q_2) & \text{if } q > 3 \text{ where } q_2 \leq 12 \end{cases}$$

Q7: Monopoly

1. True. The firm's total revenue is $TR(Q) = P(Q) \cdot Q$. For this we need the inverse demand curve:

$$Q = 45 - \frac{1}{2}P$$
$$\frac{1}{2}P = 45 - Q$$
$$P = 90 - 2Q$$

So the total revenue function is:

$$TR(Q) = (90 - 2Q)\,Q$$
$$= 90Q - 2Q^2$$

The average revenue function is:

$$AR(Q) = \frac{TR(Q)}{Q} = 90 - 2Q$$

And this is exactly the inverse demand curve!

2. The marginal revenue is:

$$MR(Q) = \frac{dTR(Q)}{dQ} = 90 - 4Q$$

Note from Q2(2), we have a linear demand curve and so the marginal revenue curve should have the same intercept and double the slope (which is exactly what we see). To see the price and quantity

effects though, we should think about the marginal revenue function as:

$$MR(Q) = \underbrace{\frac{dP(Q)}{dQ} \cdot Q}_{\text{price effect}} + \underbrace{P(Q)}_{\text{quantity effect}}$$

$$= (-2)(Q) + (90 - 2Q)$$

So the price effect is $-2Q$ and the quantity effect is $90 - 2Q$.

3. The firm optimally sets $MR = MC$. We have the marginal revenue function, so we just need the marginal cost:

$$MC(Q) = \frac{dC(Q)}{dQ} = 2Q$$

This means the optimality condition is:

$$90 - 4Q = 2Q$$
$$90 = 6Q$$
$$\therefore Q^* = 15$$

We have the equilibrium quantity. To get the price, we plug the quantity into the inverse demand function:

$$P^* = P(15) = 90 - 2(15) = 60$$

4. As we know from above, we can write $MR(Q)$ as:

$$MR(Q) = \frac{dP}{dQ} \cdot Q + P$$

Dividing this by P gives us:

$$\frac{MR(Q)}{P} = \frac{dP}{dQ} \cdot \frac{Q}{P} + \frac{P}{P} = \frac{1}{\varepsilon} + 1$$

This makes it clear where ε comes in. Therefore, we can also express the marginal revenue as:

$$MR(Q) = \left(1 + \frac{1}{\varepsilon}\right) P$$

The mark-up (MU) is equal to $P - MC$. The firm optimally sets $MC = MR$. Therefore,

$$MU = P - MC$$
$$= P - MR$$
$$= P - \left(1 + \frac{1}{\varepsilon}\right) P$$
$$= -\frac{P}{\varepsilon}$$

5. In (3), we found that $Q^* = 15$ and $P^* = 60$. The price elasticity of demand is then:

$$\varepsilon = \frac{dQ}{dP} \cdot \frac{P}{Q}$$
$$= \left(-\frac{1}{2}\right) \cdot \frac{60}{15}$$
$$= -2$$

The firm's marginal cost is:

$$MC(15) = 2(15) = 30$$

And therefore its mark-up is:

$$MU = P - MC = 60 - 30 = 30$$

Now, let's verify the relationship we derived in (4):

$$-\frac{P}{\varepsilon} = -\frac{60}{(-2)} = 30$$

Which is exactly equal to the mark-up, so everything is correct.

6.

(a) Under this new purchase, the firm's new cost function is $C(Q) = 150 + \frac{1}{4}Q^2$. The marginal cost becomes $MC(Q) = \frac{1}{2}Q$. The marginal revenue has not changed, so the optimality condition gives us:

$$90 - 4Q = \frac{1}{2}Q$$
$$90 = \frac{9}{2}Q$$
$$\therefore Q^* = 20$$

We have the equilibrium quantity. To get the price, we plug the quantity into the inverse demand function:

$$P^* = P(20) = 90 - 2(20) = 50$$

Therefore, the firm has increased quantity and lowered price.

(b) Notice that in the above calculations, the $50 did not play any role in the choice of optimal quantity and price. However, it will affect the profit of the firm. With the old technology, the firm's profit is:

$$\pi_0 = (60 \times 15) - \left(100 + 15^2\right)$$
$$= 900 - 100 - 225$$
$$= 575$$

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With the new technology, the firm's profit is: (suppose they pay a fee f for this technology)

$$\pi_1 = (50 \times 20) - \left(100 + f + \frac{1}{4}20^2\right)$$
$$= 1000 - 100 - f - 100$$
$$= 800 - f$$

So the firm will only purchase the technology if:

$$\pi_1 > \pi_0$$
$$800 - f > 575$$
$$\therefore f < 800 - 575 = 225$$

This means that the maximum willingness to pay is \$225.