Decoding: Parity-Check Matrix and Error Syndromes

The decoding operation can be described by defining the parity-check matrix:

Definition: Parity-check matrix

Let C be a code with generator matrix

$$G^T = \left(\frac{\mathbb{I}_k}{P} \right).$$

Then the parity check matrix for C is given by

$$H := \left(-P \mid \mathbb{I}_{n-k} \right).$$

Note that if $\mathcal{X}^n = \{0, 1\}^n$, then P is a matrix with binary entries, and hence $-P = P$.

The parity-check matrix shows whether or not an error has occurred, and if it did, what kind of error occurred. The information about the error is contained in the syndrome, which you get by applying the parity check to the received codeword.

Definition: Syndrome

Let C be a code with parity-check matrix H. The syndrome of a received codeword $c \in \mathcal{Y}^n$ is Hc.

For all $c \in C$, the syndrome is the all-zero vector, since $c = G^T m$ for some m, and

$$HG^T m = \left(-P \mid \mathbb{I}_{n-k} \right) \left(\frac{\mathbb{I}_k}{P} \right) m = 0^{n-k},$$

where we make use of **block matrix multiplication**. The syndrome gives a lot of valuable information about where an error occurred. Suppose a codeword c is sent over a channel, and a single bit of c is flipped, at the ith position. The output of the channel is thus $c' = c + e_i$ (where e_i is the ith unit vector, which is 1 at position i and 0 elsewhere). The syndrome for this received output is

$$Hc' = H(c + e_i) = Hc + He_i = He_i,$$

which is the ith column of H. So by comparing the syndrome to the parity-check matrix H, we can find out which bit was most likely flipped.

Example: Parity-check matrix of the $[7, 4]$ Hamming code

The following 3×7 matrix is the parity-check matrix for the $[7, 4]$ Hamming code:

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose the codeword 1010100 is received. What is the syndrome? Which errors occurred, if any? And what is the correct decoding?

Show solution

The syndrome is 110: the result of applying H to the received codeword. This syndrome matches He_2 (the second column of H), and so it is most likely the second codeword bit that was flipped. The decoding is therefore 1110 (corrected from 1010).

Notice that all columns of H are different, and that there are $2^3 = 8$ possible syndromes we might observe. This aligns well with the fact that there are three parity bits that can all either be correct or incorrect.