Convergence of Random Variables

The following definition of converging random variables may remind you of a converging sequence of numbers. Recall that a sequence $x_1, x_2, x_3, \ldots$ of numbers converges to x if $\forall \epsilon > 0\ \exists n_0\ \forall (n \geq n_0) : |x_n - x| < \epsilon$. We denote this by writing $x_n \xrightarrow{n\to\infty} x$.

Definition: Converging random variables

A sequence $X_1, X_2, X_3, \ldots$ of real random variables converges to a random variable X, if it satisfies one of the following definitions:

in probability	(notation $X_n \xrightarrow{p} X$)	if $\forall \epsilon > 0$, $P[\lvert X_n - X\rvert > \epsilon] \xrightarrow{n\to\infty} 0$
		"As n increases, the distribution of X_n gets closer and closer to that of X."
in mean square	(notation $X_n \xrightarrow{m.s.} X$)	if $\mathbb{E}[(X_n - X)^2] \xrightarrow{n\to\infty} 0$
		"As n increases, the expected (square of the) difference between X_n and X diminishes."
almost surely	(notation $X_n \xrightarrow{a.s.} X$)	if $P[\lim_{n\to\infty} X_n = X] = 1$
		"Any event ω for which X_n does *not* approach the distribution X has zero probability."

The definition of $X_n \xrightarrow{a.s.} X$ can be interpreted as $P[\{\omega \in \Omega \mid X_n(\omega) \xrightarrow{n\to\infty} X(\omega)\}] = 1$.

In general, the following implications hold (although their converses do not):

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\begin{aligned}
X_n \xrightarrow{m.s.} X \quad &\Rightarrow \quad X_n \xrightarrow{p} X \\
X_n \xrightarrow{a.s.} X \quad &\Rightarrow \quad X_n \xrightarrow{p} X
\end{aligned}
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