Definition: Confusability Graph

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Let $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ be a channel. The confusability graph G for the channel consists of the set of input symbols of the channel:

$$V(G) := \mathcal{X},$$

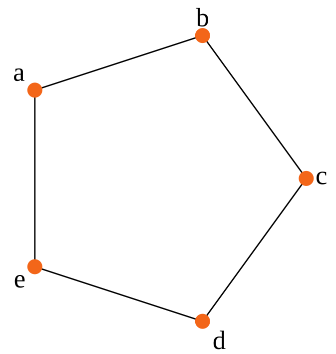
and

$$E(G) := \{\{x, x'\} \subset \mathcal{X} \mid x \neq x' \text{ and } \exists y \in \mathcal{Y} \text{ s.t. } P_{Y|X}(y|x) \cdot P_{Y|X}(y|x') > 0\}$$

is the set of input pairs that are confusable (because they reach a shared output symbol $y \in \mathcal{Y}$).

Example: Confusability graph of the noisy typewriter

Consider again the **noisy typewriter channel**. The confusability graph for that channel is:



This graph is also known as C_5, the circle of size 5. Its independence number is $\alpha(C_5) = 2$.

In the above example, the independence number of the confusability graph is exactly the number of messages that can be sent over the channel perfectly. This is no coincidence:

Proposition

Given a channel with confusability graph G, the maximal message set $[M]$ that can be communicated perfectly in a single channel use is of size $\alpha(G)$.

Proof

Let $(x, x') \in E(G)$, i.e., x and x' are confusable. They cannot both be used to send different messages, for suppose there are messages $m \neq m'$ such that $\mathtt{enc}(m) = x$ and $\mathtt{enc}(m') = x'$, then by definition of the confusability graph, there is a $y \in \mathcal{Y}$ such that x and x' are both mapped to y with nonzero probability. In order to correctly decode in all cases, it must therefore be that $\mathtt{dec}(y) = m$ and $\mathtt{dec}(y) = m'$, contradicting the assumption that $m \neq m'$. Therefore, the number of messages that can be sent over the channel cannot exceed the independence number $\alpha(G)$.

For the other direction, it is easy to find an encoding and decoding function for $\alpha(G)$ different messages. Let $\{x_1, x_2, \ldots, x_{\alpha(G)}\}$ be a largest independent set of G. Define $\mathtt{enc}(m_i) = x_i$ for all $i \in [\alpha(G)]$. Then for all $y \in \mathcal{Y}$, by definition of the confusability graph and the independent set, there is exactly one i such that $P_{Y|X}(y|x_i) > 0$. Define $\mathtt{dec}(y) = m_i$. This code can send $\alpha(G)$ different messages over the channel without error.