1951724 陈垲昕 Assignment2

1. **(Math)** In the augmented Euclidean plane, there is a line $x-3y+4=0$, what is the homogeneous coordinate of the infinity point of this line?

first we transform the line into homogeneous form:

$x-3y+4=0 \Rightarrow \frac{x}{z}-3\frac{y}{z}+4=0 \Rightarrow x-3y+4z=0$

therefore $x-3y+4=0$ can be written to homogeneous coordinates form as $(1,-3,4)^T$

since all lines intersect with infinite line $(0,0,1)^T$, which contains all the infinite points because $k(x,y,0)^T(0,0,1)=0$

the infinity point of $x-3y+4=0$ is the point that lives on $(0,0,1)^T$ and $(1,-3,4)^T$

coordinate: $x = l \times l' = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -3 & 4 \end{vmatrix} x + \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} y + \begin{vmatrix} 0 & 0 \\ 1 & -3 \end{vmatrix} z$

$= 3x+y$

the homogeneous form of point x is $(3,1,0)^T$

2. **(Math)** On the normalized retinal plane, suppose that $\mathbf{p}_n$ is an ideal point of projection without considering distortion. If distortion is considered, $\mathbf{p}_n=(x, y)^T$ is mapped to $\mathbf{p}_d=(x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is,

$$\begin{cases} x_d = x\left(1+k_1r^2+k_2r^4\right)+2\rho_1xy+\rho_2\left(r^2+2x^2\right)+xk_3r^6 \\ y_d = y\left(1+k_1r^2+k_2r^4\right)+2\rho_2xy+\rho_1\left(r^2+2y^2\right)+yk_3r^6 \end{cases}$$

where $r^2=x^2+y^2$

For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of $\mathbf{p}_d$ w.r.t $\mathbf{p}_n$, i.e.,

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$$

It should be noted that in this question $\mathbf{p}_d$ is the function of $\mathbf{p}_n$ and all the other parameters can be regarded as constants.

$$\frac{dp_d}{dp_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

$r^4=(x^2+y^2)^2= x^4+2x^2y^2+y^4$

$r^6=(x^2+y^2)^3= x^6+3x^4y^2+3x^2y^4+y^6$

thus:

$x_d = x\left(k_2x^4+k_2y^4+k_1x^2+k_1y^2+2k_2x^2y^2+1\right)+2\rho_1xy+\rho_2(3x^2+y^2)+$
$x\left(k_3x^6+3k_3x^4y^2+3k_3x^2y^4+k_3y^6\right)$

$$= k_3 x^7 + (k_2 + 3k_3 y^2) x^5 + (k_1 + 2k_2 y^2 + 3k_3 y^4) x^3 + 3\rho_2 x^2$$
$$+ (k_2 y^4 + k_1 y^2 + 2\rho_1 y + k_3 y^6 + 1) x$$

$$y_d = y(k_2 x^4 + k_2 y^4 + k_1 x^2 + k_1 y^2 + 2k_2 x^2 y^2 + 1) + 2\rho_2 xy + \rho_1 (x^2 + 3y^2) +$$
$$y k_3 (x^6 + 3x^4 y^2 + 3x^2 y^4 + y^6)$$

$$\frac{\partial x_d}{\partial x} = 7k_3 x^6 + (5k_2 + 15k_3 y^2) x^4 + (3k_1 + 6k_2 y^2 + 9k_3 y^4) x^2$$
$$+ 6\rho_2 x + k_2 y^4 + k_1 y^2 + 2\rho_1 y + k_3 y^6 + 1$$

$$\frac{\partial y_d}{\partial y} = 7k_3 y^6 + (5k_2 + 15k_3 x^2) y^4 + (3k_1 + 6k_2 x^2 + 9k_3 x^4) y^2$$
$$+ 6\rho_1 y + k_2 x^4 + k_1 x^2 + 2\rho_2 x + k_3 x^6 + 1$$

$$\frac{\partial x_d}{\partial y} = 6k_3 y^5 + (4k_2 x + 12k_3 x^3) y^3 + (6k_3 x^5 + 4k_2 x^3 + 2k_1 x + 2\rho_2) y$$
$$+ 2\rho_1 x$$

$$\frac{\partial y_d}{\partial yx} = 6k_3 x^5 + (4k_2 x + 12k_3 x^3) y^3 + (6k_3 y^5 + 4k_2 y^3 + 2k_1 y + 2\rho_1) x$$
$$+ 2\rho_2 y$$

then $\dfrac{dp_d}{dp_n^T} = \begin{bmatrix} \dfrac{\partial x_d}{\partial x} & \dfrac{\partial x_d}{\partial y} \\ \dfrac{\partial y_d}{\partial x} & \dfrac{\partial y_d}{\partial y} \end{bmatrix}$ is solved

3. **(Math)** In our lecture, we mentioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix (represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix. Suppose that

$\mathbf{r} = \theta \mathbf{n} \in \mathbb{R}^{3\times 1}$, where $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a 3D unit vector and θ is a real number denoting the rotation angle.

With Rodrigues formula, $\mathbf{r}$ can be converted to its rotation matrix form,

$$\mathbf{R} = \cos\theta \mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta \mathbf{n}^\wedge$$

and obviously $\mathbf{R} \triangleq \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote $\mathbf{u}$ by the vectorized form of $\mathbf{R}$, i.e.,

$$\mathbf{u} \triangleq (R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}, R_{33})^T$$

Please give the concrete form of Jacobian matrix of $\mathbf{u}$ w.r.t $\mathbf{r}$, i.e., $\frac{d\mathbf{u}}{d\mathbf{r}^T} \in \mathbb{R}^{9\times 3}$.

In order to make it easy to check your result, please follow the following notation requirements,

$\alpha \triangleq \sin\theta, \beta \triangleq \cos\theta, \gamma \triangleq 1-\cos\theta$

In other words, the ingredients appearing in your formula are restricted to $\alpha, \beta, \gamma, \theta, n_1, n_2, n_3$.

$$R = \beta I + \gamma n n^T + \alpha n^\wedge$$

$$= \begin{bmatrix} \beta & & \\ & \beta & \\ & & \beta \end{bmatrix} + \gamma \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} + \alpha \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$R_{11}\ R_{22}\ R_{33}$ are in the same form

$R_{12}\ R_{23}\ R_{31}$ are in the same form

$R_{13}\ R_{21}\ R_{32}$ are in the same form

$$= \begin{bmatrix} \gamma n_1 n_1 + \beta & \gamma n_1 n_2 - \alpha n_3 & \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_2 n_1 + \alpha n_3 & \gamma n_2 n_2 + \beta & \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_3 n_1 - \alpha n_2 & \gamma n_3 n_2 + \alpha n_1 & \gamma n_3 n_3 + \beta \end{bmatrix} = [R_{ij}] \ (i=1,2,3,\ j=1,2,3)$$

we know: $\theta = \sqrt{r_1^2 + r_2^2 + r_3^2}$

therefore

$$\frac{du}{dr^T} = \frac{1}{\theta} \begin{bmatrix} \frac{dR_{11}}{dr_1} & \frac{dR_{11}}{dr_2} & \frac{dR_{11}}{dr_3} \\ \frac{dR_{12}}{dr_1} & \frac{dR_{12}}{dr_2} & \frac{dR_{12}}{dr_3} \\ \vdots & \vdots & \vdots \\ \frac{dR_{33}}{dr_1} & \frac{dR_{33}}{dr_2} & \frac{dR_{33}}{dr_3} \end{bmatrix}_{9\times 3};$$

Let $r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, then $n = \frac{1}{\theta} r = \frac{1}{\|r\|_2} r \Rightarrow \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{\sqrt{r_1^2+r_2^2+r_3^2}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

then: $\frac{dn_1}{dr_1} = \frac{\sqrt{r_1^2+r_2^2+r_3^2} - \frac{r_1^2}{\sqrt{r_1^2+r_2^2+r_3^2}}}{r_1^2+r_2^2+r_3^2} = \frac{\theta - \frac{\theta^2 n_1^2}{\theta}}{\theta^2} = \frac{1-n_1^2}{\theta}$, $\frac{dn_2}{dr_2} = \frac{1-n_2^2}{\theta}$, $\frac{dn_3}{dr_3} = \frac{1-n_3^2}{\theta}$

$\frac{d\theta}{dr_1} = \frac{r_1}{\sqrt{r_1^2+r_2^2+r_3^2}} = n_1$. $\frac{d\theta}{dr_2} = n_2$, $\frac{d\theta}{dr_3} = n_3$

$\frac{dn_1}{dr_2} = \frac{-\frac{\theta^2 n_1 n_2}{\theta}}{\theta^2} = -\frac{n_1 n_2}{\theta} = \frac{dn_2}{dr_1}$, $\frac{dn_2}{dr_3} = \frac{dn_3}{dr_2} = \frac{-n_2 n_3}{\theta}$, $\frac{dn_1}{dr_3} = \frac{dr_3}{dn_1} = \frac{-n_1 n_3}{\theta}$

$$\frac{d\alpha}{dr_1} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{dr_1} = \cos\theta \cdot n_1 = \beta n_1 \quad \frac{d\alpha}{dr_2} = \beta n_2, \quad \frac{d\alpha}{dr_3} = \beta n_3$$

$$\frac{d\beta}{dr_1} = \frac{d\beta}{d\theta}\cdot\frac{d\theta}{dr_1} = -\sin\theta\, n_1 = -\alpha n_1,\quad \frac{d\beta}{dr_2} = -\alpha n_2,\quad \frac{d\beta}{dr_3} = -\alpha n_3$$

$$\frac{d\gamma}{dr_1} = \frac{d\gamma}{d\theta}\cdot\frac{d\theta}{dr_1} = \sin\theta\, n_1 = \alpha n_1,\quad \frac{d\gamma}{dr_2} = \alpha n_2,\quad \frac{d\gamma}{dr_3} = \alpha n_3$$

R_{11}:

$$\frac{dR_{11}}{dr_1} = \frac{dR_{11}}{d\gamma}\frac{d\gamma}{dr_1} + \frac{dR_{11}}{dn_1}\cdot\frac{dn_1}{dr_1} + \frac{dR_{11}}{d\beta}\frac{d\beta}{dr_1} = \alpha n_1^3 + 2\gamma n_1\cdot\left(\frac{1-n_1^2}{\theta}\right) - \alpha n_1 = \boxed{\frac{2\gamma n_1(1-n_1^2)}{\theta} + \alpha n_1(n_1^2-1)}$$

$$\frac{dR_{11}}{dr_2} = \frac{dR_{11}}{d\gamma}\frac{d\gamma}{dr_2} + \frac{dR_{11}}{dn_1}\cdot\frac{dn_1}{dr_2} + \frac{dR_{11}}{d\beta}\frac{d\beta}{dr_2} = \alpha n_1^2 n_2 - \frac{2\gamma n_1 n_1 n_2}{\theta} - \alpha n_2 = \boxed{-\frac{2\gamma n_1^2 n_2}{\theta} + \alpha n_2(n_1^2-1)}$$

$$\frac{dR_{11}}{dr_3} = \frac{dR_{11}}{d\gamma}\frac{d\gamma}{dr_3} + \frac{dR_{11}}{dn_1}\cdot\frac{dn_1}{dr_3} + \frac{dR_{11}}{d\beta}\frac{d\beta}{dr_3} = \alpha n_1^2 n_3 - \frac{2\gamma n_1 n_1 n_3}{\theta} - \alpha n_3 = \boxed{-\frac{2\gamma n_1^2 n_3}{\theta} + \alpha n_3(n_1^2-1)}$$

similar to R_{11}, we can quickly write the answer for R_{22}, R_{33} because of the same form, and their difference can be considered as only switching the name of n_1, n_2 and n_3

R_{22}

$$\frac{dR_{22}}{dr_1} = \boxed{\frac{-2\gamma n_1 n_2^2}{\theta} + \alpha n_1(n_2^2-1)} \quad \frac{dR_{22}}{dr_2} = \boxed{\frac{2\gamma n_2(1-n_2^2)}{\theta} + \alpha n_2(n_2^2-1)} \quad \frac{dR_{22}}{dr_3} = \boxed{\frac{-2\gamma n_2^2 n_3}{\theta} + \alpha n_3(n_2^2-1)}$$

R_{33}

$$\frac{dR_{33}}{dr_1} = \boxed{\frac{-2\gamma n_1 n_3^2}{\theta} + \alpha n_1(n_3^2-1)} \quad \frac{dR_{33}}{dr_2} = \boxed{\frac{-2\gamma n_2 n_3^2}{\theta} + \alpha n_2(n_3^2-1)} \quad \frac{dR_{33}}{dr_3} = \boxed{\frac{2\gamma n_3(1-n_3^2)}{\theta} + \alpha n_3(n_3^2-1)}$$

R_{12}

$$\frac{dR_{12}}{dr_1} = \frac{dR_{12}}{d\gamma}\frac{d\gamma}{dr_1} + \frac{dR_{12}}{dn_1}\frac{dn_1}{dr_1} + \frac{dR_{12}}{dn_2}\frac{dn_2}{dr_1} + \frac{dR_1}{d\alpha}\cdot\frac{d\alpha}{dr_1} + \frac{dR_{12}}{dn_3}\frac{dn_3}{dr_1}$$

$$= \alpha n_1^2 n_2 + n_2\gamma\frac{(1-n_1^2)}{\theta} - n_1\gamma\frac{n_1 n_2}{\theta} - n_3\beta n_1 + \frac{\alpha n_1 n_3}{\theta}$$

$$= \boxed{n_1(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_2(1-2n_1^2) + \alpha n_1 n_3}{\theta}}$$

$$\frac{dR_{12}}{dr_2} = \frac{dR_{12}}{d\gamma}\frac{d\gamma}{dr_2} + \frac{dR_{12}}{dn_1}\frac{dn_1}{dr_2} + \frac{dR_{12}}{dn_2}\frac{dn_2}{dr_2} + \frac{dR_1}{d\alpha}\cdot\frac{d\alpha}{dr_2} + \frac{dR_{12}}{dn_3}\frac{dn_3}{dr_2}$$

$$= \alpha n_1 n_2^2 + \gamma n_1\frac{(1-n_2^2)}{\theta} - n_2\gamma\frac{n_1 n_2}{\theta} - n_3\beta n_2 + \frac{\alpha n_2 n_3}{\theta}$$

$$= \boxed{n_2(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_1(1-2n_2^2) + \alpha n_2 n_3}{\theta}}$$

$$\frac{dR_{12}}{dr_3} = \frac{dR_{12}}{d\gamma}\frac{d\gamma}{dr_3} + \frac{dR_{12}}{dn_1}\frac{dn_1}{dr_3} + \frac{dR_{12}}{dn_2}\frac{dn_2}{dr_3} + \frac{dR_1}{d\alpha}\cdot\frac{d\alpha}{dr_3} + \frac{dR_{12}}{dn_3}\frac{dn_3}{dr_3}$$

$$= \alpha n_1 n_2 n_3 - \frac{\gamma n_2 n_1 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} - \beta n_3^2 - \alpha\frac{(1-n_3^2)}{\theta}$$

$$= \boxed{n_3(\alpha n_1 n_2 - \beta n_3) + \frac{\alpha(n_3^2-1) - 2\gamma n_1 n_2 n_3}{\theta}}$$

similar to R_{12}. we can quickly write the answer for R_{23}, R_{31} because of the same form, and their difference can be considered as only switching the name of n_1, n_2 and n_3

R_{23}

$$\frac{dR_{23}}{dr_1} = \boxed{n_1(\alpha n_2 n_3 - \beta n_1) - \frac{\alpha(1-n_1^2) + 2\gamma n_1 n_2 n_3}{\theta}}$$

$$\frac{dR_{23}}{dr_2} = \boxed{n_2(\alpha n_2 n_3 - \beta n_1) + \frac{\gamma n_3(1-2n_2^2) + \alpha n_1 n_2}{\theta}}$$

$$\frac{dR_{23}}{dr_3} = \boxed{n_3(\alpha n_2 n_3 - \beta n_1) + \frac{\alpha n_1 n_3 + \gamma n_2(1-2n_3^2)}{\theta}}$$

R_{31}

$$\frac{dR_{31}}{dr_1} = \boxed{n_1(\alpha n_1 n_3 - \beta n_2) + \frac{\alpha n_1 n_2 + \gamma n_3(1-2n_1^2)}{\theta}}$$

$$\frac{dR_{31}}{dr_2} = \boxed{n_2(\alpha n_1 n_3 - \beta n_2) - \frac{\alpha(1-n_2^2) + 2\gamma n_1 n_2 n_3}{\theta}}$$

$$\frac{dR_{31}}{dr_3} = \boxed{n_3(\alpha n_1 n_3 - \beta n_2) + \frac{\alpha n_2 n_3 + \gamma n_1(1-2n_3^2)}{\theta}}$$

since R_{13}, R_{21}, R_{32} are in the similar form with R_{12}, R_{23}, R_{31} but the $-\alpha n_i$ item are negative with corresponding item in R_{13}, R_{21}, R_{32}, so we only need to change the valuence of the derivative result $\frac{dR}{d\alpha}\frac{d\alpha}{dr_j}$, $\frac{dR}{dn_i}\frac{dn_i}{dr_j}$ and switching n_1, n_2, n_3, in this way we quickly get:

R_{13}

$$\frac{dR_{13}}{dr_1} = n_1(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_3(1-2n_1^2) - \alpha n_1 n_2}{\theta}$$

$$\frac{dR_{13}}{dr_2} = n_2(\alpha n_1 n_3 + \beta n_2) + \frac{\alpha(1-n_2^2) - 2\gamma n_1 n_2 n_3}{\theta}$$

$$\frac{dR_{13}}{dr_3} = n_3(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_1(1-2n_3^2) - \alpha n_1 n_3}{\theta}$$

R_{21}

$$\frac{dR_{21}}{dr_1} = n_1(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_2(1-2n_1^2) - \alpha n_1 n_3}{\theta}$$

$$\frac{dR_{21}}{dr_2} = n_2(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_1(1-2n_2^2) - \alpha n_2 n_3}{\theta}$$

$$\frac{dR_{21}}{dr_3} = n_3(\alpha n_1 n_2 + \beta n_3) + \frac{\alpha(1-n_3^2) - 2\gamma n_1 n_2 n_3}{\theta}$$

R_{32}

$$\frac{dR_{32}}{dr_1} = n_1(\alpha n_2 n_3 + \beta n_1) + \frac{\alpha(1-n_1^2) - 2\gamma n_1 n_2 n_3}{\theta}$$

$$\frac{dR_{32}}{dr_2} = n_2(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_3(1-2n_2^2) - \alpha n_1 n_2}{\theta}$$

$$\frac{dR_{32}}{dr_3} = n_3(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_2(1-2n_3^2) - \alpha n_1 n_3}{\theta}$$

then

$$\frac{du}{dr^T} = \frac{1}{\theta}\begin{bmatrix} \frac{dR_{11}}{dr_1} & \frac{dR_{11}}{dr_2} & \frac{dR_{11}}{dr_3} \\ \frac{dR_{12}}{dr_1} & \frac{dR_{12}}{dr_2} & \frac{dR_{12}}{dr_3} \\ \vdots & \vdots & \vdots \\ \frac{dR_{33}}{dr_1} & \frac{dR_{33}}{dr_2} & \frac{dR_{33}}{dr_3} \end{bmatrix}_{9\times 3} \quad \text{is} \quad \text{solved.}$$