Stationary Process

Often, we will be interested in stochastic processes with specific properties, such as processes where all X_i are independent, or processes with a Markov-like property. We review several such properties.

Definition: Stationary process

A stochastic process is stationary if for all $n, k \in \mathbb{N}_+$,

$$P_{X_1 \cdots X_n} = P_{X_{1+k} \cdots X_{n+k}}.$$

Stationary processes are invariant under time shifts: when observing a subsequence of length n, it does not matter where in the process you look exactly.

Example: i.i.d. process

Let X be a random variable. Consider a stochastic process $\{X_i\}$ where $P_{X_i} = P_X$ for all i. That is, the random variables in the sequence are all independent and identically distributed. This process is stationary, since for any n, k, it holds that

$$P_{X_1 \cdots X_n} = \prod_{i=1}^{n} P_{X_i} = \prod_{i=1+k}^{n+k} P_{X_i} = P_{X_{1+k} \cdots X_{n+k}}.$$

Example: Ten fair coins

The following is another example of a stationary process. Throw a fair coin 10 times: this can be described by the finite sample space $\{\mathtt{H}, \mathtt{T}\}^{10}$. Define the stochastic process $\{X_i\}_{i \in \mathbb{N}_0}$ by setting

$$X_i = \begin{cases} 1 & \text{if the } [i \bmod 10] \text{th coin lands on heads} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $[k \bmod N]$ is defined to be an element of $\{0, 1, 2, \ldots, N-1\}$. If we want the first variable in the process to be X_1 instead of X_0, we can determine the value of X_i based on the $[((i-1) \bmod 10) + 1]$th coin.

As an exercise, show that this process is indeed stationary.

Hint

Observe that for all i, $X_i = X_{i+10} = X_{i+20} = \ldots$