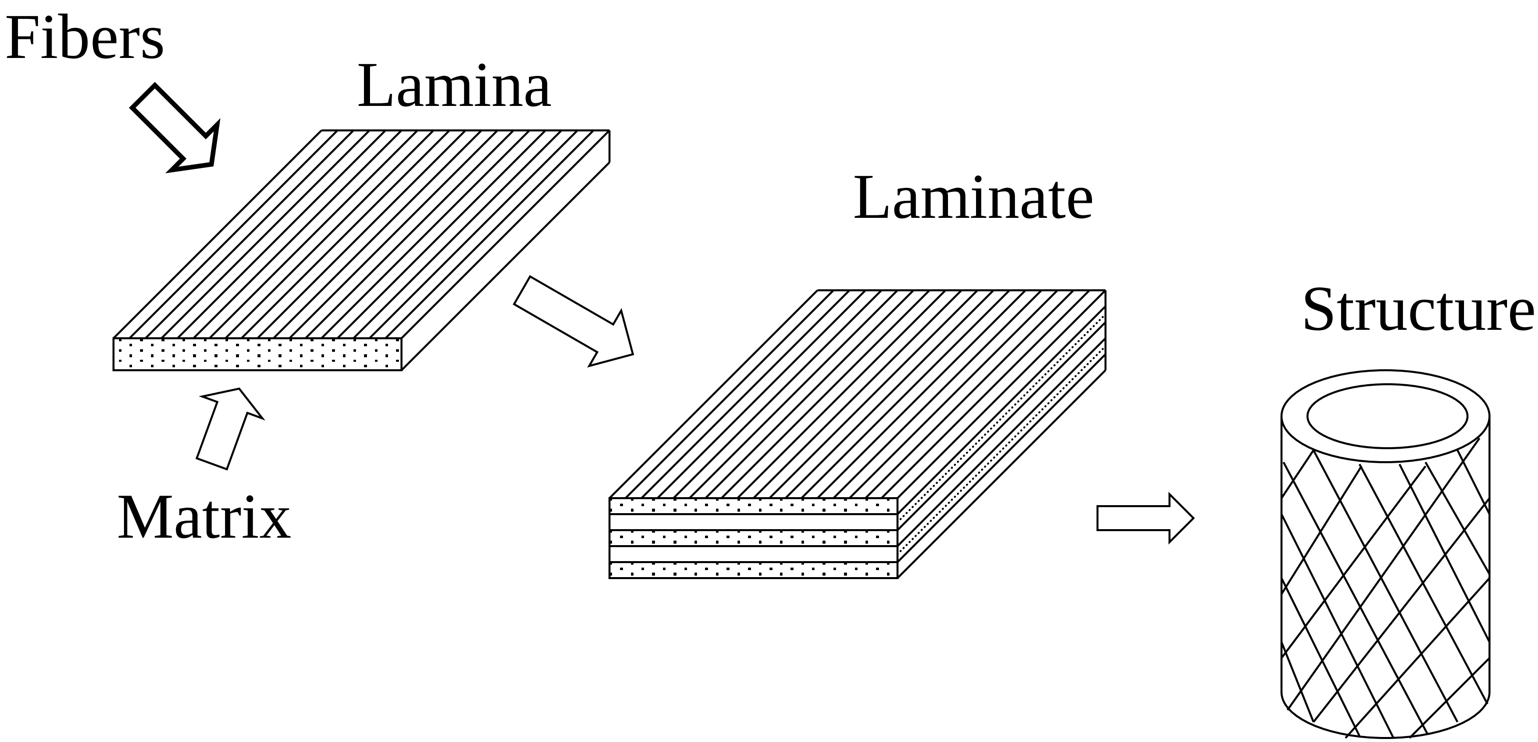
Micromechanics

see page 55

Macromechanics



Macromechanics

Study of stress-strain behavior of composites using effective properties of an equivalent homogeneous material. Only the globally averaged stresses and strains are considered, not the local fiber and matrix values.

Stress-Strain Relationships for Anisotropic Materials

First, we discuss the form of the stress-strain relationships at a point within the material, then discuss the concept of effective moduli for heterogeneous materials where properties may vary from point-to-point.

General Form of Elastic σ-ε Relationships for Constant Environmental Conditions

$$\sigma_{ij} = F_{ij}(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \ldots), \qquad i, j = 1, 2, 3\ldots \quad (2.1)$$

Each component of stress, σ_{ij}, is related to each of nine strain components, ε_{ij}

(Note: These relationships may be nonlinear)

Expanding F_{ij} in a Taylor's series and Retaining only the first order terms,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \qquad i, j, k, l = 1,2,3$$

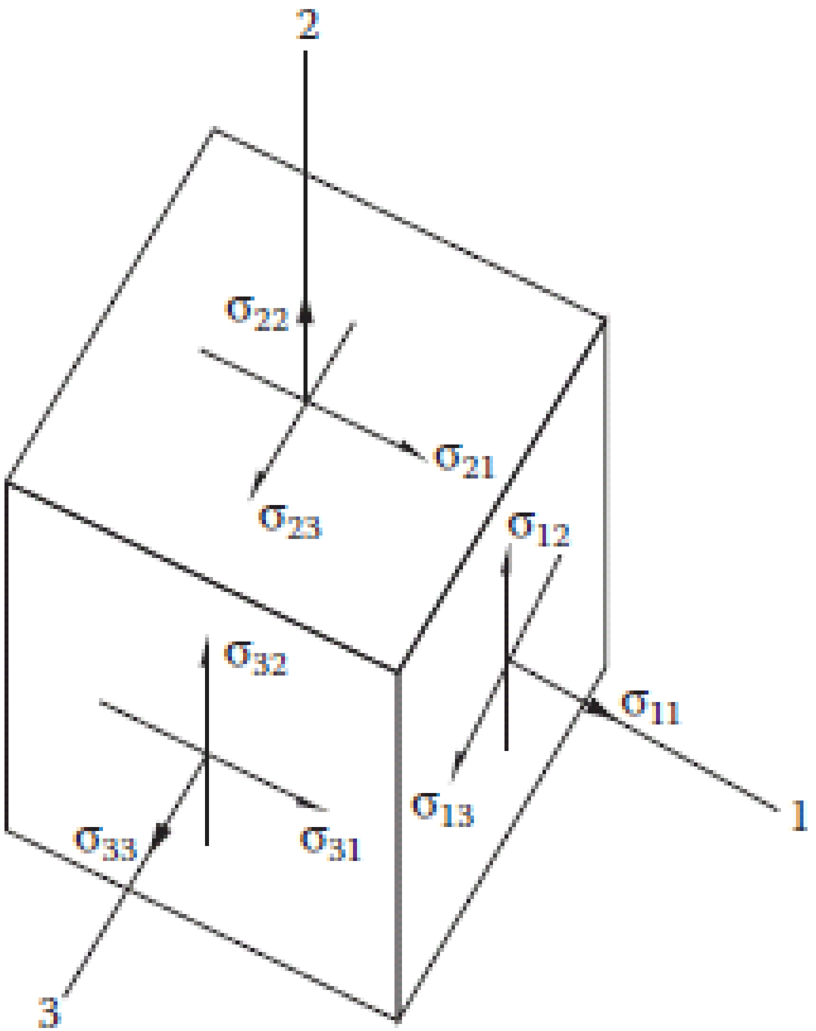
for a linear elastic material

$$\sigma_{ij} - 9 \ components$$

$$\varepsilon_{kl} - 9 \ components$$

$$C_{ijkl} - 81 \ components$$

3D state of stress

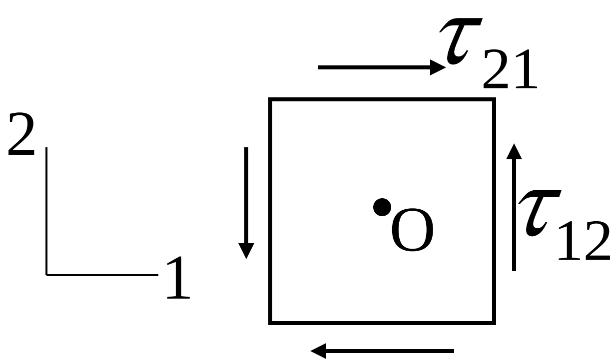


Generalized Hooke's Law for Anisotropic Material

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\\\sigma_{32}\\\sigma_{13}\\\sigma_{21}\end{bmatrix}=\begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & \vdots & C_{1132} & C_{1113} & C_{1121}\\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & \vdots & C_{2232} & C_{2213} & C_{2221}\\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & \vdots & C_{3332} & C_{3313} & C_{3321}\\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} & \vdots & C_{2332} & C_{2313} & C_{2321}\\
C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} & \vdots & C_{3132} & C_{3113} & C_{3121}\\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} & \vdots & C_{1232} & C_{1213} & C_{1221}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3231} & C_{3212} & \vdots & C_{3232} & C_{3213} & C_{3221}\\
C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1331} & C_{1312} & \vdots & C_{1332} & C_{1313} & C_{1321}\\
C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & \vdots & C_{2132} & C_{2113} & C_{2121}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{31}\\\varepsilon_{12}\\\varepsilon_{32}\\\varepsilon_{13}\\\varepsilon_{21}\end{bmatrix} \tag{2.2}$$

Symmetry Simplifies the Generalized Hooke's Law

1. Symmetry of shear stresses and strains:



$$\text{Static Equlibrium}$$

$$\sum M_0 = 0 \text{ implies } \tau_{12} = \tau_{21}$$

$$\text{or in general,}$$

$$\tau_{ij} = \tau_{ji} \text{ or } \sigma_{ij} = \sigma_{ji}$$

Same condition for shear strains, $\varepsilon_{ij} = \varepsilon_{ji}$

2. Material property symmetry – several types will be discussed.

Symmetry of shear stresses and shear strains:

$$\sigma_{ij} = \sigma_{ji} \quad and \quad \varepsilon_{ij} = \varepsilon_{ji}$$

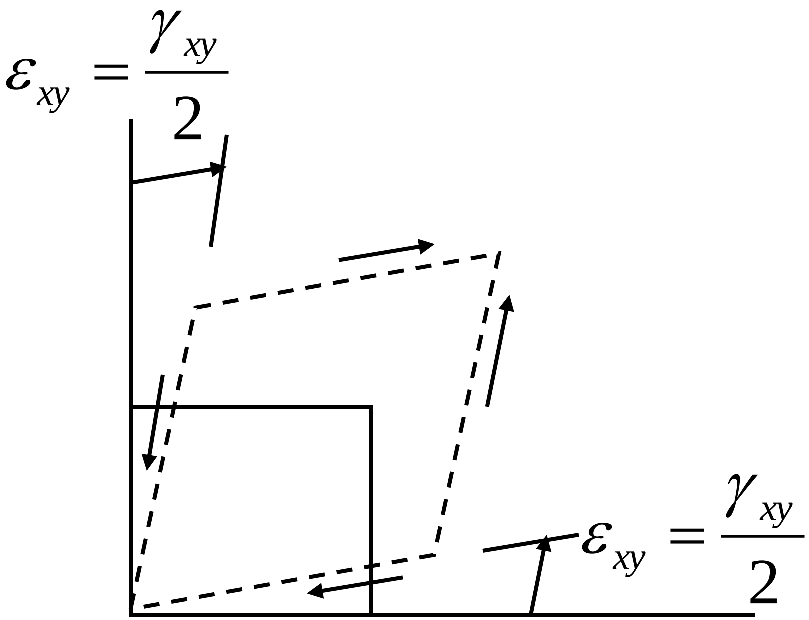
Thus, only 6 components of σ_{ij} are independent, and likewise for ε_{ij}.

This leads to a contracted notation.

Stresses	
Tensor Notation	Contracted Notation
σ_{11}	σ_1
σ_{22}	σ_2
σ_{33}	σ_3
$\sigma_{23} = \sigma_{32}$	σ_4
$\sigma_{13} = \sigma_{31}$	σ_5
$\sigma_{12} = \sigma_{21}$	σ_6

Strains	
Tensor Notation	Contracted Notation
ε_{11}	ε_{1}
ε_{22}	ε_{2}
ε_{33}	ε_{3}
$2\varepsilon_{23} = 2\varepsilon_{32} = \gamma_{23} = \gamma_{32}$	ε_{4}
$2\varepsilon_{13} = 2\varepsilon_{31} = \gamma_{13} = \gamma_{31}$	ε_{5}
$2\varepsilon_{12} = 2\varepsilon_{21} = \gamma_{12} = \gamma_{21}$	ε_{6}

Geometry of Shear Strain



γ_{xy} = Engineering Strain

ε_{xy} = Tensor Strain

Total change in original angle = γ_{xy}

Amount each edge rotates = $\gamma_{xy}/2 = \varepsilon_{xy}$

Using contracted notation

$$\sigma_i = C_{ij}\varepsilon_j \qquad i, j = 1,2,...,6 \tag{2.3}$$

or in matrix form $\{\sigma\} = [C]\{\varepsilon\}$ (2.4)

where $\{\sigma\}$ and $\{\varepsilon\}$ are column vectors and $[C]$ is a 6x6 matrix (the stiffness matrix)

Alternatively,

$$\varepsilon_i = S_{ij}\sigma_j \qquad i, j = 1,2,...,6 \tag{2.5}$$

or $$\{\varepsilon\} = [S]\{\sigma\} \tag{2.6}$$

where [S] = compliance matrix

and $$[S] = [C]^{-1}$$

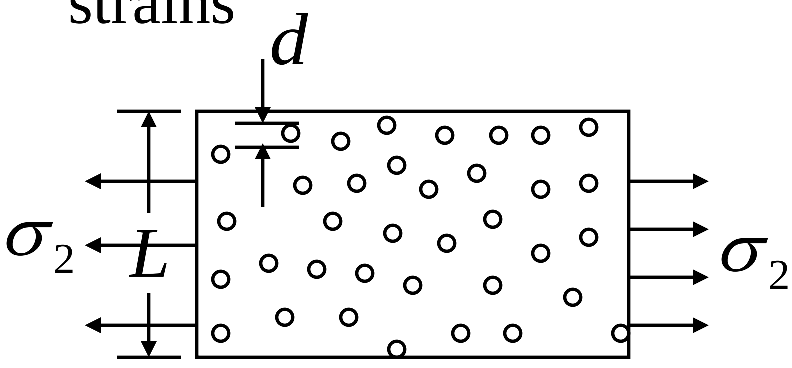
Expanding:

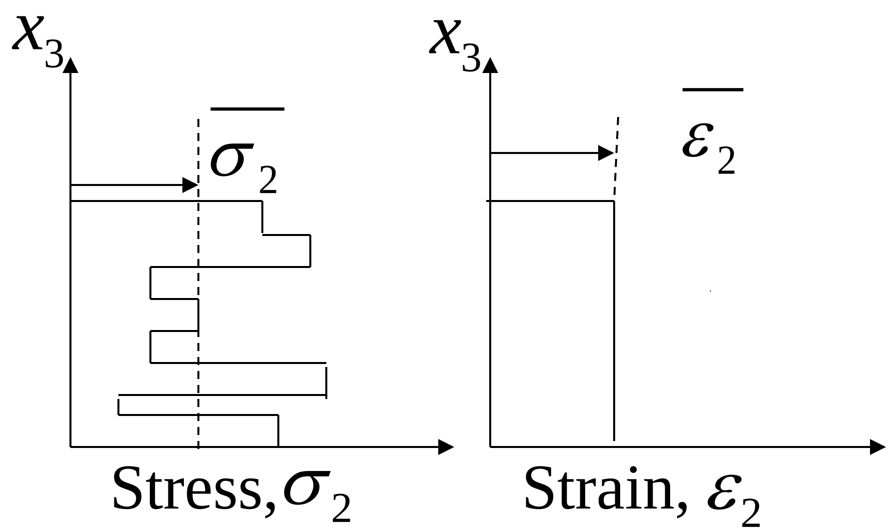
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

- Up to now, we only considered the stresses and strains at a point within the material, and the corresponding elastic constants at a point.
- What do we do in the case of a composite material, where the properties may vary from point to point?
- Use the concept of effective moduli of an equivalent homogeneous material.

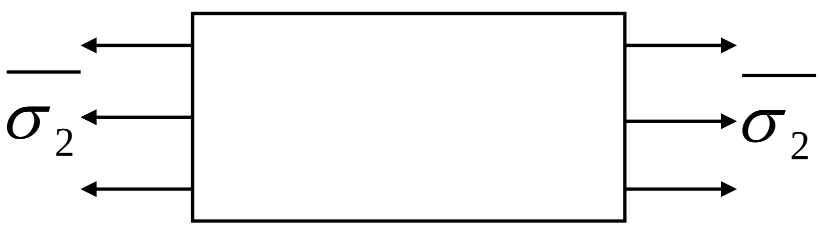
Concept of an Effective Modulus of an Equivalent Homogeneous Material

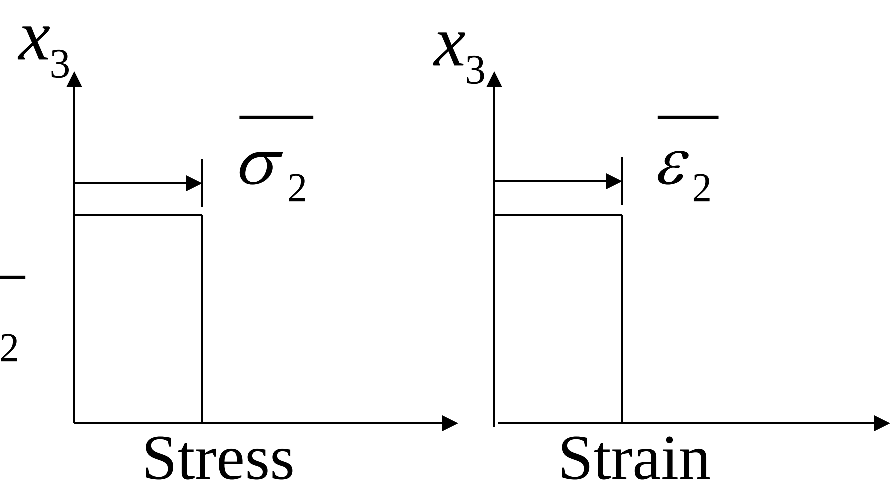
Heterogeneous composite under varying stresses and strains





Equivalent homogeneous material under average stresses and strains





Effective moduli, C_{ij}

$$\overline{\sigma_i} = C_{ij}\overline{\varepsilon_j} \tag{2.9}$$

where,

$$\overline{\sigma_i} = \text{average stress} = \frac{\int_v \sigma_i dv}{\int_v dv} \tag{2.7}$$

$$\overline{\varepsilon_i} = \text{average strain} = \frac{\int_v \varepsilon_i dv}{\int_v dv} \tag{2.8}$$

3-D Case
General Anisotropic Material

- [C] and [S] each have 36 coefficients, but only 21 are independent due to symmetry.
- Symmetry shown by consideration of strain energy.
- Proof of symmetry:

Define strain energy density

$$W = \frac{1}{2}\sigma_i \varepsilon_i \quad (i = 1,2,...,6)$$

$$W = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_1\varepsilon_1 + ... + \frac{1}{2}\sigma_6\varepsilon_6$$

$$but \qquad \sigma_i = C_{ij}\varepsilon_j$$

$$\therefore W = \frac{1}{2}C_{ij}\varepsilon_i\varepsilon_j \qquad (2.12)$$

Now, differentiate:

$$\frac{\partial W}{\partial \varepsilon_i} = \frac{1}{2}C_{ij}\varepsilon_j + \frac{1}{2}C_{ij}\varepsilon_i\frac{\partial \varepsilon_j}{\partial \varepsilon_i}$$

but $\frac{\partial \varepsilon_j}{\partial \varepsilon_i} = \delta_{ij} = Kronec\text{ker}\, delta = 1$ if i=j

0 if i≠j

$$\therefore \delta_{ij}\varepsilon_i = \varepsilon_j \qquad \text{(show)}$$

$$\therefore \frac{\partial W}{\partial \varepsilon_i} = C_{ij}\varepsilon_j \tag{2.11}$$

$$\therefore \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij} \tag{2.13}$$

But if the order of differentiation is reversed,

$$\therefore \frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = C_{ji} \tag{2.14}$$

Since order of differentiation is immaterial,

$$C_{ij} = C_{ji} \quad \text{(Symmetry)}$$

Similarly,

$$W = \frac{1}{2} S_{ij} \sigma_i \sigma_j$$

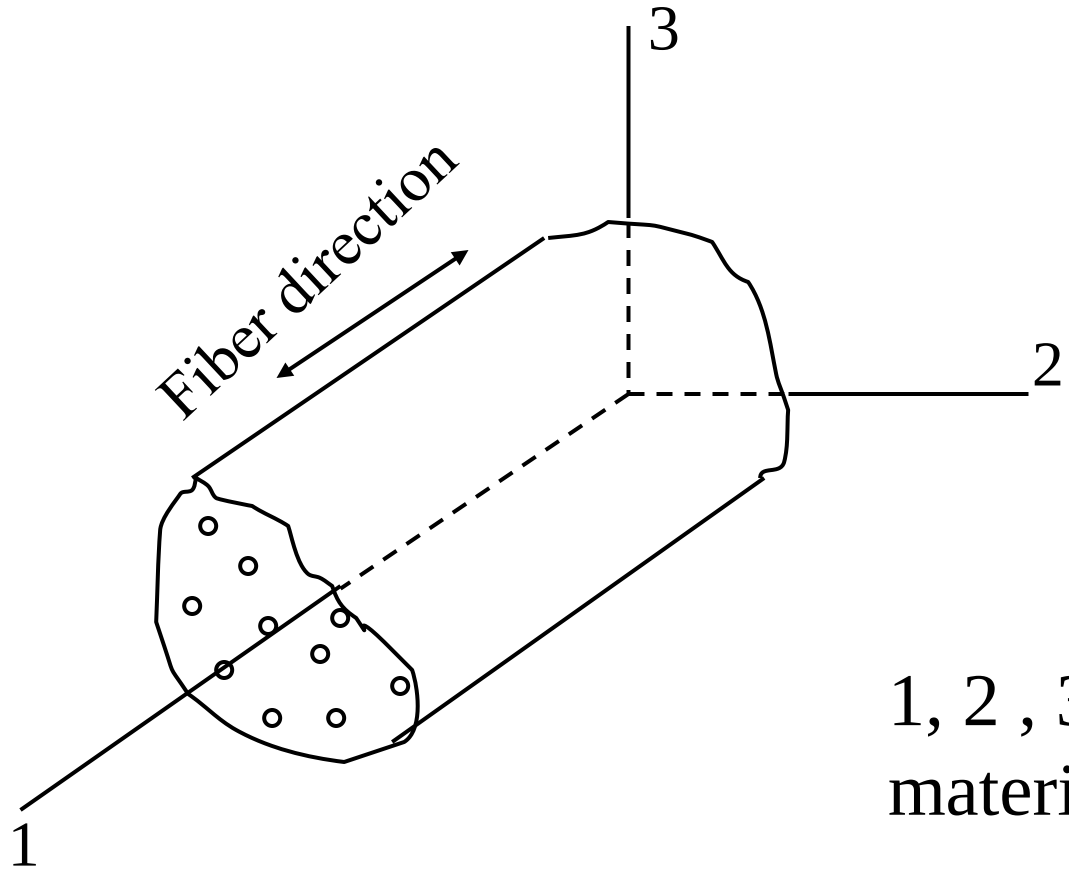
and $$S_{ij} = S_{ji}$$

$\therefore$ Only 21 of 36 coefficients are independent for anisotropic material.

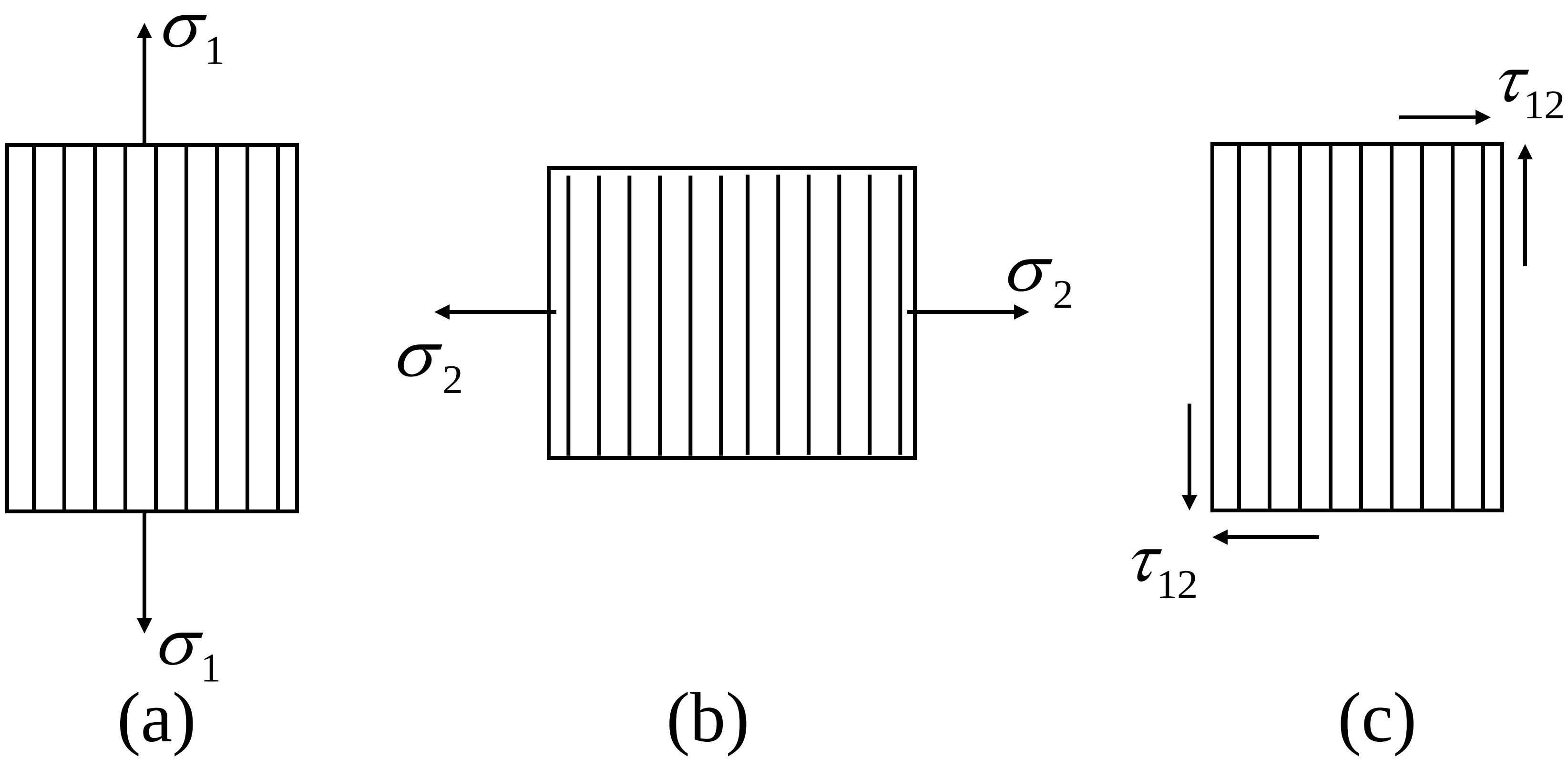
Stiffness matrix for linear elastic anisotropic material with no material property symmetry

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & \text{SYM} & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \qquad (2.15)$$

3-D Case, Specially Orthotropic

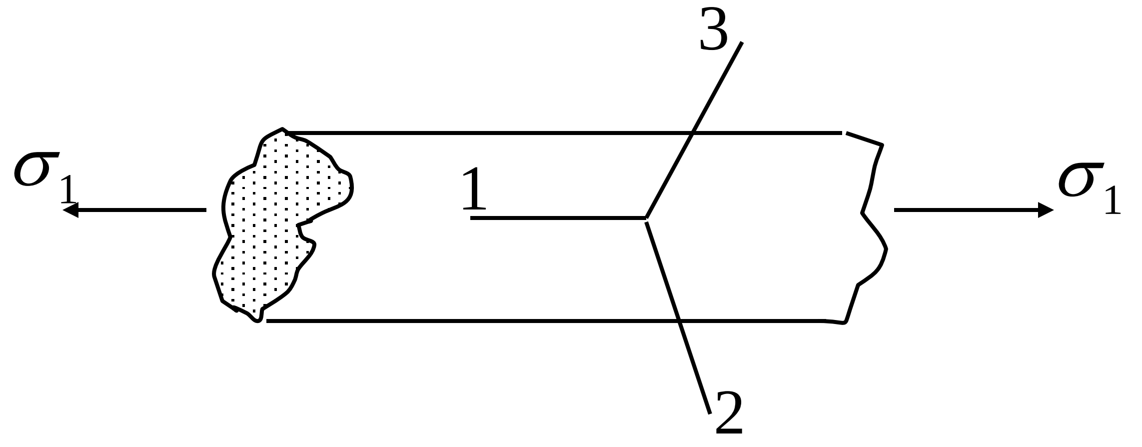


1, 2 , 3 principal
material coordinates



Simple states of stress used to define lamina engineering constants for specially orthotropic lamina.

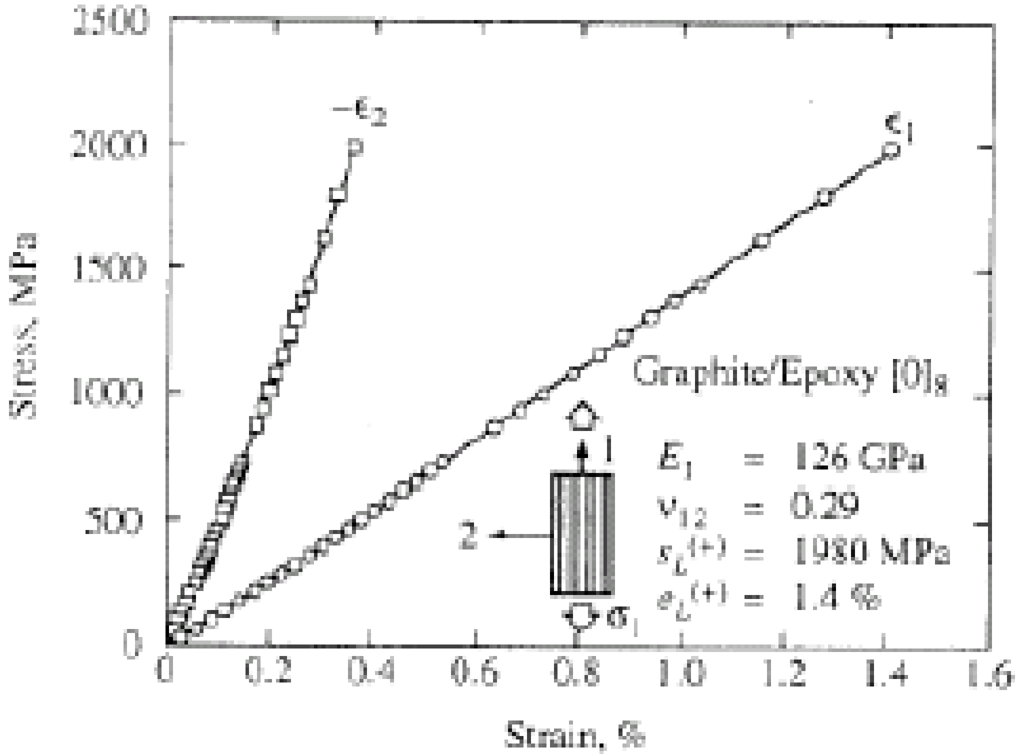
Consider normal stress σ_1 alone:



Resulting strains,

$$\varepsilon_1 = \frac{\sigma_1}{E_1}; \quad \varepsilon_2 = -\upsilon_{12}\varepsilon_1 = -\upsilon_{12}\frac{\sigma_1}{E_1} \qquad (2.19)$$

Typical stress-strain curves from ASTM D3039 tensile tests



Stress-strain data from longitudinal tensile test of carbon/epoxy composite. Reprinted from ref. [8] with permission from CRC Press.

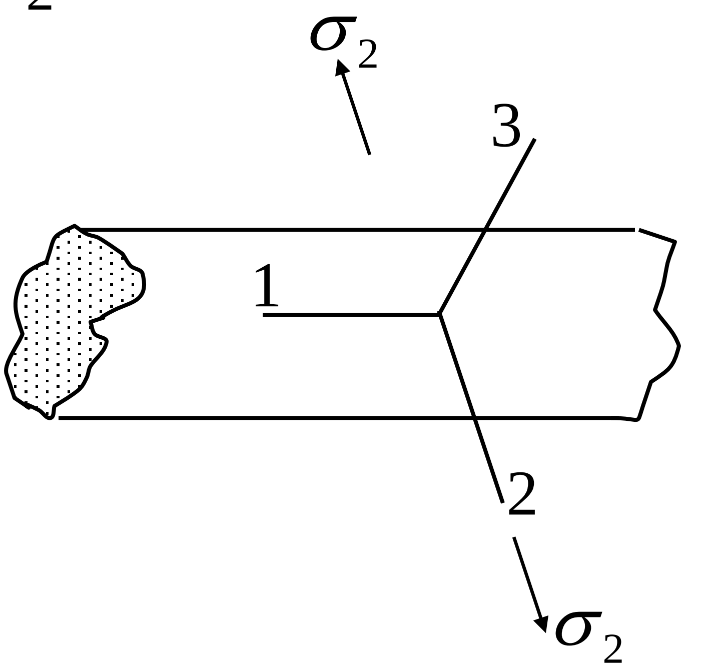
Similarly,

$$\varepsilon_3 = -\upsilon_{13}\varepsilon_1 = -\upsilon_{13}\frac{\sigma_1}{E_1}$$

where E_1 = longitudinal modulus

υ_{ij} = Poisson's ratio for strain along j direction due to loading along i direction

Now consider normal stress σ_2 alone:



Strains:

$$\varepsilon_2 = \frac{\sigma_2}{E_2};$$

$$\varepsilon_1 = -\upsilon_{21}\varepsilon_2 = -\upsilon_{21}\frac{\sigma_2}{E_2}$$

$$\varepsilon_3 = -\upsilon_{23}\varepsilon_2 = -\upsilon_{23}\frac{\sigma_2}{E_2} \qquad (2.20)$$

Where E_2 = transverse modulus

Similar result for σ_3 alone

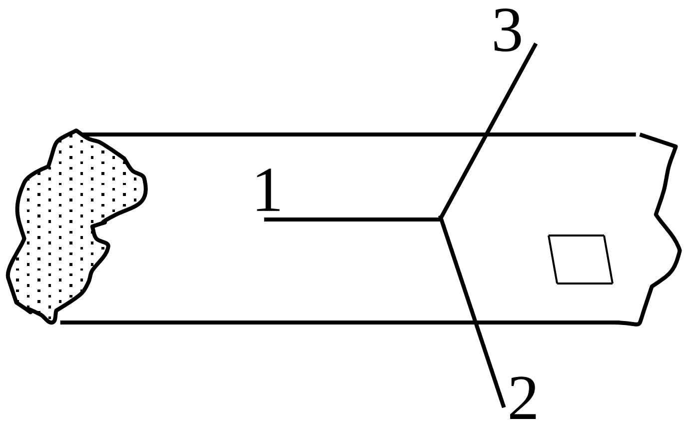
- <u>Observation:</u>

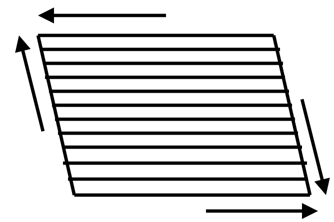
  All shear strains are zero under pure normal stress (no shear coupling).

$$\therefore \gamma_{12} = \gamma_{13} = \gamma_{23} = 0$$

For $\sigma_1, \ \sigma_2, \ \sigma_3$ alone

Now, consider shear stress τ_{12} alone,





$$\tau_{12} = \sigma_{12}$$

Strain $\gamma_{12} = \dfrac{\tau_{12}}{G_{12}}$

Where G_{12} = Shear modulus in 1-2 plane

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_{13} = \gamma_{23} = 0 \qquad (2.21)$$

(No shear coupling)

Similarly, for τ_{13} alone

$$\gamma_{13} = \frac{\tau_{13}}{G_{13}};\ \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_{12} = \gamma_{23} = 0$$

and for τ_{23} alone

$$\gamma_{23} = \frac{\tau_{23}}{G_{23}};\ \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \gamma_{13} = \gamma_{12} = 0$$

Now add strains due to all stresses using superposition

Specially Orthotropic 3D Case

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\upsilon_{21}}{E_2} & -\frac{\upsilon_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\upsilon_{12}}{E_1} & \frac{1}{E_2} & -\frac{\upsilon_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\upsilon_{13}}{E_1} & -\frac{\upsilon_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} \tag{2.22}$$

12 coefficients, but only are 9 independent

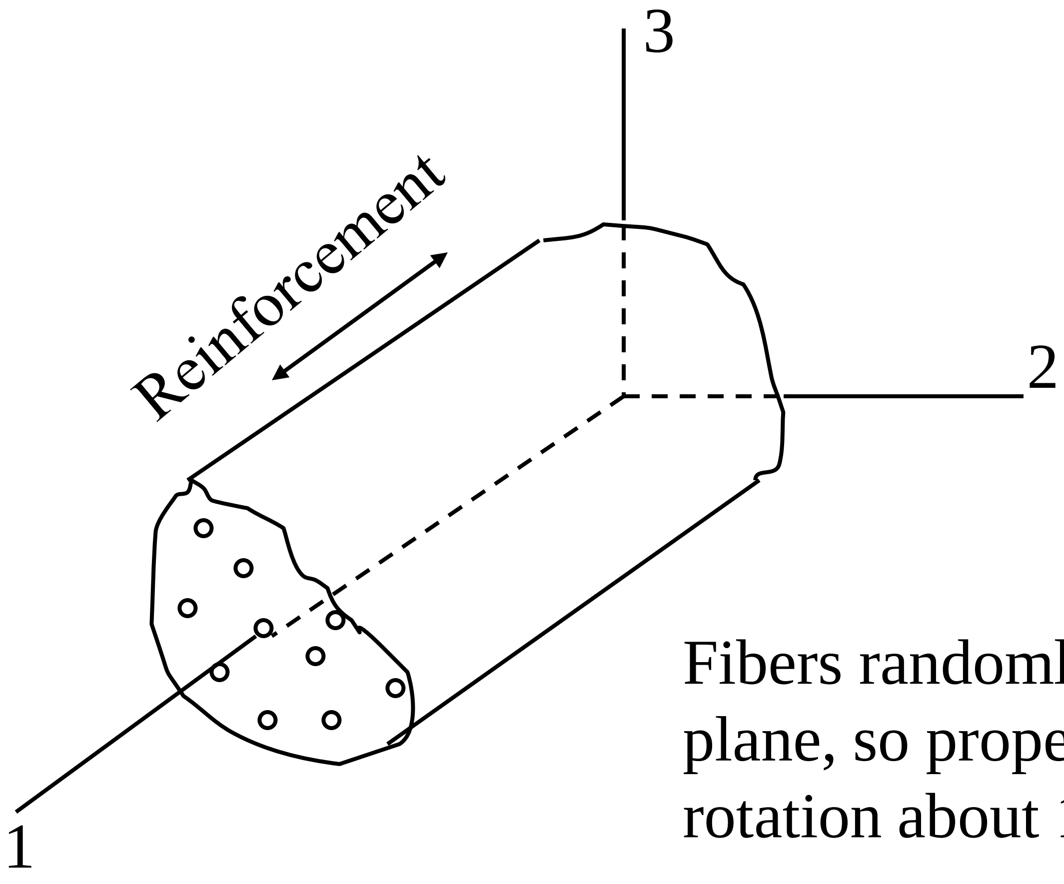
Symmetry: $S_{ij} = S_{ji}$

$$\therefore \frac{\upsilon_{ij}}{E_i} = \frac{\upsilon_{ji}}{E_j}$$

$\therefore$ Only 9 independent coefficients.

Generally orthotropic 3-D case –
similar to anisotropic with 36 nonzero coefficients, but 9 are independent as with specially orthotropic case

Specially Orthotropic – Transversely Isotropic



Fibers randomly packed in 2-3 plane, so properties are invariant to rotation about 1-axis (2 same as 3)

Specially orthotropic, transversely isotropic (2 and 3 interchangeable)

$$G_{13} = G_{12}, \quad E_2 = E_3, \quad \upsilon_{21} = \upsilon_{31}$$

$$G_{23} = \frac{E_2}{2(1+\upsilon_{32})} \qquad (2.23)$$

Now, only 5 coefficients are independent.

Isotropic

$$G_{13} = G_{23} = G_{12} = G$$

$$E_1 = E_2 = E_3 = E$$

$$\upsilon_{12} = \upsilon_{23} = \upsilon_{13} = \upsilon$$

$$G = \frac{E}{2(1+\upsilon)}$$

2 independent coefficients

Usually measure E, υ – calculate G

Isotropic – 3D case

$$\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{Bmatrix}=\begin{bmatrix}\frac{1}{E} & -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0\\ -\frac{\upsilon}{E} & \frac{1}{E} & -\frac{\upsilon}{E} & 0 & 0 & 0\\ -\frac{\upsilon}{E} & -\frac{\upsilon}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{bmatrix}\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{Bmatrix}$$

Same form for any set of coordinate axes

3-D Isotropic – stresses in terms of strains

$$\sigma_x = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_x + \upsilon(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_y + \upsilon(\varepsilon_x + \varepsilon_z)\right]$$

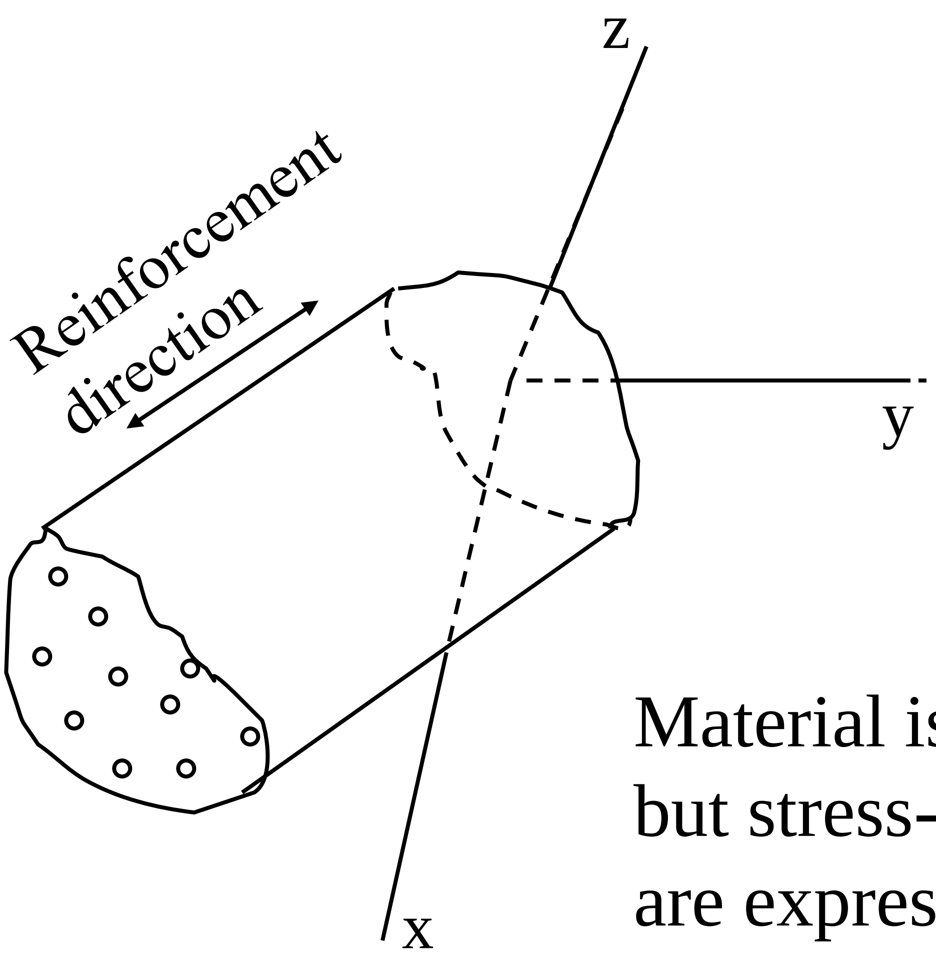
$$\sigma_z = \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\varepsilon_z + \upsilon(\varepsilon_x + \varepsilon_y)\right]$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

3-D Case, Generally Orthotropic



Material is still orthotropic, but stress-strain relations are expressed in terms of non-principal xyz axes

Generally Orthotropic

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{S}_{15} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} & \bar{S}_{25} & \bar{S}_{26} \\ \bar{S}_{31} & \bar{S}_{32} & \bar{S}_{33} & \bar{S}_{34} & \bar{S}_{35} & \bar{S}_{36} \\ \bar{S}_{41} & \bar{S}_{42} & \bar{S}_{43} & \bar{S}_{44} & \bar{S}_{45} & \bar{S}_{46} \\ \bar{S}_{51} & \bar{S}_{52} & \bar{S}_{53} & \bar{S}_{54} & \bar{S}_{55} & \bar{S}_{56} \\ \bar{S}_{61} & \bar{S}_{62} & \bar{S}_{63} & \bar{S}_{64} & \bar{S}_{65} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

Same <u>form</u> as anisotropic, with 36 coefficients, but 9 are independent as with specially orthotropic case

Elastic coefficients in the stress-strain relationship for different materials and coordinate systems

Material and coordinate system	Number of nonzero coefficients	Number of independent coefficients
Three – dimensional case		
Anisotropic	36	21
Generally Orthotropic (nonprincipal coordinates)	36	9
Specially Orthotropic (Principal coordinates)	12	9
Specially Orthotropic, transversely isotropic	12	5
Isotropic	12	2
Two – dimensional case (lamina)		
Anisotropic	9	6
Generally Orthotropic (nonprincipal coordinates)	9	4
Specially Orthotropic (Principal coordinates)	5	4
Balanced orthotropic, or square symmetric (principal coordinates)	5	3
Isotropic	5	2

2-D Cases

Use 3-D equations with,

$$\sigma_3 = \tau_{13} = \tau_{23} = 0$$

Plane stress,

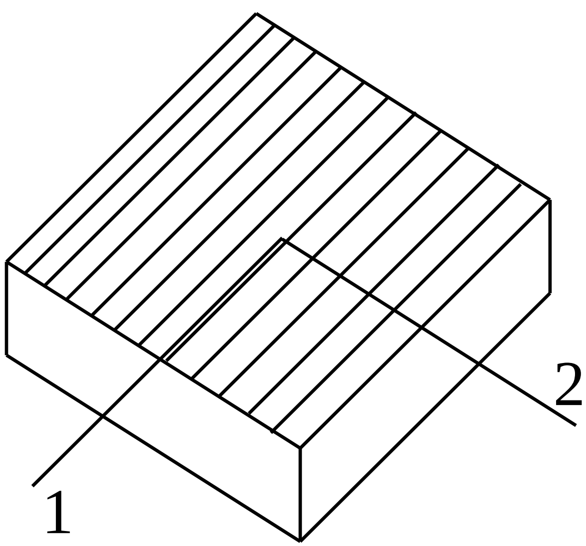
$$\sigma_1, \quad \sigma_2, \quad \tau_{12}, \neq 0$$

Or

$$\sigma_x, \quad \sigma_y, \quad \tau_{xy}, \neq 0$$

Specially Orthotropic Lamina

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (2.24)$$



Or

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (2.26)$$

5 Coefficients - 4 independent

Specially Orthotropic Lamina in Plane Stress

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad (2.24)$$

5 nonzero coefficients
4 independent coefficients

Or in terms of ‘engineering constants’

$$S_{11} = \frac{1}{E_1} \qquad S_{22} = \frac{1}{E_2}$$

$$S_{12} = S_{21} = -\frac{\upsilon_{21}}{E_2} = -\frac{\upsilon_{12}}{E_1} \tag{2.25}$$

$$S_{66} = \frac{1}{G_{12}}$$

TABLE 2.2

Typical Values of Lamina Engineering Constants for Several Composites Having Fiber Volume Fraction υ_f

Material	E_1 (Msi [GPa])	E_2 (Msi [GPa])	G_{12} (Msi [GPa])	ν_{12}	υ_f
T300/934 carbon/epoxy	19.0 (131)	1.5 (10.3)	1.0 (6.9)	0.22	0.65
AS/3501 carbon/epoxy	20.0 (138)	1.3 (9.0)	1.0 (6.9)	0.3	0.65
P-100/ERL 1962 pitch/ carbon/epoxy	68.0 (468.9)	0.9 (6.2)	0.81 (5.58)	0.31	0.62
IM7/8551-7 carbon/ toughened epoxy	23.5(162)	1.21(8.34)	0.3(2.07)	0.34	0.6
AS4/APC2 carbon/ PEEK	19.1(131)	1.26(8.7)	0.73(5.0)	0.28	0.58
Boron/6061 boron/ aluminum	34.1(235)	19.9(137)	6.8(47.0)	0.3	0.5
Kevlar® 49/934 aramid/ epoxy	11.0 (75.8)	0.8 (5.5)	0.33 (2.3)	0.34	0.65
Scotchply® 1002 E-glass/epoxy	5.6 (38.6)	1.2 (8.27)	0.6 (4.14)	0.26	0.45
Boron/5505 boron/ epoxy	29.6 (204.0)	2.68 (18.5)	0.81 (5.59)	0.23	0.5
Spectra® 900/826 polyethylene/epoxy	4.45 (30.7)	0.51 (3.52)	0.21 (1.45)	0.32	0.65
E-glass/470-36 E-glass/vinylester	3.54 (24.4)	1.0 (6.87)	0.42 (2.89)	0.32	0.30

Kevlar® is a registered trademark of DuPont Company, Wilmington, Delaware; Scotchply® is a registered trademark of 3M Company, St. Paul, Minnesota; and Spectra® is a registered trademark of Honeywell International, Inc.

Experimental Characterization of Orthotropic Lamina

- Need to measure 4 independent elastic constants
- Usually measure E_1, E_2, υ_{12}, G_{12}
  (see ASTM test standards later in Chap. 10)

Stresses in terms of <u>tensor</u> strains,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} \quad (2.26)$$

$$where\ [Q] = [S]^{-1}$$

Inverting [S]:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}{}^{2}} = \frac{E_1}{1 - \upsilon_{12}\upsilon_{21}}$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}{}^{2}} = \frac{\upsilon_{12}E_2}{1 - \upsilon_{12}\upsilon_{21}}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}{}^{2}} = \frac{E_2}{1 - \upsilon_{12}\upsilon_{21}}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

Off – Axis Compliances:

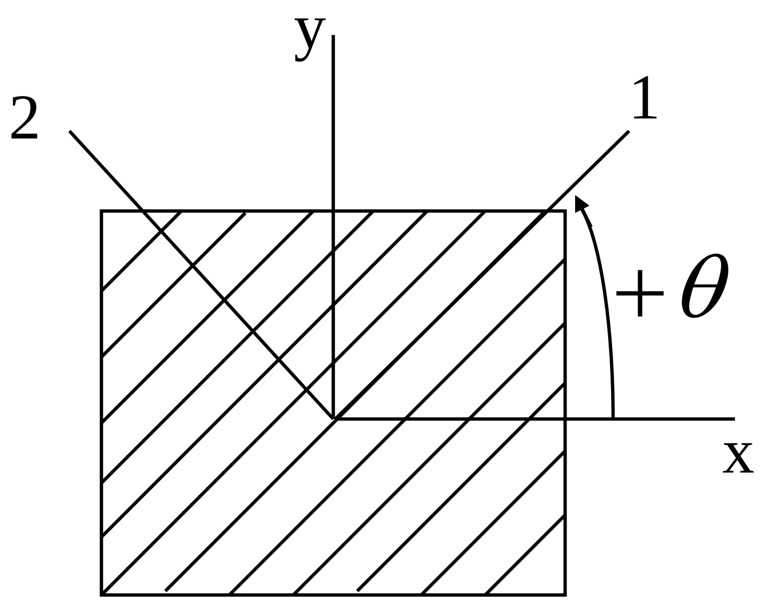
$$\overline{S}_{ij} = f_{ij}\left(all\ S_{ij}\ and\ angle\ \theta\right)$$

Off – Axis Stiffnesses:

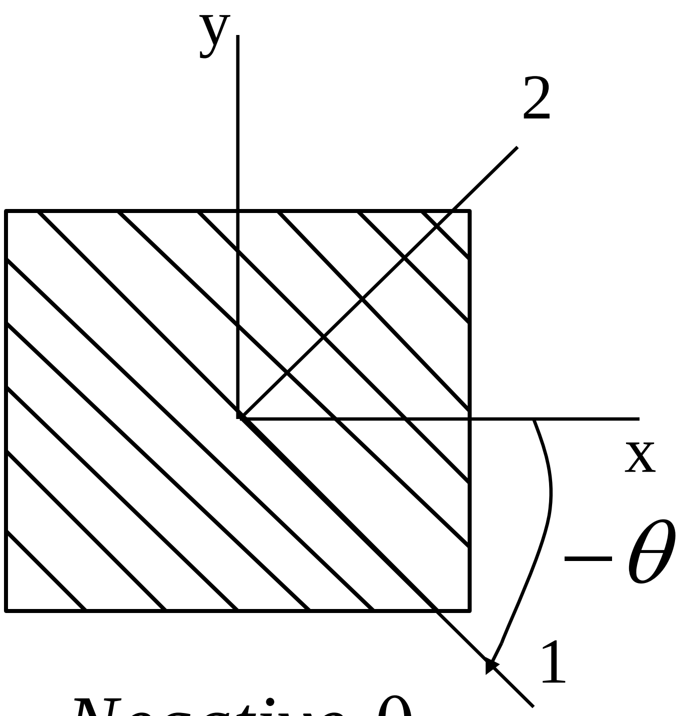
$$\overline{Q}_{ij} = f_{ij}'\left(all\ Q_{ij}\ and\ angle\ \theta\right)$$

Where f_{ij} and f_{ij}' are found from transformations of stress and strain components from 1,2 axes to x, y axes

Sign convention for lamina orientation

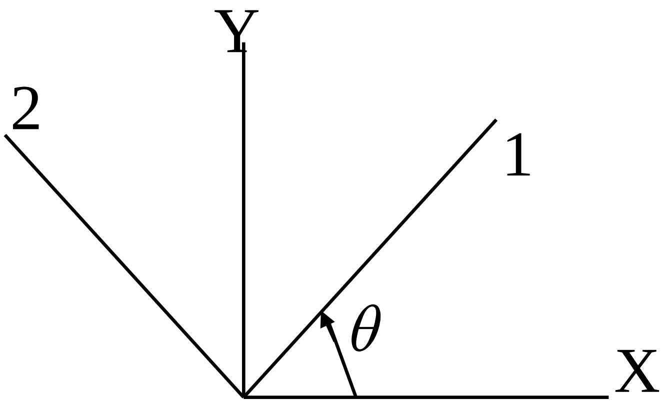


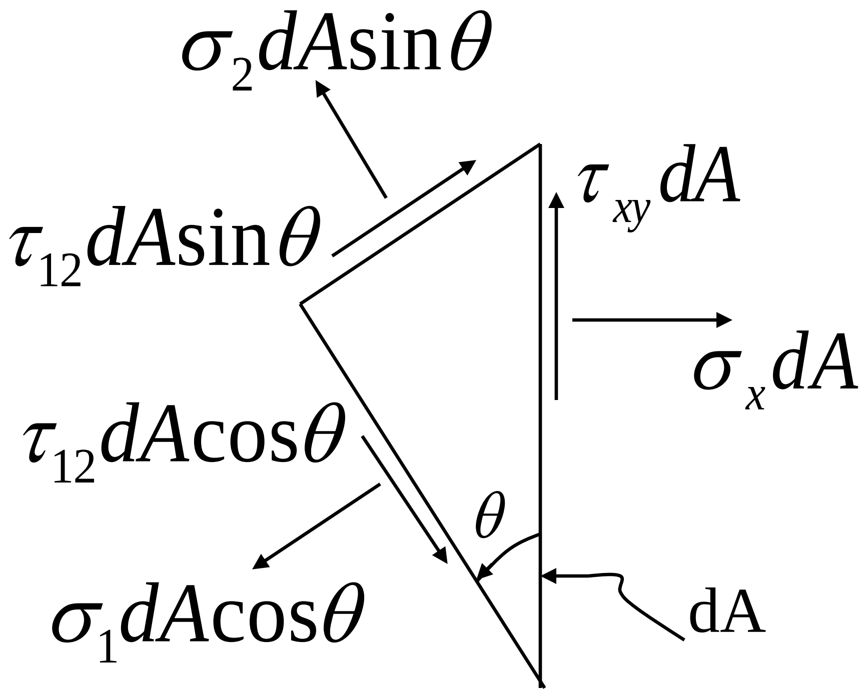
Positive θ



Negative θ

Stress Transformation:





$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

$$\sum F_x = \sigma_x dA - \sigma_1 dA\cos^2\theta - \sigma_2 dA\sin^2\theta$$

$$+ 2\tau_{12} dA\sin\theta\cos\theta = 0$$

$$\boxed{\therefore \sigma_x = \sigma_1\cos^2\theta + \sigma_2\sin^2\theta - 2\tau_{12}\sin\theta\cos\theta} \quad (2.29)$$

$$\sum F_y = 0$$

$$\boxed{\begin{aligned}\tau_{xy} &= \sigma_1\cos\theta\sin\theta - \sigma_2\cos\theta\sin\theta \\ &+ \tau_{12}(\cos^2\theta - \sin^2\theta)\end{aligned}}$$

Equations used to generate Mohr’s circle.

Resulting stress Transformation:

1 & 2 are the material directions, X & Y are the force directions

$$\begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (2.30)$$

Where $c = \cos\theta, \quad s = \sin\theta$

or

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_X \\ \sigma_Y \\ \tau_{XY} \end{Bmatrix} \quad (2.31)$$

Where

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \qquad (2.32)$$

Strain Transformation:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} \qquad (2.33)$$

Recall: Tensor shear strain

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$$

Where γ_{xy} = engineering shear strain

or

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{Bmatrix}$$

Substituting (2.33) into (2.26), then substituting the resulting equations into (2.30)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \boxed{[T]^{-1}[Q][T]} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{Bmatrix} \qquad (2.34)$$

Carrying out matrix multiplications and converting back to engineering strains,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (2.35)$$

Where

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2$$

$$\bar{Q}_{11} = \ldots$$

$$\vdots \qquad (2.36)$$

$$\bar{Q}_{66} = \ldots$$

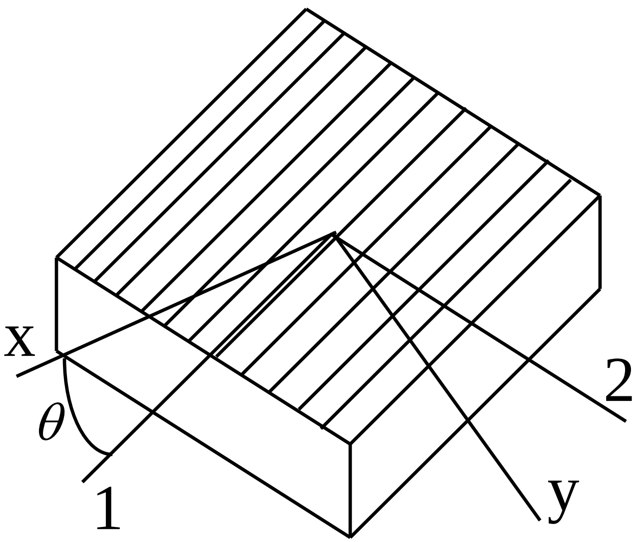
Alternatively

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (2.37)$$

Where $\qquad \left[\bar{S}\right]^{-1} = [\bar{Q}]$

Generally Orthotropic Lamina (Off Axis)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (2.37)$$



Or

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

9 Coefficients - 6 independent

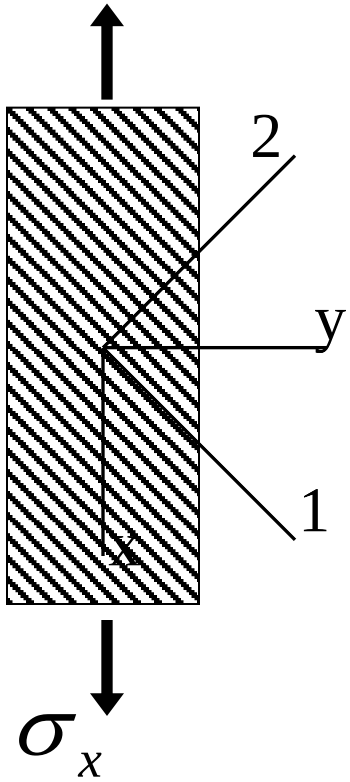
In expanded form:

$$
\begin{aligned}
\bar{S}_{11} &= S_{11}c^4 + \left(2S_{12} + S_{66}\right)s^2c^2 + S_{22}s^4 \\
\bar{S}_{12} &= S_{12}\left(s^4 + c^4\right) + \left(S_{11} + S_{22} - S_{66}\right)s^2c^2 \\
\bar{S}_{22} &= S_{11}s^4 + \left(2S_{12} + S_{66}\right)s^2c^2 + S_{22}c^4 \\
\bar{S}_{16} &= \left(2S_{11} - 2S_{12} - S_{66}\right)sc^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)s^3c \\
\bar{S}_{26} &= \left(2S_{11} - 2S_{12} - S_{66}\right)s^3c - \left(2S_{22} - 2S_{12} - S_{66}\right)sc^3 \\
\bar{S}_{66} &= 2\left(2S_{11} + 2S_{22} - 4S_{12} - S_{66}\right)s^2c^2 + S_{66}\left(s^4 + c^4\right)
\end{aligned}
\tag{2.38}
$$

Off-axis lamina engineering constants

Young's modulus, E_x

$$E_x = \frac{\sigma_x}{\varepsilon_x}$$



When $\sigma_x \neq 0, \quad \sigma_y = \tau_{xy} = 0$

$$\therefore E_x = \frac{\sigma_x}{\bar{S}_{11}\sigma_x} = \frac{1}{\bar{S}_{11}} \qquad (2.39)$$

or

$$E_x = \frac{1}{\dfrac{1}{E_1}c^4 + \left[-\dfrac{2\upsilon_{12}}{E_1} + \dfrac{1}{G_{12}}\right]c^2 s^2 + \dfrac{1}{E_2}s^4} \qquad (2.40)$$

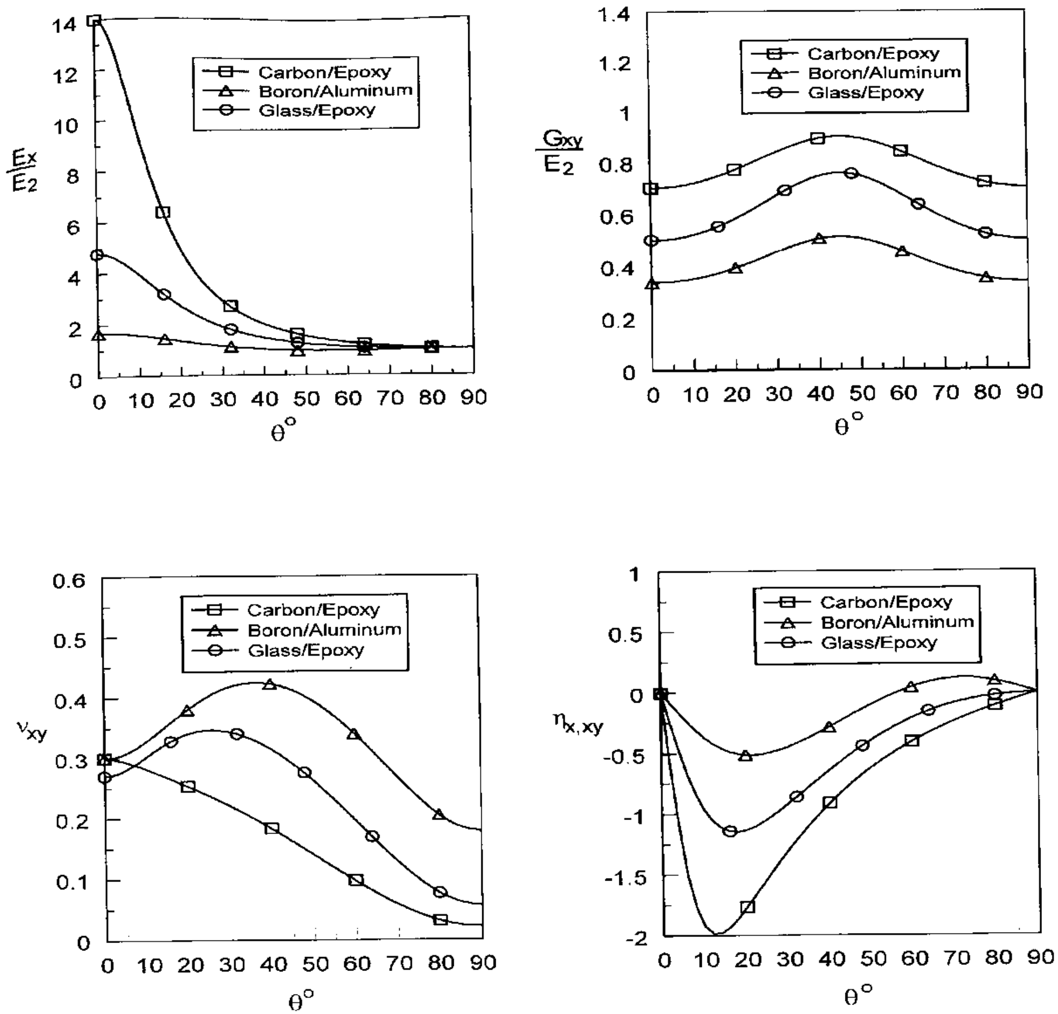
Complete set of transformation equations for lamina engineering constants

$$E_x = \left[\frac{1}{E_1} c^4 + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_1} \right) s^2 c^2 + \frac{1}{E_2} s^4 \right]^{-1}$$

$$E_y = \left[\frac{1}{E_1} s^4 + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_1} \right) s^2 c^2 + \frac{1}{E_2} c^4 \right]^{-1} \qquad (2.40)$$

$$G_{xy} = \left[\frac{1}{G_{12}} \left(s^4 + c^4 \right) + 4 \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\upsilon_{12}}{E_1} - \frac{1}{2G_{12}} \right) s^2 c^2 \right]^{-1}$$

$$\upsilon_{xy} = E_x \left[\frac{\upsilon_{12}}{E_1} \left(s^4 + c^4 \right) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) s^2 c^2 \right]$$



Variations of off-axis engineering constants with lamina orientation for unidirectional carbon/epoxy, boron/aluminum and glass/epoxy composites. (From Sun, C.T. 1998. *Mechanics of Aircraft Structures*. John Wiley & Sons, New York. With permission.)

Shear Coupling Ratios, or Mutual Influence Coefficients

- Quantitative measures of interaction between normal and shear response.
- Example: when $\sigma_x \neq 0$,

$$\sigma_y = \tau_{xy} = 0,$$

Shear Coupling Ratio

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\varepsilon_x} = \frac{\bar{S}_{16}\sigma_x}{\bar{S}_{11}\sigma_x} = \frac{\bar{S}_{16}}{\bar{S}_{11}} \qquad (2.41)$$

Analogous to Poisson's Ratio

Example of off-axis strain in terms of off-axis engineering constants

$$\varepsilon_x = \frac{1}{E_x}\sigma_x - \frac{\upsilon_{yx}}{E_y}\sigma_y + \frac{\eta_{xy,x}}{G_{xy}}\tau_{xy} \qquad (2.43)$$

Compliance matrix is still symmetric for off-axis case, so that, for example

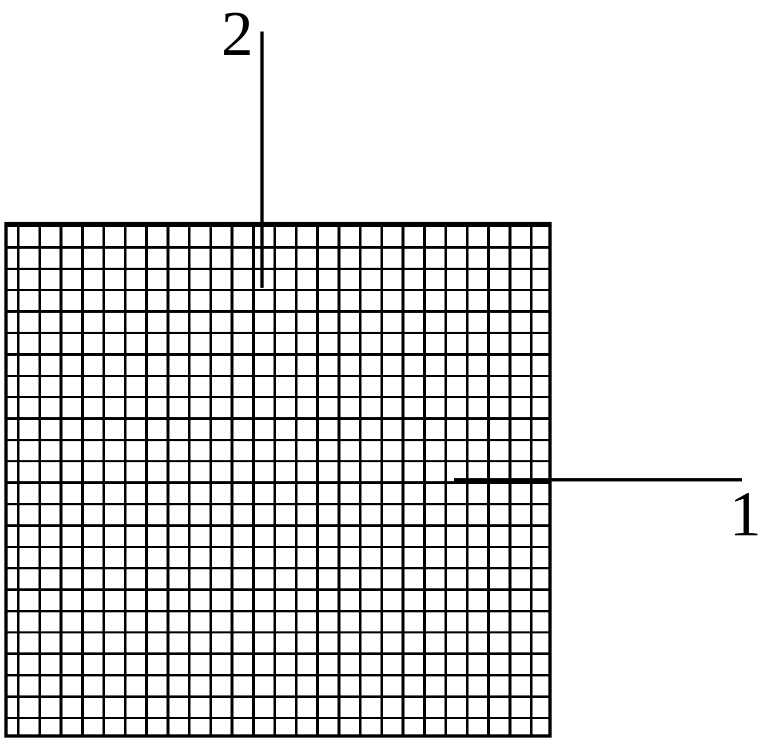
$$\bar{S}_{12} = \bar{S}_{21}$$

and

$$\frac{\upsilon_{yx}}{E_y} = \frac{\upsilon_{xy}}{E_x}$$

Balanced Orthotropic Lamina

(Ex: Woven cloth, cross-ply)



$$E_1 = E_2$$

$$Q_{11} = Q_{22}$$

$$S_{11} = S_{22}$$

Only 3 independent coefficients

Lamina Stiffness Transformations

$$\begin{Bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{12} \\ \overline{Q}_{66} \\ \overline{Q}_{16} \\ \overline{Q}_{26} \end{Bmatrix} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & 4c^2s^2 \\ s^4 & c^4 & 2c^2s^2 & 4c^2s^2 \\ c^2s^2 & c^2s^2 & c^4+s^4 & -4c^2s^2 \\ c^2s^2 & c^2s^2 & -2c^2s^2 & (c^2-s^2)^2 \\ c^3s & -cs^3 & cs^3-c^3s & 2(cs^3-c^3s) \\ cs^3 & -c^3s & c^3s-cs^3 & 2(c^3s-cs^3) \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{Bmatrix}$$

Use of Invariants

The lamina stiffness transformations can be written as:

$$\begin{aligned}
\bar{Q}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\
\bar{Q}_{12} &= U_4 - U_3 \cos 4\theta \\
\bar{Q}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\
\bar{Q}_{16} &= \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta \\
\bar{Q}_{26} &= \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta
\end{aligned} \tag{2.44}$$

Where the invariants are

$$
\begin{aligned}
U_1 &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \\
U_2 &= \frac{1}{8}(Q_{11} - Q_{22}) \\
U_3 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) \\
U_4 &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}) \\
U_5 &= \frac{1}{8}(Q_{11} + Q_{22} + 4Q_{66} - 2Q_{12})
\end{aligned}
\tag{2.45}
$$

Alternatively, the off-axis compliances can be expressed as

$$\bar{S}_{11} = V_1 + V_2 \cos 2\theta + V_3 \cos 4\theta$$

$$\bar{S}_{12} = V_4 - V_3 \cos 4\theta$$

$$\bar{S}_{22} = V_1 - V_2 \cos 2\theta + V_3 \cos 4\theta \qquad (2.46)$$

$$\bar{S}_{16} = V_2 \sin 2\theta + 2V_3 \sin 4\theta$$

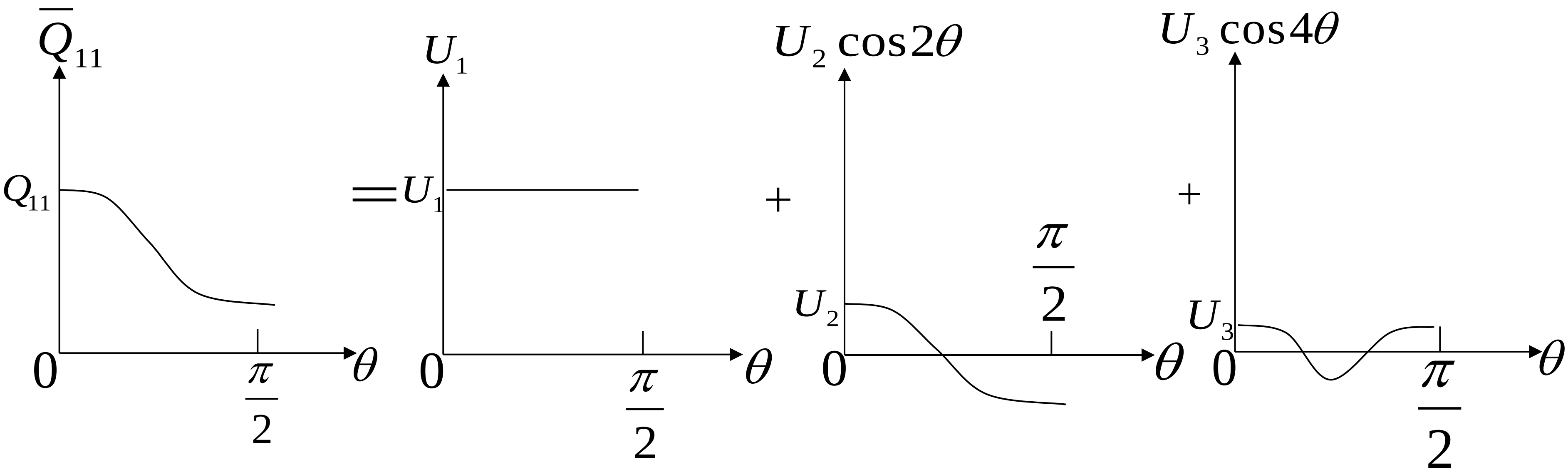
$$\bar{S}_{26} = V_2 \sin 2\theta - 2V_3 \sin 4\theta$$

$$\bar{S}_{66} = 2(V_1 - V_4) - 4V_3 \cos 4\theta$$

where the invariants are

$$
\begin{aligned}
V_1 &= \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}) \\
V_2 &= \frac{1}{2}(S_{11} - S_{22}) \\
V_3 &= \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}) \\
V_4 &= \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66})
\end{aligned}
\qquad (2.47)
$$

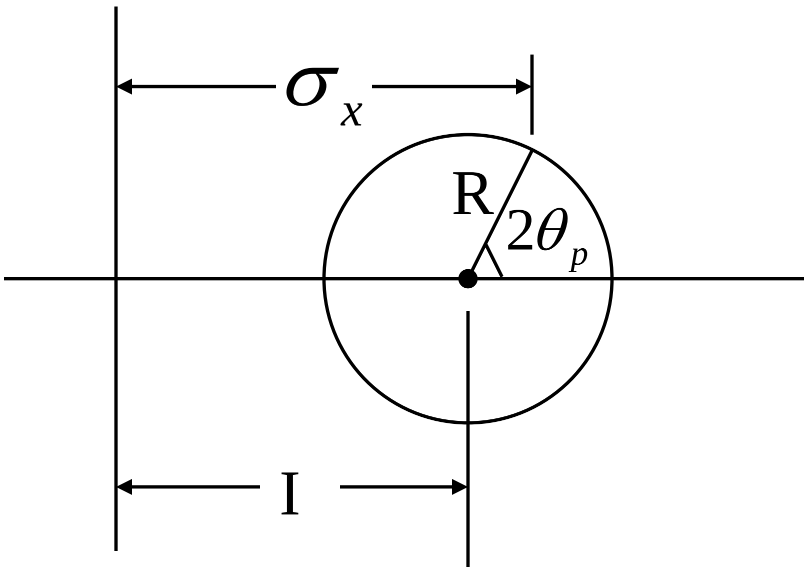
Example: decomposition of $\overline{Q}_{11}$ using invariants



References: Mechanics of Composite Materials, Jones

Introduction to Composite Materials, Tsai & Hahn

Invariants in transformation of stresses: Mohr's circle



$Ex:$

$$\sigma_x = I + R\cos 2\theta_p$$

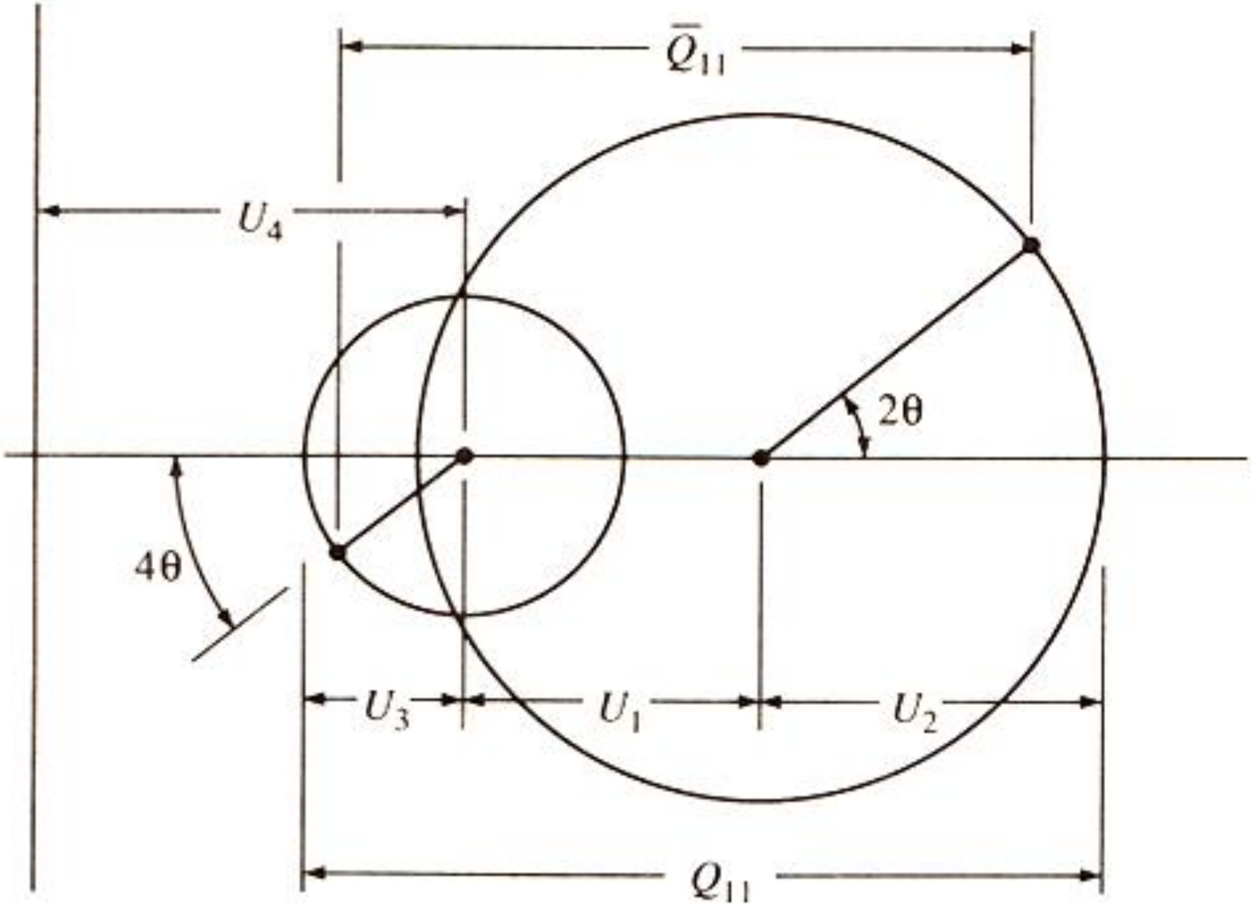
Where

$$I = \frac{\sigma_x + \sigma_y}{2} = \text{Invariant} \tag{2.48}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}} = \text{Invariant}$$

Similar graphical interpretation of stiffness transformations

Ex: $$\overline{Q}_{11} = \underbrace{U_1}_{\text{Isotropic Part}} + \underbrace{U_2 \cos 2\theta + U_3 \cos 4\theta}_{\text{Orthotropic Part}} \qquad (2.49)$$



U_1 and U_4: First order invariants

U_2 and U_3: Second order invariants

Radii of circles indicates degree of orthotropy.

(i.e., if U_2=U_3=0, we have isotropic material)