Formulas and Equations for the Classical Laminate Theory

Author: Vincent Calard
Version: 0.6
Datum: 19.09.2011

In this document, we prepared a collection of formulas and equations for the classical laminate theory. The first objective was to propose a quick reference booklet with the most important formulas to calculate the stiffness and the compliance matrix of a laminated composite. The document is not yet finished but in a pre-version release. Any comments and/or corrections are also welcome.

Notations

Coordinate axes (xyz) = subscripts (123)

Stress $[\sigma]$ and strain $[\varepsilon]$ contracted notations

The stress tensor $[\sigma]$ and strain tensor $[\varepsilon]$ are represented by two (3x3) symmetric matrices (denoted by $[.]$):

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ \text{Sym.} & & \varepsilon_{33} \end{bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{Sym.} & & \sigma_{33} \end{bmatrix} \tag{1}$$

We chose the following contracted notation in a vector form (denoted by $\{.\}$):

$$\{\epsilon\} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \mathbf{2}\varepsilon_{23} \\ \mathbf{2}\varepsilon_{13} \\ \mathbf{2}\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \qquad \{\sigma\} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \tag{2}$$

From the above relations, we note that the components with subscripts (4,5,6) are associated with the shear and that the stress and strain state in the (xy) plane is associated with the subscripts (1,2,6).

Hooke's law, Compliance matrix $\{\epsilon\} = [S]\{\sigma\}$, 3D

21 independent parameters are sufficient to describe the 36 parameters:

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & \text{Symmetric} & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \tag{3}$$

Hooke's law, Stiffness matrix $\{\sigma\} = [S]^{-1}\{\epsilon\} = [C]\{\epsilon\}$, 3D

21 independent parameters are sufficient to determine the 36 components:

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{Symmetric} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{pmatrix} \tag{4}$$

Hooke's law in plane stress condition, $\sigma_3 = \sigma_4 = \sigma_5 = 0$, 2D

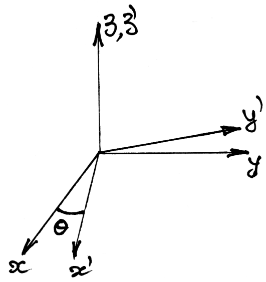
$$\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_6\end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16}\\ & S_{22} & S_{26}\\ \text{Sym.} & & S_{66}\end{bmatrix} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16}\\ & Q_{22} & Q_{26}\\ \text{Sym.} & & Q_{66}\end{bmatrix} \begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_6\end{pmatrix} \tag{5}$$

The terms of the stiffness matrix $[Q]$ will not be equal to the corresponding terms of the stiffness matrix $[C]$ in 3D.

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16}\\ & Q_{22} & Q_{26}\\ \text{Sym.} & & Q_{66}\end{bmatrix} \neq \begin{bmatrix} C_{11} & C_{12} & C_{16}\\ & C_{22} & C_{26}\\ \text{Sym.} & & C_{66}\end{bmatrix} \tag{6}$$

The components C_{16} and C_{26}, respectively S_{16} and S_{26}, are the (σ-γ) coupling components. They are responsible of distortions of the material geometry, even if an unidirectional stress or strain condition is applied to the material.

Rotation of axe (z) with angle θ of (xyz) gives $(x'y'z')$



The $[\sigma'] = [P][\sigma][P]^{-1}$ equation gives the rotation transformation of the stress tensor from the system (xyz) to the system $(x'y'z')$ and is written as follow with $c = \cos(\theta)$ and $s = \sin(\theta)$:

$$\begin{bmatrix} \sigma'_1 & \sigma'_6 & \sigma'_5\\ & \sigma'_2 & \sigma'_4\\ \text{Sym.} & & \sigma'_3\end{bmatrix} = \begin{bmatrix} c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5\\ & \sigma_2 & \sigma_4\\ \text{Sym.} & & \sigma_3\end{bmatrix} \begin{bmatrix} c & -s & 0\\ s & c & 0\\ 0 & 0 & 1\end{bmatrix} \tag{7}$$

Using the contracted notation, the rotation transformation can be written in a simple form:

$$\begin{pmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_3\\ \sigma'_4\\ \sigma'_5\\ \sigma'_6\end{pmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs\\ s^2 & c^2 & 0 & 0 & 0 & -2cs\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & c & s & 0\\ 0 & 0 & 0 & -s & c & 0\\ -cs & cs & 0 & 0 & 0 & c^2 - s^2\end{bmatrix}}_{[J_{z,\sigma}]} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} \tag{8}$$

The rotation transformation of the strain tensor with the contracted notation is:

$$\begin{pmatrix}\epsilon'_1\\ \epsilon'_2\\ \epsilon'_3\\ \epsilon'_4\\ \epsilon'_5\\ \epsilon'_6\end{pmatrix} = \begin{pmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_3\\ \mathbf{2}\varepsilon'_4\\ \mathbf{2}\varepsilon'_5\\ \mathbf{2}\varepsilon'_6\end{pmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & \frac{\mathbf{1}}{\mathbf{2}}2cs \\ s^2 & c^2 & 0 & 0 & 0 & -\frac{\mathbf{1}}{\mathbf{2}}2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ -\mathbf{2}cs & \mathbf{2}cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix}}_{[J_{z,\epsilon}]} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \mathbf{2}\varepsilon_4\\ \mathbf{2}\varepsilon_5\\ \mathbf{2}\varepsilon_6\end{pmatrix} \tag{9}$$

The rotation transformations of the stiffness and the compliance matrices are given by:

$$[C'] = [J_{z,\sigma}][C][J_{z,\epsilon}]^{-1} \quad \text{and} \quad [S'] = [J_{z,\epsilon}][S][J_{z,\sigma}]^{-1} \tag{10}$$

Rotation θ in 2D, (xy) plane associated with the subscripts (126)

$$\begin{pmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{pmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2-s^2 \end{bmatrix}}_{[J_{2,\sigma}]} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\epsilon'_1\\ \epsilon'_2\\ \epsilon'_6\end{pmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & \frac{\mathbf{1}}{\mathbf{2}}2cs \\ s^2 & c^2 & -\frac{\mathbf{1}}{\mathbf{2}}2cs \\ -\mathbf{2}cs & \mathbf{2}cs & c^2-s^2 \end{bmatrix}}_{[J_{2,\epsilon}]} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \mathbf{2}\varepsilon_6\end{pmatrix} \tag{11}$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$.
The rotation transformation of the stiffness matrix is given by $[Q'] = [J_{2,\sigma}][Q][J_{2,\epsilon}]^{-1}$:

$$\begin{pmatrix}Q'_{11}\\ Q'_{12}\\ Q'_{16}\\ Q'_{22}\\ Q'_{26}\\ Q'_{66}\end{pmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & 4c^3s & s^4 & 4cs^3 & 4c^2s^2 \\ c^2s^2 & s^4+c^4 & 2cs^3-2c^3s & c^2s^2 & 2c^3s-2cs^3 & -4c^2s^2 \\ -c^3s & c^3s-cs^3 & c^4-3c^2s^2 & cs^3 & 3c^2s^2-s^4 & 2c^3s-2cs^3 \\ s^4 & 2c^2s^2 & -4cs^3 & c^4 & -4c^3s & 4c^2s^2 \\ -cs^3 & cs^3-c^3s & 3c^2s^2-s^4 & c^3s & c^4-3c^2s^2 & 2cs^3-2c^3s \\ c^2s^2 & -2c^2s^2 & 2cs^3-2c^3s & c^2s^2 & 2c^3s-2cs^3 & (s^2-c^2)^2 \end{bmatrix} \begin{pmatrix}Q_{11}\\ Q_{12}\\ Q_{16}\\ Q_{22}\\ Q_{26}\\ Q_{66}\end{pmatrix} \tag{12}$$

In case of a stiffness matrix without (σ-γ) coupling components ($Q_{16} = Q_{26} = 0$), we have:

$$\begin{pmatrix}Q'_{11}\\ Q'_{12}\\ Q'_{16}\\ Q'_{22}\\ Q'_{26}\\ Q'_{66}\end{pmatrix} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & 4c^2s^2 \\ c^2s^2 & c^2s^2 & s^4+c^4 & -4c^2s^2 \\ -c^3s & cs^3 & c^3s-cs^3 & 2c^3s-2cs^3 \\ s^4 & c^4 & 2c^2s^2 & 4c^2s^2 \\ -cs^3 & c^3s & cs^3-c^3s & 2cs^3-2c^3s \\ c^2s^2 & c^2s^2 & -2c^2s^2 & s^4-2c^2s^2+c^4 \end{bmatrix} \begin{pmatrix}Q_{11}\\ Q_{22}\\ Q_{12}\\ Q_{66}\end{pmatrix} \tag{13}$$

For the compliance matrix, the rotation transformation is:

$$\begin{pmatrix}S'_{11}\\ S'_{12}\\ S'_{16}\\ S'_{22}\\ S'_{26}\\ S'_{66}\end{pmatrix} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & c^2s^2 \\ c^2s^2 & c^2s^2 & s^4+c^4 & -c^2s^2 \\ -2c^3s & 2cs^3 & 2c^3s-2cs^3 & c^3s-cs^3 \\ s^4 & c^4 & 2c^2s^2 & c^2s^2 \\ -2cs^3 & 2c^3s & 2cs^3-2c^3s & cs^3-c^3s \\ 4c^2s^2 & 4c^2s^2 & -8c^2s^2 & s^4-2c^2s^2+c^4 \end{bmatrix} \begin{pmatrix}S_{11}\\ S_{22}\\ S_{12}\\ S_{66}\end{pmatrix} \tag{14}$$

The stiffness components Q'_{11}, Q'_{12}, Q'_{22} and Q'_{66}, and the compliance components S'_{11}, S'_{12}, S'_{22} and S'_{66} are **even** functions of θ:

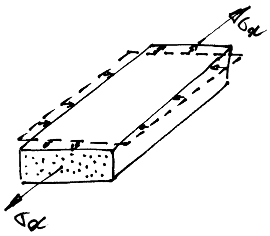
$$Q'_{11}(-\theta) = Q'_{11}(\theta)\dots \tag{15}$$

The coupling components S'_{16}, S'_{26}, Q'_{16} and Q'_{26} are **odd** functions of θ:

$$Q'_{16}(-\theta) = -Q'_{16}(\theta)\dots \tag{16}$$

Engineering constants

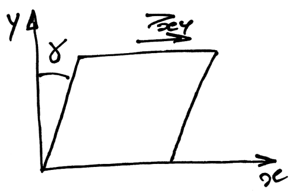
In mechanics, two engineering constants, the Young's modulus E and the Poisson's ratio ν, are measured from tensile testing. A third engineering constant, the shear modulus G, is obtained by a pure shear test. In case of a pure tensile test in x direction, we have:



$$\begin{cases} \sigma_x \neq 0 \quad , \sigma_y = \tau_{xy} = 0 \\ \epsilon_x \text{ and } \epsilon_y \text{ proportionnal to } \sigma_x \end{cases}$$

$$\Rightarrow \begin{cases} \epsilon_x = \frac{1}{E_x}\sigma_x \\ \epsilon_y = -\nu_{xy}\epsilon_x = -\frac{\nu_{xy}}{E_x}\sigma_x \\ \gamma_{xy} = 0 \end{cases}$$

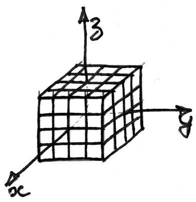
The Young's modulus is defined by the ratio between the applied longitudinal stress and the longitudinal strain, and has the dimension of a stress. The Poisson's ratio is defined by the ratio between the longitudinal and the transversal strain, and is without dimension. In a pure shear test, we have:



$$\begin{cases} \tau_{xy} \neq 0 \quad , \sigma_x = \sigma_y = 0 \\ \gamma_{xy} \text{ proportionnal to } \tau_{xy} \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \epsilon_x = 0 \text{ and } \epsilon_y = 0 \end{cases}$$

Orthotropic, 3D



The compliance matrix in the symmetry axes of the material is given by:

$$[S_{ortho}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \tag{17}$$

9 engineering constants (E_1, E_2, E_3, ν_{12}, ν_{23}, ν_{13}, G_{23}, G_{13}, G_{12}) are necessary to write the matrix in the symmetry axes:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \tag{18}$$

The symmetry relations are:

$$\frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1}; \qquad \frac{-\nu_{31}}{E_3} = \frac{-\nu_{13}}{E_1}; \qquad \frac{-\nu_{32}}{E_3} = \frac{-\nu_{23}}{E_2} \tag{19}$$

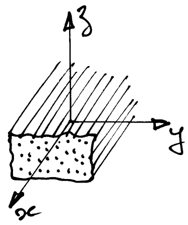
We have also the following relation $\nu_{13}\nu_{21}\nu_{32} = \nu_{12}\nu_{23}\nu_{31}$. The stiffness matrix is obtained by inverting the compliance matrix which yields the following components:

$$
\begin{aligned}
C_{11} &= \frac{(1-\nu_{23}\nu_{32})}{\Delta}E_1; \\
C_{12} &= \frac{(\nu_{21}+\nu_{31}\nu_{23})}{\Delta}E_1 = \frac{(\nu_{12}+\nu_{32}\nu_{13})}{\Delta}E_2; \\
C_{13} &= \frac{(\nu_{31}+\nu_{21}\nu_{32})}{\Delta}E_1 = \frac{(\nu_{13}+\nu_{12}\nu_{23})}{\Delta}E_3; \\
C_{22} &= \frac{(1-\nu_{13}\nu_{31})}{\Delta}E_2; \\
C_{23} &= \frac{(\nu_{32}+\nu_{12}\nu_{31})}{\Delta}E_2 = \frac{(\nu_{23}+\nu_{12}\nu_{13})}{\Delta}E_3; \\
C_{33} &= \frac{(1-\nu_{12}\nu_{21})}{\Delta}E_3; \\
C_{44} &= G_{23}; \qquad C_{55} = G_{13}; \qquad C_{66} = G_{12}
\end{aligned}
\tag{20}
$$

with the parameter $\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}$.
We have the condition $\Delta > 0$ to have positiv C_{ii} modulus.

$$
\Delta = 1 - \nu_{12}^2\frac{E_2}{E_1} - \nu_{23}^2\frac{E_3}{E_2} - \nu_{13}\frac{E_3}{E_1} - 2\nu_{12}\nu_{23}\nu_{13}\frac{E_3}{E_1} \tag{21}
$$

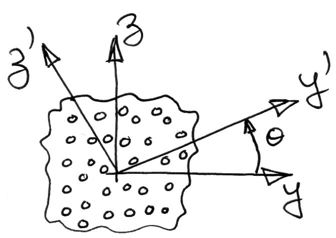
Transversely isotropic, 3D



The compliance matrix in the symmetry axes of the material is given by:

$$
[S] = \left[\begin{array}{c|ccc|cc}
S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\
\hline
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\
0 & \text{Plane(yz)} & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & S_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{array}\right] \tag{22}
$$

The isotropy property in the (yz) plane, subscript (234), yields with equation (14):



$$
[S']_{(yz)plane} = [S]_{(yz)plane}
$$

$$
\Rightarrow S_{44} = 2(S_{22}-S_{23})
$$

5 engineer constants (E_1, E_2, ν_{12}, ν_{23}, G_{12}) are necessary to write the stiffness matrix:

$$
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \quad \begin{array}{c} \text{and the symmetry condition} \\ \frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1} \end{array} \qquad (23)
$$

The inversion of the compliance matrix yields the stiffness matrix:

$$
[C] = \begin{bmatrix} \frac{(1-\nu_{23})}{\delta} E_1 & \frac{\nu_{12}}{\delta} E_2 & \frac{\nu_{12}}{\delta} E_2 & 0 & 0 & 0 \\ & \frac{(1-\nu_{12}\nu_{21})}{(1+\nu_{23})\delta} E_2 & \frac{(\nu_{23}+\nu_{12}\nu_{21})}{(1+\nu_{23})\delta} E_2 & 0 & 0 & 0 \\ & & \frac{(1-\nu_{12}\nu_{21})}{(1+\nu_{23})\delta} E_2 & 0 & 0 & 0 \\ & & & \frac{E_2}{2(1+\nu_{23})} & 0 & 0 \\ & \text{Sym.} & & & G_{12} & 0 \\ & & & & & G_{12} \end{bmatrix} \qquad (24)
$$

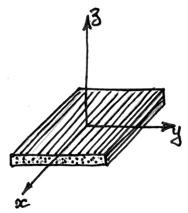
with the parameter $\delta = (1 - \nu_{23} - 2\nu_{12}\nu_{21})$

We can also introduce the 5 engineering constants E_1, E_2, G_{12}, ν_{12} et ν_{23}:

$$
[C] = \begin{bmatrix} \frac{(1-\nu_{23})}{\delta} E_1 & \frac{\nu_{12}}{\delta} E_2 & \frac{\nu_{12}}{\delta} E_2 & 0 & 0 & 0 \\ & \frac{(1-\nu_{12}^2 E_2/E_1)}{(1+\nu_{23})\delta} E_2 & \frac{(\nu_{23}+\nu_{12}^2 E_2/E_1)}{(1+\nu_{23})\delta} E_2 & 0 & 0 & 0 \\ & & \frac{(1-\nu_{12}^2 E_2/E_1)}{(1+\nu_{23})\delta} E_2 & 0 & 0 & 0 \\ & & & \frac{E_2}{2(1+\nu_{23})} & 0 & 0 \\ & \text{Sym.} & & & G_{12} & 0 \\ & & & & & G_{12} \end{bmatrix} \qquad (25)
$$

with the parameter $\delta = (1 - \nu_{23} - 2\nu_{12}^2 \frac{E_2}{E_1})$. We have the condition $\delta > 0$.

UD Laminate in plane stress and in symmetry axes, 2D



The compliance matrix in the symmetry axes of the material has no (σ-γ) coupling components ($S_{16} = S_{26} = 0$) and depends of 4 engineering constants E_1, E_2, G_{12} and ν_{12}:

$$
\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}}_{[S]} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} \quad \begin{array}{c} \text{and symmetry condition} \\ \frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1} \end{array} \qquad (26)
$$

The inversion of the compliance matrix yields the stiffness matrix:

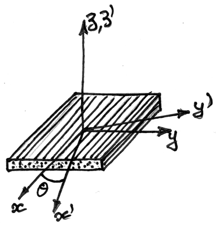
$$
\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{(1-\nu_{12}\nu_{21})} E_1 & \frac{\nu_{21}}{(1-\nu_{12}\nu_{21})} E_1 & 0 \\ \frac{\nu_{12}}{(1-\nu_{12}\nu_{21})} E_2 & \frac{1}{(1-\nu_{12}\nu_{21})} E_2 & 0 \\ 0 & 0 & G_{12} \end{bmatrix}}_{[Q]} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} \qquad (27)
$$

We can also introduce the 4 engineering constants E_1, E_2, G_{12} and ν_{12}:

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{(1-\nu_{12}^2 E_2/E_1)}E_1 & \frac{\nu_{12}}{(1-\nu_{12}^2 E_2/E_1)}E_2 & 0\\ & \frac{1}{(1-\nu_{12}^2 E_2/E_1)}E_2 & 0\\ \text{Sym.} & & G_{12}\end{bmatrix}}_{[Q]} \begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_6\end{pmatrix} \tag{28}$$

We have the condition $(1-\nu_{12}^2 E_2/E_1) > 0$.

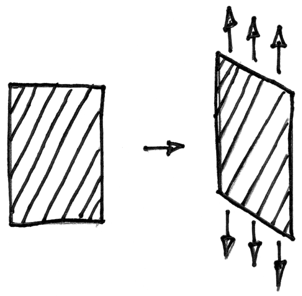
UD Laminate in plane stress and out of symmetry axes, 2D



The compliance matrix out of the symmetry axes of the material has two (σ-γ) coupling components ($S'_{16} \neq 0$, $S'_{26} \neq 0$) and depends of 6 engineering constants E'_1, E'_2, G'_{12}, ν'_{12}, μ'_{12}, η'_{12}:

$$\begin{pmatrix}\epsilon'_1\\ \epsilon'_2\\ \epsilon'_6\end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{E'_1} & -\frac{\nu'_{21}}{E'_2} & \frac{\eta'_{12}}{G'_{12}}\\ & \frac{1}{E'_2} & \frac{\mu'_{12}}{G'_{12}}\\ \text{Sym.} & & \frac{1}{G'_{12}}\end{bmatrix}}_{[S']} \begin{pmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{pmatrix} \tag{29}$$

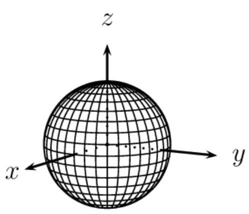
From the components of the laminate in the symmetry axes, the components are calculated with equation (14). The coupling components are equal to zero in the symmetry axes, and they are odd functions of θ:



$$\begin{aligned} \frac{\eta'_{12}}{G'_{12}} &= -2cs\left(\frac{c^2}{E_1} - \frac{s^2}{E_2} + (c^2 - s^2)\left(\frac{\nu_{21}}{E_2} - \frac{1}{2G_{12}}\right)\right)\\ \frac{\mu'_{12}}{G'_{12}} &= -2cs\left(\frac{c^2}{E_1} - \frac{s^2}{E_2} - (c^2 - s^2)\left(\frac{\nu_{21}}{E_2} - \frac{1}{2G_{12}}\right)\right) \end{aligned} \tag{30}$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$.

Isotropic material, 3D



$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0\\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0\\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \tag{31}$$

Two parameters E and ν are necessary to describe the compliance and stiffness matrices:

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1}{G} & 0 & 0 \\ & \text{Sym.} & & & \frac{1}{G} & 0 \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} \quad \begin{matrix} \text{and isotropy condition} \\ G = \frac{E}{2(1+\nu)} \end{matrix} \tag{32}$$

The stiffness matrix is given by:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{bmatrix} \frac{(1-\nu)}{(1+\nu)(1-2\nu)}E & \frac{\nu}{(1+\nu)(1-2\nu)}E & \frac{\nu}{(1+\nu)(1-2\nu)}E & 0 & 0 & 0 \\ & \frac{(1-\nu)}{(1+\nu)(1-2\nu)}E & \frac{\nu}{(1+\nu)(1-2\nu)}E & 0 & 0 & 0 \\ & & \frac{(1-\nu)}{(1+\nu)(1-2\nu)}E & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & \text{Sym.} & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \tag{33}$$

We can define a parameter $d = (1+\nu)(1-2\nu) = (1-\nu-2\nu^2)$.

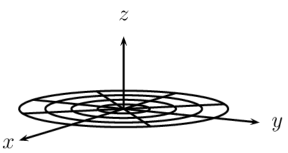
Considering that the diagonal C_{ii} constants must be positive, We have $d > 0$, which means $\nu < 0.5$. A value of ν close to 0.5 is characteristic of an incompressible material. The compression dilatation is given by:

$$\frac{\Delta V}{V} = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) \tag{34}$$

Another representation can be obtained by using the 2 Lame coefficients λ and μ.

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{Sym.} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \quad \begin{matrix} \text{Lamé relations:} \\ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu = \frac{E}{2(1+\nu)} \end{matrix} \tag{35}$$

Isotropic material in plane stress, 2D



Under plane stress condition $\sigma_{33} = \sigma_{31} = \sigma_{23} = 0$, and the Hooke's law takes the form:

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ & \frac{1}{E} & 0 \\ \text{Sym.} & & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} \quad \begin{matrix} \text{and isotropy condition} \\ G = \frac{E}{2(1+\nu)} \end{matrix} \tag{36}$$

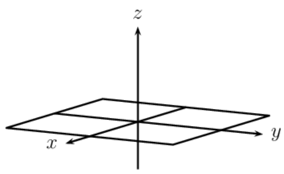
The inversion of the compliance matrix yields:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{(1-\nu^2)}E & \frac{\nu}{(1-\nu^2)}E & 0 \\ & \frac{1}{(1-\nu^2)}E & 0 \\ \text{Sym.} & & G \end{bmatrix}}_{\text{Isotropy condition:}[Q]=[Q']} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} \tag{37}$$

Summary table of all independent parameters of the $[S]$ and $[C]$ matrices

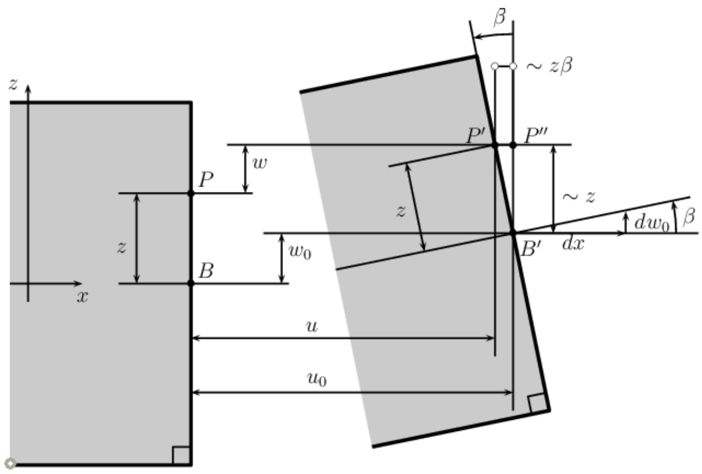
	Number of Independent Parameters	Parameters	Non zero components in Sym. Axes	Non zero components outside Sym. Axes
Hook's law	21	C_{ij}	36	36
Othotropic	9	$E_1, E_2, E_3, \nu_{12}, \nu_{13}, \nu_{23}, G_{12}, G_{13}, G_{23}$	12	36
Trans. Iso.	5	$E_1, E_2, \nu_{12}, \nu_{23}, G_{12}$	12	36
UD Laminate	4	$E_1, E_2, \nu_{12}, G_{12}$	5	9
Isotropic	2	E, ν	12	12

Theory of Love-Kirchhoff for thin shell



The following kinematic assumptions are made in Love-Kirchhoff theory:

- Straight lines normal to the mid-surface remain straight after deformation.
- Straight lines normal to the mid-surface remain normal to the mid-surface after deformation.
- The thickness of the plate does not change during a deformation.



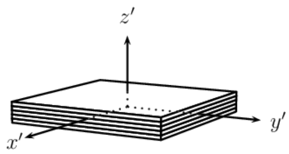
The plane $(z = 0)$ is the mid-plane surface. The displacement of a point $(u\ ,v,\ w)$ outside the mid-plane surface $(u_0,\ v_0,\ w_0)$ is calculated relatively to the mid-plane displacement and its position z, which allows then the determination of the strain tensor on the whole shell (plane condition):

$$\begin{cases} u(x,y,z\neq 0) = u_0 - z\frac{\partial w_0}{\partial x} \\ v(x,y,z\neq 0) = u_0 - z\frac{\partial w_0}{\partial y} \\ w(x,y,z\neq 0) = w_0 \end{cases} \Rightarrow \begin{cases} \epsilon_1 = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \\ \epsilon_2 = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \\ \epsilon_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - z2\frac{\partial^2 w_0}{\partial x \partial y} \end{cases} \tag{38}$$

Then if we introduce a curvature vector $\{k_0\}$ of the mid-plane and the strain vector $\{\epsilon_0\}$ of the mid-plane, the vector $\{\epsilon\}$ representation of the strain tensor for a point on z coordinate is given by :

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} = \begin{pmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_6^0 \end{pmatrix} + z\begin{pmatrix} k_1^0 \\ k_2^0 \\ k_6^0 \end{pmatrix} \Rightarrow \begin{cases} \{\epsilon\} = \{\epsilon_0\} + z\{k_0\} \\ \{\sigma\} = [Q]\{\epsilon\} = [Q]\{\epsilon_0\} + z[Q]\{k_0\} \end{cases} \tag{39}$$

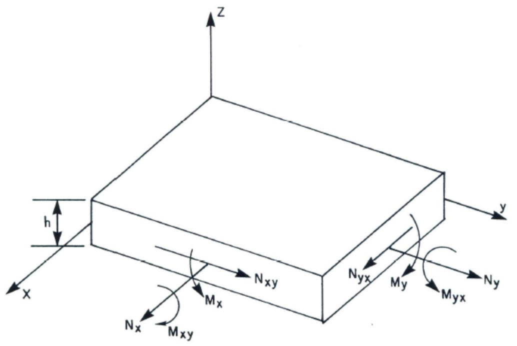
Classical Laminate Theory of thin plate



The classical Laminate Theory is based on the Love-Kirchhoff theorie with the following hypothesis:

- The plate is constructed of an arbitrary number of layers of orthotropic sheets bonded together. However, the orthotropic axes of material symmetry of an individual layer need not coincide with the x-y axes of the plate.
- The plate is thin, i.e., the thickness h is much smaller than the other physical dimensions.
- The displacements u, v, and w are small compared to the plate thickness.
- Properties of plate are represented by reference plane: the CLT is applicable preferably to symmetric laminates
- In-plane strains σ_x , σ_y and σ_{xy} are small compared to unity.
- Transverse shear strains σ_{xz} and σ_{yz} are negligible.
- Tangential displacements u and v are linear functions of the z coordinate.
- The transverse normal strain σ_z is negligible.
- Each ply obeys Hooke's law.
- No slip between layers
- The plate has constant thickness.

The forces applied to a small part of the laminate, can be described by 6 components in classical shell theory: 3 inplane forces and 3 moments. In a practical notation, the inplane forces and the momnets are calculated per unit length. The inplane forces are denoted by N_i, $(i = 1, 2, 6)$, for all component of the plane stress condition in i direction. The 3 moments per length are denoted M_i, $(i = 1, 2, 6)$ for the i direction. M_{xy} is a *twisting* moment and can be associated to a distortion of the laminate in the bending mode due to shear stress gradient. N_i and M_i are sometimes called respectively *stress resultants* and *moment resultants*.

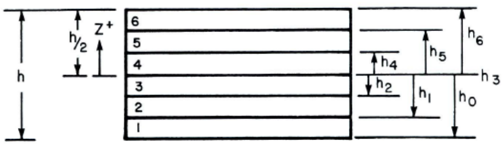


$$N_i = \int_{-h/2}^{h/2} \sigma_i \, dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \sigma_i^k \, dz \quad (40)$$

$$M_i = \int_{-h/2}^{h/2} z\sigma_i \, dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} z\sigma_i^k \, dz \quad (41)$$

The matrix notation is:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{pmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{pmatrix} \epsilon_1^0 \\ \epsilon_2^0 \\ \epsilon_6^0 \\ k_1^0 \\ k_2^0 \\ k_6^0 \end{pmatrix} \quad (42)$$

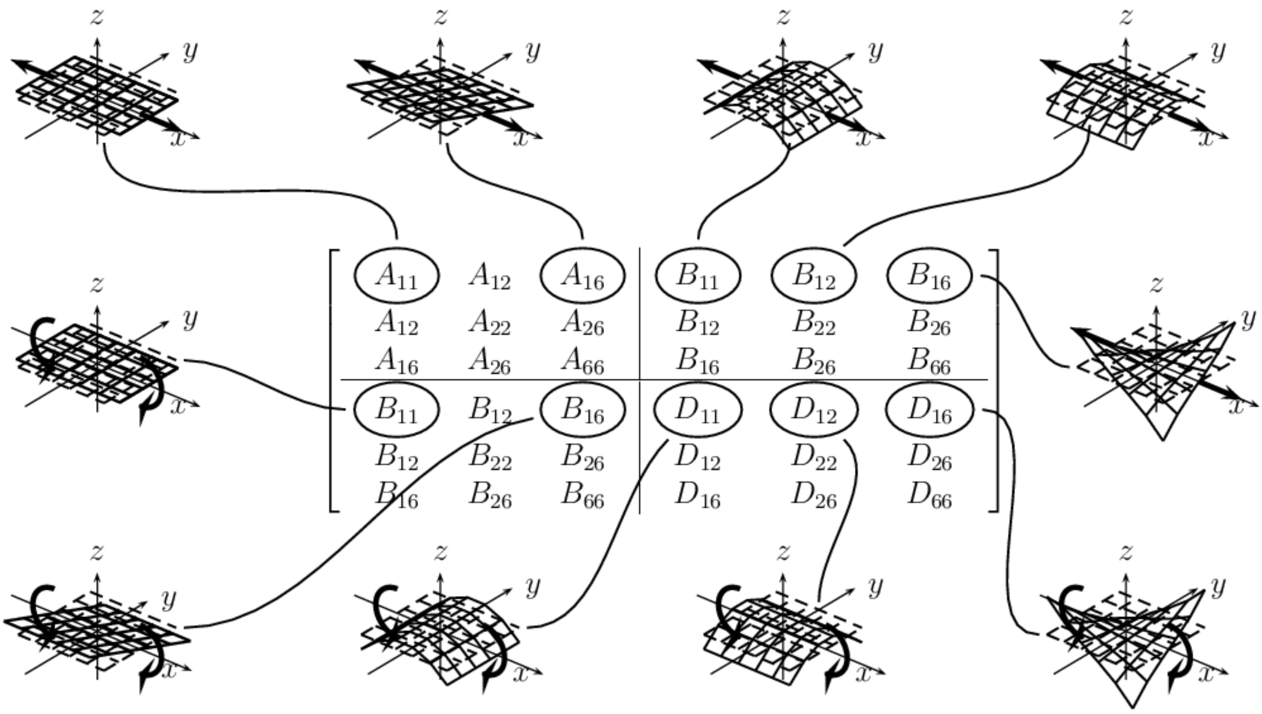


$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{k} \left(h_k - h_{k-1}\right) \tag{43}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^{k} \left(h_k^2 - h_{k-1}^2\right) \tag{44}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{k} \left(h_k^3 - h_{k-1}^3\right) \tag{45}$$

ABD matrix and mechanical behaviors



Laminate notation

Square Brackets [. . .] denote a sequence of ply orientations: $[0, 90]_s$ = 0/90/90/0.
Subscript s $[\dots]_s$ indicates symmetry: $[0, 90]_s$ = 0/90/90/0. In this case, each layer is exactly minored about the geometric mid-plane in terms of its properties, thickness and orientation.
Repeated plies $(\dots)_n$ or repeated sequences $[\dots]_n$ are denoted by a numerical subscript n.

$[0_3, 90_2]_s$ = 0/0/0/90/90/90/90/0/0/0
$[0, (+45,-45)_2]_s$ = 0/+45/-45/+45/-45/-45/+45/-45/+45/0
$[+45,-45]_2$ = +45/-45/+45/-45

When no Subscript is present, the total description of the laminate lay-up is given by the sequence:

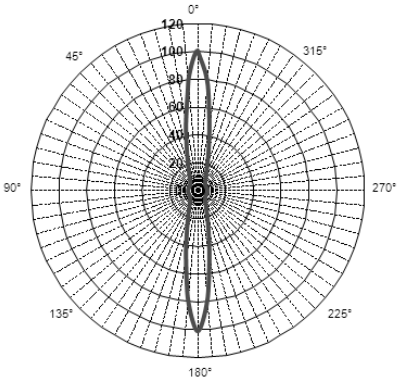
[0,+60,+120] = 0/+60/+120.

Cross-ply laminate: only 0° and 90° plies for the laminate.
Angled-ply laminate: only two angles θ and $-\theta$ for the ply orientation.

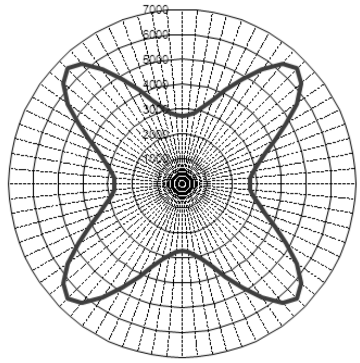
$[A]$ matrix components in relation to θ for an UD laminate

The components of the A matrix can be represented in a circular graphic with the variation of the θ angle of the laminate. The principal Young'modulus, the shear modulus and the Poisson's ratio are represented respectively by A_{11}, A_{33} and A_{12}/A_{11}. Calculations have been performed with 'LamiCens' (source: http://www.r-g.de/laminatberechnung.html).

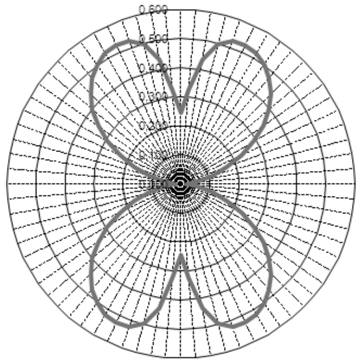
A_{11}



A_{33}

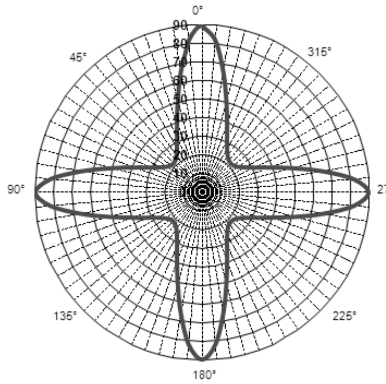


A_{12}/A_{11}

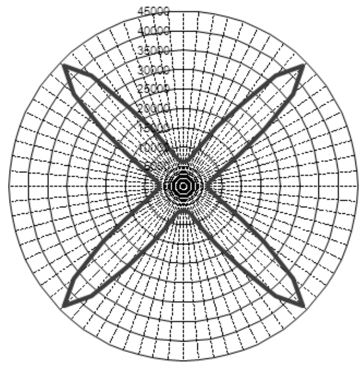


$[A]$ matrix components in relation to θ for a specially orthotropic laminate $[(0,90)_n]_s$

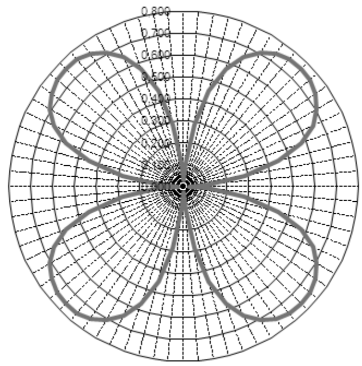
A_{11}



A_{33}

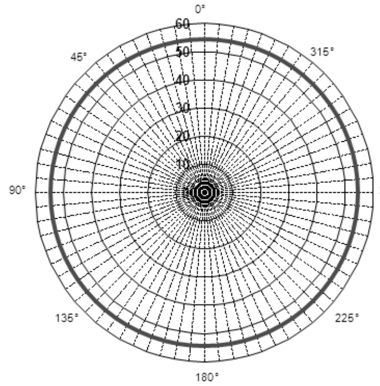


A_{12}/A_{11}

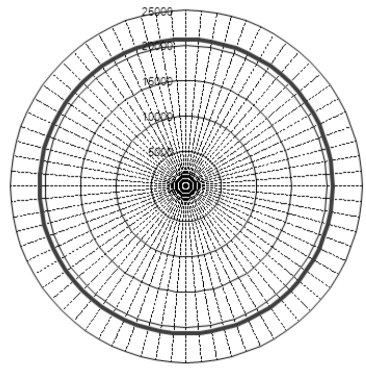


$[A]$ matrix components in relation to θ for a quasi-isotropic laminate $[0,60,-60]_s$

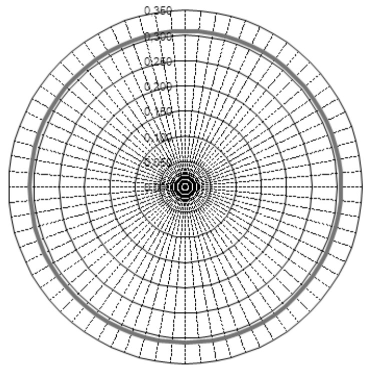
A_{11}



A_{33}



A_{12}/A_{11}



References

- P. Bompard, "Matériaux composites", Ecole Centrale Paris, in French, Master's Degree Lecture, (1993).
- D. Gay and S.V. Hoa, "Composite materials, Design and application", CRC Press, second edition, ISBN: 978-1-4200-4519-2, (2007).
- E. Barbero, "Finite element analysis of composite materials", CRC Press, ISBN:978-1-4200-5433-0, (2008).
- L. Gornet, "Généralités sur les matériaux composites", Ecole Centrale Nantes, in French, 43 pages, *http://cel.archives-ouvertes.fr*, (2008).
- J.M. Berthelot, "Matériaux composites : Comportement mécanique et analyse des structures", 4ème édition , Paris : Editions Tec& Doc, ISBN: 978-2743007713, (2005).
- "Faserverbundwerkstoffe Handbuch, Composite materials Handbook", R&G Faserverbundwerkstoffe GmbH, *www.r-g.de*, (2009).
- M.J. Hinton, A.S. Kaddour, P.D. Soden, "Failure Criteria in Fibre Reinforced Polymer Composites: The World-Wide Failure Exercise (WWFE)", Elsevier, First edition 2004, ISBN: 0-08-044475-X

Softwares

- **Maxima**: Computer algebra system, matrix manipulation, *http://maxima.sourceforge.net*.
- **Alfalam**: Laminate theory, MS Excel, $[ABD]$ matrix, Puck failure criteria, *www.klub.tu-darmstadt.de*.
- **eLamX**: Laminate theory, Java, $[ABD]$ matrix, 3D failure envelope plots, *http://tu-dresden.de*.
- **LamiCens**: Laminate theory, MS Excel, $[A]$ Matrix, Graphics, *www.r-g.de*.
- **Texgen**: Generation of texture, *http://texgen.sourceforge.net*.
- **Code_Aster**: Finite Element Modeling, FEM, *www.caelinux.com*.
- **MyRTM**: RTM simulation, *www.iwk.hsr.ch*.
- **CADEC**: Computer Aided Design Environment for Composites, Laminate theory, $[ABDH]$ matrix, FEM application, *www.mae.wvu.edu/∼barbero/cadec.html*

Author

Dr. Vincent Calard
vincent.calard@centraliens.net

Extended version available soon at *www.lulu.com*