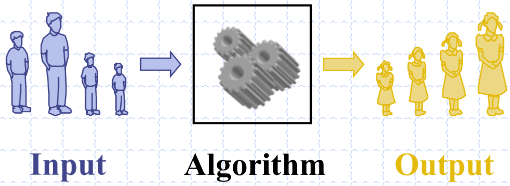
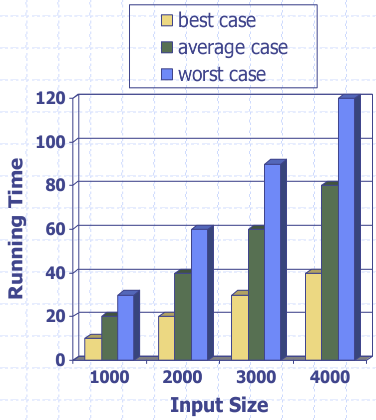
Analysis of Algorithms



An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

Running Time (§1.1)

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics

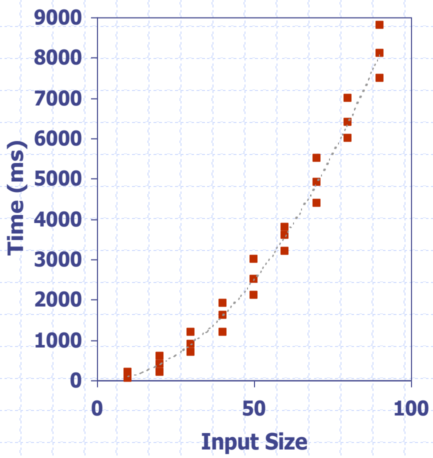


Question

◆ How do we measure running time?

Experimental Studies (§ 1.6)

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time
- Plot the results



Question

- What are the limitations of the experimental studies?

Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Question

- What is solution?

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode (§1.1)

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

Algorithm *arrayMax*(*A*, *n*)
  Input array *A* of *n* integers
  Output maximum element of *A*

  currentMax ← *A*[0]
  for *i* ← 1 **to** *n* − 1 **do**
    if *A*[*i*] > *currentMax* **then**
      currentMax ← *A*[*i*]
  return *currentMax*

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

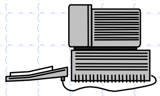
  Algorithm *method* (*arg* [, *arg*…])
    Input …
    Output …
- Method call

  var.method (*arg* [, *arg*…])
- Return value

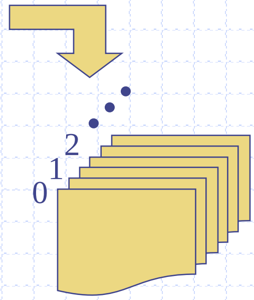
  return *expression*
- Expressions
  - ← Assignment (like = in Java)
  - = Equality testing (like == in Java)
  - n^2 Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

- A **CPU**



- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character



- Memory cells are numbered and accessing any cell in memory takes unit time.

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

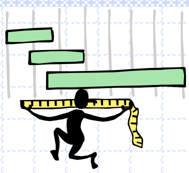
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

Counting Primitive Operations (§1.1)

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm *arrayMax*(A, n)	# operations
$currentMax \leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	$2 + n$
if $A[i] > currentMax$ **then**	$2(n - 1)$
$currentMax \leftarrow A[i]$	$2(n - 1)$
{ increment counter i }	$2(n - 1)$
return *currentMax*	1
Total	$7n - 1$

Estimating Running Time



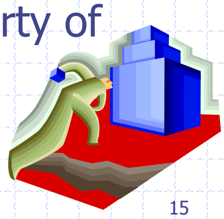
- Algorithm ***arrayMax*** executes $7n - 1$ primitive operations in the worst case. Define:
  - a = Time taken by the fastest primitive operation
  - b = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of ***arrayMax***. Then

$$a\,(7n - 1) \le T(n) \le b(7n - 1)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

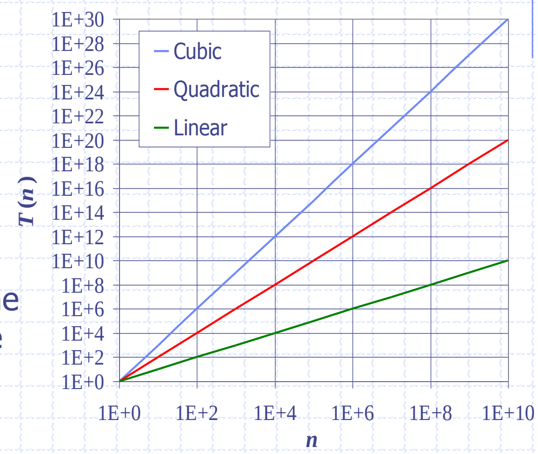
Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***



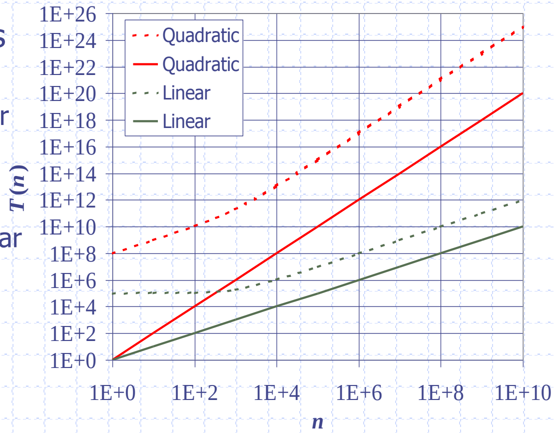
Growth Rates

- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function



Constant Factors

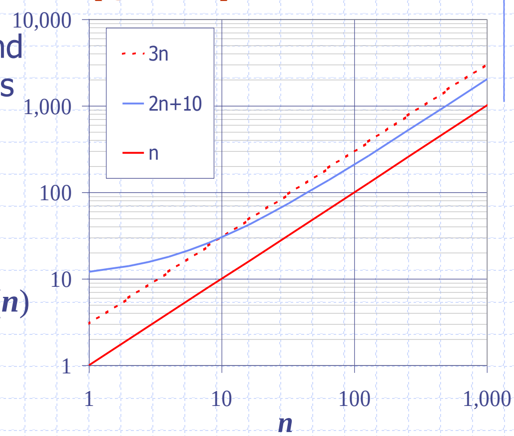
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2n + 10^5$ is a linear function
  - $10^5n^2 + 10^8n$ is a quadratic function



Big-Oh Notation (§1.2)

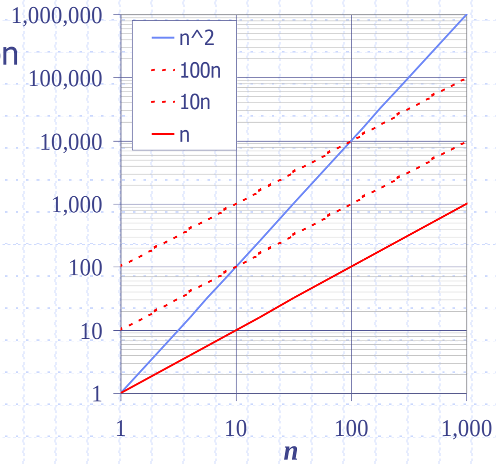
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \le cn$
  - $n \le c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2

  7n-2 is O(n)

  need c > 0 and $n_0 \ge 1$ such that $7n-2 \le c \cdot n$ for $n \ge n_0$

  this is true for c = 7 and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need c > 0 and $n_0 \ge 1$ such that $3n^3 + 20n^2 + 5 \le c \cdot n^3$ for $n \ge n_0$

  this is true for c = 4 and $n_0 = 21$

- 3 log n + log log n

  3 log n + log log n is O(log n)

  need c > 0 and $n_0 \ge 1$ such that $3 \log n + \log \log n \le c \cdot \log n$ for $n \ge n_0$

  this is true for c = 4 and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

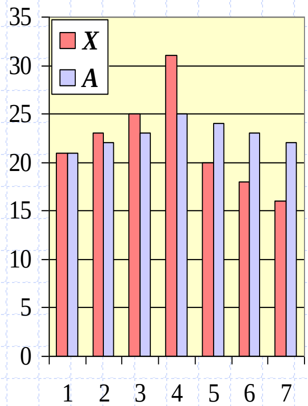
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $7n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i + 1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



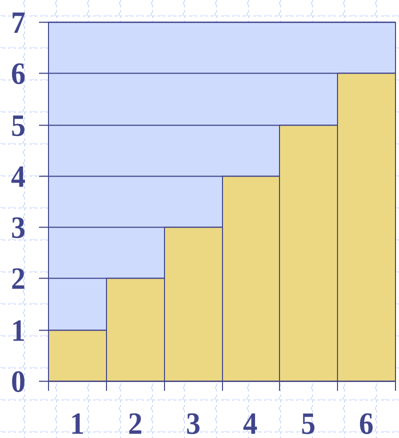
Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$A[i] \leftarrow s / (i + 1)$	n
return A	1

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\quad s \leftarrow s + X[i]$	n
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

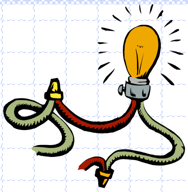
Math you need to Review

- Summations (Sec. 1.3.1)
- Logarithms and Exponents (Sec. 1.3.2)
  - **properties of logarithms:**
    $\log_b(xy) = \log_b x + \log_b y$
    $\log_b (x/y) = \log_b x - \log_b y$
    $\log_b xa = a\log_b x$
    $\log_b a = \log_x a / \log_x b$
  - **properties of exponentials**:
    $a^{(b+c)} = a^b a^c$
    $a^{bc} = (a^b)^c$
    $a^b / a^c = a^{(b-c)}$
    $b = a^{\log_a b}$
    $b^c = a^{c*\log_a b}$
- Proof techniques (Sec. 1.3.3)
- Basic probability (Sec. 1.3.4)

Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$
- **little-oh**
  - f(n) is $o(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \leq c \bullet g(n)$ for $n \geq n_0$
- **little-omega**
  - f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

Intuition for Asymptotic Notation



Big-Oh
- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

big-Omega
- f(n) is $\Omega(g(n))$ if f(n) is asymptotically **greater than or equal** to g(n)

big-Theta
- f(n) is $\Theta(g(n))$ if f(n) is asymptotically **equal** to g(n)

little-oh
- f(n) is o(g(n)) if f(n) is asymptotically **strictly less** than g(n)

little-omega
- f(n) is $\omega(g(n))$ if is asymptotically **strictly greater** than g(n)

Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\omega(n)$**

  f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

  need $5n_0^2 \geq c \bullet n_0 \rightarrow$ given c, the n_0 that satifies this is $n_0 \geq c/5 \geq 0$