ICE 304

Digital Signal

and

Image Processing

ICE 304

Course Teacher

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Text and Reference

- Text
  - Signals and Systems

    By

    Simon Haykin and Barry Van Veen

- Reference
  - Analysis of Linear Systems

    By

    David K. Cheng

Course Outline

- Mostly based on Simon Haykin Book.
- May not always correspond to those that are listed on printed syllabus.
- Content will be clearer as we progress.

Signal

- **What is a Signal?**

  Is a basic ingredient of our life.

  - Speech : Voice Communication
  - Image : Visual Communication
  - Heartbeat : Biological Information Bearer
  - Prices of Stocks/Commodities : Financial Forecaster
  - Email, Internet : Carry Information Bearing Signal

- **Definition of a Signal**

  A signal is defined as a function of <u>one</u> or <u>more</u> variables which conveys information on the nature of a physical phenomenon.

Signal contd.

- ## One-dimensional Signal

  The function depends only on one variable.

  - Voice : Amplitude varies with time.

- ## Multidimensional

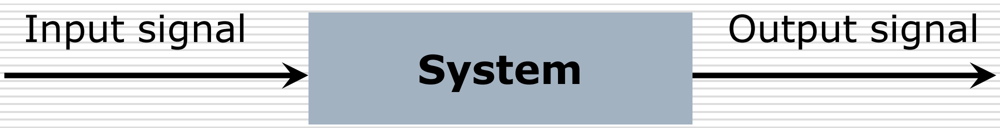
  The function depends on more than one variable.

  - Image : Represented by horizontal and vertical coordinates.

System

❑ What is a System?

Is an entity that manipulates one or more signal to accomplish a function, thereby yielding new signal(s).



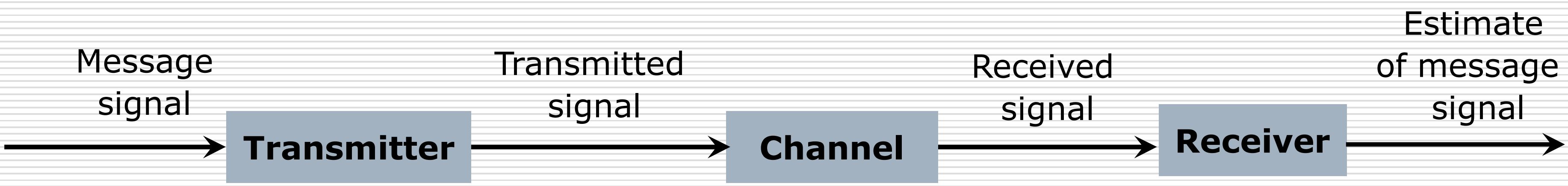
❑ Nature of Input/Output

Depends on the intended application of the system.

Overview of Specific Systems

- Communication Systems
- Control Systems
- Remote Sensing Systems
- Biomedical Signal Processing Systems
- Auditory System

Overview of Communication Systems



- Basic Elements
  - Transmitter
  - Channel
  - Receiver

- Each element can be viewed as a system with associated signals of its own.

Overview of Communication Systems contd.

- **Transmitter**

  Converts the message signal into a form suitable for transmission over the channel.

- **Channel**

  The physical medium that connects the transmitter and the receiver.

  - Can be wired or wireless.
  - Physical characteristics of the channel, noise and interference from other signals all can distort the signal in propagation.

- **Receiver**

  Receives the corrupted version of the transmitted signal and reconstructs it to a recognizable form (i.e. produce an estimate) of the original message signal.

  i.e. it has two fundamental responsibilities

  1. Performing the reverse operations of the transmitter and
  2. Reversing the effect of the channel.

Overview of Communication Systems contd.

- **Analog Communication**
  - Modulator (in transmitter) : Converts message signal into a form that is compatible with transmission characteristics of the channel.
  - Demodulator (in receiver): Reverse operation of Modulator.

- **Digital Communication : More complex than its analog counterpart**

Fundamental Operations (in transmitter) when the message signal is analog

- Sampling : Converts the message signal into a sequence of number.
- Quantization : Representing the sampled values to the nearest level of pre-selected values.
- Coding : Representing each quantized value by a code word.
- Modulation : Carrier wave modulation for transmission.

Additional Operations (in transmitter)

- Data Compression : Removing redundant information from the signal for efficient utilization pf the channel capacity to meet a bit rate requirement.
- Channel Coding : Inserting redundant elements into message to provide protection against channel noise.

Overview of Communication Systems contd.

- Analog (Continuous-time Approach) Signal Processing
  - Was dominant for many years.
  - Relies on analog circuit elements like resistors, capacitors, inductors etc.
  - Based on solving differential equation that describe natural systems
  - Real time solutions can be obtained.

- Digital (Discrete-time Approach) Signal Processing
  - Is the present trend in signal processing.
  - Based on numerical computation and hence relies on basic digital computer elements like adders, multipliers, memory elements etc.
  - Requires greater circuit complexity and yet no assurance of real time output.

Overview of Communication Systems contd.

- Advantages of Digital Approach over Analog Signal Processing
  - Flexibility

    Only change of software can change the functionality of a hardware. (e.g. filter)
  - Repeatability

    Normally does not suffer from external effects like supply voltage or room temperature when repeating the same tasks again and again.
- Cost and size also have been remarkably reduced due the phenomenal developments in VLSI technology.

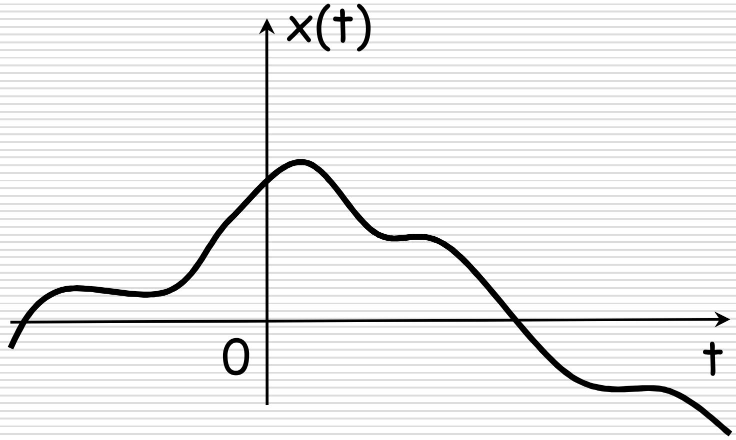
Classification of Signals

- Note : Discussion will be Restricted to One Dimensional Single Valued only.
- Signals can be classified as
  - Continuous-time and Discrete-time Signals
  - Even and Odd Signals
  - Periodic and Non-periodic Signals
  - Deterministic and Random Signals
  - Energy Signals and Power Signals

Continuous-time and Discrete-time Signals

Continuous-time signal

A signal $x(t)$ that is defined for all values of time t.



- Arises naturally i.e. speech in voice, conversion of sound or light into electrical signal (by means of transducers, photocell) etc.

Continuous-time and Discrete-time Signals contd.

Discrete-time signal

A signal that is defined only at discrete instants of time.

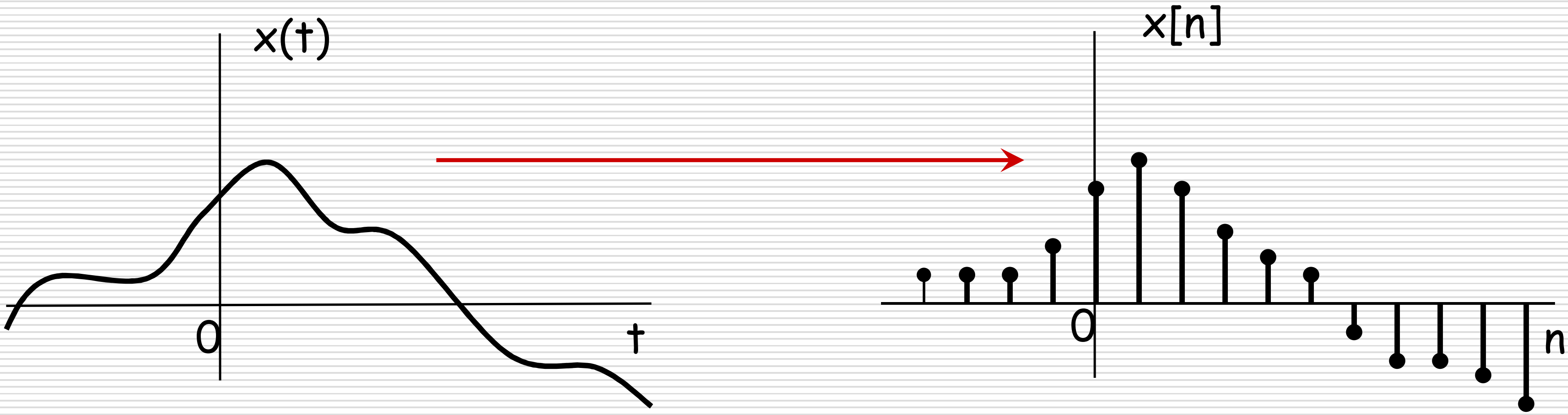
- The independent variable only has discrete values which are usually equally placed
- Often derived from a continuous time-signal by sampling it at a uniform rate
- Way of Representation

$$x[n] = x(mt), n = 0, \pm 1, \pm 2, \ldots$$

$$m = \text{Sampling interval}$$

Continuous-time and Discrete-time Signals contd.

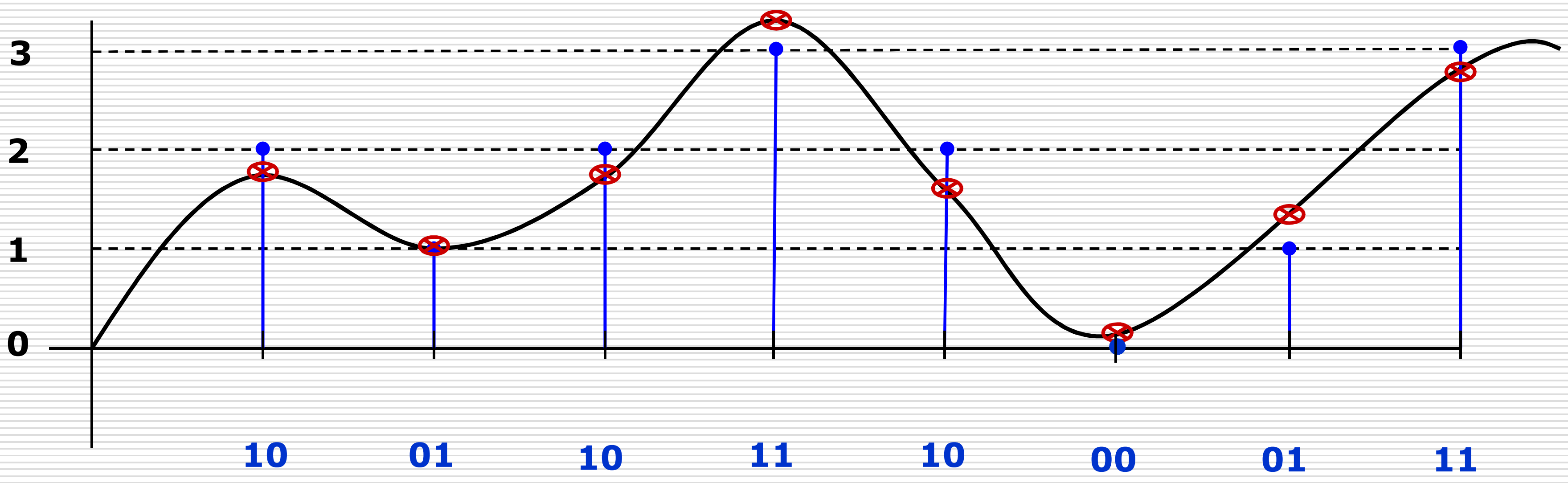
Sampling



Continuous-time and Discrete-time Signals contd.

- Coding
  - Sampled values are converted to codes.
    - Example : Converting to binary numbers.
- Binary Coding
  - Number of bits for representing one sample value is at first decided e.g. 8 bit coding, 16 bit coding etc.
  - Assuming unsigned sampled values (non-negative values), an n-bit coding scheme can represent maximum of 2^n number of distinct sampled values.
    - Example : 2 bit coding can represent 0,1,2 and 3, these 4 sampled values only.

Continuous-time and Discrete-time Signals contd.



Quantization : Representing each sampled value to the nearest level selected from a finite number of amplitude levels.

Classification of Signals Revisited

Signals can be classified as

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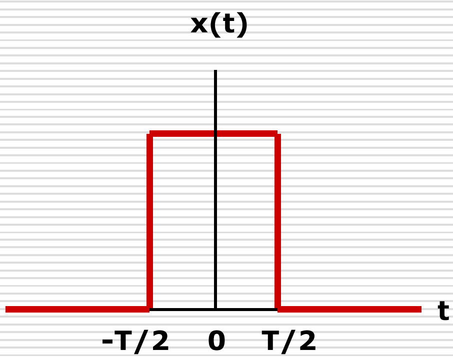
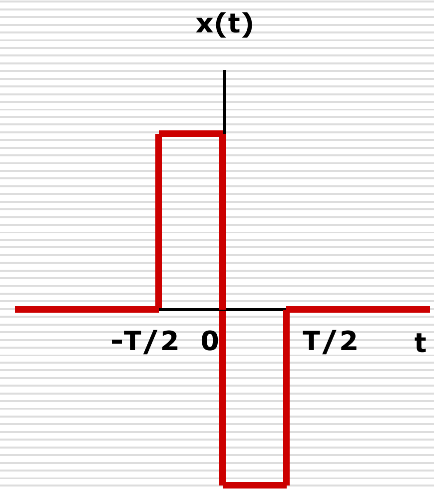
Even and Odd Signal

- for Continuous-time Signal
  - if x(-t)=x(t) for all t, it's an even signal
  - if x(-t)=-x(t) for all t, it's an odd signal



Even Signal



Odd Signal

Even and Odd Signal contd.

- Even/odd decomposition of a general signal $x(t)$.

Let $x(t)$ be defined as

$$x(t) = x_e(t) + x_o(t) \quad \dots \dots (1)$$

where $x_e(t)$ is even and $x_o(t)$ is odd, i.e.,

$$x_e(-t) = x_e(t)$$

and

$$x_o(-t) = -x_o(t)$$

Replacing t by $-t$ in (1),

$$\begin{aligned} x(-t) &= x_e(-t) + x_o(-t) \\ &= x_e(t) - x_o(t) \quad \dots \dots (2) \end{aligned}$$

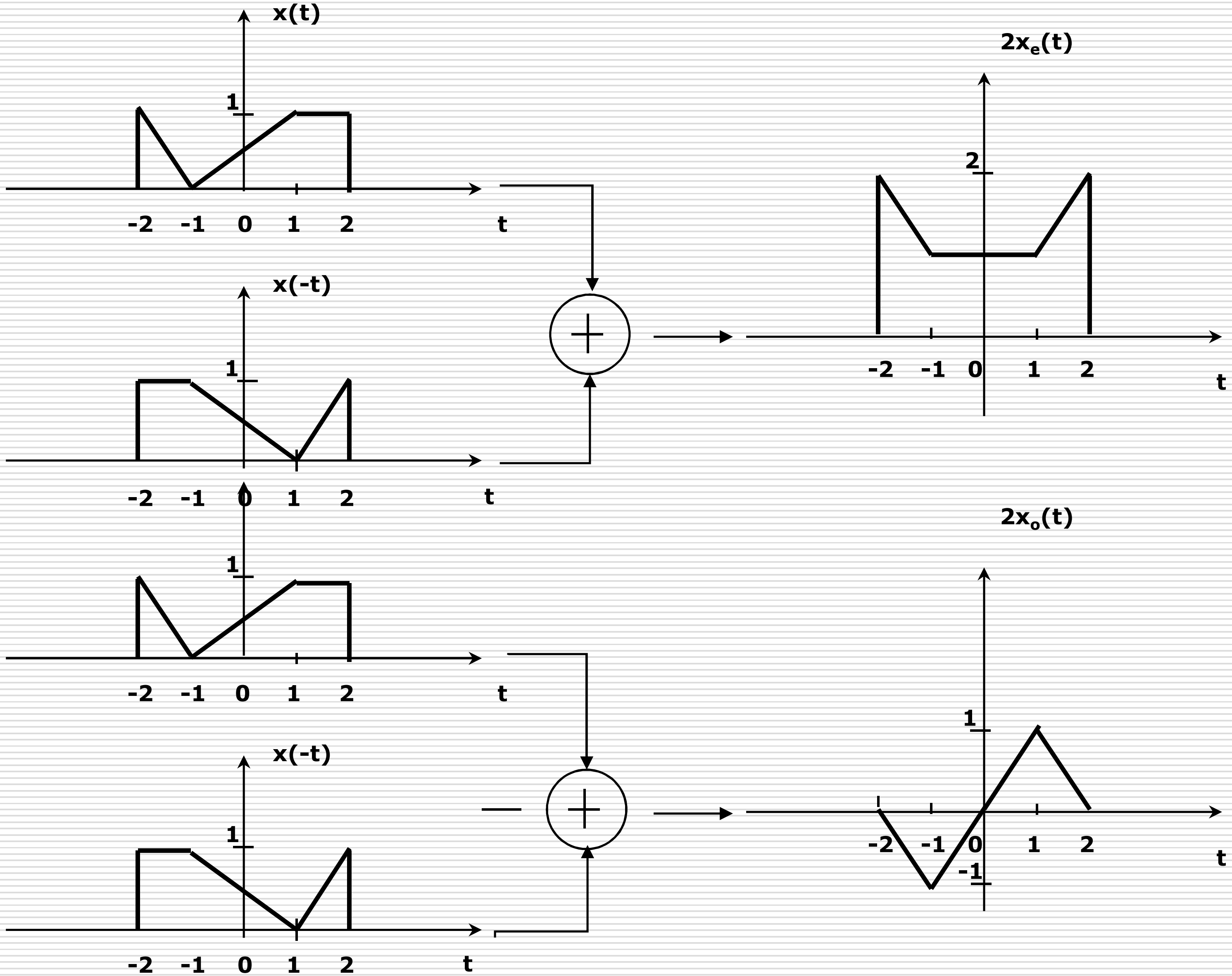
Solving (1) and (2)

$$x_e(t) = \tfrac{1}{2}\{x(t) + x(-t)\}$$

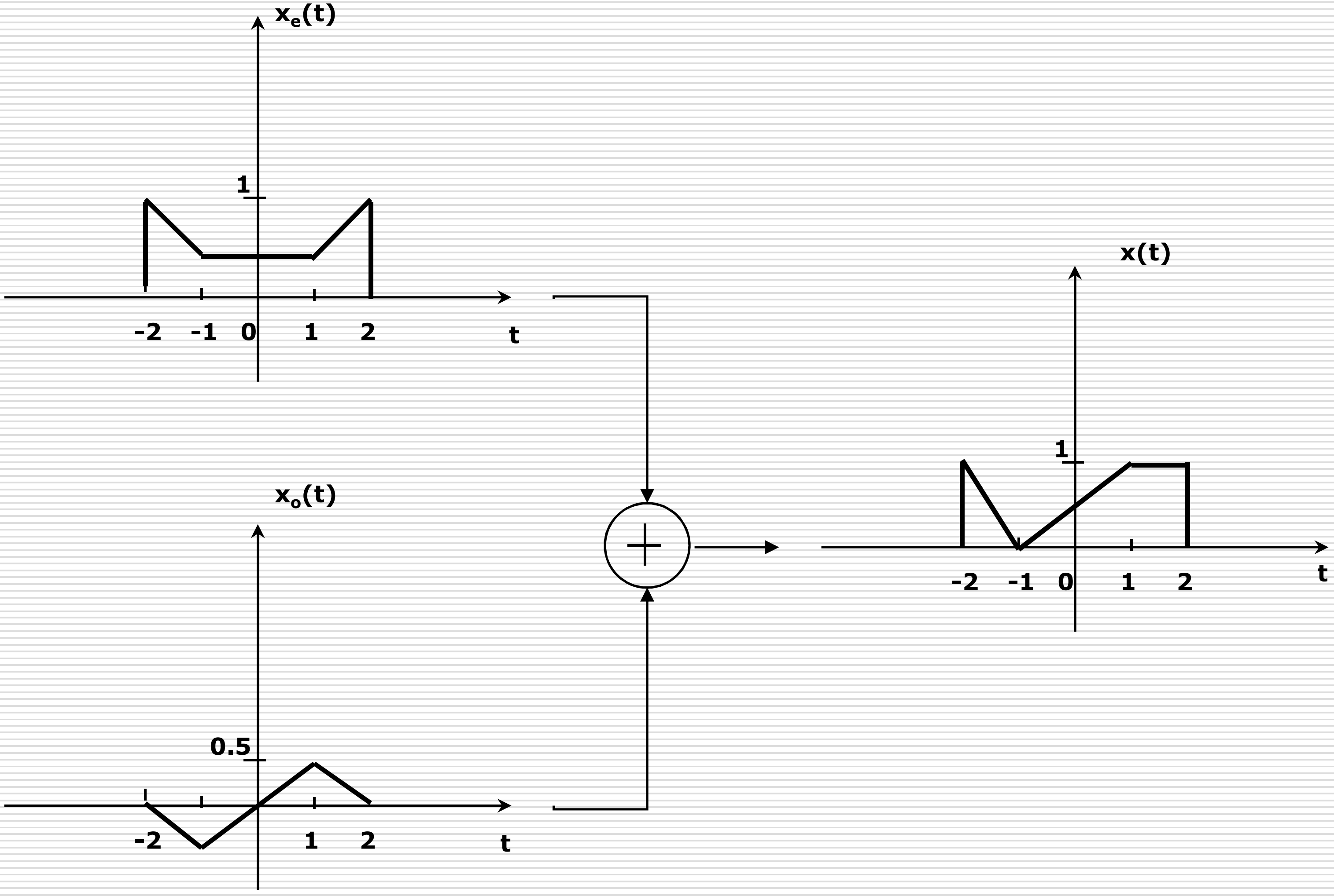
and

$$x_o(t) = \tfrac{1}{2}\{x(t) - x(-t)\}$$

Even/odd decomposition



Even/odd decomposition contd.



Even and Odd Signal contd.

❑ Complex Valued Signal

Let

$$x(t) = a(t) + jb(t)$$

x (t) is conjugate symmetric,

$$\text{if } x(-t) = x^*(t)$$

, i.e.,

$$a(-t) + jb(-t) = a(t) - jb(t)$$

Which implies

$$a(-t) = a(t) \text{ , i.e. } a(t) \text{ is even}$$

$$\text{and } b(-t) = -b(t) \text{ , i.e. } b(t) \text{ is odd}$$

A complex signal is conjugate symmetric if its real part is even and imaginary part is odd.

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Periodic and Non-periodic Signals

❑ Periodic Signal

- Continuous-time

  $x(t)$ is periodic if $x(t) = x(t+T)$ for all t, where T is a positive constant.

  If the above condition is satisfied for $T = T_0$, it is also satisfied for $T = 2T_0$, $T = 3T_0$ and so on.

  The smallest possible value of T is called fundamental period or period of $x(t)$ and the reciprocal of that T is known as the fundamental frequency of $x(t)$.

  i.e.

  $$f = 1/T \text{ measured in hertz or cps.}$$

  Angular frequency is measured in radians per second and is defined as

  $$\omega = 2\pi/T$$

Periodic and Non-periodic Signals contd.

❑ Periodic Signal

➢ Discrete-time

x[n] is periodic if x[n] = x[n+N] for all integer n, where N is a positive <u>integer</u> constant.

Fundamental angular frequency is measured in radians per second and is defined as

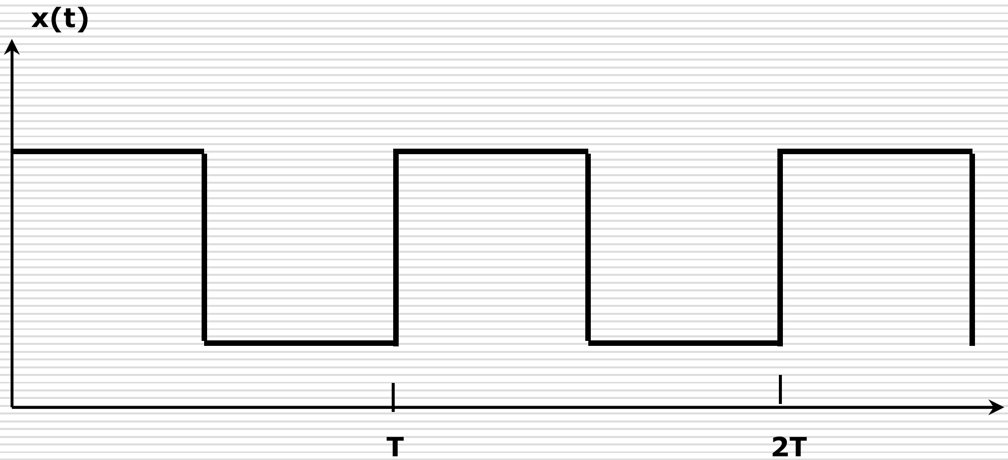
$$\Omega = 2\pi/N$$

❑ Non-periodic or Aperiodic signal

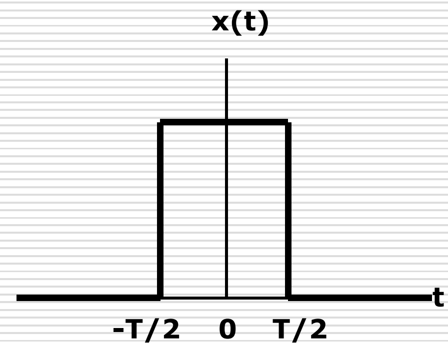
x(t) (or x[n]) is non-periodic/aperiodic if

no value of T (or N) exists such that x(t) = x(t+T) (or x[n] = x[n+N])

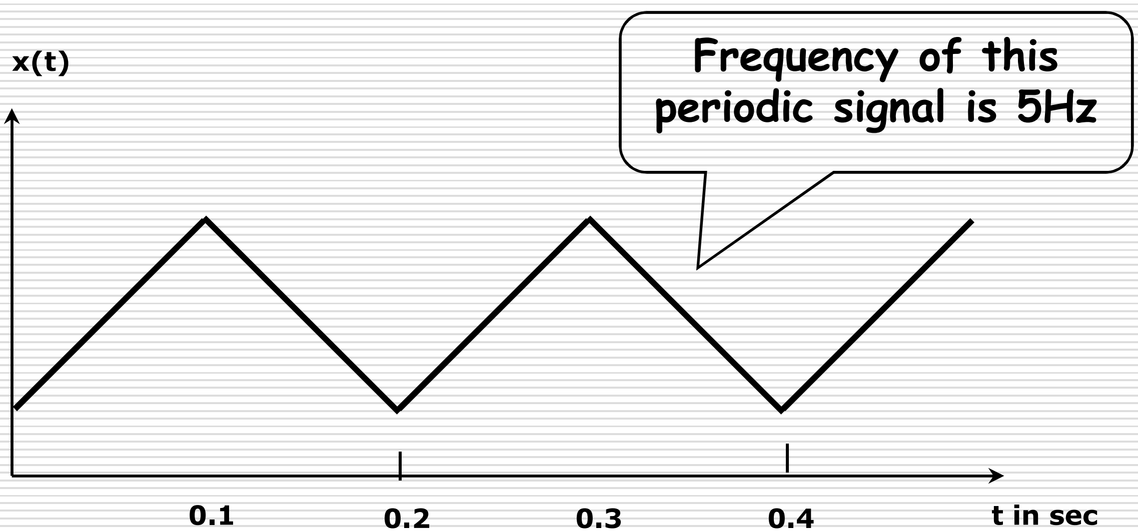
Examples of Periodic and Non-periodic Signals



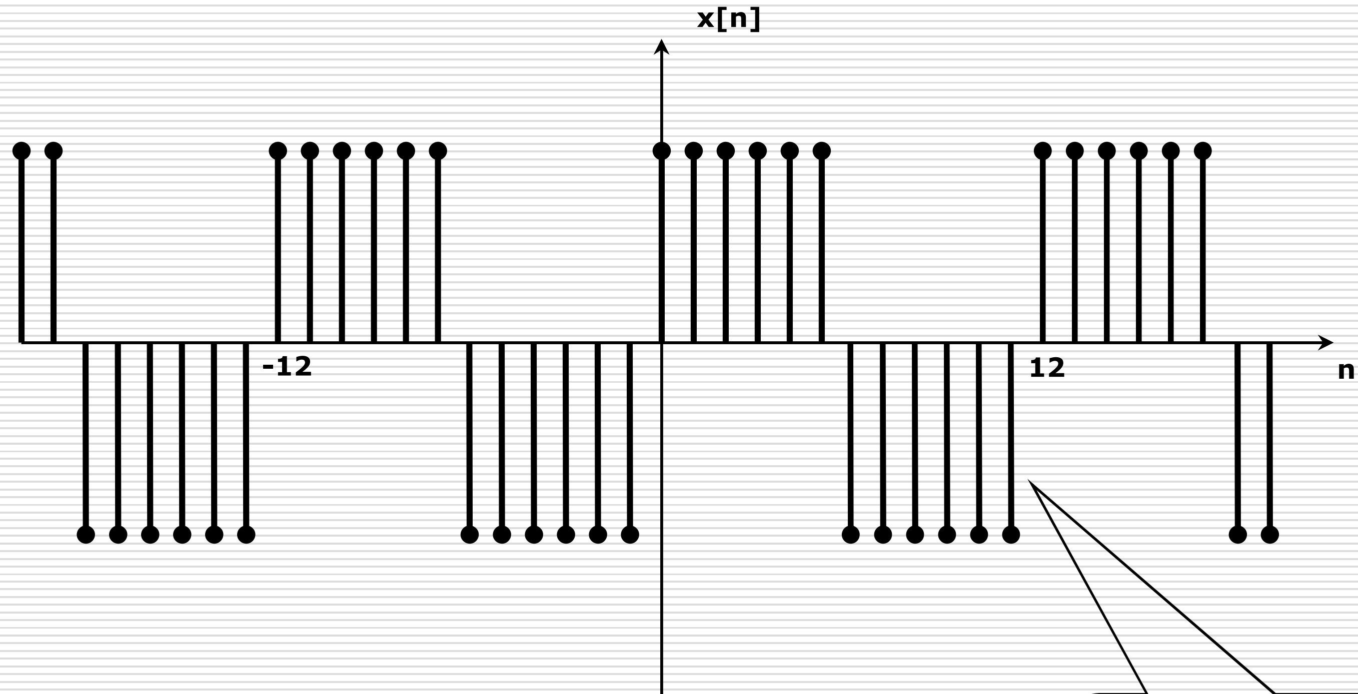
A Periodic signal with a period of T



A Non-periodic signal

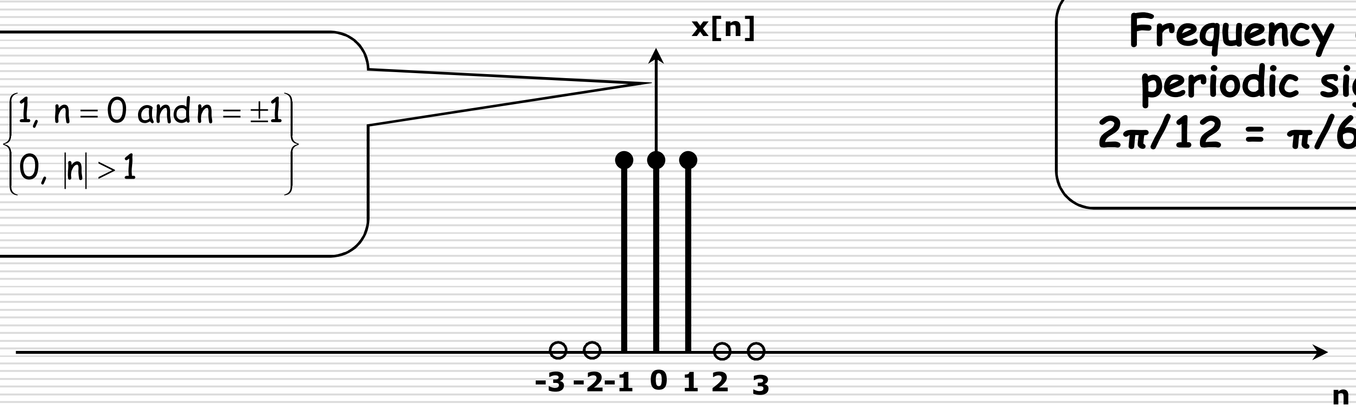


Examples of Periodic and Non-periodic Signals contd.



Frequency of this periodic signal is $2\pi/12 = \pi/6$ radians

$$x[n] = \begin{cases} 1, & n = 0 \text{ and } n = \pm 1 \\ 0, & |n| > 1 \end{cases}$$



Classification of Signals Revisited

Signals can be classified as

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Deterministic and Random Signals

Deterministic Signal

- A signal about which there is no uncertainty with respect to its value at any time, i.e. they are completely specified function of time.
- Examples : All periodic signals.

Random Signal

- A signal about which there is uncertainty before its actual occurrence.
- They often belong to ensemble or group of signals of different waveforms.
- Example : Noise generated in amplifier, wireless channel ECG signal etc.

Classification of Signals Revisited

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Energy Signals and Power Signals

- Instantaneous Power $$p(t) = x^2(t)$$

- Total Energy of a Signal $$E = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

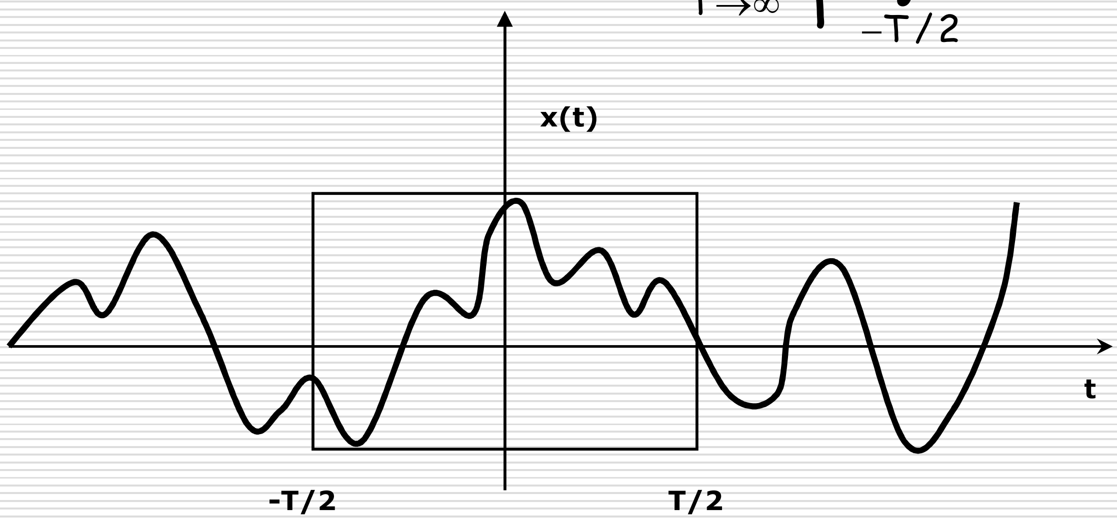
Limitation of the definition of Total Energy:

If the signal does not decay with time, or if it is a periodic signal, it means the signal has infinite energy. A signal having infinite energy has no meaning as far as comparing it with another signal is concerned. In these cases the concept of average power helps.

Energy Signals and Power Signals contd.

- **Average Power**

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$



Computing the energy for a specific unit of time, then allow that time to go to infinity.

- **Average Power of a Periodic signal**

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

T/2

Energy Signals and Power Signals contd.

In Discrete-time approach

- Total Energy

$$E = \sum_{n=-\infty}^{n=+\infty} x^2[n]$$

- Average Power

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x^2[n]$$

- Average Power of a Periodic Signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

Energy Signals and Power Signals contd.

Power Signal

A signal that have non-zero finite average power, i.e., $0 < P < \infty$

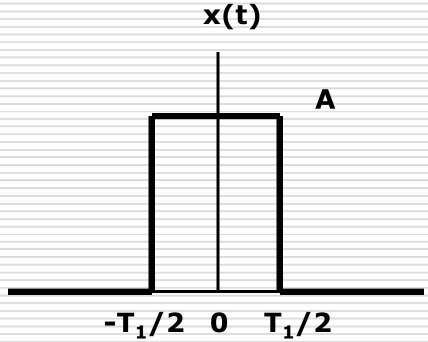
- It has infinite energy.
- Examples : periodic signal, random signal

Energy Signal

A signal that have non-zero finite energy, i.e., $0 < E < \infty$

- It has zero average power.
- Example: both non-periodic and deterministic signal

Energy Signals and Power Signals contd.



$$x(t) = \begin{cases} A, & -T_1/2 \le t \le T_1/2 \\ 0, & \text{Otherwise} \end{cases}$$

$$E = \int_{-T_1/2}^{T_1/2} A^2 \, dt$$

$$= A^2 T_1$$

An Energy Signal

$$P = \lim_{T \to \infty} \frac{A^2 T_1}{T} \, dt$$

$$= 0$$

Energy Signals and Power Signals contd.

$$\text{Avg. Power, } P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

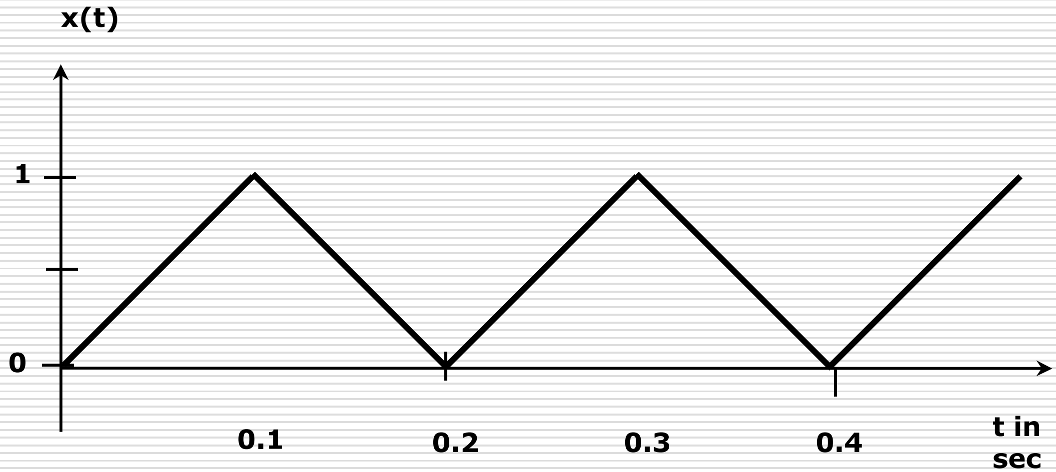
$$= \frac{1}{T}\int_{0}^{T} x^2(t)\,dt$$

$$= \frac{1}{T}2 * \int_{0}^{T/2} x^2(t)\,dt$$

$$= \frac{2}{0.2}\int_{0}^{0.1}\left(\frac{1}{0.1}t\right)^2 dt$$

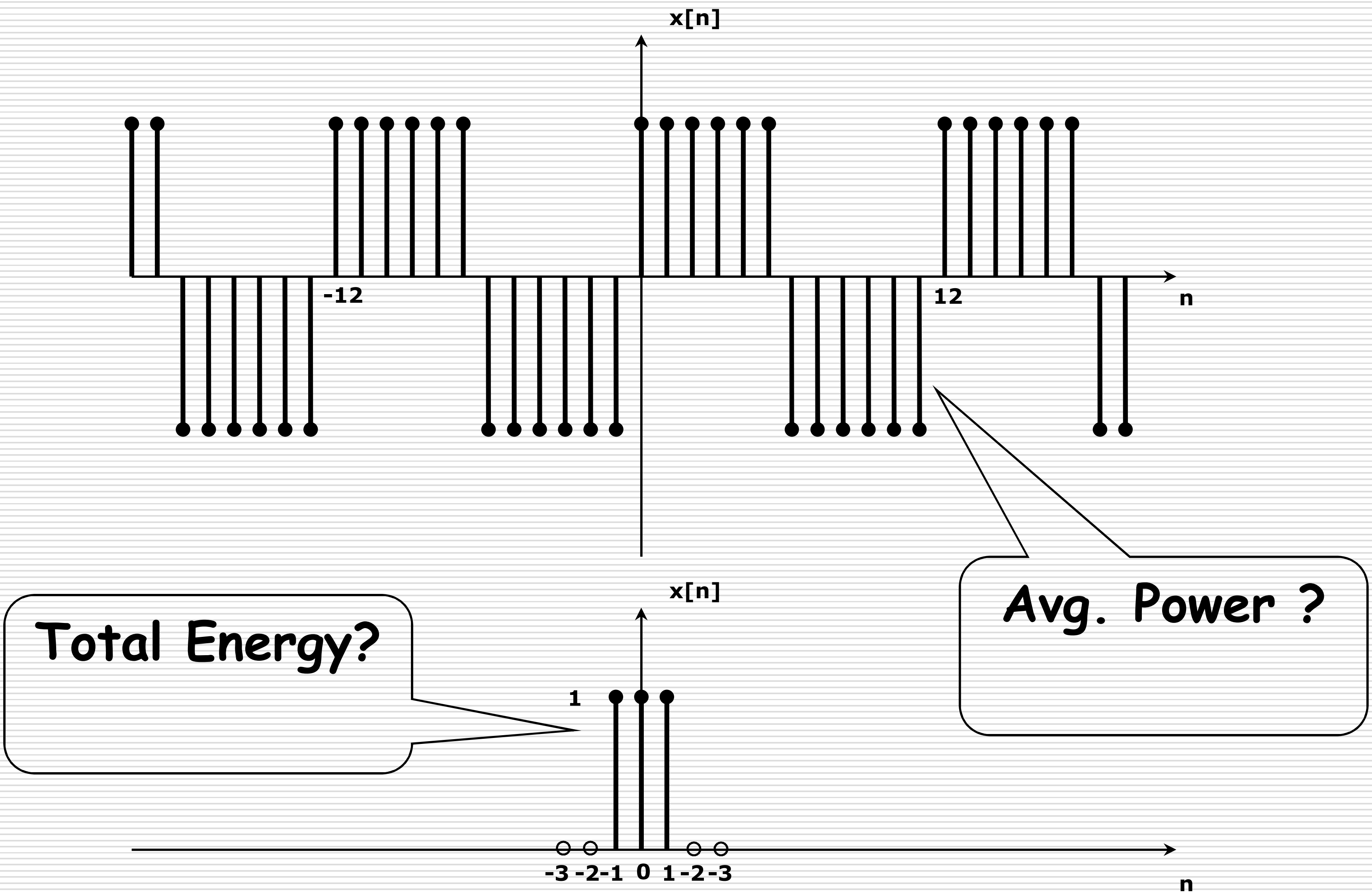
$$= 1000 * \left[\frac{t^3}{3}\right]_0^{0.1}$$

$$P = \frac{1}{3}$$



A Power Signal

Energy Signals and Power Signals contd



Basic Operations on Signals

Operations performed on dependent variable

- Amplitude Scaling
- Addition
- Multiplication
- Differentiation
- Integration

Operations performed on independent variable

- Time Scaling
- Reflection
- Time Shifting

Amplitude Scaling

- For Continuous-time signal

Applying amplitude scaling on x (t), yields

$$y(t)=cx(t),$$

where c is a scaler quantity known as the scaling factor.

- For Discrete-time signal,

$$y[n]=cx[n]$$

- Example : Electronic Amplifier, A resistor (i.e. x(t) current, c resistance, y(t) voltage)

Addition

- For Continuous-time signal

if $x_1(t)$ and $x_2(t)$ are two signals , the signal $y(t)$ obtained by the addition of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t)= x_1(t) + x_2(t)$$

- For Discrete-time signal,

$$y[n]= x_1[n] + x_2[n]$$

- Example : Audio mixer

Multiplication

- For Continuous-time signal

if $x_1(t)$ and $x_2(t)$ are two signals , the signal $y(t)$ obtained by the multiplication of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t)= x_1(t)\, x_2(t)$$

- For Discrete-time signal,

$$y[n]= x_1[n]\, x_2[n]$$

- Example : AM Modulation, Sampling, Spreading

Differentiation

If x (t) is continuous-time signal, the derivative of x(t) with respect to time is given by,

$$y(t) = \frac{d}{dt}x(t)$$

- Example : Voltage across an inductor of inductance L due to the flow of current i(t)

$$v(t) = L\frac{d}{dt}i(t)$$

Integration

If x (t) is continuous-time signal, the integral of x(t) with respect to time is given by,

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- Example : Voltage across a capacitor of capacitance of C due to the flow of current i(t),

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

Time Scaling

- **For Continuous-time signal**

If $x(t)$ is a continuous-time signal, the signal $y(t)$ obtained by the scaling of independent variable t, by the factor a is defined by,

$$y(t) = x(at)$$

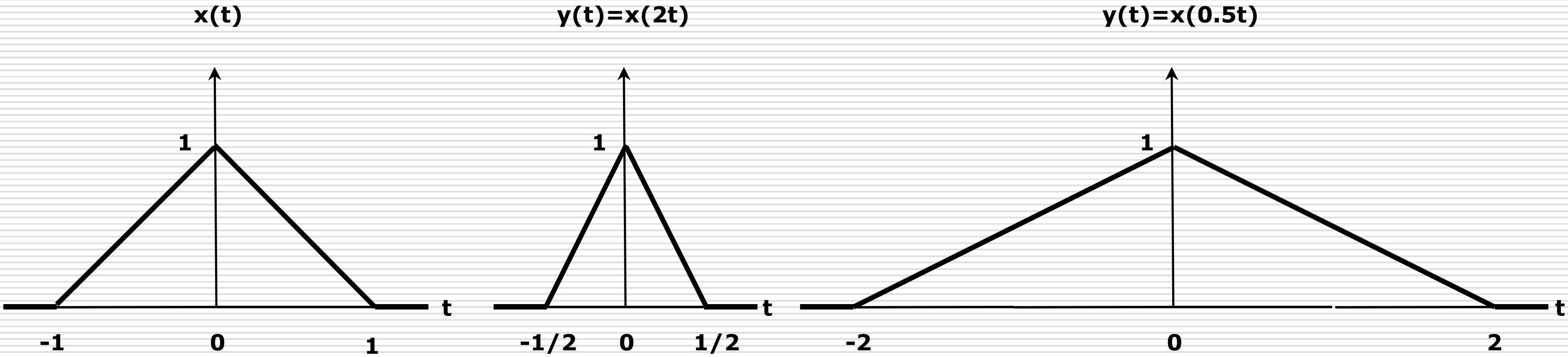
- If $a > 1$, $y(t)$ is the compressed version of $x(t)$.
- If $0 < a < 1$, $y(t)$ is the expanded (or stretched) version of $x(t)$.

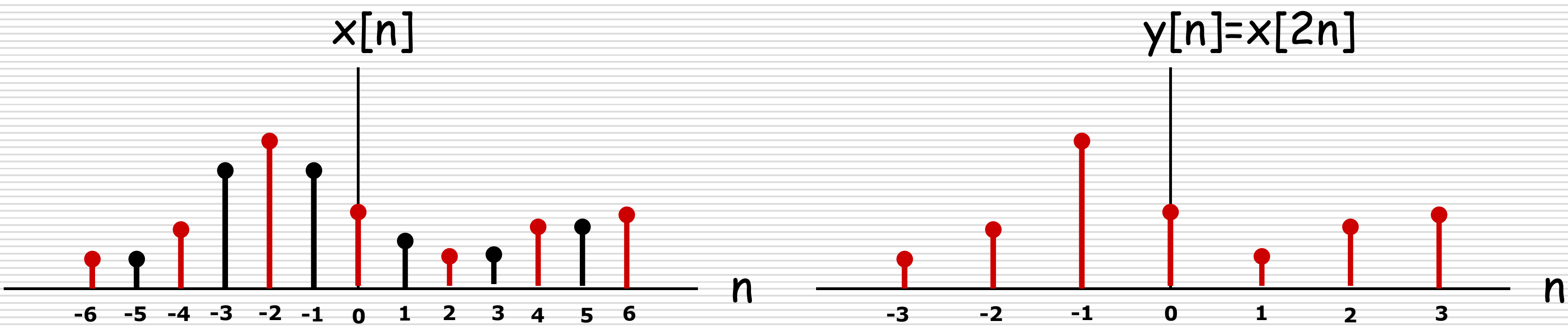
- **For Discrete-time signal**

$$y[n] = x[kn],\ k \geq 1$$

- If $k > 1$, $y[n]$ is the compressed version of $x[n]$ and some values of the signal is lost.

Examples of Time Scaling





Reflection

- For Continuous-time signal

If $x(t)$ is a continuous-time signal, the signal $y(t)$ obtained by replacing t by $-t$ is called the reflected version of $x(t)$, i.e.

$$y(t) = x(-t)$$

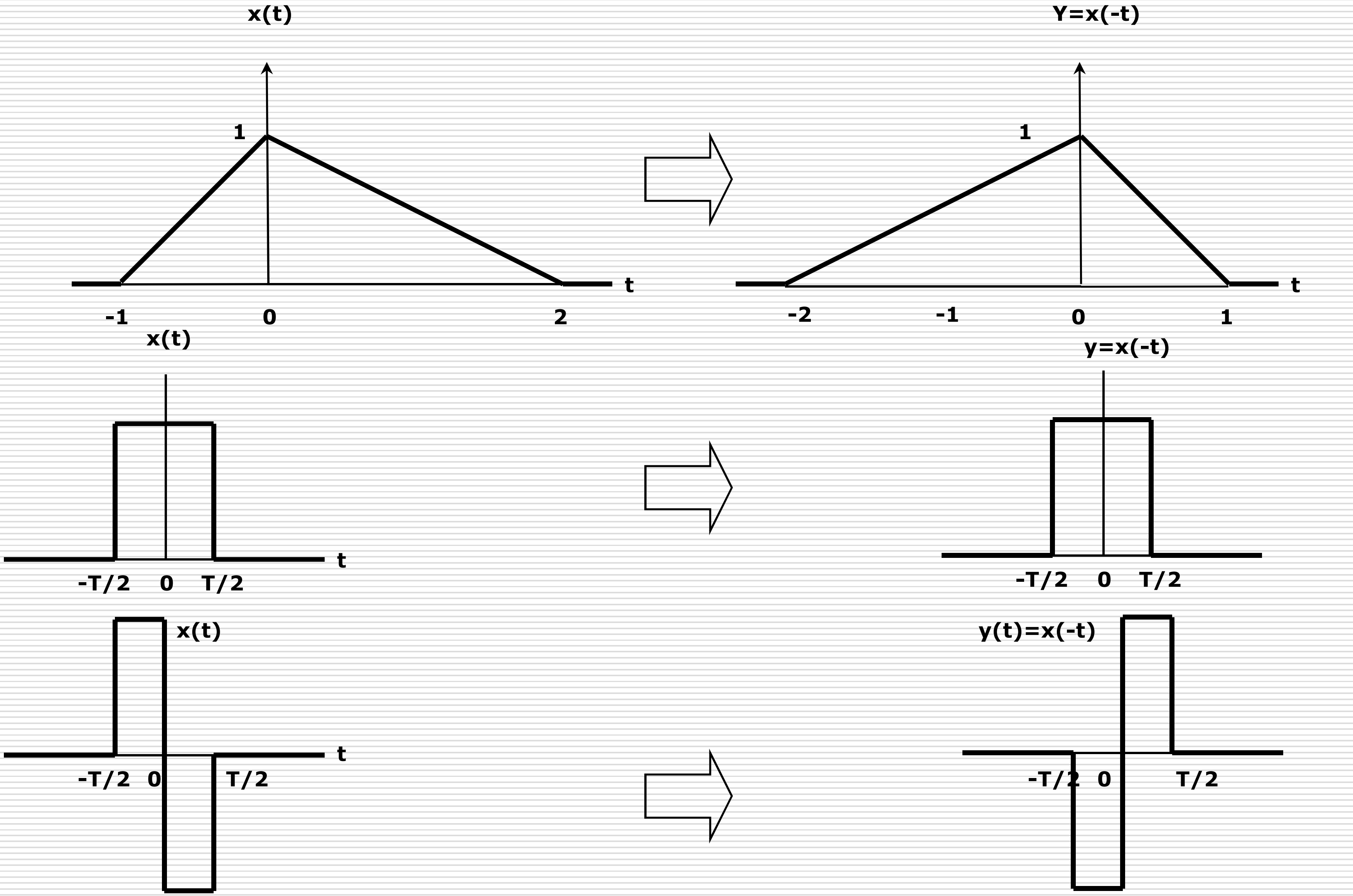
Special Cases :

- For even-signals, since $x(-t)=x(t)$ for all t, an even signal is same as its reflected version
- For Odd-signals, since $x(-t)=-x(t)$ for all t, an odd signal is '–'ve of its reflected version.

- For Discrete-time signal

Similar observations are applicable

Example of Reflection



Time Shifting

- For Continuous-time signal

If $x(t)$ is a continuous-time signal, the time shifted version of $x(t)$ defined by,

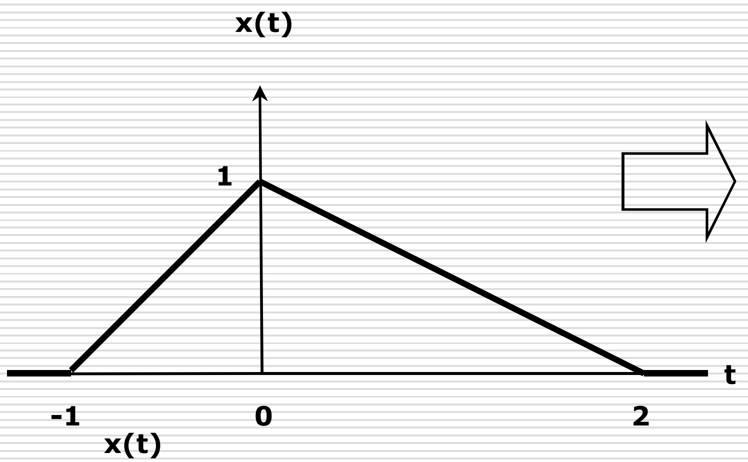
$$y(t) = x(t-t_0), \text{ here } t_0 \text{ is the time shift.}$$

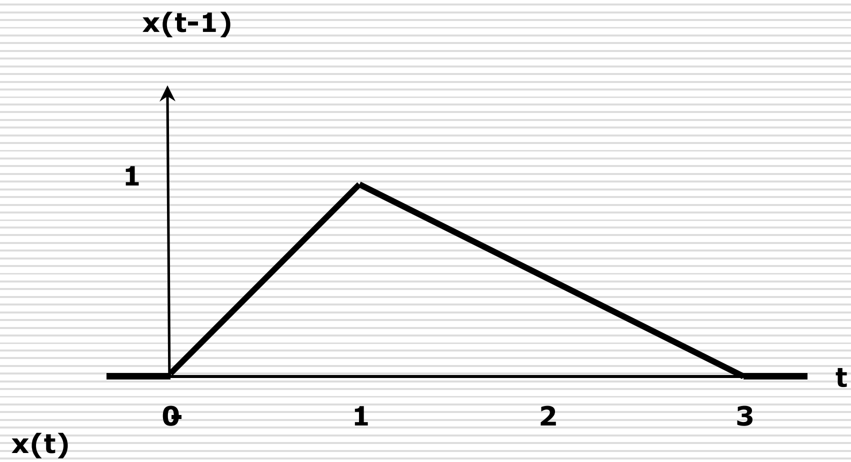
- If $t_0 > 0$, the waveform of $x(t)$ is shifted intact to the right.
- If $t_0 < 0$, the waveform of $x(t)$ is shifted intact to the left.

- For Discrete-time signal

$$y[n] = x[n-m], \text{ here } m \text{ is the}$$
shift which must be an integer.

Example of Time Shifting

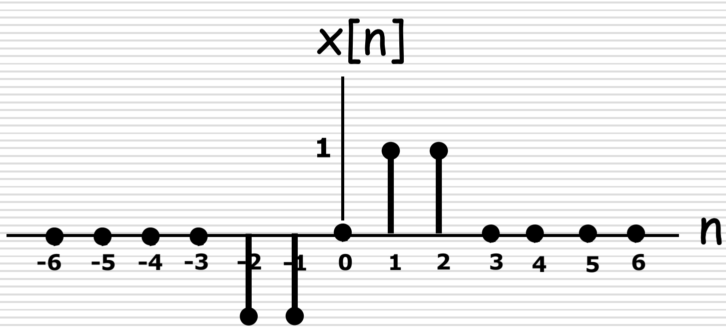


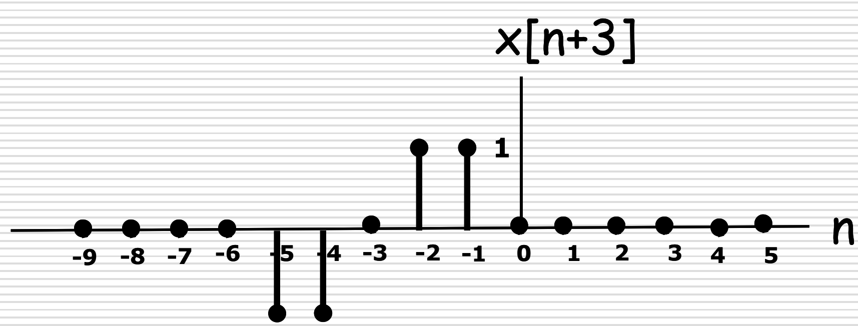


$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}$$

$$\Rightarrow$$

$$x[n+3] = \begin{cases} 1, & n = -1, -2 \\ -1, & n = -4, -5 \\ 0, & n = -3, n < -5 \text{ and } n > -1 \end{cases}$$





Time Shifting And Time Scaling

- For Continuous-time signal

If $x(t)$ is a continuous-time signal, the time shifted and time scaled version of $x(t)$ is defined by,

$$y(t) = x(at-b)$$

Relation between $y(t)$ and $x(t)$ satisfies the following conditions

1. $y(0) = x(-b)$
2. $y(b/a) = x(0)$

- These provide useful checks on $y(t)$ in terms of corresponding values of $x(t)$.

Time Shifting And Time Scaling contd.

- **The Precedence rule**

  First time shifting and then time scaling

- **Reason**

  Time scaling always replaces t by at

  Time shifting always replaces t by t-b

- **Implementation**

- Apply time shift operation that results in intermediate signal v(t),

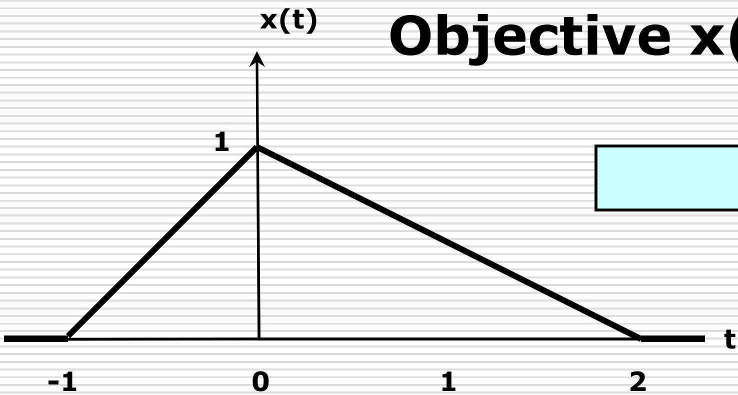
$$v(t) = x(t-b)$$

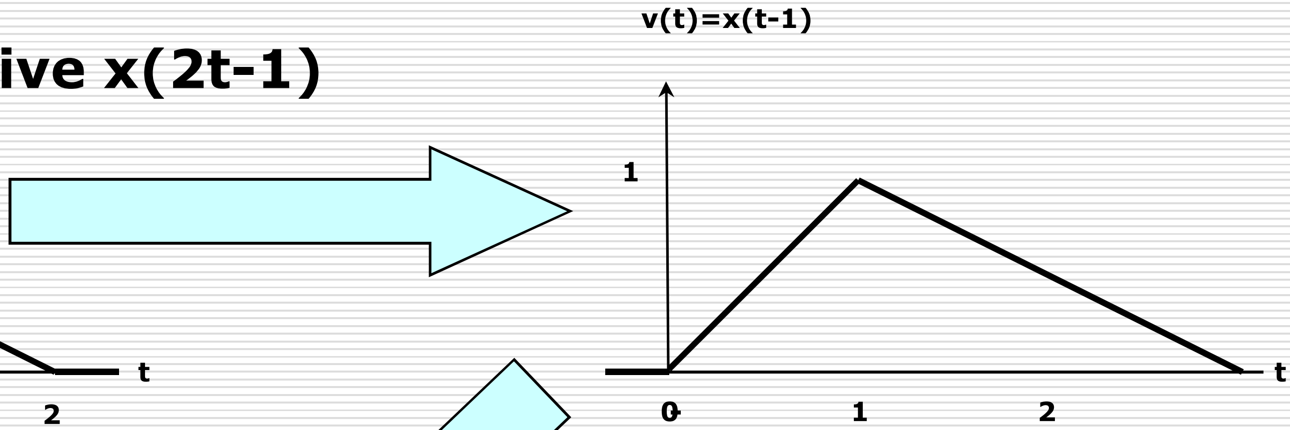
- Then apply time scaling on v(t) which will result in y(t)

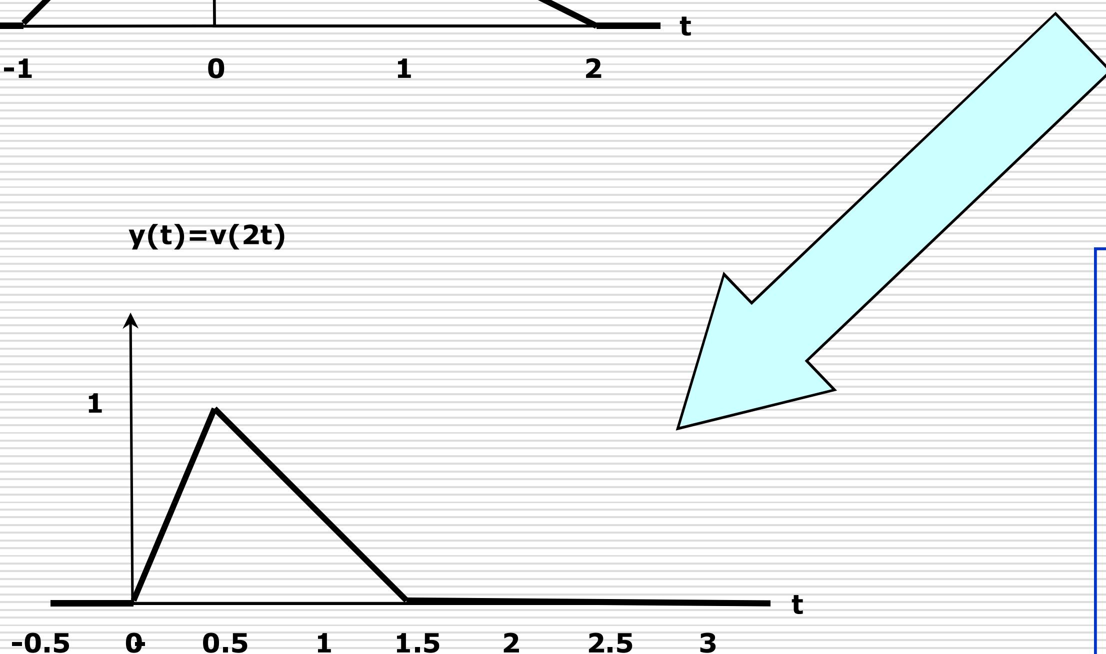
$$y(t) = v(at)$$

Time Shifting And Time Scaling contd.

Objective x(2t-1)







Checks

y(0)=0,x(-1)=0

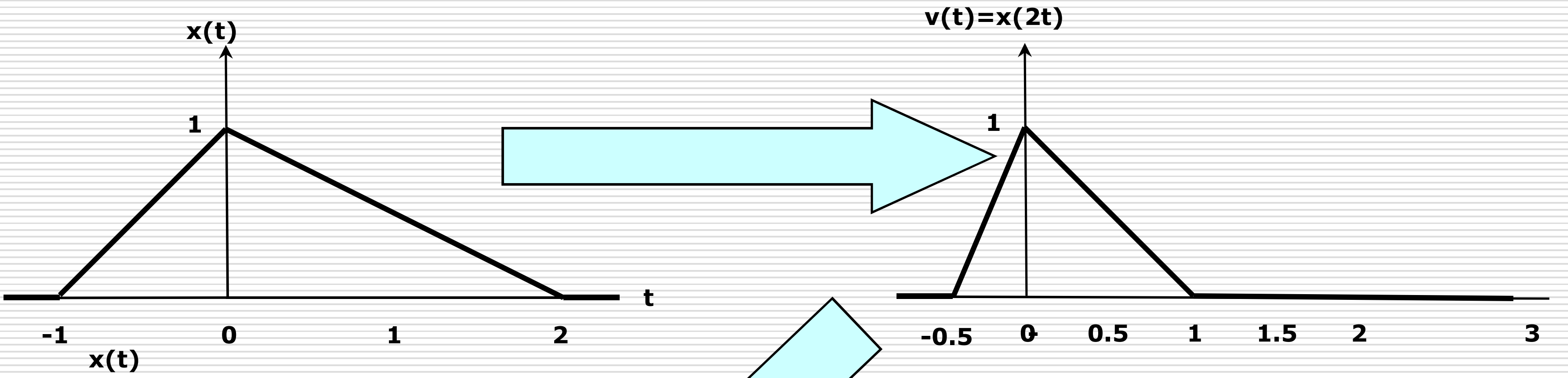
i.e. check 1 is satisfied

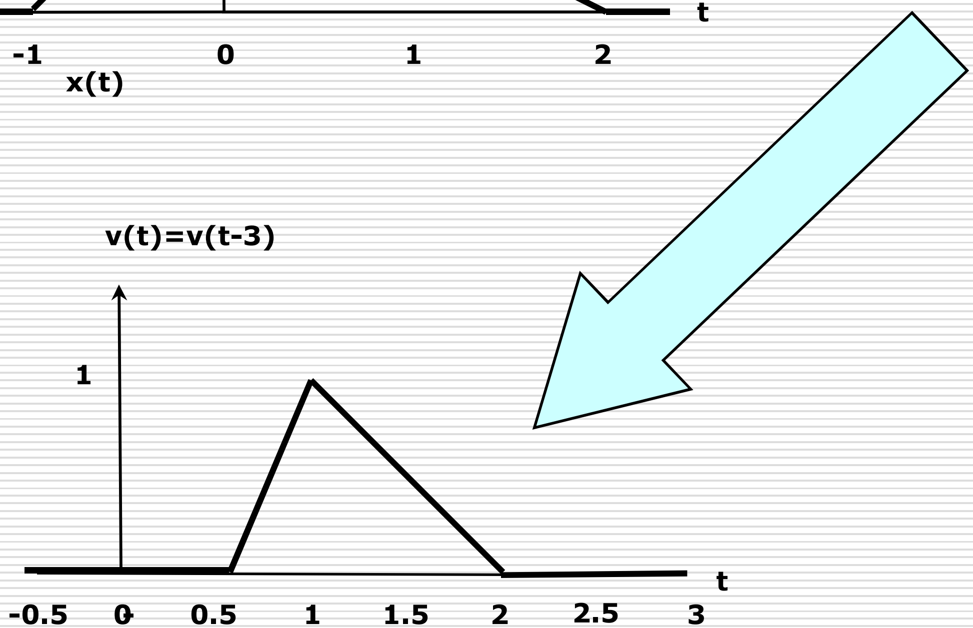
y(1/2)=1,x(0)=1

i.e. check 2 is also satisfied

Following precedence rule

Time Shifting And Time Scaling contd.





Checks

y(0)=0,x(-1)=0

i.e. check 1 is satisfied

y(1/2)=0,x(0)=1

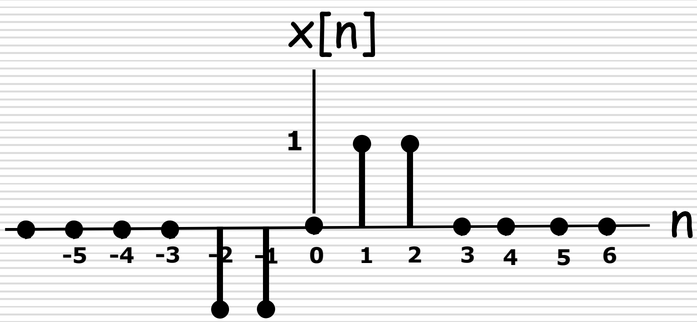
i.e. check 2 is not satisfied

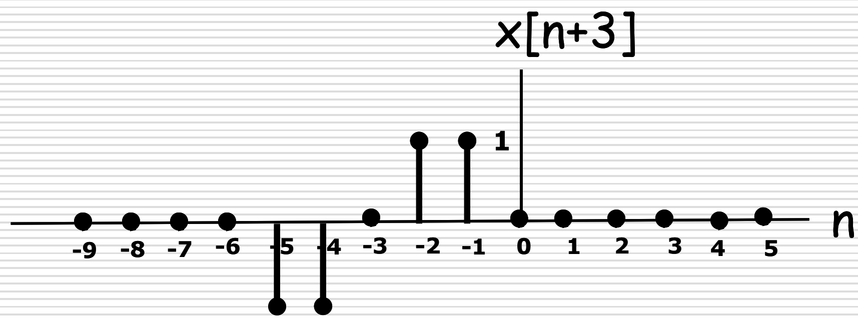
Precedence rule if not followed

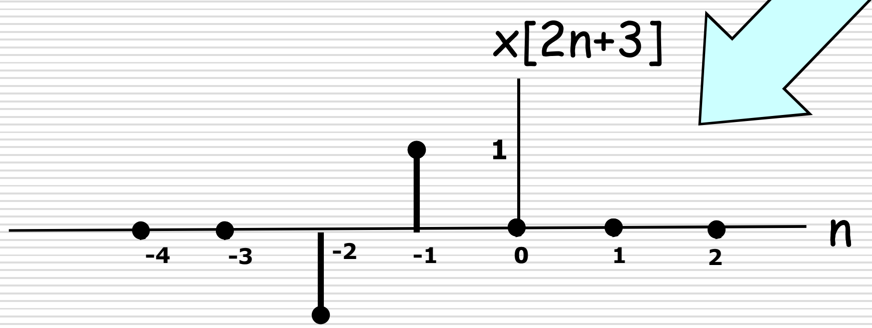
Time Shifting And Time Scaling contd.

$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}$$

$$x[n+3] = \begin{cases} 1, & n = -1, -2 \\ -1, & n = -4, -5 \\ 0, & n = -3,\ n < -5 \text{ and } n > -1 \end{cases}$$







Elementary Signals

Some elementary yet predominant signals

- Exponential Signals
- Sinusoidal Signals
- Exponentially Damped Sinusoidal Signals
- Impulse Function
- Ramp Function

These signals

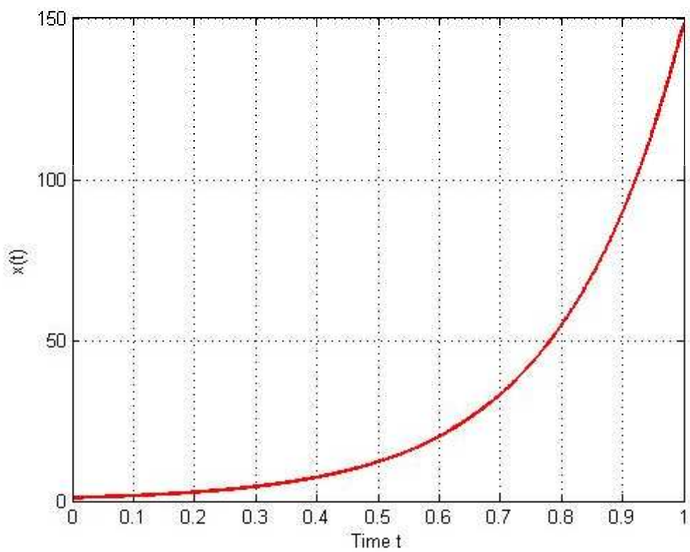
- Serve as the building blocks for the construction of more complex signals.
- Can be used to model many physical signals that occur in nature.

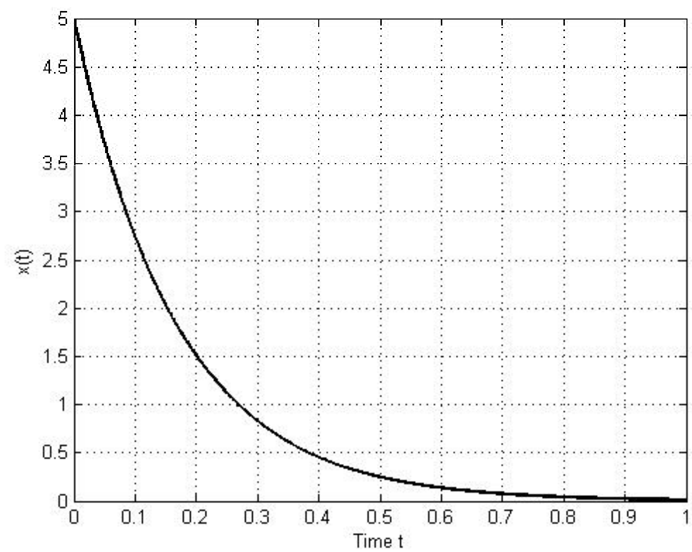
Exponential Signals

A real exponential signal in its most general form is

$$x(t) = Be^{at}$$

where B (amplitude at t=0) and a are both real.



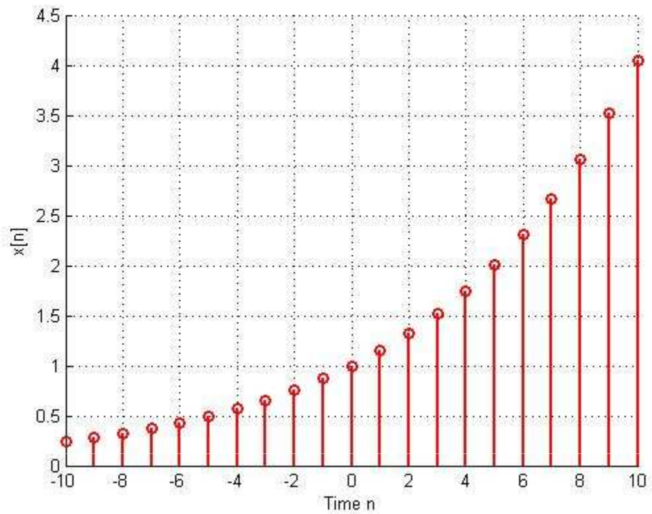


B=1, a=5

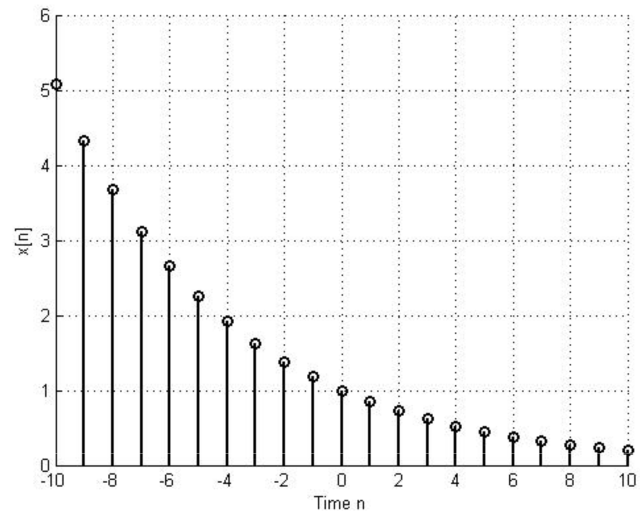
B=5, a=-6

Exponential Signals contd.

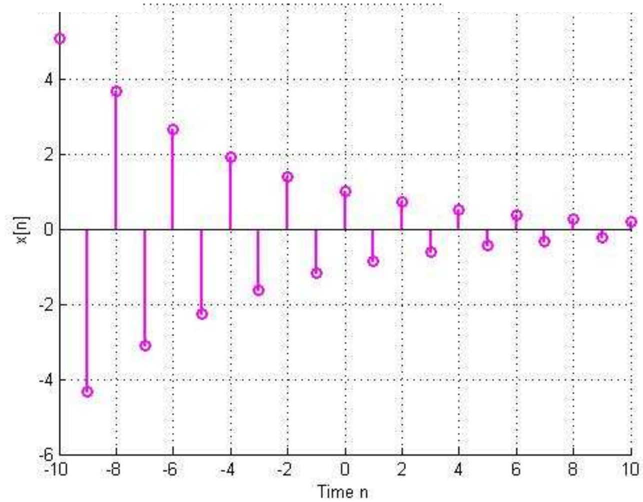
- Discrete-time approach, $x[n] = Br^n$, r can be defined as e^{α}



B = 1, r = 1.15



B = 1, r = 0.85



B = 1, r = -0.85

Sinusoidal Signals

- Continuous-time approach,

$$x(t) = A\cos(wt + \varphi)$$

- Discrete-time approach,

$$x[n] = A\cos(\Omega n + \varphi)$$

Again $x[n+N] = A\cos(\Omega n + \Omega N + \varphi)$

To be periodic with period of N samples,

$x[n+N]$ must be equal to $x[n]$

i.e. $\Omega N = 2\pi m$, where m is an integer,

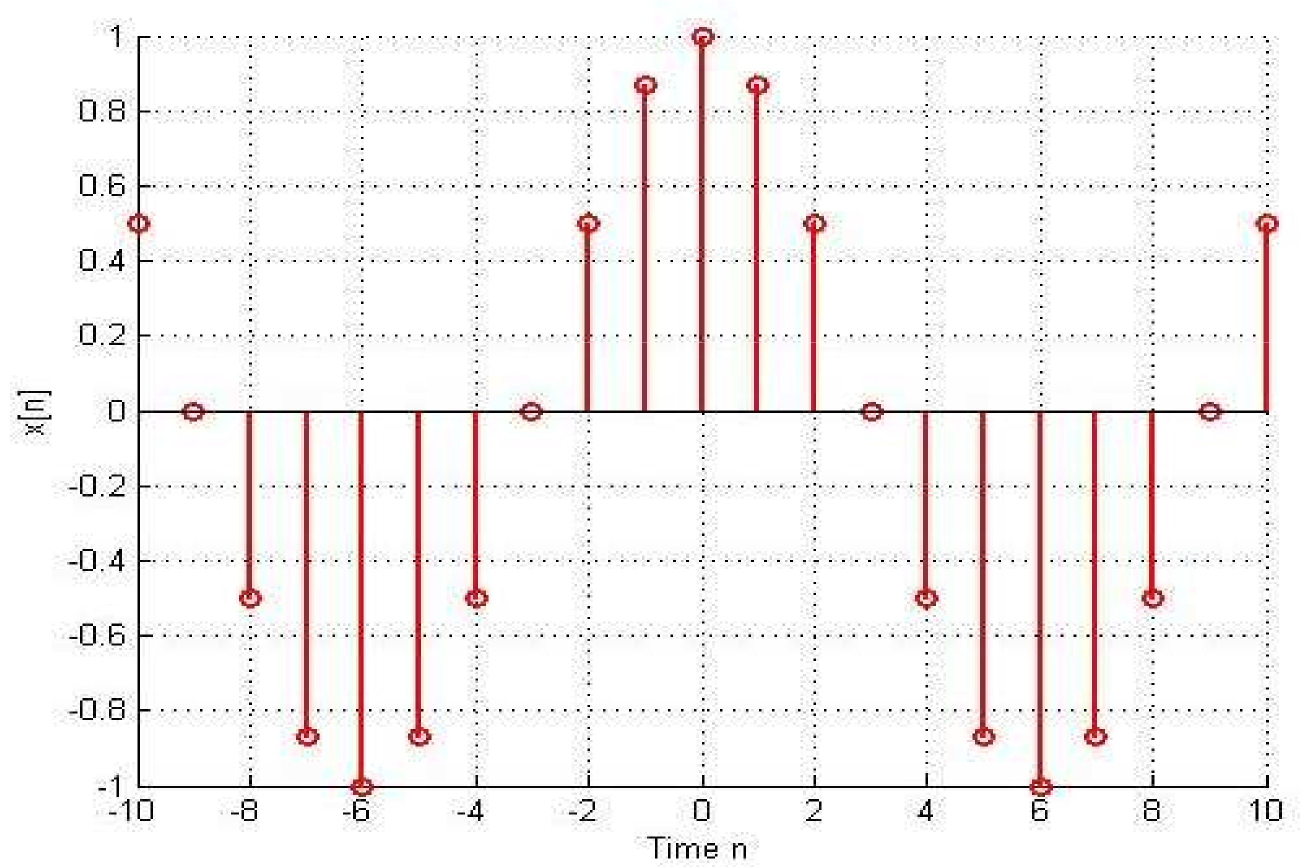
means $N = 2\pi m/\Omega$

Since N must be an integer, only those value of Ω are allowed that makes $2\pi m/\Omega$ integer.

- Not all discrete-time sinusoids with arbitrary value of Ω are periodic.

Sinusoidal Signals contd.

- Example : cos(2*pi/12*n)



Sinusoidal Signals contd.

- $x[n] = 5\sin(2n)$

  Here, $N=2\pi m/2$,

  N can not be integer for any integer value of m

  Hence, nonperiodic.

- $x[n] = 5\sin(6\pi n/35)$

  $N = 2\pi m/(6\pi/35)$,

  $= 35m/3$

  N is integer for m=3,6,9…

  Periodic, Fundamental period N = 35.

- $x[n] = 5\sin(0.2\pi n)$

  $N=2\pi m/0.2\pi$, N is integer for m=1,2,3…

  Periodic, Fundamental period N = 10.

Sinusoidal Signals contd.

Relation between Sinusoidal and Complex Exponential Signals.

From Euler's Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Rightarrow \cos\theta = \mathrm{Re}[e^{j\theta}]$$

Similarly,

$$A\cos(\omega t + \varphi) = A\,\mathrm{Re}[e^{j(\omega t + \varphi)}]$$

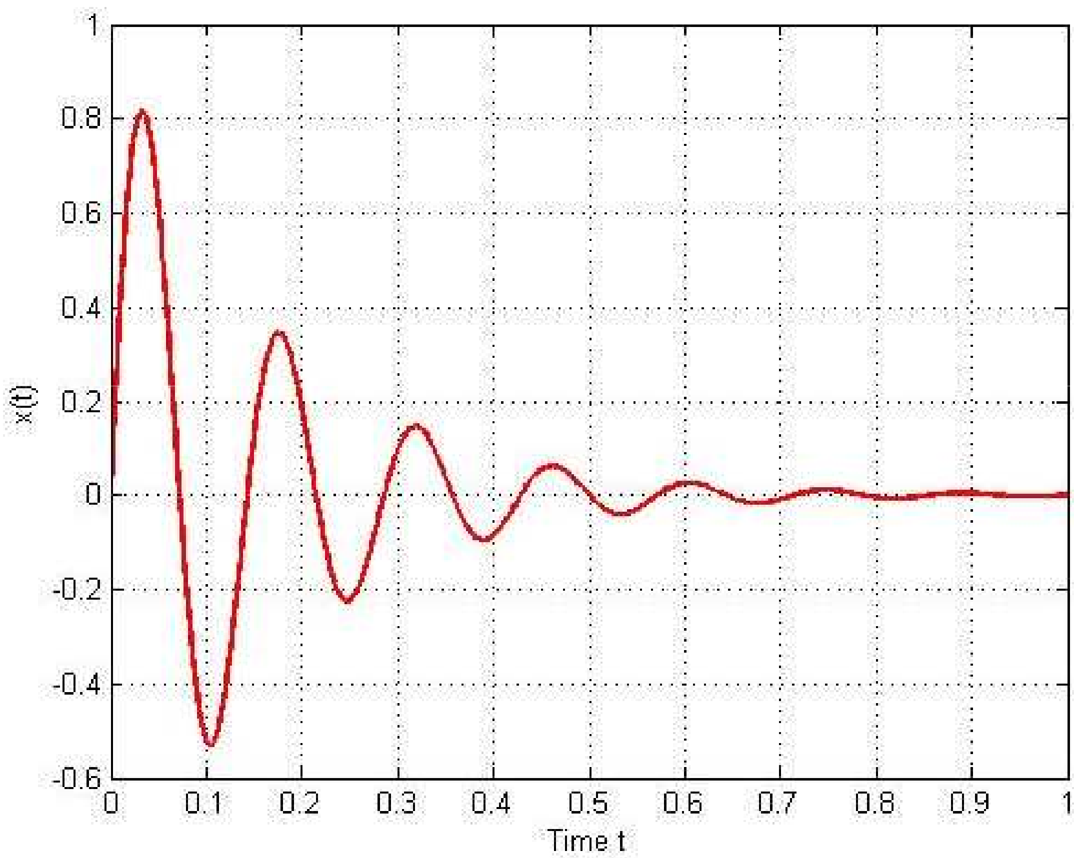
$$\Rightarrow A\cos(\omega t + \varphi) = \mathrm{Re}[Be^{j\omega t}]$$

where, $B = Ae^{j\varphi}$

Similarly, $A\sin(\omega t + \varphi) = \mathrm{Im}[Be^{j\omega t}]$

Exponentially Damped Sinusoid

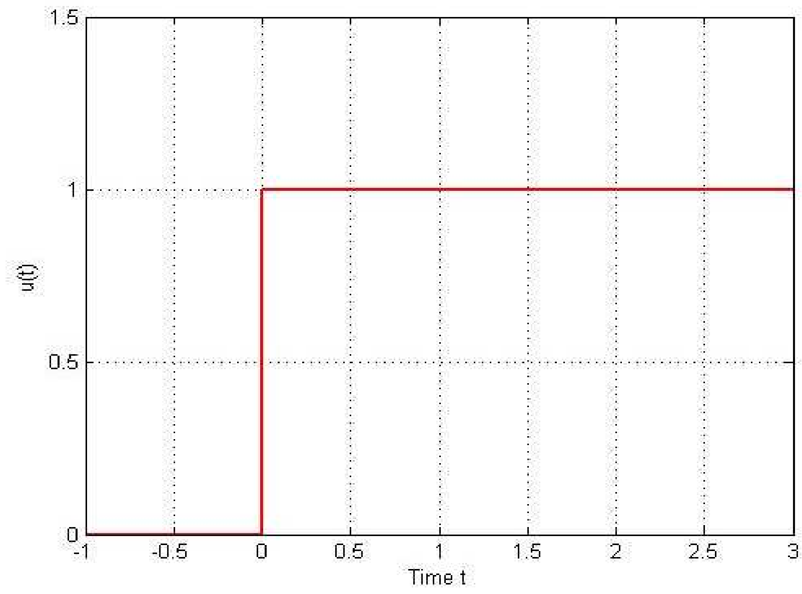
$x(t) = Ae^{-\alpha t}\sin(\omega t)$

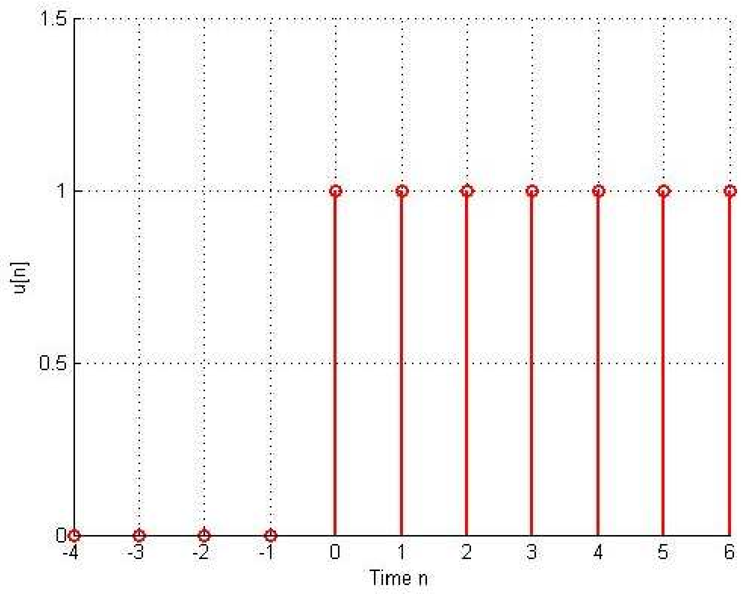


Example : Response of an RLC ckt.

Unit Step Function

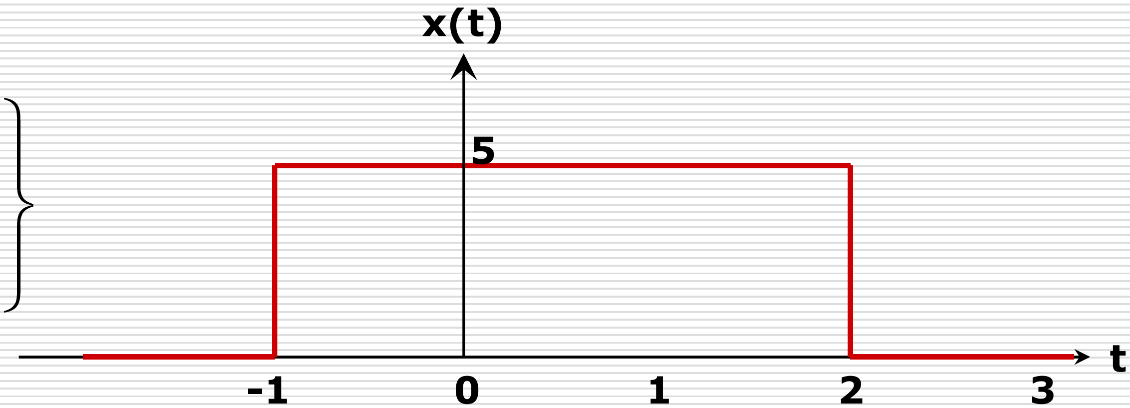
- Defined by
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$
- Used for testing and defining other signals.
- For example, when different shifted versions of the step function are multiplied by other signals, one can select a certain portion of the signal and zero out the rest.



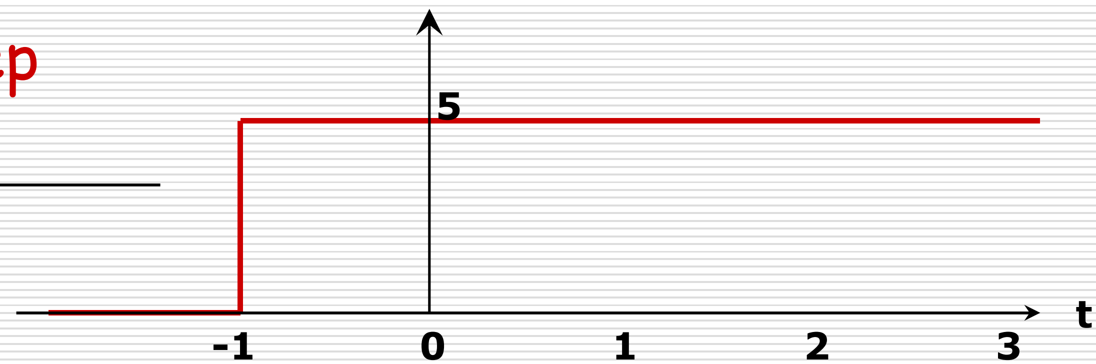


Unit Step Function contd.

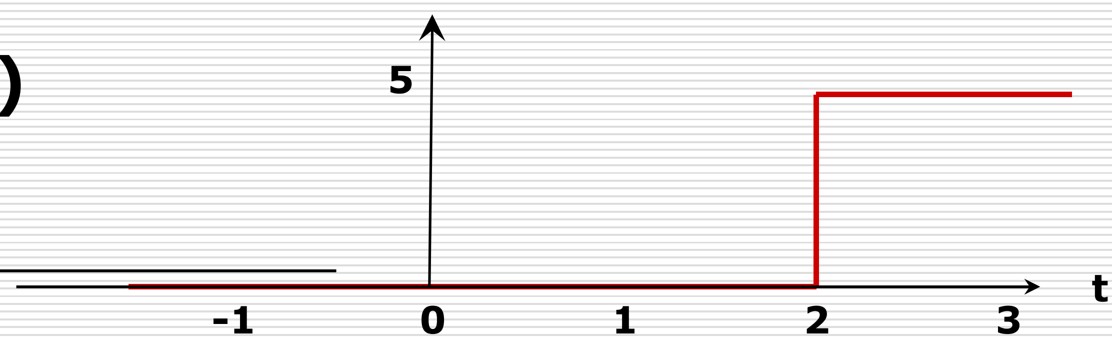
$$x(t) = \begin{cases} 5, & -1 \le t \le 2 \\ 0, & \text{Otherwise} \end{cases}$$



- Expressing it with step functions



5*u(t+1) – 5*u(t-2)



Impulse Function

- Unit impulse function (Dirac function): *A* function that has zero duration, infinite amplitude and unit area under it.
- Designated by δ(t), where

$$\delta(t) = 0, \quad \text{for } t \neq 0$$

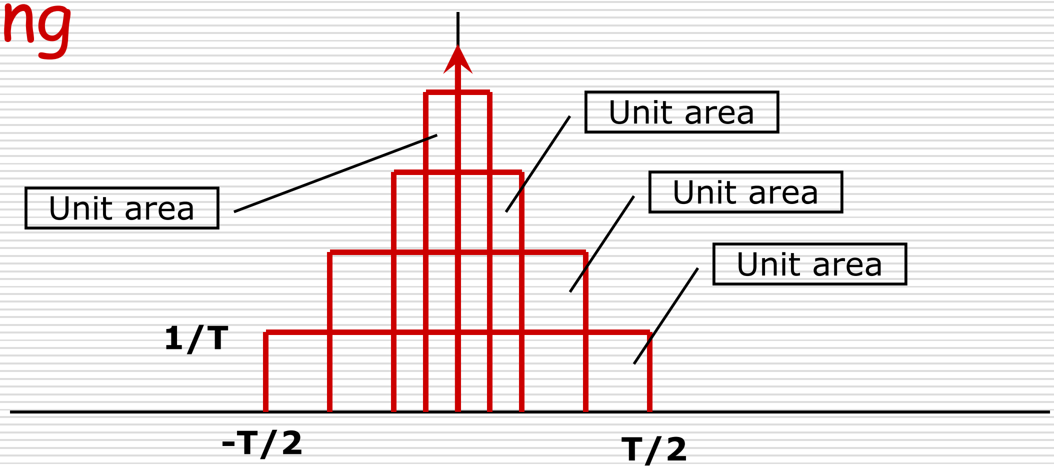
and

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

- Used to quantitatively define a signal that occurred momentarily.
- Used heavily in engineering while classical mathematics still doubt about its claim being a function.

Impulse Function

- Visualizing



$$\delta(t) = \lim_{T \to 0} g_T(t)$$

- Area under the pulse is called the strength of the impulse. It is unity for unit impulse.

Impulse Function contd.

The Sifting (or Sampling) Property

If a continuous signal $x(t)$ is multiplied by an impulse $\delta(t)$ at $t=0$, we get

$$x(0)\,\delta(t)$$

Integrating

$$\int_{-\infty}^{+\infty} x(0)\delta(t)dt$$

$$= x(0)\int_{-\infty}^{+\infty} \delta(t)dt$$

$$= x(0)$$

Impulse Function contd.

❑ Unit Impulse Properties

$$1.\ \delta(t) = \delta(-t)$$

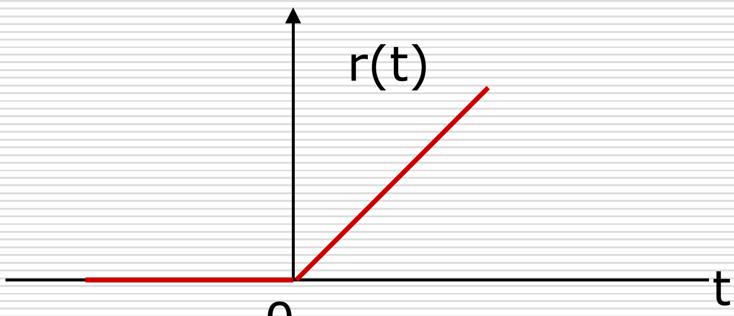
$$2.\ \delta(at) = \frac{1}{a}\delta(t), a > 0$$

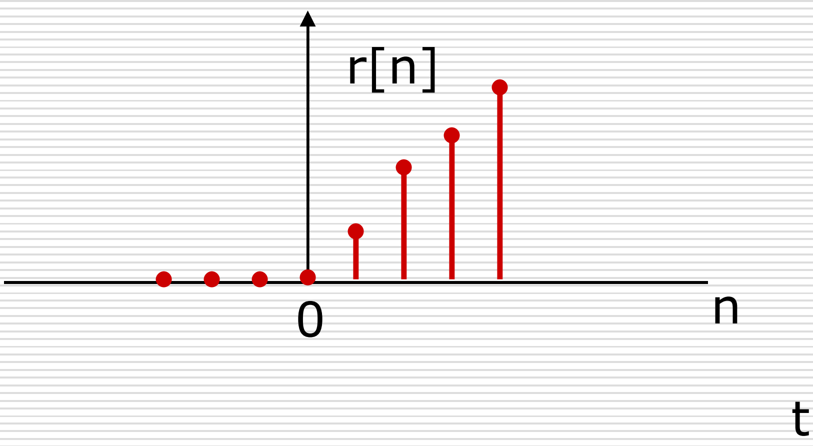
$$3.\ \delta(t) = \frac{d}{dt}u(t),\ \ t \neq 0$$

Ramp Function

- Designated by r(t) or r[n], where

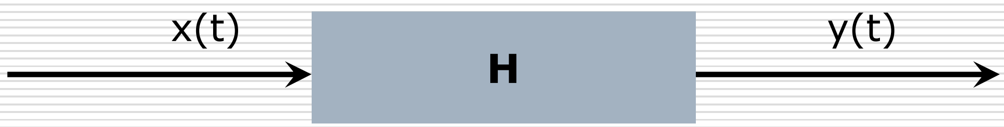
$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$





System Viewed as Interconnections of Operations

A System can be viewed as interconnections of operations that transform and input signal into an output signal having different properties than the input signal.



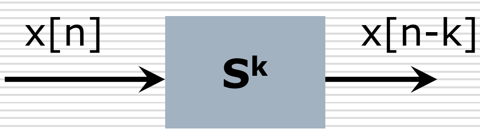
H is the overall operator
It denotes the action of the system

System Viewed as Interconnections of Operations

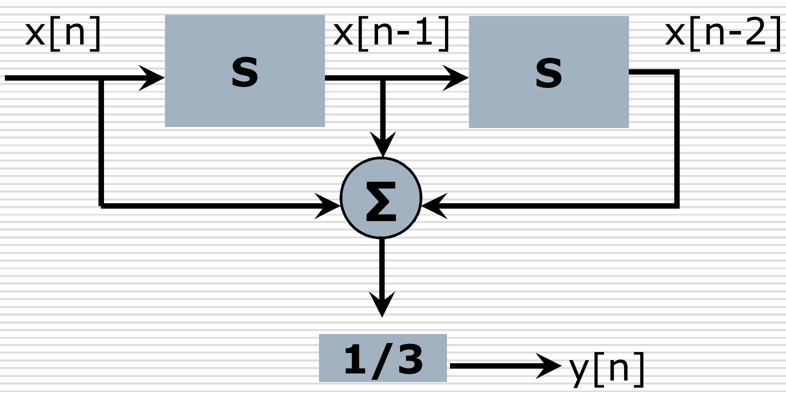
Formulation of the operator H for discrete-time system that has the following output- input relationship.

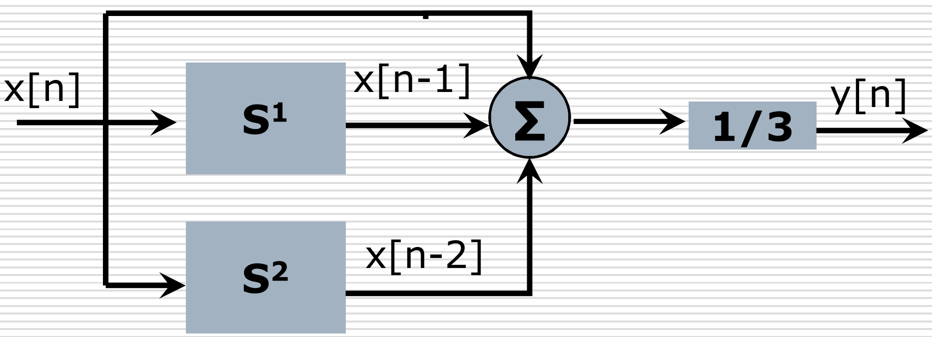
$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

Discrete time shift operator s^k



$$H = \frac{1}{3}(s^0 + s^1 + s^2)$$





Properties of Systems

- Stability
- Memory
- Causality
- Invertibility
- Time invariance
- Linearity

Stability

A System is bounded input bounded output (BIBO) stable if every bounded input results in bounded output, i.e. the output does not diverge as long as the input does not diverge.

$$\text{output } |y(t)| \le M_y < \infty \text{ for all } t$$

$$\text{when input } |x(t)| \le M_x < \infty \text{ for all } t$$

where M_x and M_y are some finite positive numbers

Stability contd.

- BIBO stable

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- BIBO unstable

$$y[n] = r^n x[n], \quad r > 1$$

Memory

- A System is said to possess memory if the output signal depends on the past values of the input signal.
  - Examples
    - Voltage across a capacitor or current through an inductor. $$v(t) = \frac{1}{C}\int_{-\infty}^{t} i\,d\tau$$
    - System having the following output-input relationship $$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- In a memoryless system the output signal depends only on the present value of the input signal.
  - Current or voltage of a resistor.
  - System having the following output-input relationship $$y[n] = 3x[n]$$

Causality

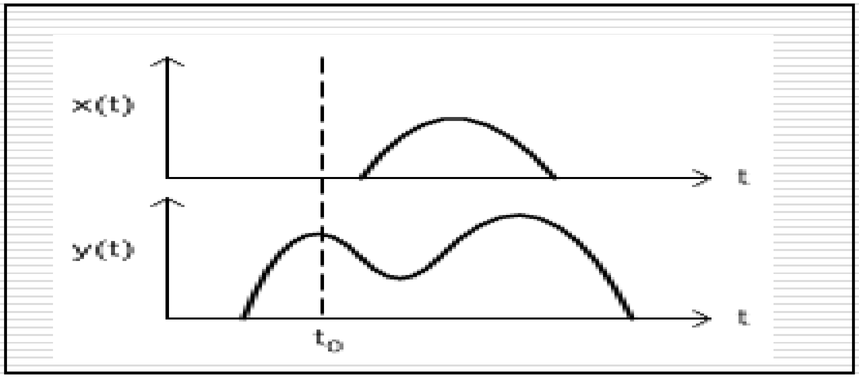
- Causal System : The output signal depends only on the present or past values of the input signal.
  - Examples
    - Voltage across a capacitor or current through an inductor.
    - System having the following output-input relationship

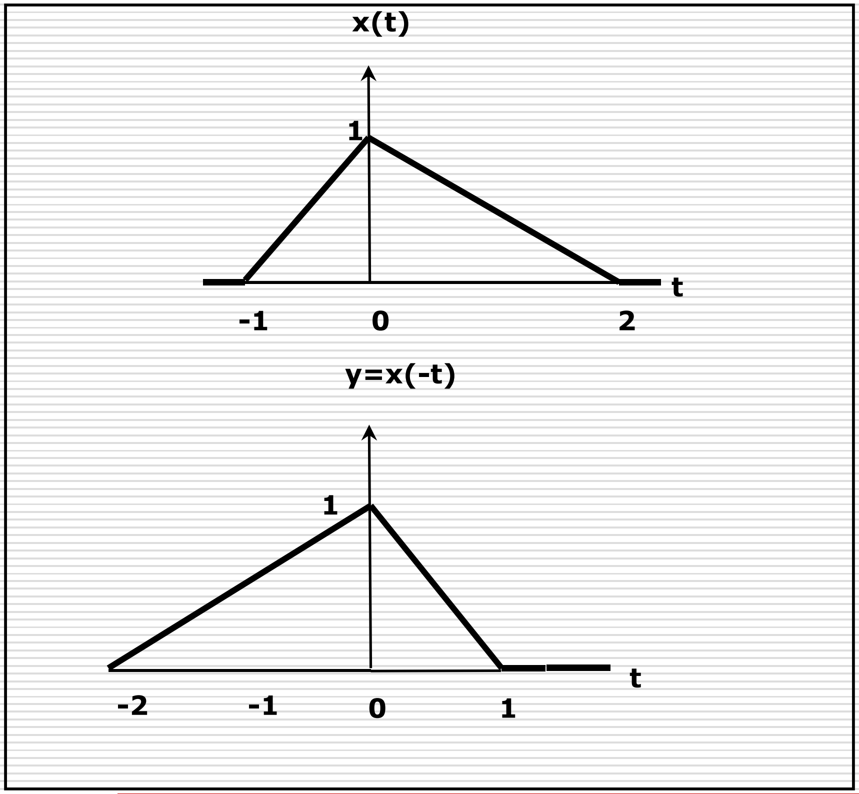
$$y[n] = \tfrac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- Noncausal System : The output signal depends on future values of the input signal. No physical or real systems are noncausal.
  - Example
    - System having the following output-input relationship

$$y[n] = \tfrac{1}{3}(x[n] + x[n+1] + x[n-2])$$

Causality contd.



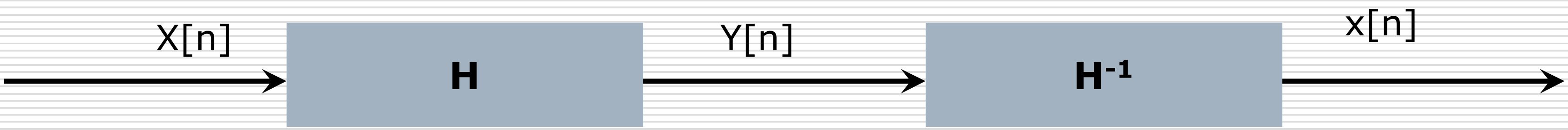


Causal?

Noncausal?

Invertibility

- A system is invertible if the input can be reconstructed from the output.
- Can be visualized as a cascaded system as below



$$H^{-1}\{y([n]\} = H^{-1}\{H\{x([n]\}\}$$

$$= H^{-1}H\{x([n]\}$$

- To be invertible we require that

$$H^{-1}H = I$$

where I is the Identity operator.

Invertibility contd.

- For a system to be invertible, distinct input must yield distinct output.

$$y[n] = x^2[n]$$

Noninvertible

$$y(t) = \frac{d}{dt}\{x(t)\}$$

Noninvertible

$$y[n] = x^3[n]$$

Invertible

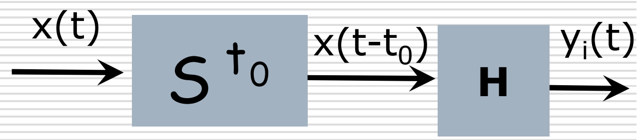
Time Invariance

- A system is time-invariant if a time shift in the input signal leads to an identical time shift in the output signal.
- Stated otherwise, the characteristics of a time-invariant system do not change with time.

➢ Considering a system where $y(t) = H\{x(t)\}$

Now, if the input signal is delayed by t_0, the new input will be $x(t-t_0)$

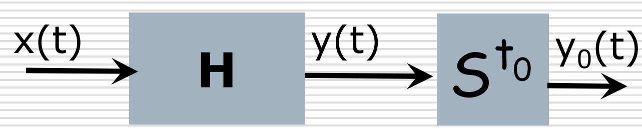
Let $y_i(t)$, be the new output



$$y_i(t) = H\{S^{t_0}\{x(t)\}\}$$

$$y_i(t) = HS^{t_0}\{x(t)\}$$

➢ Now let's assume $y_0(t)$ is the output of the original system shifted by t_0



$$y_o(t) = S^{t_0}\{y(t)\}$$

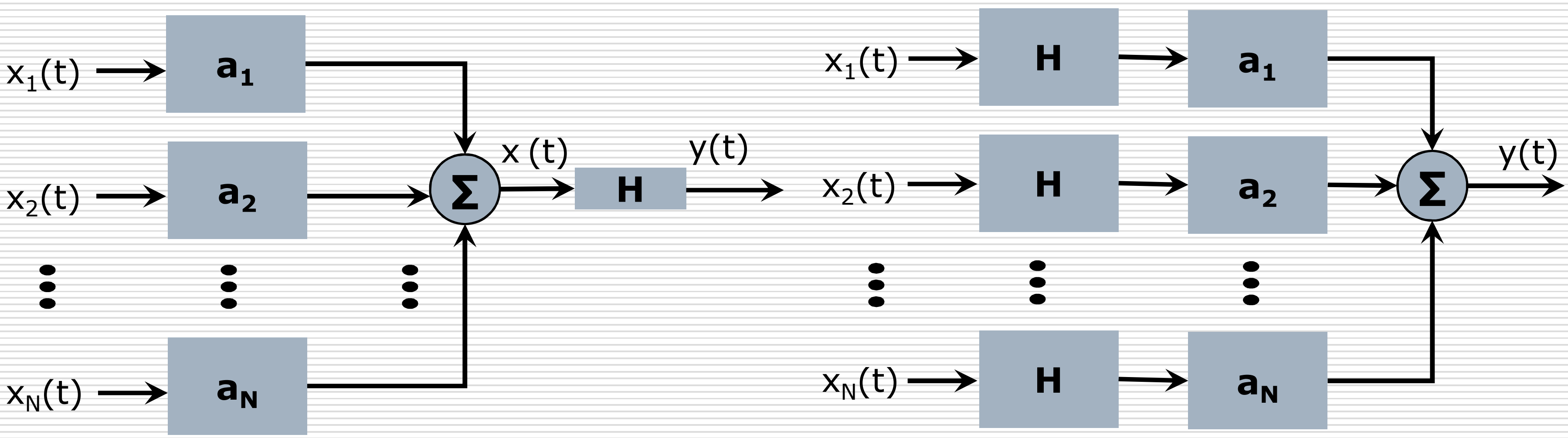
$$y_o(t) = S^{t_0}\{H\{x(t)\}\}$$

$$y_o(t) = S^{t_0}H\{x(t)\}$$

❖ To be time invariant, $HS^{t_0} = S^{t_0}H$, i.e., H and S^{t_0} must commute.

Linearity

- A system is linear if it satisfies the principle of superposition.



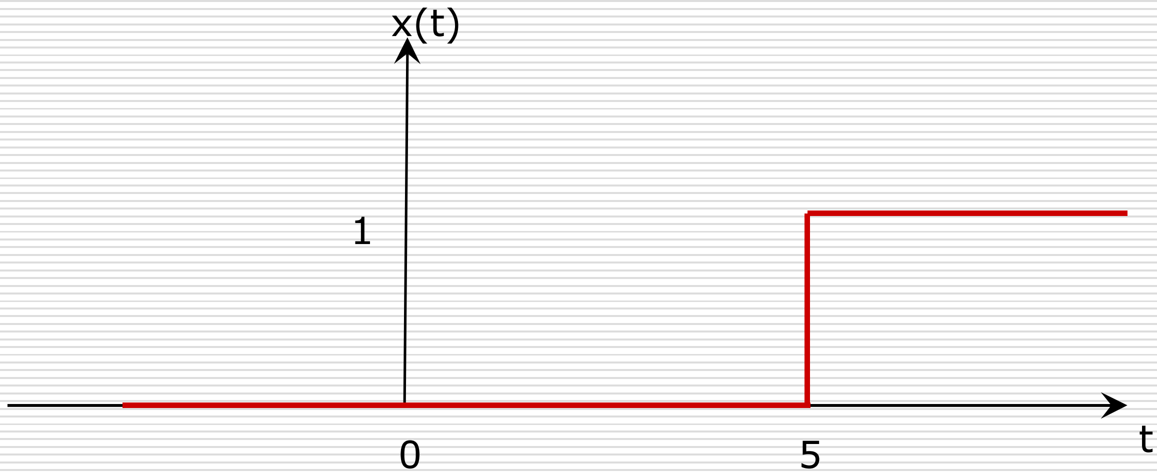
$$y(t) = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\}$$

$$y(t) = \sum_{i=1}^{N} a_i H\{x_i(t)\}$$

- For linearity H must commute with summation and amplitude scaling.

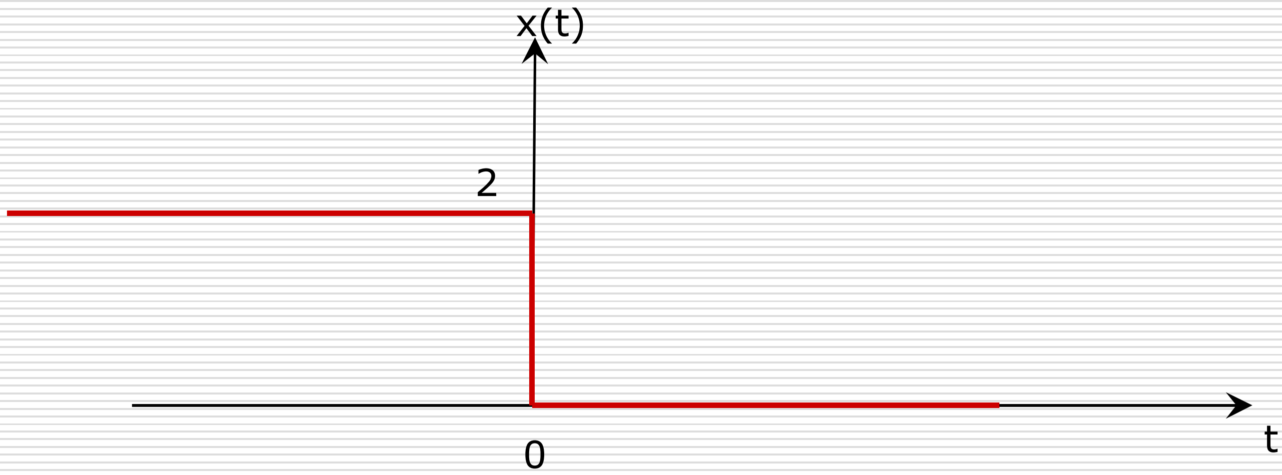
Review of Chapter 1

- **Plot u(t-5)**



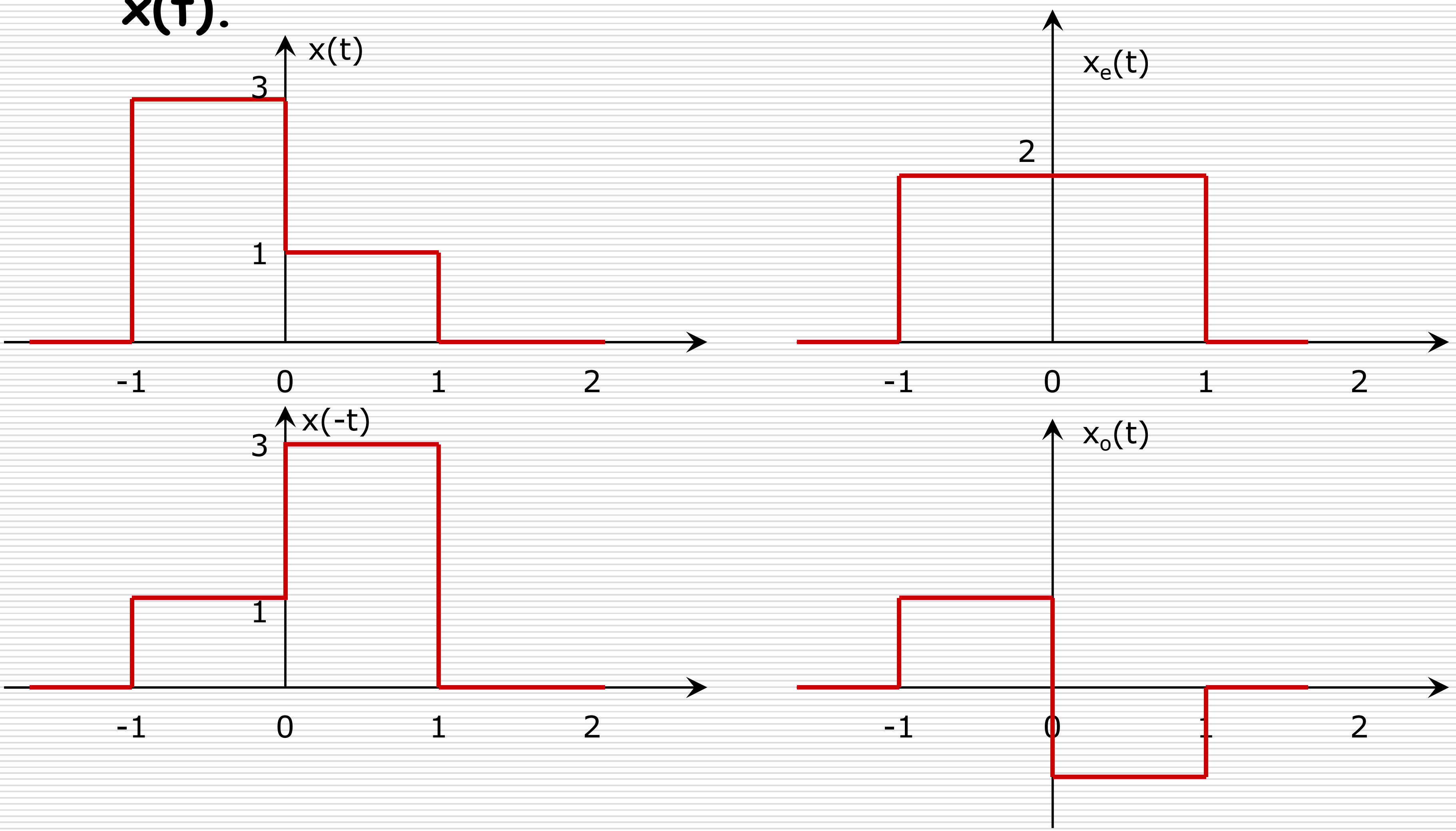
Review of Chapter 1 contd.

- Plot $x(t)=2u(-t)$

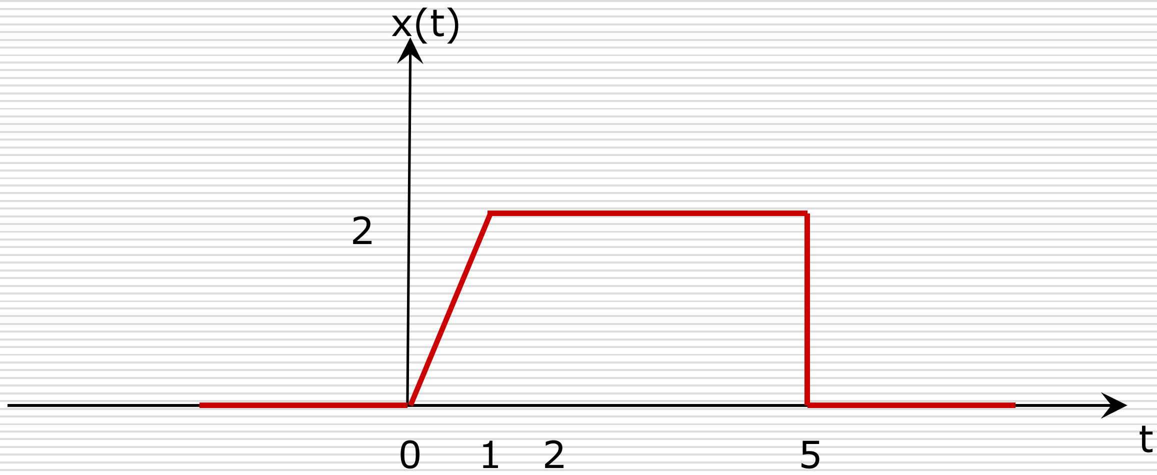


Review of Chapter 1

- Develop the Even and Odd decomposition of x(t).



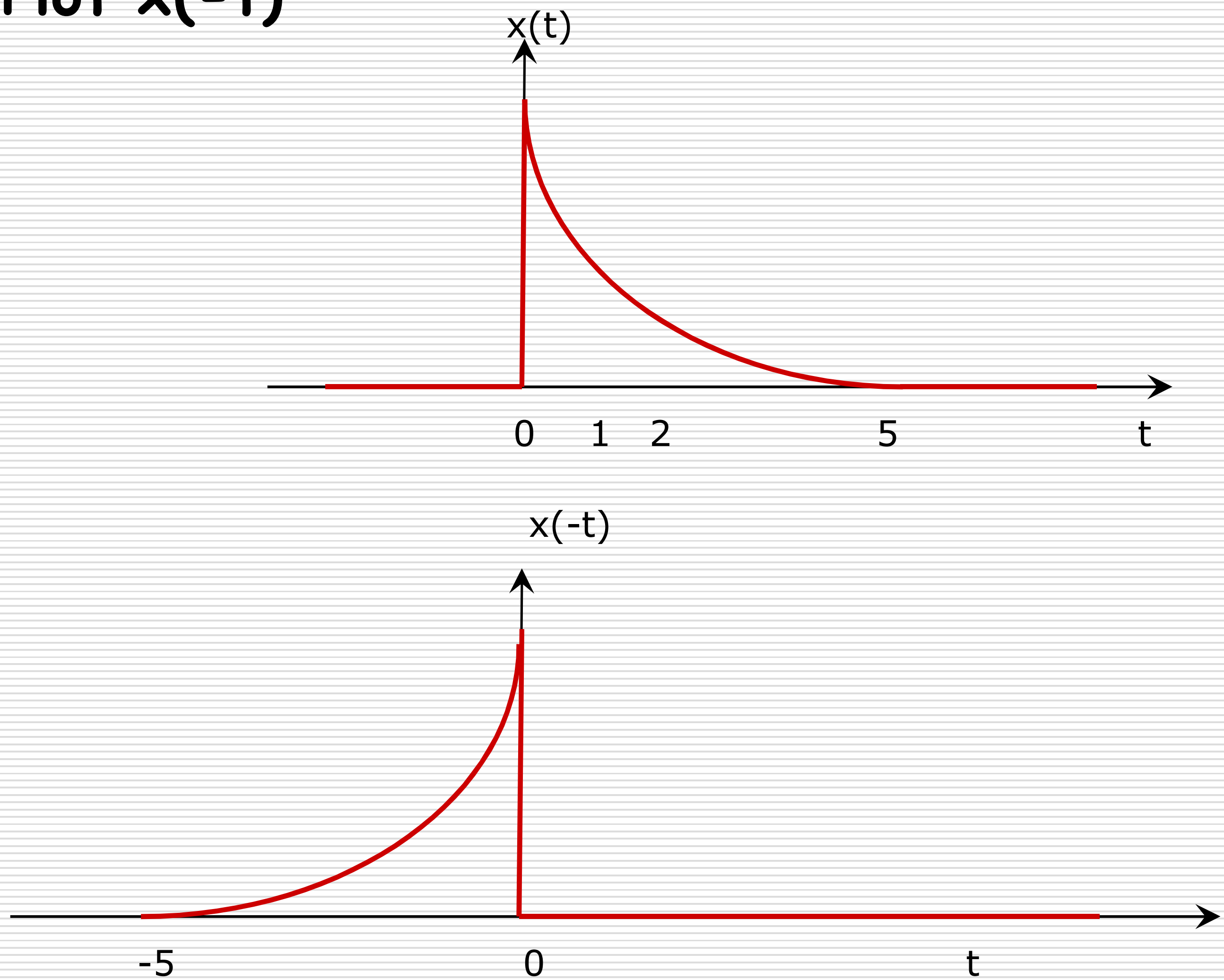
- Express the signal with unit step and ramp function.



x(t) = 2r(t) - 2r(t-1) -2u(t-5)

Review of Chapter 1 contd.

- Plot x(-t)



Review of Chapter 1 contd.

- $x[n]=\cos(2n)$…Periodic or non periodic?
- $x[n]=(-1)^n$ …Periodic or non periodic?

$x[n] = \cos(2n)$

Here, $N=2\pi m/2$,[$\Omega=2$]

N can not be integer for any integer value of m.

Hence, nonperiodic

x[n]

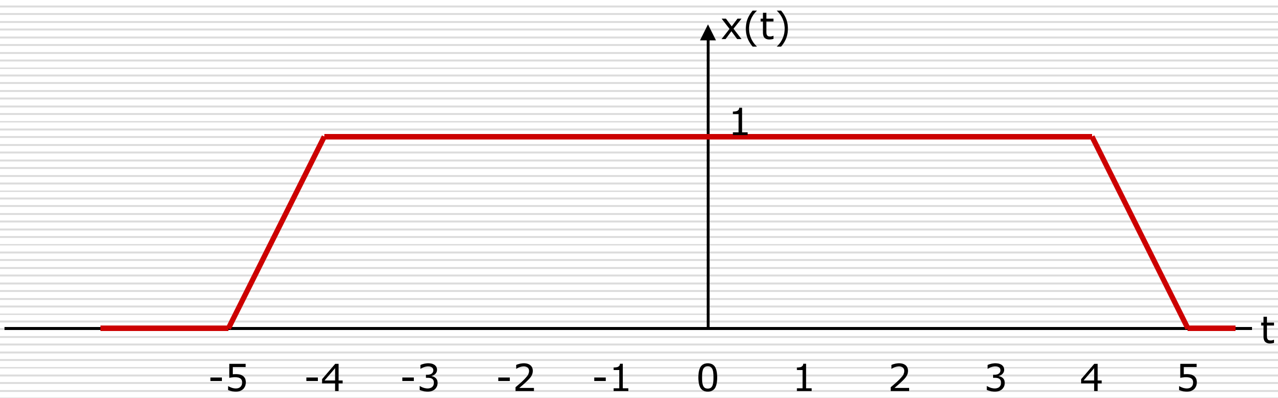
Review of Chapter 1 contd.

1. sin(2t)… Energy signal or Power Signal?
2. sin(2t)… What is the average power?

$$
\begin{aligned}
P_{av} &= \frac{1}{T}\int_0^T \sin^2 t\,dt \\
&= \frac{1}{2T}\int_0^T (1-\cos 2t)\,dt \\
&= \frac{1}{2T}\left[t-\frac{1}{2}\sin 2t\right]_0^T \\
&= \frac{1}{2T}\left[T-\frac{1}{2}\sin 2T\right] \\
&= \frac{1}{2\pi}\left[\pi-\frac{1}{2}\sin 2\pi\right] \\
&= \frac{1}{2}
\end{aligned}
$$

Review of Chapter 1 contd.

- Express x(t) with respect to t.



$$x(t) = \begin{cases} 5+t, & -5 \le t \le -4 \\ 1, & -4 \le t \le 4 \\ 5-t, & 4 \le t \le 5 \\ 0, & \text{Otherwise} \end{cases}$$

- Determine total Energy of x(t)

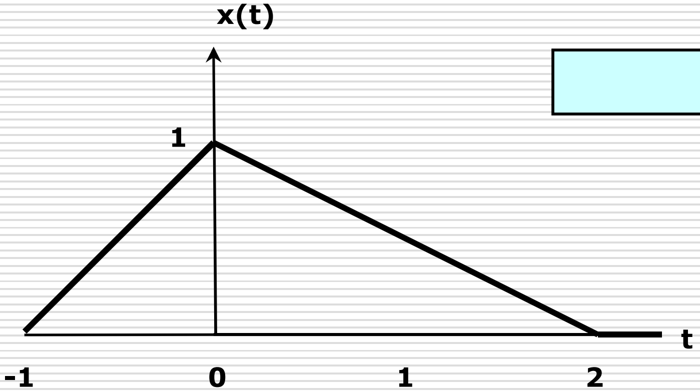
$$E = 2 * \int_{-5}^{-4} (5+t)^2 \, dt + \int_{-4}^{4} 1dt$$

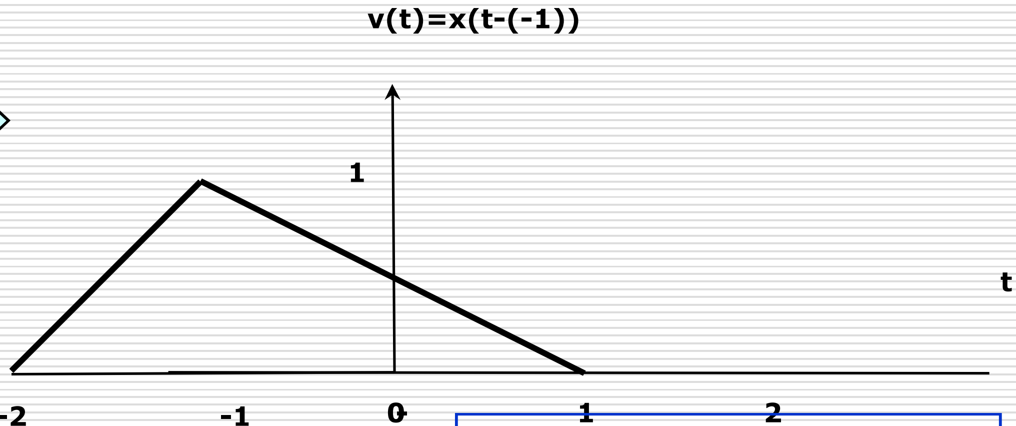
$$= 2 * [25t + 5t^2/2 + t^3/3]_{-5}^{-4} + 8$$

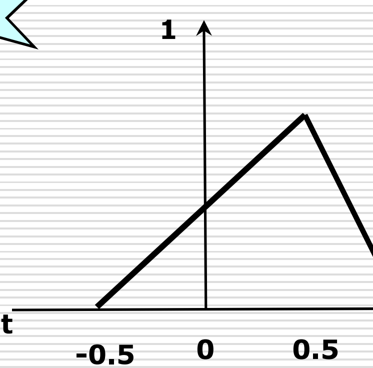
$$= 26/3$$

Review of Chapter 1 contd.

- Given x(t), draw x(1-2t)







Checks

y(0)=x(1) **[y(0)=x(-b)]**

i.e. check 1 is satisfied

y(-1/-2)=1,x(0)=1)
[y(b/a)=x(0)]

i.e. check 2 is also satisfied

Linear Time Invariant (LTI) System

- The main objective is to study the relationships between input and output of LTI systems in time domain.
- In fulfilling that objective we will study
- Impulse response of LTI systems
  - Applying the knowledge of impulse response, we are interested in determining the output of the system for any arbitrary input. [Convolution]
- Solving constant coefficient differential and difference equations.
- Block diagram representation.
- State variable description

Convolution: Impulse Response Representation for LTI Systems

- Impulse Response : Output of a system due to an impulse input applied at t=0 or n=0.
- Impulse response completely characterizes the behavior of a LTI system.
- Impulse response can be determined from the knowledge of system configuration or can be measured by applying approximate impulse as input.
- For Linear systems, if the input can be expressed as weighted superposition of time shifted impulses, then the output is a weighted superposition of the system response to each time shifted impulse.
- Again for TI system, system response to time shifted impulse is the time shifted version of impulse response.

The Convolution Sum

- From the sifting property of unit impulse ,

$$x[n]\delta[n] = x[0]\delta[n]$$

$$\Rightarrow x[n]\delta[n-k] = x[k]\delta[n-k]$$

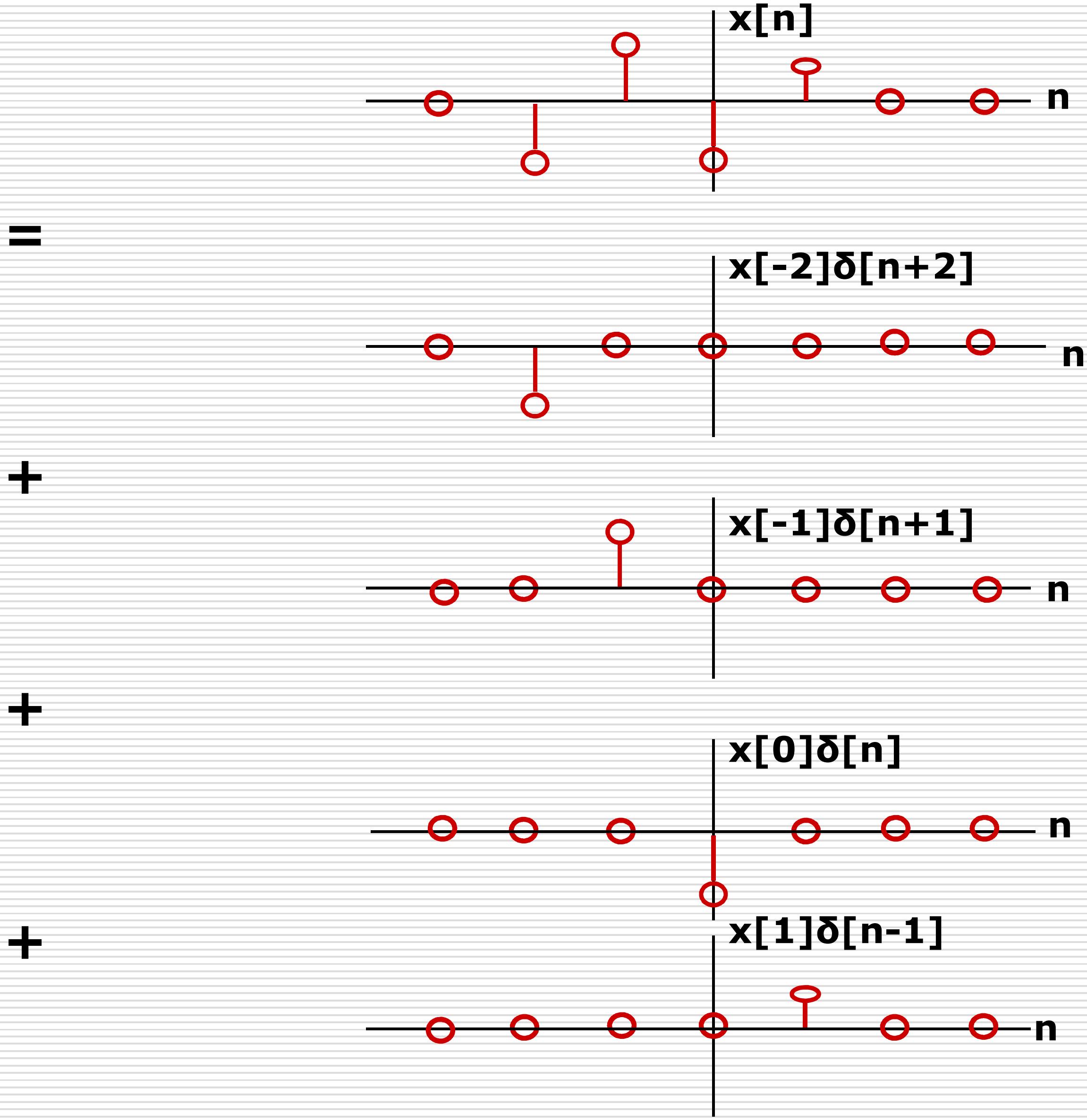
So, we can write,

$$x[n] = \ldots. + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

- Expressing discrete-time signal with impulse

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

The Convolution Sum contd.



The Convolution Sum contd.

- Corresponding output for a system with operator H

$$y[n] = H\left\{ \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \right\}$$

- Exchanging H with ∑ and x[k]

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]H\delta[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

where $h_k[n] = H\{\delta[n-k]\}$, is the response of the system to a time shifted impulse.

The Convolution Sum contd.

- Again for time invariant system, $h_k[n] = h_0[n-k]$,
- Letting $h_0[n] = h[n]$

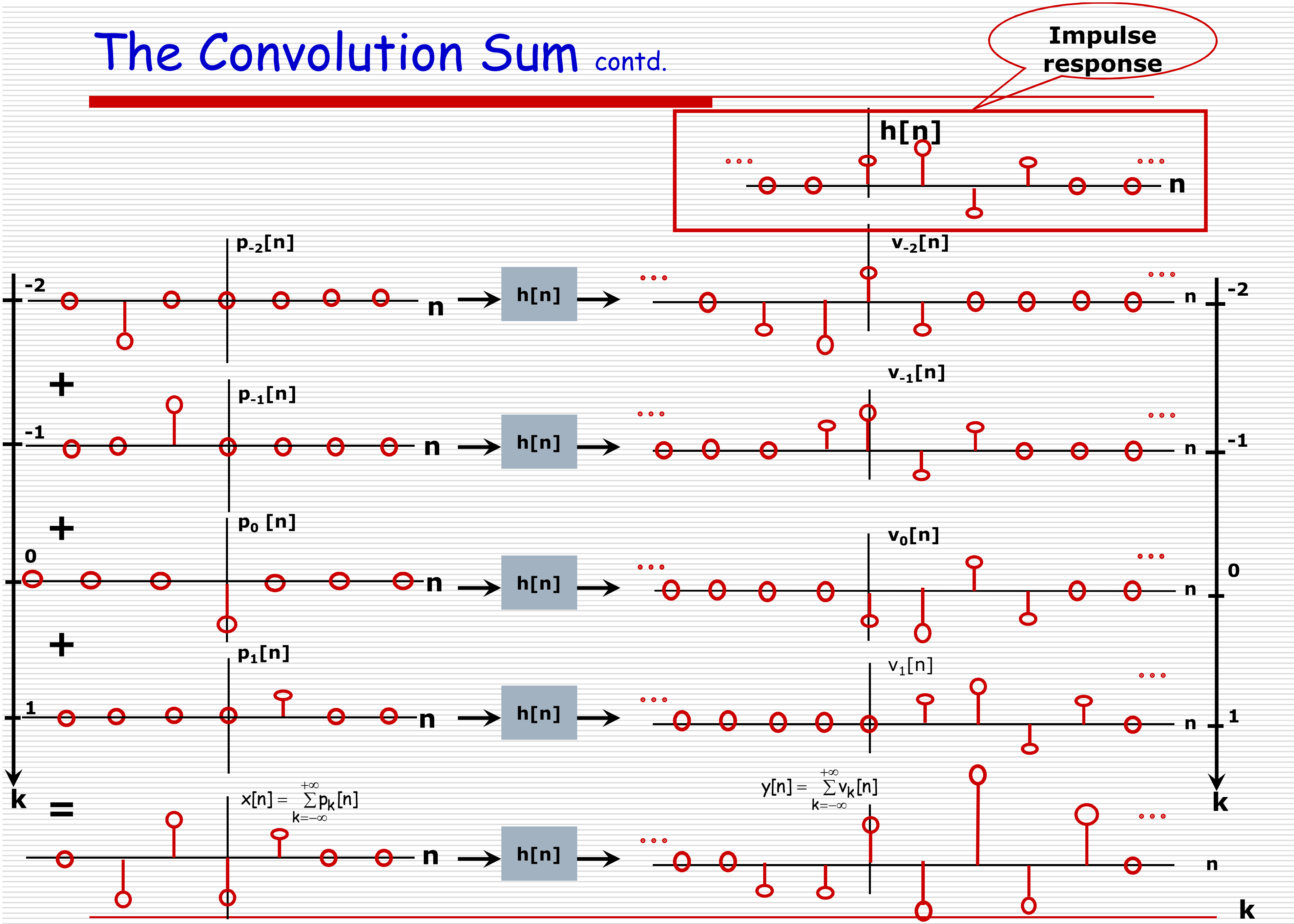
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Thus the output of a LTI system is given by the weighted sum of time shifted impulse responses.

- The convolution sum,

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

The Convolution Sum contd.



The Convolution Sum…Direct Evaluation

An LTI system has impulse response

$$h[n]=\begin{cases} 1, & n=\pm 1 \\ 2, & n=0 \\ 0, & otherwise \end{cases}$$

Determine the output for this system in response to

$$x[n]=\begin{cases} 2, & n=0 \\ 3, & n=1 \\ -2, & n=2 \\ 0, & otherwise \end{cases}$$

The Convolution Sum…Direct Evaluation..contd.

- Step 2 : Establish as expression for y[n]

Since a weighted, time-shifted impulse input, aδ[n-k], results in a weighted, time-shifted, impulse response output, ah[n-k], the system output may be written as,

$$y[n] = 2h[n] + 3h[n-1] - 2h[n-2]$$

here, $v_0[n] = 2h[n]$, $v_1[n] = 3h[n-1]$, $v_2[n] = -2h[n-2]$
and all other $v_k[n] = 0$.

The Convolution Sum…Direct Evaluation..contd.

- Step 3 : Calculate $v_k[n]$ for different values of n.

$$y[n]=v_0[n]+v_1[n]+v_2[n]$$

here, $v_0[n]=2h[n],\ v_1[n]=3h[n-1],\ v_2[n]=-2h[n-2]$

and all other $v_k[n]=0.$

$$h[n]=\begin{cases}1, & n=\pm 1\\ 2, & n=0\\ 0, & otherwise\end{cases}$$

$$v_0[0]=2h[0]=4$$
$$v_1[0]=3h[0-1]=3h[-1]=3$$
$$v_2[0]=-2h[0-2]=-2h[-2]=0$$
$$y[0]=4+3+0=7$$

$$v_0[1]=2h[1]=2$$
$$v_1[1]=3h[1-1]=3h[0]=6$$
$$v_2[1]=-2h[1-2]=-2h[-1]=-2$$
$$y[1]=2+6-2=6$$

$$v_0[2]=2h[2]=0$$
$$v_1[2]=3h[2-1]=3h[1]=3$$
$$v_2[2]=-2h[2-2]=-2h[0]=-4$$
$$y[2]=0+3-4=-1$$

- How many values of n to be considered?
- All those values of n for which any one of h[n],h[n-1] and h[n-2] is non-zero.

The Convolution Sum…Direct Evaluation..contd.

- Step 3 contd….

In this case range of n=-2 to n=4 is enough

$$v_0[-1]=2h[-1]=2$$
$$v_1[-1]=3h[-1-1]=3h[-2]=0$$
$$v_2[-1]=-2h[-1-2]=-2h[-3]=0$$
$$y[-1]=2+0+0=0$$

$$v_0[3]=2h[3]=0$$
$$v_1[3=3h[3-1]=3h[2]=0$$
$$v_2[3]=-2h[3-2]=-2h[1]=-2$$
$$y[3]=0+0-2=-2$$

- Finally,

$$y[n]=\begin{cases} 0, & n\le -2 \\ 2, & n=-1 \\ 7, & n=0 \\ 6, & n=1 \\ -1, & n=2 \\ -2, & n=3 \\ 0, & n\ge 4 \end{cases}$$

The Convolution Sum…Alternative Approach.

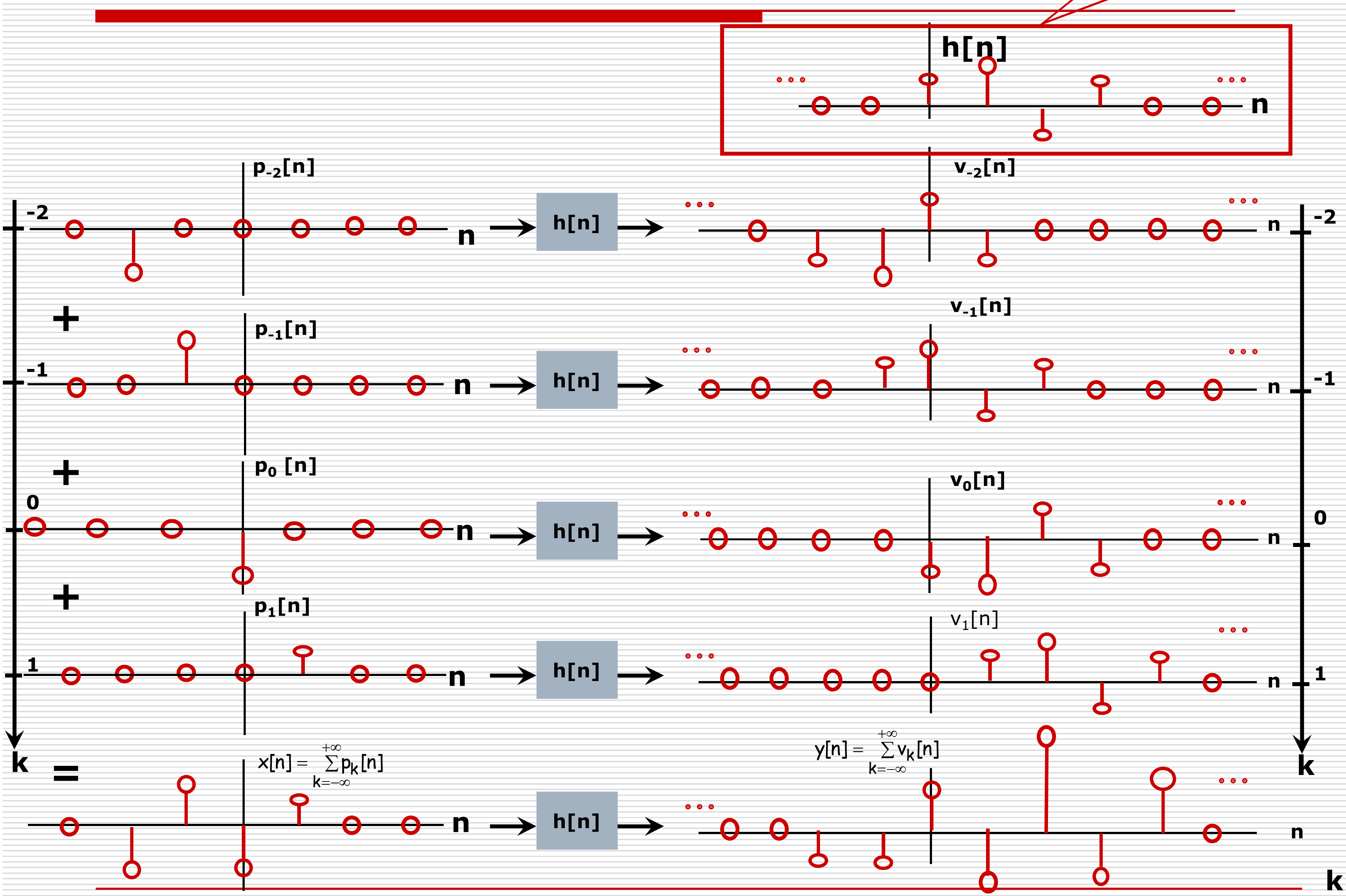
- When the signal is of short duration, only a few number of $v_k[n]$ are needed to be summed up and thus the approach just shown is very effective.
- But when signal persists for a long duration, large number of $v_k[n]$'s have to be summed up which can be a cumbersome job.
- An alternative approach is to calculate y[n], only at particular values of n, e.g., n_0, n_1 etc.

$$y[n_0] = \sum_{k=-\infty}^{+\infty} v_k[n_0]$$

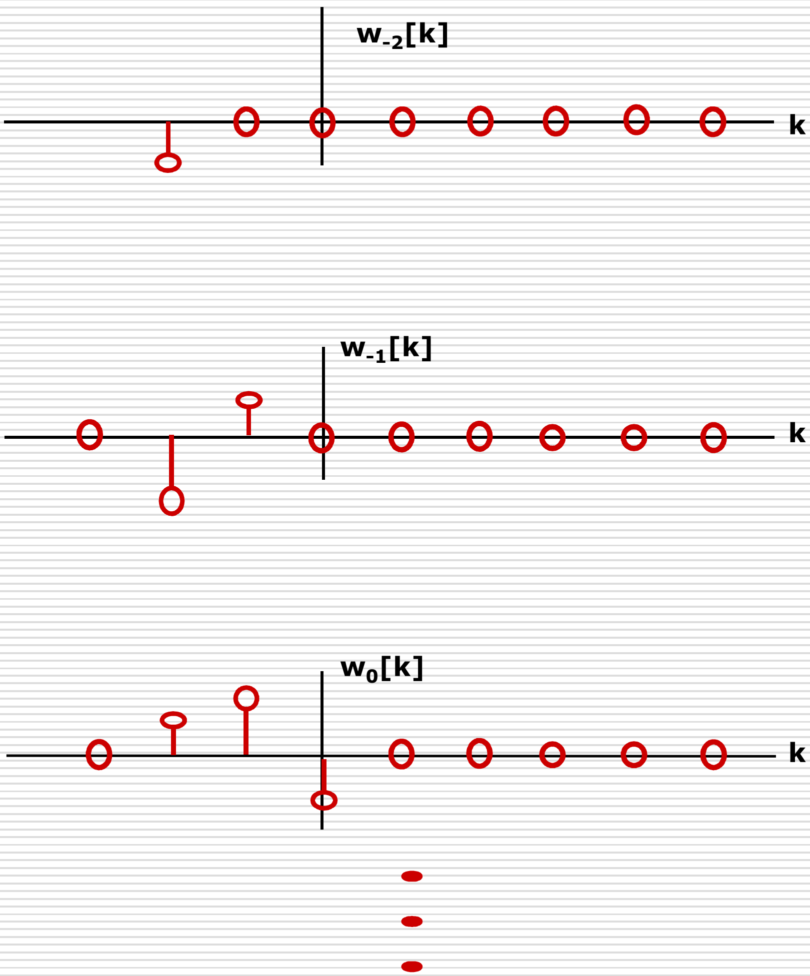
$$y[n_0] = \sum_{k=-\infty}^{+\infty} w_{n_0}[k]$$

- That is we sum along k axis at a fixed value of n

The Convolution Sum…Alternative Approach contd.



The Convolution Sum contd.



$$y[-2] = \sum_{k=-\infty}^{+\infty} w_{-2}[k]$$

$$y[-1] = \sum_{k=-\infty}^{+\infty} w_{-1}[k]$$

$$y[0] = \sum_{k=-\infty}^{+\infty} w_{0}[k]$$

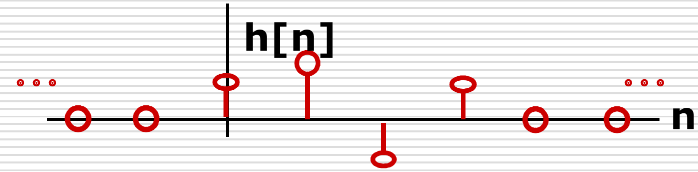
The Convolution Sum contd.

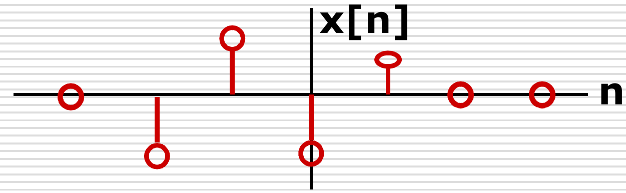
- Again $w_n[k]=x[k]h[n-k]$.
- Here k is the independent variable and n is a constant.
- Hence $h[n-k]$ is a reflected and time shifted (by –n) version of $h[k]$.
- In general,

$$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$$

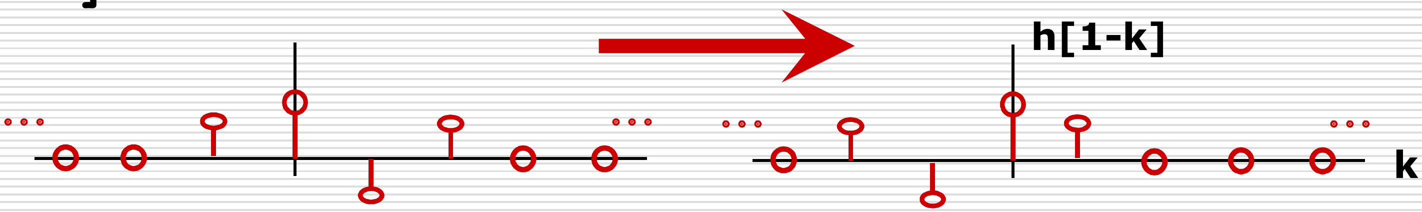
The Convolution Sum…Alternative Approach contd.

Given x[n] and h[n] calculate and y[1]

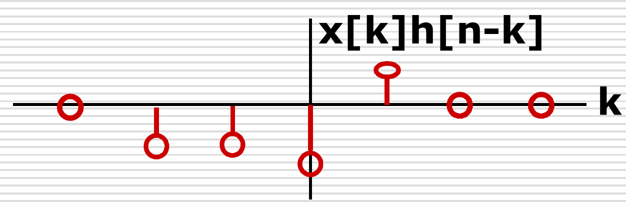




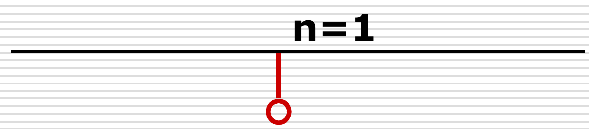
- Time shift (by -1) and reflect h[n] to get h[1-k]



- Calculate $w_1[k]$ by multiplying x[k] and h[1-k]

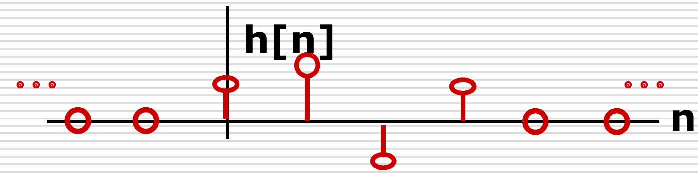


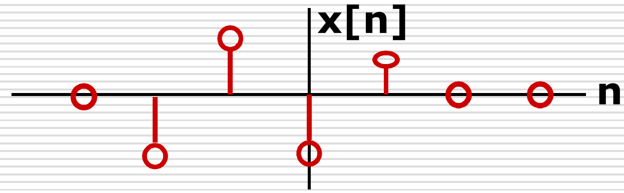
- Calculate y[1] by summing up $w_1[k]$ for all k's



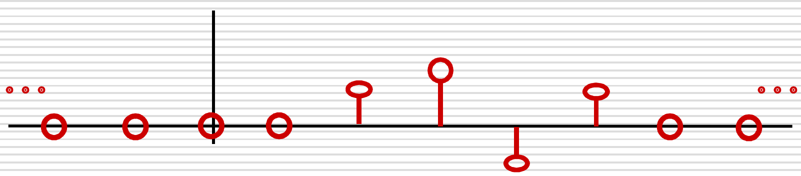
The Convolution Sum…Alternative Approach contd.

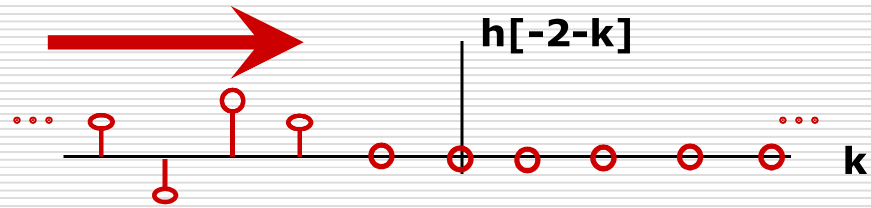
Given x[n] and h[n] calculate and y[-2]



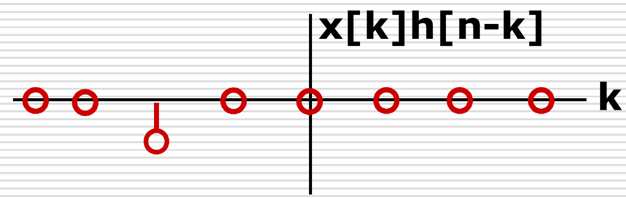


❑Time shift (by 2) and reflect h[n] to get h[-2-k]

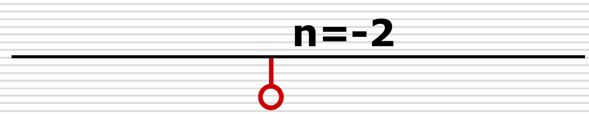




❑Calculate $w_{-2}[k]$ by multiplying x[k] and h[-2-k]



❑Calculate y[-2] by summing up $w_{-2}[k]$ for all k's



The Convolution Integral

- The process of determining the output of a continuous time LTI system solely from the input and the system's impulse response.
- Unlike summation in discrete time LTI system, superposition here is implemented through integration.
- In contrast to discrete time shift variable (k) the time shift variable (τ) is continuous here.

Comparison of Expressions

Entities	Discrete Time LTI	Continuous time LTI
input	$x[n]=\sum_{k=-\infty}^{+\infty}x[k]\delta[n-k]$	$x(t)=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau$
output associated with each input	$v_k[n]=x[k]h[n-k]$	$v_\tau(t)=x(\tau)h(t-\tau)$
output	$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$	$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$
convolution	$x[n]*h[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$	$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$

Comparison of Expressions contd.

Entities	Discrete Time LTI	Continuous time LTI
Output at fixed n or t, e.g., n_0 or t_0	$y[n_0] = \sum_{k=-\infty}^{+\infty} v_k[n_0]$ $y[n_0] = \sum_{k=-\infty}^{+\infty} w_{n_0}[k]$	$y(t_0) = \int_{-\infty}^{\infty} v_\tau(t_0)d\tau$ $y(t_0) = \int_{-\infty}^{\infty} w_{t_0}(\tau)d\tau$
Output	$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$ $w_n[k] = x[k]h[n-k]$	$y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau$ $w_t(\tau) = x(t)h(t-\tau)$
Comments	h[n-k] is the reflected and time shifted version of h[k]	h(t-τ) is the reflected and time shifted version of h[τ]

Graphical Estimation Procedure of Convolution Integral

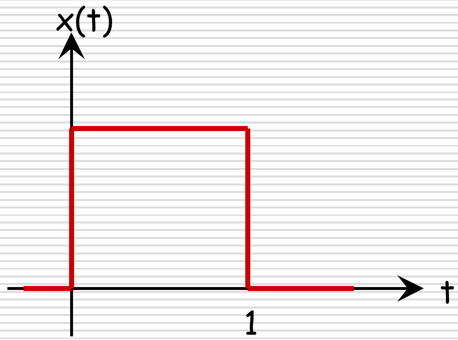
- Step 1 : Plotting $x(\tau)$ and $h(t-\tau)$ from given $x(t)$ and $h(t)$.
- Step 2: Sliding $h(t-\tau)$ across $x(\tau)$ from left to right and writing all distinct functional form of $w_t(\tau)$. [i.e., $x(t)h(t-\tau)$]

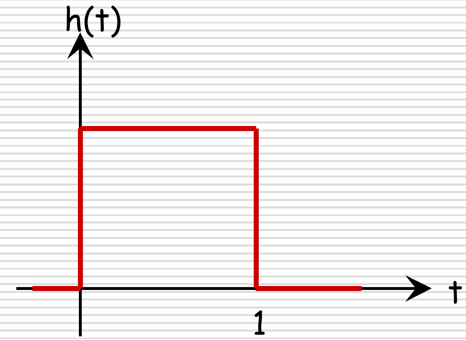
  By

  - Starting with large negative value of time shift t and checking corresponding $w_t(\tau)$.
  - Increasing t until functional form of $w_t(\tau)$ changes.
- Step 3: Integrating $w_t(\tau)$ from $\tau=-\infty$ to $\tau=+\infty$ for each interval of time shift t to obtain $y(t)$ for that interval.

Example 1

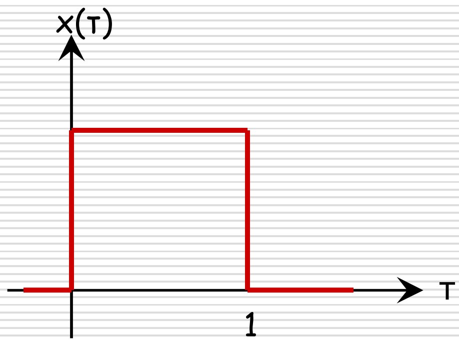
Given

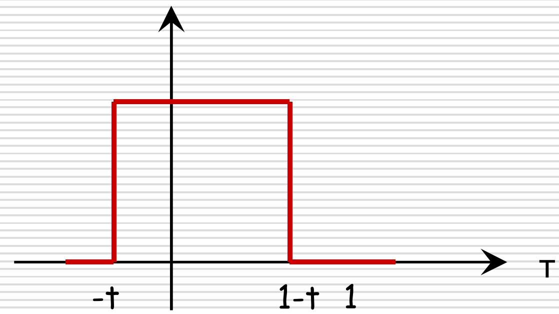


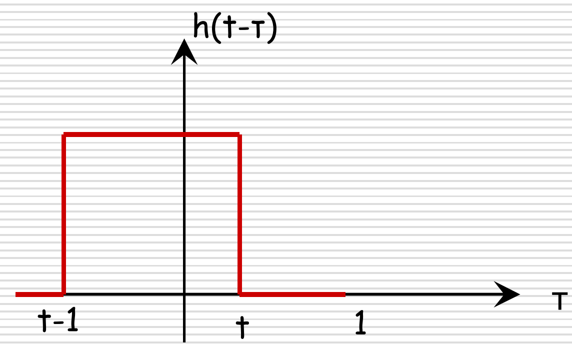


Step 1

Plotting x(τ) and h(t- τ)



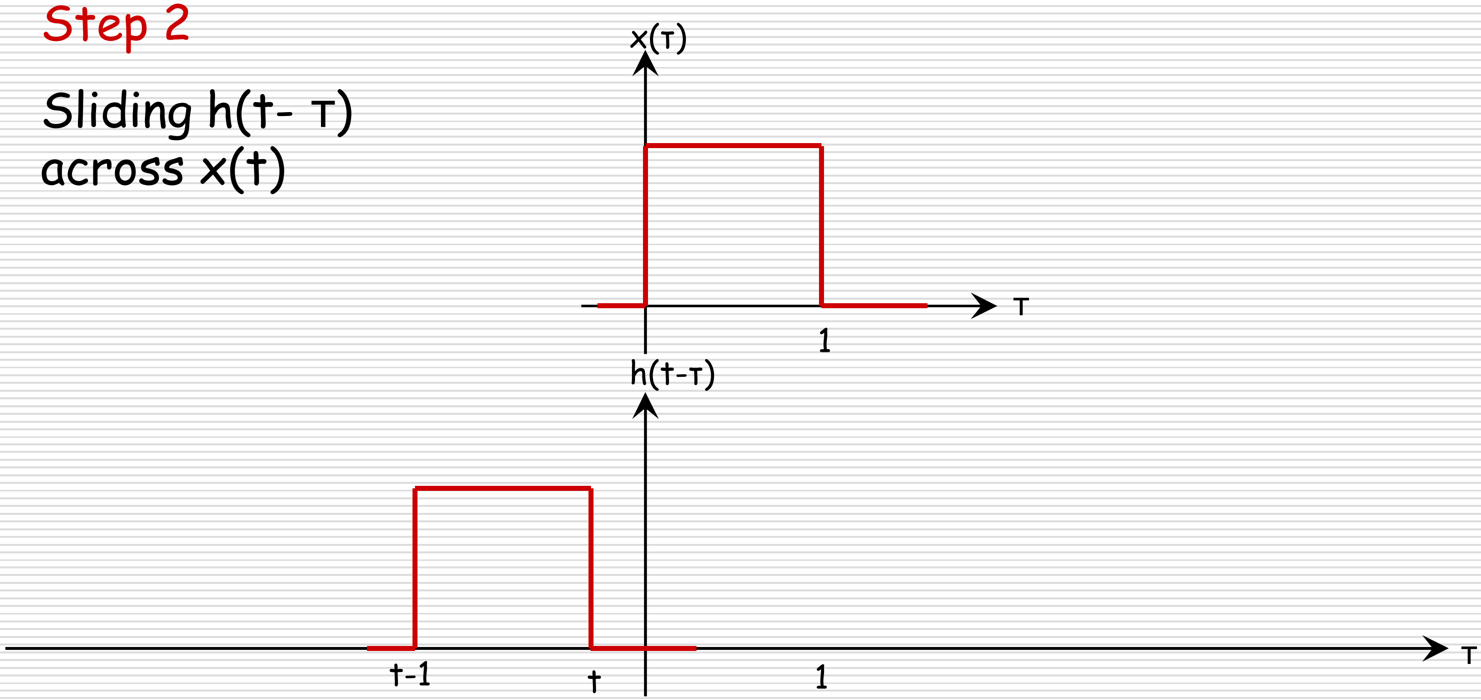




Example 1 contd.

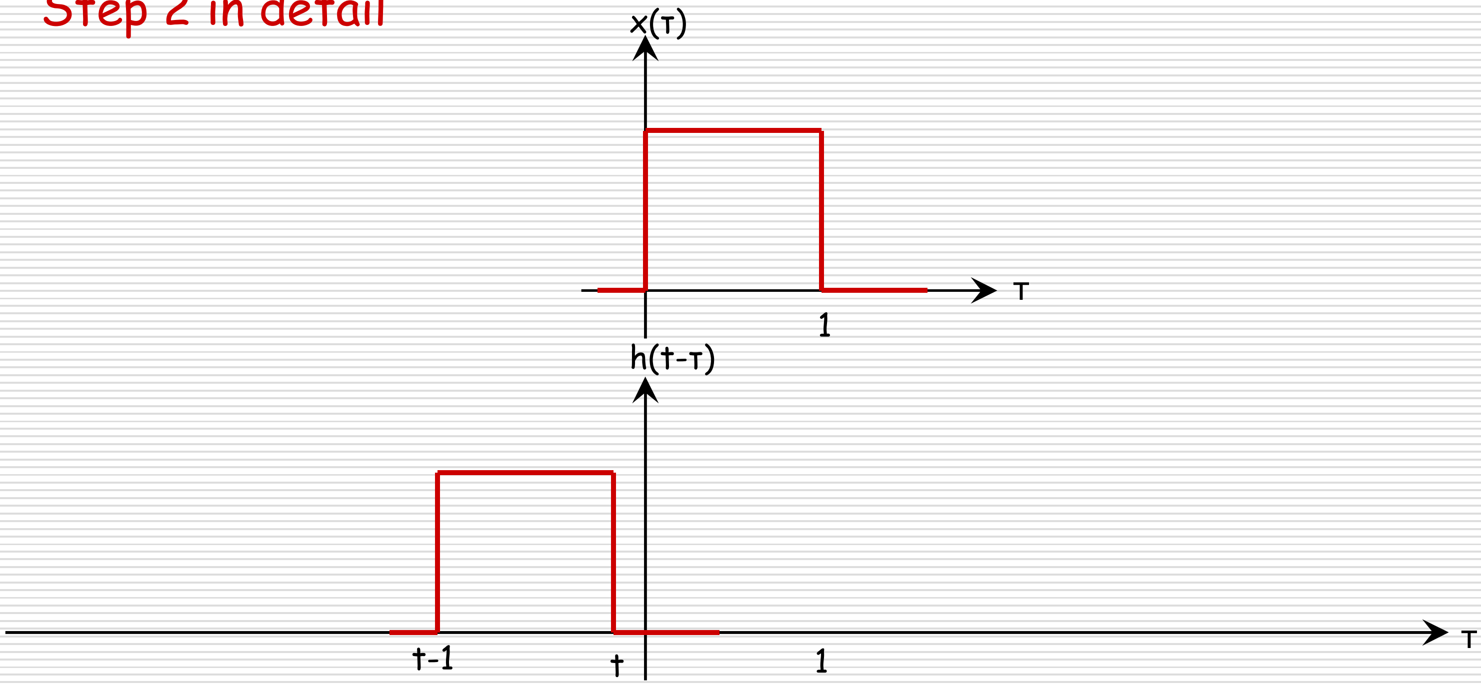
Step 2

Sliding h(t- τ) across x(t)



Example 1 contd.

Step 2 in detail

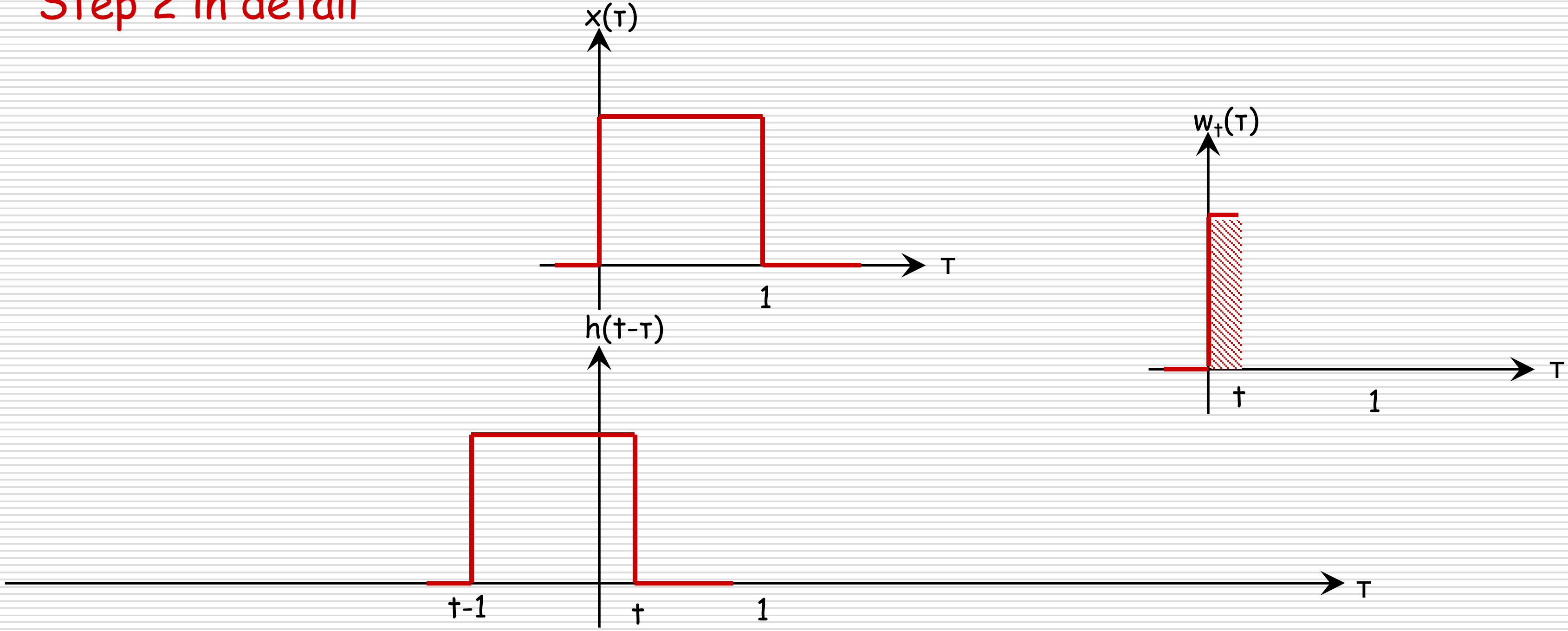


When the right edge of $h(t-\tau)$ is on the left of the non-zero portion of $x(\tau)$, i.e., $t<0$,

$$\boxed{w_t(\tau) = 0, \quad t<0}$$

Example 1 contd.

Step 2 in detail

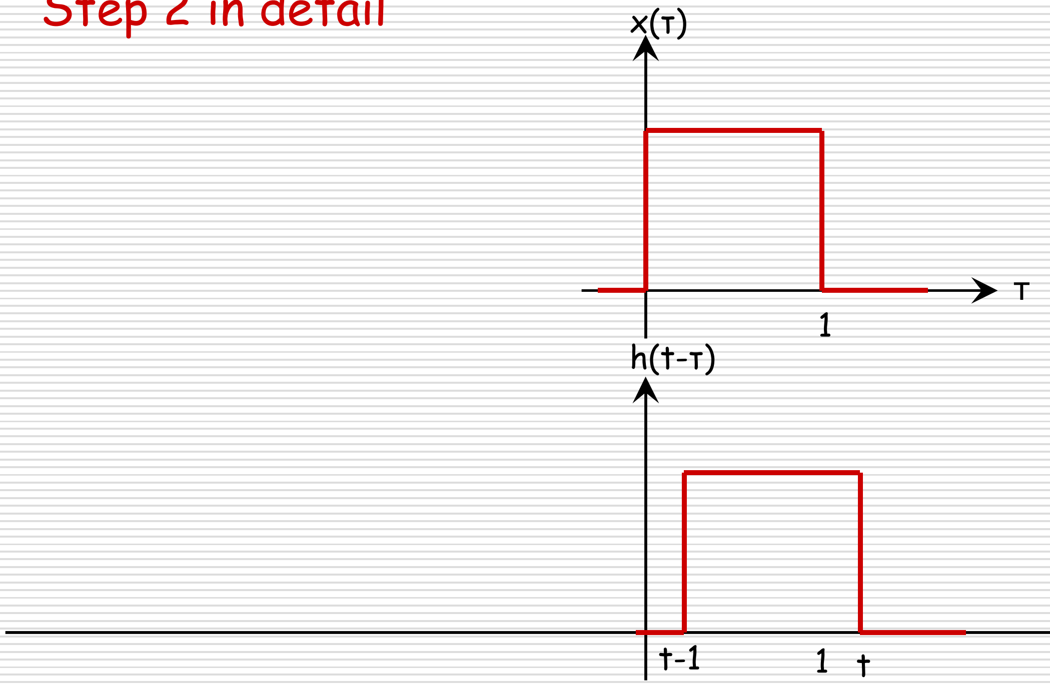


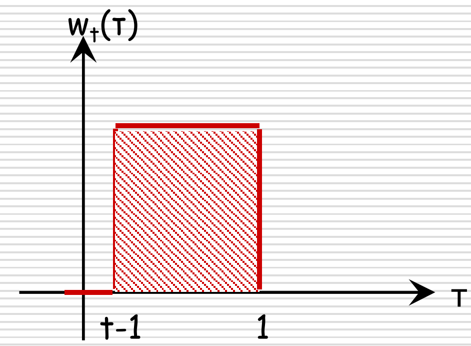
When the right edge of h(t-τ) overlaps with the non-zero portion of x(τ), i.e., t>0,

$$\boxed{\begin{aligned} w_t(\tau)&=1,\ 0<t<1 \\ w_t(\tau)&=1,\ 0<\tau<t \end{aligned}}$$

Example 1 contd.

Step 2 in detail



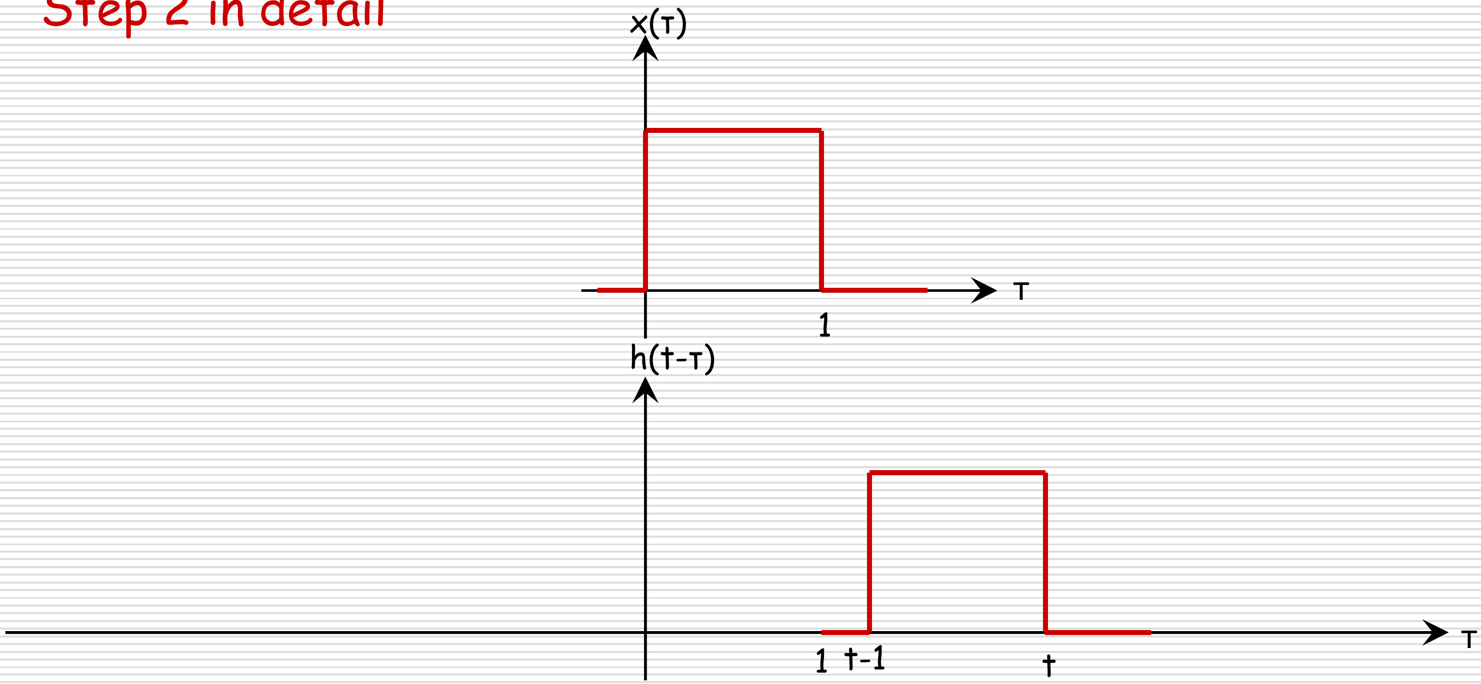


When the right edge of $h(t-\tau)$ is on the right of the non-zero portion of $x(\tau)$, i.e., $t-1>0$ or $t>1$,

$$\boxed{\begin{array}{l} w_t(\tau)=1,\ 1<t<2 \\ w_t(\tau)=1,\ t-1<\tau<1 \end{array}}$$

Example 1 contd.

Step 2 in detail



When the left edge of $h(t-\tau)$ is on the right of the non-zero portion of $x(\tau)$, i.e., $t-1>1$ or $t>2$,

$$\boxed{w_t(\tau)=0,\ t>2}$$

Example 1 contd.

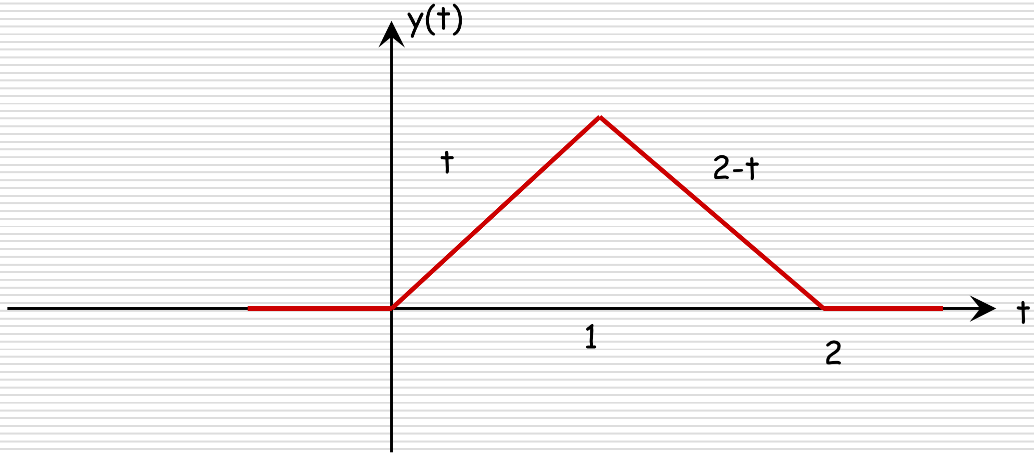
Step 3
Integrating

$$\text{for } t < 0,\quad y(t) = 0$$

$$\text{for } 0 < t < 1,\quad y(t) = \int_0^t 1\,d\tau = t$$

$$\text{for } 1 < t < 2,\quad y(t) = \int_{t-1}^{1} 1\,d\tau = 2 - t$$

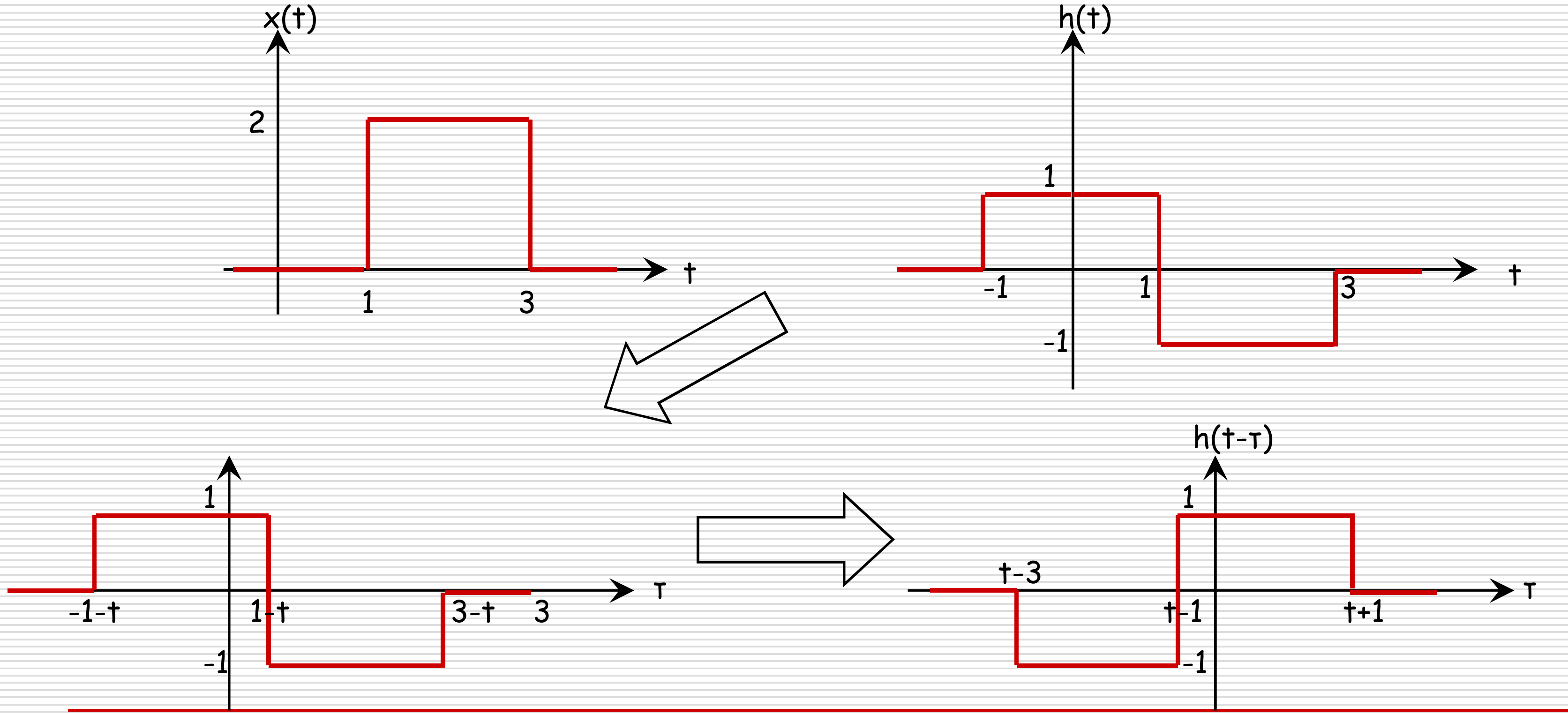
$$\text{for } t > 2,\quad y(t) = 0$$



Example 2

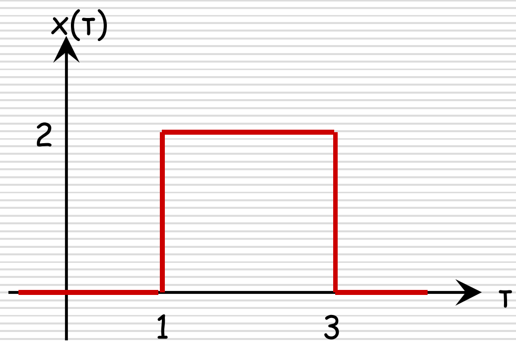
Given, $x(t) = 2u(t-1) - 2u(t-3)$ and

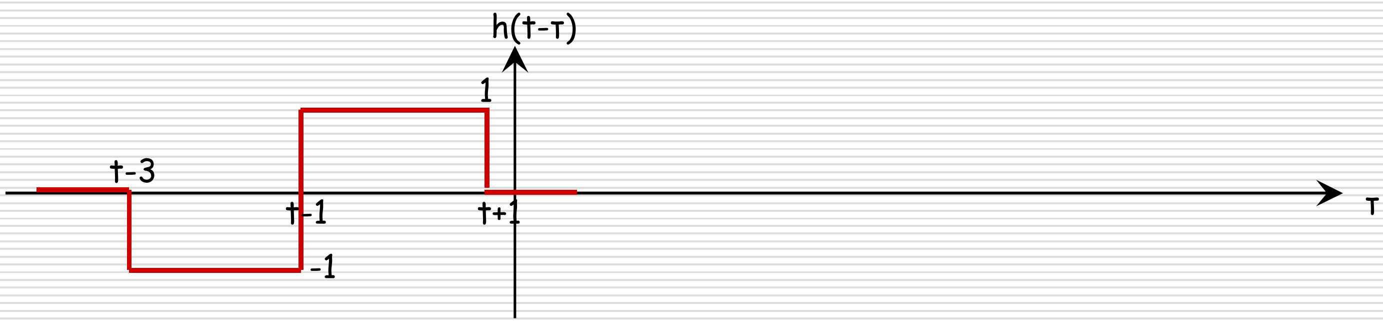
$$h(t) = u(t+1) - 2u(t-1) + u(t-3)$$



Example 2 contd.

Step 2 in detail



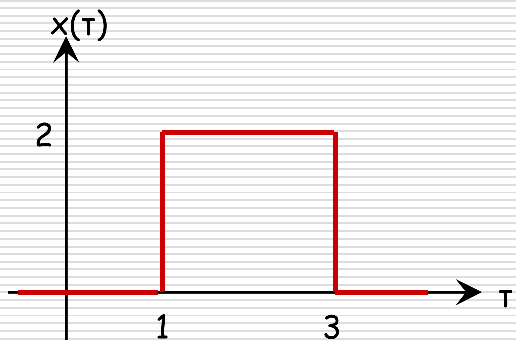


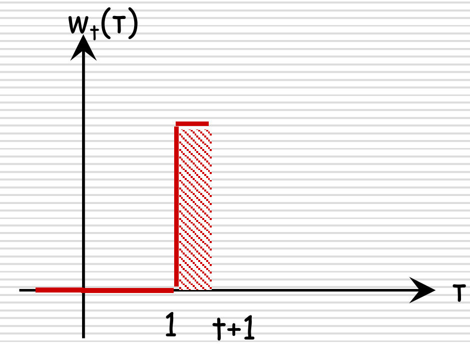
When the right edge of $h(t-\tau)$ is on the left of the non-zero portion of $x(\tau)$, i.e., $t+1<1$ or $t<0$

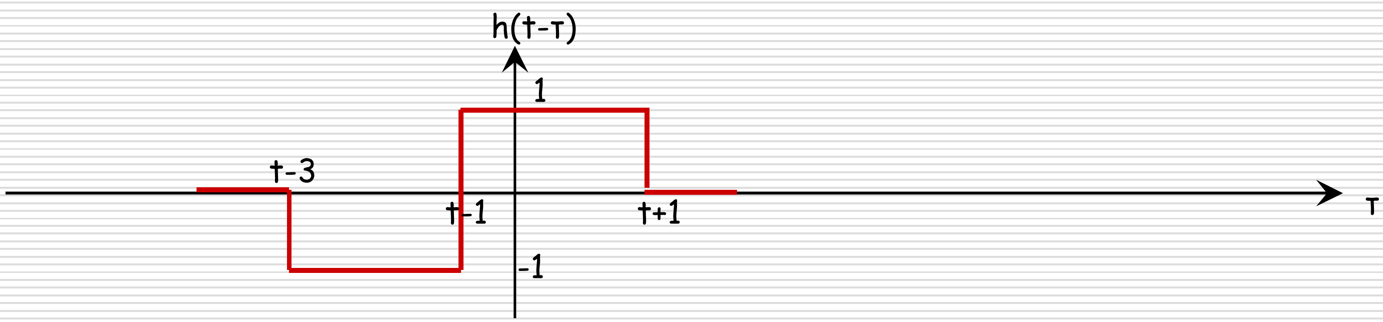
$$\boxed{w_t(\tau) = 0, \quad t<0}$$

Example 2 contd.

Step 2 in detail





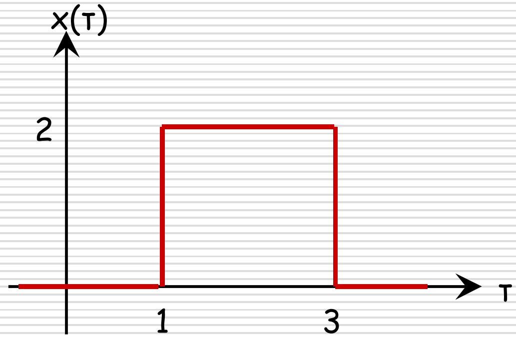


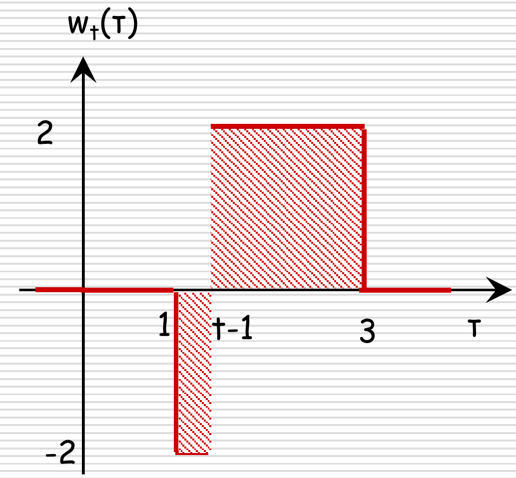
When the right edge of $h(t-\tau)$ overlaps with the non-zero portion of $x(\tau)$, i.e., $t+1>1$ or $t>0$

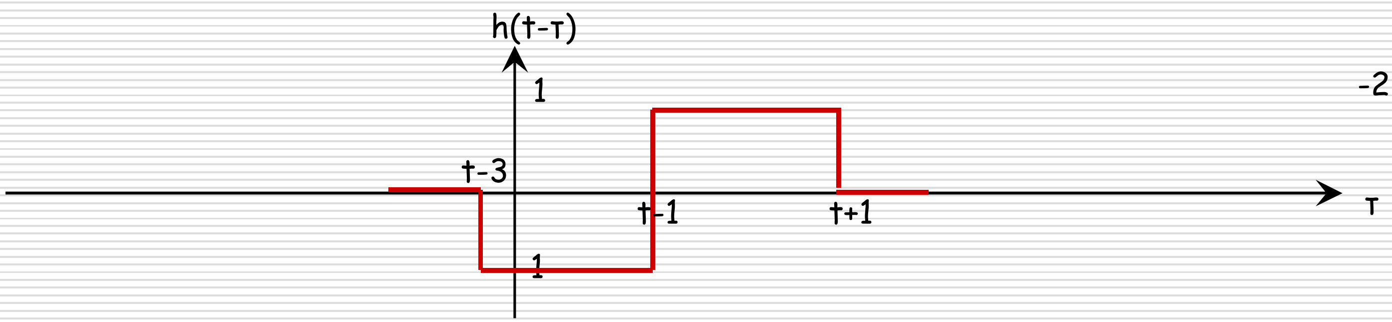
$$\boxed{\begin{array}{l} w_t(\tau)=2,\ 0<t<2 \\ w_t(\tau)=2,\ 1<\tau<t+1 \end{array}}$$

Example 2 contd.

Step 2 in detail







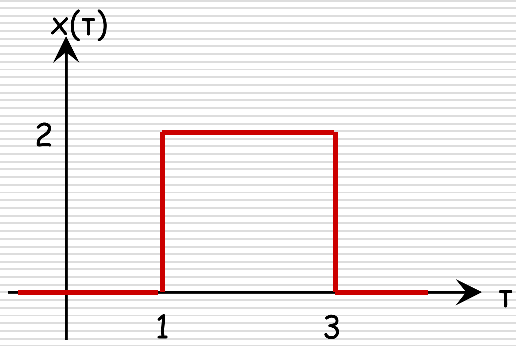
When the right edge of $h(t-\tau)$ is on the right of the non-zero portion of $x(\tau)$, i.e., $t-1>1$ or $t>2$

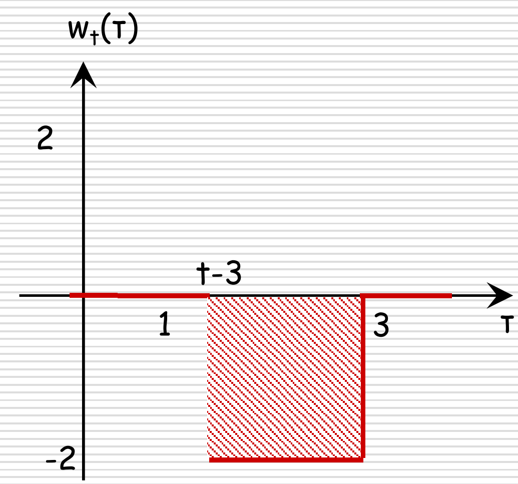
$w_t(\tau)=-2$ or 2, $2<t<4$

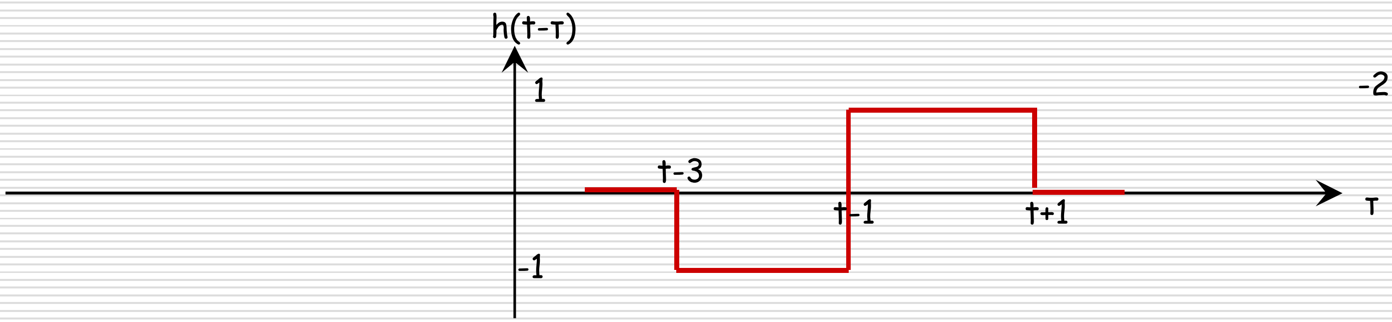
$w_t(\tau)=-2$, $1<\tau<t-1$; $w_t(\tau)=2$, $t-1<\tau<3$

Example 2 contd.

Step 2 in detail







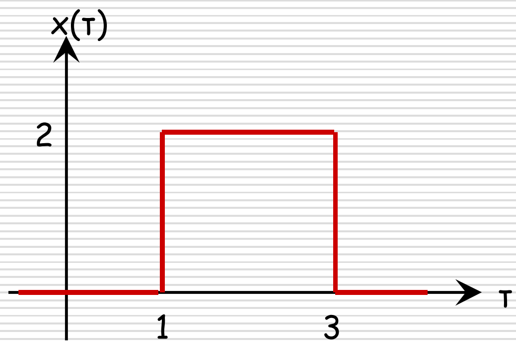
When the left edge of $h(t-\tau)$ overlaps with the non-zero portion of $x(\tau)$, i.e., $t-3>1$ or $t>4$

$$w_t(\tau)=-2,\ 4<t<6$$

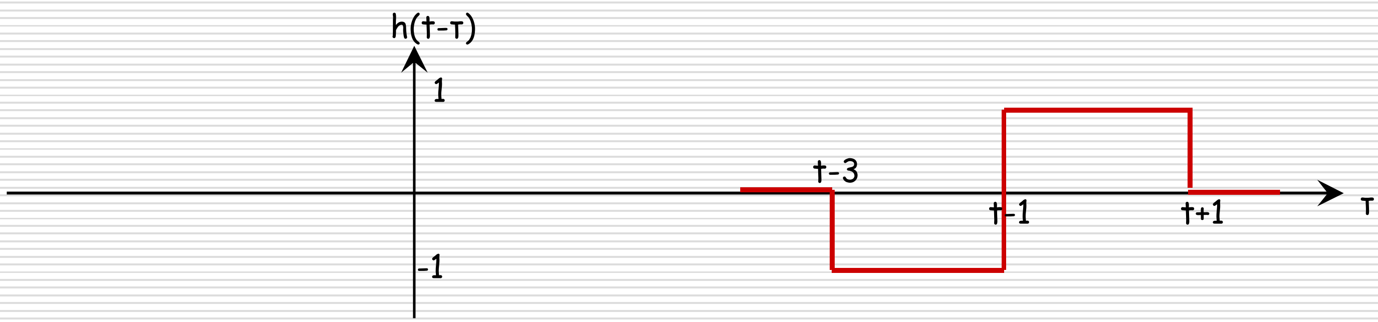
$$w_t(\tau)=-2,\ t-3<\tau<3$$

Example 2 contd.

Step 2 in detail



τ



When the left edge of h(t-τ) is on the right of the non-zero portion of x(τ), i.e., t-3>3 or t>6

$\boxed{w_t(\tau)=0,\ t>6}$

Example 2 contd.

Step 3
Integrating

$$for\ t<0,\quad y(t)=0$$

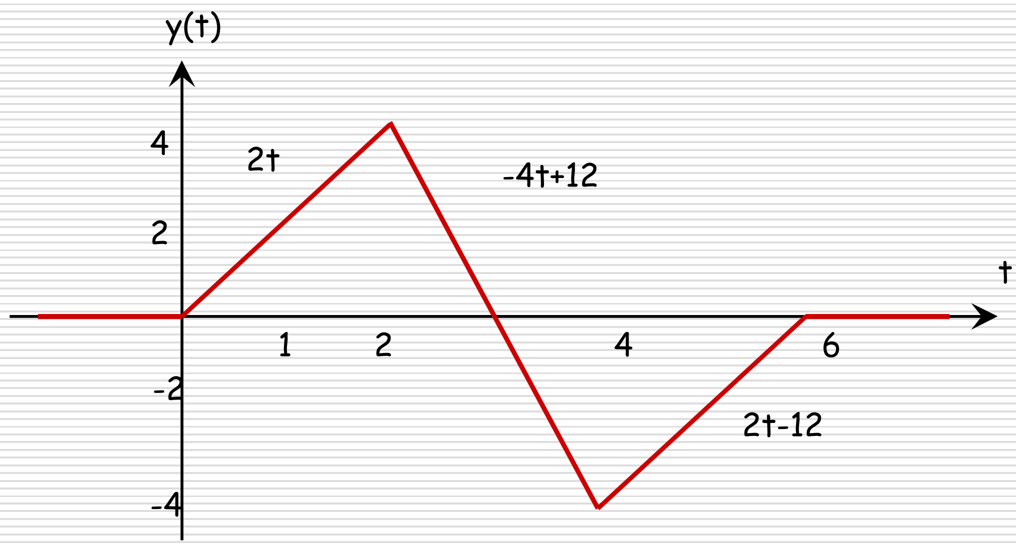
$$for\ 0\le t<2,\quad y(t)=\int_{1}^{t+1}2d\tau=2t$$

$$for\ 2\le t<4,\quad y(t)=\int_{1}^{t-1}-2d\tau+\int_{t-1}^{3}2d\tau=-4t+12$$

$$for\ 4\le t<6,\quad y(t)=\int_{t-3}^{3}-2d\tau=2t-12$$

$$for\ t\ge 6,\quad y(t)=0$$

$$y(t)=\begin{cases}0, & t<0\\ 2t, & 0\le t<2\\ -4t+12, & 2\le t<4\\ 2t-12, & 4\le t<6\\ 0, & t>6\end{cases}$$



Assignment

- Questions of Class Test-1 given as Assignment-1
- Subsequent lectures will be consized and will focus mainly on Discrete-time signals and systems
- Students are asked to review previous lectures.

Fourier Representation of Signals

- Signals can be represented as superposition of weighted complex sinusoids.
- Applying such a signal on an LTI systems gives an output which is a weighted superposition of the system response to individual complex sinusoids
- This is an alternative expression for LTI system input-output compared to the convolution approach.
- The study of signals and systems using sinusoidal representation is called "Fourier Analysis."

Complex Sinusoids and Frequency Response of LTI Sys.

- Response of an LTI system to a sinusoid input characterizes the behavior of the system.
- It is obtained in terms of impulse response by using convolution and a complex sinusoid.
- Considering the output of a LTI system having a impulse response h[n] for an unit amplitude complex sinusoid $x[n]=e^{j\Omega n}$, we can write

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{+\infty} h[k]e^{-j\Omega k}$$

$$= H(e^{j\Omega})e^{j\Omega n}$$

$$\text{where, } H(e^{j\Omega}) = \sum_{k=-\infty}^{+\infty} h[k]e^{-j\Omega k}$$

H is only a function of frequency not of time and is called the frequency response of the system

Categories of Fourier Representation

Time Property	Periodic	Non-periodic
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

DTFS

- If x[n] is a discrete-time signal with fundamental period N (i.e., fundamental frequency $\Omega_0=2\pi/N$), then we can represent x[n] by the DTFS as

$$\hat{x}[n] = \sum_{k} A[k] e^{jk\Omega_0 n}$$

- How many terms and weights?
  - Complex sinusoids with discrete frequencies are not always distinct

$$e^{j(N+k)\Omega_0 n} = e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{j2\pi n} e^{jk\Omega_0 n} = e^{jk\Omega_0 n}$$

  - Hence k=0 to N-1 is sufficient
  - Again, for even or odd signal k=-(N-1)/2 to +(N-1)/2 is ok.,if N is odd.

DTFS contd.

- DTFS Weights or coefficients are calculated using MSE method.
- The DTFS representation of periodic signal x[n] with fundamental period N and fundamental frequency $\Omega_0=2\pi/N$ is given by

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$$

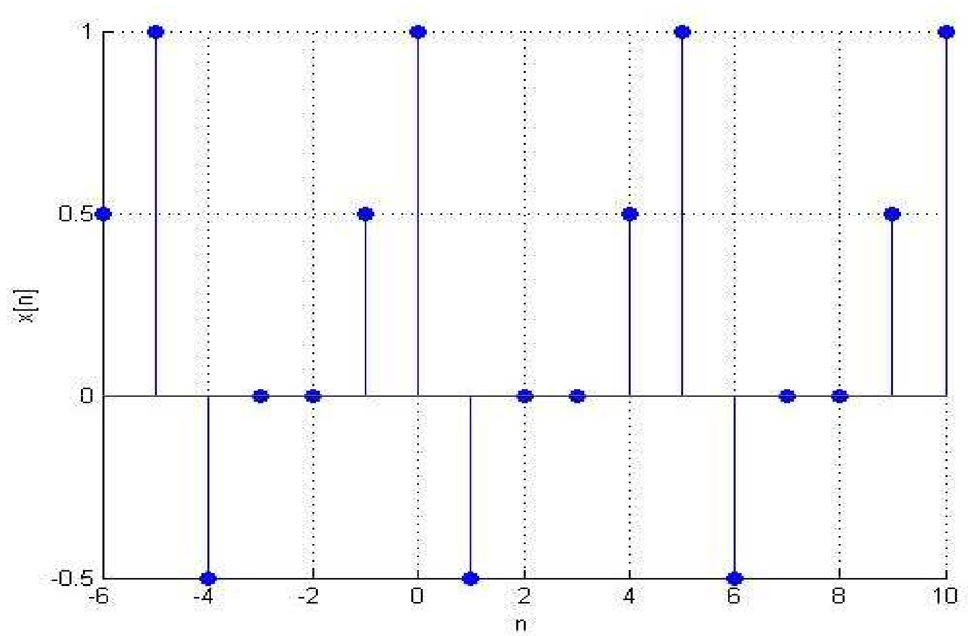
where,

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$$

are the DTFS coefficients of x[n]. x[n] are X[k] are known as DTFS pair.

DTFS contd.

- Example



- N=5, so $\Omega_0=2\pi/5$

DTFS Example contd.

$$X[k] = \frac{1}{5}\sum_{n=0}^{5-1} x[n]e^{-jk2\pi n/5}$$

$$= \frac{1}{5}\left\{x[0]e^{0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-jk8\pi/5}\right.$$

$$= \frac{1}{5}\left\{1 - 0.5e^{-jk2\pi/5} + 0.5e^{-jk8\pi/5}\right\}$$

$$= \frac{1}{5}\left\{1 - 0.5e^{-jk2\pi/5} + 0.5e^{-jk2\pi}.e^{jk2\pi/5}\right\}$$

$$= \frac{1}{5}\left\{1 - 0.5e^{-jk2\pi/5} + 0.5\{\cos(k2\pi) - j\sin(k2\pi)\}.e^{jk2\pi/5}\right\}$$

$$= \frac{1}{5}\left\{1 - 0.5e^{-jk2\pi/5} + 0.5e^{jk2\pi/5}\right\}$$

$$\boxed{X[k] = \frac{1}{5}\{1 + j\sin(2\pi k/5)\}}$$

DTFS Example contd.

$$\boxed{X[k] = \frac{1}{5}\{1 + j\sin(2\pi k/5)\}}$$

$$X[0] = 0.2e^{0}$$

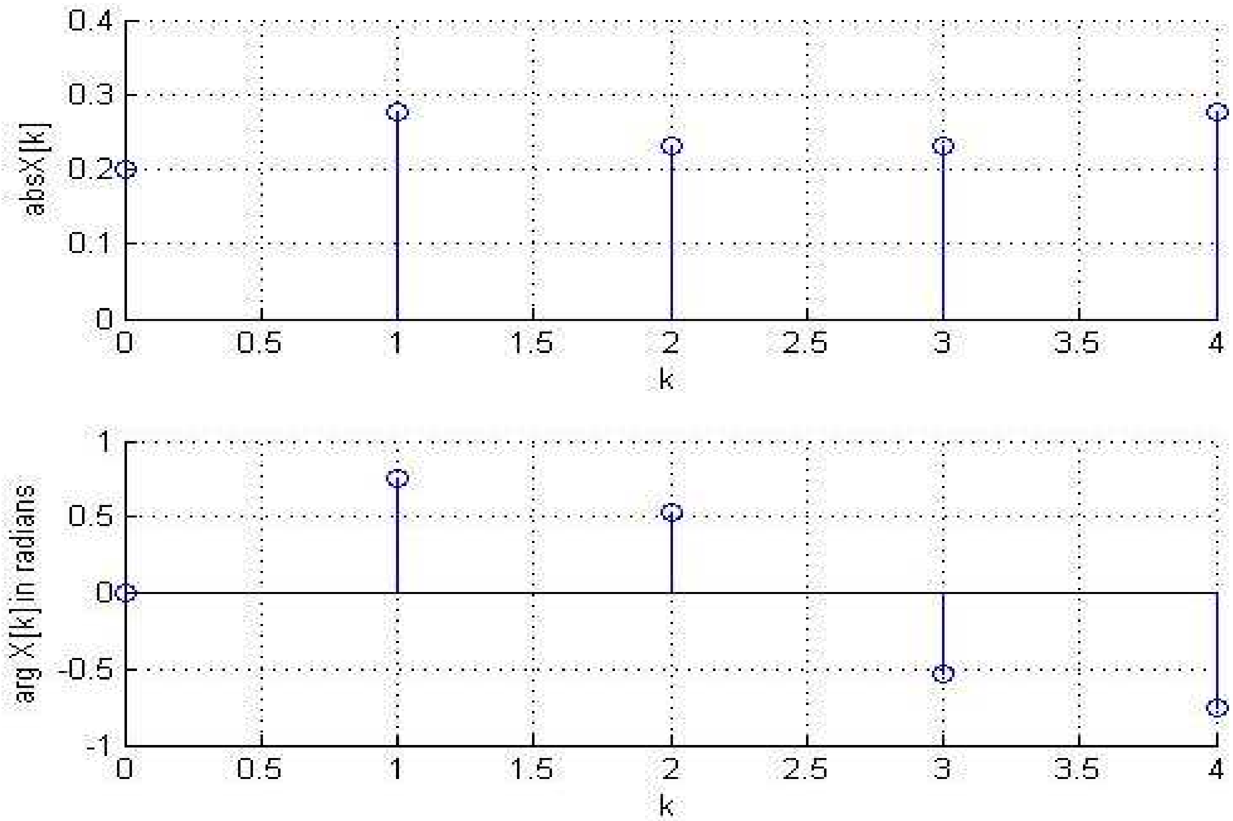
$$X[1] = 0.2 + j0.951 = 0.276e^{j0.760}$$

$$X[2] = 0.232e^{j0.531}$$

$$X[3] = 0.232e^{-j0.531}$$

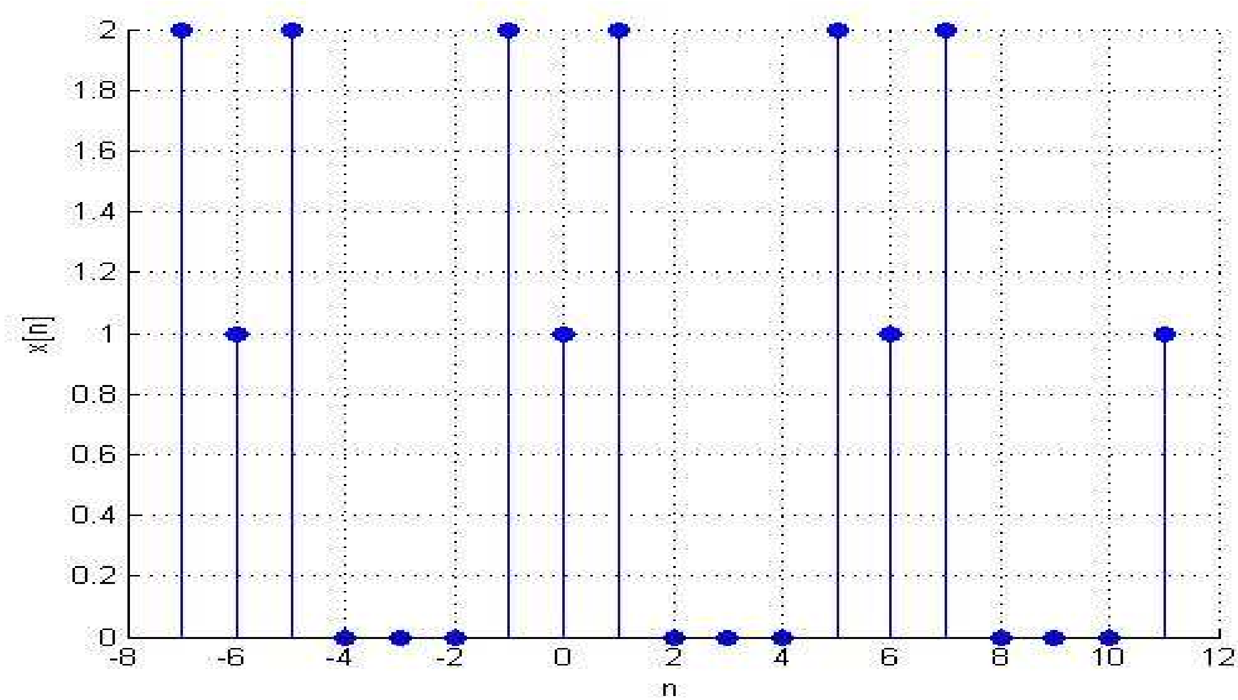
$$X[4] = 0.276e^{-j0.760}$$

DTFS Example contd.



DTFS Example 2.

- Example



- N=6, so $\Omega_0=2\pi/6=\pi/3$

DTFS Example 2 contd.

$$X[k] = \frac{1}{6}\sum_{n=0}^{6-1} x[n]e^{-jk\pi n/3}$$

$$= \frac{1}{6}\left\{x[0]e^{0} + x[1]e^{-jk\pi/3} + x[2]e^{-jk2\pi/3} + x[3]e^{-jk\pi} + x[4]e^{-jk4\pi/3} + x[5]e^{-jk5\pi/3}\right.$$

$$= \frac{1}{6}\left\{1 + 2e^{-jk2\pi/3} + 0 + 0 + 0 + 2e^{-jk5\pi/3}\right\}$$

$$= \frac{1}{6}\left\{1 + 2e^{-jk2\pi/5} + 2e^{-jk2\pi}.e^{jk\pi/3}\right\}$$

$$\boxed{X[k] = \frac{1}{6}\left\{1 + 4\cos(k\pi/3)\right\}}$$

DTFS Determination…Alterative Approach

- Determination of X[k] by inspection.
- Applicable when x[n], i.e., the original time-domain signal is a real or complex sinusoids.
- Method :
  - Step 1: Expand x[n] in terms of complex sinusoids
  - Step 2: Compare outcome of Step 2 with each term of of the following equation

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

DTFS Determination...Alterative Approach contd.

- Example
  - Time domain signal, $x[n]=\cos(\pi n/3+\varphi)$
- Here $\Omega_0=\pi/3$,so $N=2\pi/(\pi/3)=6$

$$x[n]=\frac{1}{2}\{e^{j(\pi n/3+\varphi)}+e^{-j(\pi n/3+\varphi)}\} \Rightarrow x[n]=\frac{1}{2}e^{j\varphi}e^{j\pi n/3}+\frac{1}{2}e^{-j\varphi}e^{-j\pi n/3}$$

$$\boxed{x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}}$$

$$=X[0]e^{0}+X[1]e^{j\pi n/3}+X[2]e^{j2\pi n/3}+X[3]e^{j\pi n}+X[4]e^{j4\pi n/3}+X[5]e^{j5\pi n/3}\dots$$

$$\boxed{\begin{aligned} x[0]&=0\\ x[1]&=\frac{1}{2}e^{j\varphi}\\ x[2]&=0\\ x[3]&=0\\ x[4]&=0\\ x[5]&=\frac{1}{2}e^{-j\varphi}\end{aligned}}$$

Discrete Time Fourier Transform (DTFT)

- DTFT is used to represent a discrete time non-periodic signal as a superposition of complex sinusoids.
- Unlike DTFS, DTFT has no restrictions on the period of the sinusoids and thus involves a continuum of frequencies on the interval $-\pi < \Omega < +\pi$ where Ω have units of radians.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

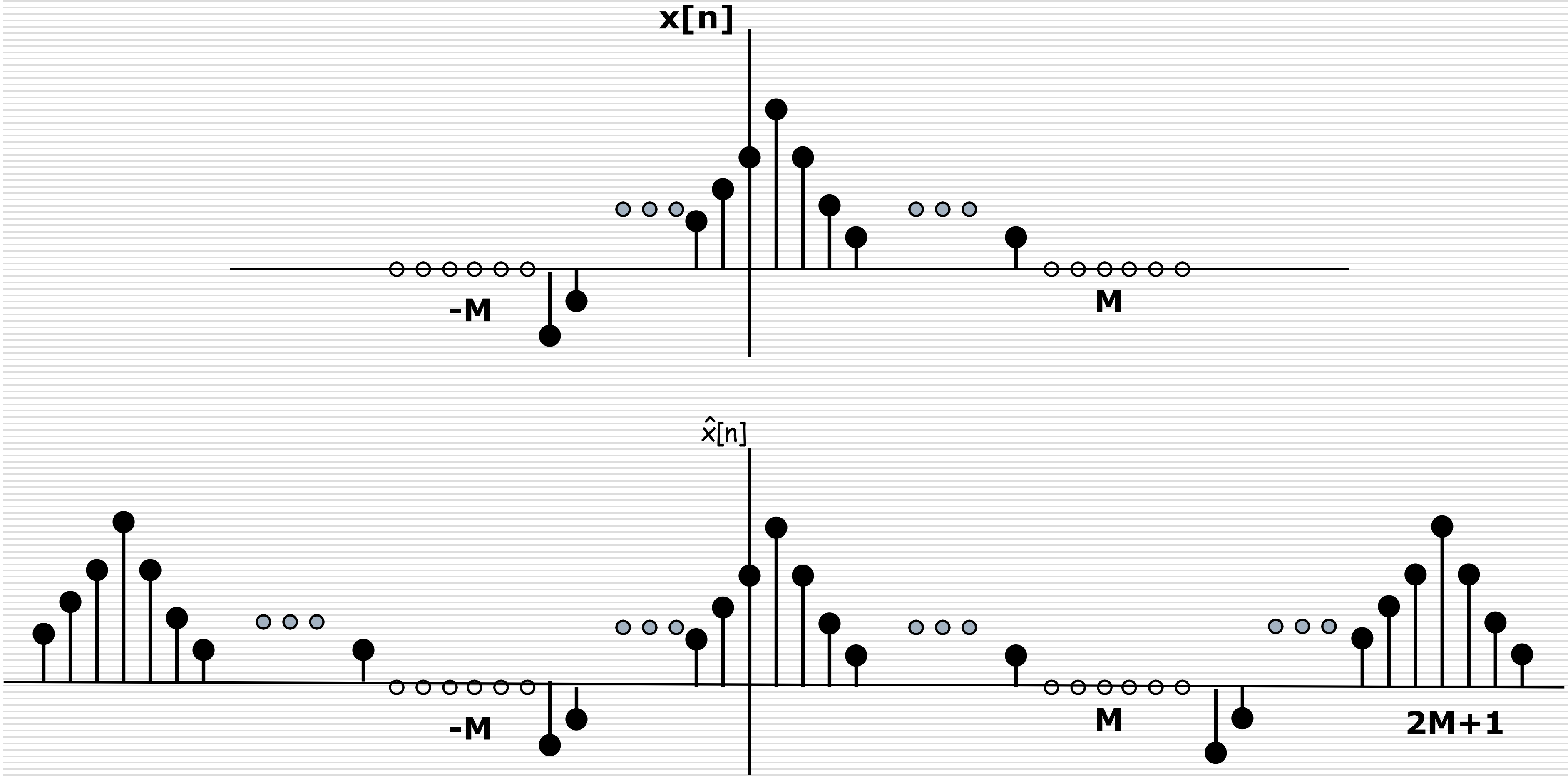
DTFT contd.

- Applicable for non-periodic discrete signal
- Can be developed from DTFS by describing a non-periodic signal as the limit of a periodic signal whose fundamental period N is infinity.

- We consider a non-periodic signal represented by one period of a periodic signal which is centered at origin and the fact that the value of N approaching infinity is taken in a symmetric manner.
- Let x[n] is defined as a non-periodic signal consisting of one period of a periodic signal with period N=2M+1 as described by

$$x[n] = \begin{cases} \hat{x}[n], & -M \le n \le M \\ 0, & |n| > M \end{cases}$$

DTFT contd.



$$x[n] = \lim_{M \to \infty} \hat{x}[n]$$

DTFT contd.

- The DTFS representation of the periodic signal

$$\hat{x}[n] = \sum_{k=-M}^{M} X[k] e^{jk\Omega_0 n}$$

$$X[k] = \frac{1}{2M+1} \sum_{n=-M}^{M} \hat{x}[n] e^{-jk\Omega_0 n}$$

- The DTFS representation of the periodic signal can be rewritten as

$$X[k] = \frac{1}{2M+1} \sum_{n=-M}^{M} x[n] e^{-jk\Omega_0 n}$$

$$X[k] = \frac{1}{2M+1} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n}$$

DTFT contd.

- Now a continuous function of Frequency ,$X(e^{j\Omega})$ is defined whose samples at $k\Omega_0$ are equal to DTFS coefficients normalized by 2M+1

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad X[k] = X(e^{jk\Omega_0})/(2M+1)$$

- So we get

$$\hat{x}[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} X(e^{jk\Omega_0})e^{jk\Omega_0 n}$$

- And using the relationship $\Omega_0=2\pi/(2M+1)$

$$\hat{x}[n] = \frac{1}{2\pi}\sum_{k=-M}^{M} X(e^{jk\Omega_0})e^{jk\Omega_0 n}\Omega_0$$

- Applying limit

$$x[n] = \lim_{M\to\infty}\hat{x}[n] = \lim_{M\to\infty}\frac{1}{2\pi}\sum_{k=-M}^{M} X(e^{jk\Omega_0})e^{jk\Omega_0 n}\Omega_0$$

DTFT contd.

- Applying rectangular rule of approximation to an integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

- Finally

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega}$$

$$\boxed{X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}}$$

Condition for existence of DTFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

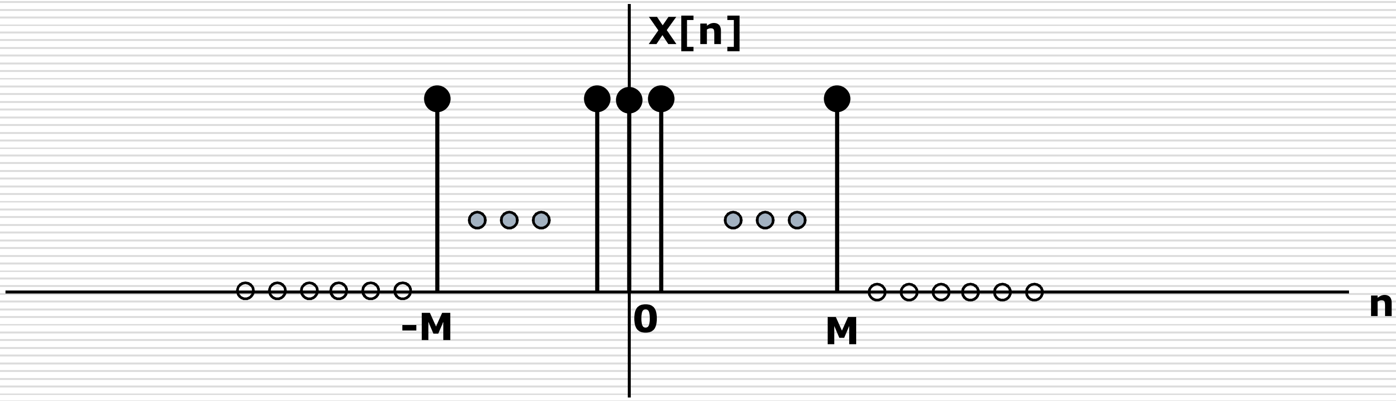
- If

$$\sum_{n=-\infty}^{\infty} |\, x[n] \,| < \infty$$

- Or If

$$\sum_{n=-\infty}^{\infty} |\, x[n] \,|^2 < \infty$$

DTFT of a Rectangular Pulse



- Defined by

$$x[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & |n| > M \end{cases}$$

DTFT of a Rectangular Pulse contd.

- The DTFT representation

$$X(e^{j\Omega}) = \sum_{n=-M}^{M} 1 e^{-j\Omega n}$$

- Using change of variable as m=n+M

$$X(e^{j\Omega}) = \sum_{m=0}^{2M} e^{-j\Omega(m-M)}$$

$$X(e^{j\Omega}) = e^{j\Omega M} \sum_{m=0}^{2M} e^{-j\Omega m}$$

$$\sum_{n=0}^{N-1} \beta^n = \begin{cases} \dfrac{1-\beta^N}{1-\beta}, \beta \neq 1 \\ N, \ \beta = 1 \end{cases}$$

$$X(e^{j\Omega}) = \begin{cases} e^{j\Omega M} \dfrac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}, \Omega \neq 0, \pm 2\pi, \pm 4\pi \dots \\ 2M+1, \Omega = 0, \pm 2\pi, \pm 4\pi \dots \end{cases}$$

DTFT of a Rectangular Pulse contd.

- Only considering the first scenario

$$X(e^{j\Omega}) = e^{j\Omega M}\frac{1-e^{-j\Omega(2M+1)}}{1-e^{-j\Omega}}$$

$$X(e^{j\Omega}) = e^{j\Omega M}\frac{e^{-j\Omega(2M+1)/2}(e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2})}{e^{-j\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})}$$

$$X(e^{j\Omega}) = \frac{e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2}}{(e^{j\Omega/2}-e^{-j\Omega/2})}$$

$$\boxed{X(e^{j\Omega}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)}}$$

Using L'Hospital's rule $\lim\limits_{\Omega\to 0,\pm 2\pi,\pm 4\pi,\ldots} X(e^{j\Omega}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)} = 2M+1$

DTFT Example

- Find the DTFT of $x[n]=a^n u[n]$, where $a=0.5$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\Omega n}$$

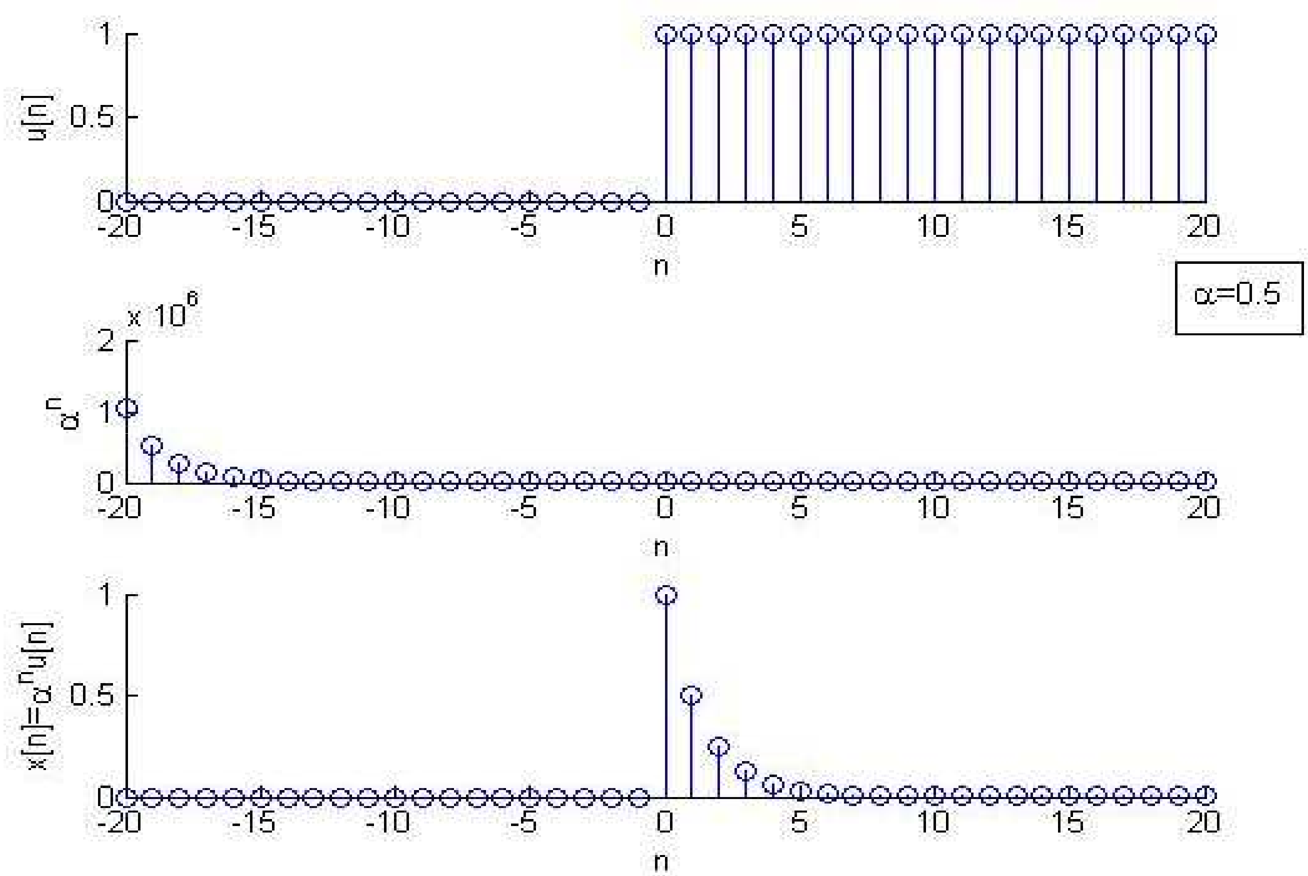
$$= \frac{1}{1 - a e^{-j\Omega}}, \text{for } |a| < 1$$

$$= \frac{1}{1 - a\cos\Omega + ja\sin\Omega}$$

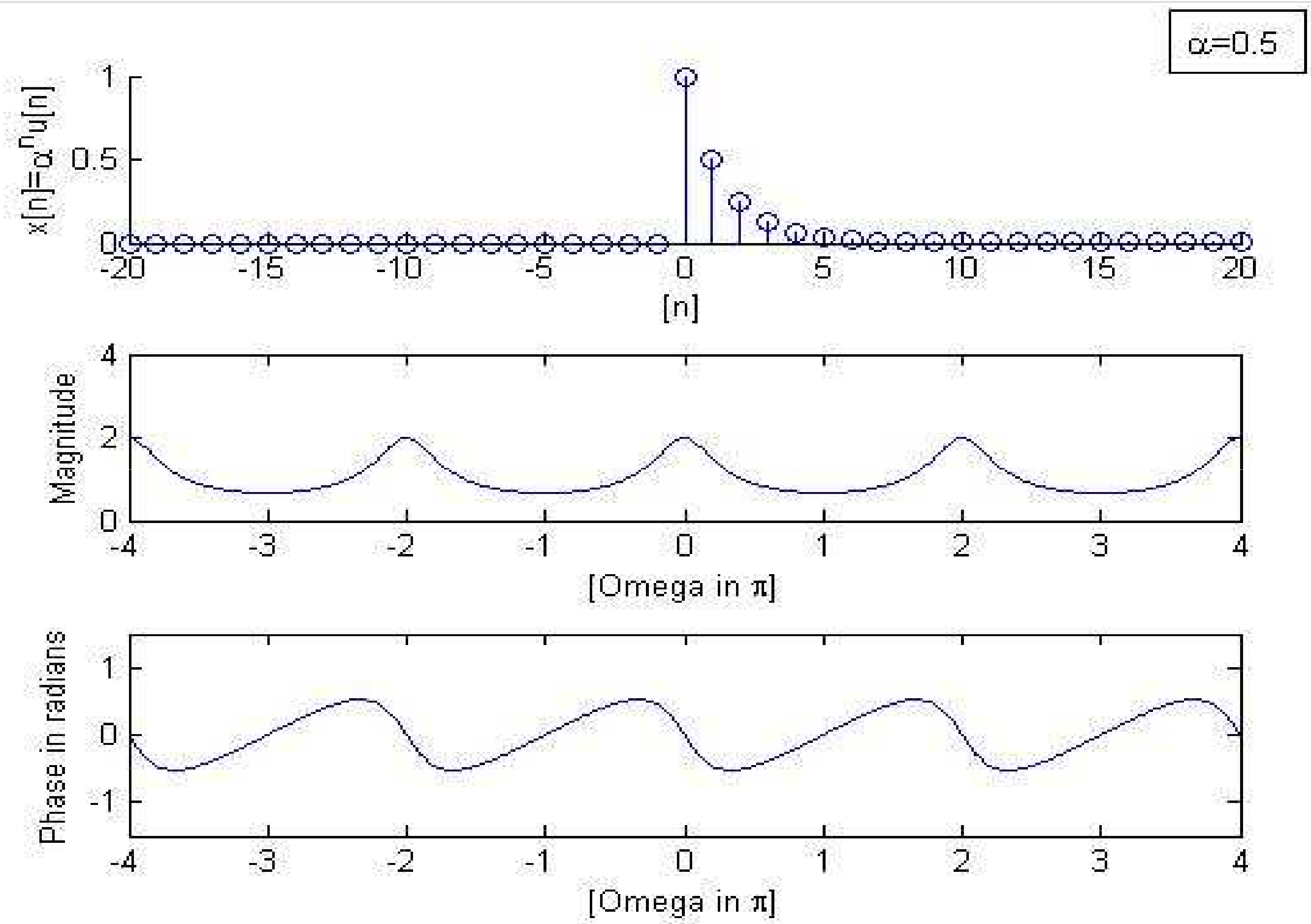
$$\left|X(e^{j\Omega})\right| = \frac{1}{(1+a^2-2\cos\Omega)^{1/2}}$$

$$\arg\{X(e^{j\Omega})\} = -\tan^{-1}\frac{a\sin\Omega}{1-a\cos\Omega}$$

DTFT Example



DTFT Example



DTFT Example

- Find the DTFT of $x[n]=2(3)^n u[-n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} 2(3)^n u[-n] e^{-j\Omega n}$$

$$= 2\sum_{n=-\infty}^{0} (3e^{-j\Omega})^n$$

$$= 2\frac{(3e^{-j\Omega})^{-\infty} - (3e^{-j\Omega})^1}{1-3e^{-j\Omega}}, \quad \text{using } \sum_{k=n_1}^{n_2} a^k = \frac{a^{n_1} - a^{n_2+1}}{1-a}, a \neq 1$$

$$= 2\frac{-3e^{-j\Omega}}{1-3e^{-j\Omega}}$$

$$= \frac{2}{1-\frac{1}{3}e^{j\Omega}}$$

Fourier Representation Properties

Periodicity

- DTFS and DTFT are periodic
- Since complex sinusoids are 2π-periodic functions of frequency.
- That is, discrete-time sinusoids whose frequency differ by integer multiples of 2π are identical.

Fourier Representation Properties contd.

Linearity

- All Fourier representations are linear in nature

$$z[n] = ax[n] + by[n] \overset{DTFT}{\longleftrightarrow} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \overset{DTFS,\Omega_0}{\longleftrightarrow} Z(k) = aX[k] + bY[k]$$

This property is used to find Fourier representations of signals that are constructed as sums of signals whose representations are already known.

Fourier Representation Properties contd.

- Symmetry

Fourier Representation	Real-valued Time Signal	Imaginary Valued Time Signal
DTFS and DTFT	Magnitude Spectrum even function and Phase spectrum odd function	Magnitude Spectrum odd function and Phase spectrum even function

Fourier Representation Properties contd.

- Convolution Property

Considering the convolution of two non-periodic continuous-time signal x(t) and h(t), we define y(t) as

$$y(t) = h(t) * x(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Now, expressing x(t-τ) in terms of its FT, we can write

$$x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-\tau)}d\omega$$

Substituting Fourier representation of x(t-τ) into the expression of y(t), we get

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]X(j\omega)e^{j\omega t}d\omega$$

Fourier Representation Properties contd.

Convolution Property

Now, remembering that the inner integral is FT of $h(\tau)$, we can write

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega \rightarrow \text{Since } H(j\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

$$\text{Recalling, } y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega$$

$$\text{We find, } Y(j\omega)=H(j\omega)X(j\omega)$$

Finally,

$$\boxed{y(t)=h(t)*x(t)\overset{FT}{\longleftrightarrow}Y(j\omega)=X(j\omega)H(j\omega)}$$

Convolution in time domain is equivalent to multiplication in frequency domain

Fourier Representation Properties contd.

- Convolution Properties

$$x(t) * h(t) \overset{FT}{\longleftrightarrow} X(j\omega)H(j\omega)$$

$$x(t) \otimes h(t) \overset{FS,\omega_0}{\longleftrightarrow} TX[k]H[k]$$

$$x[n] * h[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\Omega})H(e^{j\Omega})$$

$$x[n] \otimes h[n] \overset{DTFS,\Omega_0}{\longleftrightarrow} NX[k]H[k]$$

- Application of Convolution Property

- When estimation of time domain convolution is too complicated, the convolution property can help in reducing computational complexity.

- Example, Given $h[n]=(1/2)^n u[n]$ and $x[n]=(1/3)^n u[n]$, derive the expression for $y[n]$

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{1}{1 - \frac{1}{3} e^{-j\Omega}}$$

$$h[n] \otimes x[n] \stackrel{DTFT}{\leftrightarrow} H(e^{j\Omega})X(e^{j\Omega})$$

Fourier Representation Properties contd.

$$Y(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}}\frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = \frac{3}{1-\frac{1}{2}e^{-j\Omega}} - \frac{2}{1-\frac{1}{3}e^{-j\Omega}}$$

$$y[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

Fourier Transforms on Digital Computers

- Application of Fourier Transforms
  - Spectral Analysis
  - Analysis of Systems with respect to frequency response.
- Transforms that are suited for computation on digital computers are called Discrete Fourier Transform (DFT).
- Can be viewed as a logical extension to Fourier Transform.

Fourier Transforms on Digital Computers

Problem: Fourier transform of an analog signal x(t) needs to be done on digital computers.

Constraints: Digital computers can store and manipulate only finite set of values.

Requirement: x(t) must be represented with finite set of values. In doing so, the steps are,

- Step 1: Sampling x(t) at pre-decided intervals and thus obtain a discrete sequence $x_a[n]$.
- Step 2: Since x(t) may not be time limited, obtaining a finite set of samples of $x_a[n]$ by means of truncation.

Fourier Transforms on Digital Computers

- Step 2 continues: Let us assume these samples, x[n], are defined for n in the range [0, N-1].
  - x[n] can be obtained from $x_a[n]$ by multiplying $x_a[n]$ with a window function w[n] where w[n] is defined as,

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{Otherwise} \end{cases}$$

So that,

$$x[n] = x_a[n]w[n]$$

Fourier Transforms on Digital Computers

- Step 2 continues: Since x[n] is discrete, DTFT can be applied on x[n] which is represented as,

$$X(e^{j\Omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

- But it is still not in a suitable form for machine computation as because Ω is a continuous variable taking all possible values between 0 and 2π. Therefore, another step is needed.

Fourier Transforms on Digital Computers

- Step 3 : Evaluating $X(e^{j\Omega})$ only at finite number of values Ω_k, by uniform sampling in the range 0 to 2π. This can be represented as,

$$X(e^{j\Omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\Omega_k n}, \quad k = 0,\ 1,\ 2,....,\ \ M-1$$

where,

$$\Omega_k = \frac{2\pi}{M}k$$

- M, the number of frequency samples can be any value but is often chosen as equal to the number of time samples N. Using M=N and writing $X(\Omega_k)$ as X[k], we get,

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}}$$

Discrete Fourier Transform (DFT)

Notes

- Theoretically x(t) can take any value within -∞ to +∞, but since computer representation must be a finite number, quantization is needed.

- X[k] is an approximation of continuous time Fourier transform of signal x(t). More accurately speaking, it defines the Discrete Fourier Transform (DFT) of the N point sequence x[n].

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}}$$

The N-point DFT of x[n].

Discrete Fourier Transform (DFT)

- DFT is similar to DTFT but with some differences.
- Use of DFT is widespread since fast and efficient algorithms for implementing DFT on digital computer exist.
- This algorithms collectively are known as Fast Fourier Transform (FFT) algorithms.

DFT and Its Inverse

- Considering x[n] as an N-point sequence (i.e., n=0,1 ,2,…, N-1), DFT of x[n] is given by

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}}$$

- And the inverse of DFT, called IDFT is given by

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}}$$

The DFT and Its Inverse

- Both DFT and IDFT are periodic with period N.
- That is both DFT and IDFT replaces the finite sequence x[n] with its periodic extension.
- There is connection between DTFS and DFT.
- Since DFT is evaluated in the range 0 to 2π spaced apart by $2\pi/N$, in considering DFT of two signals simultaneously, the frequencies corresponding to the DFT must be same for any operation to be meaningful. This means N should be equal.
- If needed zeros should be added to make N equal.

The Matrix Representation of The DFT

- Denoting $e^{-j2\pi/N}$ by W_N ,the DFT of x[n] can be written as,

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{k*n} \quad , k = 0,1,\ldots, N-1$$

- Now, Let $\hat{W}$ be the matrix whose (k,n)-th element $[\hat{W}]_{kn}$ is equal to W_N^{k*n}

$$\hat{W} = \begin{bmatrix} W_N^{0*0} & W_N^{0*1} & \cdots & W_N^{0*(N-1)} \\ W_N^{1*0} & W_N^{1*1} & \cdots & W_N^{1*(N-1)} \\ \cdots & \cdots & \cdots & \cdots \\ W_N^{(N-1)*0} & W_N^{(N-1)*1} & \cdots & W_N^{(N-1)*(N-1)} \end{bmatrix}$$

- The transform vector $\hat{X} = [X[0], X[1], \ldots X[N-1]]^T$ can be obtained as

  $\hat{X} = \hat{W}\hat{x}$, where $\hat{x}$ is the data vector given by $\hat{x} = [x[0], x[1], \ldots x[N-1]]^T$

The Matrix Representation of The DFT

- In detail,

$$\hat{W}\hat{x} = \begin{bmatrix} W_N^{0*0} & W_N^{0*1} & .... & W_N^{0*(N-1)} \\ W_N^{1*0} & W_N^{1*1} & .... & W_N^{1*(N-1)} \\ .... & .... & .... & .... \\ W_N^{(N-1)*0} & W_N^{(N-1)*1} & .... & W_N^{(N-1)*(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ ... \\ x[N-1] \end{bmatrix}$$

$$= \begin{bmatrix} W_N^{0*0}x[0] + W_N^{0*1}x[1] + ... + W_N^{0*(N-1)}x[N-1] \\ W_N^{1*0}x[0] + W_N^{1*1}x[1] + ... + W_N^{1*(N-1)}x[N-1] \\ .... \\ W_N^{(N-1)*0}x[0] + W_N^{(N-1)*1}x[1] + ... + W_N^{(N-1)*(N-1)}x[N-1] \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=0}^{N-1} W_N^{0*n}x[n] \\ \sum_{n=0}^{N-1} W_N^{1*n}x[n] \\ .... \\ \sum_{n=0}^{N-1} W_N^{(N-1)*n}x[n] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ .... \\ X[N-1] \end{bmatrix} = \begin{bmatrix} X[0] & X[1] & ..... & X[N-1] \end{bmatrix}^T = \hat{X}$$

Fast Fourier Transform (FFT)

- Recalling, $X[k]=\sum_{n=0}^{N-1} x[n]W_N^{kn} \quad ,k=0,1,....,N-1$
- Which can be written as,

$$X[k]=x[0]W_N^0+x[1]W_N^k+x[2]W_N^{2k}+.....+x[N-1]W_N^{(N-1)k}$$

- It follows that determination of each X[k] requires N complex multiplication and N-1 complex addition.

- And considering all values of X(k) where k=0,1,2,…N-1, it follows that $N*N=N^2$ complex multiplications and N*(N-1) complex additions are required. Total computational burden $=2N^2-N$.

- Since N is only 0.78% and 0.39% of $2N^2$ when N=64 and 128 respectively, for large values of N total computational burden can be considered as $2N^2$.

Fast Fourier Transform (FFT)

- This computational complexity is often denoted by "Big O" notation. Here Complexity= $O(N^2)$.
- Procedures that can reduce computational burden are of considerable interest.
- These procedures are known as Fast Fourier Transform (FFT) algorithms.
- The basic idea is to divide the given sequences into subsequences of smaller lengths, evaluate their DFTs and then combine them to obtain the DFT of the original sequence.

FFT Algorithms

- The Decimation-in-Time (DIT) Algorithm
  - Input x[n] is divided into smaller sequences (subsequences).
- The Decimation-in-Frequency (DIF) Algorithm
  - Output X[K] is divided into smaller sequences (subsequences).

Note: These algorithms are applicable only when the Data Length (N) is a power of 2, i.e. $N=2^P$, where P is a positive integer, e.g., N=2, 4, 8, 16, 32 etc, and hence they are commonly known as radix-2 algorithms.

The DIT Algorithm Example 8

- Here x[n] is at first divided into two subsequences of length N/2 by grouping the even-indexed samples and the odd-indexed samples together.
  - Recalling, $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \quad ,k = 0,1,....,\ N-1$$
  - We can write,

$$X[k] = \sum_{n\ \text{even}} x[n] W_N^{nk} + \sum_{n\ \text{odd}} x[n] W_N^{nk}$$

- Letting n=2r in the first sum and n=2r+1 in the second, we get,

$$X[k] = \sum_{r=0}^{N/2-1} x[2r] W_N^{2rk} + \sum_{r=0}^{N/2-1} x[2r+1] W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{N/2-1} g[r] W_N^{2rk} + W_N^{k} \sum_{r=0}^{N/2-1} h[r] W_N^{2rk}$$

where, g[r]=x[2r] and h[r]=x[2r+1]

The DIT Algorithm

- Again remembering that for an N/2 point sequence Y[n], the DFT is given by,

$$Y[k] = \sum_{n=0}^{\frac{N}{2}-1} y[n] W_{N/2}^{nk}$$

$$= \sum_{n=0}^{N/2-1} y[n] \left(e^{-j\frac{2\pi}{N/2}} \right)^{nk}, \quad \text{since} \;\; W_N = e^{-j\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y[n] \left(e^{-j\frac{2\pi}{N}} \right)^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} y[n] W_N^{2nk}$$

- Thus from
- We can write

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} g[r] W_N^{2rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} h[r] W_N^{2rk}$$

$$X[k] = G[k] + W_N^K H[k]$$

where, G[K] and H[K] are N/2 point DFTs of g[r] and h[r] respectively.

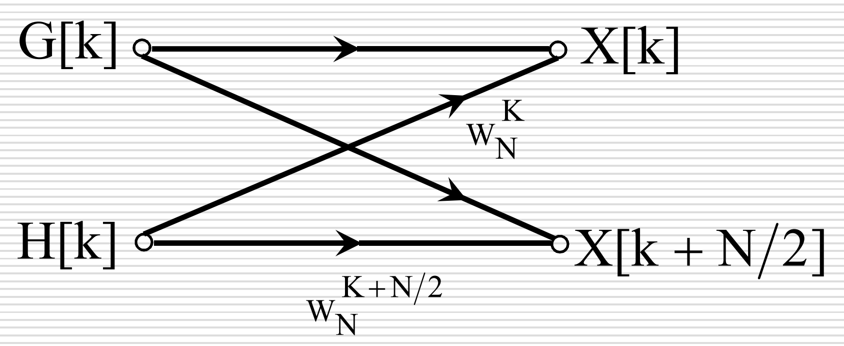
The DIT Algorithm

- Since, G[K] and H[K] are N/2 periodic, we can write the previous expression as

$$X[k] = G[k] + W_N^k H[k], \quad k = 0,\ 1,\ 2,\ \dots,\ \frac{N}{2} - 1$$

$$X[k + N/2] = G[k] + W_N^{k+N/2} H[k]$$

- The Signal Flow Graph



The DIT Algorithm

- Example: 4-point Sequence, i.e., N=4, k=0,1

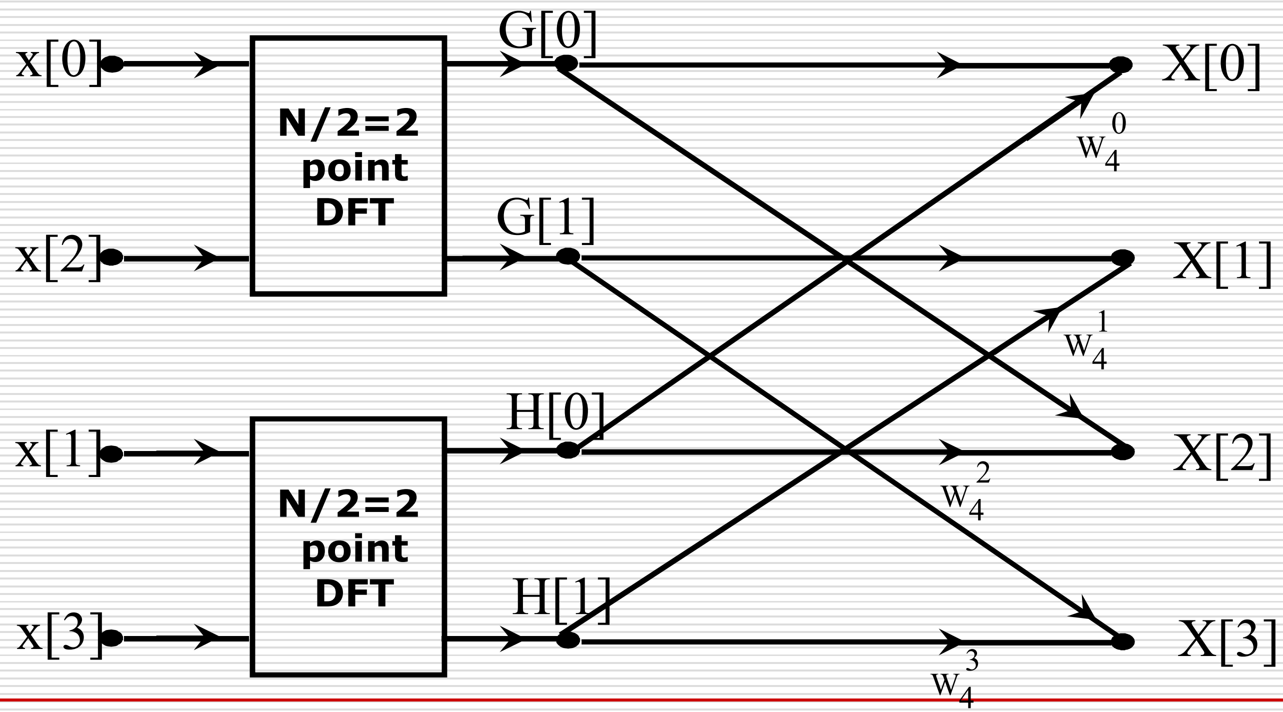
1. K=0

$$X[0] = G[0] + W_4^0 H[0]$$
$$X[2] = G[0] + W_4^2 H[0]$$

2. K=1

$$X[1] = G[1] + W_4^1 H[1]$$
$$X[3] = G[1] + W_4^3 H[1]$$

$$X[k] = G[k] + W_N^k H[k]$$
$$X[k + N/2] = G[k] + W_N^{k+N/2} H[k]$$



The DIT Algorithm

- Example: 2-point Sequence, i.e., N=2, k=0

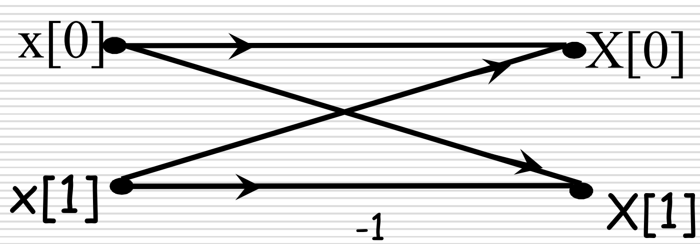
1. K=0

$$X[0] = G[0] + W_2^0 H[0]$$
$$= G[0] + H[0]$$
$$= \sum_{r=0}^{\frac{N}{2}-1} g[r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} h[r] W_N^{2rk}$$
$$= g[0] + h[0] = x[0] + x[1]$$

$$X[k] = G[k] + W_N^k H[k]$$
$$X[k + N/2] = G[k] + W_N^{k+N/2} H[k]$$

since, $g[r]=x[2r]$
and $h[r]=x[2r+1]$

$$X[1] = x[0] + W_2^1 x[1]$$
$$= x[0] + e^{-j\frac{2\pi}{2}} x[1]$$
$$= x[0] + (-1) * x[1]$$



Butterfly Computation

The DIT Algorithm

- Example: 8-point Sequence, i.e., N=8,
- Dividing the sequence into two N/2=4 point sequences and grouping the even-indexed and odd-indexed input sequences together, i.e., x[0],x[2],x[4],x[6] and x[1],x[3],x[5],x[7] together.

1. K=0

$$X[0] = G[0] + W_8^0 H[0]$$
$$X[4] = G[0] + W_8^4 H[0]$$

2. K=1

$$X[1] = G[1] + W_8^1 H[0]$$
$$X[5] = G[1] + W_8^5 H[0]$$

$$X[k] = G[k] + W_N^k H[k]$$
$$X[k + N/2] = G[k] + W_N^{k+N/2} H[k]$$

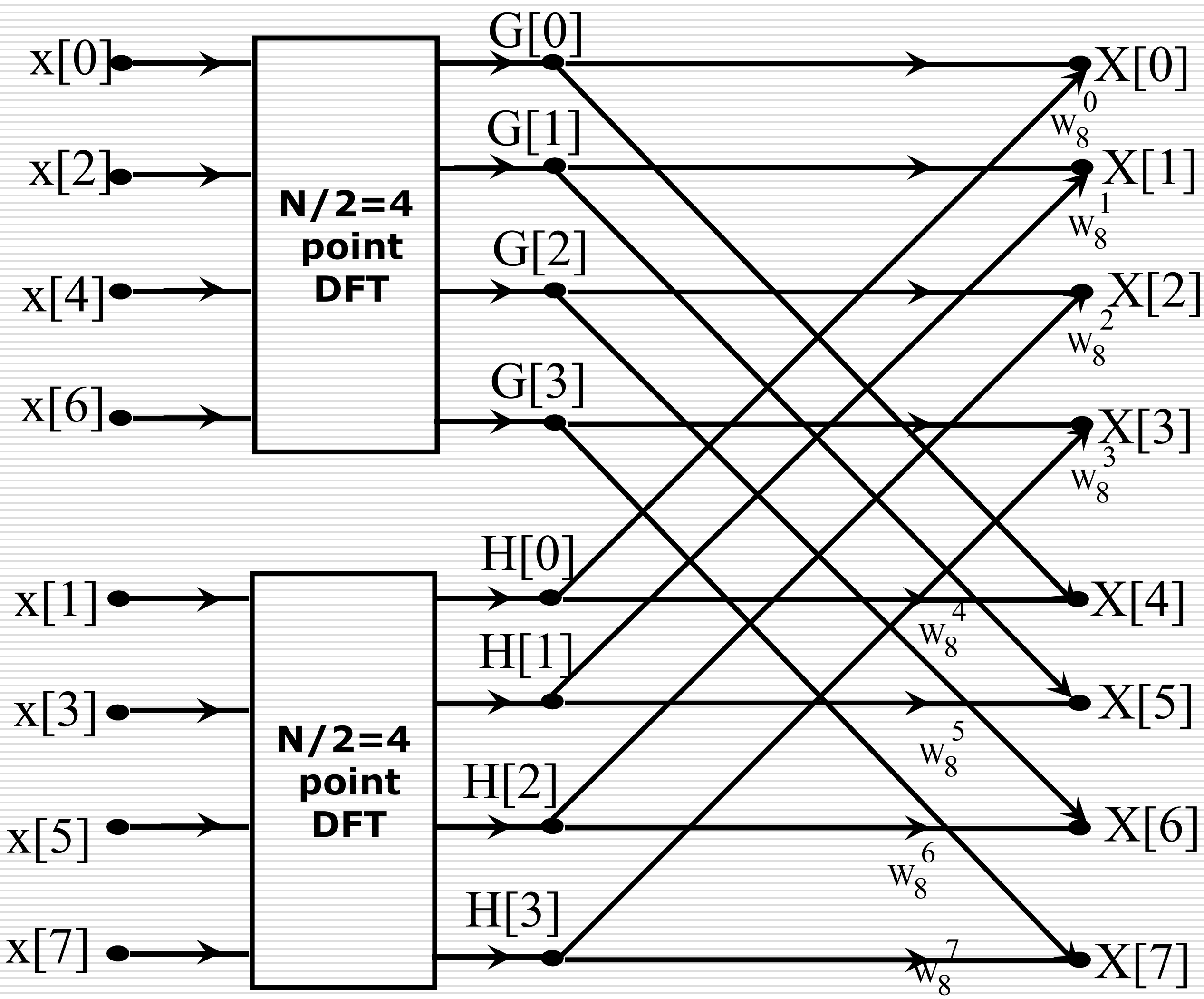
3. K=2

$$X[2] = G[2] + W_8^2 H[2]$$
$$X[6] = G[2] + W_8^6 H[2]$$

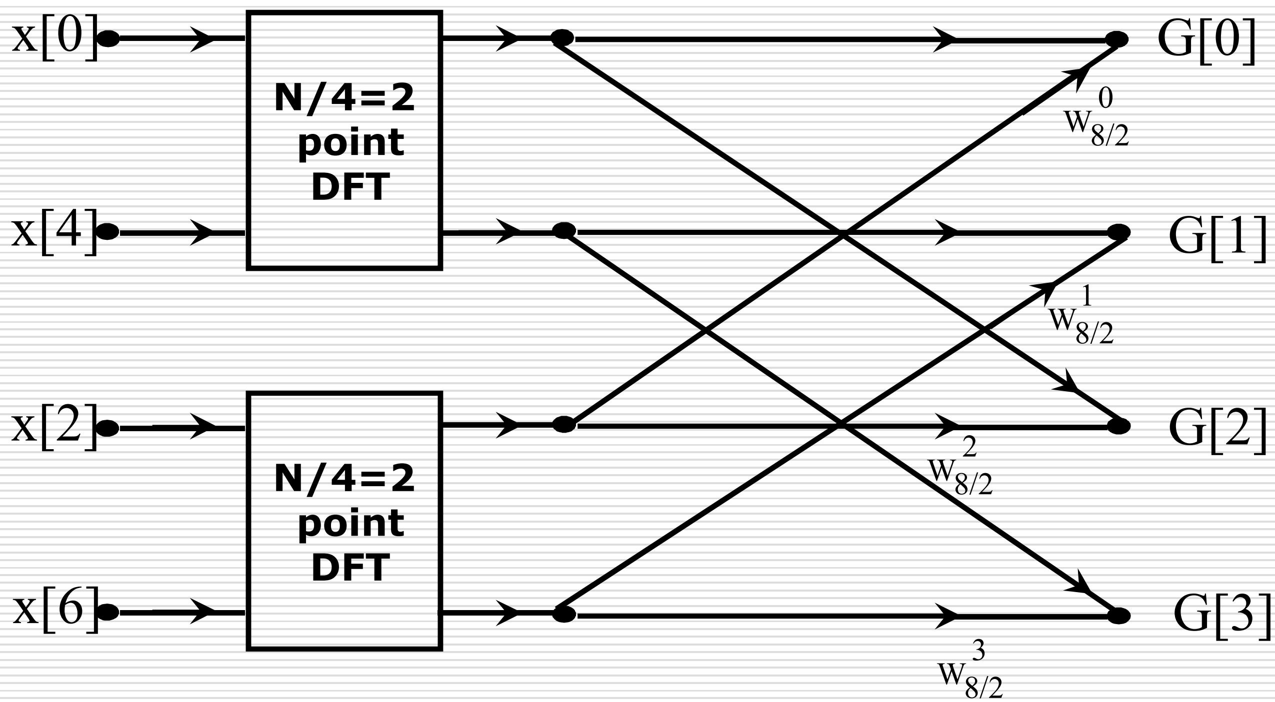
4. K=3

$$X[7] = G[3] + W_8^7 H[3]$$
$$X[5] = G[1] + W_8^5 H[0]$$

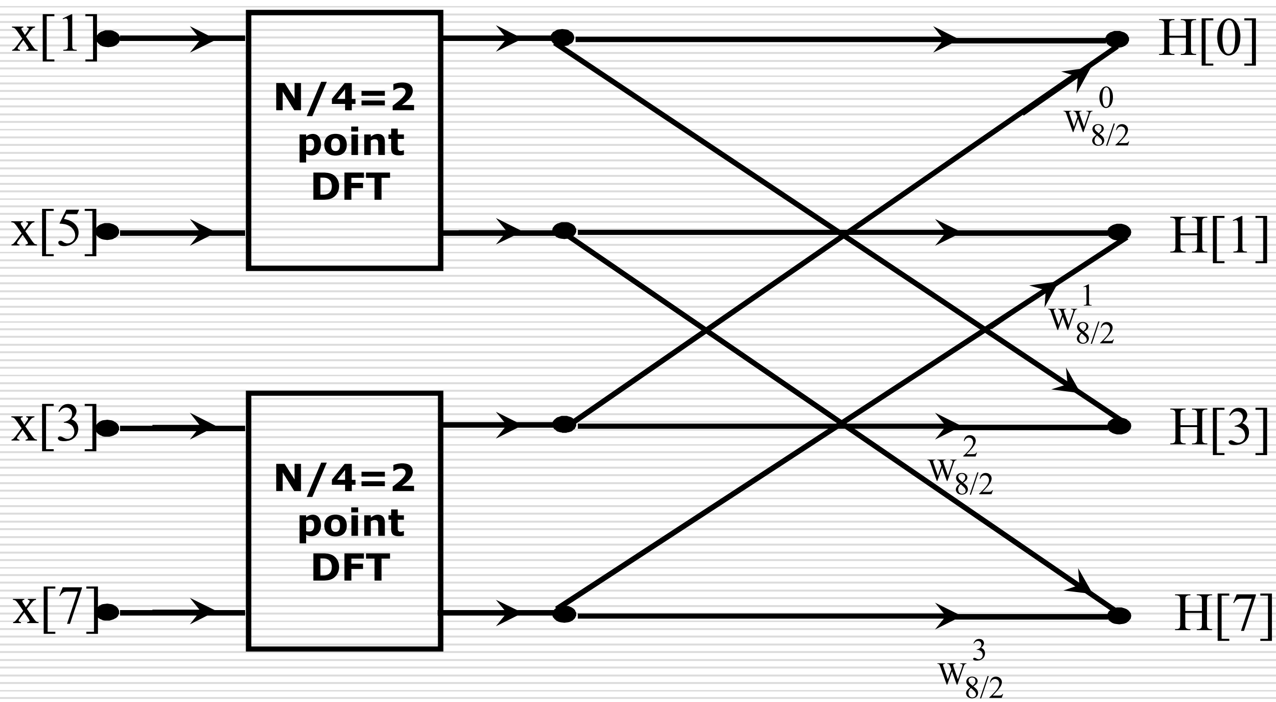
Signal Flow Graph



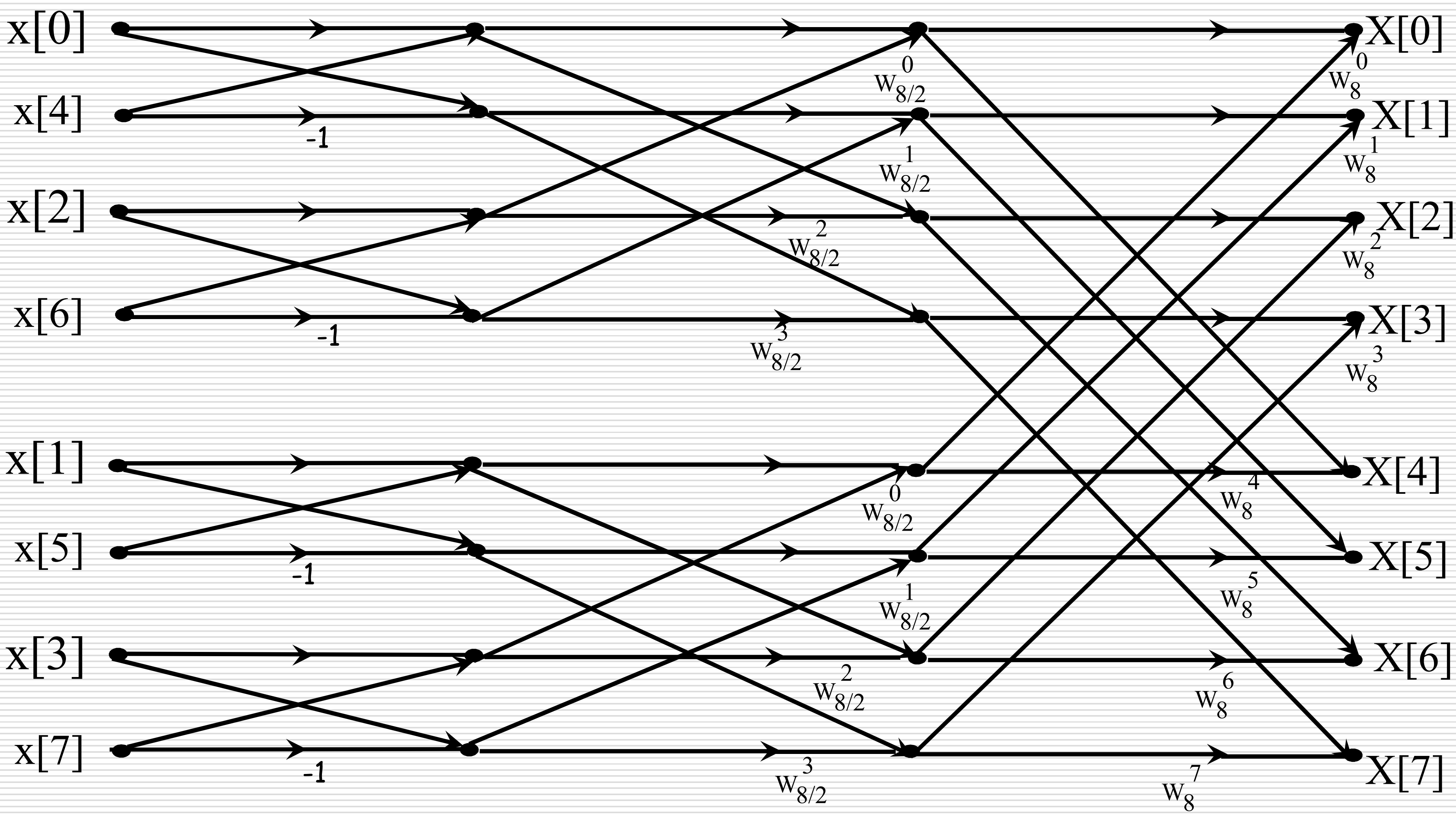
4 N/4 Point DFTs (First 2)



4 N/4 Point DFTs (Second 2)



The Complete Signal Flow Graph



The DIT Algorithm

- **Observations**
  - Total number of stages is $\log_2 N$
  - Computational Complexity $N \log_2 N$
  - In-place computation is possible, i.e., the results of the computations at any stage can be stored in the same locations as those of the input to that stage.
  - Ordering of the input is determined by bit reversing the natural numbers 1 to N as shown in the next slide.

Input Ordering

Decimal Number	Binary Representation	Bit-reversed Representation	Decimal Equivalent.
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7