MODULE 2 LESSON 1

THE KALMAN FILTER

Module 2 | Linear and Nonlinear Kalman Filters

In this module…

- A brief history and overview of the (linear) Kalman filter
- Kalman filter as the BLUE
- Extending the Kalman filter to nonlinear systems through linearization
- Limitations of linearization
- Unscented Kalman filter

Module 2 | Linear and Nonlinear

By the end of this video, you will be able to…

1. Describe the Kalman filter as a two stage filter: (1) prediction and (2) correction

2. Understand the difference between motion and measurement models

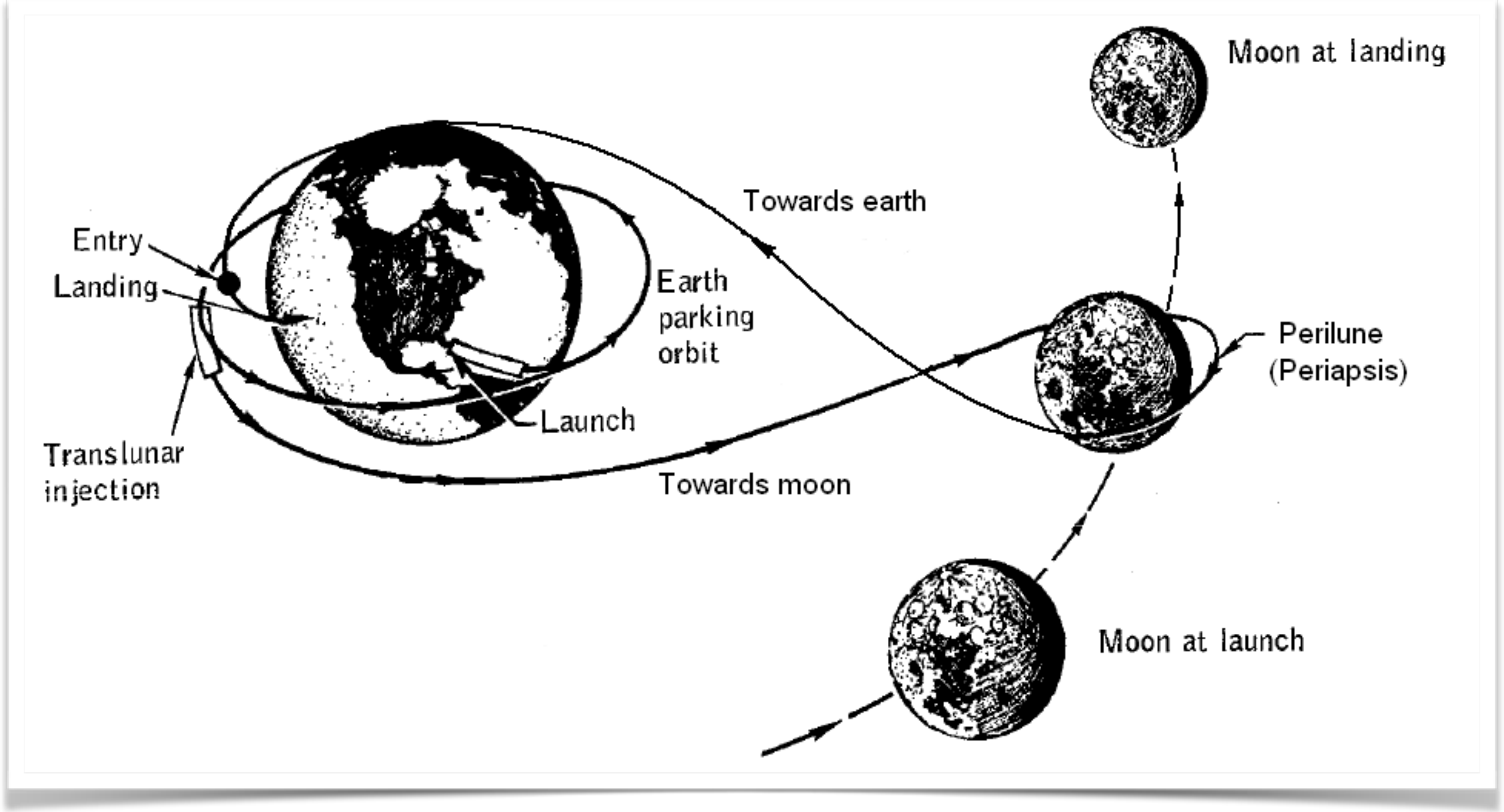
3. Use the Kalman filter in a simple 1D localization example

The Kalman Filter

Rudolf E. Kálmán receiving the National Medal of Science

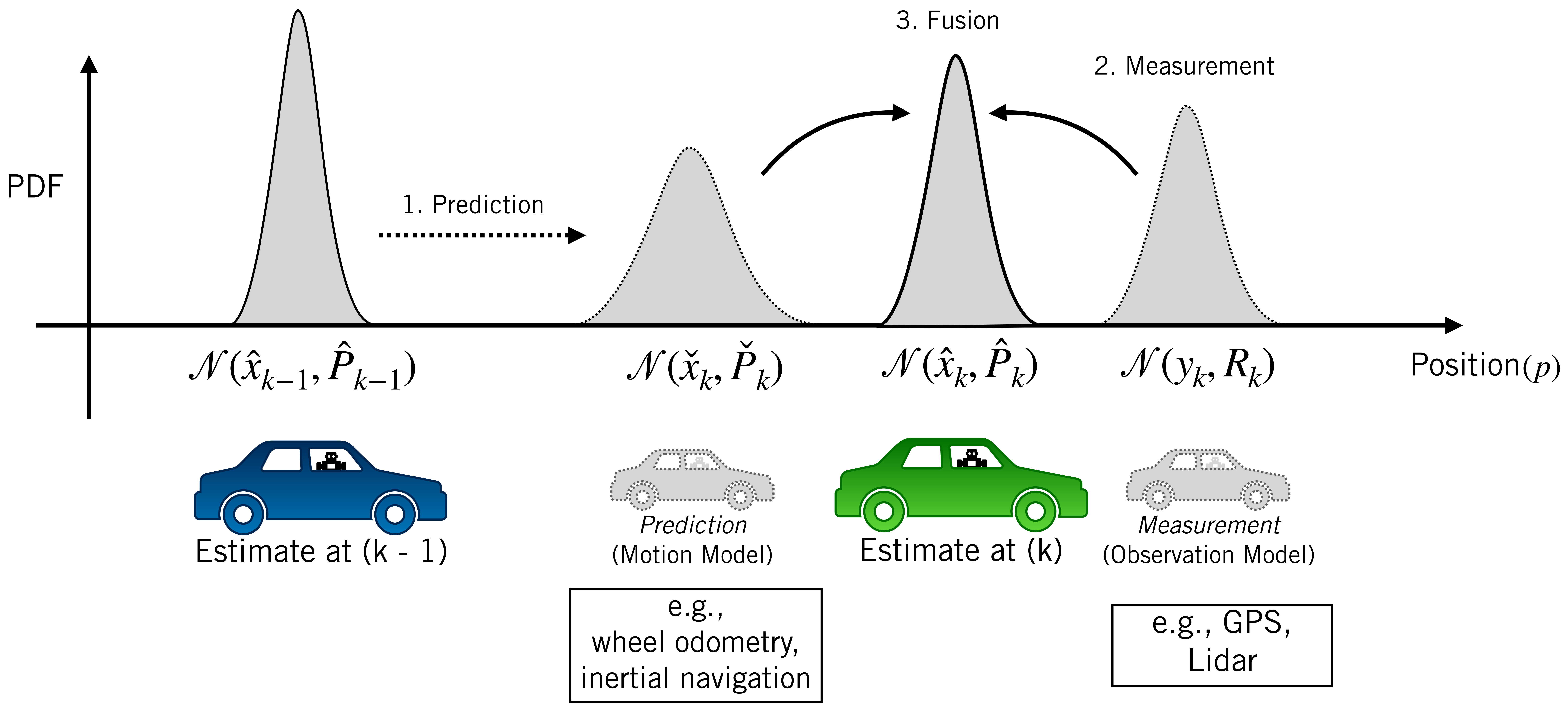
The Kalman Filter

Apollo Guidance Computer



The (extended) Kalman Filter became widely known after its use in the Apollo Guidance Computer for circumlunar navigation.

The Kalman Filter | Prediction and Correction

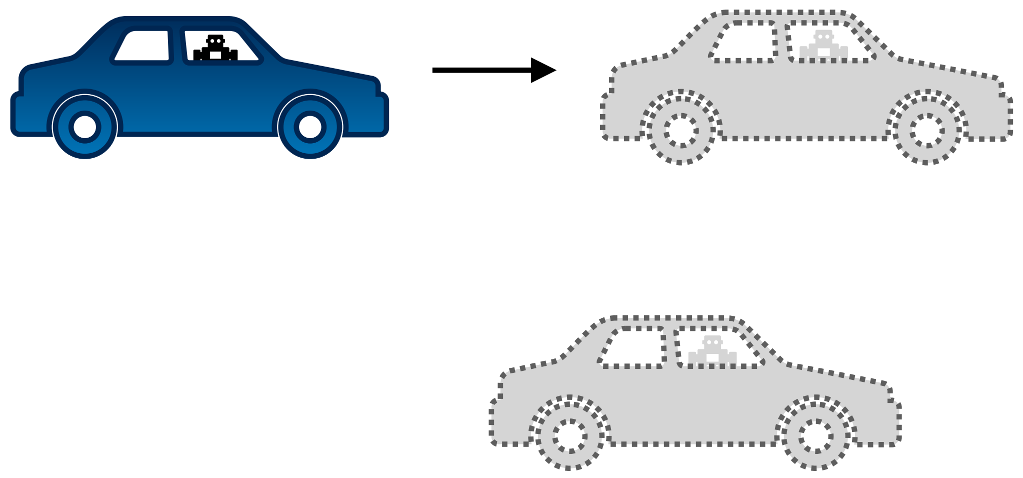


The Kalman Filter | Linear Dynamical System

- The Kalman Filter requires the following motion and measurement models:

Motion model: $$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \underbrace{\mathbf{G}_{k-1}\mathbf{u}_{k-1}}_{\textit{input}} + \underbrace{\mathbf{w}_{k-1}}_{\textit{noise}}$$

Measurement model: $$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \underbrace{\mathbf{v}_k}_{\textit{noise}}$$



- With the following noise properties:

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

Measurement Noise

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$$

Process or Motion Noise

The Kalman Filter | Recursive Least Squares + Process Model

- The Kalman filter is a recursive least squares estimator that also includes a motion model

1 Prediction

$$\check{\mathbf{x}}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1}$$

$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}$$

2a Optimal Gain

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

2b Correction

$$\hat{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k)$$

$$\hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{P}}_k$$

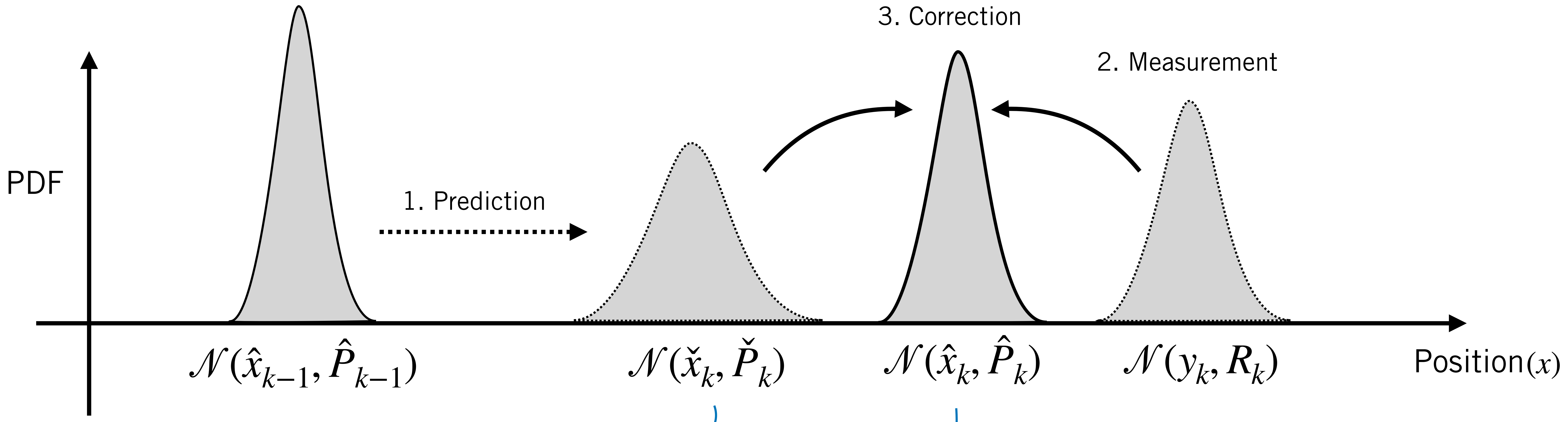
$$(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k)$$

is often called the 'innovation'

$\check{\mathbf{x}}_k$ **Prediction** *(given motion model) at time k*

$\hat{\mathbf{x}}_k$ **Corrected prediction** (given measurement) at time k

The Kalman Filter | Prediction & Correction



1. Prediction

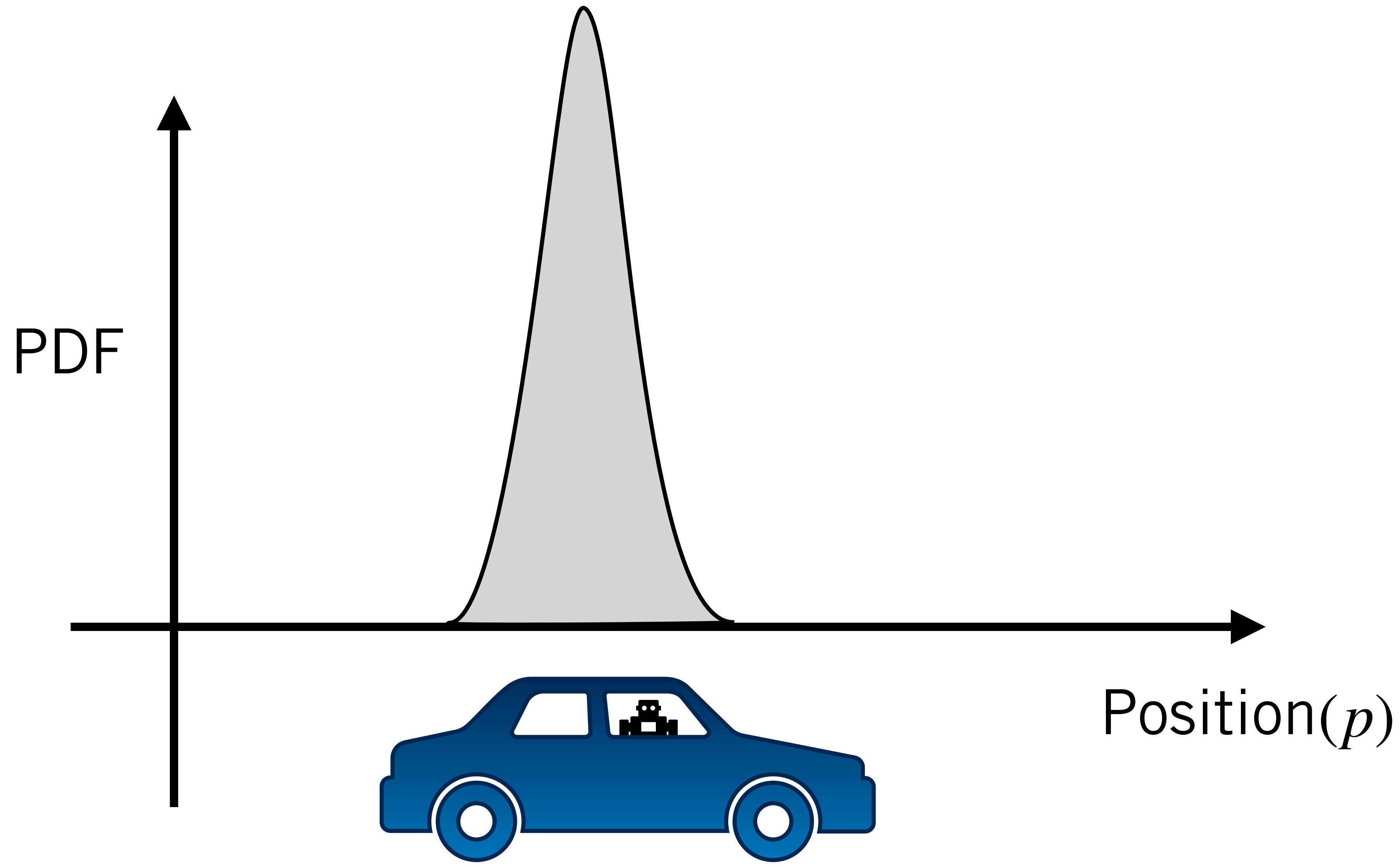
$$\left\{\begin{aligned} \check{\mathbf{x}}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} \\ \check{\mathbf{P}}_k &= \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1} \end{aligned}\right.$$

2,3. Measurement & Correction

$$\mathbf{K}_k = \check{\mathbf{P}}_k\mathbf{H}_k^T(\mathbf{H}_k\check{\mathbf{P}}_k\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\left\{\begin{aligned} \hat{\mathbf{x}}_k &= \check{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\check{\mathbf{x}}_k) \\ \hat{\mathbf{P}}_k &= (\mathbf{1} - \mathbf{K}_k\mathbf{H}_k)\check{\mathbf{P}}_k \end{aligned}\right.$$

The Kalman Filter | Short Example



$$\mathbf{x} = \begin{bmatrix} p \\ \frac{dp}{dt} = \dot{p} \end{bmatrix} \qquad \mathbf{u} = a = \frac{d^2p}{dt^2}$$

Motion/Process Model

$$\mathbf{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} \mathbf{u}_{k-1} + \mathbf{w}_{k-1}$$

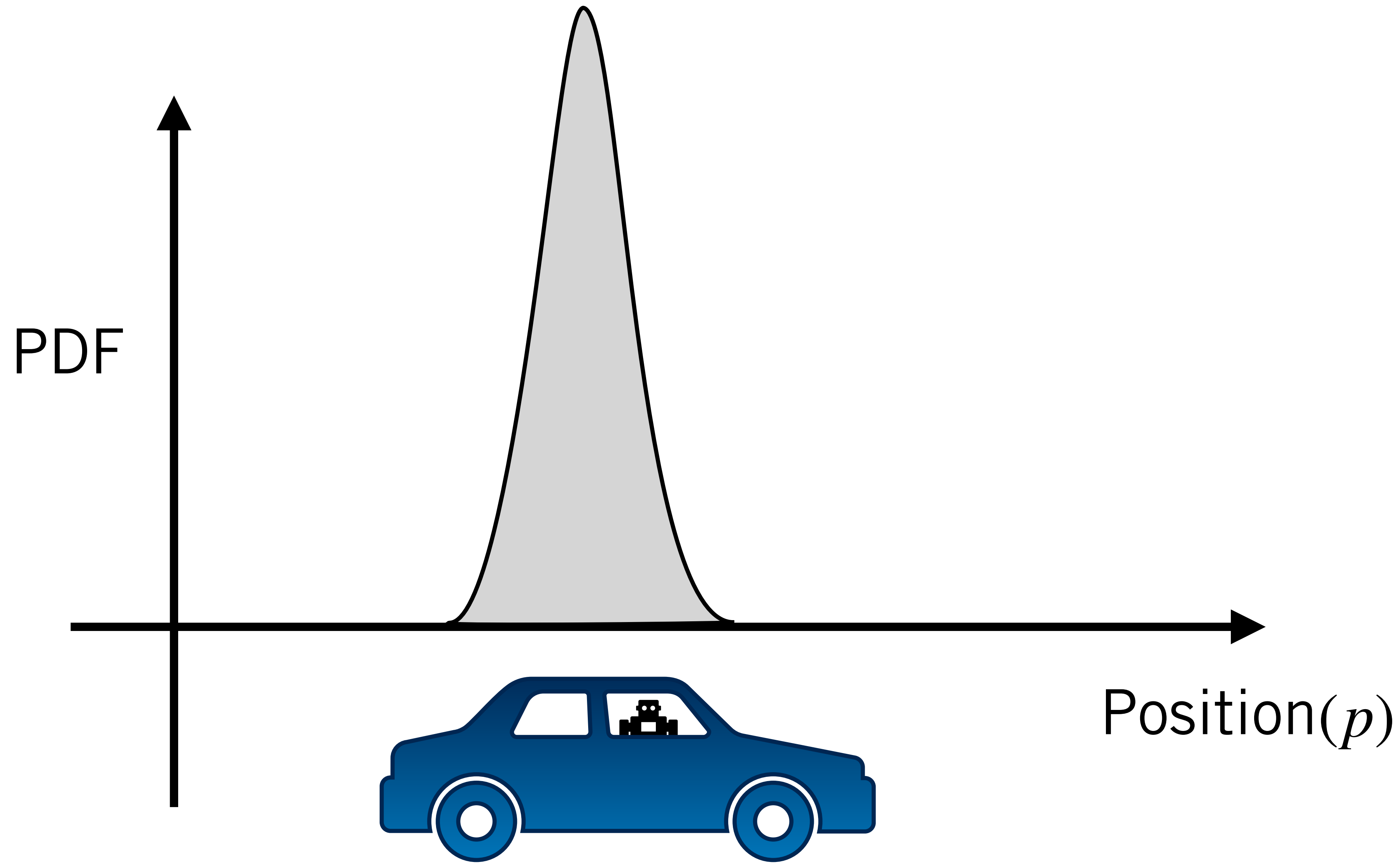
Position Observation

$$y_k = [1 \quad 0]\, \mathbf{x}_k + v_k$$

Noise Densities

$$v_k \sim \mathcal{N}(0,\, 0.05) \qquad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, (0.1)\mathbf{1}_{2\times 2})$$

The Kalman Filter | Short Example



Data

$$\hat{\mathbf{x}}_0 \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\Delta t = 0.5\text{s}$$

$$u_0 = -2\ [m/s^2] \qquad y_1 = 2.2\ [m]$$

The Kalman Filter | Short Example Solution

Prediction

$$\check{\mathbf{x}}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1}$$

$$\begin{bmatrix} \check{p}_1 \\ \check{\dot{p}}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} (-2) = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}$$

$$\check{\mathbf{P}}_k = \mathbf{F}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1}$$

$$\check{\mathbf{P}}_1 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

The Kalman Filter | Short Example Solution

Correction

$$\mathbf{K}_1 = \check{\mathbf{P}}_1 \mathbf{H}_1^T (\mathbf{H}_1 \check{\mathbf{P}}_1 \mathbf{H}_1^T + \mathbf{R}_1)^{-1}$$

$$= \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.36 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.05 \right)^{-1}$$

$$= \begin{bmatrix} 0.88 \\ 1.22 \end{bmatrix}$$

$$\hat{\mathbf{x}}_1 = \check{\mathbf{x}}_1 + \mathbf{K}_1 (\mathbf{y}_1 - \mathbf{H}_1 \check{\mathbf{x}}_1)$$

$$\begin{bmatrix} \hat{p}_1 \\ \hat{\dot{p}}_1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 4 \end{bmatrix} + \begin{bmatrix} 0.88 \\ 1.22 \end{bmatrix} \left(2.2 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2.5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2.24 \\ 3.63 \end{bmatrix}$$

Bonus!

$$\hat{\mathbf{P}}_1 = (\mathbf{1} - \mathbf{K}_1 \mathbf{H}_1) \check{\mathbf{P}}_1$$

$$= \begin{bmatrix} 0.04 & 0.06 \\ 0.06 & 0.49 \end{bmatrix}$$

Summary | The Kalman Filter

- The Kalman Filter is very similar to RLS but includes a *motion model* that tells us how the state evolves over time

- The Kalman Filter updates a state estimate through two stages:

    1. *prediction* using the motion model

    2. *correction* using the measurement model