Definitions: Code, Rate, and Error Probability

In order to get as much information through a channel as possible, we can encode messages before sending them through the channel.

Definition: Code

Let $M, n \in \mathbb{N}$. An (M, n)-code for the channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ consists of

- An index set $[M] = \{1, \ldots, M\}$ representing the set of possible messages.
- A (possibly probabilistic) encoding function $\texttt{enc} : [M] \to \mathcal{X}^n$. This encoding function should be injective. n represents the number of channel uses we need to send a single message.
- A deterministic decoding function $\texttt{dec} : \mathcal{Y}^n \to [M]$. The set of all codewords, $\{\texttt{enc}(1), \ldots, \texttt{enc}(M)\}$ is called the **codebook**.

An alternative notation for codes is $[n, k]$ **code**, using box brackets instead of round brackets: such a code encodes a k-bit message into n bits. In the notation of the above definition, an $[n, k]$ code would be an $(2^k, n)$ code.

The number of bits of information that are transmitted per channel use is captured by the following notion:

Definition: Rate

The rate of an (M, n)-code is defined as

$$R := \frac{\log M}{n}.$$

Given a code for a specific channel, we can study the probability that an error occurs while transmitting a message.

Definition: Probability of error

Given an (M, n) code for a channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$, the probability of error λ_m is the probability that the decoded output is not equal to the input message m. More formally,

$$\lambda_m^{(n)} := P[\mathtt{dec}(Y^n) \neq m \mid X^n = \mathtt{enc}(m)].$$

Given this quantity, the **maximal probability of error** is defined as

$$\lambda^{(n)} := \max_{m \in [M]} \lambda_m^{(n)}.$$

Similarly, the **average probability of error** is defined as

$$p_e^{(n)} := \frac{1}{M} \sum_{m=1}^{M} \lambda_m^{(n)}.$$