## **Properties of Typical Sets**

In the coin-flipping example on the previous page, the typical set eventually contained almost all of the probability. The following proposition states that this is a general property of typical sets. The proposition also bounds the size of the typical set.

## **Proposition**

A typical set  $A_{\epsilon}^{(n)}$  satisfies the following:

1. For all  $(x_1,\ldots,x_n)\in A_\epsilon^{(n)}$  ,

$$H(X) - \epsilon \leq -rac{1}{n} {
m log}\, P_{X^n}(x_1,\ldots,x_n) \leq H(X) + \epsilon.$$

2.  $P[A_\epsilon^{(n)}]>1-\epsilon$  (for large enough n). 3.  $|A_\epsilon^{(n)}|\leq 2^{n(H(X)+\epsilon)}$  .

4.  $|A_{\epsilon}^{(n)}| \geq (1-\epsilon)2^{n(H(X)-\epsilon)}$  (for large enough n).

Proof

- 1. This is immediate from the definition (take the logarithm and divide by -n, thereby reversing the inequalities).
- 2. This follows from the Asymptotic Equipartition Property: for all  $\epsilon > 0$ ,

$$P[|-rac{1}{n}{
m log}\, P_{X^n}(X_1,\ldots,X_n)-H(X)|>\epsilon]\stackrel{n o\infty}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} 0,$$

that is,

$$orall (\epsilon>0) \; orall (\delta>0) \; \exists n_0 \; orall (n\geq n_0) \; P[|-rac{1}{n} \log P_{X^n}(X_1,\ldots,X_n) - H(X)| \leq \epsilon] > 1-\delta.$$

By choosing  $\delta := \epsilon$ , the result follows from the first property.

3. First, observe that

$$1 = \sum_{ec{x} \in \mathcal{X}^n} P_{X^n}(ec{x}) \geq \sum_{ec{x} \in A^{(n)}_{\epsilon}} P_{X^n}(ec{x}) \geq |A^{(n)}_{\epsilon}| \cdot 2^{-n(H(X) + \epsilon},$$

where the last inequality follows by the definition of typicality. The claim follows by multiplying both sides of the equation by  $2^{n(H(X)+\epsilon)}$ 

4. By Property 2, we can choose an n large enough so that

$$1-\epsilon < P[A_\epsilon^{(n)}] = \sum_{ec{x} \in A_\epsilon^{(n)}} P_{X^n}(ec{x}) \leq |A_\epsilon^{(n)}| \cdot 2^{-n(H(X)-\epsilon)},$$

where again, the last inequality follows by the definition of typicality.

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