Definition: Conditional Mutual Information

Applying the definition of mutual information to the conditional distribution $P_{XY|\mathcal{A}}$ naturally defines $I(X;Y|\mathcal{A})$, the mutual information of X and Y conditioned on the event $\mathcal{A}$:

Definition: Conditional mutual information

Let X, Y, Z be random variables. Then the conditional mutual information of X and Y given Z is defined as

$$I(X;Y|Z) = \sum_z P_Z(z) I(X;Y|Z{=}z)\,,$$

with the convention that the corresponding argument in the summation is 0 for z with $P_Z(z) = 0$.

Conditional mutual information has properties similar to the ones we saw for mutual information:

$$\begin{aligned}
&I(X;Y|Z) = I(Y;X|Z)\\
&I(X;Y|Z) \geq 0\\
&I(X;Y|Z) = 0 \text{ iff } X \text{ and } Y \text{ are independent given } Z.
\end{aligned}$$

Furthermore, the previous bounds $H(X) \geq 0$, $H(X|Y) \geq 0$, and $I(X;Y) \geq 0$, can all be seen as special cases of $I(X;Y|Z) \geq 0$. These bounds, and any bound they imply, are called **Shannon inequalities**. It is important to realize that $I(X;Y|Z)$ may be larger or smaller than (or equal to) $I(X;Y)$.

The following is sometimes used as definition of $I(X;Y|Z)$: verify it for yourself using the definition above.

Alternative definition

Let X, Y, Z be random variables. Then

$$I(X;Y|Z) = H(X|Z) - H(X|YZ)\,.$$

Proof

$$\begin{aligned}
I(X;Y|Z) &= \sum_{z\in\mathcal{Z}} P_Z(z) I(X;Y|Z=z) && \text{(by defini}\\
&= \sum_{z\in\mathcal{Z}} P_Z(z)(H(X|Z=z) - H(X|Y,Z=z)) && \text{(definition of mutual informa}\\
&= \sum_{z\in\mathcal{Z}} P_Z(z)H(X|Z=z) - \sum_{z\in\mathcal{Z}} P_Z(z)H(X|Y,Z=z)\\
&= H(X|Z) - H(X|YZ).
\end{aligned}$$