

# Stationary Process

Often, we will be interested in stochastic processes with specific properties, such as processes where all  $X_i$  are independent, or processes with a Markov-like property. We review several such properties.

## Definition: Stationary process

A stochastic process is stationary if for all  $n, k \in \mathbb{N}_+$ ,

$$P_{X_1 \dots X_n} = P_{X_{1+k} \dots X_{n+k}}.$$

Stationary processes are invariant under time shifts: when observing a subsequence of length  $n$ , it does not matter where in the process you look exactly.

## Example: i.i.d. process

Let  $X$  be a random variable. Consider a stochastic process  $\{X_i\}$  where  $P_{X_i} = P_X$  for all  $i$ . That is, the random variables in the sequence are all independent and identically distributed. This process is stationary, since for any  $n, k$ , it holds that

$$P_{X_1 \dots X_n} = \prod_{i=1}^n P_{X_i} = \prod_{i=1+k}^{n+k} P_{X_i} = P_{X_{1+k} \dots X_{n+k}}.$$

## Example: Ten fair coins

The following is another example of a stationary process. Throw a fair coin 10 times: this can be described by the finite sample space  $\{\text{H}, \text{T}\}^{10}$ . Define the stochastic process  $\{X_i\}_{i \in \mathbb{N}_0}$  by setting

$$X_i = \begin{cases} 1 & \text{if the } [i \bmod 10]\text{th coin lands on heads} \\ 0 & \text{otherwise.} \end{cases}$$

Here,  $[k \bmod N]$  is defined to be an element of  $\{0, 1, 2, \dots, N-1\}$ . If we want the first variable in the process to be  $X_1$  instead of  $X_0$ , we can determine the value of  $X_i$  based on the  $[((i-1) \bmod 10) + 1]$ th coin.

As an exercise, show that this process is indeed stationary.

Hint

Observe that for all  $i$ ,  $X_i = X_{i+10} = X_{i+20} = \dots$