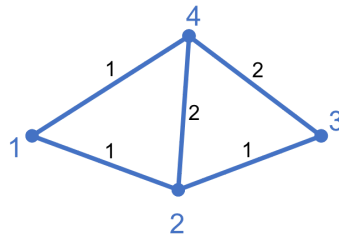


# Random Walks on Graphs

An important and widely applicable example of a time-invariant Markov process is a random walk on a connected graph  $G$  with strictly positive symmetric edge weights  $W_{ij} = W_{ji}$ . The random walk is defined as follows: at node  $i$ , walk to node  $j$  with probability  $\frac{W_{ij}}{W_i}$  where  $W_i := \sum_j W_{ij}$  is the sum of the weights of all edges involving node  $i$ , and  $W := \frac{1}{2} \sum_i W_i$  is the total of all edge weights.

## Example



For the following graph, we have that  $W_1 = 2, W_2 = 4, W_3 = 3, W_4 = 5$  and  $2 \cdot W = \sum_i W_i = 2 \cdot 7$ .

The stationary distribution of this random walk is given by  $\mu_i := \frac{W_i}{2W}$ , because indeed, at every node  $i$ , we have that the sum of all incoming weight is

$$\sum_j \mu_j \frac{W_{ij}}{W_j} = \sum_j \frac{W_j}{2W} \frac{W_{ij}}{W_j} = \frac{W_i}{2W} = \mu_i.$$

We continue to compute the entropy rate of this random walk. Assuming we start in the stationary distribution, we can compute the entropy rate as follows.

$$\begin{aligned} H(\{X_i\}) &= \sum_i \mu_i H(\dots \frac{W_{ij}}{W_i} \dots) = - \sum_i \mu_i \sum_j \frac{W_{ij}}{W_i} \log \frac{W_{ij}}{W_i} \\ &= - \sum_{i,j} \frac{W_i}{2W} \cdot \frac{W_{ij}}{W_i} \log \left( \frac{W_{ij}}{2W} \cdot \frac{2W}{W_i} \right) \\ &= - \sum_{i,j} \frac{W_{ij}}{2W} \log \left( \frac{W_{ij}}{2W} \right) + \sum_{i,j} \frac{W_{ij}}{2W} \log \left( \frac{W_i}{2W} \right) \\ &= - \sum_{i,j} \frac{W_{ij}}{2W} \log \left( \frac{W_{ij}}{2W} \right) + \sum_i \frac{W_i}{2W} \log \left( \frac{W_i}{2W} \right) \\ &= H(\dots \frac{W_{ij}}{2W} \dots) - H(\dots \frac{W_i}{2W} \dots) \end{aligned}$$

which is the difference of the entropy of the edge distribution and the entropy of the stationary distribution.

**Example, continued**

In the example above, the edge distribution is  $\frac{1}{14}(1, 1, 1, 2, 2, 1, 1, 1, 2, 2)$  and the stationary distribution is  $\frac{1}{14}(2, 4, 3, 5)$ , resulting in

$$H(\{X_i\}) = H\left(\frac{1}{14}(1, 1, 1, 2, 2, 1, 1, 1, 2, 2)\right) - H\left(\frac{1}{14}(2, 4, 3, 5)\right) \approx 1.312$$