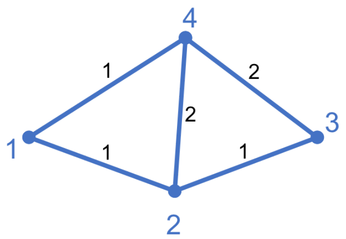
Random Walks on Graphs

An important and widely applicable example of a time-invariant Markov process is a random walk on a connected graph G with strictly positive symmetric edge weights $W_{ij} = W_{ji}$. The random walk is defined as follows: at node i, walk to node j with probability $\frac{W_{ij}}{W_i}$ where $W_i := \sum_j W_{ij}$ is the sum of the weights of all edges involving node i, and $W := \frac{1}{2}\sum_i W_i$ is the total of all edge weights.

Example

For the following graph  , we have that $W_1 = 2, W_2 = 4, W_3 = 3, W_4 = 5$ and $2 \cdot W = \sum_i W_i = 2 \cdot 7$.

The stationary distribution of this random walk is given by $\mu_i := \frac{W_i}{2W}$, because indeed, at every node i, we have that the sum of all incoming weight is

$$\sum_j \mu_j \frac{W_{ij}}{W_j} = \sum_j \frac{W_j}{2W}\frac{W_{ij}}{W_j} = \frac{W_i}{2W} = \mu_i \,.$$

We continue to compute the entropy rate of this random walk. Assuming we start in the stationary distribution, we can compute the entropy rate as follows.

$$\begin{aligned}
H(\{X_i\}) = \sum_i \mu_i H(\dots \frac{W_{ij}}{W_i} \dots) &= -\sum_i \mu_i \sum_j \frac{W_{ij}}{W_i} \log \frac{W_{ij}}{W_i} \\
&= -\sum_{i,j} \frac{W_i}{2W} \cdot \frac{W_{ij}}{W_i} \log\left(\frac{W_{ij}}{2W} \cdot \frac{2W}{W_i}\right) \\
&= -\sum_{i,j} \frac{W_{ij}}{2W} \log\left(\frac{W_{ij}}{2W}\right) + \sum_{i,j} \frac{W_{ij}}{2W} \log\left(\frac{W_i}{2W}\right) \\
&= -\sum_{i,j} \frac{W_{ij}}{2W} \log\left(\frac{W_{ij}}{2W}\right) + \sum_{i} \frac{W_i}{2W} \log\left(\frac{W_i}{2W}\right) \\
&= H(\dots \frac{W_{ij}}{2W} \dots) - H(\dots \frac{W_i}{2W} \dots)
\end{aligned}$$

which is the difference of the entropy of the edge distribution and the entropy of the stationary distribution.

Example, continued

In the example above, the edge distribution is $\frac{1}{14}(1,1,1,2,2,1,1,1,2,2)$ and the stationary distribution is $\frac{1}{14}(2,4,3,5)$, resulting in
$H(\{X_i\}) = H(\frac{1}{14}(1,1,1,2,2,1,1,1,2,2)) - H(\frac{1}{14}(2,4,3,5)) \approx 1.312$