Probability Spaces and Events

For this course, we will only be concerned with discrete probabilities. This section formalizes some notions you should already be familiar with: probability spaces, events and probability distributions.

Definition: Probability space

A (discrete) probability space (Ω, \mathcal{F}, P) consists of a discrete, non-empty sample space Ω , an event space $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ (where $\mathcal{P}(\Omega)$ is the **powerset** of Ω) and a probability measure P which is a function $P:\Omega \to \mathbb{R}_{\geq 0}$ that satisfies

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

The event space $\mathcal F$ is required to be non-empty and closed under intersection, union and complements. For convenience, we will most often assume that $\mathcal F$ equals the powerset $\mathcal P(\Omega)$ of Ω , i.e., it contains all possible subsets of events, and therefore fulfils the required properties.

Definition: Event

An event \mathcal{A} is an element of the event space $\mathcal{F}\subseteq\mathcal{P}(\Omega)$, i.e., a subset \mathcal{A} of the sample space Ω . Its probability is defined as

$$P[\mathcal{A}] := \sum_{\omega \in \mathcal{A}} P(\omega),$$

where by convention $P[\emptyset] = 0$.

As a notational convention, we write $P[\mathcal{A},\mathcal{B}]$ for $P[\mathcal{A}\cap\mathcal{B}]$, and $P[\overline{\mathcal{A}}]$ for $P[\Omega \setminus \mathcal{A}]$. The following identities hold for arbitrary events $\mathcal{A},\mathcal{B}\subseteq\Omega$ (try to prove them for yourself):

- $P[\overline{\mathcal{A}}] = 1 P[\mathcal{A}]$
- $P[A \cup B] = P[A] + P[B] P[A, B]$
- $P[A] = P[A, B] + P[A, \overline{B}].$

It is often useful to consider the probability of an event *given* that some other event happened:

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For events $\mathcal A$ and $\mathcal B$ with $P[\mathcal A]>0$, the conditional probability of $\mathcal B$ given $\mathcal A$ is defined as

$$P[\mathcal{B}|\mathcal{A}] := \frac{P[\mathcal{A},\mathcal{B}]}{P[\mathcal{A}]}.$$

Example: Fair die

We throw a six-sided fair die once, and consider the number that comes up. The sample space for this experiment is $\Omega=1,2,3,4,5,6$, with event space $\mathcal{F}=\mathcal{P}(\Omega)$ and probability measure $P[i]=\frac{1}{|\Omega|}=\frac{1}{6}$ for all $i\in\Omega$ (this is a

uniform probability measure). Consider the events $\mathcal{A}=2,4,6$ and $\mathcal{B}=3,6$. Using the formulas in the definitions of events and conditional probabilities, we can compute the following probabilities:

$$P[\mathcal{A}] = \frac{1}{2} \text{(the outcome is even)}$$

$$P[\mathcal{B}] = \frac{1}{3} \text{(the outcome is a multiple of 3)}$$

$$P[\mathcal{A}, \mathcal{B}] = P[6] = \frac{1}{6} \text{(the roll is even and a multiple of 3)}$$

$$P[\mathcal{A}|\mathcal{B}] = \frac{1/6}{1/3} = \frac{1}{2} \text{(the roll is even, given that it is a multiple of 3)}$$

$$P[\mathcal{B}|\mathcal{A}] = \frac{1/6}{1/2} = \frac{1}{3} \text{(the roll is a multiple of 3, given that it is even)}$$

This example shows that in general, $P[\mathcal{A}|\mathcal{B}]$ is not necessarily equal to $P[\mathcal{B}|\mathcal{A}]$. In fact, they are related through Bayes' rule.

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