Multiple Channel Uses

Earlier in this module we have found that the independence number of the confusability graph is a conclusive upper bound for the number of messages that can be sent over a channel *in a single use*. But it turns out that if the same channel is allowed to be used more than once, it is sometimes possible to achieve a higher rate, still error-free!

Example: Two uses of the noisy typewriter

Let us reconsider the noisy typewriter, but this time we are allowed to use the channel twice. We can of course use the following encoding function to encode $2 \times 2 = 4$ different messages:

$$\begin{aligned}
\mathtt{enc}(m_1) &= \mathtt{aa} \\
\mathtt{enc}(m_2) &= \mathtt{ac} \\
\mathtt{enc}(m_3) &= \mathtt{ca} \\
\mathtt{enc}(m_4) &= \mathtt{cc}
\end{aligned}$$

This encoding function is defined by simply concatenating the encoding function for the single-use example. It has a rate of $\log(4)/2 = 1$, just like the single-use noisy typewriter channel code. There is, however, a code that is able to encode and correctly decode *five* different messages! It uses codewords $\mathtt{aa}, \mathtt{bc}, \mathtt{ce}, \mathtt{db}, \mathtt{ed}$ as codewords for messages m_1 through m_5. To see that the decoding is error-free, observe that the channel maps the codewords to the following sets of possible outputs:

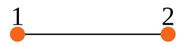
$$\begin{aligned}
\mathtt{aa} &\mapsto \{\mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}\} \\
\mathtt{bc} &\mapsto \{\mathtt{bc}, \mathtt{bd}, \mathtt{cc}, \mathtt{cd}\} \\
\mathtt{ce} &\mapsto \{\mathtt{ce}, \mathtt{ca}, \mathtt{de}, \mathtt{da}\} \\
\mathtt{db} &\mapsto \{\mathtt{db}, \mathtt{dc}, \mathtt{eb}, \mathtt{ec}\} \\
\mathtt{ed} &\mapsto \{\mathtt{ed}, \mathtt{ee}, \mathtt{ad}, \mathtt{ae}\}
\end{aligned}$$

All of these sets are non-overlapping, so none of the inputs are confusable. The rate of this code is $\log(5)/2 \approx 1.161$, which is strictly better than the single-use channel!

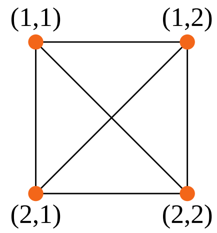
The above example shows that there are optimal codes that make better use of the channel if the channel is used more than once. We can determine whether this is the case by looking at the strong graph product of the channel's confusability graph with itself. An edge in the strong graph product reflects that the two input pairs are confusable: there exists an output pair to which they are both mapped with nonzero probability.

Example: Binary symmetric channel

The confusability graph of the non-trivial binary symmetric channel (BSC) is C_2:



This graph is also known as K_2, the **complete graph** of size 2. $K_2 \boxtimes K_2$ is the complete graph on 4 vertices K_4:



In this case, using the channel twice does not help: the independence number of both graphs is 1, and the rate of any error-free code is therefore zero.