

Convergence of Random Variables

The following definition of converging random variables may remind you of a converging sequence of numbers. Recall that a sequence x_1, x_2, x_3, \dots of numbers converges to x if $\forall \epsilon > 0 \exists n_0 \forall (n \geq n_0) : |x_n - x| < \epsilon$. We denote this by writing $x_n \xrightarrow{n \rightarrow \infty} x$.

Definition: Converging random variables

A sequence X_1, X_2, X_3, \dots of real random variables converges to a random variable X , if it satisfies one of the following definitions:

in probability	(notation $X_n \xrightarrow{p} X$)	if $\forall \epsilon > 0, P[X_n - X > \epsilon] \xrightarrow{n \rightarrow \infty} 0$
		"As n increases, the distribution of X_n gets closer and closer to that of X ."
in mean square	(notation $X_n \xrightarrow{m.s.} X$)	if $\mathbb{E}[(X_n - X)^2] \xrightarrow{n \rightarrow \infty} 0$
		"As n increases, the expected (square of the) difference between X_n and X diminishes."
almost surely	(notation $X_n \xrightarrow{a.s.} X$)	if $P[\lim_{n \rightarrow \infty} X_n = X] = 1$
		"Any event ω for which X_n does <i>not</i> approach the distribution X has zero probability."

The definition of $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$ can be interpreted as $P[\{\omega \in \Omega \mid X_n(\omega) \xrightarrow{n \rightarrow \infty} X(\omega)\}] = 1$.

In general, the following implications hold (although their converses do not):

$$\begin{aligned} X_n \xrightarrow{m.s.} X &\Rightarrow X_n \xrightarrow{p} X \\ X_n \xrightarrow{a.s.} X &\Rightarrow X_n \xrightarrow{p} X \end{aligned}$$

