

# Definition: Prefix-freeness

One convenient way to guarantee that a code is unique decodable is to require it to be prefix-free:

## Definition: Prefix-free code

A binary symbol code  $C : \mathcal{X} \rightarrow \{0, 1\}^*$  is prefix-free (or: **instantaneous**) if for all  $x, x' \in \mathcal{X}$  with  $x \neq x'$ ,  $C(x)$  is *not* a **prefix** of  $C(x')$ .

With a prefix-free encoding, the elements  $x_1, \dots, x_m$  can be uniquely recovered from  $C(x_1) | \dots | C(x_m)$ , simply by reading the encoding from left to right one bit at a time: by prefix-freeness it will remain unambiguous as reading continues when the current word terminates and the next begins. This is a loose argument for the following proposition:

## Proposition

If a code  $\mathcal{C}$  is prefix-free and  $\mathcal{C} \neq \perp$  then  $\mathcal{C}$  is uniquely decodable.

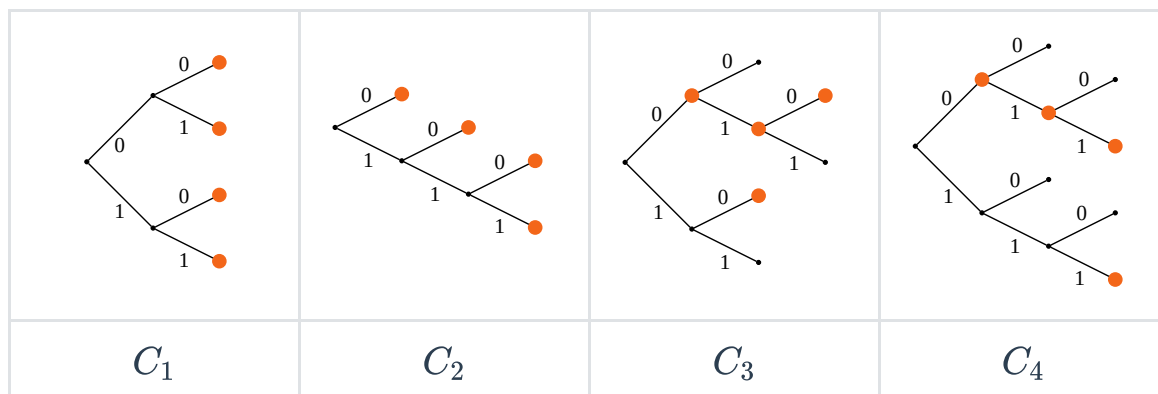
The other direction does not hold: uniquely decodable codes need not be prefix-free. A prefix-free code is appealing from an efficiency point of view, as it allows to decode "on the fly". For a general uniquely decodable code one may possibly have to inspect all bits in the entire string before being able to even recover the first word.

## Example

The following are all codes for the source  $P_X$ , with  $\mathcal{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ :

$x$	$P_X(x)$	$C_1(x)$	$C_2(x)$	$C_3(x)$	$C_4(x)$
<b>a</b>	0.5	00	0	0	0
<b>b</b>	0.25	01	10	010	01
<b>c</b>	0.125	10	110	01	011
<b>d</b>	0.125	11	111	10	111

These codes can be visualised as binary trees, with marked codewords, as follows:



Which of these codes are uniquely decodable? Which are prefix-free?

Show solution

$C_1$  and  $C_2$  are prefix-free, and therefore also uniquely decodable.  $C_3$  is not uniquely decodable, as  $C_3(ad) = C_3(b)$ .  $C_4$  is not prefix-free, but it is uniquely decodable, since it can be decoded from right to left (it is ``**suffix**-free"). Note that the binary trees for the prefix-free codes  $C_1, C_2$  only have codewords at the leaves.