

# Definition: Confusability Graph

## Definition: Confusability graph

Let  $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$  be a channel. The confusability graph  $G$  for the channel consists of the set of input symbols of the channel:

$$V(G) := \mathcal{X},$$

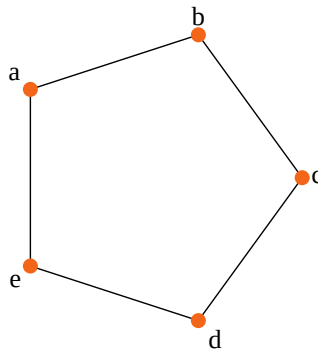
and

$$E(G) := \{\{x, x'\} \subset \mathcal{X} \mid x \neq x' \text{ and } \exists y \in \mathcal{Y} \text{ s.t. } P_{Y|X}(y|x) \cdot P_{Y|X}(y|x') > 0\}$$

is the set of input pairs that are confusable (because they reach a shared output symbol  $y \in \mathcal{Y}$ ).

## Example: Confusability graph of the noisy typewriter

Consider again the **noisy typewriter channel**. The confusability graph for that channel is:



This graph is also known as  $C_5$ , the circle of size 5. Its independence number is  $\alpha(C_5) = 2$ .

In the above example, the independence number of the confusability graph is exactly the number of messages that can be sent over the channel perfectly. This is no coincidence:

## Proposition

Given a channel with confusability graph  $G$ , the maximal message set  $[M]$  that can be communicated perfectly in a single channel use is of size  $\alpha(G)$ .

Proof

### Information Theory | Definition: Confusability Graph

Let  $(x, x') \in E(G)$ , i.e.,  $x$  and  $x'$  are confusable. They cannot both be used to send different messages, for suppose there are messages  $m \neq m'$  such that  $\text{enc}(m) = x$  and  $\text{enc}(m') = x'$ , then by definition of the confusability graph, there is a  $y \in \mathcal{Y}$  such that  $x$  and  $x'$  are both mapped to  $y$  with nonzero probability. In order to correctly decode in all cases, it must therefore be that  $\text{dec}(y) = m$  and  $\text{dec}(y) = m'$ , contradicting the assumption that  $m \neq m'$ . Therefore, the number of messages that can be sent over the channel cannot exceed the independence number  $\alpha(G)$ .

For the other direction, it is easy to find an encoding and decoding function for  $\alpha(G)$  different messages. Let  $\{x_1, x_2, \dots, x_{\alpha(G)}\}$  be a largest independent set of  $G$ . Define  $\text{enc}(m_i) = x_i$  for all  $i \in [\alpha(G)]$ . Then for all  $y \in \mathcal{Y}$ , by definition of the confusability graph and the independent set, there is exactly one  $i$  such that  $P_{Y|X}(y|x_i) > 0$ . Define  $\text{dec}(y) = m_i$ . This code can send  $\alpha(G)$  different messages over the channel without error.