Definition: Convexity and Concavity

In the following, let $\mathcal{D}$ be an interval in $\mathbb{R}$.

Definition: Convex and concave functions

The function $f : \mathcal{D} \to \mathbb{R}$ is convex if for all $x_1, x_2 \in \mathcal{D}$ and all $\lambda \in [0, 1] \subset \mathbb{R}$:

$$\lambda f(x_1) + (1 - \lambda) f(x_2) \geq f(\lambda x_1 + (1 - \lambda) x_2).$$

The function f is *strictly* convex if equality only holds when $\lambda \in \{0, 1\}$ or when $x_1 = x_2$. The function f is (strictly) concave if the function $-f$ is (strictly) convex.

Intuitively, a function is convex if any straight line drawn between two points $f(x_1)$ and $f(x_2)$ lies above the graph of f entirely. For a concave function, such a line must lie entirely beneath the graph. Play around with the interactive figure below to understand the formulas above! Note that you can move the red slider for λ, and you can also adjust x_0, x_1 directly in the graph.

Convex or concave? MacKay's Mnemonic

David MacKay has a great way of remembering "which way" convex and concave functions go, namely by noticing the following. When pronouncing the word "convex", one could continue to say "smile", while pronouncing the word "concave", it could be followed by the word "frown".

Video-2018-05-16-12-30-51_MacKays mnemonic.MP4

The following proposition establishes a formal method of proving the convexity of a function.

Proposition

Let $f : \mathcal{D} \to \mathbb{R}$. If $\mathcal{D}$ is open, and for all $x \in \mathcal{D}$, the second-order derivative $f''(x)$ exists and is non-negative (positive), then f is convex (strictly) convex.

Proof

We omit the proof, which can be found in, for example, [CF] (Lemma 1).