

Control of Mobile Robots: Glue Lectures



Instructor:

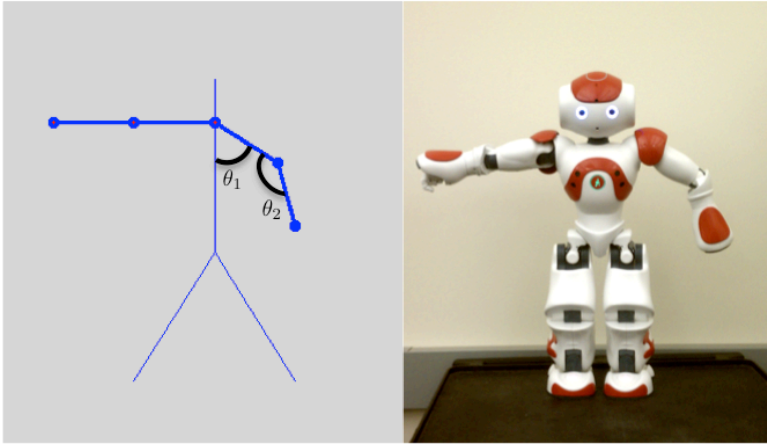


Smriti Chopra
ECE Ph.D. Candidate
Georgia Tech

Glue Lecture 4: Controllability and Observability

Pay attention, this lecture will help you all with Quiz 4!

System Stability



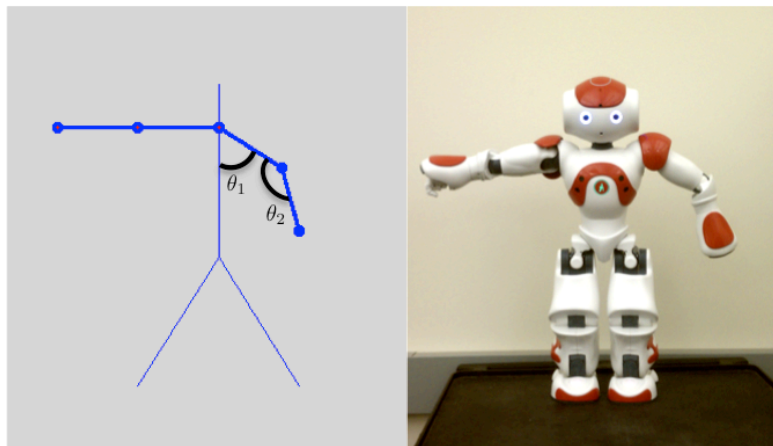
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{x} = Ax$$

In Matlab: `>> eig(A)`

All eigen values of A are zero! **Unstable !!!**

Introduce Control...



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

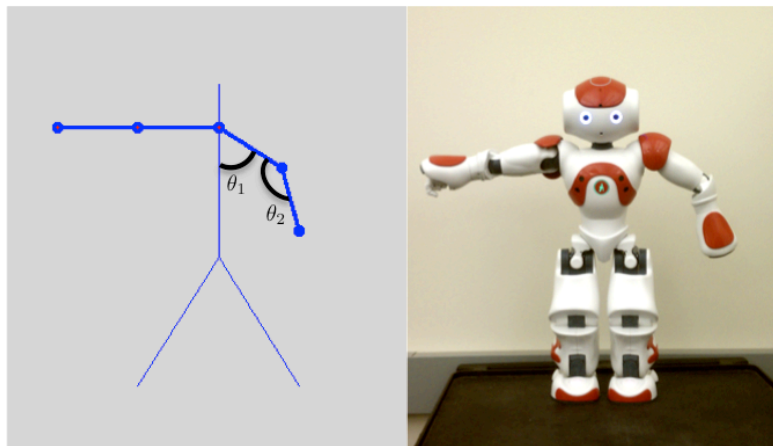
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu \quad \text{Controllable ??}$$

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(\Gamma) = 2 \neq 4$$

Introduce Control...



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu \quad \text{Uncontrollable !!!}$$

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank}(\Gamma) = 2 \neq 4$$

In Matlab...

```
EDU>> A=[0 1 0 0; 0 0 0 0; 0 0 0 1; 0 0 0 0]
```

```
A =
```

```

0     1     0     0
0     0     0     0
0     0     0     1
0     0     0     0

```

```
EDU>> B=[0; 1; 0; 0]
```

```
B =
```

```

0
1
0
0

```

```
EDU>> ctrb(A,B)
```

```
ans =
```

```

0     1     0     0
1     0     0     0
0     0     0     0
0     0     0     0

```

```
EDU>> rank(ctrb(A,B))
```

```
ans =
```

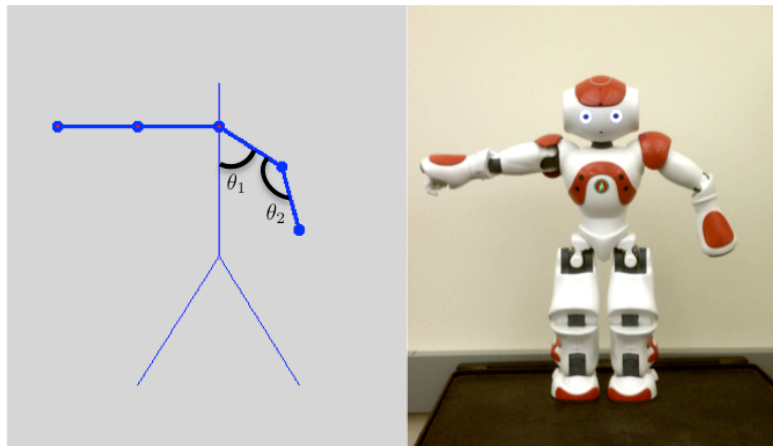
```
2
```

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Uncontrollable !!!

Need “more” control...



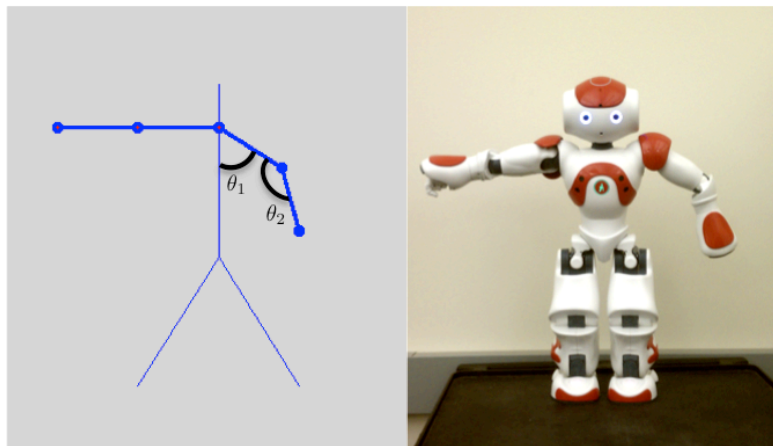
$$\dot{x} = Ax + Bu$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Need “more” control...



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$ and $\ddot{\theta}_2$

$$\dot{x} = Ax + Bu \quad \text{Controllable ??}$$

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

In Matlab again ☺...

```
EDU>> A=[0 1 0 0; 0 0 0 0; 0 0 0 1; 0 0 0 0]
```

```
A =
```

```

0     1     0     0
0     0     0     0
0     0     0     1
0     0     0     0

```

```
EDU>> B=[0 0; 1 0; 0 0; 0 1]
```

```
B =
```

```

0     0
1     0
0     0
0     1

```

```
EDU>> ctrb(A,B)
```

```
ans =
```

```

0     0     1     0     0     0     0     0
1     0     0     0     0     0     0     0
0     0     0     1     0     0     0     0
0     1     0     0     0     0     0     0

```

```
EDU>> rank(ctrb(A,B))
```

```
ans =
```

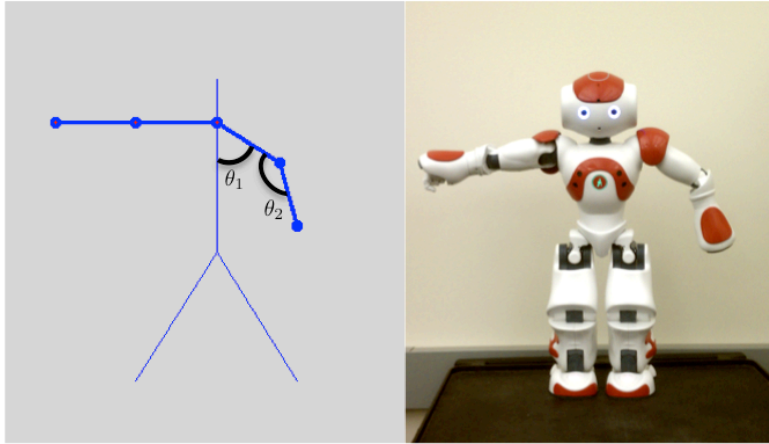
```
4
```

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Controllable !!!

State feedback...



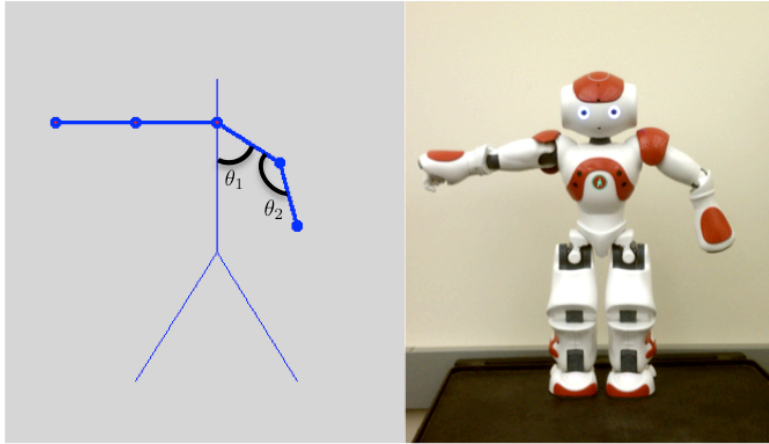
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \quad \text{Controllable}$$

$$u = -Kx$$

State feedback... on a simpler system

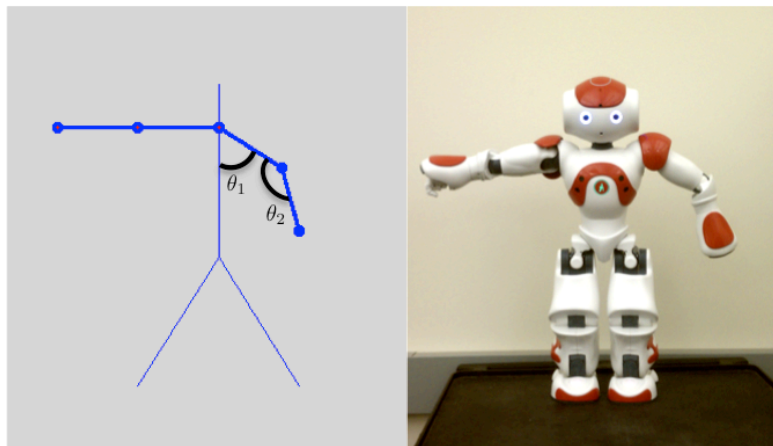


$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{x} = Ax$$

Unstable !!!

State feedback... on a simpler system



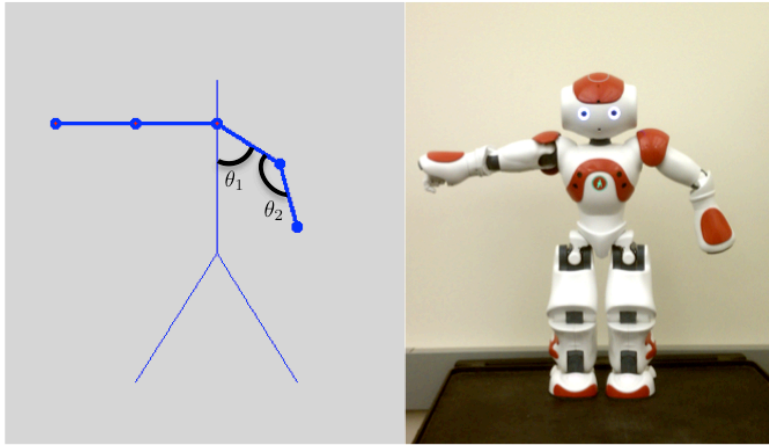
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu \quad \text{Controllable ???}$$

State feedback... on a simpler system



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \quad \text{Controllable !!!}$$

$$u = -Kx$$

State feedback... on a simpler system

$$\begin{aligned} \dot{x} &= Ax + Bu & \text{Controllable} & & x &= \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} & A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ u &= -Kx & & & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \dot{x} &= (A - BK)x \\ \dot{x} &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) x \\ \dot{x} &= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x \end{aligned}$$

State feedback... on a simpler system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x \quad \rightarrow A'$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A' - \lambda I)$$

$$\lambda^2 + k_2\lambda + k_1$$

Pick your favorite 2 eigen values (LHP) :

$$\lambda_1 = -1$$

$$\lambda_2 = -2$$

$$(\lambda + 1)(\lambda + 2)$$

State feedback... on a simpler system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x \quad \rightarrow A'$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial:

$$\det(A' - \lambda I)$$

$$\lambda^2 + k_2\lambda + k_1$$

Pick your favorite 2 eigen values (LHP) :

$$\lambda_1 = -1$$

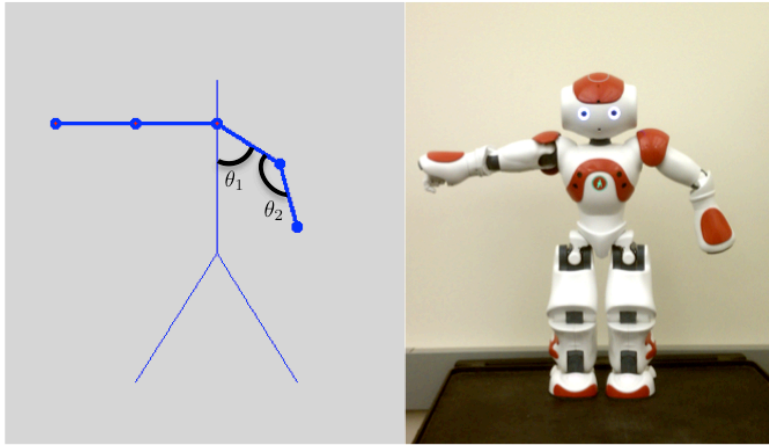
$$\lambda_2 = -2$$

$$\lambda^2 + 3\lambda + 2$$

$$k_2 = 3$$

$$k_1 = 2$$

State feedback... on a simpler system



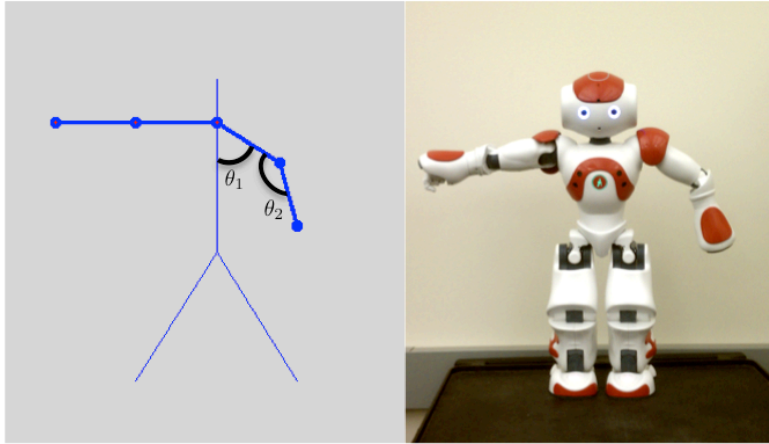
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \quad \text{Stable !!!}$$

$$u = -Kx$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

But we don't know our state...

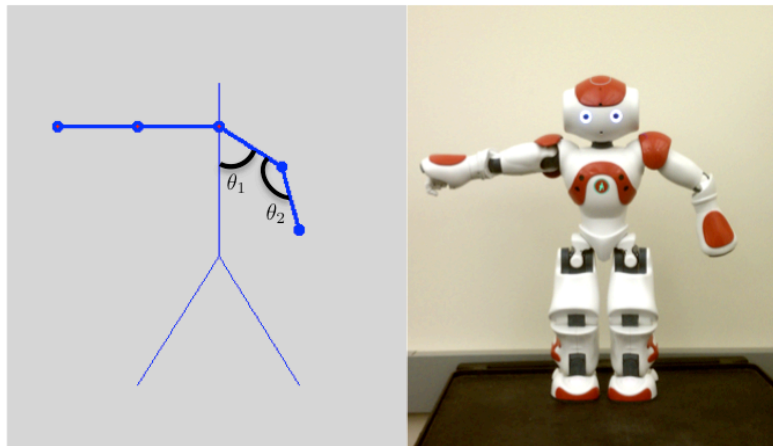


$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

Need to estimate our state...



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Say we can "see" θ_1

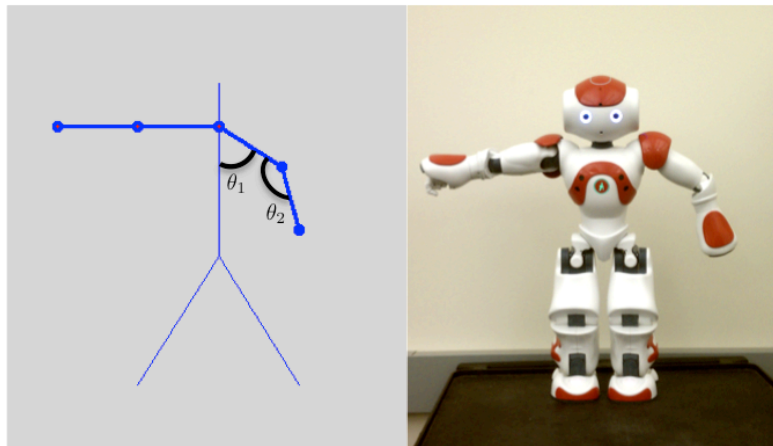
$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Observable ???

$$u = -K\hat{x}$$

Need to estimate our state...



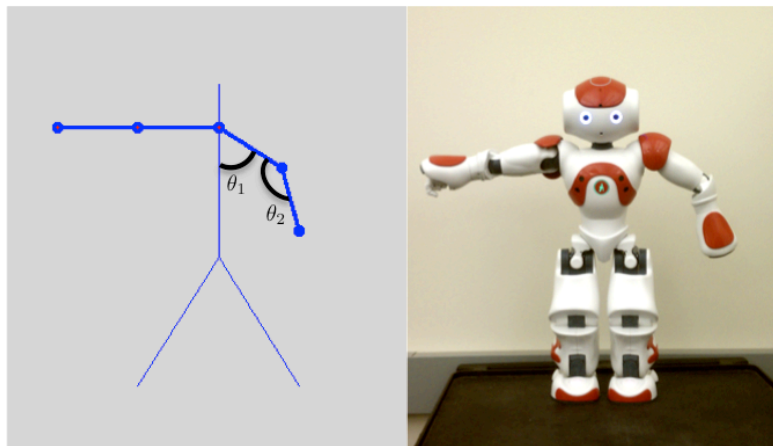
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \quad y = Cx \quad \text{Observable ???}$$

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{rank}(\Omega) = 2$$

Need to estimate our state...



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu \quad y = Cx \quad \text{Observable !!!}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$$

All together... execution

$$\dot{x} = Ax + Bu \quad y = Cx \quad x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Controllable and Observable

K

L

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a) Wake up $t = t_0$, $x = x_0$, $\hat{x} = \hat{x}_0$

b) Start Loop (dt increments)

c) Read output $y = Cx$

d) Compute control $u = -K\hat{x}$

e) Send control u

f) Update \hat{x} using dynamics $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

g) Repeat

$$+ \hat{x}_{k+1} = \hat{x}_k + dt \dot{\hat{x}}$$

Check the forums, and good luck with Quiz 4!