

1. (Math) In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous coordinate of the infinity point of this line?

first we transform the line into homogeneous form:

$$x - 3y + 4 = 0 \Rightarrow \frac{x}{z} - 3\frac{y}{z} + 4 = 0 \Rightarrow x - 3y + 4z = 0$$

therefore $x - 3y + 4 = 0$ can be written to homogeneous coordinates form as $(1, -3, 4)^T$

since all lines intersect with infinite line $(0, 0, 1)^T$, which contains all the infinite points because $k(x, y, 0)^T (0, 0, 1)^T = 0$

the infinity point of $x - 3y + 4 = 0$ is the point that lies on $(0, 0, 1)^T$ and $(1, -3, 4)^T$

$$\text{coordinate: } X = [x \ y \ z]^T = \begin{bmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x + y + 0z$$

the homogeneous form of point x is $(3, 1, 0)^T$

2. (Math) On the normalized retinal plane, suppose that \mathbf{p}_n is an ideal point of projection without considering distortion. If distortion is considered, $\mathbf{p}_n = (x, y)^T$ is mapped to $\mathbf{p}_d = (x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is,

$$\begin{cases} x_d = x(1 + k_1 r^2 + k_2 r^4) + 2\rho_1 xy + \rho_2 (r^2 + 2x^2) + xk_3 r^6 \\ y_d = y(1 + k_1 r^2 + k_2 r^4) + 2\rho_2 xy + \rho_1 (r^2 + 2y^2) + yk_3 r^6 \end{cases}$$

$$\text{where } r^2 = x^2 + y^2$$

For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of \mathbf{p}_d w.r.t \mathbf{p}_n , i.e.,

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$$

It should be noted that in this question \mathbf{p}_d is the function of \mathbf{p}_n and all the other parameters can be regarded as constants.

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix} \quad \begin{aligned} r^4 &= (x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4 \\ r^6 &= (x^2 + y^2)^3 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 \end{aligned}$$

thus:

$$\begin{aligned} x_d &= x(k_1 x^4 + k_2 y^4 + k_3 x^2 + k_4 y^2 + 2k_5 x^2 y^2 + 1) + 2\rho_1 xy + \rho_2 (3x^2 + y^2) + \\ &\quad x(k_6 x^6 + 3k_7 x^4 y^2 + 3k_8 x^2 y^4 + k_9 y^6) \end{aligned}$$

$$= k_3 x^7 + (k_2 + 3k_3 y^2) x^5 + (k_1 + 2k_2 y^2 + 3k_3 y^4) x^3 + 3p_2 x^2 \\ + (k_2 y^4 + k_1 y^2 + 2p_1 y + k_3 y^6 + 1) x$$

$$y d = y (k_2 x^4 + k_2 y^4 + k_1 x^2 + k_1 y^2 + 2k_2 x^2 y^2 + 1) + 2p_2 x y + p_1 (x^2 + 3y^2) + \\ y k_3 (x^6 + 3x^4 y^2 + 3x^2 y^4 + y^6)$$

$$\frac{\partial x d}{\partial x} = 7k_3 x^6 + (5k_2 + 15k_3 y^2) x^4 + (3k_1 + 6k_2 y^2 + 9k_3 y^4) x^2 \\ + 6p_2 x + k_2 y^4 + k_1 y^2 + 2p_1 y + k_3 y^6 + 1$$

$$\frac{\partial y d}{\partial y} = 7k_3 y^6 + (5k_2 + 15k_3 x^2) y^4 + (3k_1 + 6k_2 x^2 + 9k_3 x^4) y^2 \\ + 6p_1 y + k_2 x^4 + k_1 x^2 + 2p_2 x + k_3 x^6 + 1$$

$$\frac{\partial x d}{\partial y} = 6k_3 y^5 x + (4k_2 x + 12k_3 x^3) y^3 + (6k_3 x^5 + 4k_2 x^3 + 2k_1 x + 2p_2) y \\ + 2p_1 x$$

$$\frac{\partial y d}{\partial x} = 6k_3 x^5 y + (4k_2 y + 12k_3 y^3) x^3 + (6k_3 y^5 + 4k_2 y^3 + 2k_1 y + 2p_1) x \\ + 2p_2 y$$

then $\frac{dp_d}{dp_n^\tau} = \begin{bmatrix} \frac{\partial x d}{\partial x} & \frac{\partial x d}{\partial y} \\ \frac{\partial y d}{\partial x} & \frac{\partial y d}{\partial y} \end{bmatrix}$ is solved

3. **(Math)** In our lecture, we mentioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix (represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix. Suppose that

$$\mathbf{r} = \theta \mathbf{n} \in \mathbb{R}^{3 \times 1}, \text{ where } \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \text{ is a 3D unit vector and } \theta \text{ is a real number denoting the rotation angle.}$$

With Rodrigues formula, \mathbf{r} can be converted to its rotation matrix form,

$$\mathbf{R} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{n} \mathbf{n}^T + \sin \theta \mathbf{n}^\wedge$$

and obviously $\mathbf{R} \triangleq \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote \mathbf{u} by the vectorized form of \mathbf{R} , i.e.,

$$\mathbf{u} \triangleq (R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33})^T$$

Please give the concrete form of Jacobian matrix of \mathbf{u} w.r.t \mathbf{r} , i.e., $\frac{d\mathbf{u}}{d\mathbf{r}^T} \in \mathbb{R}^{9 \times 3}$.

In order to make it easy to check your result, please follow the following notation requirements,

$$\alpha \triangleq \sin \theta, \beta \triangleq \cos \theta, \gamma \triangleq 1 - \cos \theta$$

In other words, the ingredients appearing in your formula are restricted to $\alpha, \beta, \gamma, \theta, n_1, n_2, n_3$.

$$\begin{aligned} \mathbf{R} &= \beta \mathbf{I} + \gamma \mathbf{n} \mathbf{n}^T + \alpha \mathbf{n}^\wedge \\ &= \begin{bmatrix} \beta & & \\ & \beta & \\ & & \beta \end{bmatrix} + \gamma \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} + \alpha \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma n_1 n_1 + \beta & \gamma n_1 n_2 - \alpha n_3 & \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_2 n_1 + \alpha n_3 & \gamma n_2 n_2 + \beta & \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_3 n_1 - \alpha n_2 & \gamma n_3 n_2 + \alpha n_1 & \gamma n_3 n_3 + \beta \end{bmatrix} = [\mathbf{R}_{ij}] \quad (i, j = 1, 2, 3) \end{aligned}$$

$R_{11} \ R_{22} \ R_{33}$ are in the same form
 $R_{12} \ R_{23} \ R_{31}$ are in the same form
 $R_{13} \ R_{21} \ R_{32}$ are in the same form

we know: $\theta = \sqrt{n_1^2 + n_2^2 + n_3^2}$

therefore $\frac{d\mathbf{u}}{d\mathbf{r}^T} = \frac{1}{\theta} \begin{bmatrix} \frac{dR_{11}}{dr_1} & \frac{dR_{11}}{dr_2} & \frac{dR_{11}}{dr_3} \\ \frac{dR_{12}}{dr_1} & \frac{dR_{12}}{dr_2} & \frac{dR_{12}}{dr_3} \\ \vdots & \vdots & \vdots \\ \frac{dR_{33}}{dr_1} & \frac{dR_{33}}{dr_2} & \frac{dR_{33}}{dr_3} \end{bmatrix}_{9 \times 3}$

Let $\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, then $\mathbf{n} = \frac{1}{\theta} \mathbf{r} = \frac{1}{\|\mathbf{r}\|_2} \mathbf{r} \Rightarrow \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{\sqrt{r_1^2 + r_2^2 + r_3^2}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$$\text{then, } \frac{dn_1}{dr_1} = \frac{\frac{r_1}{\sqrt{r_1^2 + r_2^2 + r_3^2}} - \frac{r_1^2}{\sqrt{r_1^2 + r_2^2 + r_3^2}}}{r_1^2 + r_2^2 + r_3^2} = \frac{\theta - \frac{\theta^3 n_1^2}{\theta^2}}{\theta^2} = \frac{1 - n_1^2}{\theta}, \quad \frac{dn_1}{dr_2} = \frac{-n_1 n_2}{\theta}, \quad \frac{dn_1}{dr_3} = \frac{-n_1 n_3}{\theta}$$

$$\frac{d\theta}{dr_1} = \frac{r_1}{\sqrt{r_1^2 + r_2^2 + r_3^2}} = n_1, \quad \frac{d\theta}{dr_2} = n_2, \quad \frac{d\theta}{dr_3} = n_3$$

$$\frac{dn_1}{dr_2} = \frac{-\frac{\theta^3 n_1 n_2}{\theta^2}}{\theta^2} = -\frac{n_1 n_2}{\theta} = \frac{dn_2}{dr_1}, \quad \frac{dn_1}{dr_3} = \frac{-n_1 n_3}{\theta} = \frac{dn_3}{dr_1}, \quad \frac{dn_2}{dr_3} = \frac{-n_2 n_3}{\theta} = \frac{dn_3}{dr_2}$$

$$\frac{d\alpha}{dr_1} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{dr_1} = \cos \theta \cdot n_1 = \beta n_1, \quad \frac{d\alpha}{dr_2} = \beta n_2, \quad \frac{d\alpha}{dr_3} = \beta n_3$$

$$\frac{d\beta}{dr_1} = \frac{d\beta}{d\theta} \cdot \frac{d\theta}{dr_1} = -\sin\theta n_1 = -\alpha n_1, \quad \frac{d\beta}{dr_2} = -\alpha n_2, \quad \frac{d\beta}{dr_3} = -\alpha n_3$$

$$\frac{dy}{dr_1} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dr_1} = \sin\theta n_1 = \alpha n_1, \quad \frac{dy}{dr_2} = \alpha n_2, \quad \frac{dy}{dr_3} = \alpha n_3$$

R_{11} :

$$\frac{dR_{11}}{dr_1} = \frac{dR_{11}}{dy} \frac{dy}{dr_1} + \frac{dR_{11}}{dn_1} \frac{dn_1}{dr_1} + \frac{dR_{11}}{d\beta} \frac{d\beta}{dr_1} = \alpha n_1^3 + 2\gamma n_1 \cdot \frac{(1-n_1^2)}{\theta} - \alpha n_1 = \boxed{\frac{2\gamma n_1(1-n_1^2)}{\theta} + \alpha n_1(n_1^2-1)}$$

$$\frac{dR_{11}}{dr_2} = \frac{dR_{11}}{dy} \frac{dy}{dr_2} + \frac{dR_{11}}{dn_1} \frac{dn_1}{dr_2} + \frac{dR_{11}}{d\beta} \frac{d\beta}{dr_2} = \alpha n_1^2 n_2 - \frac{2\gamma n_1 n_2}{\theta} - \alpha n_2 = \boxed{-\frac{2\gamma n_1^2 n_2}{\theta} + \alpha n_2(n_1^2-1)}$$

$$\frac{dR_{11}}{dr_3} = \frac{dR_{11}}{dy} \frac{dy}{dr_3} + \frac{dR_{11}}{dn_1} \frac{dn_1}{dr_3} + \frac{dR_{11}}{d\beta} \frac{d\beta}{dr_3} = \alpha n_1^2 n_3 - \frac{2\gamma n_1 n_3}{\theta} - \alpha n_3 = \boxed{-\frac{2\gamma n_1^2 n_3}{\theta} + \alpha n_3(n_1^2-1)}$$

similar to R_{11} , we can quickly write the answer for R_{22} , R_{33} because of the same form, and their difference can be considered as only switching the name of n_1, n_2 and n_3

$$\begin{aligned} (R_{22}) \quad \frac{dR_{22}}{dr_1} &= \boxed{-\frac{2\gamma n_1 n_2^2}{\theta} + \alpha n_1(n_2^2-1)} & \frac{dR_{22}}{dr_2} &= \boxed{\frac{2\gamma n_2(1-n_2^2)}{\theta} + \alpha n_2(n_2^2-1)} & \frac{dR_{22}}{dr_3} &= \boxed{-\frac{2\gamma n_2^2 n_3}{\theta} + \alpha n_3(n_2^2-1)} \\ (R_{33}) \quad \frac{dR_{33}}{dr_1} &= \boxed{-\frac{2\gamma n_1 n_3^2}{\theta} + \alpha n_1(n_3^2-1)} & \frac{dR_{33}}{dr_2} &= \boxed{-\frac{2\gamma n_2 n_3^2}{\theta} + \alpha n_2(n_3^2-1)} & \frac{dR_{33}}{dr_3} &= \boxed{\frac{2\gamma n_3(1-n_3^2)}{\theta} + \alpha n_3(n_3^2-1)} \end{aligned}$$

R_{12}

$$\begin{aligned} \frac{dR_{12}}{dr_1} &= \frac{dR_{12}}{dy} \frac{dy}{dr_1} + \frac{dR_{12}}{dn_1} \frac{dn_1}{dr_1} + \frac{dR_{12}}{dn_2} \frac{dn_2}{dr_1} + \frac{dR_{12}}{d\alpha} \frac{d\alpha}{dr_1} + \frac{dR_{12}}{dn_3} \frac{dn_3}{dr_1} \\ &= \alpha n_1^2 n_2 + n_2 \gamma \frac{(1-n_1^2)}{\theta} - n_1 \gamma \frac{n_1 n_2}{\theta} - n_3 \beta n_1 + \frac{\alpha n_1 n_3}{\theta} \\ &= \boxed{n_1(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_2(1-2n_1^2) + \alpha n_1 n_3}{\theta}} \end{aligned}$$

$$\begin{aligned} \frac{dR_{12}}{dr_2} &= \frac{dR_{12}}{dy} \frac{dy}{dr_2} + \frac{dR_{12}}{dn_1} \frac{dn_1}{dr_2} + \frac{dR_{12}}{dn_2} \frac{dn_2}{dr_2} + \frac{dR_{12}}{d\alpha} \frac{d\alpha}{dr_2} + \frac{dR_{12}}{dn_3} \frac{dn_3}{dr_2} \\ &= \alpha n_1 n_2^2 + \gamma n_2 \frac{(1-n_2^2)}{\theta} - n_2 \gamma \frac{n_1 n_2}{\theta} - n_3 \beta n_2 + \frac{\alpha n_2 n_3}{\theta} \\ &= \boxed{n_2(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_1(1-2n_2^2) + \alpha n_2 n_3}{\theta}} \end{aligned}$$

$$\begin{aligned}
\frac{dR_{12}}{dr_3} &= \frac{dR_{12}}{dy} \frac{dy}{dr_3} + \frac{dR_{12}}{dn_1} \frac{dn_1}{dr_3} + \frac{dR_{12}}{dn_2} \frac{dn_2}{dr_3} + \frac{dR_1}{d\alpha} \cdot \frac{d\alpha}{dr_3} + \frac{dR_{12}}{dn_3} \frac{dn_3}{dr_3} \\
&= \alpha n_1 n_2 n_3 - \frac{\gamma n_2 n_1 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} - \beta n_3^2 - \alpha \frac{(1-n_3^2)}{\theta} \\
&= \boxed{n_3(\alpha n_1 n_2 - \beta n_3) + \frac{\alpha(n_3^2 - 1) - 2\gamma n_1 n_2 n_3}{\theta}}
\end{aligned}$$

similar to R_{12} , we can quickly write the answer for R_{23} , R_{31} because of the same form, and their difference can be considered as only switching the name of n_1, n_2 and n_3

$$\begin{aligned}
\textcircled{R_{23}} \quad \frac{dR_{23}}{dr_1} &= \boxed{n_1(\alpha n_2 n_3 - \beta n_1) - \frac{\alpha(1-n_1^2) + 2\gamma n_1 n_2 n_3}{\theta}} \quad \frac{dR_{23}}{dr_2} = \boxed{n_2(\alpha n_1 n_3 - \beta n_1) + \frac{\gamma n_3(1-2n_1^2) + \alpha n_1 n_2}{\theta}} \\
\frac{dR_{23}}{dr_3} &= \boxed{n_3(\alpha n_2 n_3 - \beta n_1) + \frac{\alpha n_1 n_3 + \gamma n_2(1-2n_3^2)}{\theta}}
\end{aligned}$$

$$\begin{aligned}
\textcircled{R_{31}} \quad \frac{dR_{31}}{dr_1} &= \boxed{n_1(\alpha n_1 n_3 - \beta n_2) + \frac{\alpha n_1 n_2 + \gamma n_3(1-2n_1^2)}{\theta}} \\
\frac{dR_{31}}{dr_2} &= \boxed{n_2(\alpha n_1 n_3 - \beta n_2) - \frac{\alpha(1-n_2^2) + 2\gamma n_1 n_2 n_3}{\theta}} \\
\frac{dR_{31}}{dr_3} &= \boxed{n_3(\alpha n_1 n_3 - \beta n_2) + \frac{\alpha n_2 n_3 + \gamma n_1(1-2n_3^2)}{\theta}}
\end{aligned}$$

since R_{13} , R_{21} , R_{32} are in the similar form with R_{12} , R_{23} , R_{31} but the $-\alpha n_i$ item are negative with corresponding item in R_{13} , R_{21} , R_{32} , so we only need to change the valence of the derivative result $\frac{dR}{d\alpha} \frac{d\alpha}{dr_j}$, $\frac{dR}{dn_i} \frac{dn_i}{dr_j}$ and switching n_1, n_2, n_3 , in this way we quickly get:

$$\begin{aligned}
 (R_{13}) \quad \frac{dR_{13}}{dr_1} &= n_1(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_3(1-2n_1^2) - \alpha n_1 n_2}{\theta} \\
 \frac{dR_{13}}{dr_2} &= n_2(\alpha n_1 n_3 + \beta n_2) + \frac{\alpha(1-n_2^2) - 2\gamma n_1 n_2 n_3}{\theta} \\
 \frac{dR_{13}}{dr_3} &= n_3(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_1(1-2n_3^2) - \alpha n_1 n_2}{\theta} \\
 (R_{21}) \quad \frac{dR_{21}}{dr_1} &= n_1(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_2(1-2n_1^2) - \alpha n_1 n_3}{\theta} \\
 \frac{dR_{21}}{dr_2} &= n_2(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_1(1-2n_2^2) - \alpha n_2 n_3}{\theta} \\
 \frac{dR_{21}}{dr_3} &= n_3(\alpha n_1 n_2 + \beta n_3) + \frac{\alpha(1-n_3^2) - 2\gamma n_1 n_2 n_3}{\theta} \\
 (R_{32}) \quad \frac{dR_{32}}{dr_1} &= n_1(\alpha n_2 n_3 + \beta n_1) + \frac{\alpha(1-n_1^2) - 2\gamma n_1 n_2 n_3}{\theta} \\
 \frac{dR_{32}}{dr_2} &= n_2(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_3(1-2n_2^2) - \alpha n_1 n_2}{\theta} \\
 \frac{dR_{32}}{dr_3} &= n_3(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_2(1-2n_3^2) - \alpha n_1 n_3}{\theta}
 \end{aligned}$$

then

$$\frac{du}{dr^T} = \frac{1}{\theta} \begin{bmatrix} \frac{dR_{11}}{dr_1} & \frac{dR_{11}}{dr_2} & \frac{dR_{11}}{dr_3} \\ \frac{dR_{12}}{dr_1} & \frac{dR_{12}}{dr_2} & \frac{dR_{12}}{dr_3} \\ \vdots & \vdots & \vdots \\ \frac{dR_{33}}{dr_1} & \frac{dR_{33}}{dr_2} & \frac{dR_{33}}{dr_3} \end{bmatrix}_{9 \times 3} \text{ is solved.}$$