Random Numbers

AI21BTECH11007

Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/MouktikaCherukupalli
    /Random_Numbers/blob/main/codes/1.1.c
wget https://github.com/MouktikaCherukupalli
    /Random_Numbers/blob/main/codes/
    coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/MouktikaCherukupalli
    /Random_Numbers/blob/main/codes/1.2.
    py
```

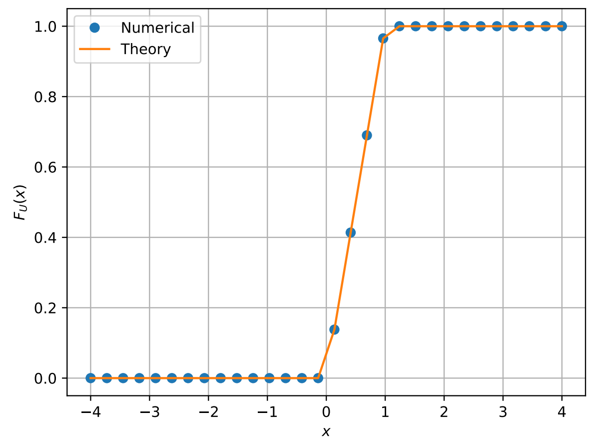

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** The PDF of U is given by

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(x)\, \mathrm{d}x \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{x} 0\, \mathrm{d}x = 0 \tag{1.4}$$

If $0 < x < 1$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{x} 1\, \mathrm{d}x \tag{1.5}$$

$$= 0 + x \tag{1.6}$$

$$= x \tag{1.7}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x$$

$$= \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{1} 1\, \mathrm{d}x + \int_{1}^{x} 0\, \mathrm{d}x \tag{1.8}$$

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = 0 + 1 + 0 \tag{1.9}$$

$$= 1 \tag{1.10}$$

Therefore, we obtain the CDF of U as

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.11}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{1.12}$$

and its variance as

$$\text{Var}[U] = E[U - E[U]]^2 \tag{1.13}$$

Write a C program to find the mean and variance of U

Solution: Download the C source code by executing the following commands

```
wget https://github.com/MouktikaCherukupalli
    /Random_Numbers/blob/main/codes/1.4.c
```

Compile and run the C program by executing the following

```
gcc 1.4.c −lm
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.500007 \tag{1.14}$$
$$\sigma^2_{\text{emp}} = 0.083301 \tag{1.15}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) dx \tag{1.16}$$

Solution: Verifying result theoritically
Given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

Mean is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.18}$$
$$= \int_{-\infty}^{\infty} x dx \tag{1.19}$$
$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.20}$$
$$= \frac{1}{2} \tag{1.21}$$

Varaiance is given by

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \tag{1.22}$$
$$E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.23}$$
$$= \int_{-\infty}^{\infty} x^2 dx \tag{1.24}$$
$$= \left[\frac{x^3}{3}\right]_0^1 \tag{1.25}$$
$$= \frac{1}{3} \tag{1.26}$$
$$E[U^2] - [E[U]]^2 = \frac{1}{3} - (\frac{1}{2})^2 \tag{1.27}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.28}$$
$$= \frac{1}{12} \tag{1.29}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/2.1.c
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/coeffs.h
```

Compile and run the C program by executing the following

```
gcc 2.1.c −lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the following Python code that plots Fig. 2.2

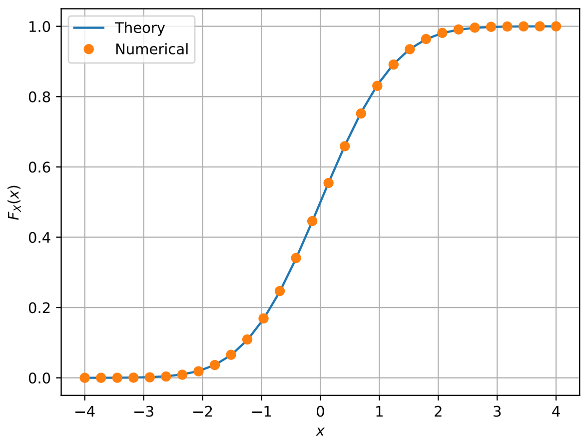

Fig. 2.2: The CDF of X

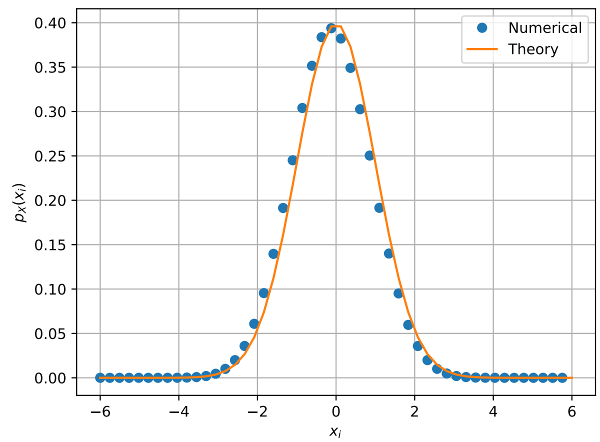

Fig. 2.3: The PDF of X

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

Every cdf is non decreasing function and bounded between 0 and 1

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have? **Solution:** Download the following Python code that plots Fig. 2.2

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/2.3.py
```

pdf is bounded between 0 and 1

2.4 Find the mean and variance of X by writing a C program
Solution: Download the C source code by executing the following commands

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob/main/
    codes/2.4.c
```

Compile and run the C program by executing the following

```
gcc 2.4.c −lm
./a.out
```

The output of the code is

$$\mu_{\text{emp}} = 0.000294 \tag{2.3}$$
$$\sigma^2_{\text{emp}} = 0.999560 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x \tag{2.6}$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.7}$$

if we consider

$$\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = f(x) \tag{2.8}$$
$$\implies f(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \tag{2.9}$$
$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.10}$$
$$= -f(x) \tag{2.11}$$

$\therefore f(x)$ is an odd function
we know that for an odd function,

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0 \implies E[X] = 0 \tag{2.12}$$

for variance

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\mathrm{d}x \tag{2.13}$$
$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.14}$$

integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.15}$$

$$= \sqrt{\frac{1}{2\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\right)\Bigg|_{-\infty}^{\infty} - \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.16}$$

Substituting $t = -\frac{x^2}{2} \implies \mathrm{d}t = -x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x = \int -\exp(t)\mathrm{d}t \tag{2.17}$$
$$= -\exp(t) \tag{2.18}$$
$$= -\exp\left(-\frac{x^2}{2}\right) \tag{2.19}$$

$$-x \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} = 0 - 0 = 0 \tag{2.20}$$
$$as, \lim_{x\to\infty} x \exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.21}$$
$$and \lim_{x\to-\infty} x \exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.22}$$

Also,

$$\int_{-\infty}^{\infty} -\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.23}$$
$$\text{by substituting} \frac{x^2}{2} = t^2 \tag{2.24}$$
$$= -\sqrt{2} \int_{-\infty}^{\infty} \exp(-t^2)\mathrm{d}t \tag{2.25}$$
$$= -\sqrt{2}\sqrt{\pi} \tag{2.26}$$
$$= -\sqrt{2\pi} \tag{2.27}$$

now,

$$E\left[X^2\right] = 0 - \sqrt{\frac{1}{2\pi}}\left(-\sqrt{2\pi}\right) \tag{2.28}$$
$$= 1 \tag{2.29}$$
$$\therefore \operatorname{var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.30}$$
$$= 1 - 0 \tag{2.31}$$
$$= 1 \tag{2.32}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF
Solution: Download the C source code by executing the following commands

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/3.1.c
```

Compile and run the C program by executing the following

```
gcc 3.1.c −lm
./a.out
```

Download the following Python code that plots Fig. 3.1

```
wget https://github.com/
    MouktikaCherukupalli/
    Random_Numbers/blob
    /main/codes/3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$
Solution: We have

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

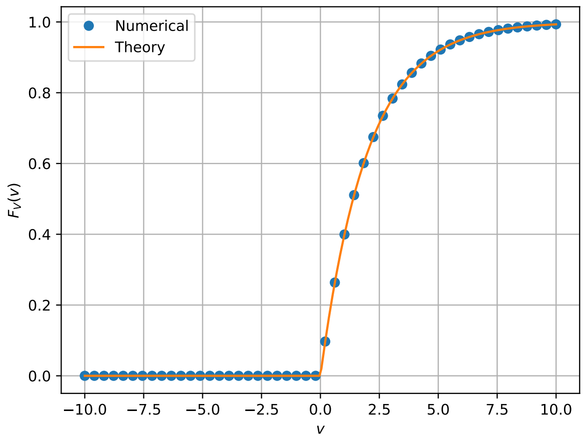

Fig. 3.1: The CDF of V

we know that ,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{3.8}$$

now,

$$0 \le 1 - \exp\left(-\frac{x}{2}\right) < 1 \qquad \text{if } x \ge 0 \tag{3.9}$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \qquad \text{if } x < 0 \tag{3.10}$$

Therefore,

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{3.11}$$