Classical Oscillator Model of Dipole Trapping

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1 Static Electric Fields and Matter

1.1 Recall that a polarizable particle exposed to an electric field will have an induced dipole moment of the form:

$$\mathbf{p} = \alpha \mathbf{E} = q\mathbf{d} \tag{1.1}$$

where α is the polarizability of the particle. If the electric field is nonuniform, then there will be a force acting on the dipole which will draw it to the region of the stronger electric field.

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} \tag{1.2}$$

If we apply eq (4.1) from the appendix

$$(\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} = \frac{1}{2}\boldsymbol{\nabla} E^2$$

$$\mathbf{F} = \frac{\alpha}{2}\boldsymbol{\nabla} E^2 \tag{1.3}$$

1.2 Suppose $\mathbf{E}^2$ had a local maximum at point P so that there is a sphere of radius $P + c$ such that:

$$E^2(P) > E^2(P + c) \Rightarrow |E(P)| > |E(P + c)|$$

However, we know that if there's no charge in the sphere, then $E_{inside} = E_{outside}$. This contradicts the inequality, so there can't be a local maximum which means that it isn't possible to trap particles in a static field.

2 Dipole Potential

2.1 What about time-varying electric fields? Since we are no longer dealing with electrostatics, $\boldsymbol{\nabla} \times \mathbf{E} \neq 0$. This means that the application of eq (4.1) from the appendix gives us:

$$(\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} = \frac{1}{2}\boldsymbol{\nabla} E^2 + (\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t})$$

Note that:

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

Since there is no current density, $\frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{B}$ so:

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{B} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

The first term is 0 which leaves:

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

Finally, the force on a dipole due to a nonuniform, time varying field is:

$$\mathbf{F} = \alpha[\frac{1}{2}\boldsymbol{\nabla} E^2 + \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})] \tag{2.1}$$

recall that:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

So the force can be rewritten as:

$$\mathbf{F} = \alpha[\frac{1}{2}\boldsymbol{\nabla} E^2 + \mu_0 \frac{\partial \mathbf{S}}{\partial t}]$$

2.2 Electric and magnetic fields oscillate at very rapid speeds, much faster than an atom can react. Thereofre a more useful value is the time averaged force which we will define to be the dipole force;

$$\mathbf{F_{dip}} \equiv \langle \mathbf{F} \rangle$$

$$= \langle \frac{\alpha}{2} \boldsymbol{\nabla} |E|^2 \rangle + \mu_0 \left\langle \frac{\partial \mathbf{S}}{\partial t} \right\rangle$$

$$= \boldsymbol{\nabla} \langle \frac{\alpha}{2} |E|^2 \rangle + \mu_0 \frac{\partial}{\partial t} \langle \mathbf{S} \rangle$$

The second term will be zero since the time average of the Poynting vector is a constant, this leaves:

$$\mathbf{F_{dip}} = \langle \frac{\alpha}{2} \boldsymbol{\nabla} |E|^2 \rangle$$

$\alpha \in \mathbb{C}$ but the time average must be real, so we average over the real parts of the argument instead:

$$\mathbf{F_{dip}} = \frac{1}{2} Re(\alpha) \boldsymbol{\nabla} \langle |E|^2 \rangle$$

$$\langle |E|^2 \rangle = \frac{1}{2} |\tilde{E}_0|^2$$

The square of the magnitude of an electric field is related to its intensity by the expression:

$$I = \frac{1}{2} c \varepsilon_0 |\tilde{E}_0|^2$$

In terms of the intensity, the dipole force is therefore:

$$\mathbf{F}_{dip} = \frac{1}{2c\varepsilon_0} Re(\alpha) \boldsymbol{\nabla} I$$

This force is conservative which means that it must be the gradient of some potential. So finally we can express the dipole potential as:

$$U_{dip} = -\frac{1}{2c\varepsilon_0} Re(\alpha) I \tag{2.2}$$

3 Scattering Rate

3.1 Recall that power is simply work per unit time:

$$P = \frac{W}{t} = q\frac{\Delta V}{t} = qE\frac{d}{\Delta t} = qEv = E\dot{p}$$

Again we take the time average to get the power absorbed:

$$P_{abs} = \langle \dot{p}E \rangle \tag{3.1}$$

A time dependent electric field is usually of the form:

$$E = \tilde{E}_0 e^{i(kz-\omega t)}$$

This means that:

$$p = \alpha \tilde{E}_0 e^{i(kz-\omega t)}$$

$$\dot{p} = -i\omega\alpha\tilde{E}_0 e^{i(kz-\omega t)} = -i\omega\alpha E$$

If we plug this result into eq (3.1):

$$P_{abs} = \omega\langle i\alpha|E|^2\rangle$$

This time, since α is being multiplied by i we need to average over the complex part to get a real value:

$$P_{abs} = \omega Im(\alpha)\langle|E|^2\rangle = \frac{\omega}{2}Im(\alpha)|\tilde{E}_0|^2$$

$$P_{abs} = \frac{\omega}{c\varepsilon_0}Im(\alpha)I \tag{3.2}$$

If we view the electric field as a stream of photons, each with energy $\hbar\omega$, then the number of photons absorbed and then re emitted by the atom over time defines the scattering rate Γ_{sc}

$$\Gamma_{sc} = \frac{P_{abs}}{\hbar\omega}$$

$$\Gamma_{sc} = \frac{1}{c\varepsilon_0\hbar}Im(\alpha)I \tag{3.3}$$

4 Polarizability

4.1 To determine the polarizability of an atom, we consider the Lorentz model of a classical oscillator. In this model, an electron is considered to be bound to the nucleus by a spring with spring constant $k = m_e\omega_0^2$ where ω^0 is corresponds to the optical transition frequency. From newton's second law.

$$F_{net} = F_{driving} - F_{damping} + F_{spring} = m_e\ddot{x}$$

The spring force is simply $-kx = -\omega_0^2 m_e x$ and the driving force comes from the electric field and is $F_{driving} = qE$ where $E(t) = \tilde{E}_0 e^{-i\omega t}$. We are assuming here that the dipole is small so that there is no significant spatial dependence to the electric field in that region.

4.2 The damping force comes from the fact that the electron will radiate energy as it accelerates. The power radiated by an accelerating charge is given by Larmor's formula:

$$P = \frac{e^2a^2}{6\pi\varepsilon_0 c^3}$$

The force is therefore:

$$F = \frac{P}{v} = \frac{e^2a^2}{6\pi\varepsilon_0 v c^3} = \Gamma m_e\dot{x}$$

Where:

$$\Gamma = \frac{e^2a^2}{6\pi\varepsilon_0 m_e v^2 c^3}$$

Note that $\frac{a^2}{v^2} = \omega^2$ so

$$F_{damping} = \Gamma_\omega \dot{x}$$

Where:

$$\Gamma_\omega = \frac{e^2\omega^2}{6\pi\varepsilon_0 m_e c^3} \tag{4.1}$$

Is the damping rate of the system.

4.3 If we plug all of this into Newton's second law we are left with;

$$qE - \Gamma_\omega m_e\dot{x} - \omega_0^2 m_e x = m_e\ddot{x}$$

$$\ddot{x} + \Gamma_\omega\dot{x} + \omega_0^2 x = \frac{q\tilde{E}_0}{m_e}e^{-i\omega t} \tag{4.2}$$

The solution will be of the form:

$$x(t) = x_0 e^{-i\omega t}$$

If we plug this result into the ODE then:

$$-x_0\omega^2 e^{-i\omega t} - i\omega\Gamma_\omega x_0 e^{-i\omega t} + x_0\omega_0^2 e^{-i\omega t} = \frac{q\tilde{E}_0}{m_e}e^{-i\omega t}$$

The exponentials will cancel and we're left with:

$$x_0 = \frac{q\tilde{E}_0}{m_e}\frac{1}{-\omega^2 - i\omega\Gamma_\omega + \omega_0^2}$$

Recall from eq (1.1) that:

$$p = qx_0 = \frac{q^2\tilde{E}_0}{m_e}\frac{1}{-\omega^2 - i\omega\Gamma_\omega + \omega_0^2} = \alpha\tilde{E}_0$$

The polarizability is therefore:

$$\alpha = \frac{q^2}{m_e} \frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma_\omega} \tag{4.3}$$

From equation (4.2)

$$\frac{e^2}{m_e^2} = \frac{6\pi\varepsilon_0 c^3}{\omega^2}\Gamma_\omega$$

To remove the dependence of Γ_ω on ω we introduce the on resonance damping rate Γ, this corresponds to the lifetime of the exited state of the atom which is an actual measurable quantity.

$$\Gamma = \Gamma_{\omega_0} = \frac{\omega_0^2}{\omega^2}\Gamma_\omega$$

Finally we conclude that the atomic polarizability will be:

$$\alpha = 6\pi\varepsilon_0 c^3 \frac{\Gamma/\omega_0^2}{\omega_0^2 - \omega^2 - i(\omega^3/\omega_0^2)\Gamma} \tag{4.4}$$

5 Putting it all together

5.1 From equation (4.4) the real part of α will be:

$$Re(\alpha) = \frac{3\pi\varepsilon_0 c^3}{\omega_0^3}\left(\frac{\Gamma}{\omega_0-\omega} + \frac{\Gamma}{\omega_0+\omega}\right)$$

While the imaginary part will be:

$$Im(\alpha) = \frac{3\pi\varepsilon_0 c^3}{2\omega_0^3}\left(\frac{\omega}{\omega_0}\right)^3\left(\frac{\Gamma}{\omega_0-\omega} + \frac{\Gamma}{\omega_0+\omega}\right)^2$$

If we plug these expressions into equations (2.2) and (3.3) we obtain the following expressions for the dipole potential and scattering rate

$$U_{dip}(\mathbf{r}) = -\frac{3\pi c^3}{2\omega_0^3}\left(\frac{\Gamma}{\omega_0-\omega} + \frac{\Gamma}{\omega_0+\omega}\right) I(\mathbf{r})$$

$$\Gamma_{sc}(\mathbf{r}) = \frac{3\pi c^2}{2\hbar\omega_0^3}\left(\frac{\omega}{\omega_0}\right)^3\left(\frac{\Gamma}{\omega_0-\omega} + \frac{\Gamma}{\omega_0+\omega}\right)^2 I(\mathbf{r})$$

In most experiments the laser is tuned close to resonance so that the detuning $\Delta = \omega - \omega_0$ is small. This means that we can apply the rotating wave approximation so that the second term in both expressions is 0 and $\omega/\omega_0 \approx 1$. From this we arrive at the final expressions for the dipole potential and scattering rates in terms of the detuning Δ and the field intensity $I(\mathbf{r})$

$$U_{dip}(\mathbf{r}) = \frac{3\pi c^3}{2\omega_0^3}\frac{\Gamma}{\Delta} I(\mathbf{r}) \tag{5.1}$$

$$\Gamma_{sc}(\mathbf{r}) = \frac{3\pi c^2}{2\hbar\omega_0^3}\left(\frac{\Gamma}{\Delta}\right)^2 \tag{5.2}$$

These two equations are related by the expression:

$$\hbar\Gamma_{sc} = \frac{\Gamma}{\Delta} U_{dip} \tag{5.3}$$

6 Appendix

6.1 Vector Product Rule (4) from Griffiths

$$\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} \tag{6.1}$$

6.2 In general, the time average of two sinusoidal/oscillatory functions with the same frequency and wave number will be (Griffiths 5th edition problem 9.12):

$$\langle fg \rangle = \frac{1}{2} Re(\tilde{f}\tilde{g}^*)$$

where $\tilde{f}$ and $\tilde{g}$ are the amplitudes of f and g respectively

7 References

Introduction to Electrodynamics 5th Edition Chapters 4, 8 and 9

Optical Dipole Traps for Neutral Atoms

Lorenz Model of a Classical Oscillator