Markov Process: Irreducibility, Periodicity, Convergence

Every time-invariant Markov process has a stationary distribution, but it might not be unique, and the process might never reach the stationary distribution. For uniqueness and convergence, we need additional requirements on the Markov process.

Definition: Irreducible Markov process

A time-invariant Markov process is irreducible if every state is reachable from any other state in a finite number of steps.

The process in Example 2 is irreducible: the state `a` is reachable from `b` and vice versa. The process in Example 1 is not irreducible. For example, the state `0` is not reachable from the state `2`.

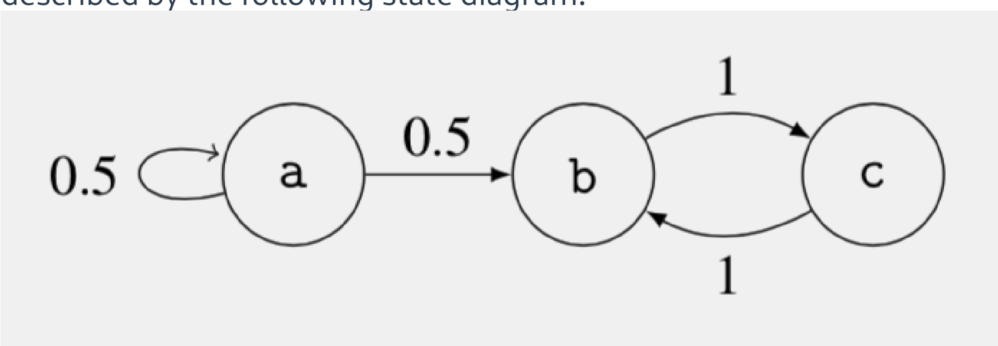
Definition: Aperiodic Markov process

A state in a time-invariant Markov process is aperiodic if the greatest common divisor of all path lengths from that state to itself is 1. The process itself is aperiodic if all states are aperiodic.

The processes in Example 1 and Example 2 are both aperiodic. In both examples, every state is reachable from itself with paths of any length, so the greatest common divisor of the path lengths is always 1. Below, we will present an example of a process that is periodic (i.e., not aperiodic).

Example 3: Periodic Markov process

Consider the process that starts in state `a` with probability 1, and is further described by the following state diagram:



The state `a` is aperiodic, but `b` and `c` are not. The paths from state `b` to itself are all

of even length, as are the paths from state **c** to itself. A typical run of this process might look like **aaaaabcbcbcbc**; the period 2 is clearly visible in this run.
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Proposition

If a time-invariant Markov chain is finite-state, irreducible and aperiodic, then there exists a unique stationary distribution. Moreover, from any initial distribution, the distribution of X_n tends to the stationary distribution as $n \to \infty$.
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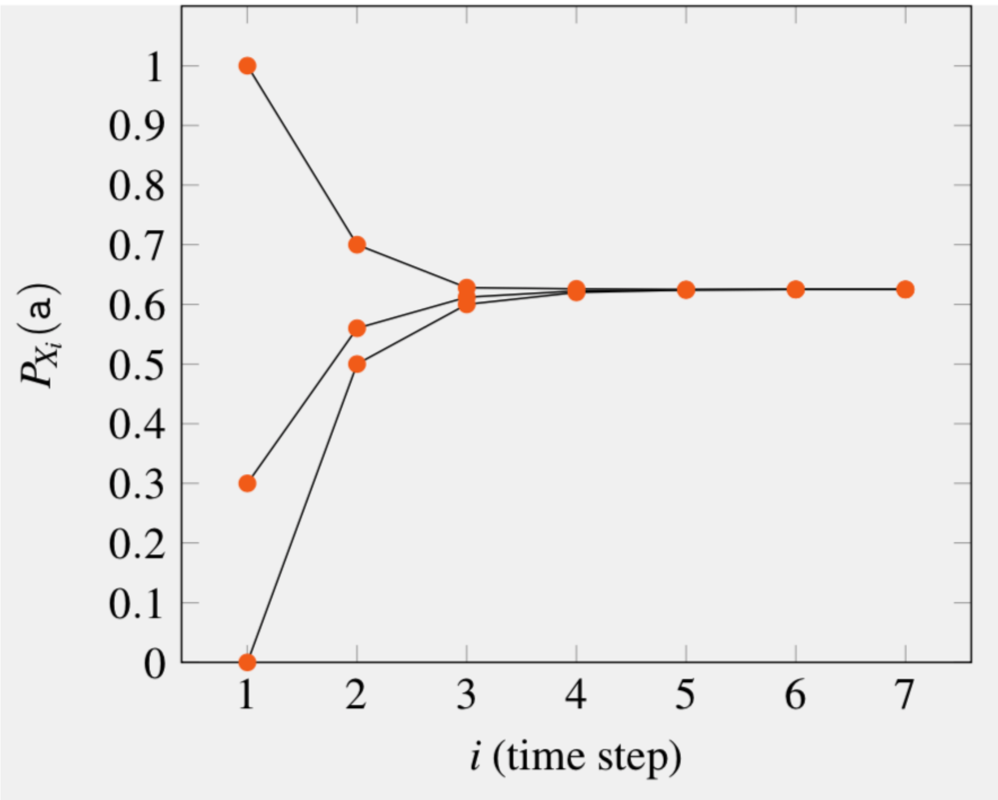
Example 2, continued

Consider again the process of Example 2. This process does have a stationary distribution, because it is finite-state, irreducible and aperiodic. Let $\mu_{\mathtt{a}}$ denote the probability of observing an **a** in the stationary distribution, and $\mu_{\mathtt{b}}$ the probability of observing a **b**. Then the stationary distribution must satisfy the following set of equations:

$$\begin{aligned} \mu_{\mathtt{a}} &= 0.7\mu_{\mathtt{a}} + 0.5\mu_{\mathtt{b}}, \\ \mu_{\mathtt{b}} &= 0.3\mu_{\mathtt{a}} + 0.5\mu_{\mathtt{b}}, \\ \mu_{\mathtt{a}} + \mu_{\mathtt{b}} &= 1. \end{aligned}$$

Solving this set of equations gives the stationary distribution $\mu_{\mathtt{a}} = 0.625$ and $\mu_{\mathtt{b}} = 0.375$. This outcome matches our observations in the figure below. The starting distribution is irrelevant, because in the limit, the stationary distribution is reached. The following plot exemplifies this for three different starting distributions ($P_{X_1}(\mathtt{a}) = 0$, $P_{X_1}(\mathtt{a}) = 0.3$, and $P_{X_1}(\mathtt{a}) = 1$).



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