

# Definitions: Code, Rate, and Error Probability

In order to get as much information through a channel as possible, we can encode messages before sending them through the channel.

## Definition: Code

Let  $M, n \in \mathbb{N}$ . An  $(M, n)$ -code for the channel  $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$  consists of

- An index set  $[M] = \{1, \dots, M\}$  representing the set of possible messages.
- A (possibly probabilistic) encoding function  $\text{enc} : [M] \rightarrow \mathcal{X}^n$ . This encoding function should be injective.  $n$  represents the number of channel uses we need to send a single message.
- A deterministic decoding function  $\text{dec} : \mathcal{Y}^n \rightarrow [M]$ . The set of all codewords,  $\{\text{enc}(1), \dots, \text{enc}(M)\}$  is called the **codebook**.

An alternative notation for codes is  $[n, k]$  **code**, using box brackets instead of round brackets: such a code encodes a  $k$ -bit message into  $n$  bits. In the notation of the above definition, an  $[n, k]$  code would be an  $(2^k, n)$  code.

The number of bits of information that are transmitted per channel use is captured by the following notion:

## Definition: Rate

The rate of an  $(M, n)$ -code is defined as

$$R := \frac{\log M}{n}.$$

Given a code for a specific channel, we can study the probability that an error occurs while transmitting a message.

## Definition: Probability of error

Given an  $(M, n)$  code for a channel  $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ , the probability of error  $\lambda_m$  is the probability that the decoded output is not equal to the input message  $m$ . More formally,

$$\lambda_m^{(n)} := P[\text{dec}(Y^n) \neq m \mid X^n = \text{enc}(m)].$$

Given this quantity, the **maximal probability of error** is defined as

$$\lambda^{(n)} := \max_{m \in [M]} \lambda_m^{(n)}.$$

Similarly, the **average probability of error** is defined as

$$p_e^{(n)} := \frac{1}{M} \sum_{m=1}^M \lambda_m^{(n)}.$$