1951724 陈垲昕 Assignment-1

1. (Math) In our lectures, we mentioned that matrices that can represent Euclidean transformations can form a group. Specifically, in 3D space, the set comprising matrices $\{M_i\}$ is actually a group, where

$M_i = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$, $R_i \in \mathbb{R}^{3\times3}$ is an orthonormal matrix, $\det(\mathbf{R}_i)=1$, and $\mathbf{t}_i \in \mathbb{R}^{3\times1}$ is a vector.

Please prove that the set $\{M_i\}$ forms a group.

① proving the closure:

for $\forall M_1, M_2 \in \{M_i\} \in R^{4\times4}$

$$M = M_1 \times M_2 = \left[\begin{array}{c:c} R_{1\,3\times3} & t_{1\,3\times1} \\ \hdashline 0^T_{1\times3} & 1_{1\times1} \end{array}\right] \times \left[\begin{array}{c:c} R_{2\,3\times3} & t_{2\,3\times1} \\ \hdashline 0^T_{1\times3} & 1_{1\times1} \end{array}\right]$$

$$\overset{\text{block multi}}{=\!=} \begin{bmatrix} R_1\times R_2 + t_1\times 0^T & R_1\times t_2 + t_1\times 1 \\ 0^T\times R_2 + 1\times 0^T & 0^T\times t_2 + 1\times 1 \end{bmatrix} = \begin{bmatrix} R_1\times R_2 & U \\ 0^T & 1 \end{bmatrix}$$

where $U \in R^3$ is a vector

we know $R_1 \cdot R_1^T = R_1^T R_1 = I$, $R_2 \cdot R_2^T = R_2^T R_2 = I$

for $R = R_1 R_2$, $RR^T = R_1 R_2 \cdot R_2^T R_1^T = R_1 I R_1^T = I$, $R^T R = R_2^T R_1^T R_1 R_2 = R_2^T I R_2 = I = RR^T$

$\det(R) = \det(R_1 R_2) = \det(R_1)\det(R_2) = 1\times1 = 1$

thus R is orthonormal, thus $M \in \{M_i\} \in R^{4\times4}$

② proving the associativity

for $\forall M_1, M_2, M_3 \in \{M_i\} \in R^{4\times4}$

$M_1 = \begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix}$ $M_2 = \begin{bmatrix} R_2 & t_2 \\ 0^T & 1 \end{bmatrix}$ $M_3 = \begin{bmatrix} R_3 & t_3 \\ 0^T & 1 \end{bmatrix}$

$$(M_1 M_2) M_3 = \begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_3 & t_3 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2 t_3 + R_1 t_2 + t_1 \\ 0^T & 1 \end{bmatrix}$$

$$M_1 (M_2 M_3) = \begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} R_2 R_3 & R_2 t_3 + t_2 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 R_3 & R_1(R_2 t_3 + t_2) + t_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 R_3 & R_1 R_2 t_3 + R_1 t_2 + t_1 \\ 0^T & 1 \end{bmatrix}$$

thus $(M_1 M_2) M_3 = M_1 (M_2 M_3)$

③ proving the existence of an identity element:

Let $E = I_{4\times4}$, which can be written as $\begin{bmatrix} I_{3\times3} & 0 \\ 0^T & 1 \end{bmatrix}$, where $0 \in R_{3\times1}$ is a vector

first we should prove that $E \in \{M_i\}$

it's easy to get: $EE^T = E^TE = I_{4\times4}$, $det(E) = \prod_{i=1}^{4} e_{ii} = 1$

for $\forall M_1 \in \{M_i\}$, we have

$$EM_1 = \begin{bmatrix} I_{3\times3} & 0 \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} IR_1 & It_1+0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix} = M_1$$

$$M_1E = \begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} I_{3\times3} & 0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1I & R_10+t_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 & t_1 \\ 0^T & 1 \end{bmatrix} = M_1$$

thus E is the identity element of $\{M_i\}$

④ proving the existence of an inverse element for each M in $\{M_i\}$

for $\forall M \in \{M_i\}$, $M = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$

we know the $det(R) = 1$, so $\exists R^{-1}$, $RR^{-1} = R^{-1}R = I_{3\times3}$

Let $M^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}t \\ 0^T & 1 \end{bmatrix}$ ($-R^{-1}t \in R_{3\times1}$)

because R is orthonormal, we have $R^T = R^{-1}$

so $(R^{-1})(R^{-1})^T = (R^{-1})(R^T)^T = R^{-1}R = E$, $(R^{-1})^TR^{-1} = RR^{-1} = E$

$det(R^{-1}) = det(R^T) = det(R) = 1$

thus $M^{-1} \in \{M_i\}$

$$MM^{-1} = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R^{-1} & -R^{-1}t \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} RR^{-1} & -RR^{-1}t+t \\ 0^T & 1 \end{bmatrix} = E \quad (E \in \{M_i\})$$

$$M^{-1}M = \begin{bmatrix} R^{-1} & -R^{-1}t \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R^{-1}R & R^{-1}t-R^{-1}t \\ 0^T & 1 \end{bmatrix} = E$$

thus M^{-1} is the inverse element of M

thus, $\{M_i\}$ (with $\circ$ as the matrix multiplication) forms a group. QED

2: 2. (Math) When deriving the Harris corner detector, we get the following matrix M composed of first-order partial derivatives in a local image patch w,

$$M = \begin{bmatrix} \sum_{(x_i,y_i)\in w} (I_x)^2 & \sum_{(x_i,y_i)\in w} (I_x I_y) \\ \sum_{(x_i,y_i)\in w} (I_x I_y) & \sum_{(x_i,y_i)\in w} (I_y)^2 \end{bmatrix}$$

a) Please prove that M is positive semi-definite.

b) In practice, M is usually positive definite. If M is positive definite, prove that in the Cartesian coordinate system, $[x, y]M\begin{bmatrix} x \\ y \end{bmatrix} = 1$ represents an ellipse.

c) Suppose that M is positive definite and its two eigen-values are λ_1 and λ_2 and $\lambda_1 > \lambda_2 > 0$. For the ellipse defined by $[x, y]M\begin{bmatrix} x \\ y \end{bmatrix} = 1$, prove that the length of its semi-major axis is $\frac{1}{\sqrt{\lambda_2}}$ while the length of its semi-minor axis is $\frac{1}{\sqrt{\lambda_1}}$.

① To prove that $M_{2\times 2}$ is positive semi-definite

we need to have its eigenvalue $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$

we have $det(M) = \lambda_1 \lambda_2 = \sum_{(x_i,y_i)\in w}(I_x) \sum_{(x_i,y_i)\in w}(I_y) - \left(\sum_{(x_i,y_i)\in w}(I_x I_y)\right)^2$

$tr(M) = \lambda_1 + \lambda_2 = \sum_{(x_i,y_i)\in w}(I_x) \sum_{(x_i,y_i)\in w}(I_y)$

from Cauchy-Schwarz inequality

we have $\sum_{(x_i,y_i)\in w}(I_x) \sum_{(x_i,y_i)\in w}(I_y) \geq \left(\sum_{(x_i,y_i)\in w}(I_x I_y)\right)^2 \geq 0$

thus $\lambda_1 \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \geq 0$ $\Rightarrow$ $\lambda_1 \geq 0$, $\lambda_2 \geq 0$

thus M is positive semi-definite

② note $\sum_{(x_i,y_i)\in W}(I_x)=a$ $\quad \sum_{(x_i,y_i)\in W}(I_y)=b$ $\quad \sum_{(x_i,y_i)\in W}(I_xI_y)=c$

$M=\begin{bmatrix} a & c \\ c & b \end{bmatrix}$.

because M is positive definite.

we have $\lambda_1>0$, $\lambda_2>0$

therefore $\lambda_1+\lambda_2>0 \Rightarrow tr(M)=ab>0$

$\lambda_1\lambda_2>0 \Rightarrow \det(M)=ab-c^2>0 \Rightarrow ab>c^2$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T M \begin{bmatrix} x \\ y \end{bmatrix} = [x,y]\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [ax+cy,\ cx+by]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= ax^2+2cxy+by^2=1$$

which is the general form of an eclipse, because

$(2c)^2-4ab = 4(c^2-ab)<0$

③ we notice that M is a symmetric matrix

$M\in S^{2\times2}$, let a_1, a_2 be the eigen vector of M

with $\|a_1\|_2^2=\|a_2\|_2^2=1$, **M can be diagonalized.**

$Q^TMQ=\Lambda$, $Q=[a_1,a_2]_{2\times2}$ $\Lambda=\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}_{2\times2}$

$$\begin{bmatrix} x \\ y \end{bmatrix}^T M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T Q^T \Lambda Q \begin{bmatrix} x \\ y \end{bmatrix} = \left(Q\begin{bmatrix} x \\ y \end{bmatrix}\right)^T \Lambda \left(Q\begin{bmatrix} x \\ y \end{bmatrix}\right)=1$$

let $Y=\begin{bmatrix} \alpha \\ \beta \end{bmatrix}=Q\begin{bmatrix} x \\ y \end{bmatrix}$, we have $Y^T\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}Y=1$

which is $\lambda_1\alpha^2+\lambda_2\beta^2=1 \Rightarrow \dfrac{\alpha^2}{\left(\frac{1}{\lambda_2}\right)}+\dfrac{\beta^2}{\left(\frac{1}{\lambda_1}\right)}=1$

since $\lambda_1>\lambda_2>0$, $\frac{1}{\lambda_2}>\frac{1}{\lambda_1}>0$

by the eclipse standard form we know:

$$\begin{cases} \text{semi-major axis } \frac{1}{\sqrt{\lambda_2}} \\ \text{semi-minor axis } \frac{1}{\sqrt{\lambda_1}} \end{cases}$$

3.

3. (Math) In the lecture, we talked about the least square method to solve an over-determined linear system $A\mathbf{x} = b, A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^{n\times 1}, m > n, rank(A) = n$. The closed form solution is $\mathbf{x} = \left(A^T A\right)^{-1} A^T b$. Try to prove that $A^T A$ is non-singular (or in other words, it is invertible).

one way to prove A^TA is invertible is to prove that the only vector in A^TA's null space is zero, meaning $A^TAx=0$ has a unique solution $x=0$

to solve x:

$$A^TAx=0$$

$$x^TA^TAx=0$$

$$(Ax)^TAx=0$$

$$\|Ax\|_2^2=0$$

so the length of vector Ax is 0.

meaning $Ax=0$

We know that $rank(A)=n$, $m>n$,

so A is a column full rank matrix

so $x=0$ is the only solution, therefore

A^TA is non-signular