Properties of Relative Entropy

Even though relative entropy is always nonnegative (see the theorem below), it is not a proper distance measure, because it is not symmetric and does not satisfy the triangle inequality.

Lemma: Alternative Definition of Mutual Information

The mutual information between X and Y can be expressed in terms of the relative entropy of their distributions as follows:

$$I(X;Y) = D(P_{XY}||P_X \cdot P_Y)$$

Proof

The statement follows by writing out the definitions of mutual information and relative entropy, and rearranging terms.

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= -\sum_{x\in\mathcal{X}} P_X(x)\log P_X(x) + \sum_{y\in\mathcal{Y}} P_Y(y) \sum_{x\in\mathcal{X}} P_{X|Y}(x|y)\log P_{X|Y}(x|y) \\
&= -\sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_{XY}(x,y)\log P_X(x) + \sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_{XY}(x,y)\log P_{X|Y}(x|y) \\
&= -\sum_{x\in\mathcal{X},y\in\mathcal{Y}:P_{XY}(x,y)>0} P_{XY}(x,y)\log P_X(x) + \sum_{x\in\mathcal{X},y\in\mathcal{Y}:P_{XY}(x,y)>0} P_{XY}(x,y)\log P_{X|Y}(x|y) \\
&= \sum_{x\in\mathcal{X},y\in\mathcal{Y}:P_{XY}(x,y)>0} P_{XY}(x,y)(-\log P_X(x) + \log P_{X|Y}(x|y)) \\
&= \sum_{x\in\mathcal{X},y\in\mathcal{Y}:P_{XY}(x,y)>0} P_{XY}(x,y)\log \frac{P_{X|Y}(x|y)}{P_X(x)} \\
&= \sum_{x\in\mathcal{X},y\in\mathcal{Y}:P_{XY}(x,y)>0} P_{XY}(x,y)\log \frac{P_{XY}(x,y)}{P_X(x)P_Y(y)} \\
&= D(P_{XY}||P_X P_Y)
\end{aligned}$$

Theorem: Information Inequality

For any two probability distributions P and Q defined on the same $\mathcal{X}$,

$$D(P||Q) \geq 0.$$

Equality holds if and only if $P = Q$.

Proof

Left as an exercise. Hint: use Jensen's inequality.

The above lemma and theorem together show that the mutual information is a measure of 'how independent' the variables X and Y are: if $P_{XY} = P_X \cdot P_Y$, the variables are independent and their mutual information is zero.