Definition: (Minimal) Average Length

For efficiency reasons, we are often interested in the average (expected) length of a code C:

Definition: Average length

Let $\ell(s)$ denote the length of a string $s \in \{0,1\}^*$. The (average) length of a code C for a source P_X is defined as

$$\ell_C(P_X) := \mathbb{E}[\ell(C(X))] = \sum_{x \in \mathcal{X}} P_X(x)\ell(C(x))\,.$$

Example

The following are all codes for the source P_X, with $\mathcal{X} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$:

x	$P_X(x)$	$C_1(x)$	$C_2(x)$	$C_3(x)$	$C_4(x)$
$\mathtt{a}$	0.5	00	0	0	0
$\mathtt{b}$	0.25	01	10	010	01
$\mathtt{c}$	0.125	10	110	01	011
$\mathtt{d}$	0.125	11	111	10	111

For the codes above, we obtain the following average codeword lengths: $\ell_{C_1}(P_X) = 2, \ell_{C_2}(P_X) = \ell_{C_4}(P_X) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = \frac{7}{4} = 1.75$ and $\ell_{C_3}(P_X) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 3 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 2 = \frac{7}{4} = 1.75.$ We see that the codes C_2, C_3, C_4 have a smaller average codeword length, but C_2 and C_4 are preferred over C_3 because their unique decodability. Notice that the individual codeword lengths of codes C_2 and C_4 correspond exactly to the surprisal values of P_X in bits, e.g. $\ell(C_2(b)) = \ell(C_4(b)) = 2 = -\log P_X(b)$. Therefore, the computations of the entropy $H(X)$ and of the average code length $\ell_{C_2}(P_X)$ are exactly the same, and we have that $H(X) = \ell_{C_2}(P_X) = \ell_{C_4}(P_X)$. We will see later that this property characterizes optimal codes.

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Definition: Minimal code length

The minimal code length of a source P_X is defined as

$$\ell_{\min}(P_X) := \min_{C \in \mathfrak{C}} \ell_C(P_X)$$

where $\mathfrak{C}$ is some class of codes, for example the set of all prefix-free codes (resulting in $\ell_{\min}^{\text{p.f.}}$), or the set of all uniquely decodable codes (resulting in $\ell_{\min}^{\text{u.d.}}$).