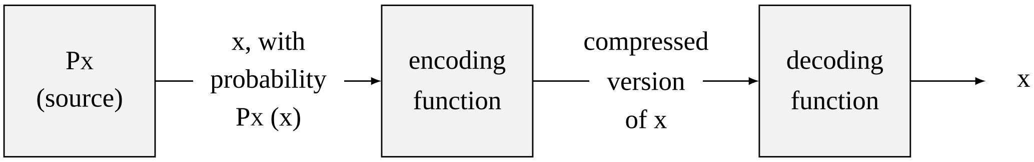
Introduction to Module 03

Suppose we sample x from a distribution P_X with image $\mathcal{X}$. In the context of data compression, P_X is typically called a **source** that emits value $x \in \mathcal{X}$ with probability $P_X(x)$. We want to compress (or encode) symbols x sampled from P_X in such a way that we can later decompress (or decode) it reliably, without losing any information about the value x.



A counting argument shows that it is possible to encode the elements of $\mathcal{X}$ by bit strings of length n, where $n = \lceil \log(|\mathcal{X}|) \rceil$: we simply list all elements of $\mathcal{X}$, and use the (binary) index of x in the list as its encoding. Thus, to store or to transmit an element $x \in \mathcal{X}$, n bits of information always suffice. However, if not all $x \in \mathcal{X}$ are equally likely according to P_X, one should be able to exploit this to achieve codes with shorter *average* length. The idea is to use encodings of varying lengths, assigning shorter codewords to the elements in $\mathcal{X}$ that have higher probabilities, and vice versa.

In the video, Alice and Bob communicate by encoding their messages (dice rolls) from $\mathcal{X} = \{2, 3, \ldots, 12\}$ into a unitary alphabet $\{1\}$, where each 1 stands for a pluck of the wire. For example, the roll 8 is encoded as 111, or three plucks.

Exercise

At the end of the video, Bob gets a better idea. He notices that they can pluck the wire in two different ways that are easy to distinguish: long or short. Can you design a code using this binary alphabet of plucks? How long are your codewords on average?

Show solution

The code on the right is an example of a code that Alice and Bob may use: 0 [illegible] pluck, 1 stands for a long one. Each die roll has a different

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codeword, and short codewords are assigned to the most likely outcomes. The expected codeword length is

$$\frac{1}{36}\cdot 3+\frac{2}{36}\cdot 3+\frac{3}{36}\cdot 2+\ldots+\frac{1}{36}\cdot 3=\frac{35}{18}\approx 1.944.$$

So on average, Alice and Bob expect to pluck the wire a little less than two times per die roll they want to communicate. However, if they want to communicate a list of die roll outcomes, they run into a problem: if Alice receives 011, how can she tell whether Bob sent the list [7,4], or [5,6], or even [2]?

Die roll	Codeword
2	011
3	001
4	11
5	01
6	1
7	0
8	00
9	10
10	000
11	010
12	100

This problem is resolved in the code on the right: confusions do not arise even when variable-length lists of messages are sent. The average codeword length is longer, however: roughly 3.306 plucks on average. In this module, you will encounter several algorithms for constructing such codes yourself for any given probability distribution.

Die roll	Codeword
2	11110
3	0010
4	0011
5	100
6	000

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Die roll	Codeword
7	010
8	011
9	101
10	110
11	1110
12	11111

The question we will answer in this module is: how short can codes be in general (on average over repeated samples x from P_X)? We explore both **lossless** codes (where we want to recover the original data with certainty) and **lossy** codes (where with small probability, the data is lost). For now, we will assume that our communication channels are perfect; later in the course, we will introduce channels with some noise.