Source-Coding Theorem for Stationary Stochastic Processes

Recall Shannon's Source-Coding Theorem: it states that an optimal code (for an i.i.d. source X) has expected codeword length approximately $H(X)$.

We can state a similar result for stochastic processes:

Theorem: Source-coding theorem (for stochastic processes)

Let $\{X_i\} = X_1, X_2, X_3, \ldots$ be a stationary stochastic source. Let $\ell_{\min}(n)$ be the expected minimal codeword length *per symbol* when encoding blocks of n source symbols, that is, $\ell_{\min}(n) := \ell_{\min}(P_{X_1 \cdots X_n})/n$. Then

$$\lim_{n \to \infty} \ell_{\min}(n) = H(\{X_i\}).$$

Proof

By Shannon's souce-coding theorem, we have that for every n,

$$H(X_1 X_2 \cdots X_n) \leq \ell_{\min}(P_{X_1 X_2 \cdots X_n}) \leq H(X_1 X_2 \cdots X_n) + 1.$$

Dividing all sides by n, and recalling that for stationary processes, $H(X_1 X_2 \cdots X_n)/n$ converges to the entropy rate $H(\{X_i\})$, the result follows.

In the limit, we can compress a stationary stochastic source down to its entropy rate.

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