The Repetition Code

A simple and intuitive code is the n-bit **repetition code** R_n: a single message bit is encoded by simply repeating the bit n times. Decoding is done by majority vote, that is, $\texttt{dec}(y) = MAJ(y_1, \ldots, y_n)$, which is 1 if and only if (strictly) more than half of the bits in y are 1s. In order to avoid ties in the decoding, repetition codes usually require that n is odd.

Example: 3-bit repetition code

Consider the 3-bit repetition code R_3. It is a (2,3) code with codebook $\{000, 111\}$. The rate of R_3 is $1/3$. The probability of error for the message $w = 0$, sent over a BSC with bit flip propability f, is

$$\begin{aligned}\lambda_0 &= P[\texttt{dec}(Y^n) \neq 0 \mid X^n = 000] \\ &= P[Y^n = 011 \;\cup\; Y^n = 101 \;\cup\; Y^n = 110 \;\cup\; Y^n = 111 \mid X^n = 000] \\ &= 3f^2(1-f) + f^3.\end{aligned}$$

A similar calculation shows that $\lambda_1 = \lambda_0$. Hence, the maximal and average probability of error are equal to λ_0 as well. As a concrete example, if $f = 0.1$, the 3-bit repetition code has an error probability of approximately 0.03. Hence, the 3-bit repetition code provides an error probability that is about three times lower (3% instead of 10%), at the expense of a rate that is a factor 3 worse than simply sending the messages through the channel without encoding.

In general, the n-bit repetition code is a $(2, n)$ code with the relatively low rate of $1/n$ and an average/maximal probability of error of

$$\sum_{k=(n+1)/2}^{n} \binom{n}{k} f^k (1-f)^{n-k}$$

when used on a binary symmetric channel with bit flip probability f.