Eton College

Computational Physics Prize

2023

Modelling the Climate of an Earth-Like Planet

Matteo Mountain (IRG) | Hugo Tollit (NPTL)

Modelling the Climate of an Earth-Like Planet

Matteo Mountain (IRG) | Hugo Tollit (NPTL)

May 4, 2023

Contents

1 Introduction

1.1 History of Climate Modelling

The history of climate modelling can be traced back to when Svante Arrhenius discovered the important role that atmospheric gases such as carbon dioxide play in the global climate[1]. In his 1896 paper, Arrhenius was the first to suggest that carbon dioxide concentrations in the atmosphere - via the greenhouse effect - directly affect earth's temperature.

In the early 1900s, the behaviour of the climate was modelled mathematically. It was not until the late 1900s when computers began to take over the field of research.

Today, extremely complex climate models are continuously used to make predictions on how the climate will change in the years ahead, crucially shaping our response to the climate crisis.

1.2 Tackling the Problem

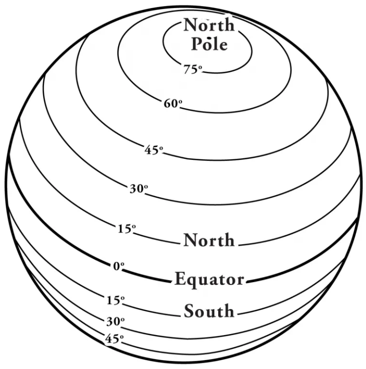
We decided to use Python to create an object-oriented solution to this challenge. Using equations that we derived ourselves, we experimented and investigated how different initial conditions affect the climate of the planet. Most importantly, however, we observed how important limiting the greenhouse effect is to sustain life on the planet.

2 Theory of the Model

2.1 The Energy-Balance Model

The energy-balance model is based on the principle of energy conservation - the total energy in the planet remains constant.

The planet is divided into regions, aka latitude belts, like so:



Lines of latitude

The model then calculates the amount of energy that enters and leaves each latitude belt, taking numerous factors into account. This energy is then used to calculate the temperature of each latitude belt.

The energy-balance model is a useful and relatively simple system for understanding how different factors affect the climate. As a result, it is used extensively in the study of the planet's climate.

2.2 Factors Included

We included the following factors into the model:

- Snow line
- Albedo
- Convective energy transfer between latitude belts
- Carbon dioxide concentration
- Obliquity of planet
- Greenhouse effect

3 Mathematics Involved

3.1 Perceived Area vs. Actual Area

We have found the need to separate perceived area and actual area. The perceived area is the area of each latitude as observed by the sun whereas actual area is the true area of the belt.

3.1.1 Perceived Area

To derive an equation for perceived area, A_p, an equation for the radius of a latitude belt is needed:

$$R = \sqrt{r^2 - x^2}$$

where r is the radius of the planet and x is the depth of the latitude belt

Thus the formula for the perceived area is:

$$\begin{aligned} A_p &= \int_{(n+1)\theta}^{n\theta} R\, dx \\ &= \int_{(n+1)\theta}^{n\theta} \sqrt{r^2 - x^2}\, dx \end{aligned}$$

By substituting $x = r \sin(\phi)$ and evaluating the integral:

$$r^2(\frac{1}{2}\sin(2\phi) - \phi) + C$$

Inputting the limits gives:

$$A_p = \frac{r^2(\sin(2n\theta) - \sin(2\theta(n+1)))}{2} + r^2\theta$$

To account for the obliquity of the planet, β, the perceived obliquity, β_p, from the sun is considered:

$$\beta_p = \beta \sin(\frac{2\pi t)}{t_o}$$

where β is the obliquity, t is the number of time steps that have passed, and t_o is the number of time steps in an orbit.

Finally, this is incorporated into the perceived area by having it shift the effective position of the pole.

$$A_p = \frac{r^2(\sin(2n(\theta - \beta_p)) - \sin(2(\theta + \beta_p)(n+1)))}{2} + r^2\theta$$

3.1.2 Actual Area

To derive an equation for the actual area, A_a, the circumference, C, of the latitude belt at a specific height is found. It is then integrated through all heights within the latitude belt.

$$C = 2\pi r^2 \sin(\phi)$$

where r is the radius of the planet and ϕ is the angle of the circumference from the north pole.

Therefore:

$$\begin{aligned} A_a &= \int_{n\theta}^{(n+1)\theta} C \, d\phi \\ &= \int_{n\theta}^{(n+1)\theta} 2\pi r^2 \sin(\phi) \, d\phi \end{aligned}$$

where n is the index of the latitude belt and θ is the angle between latitude belts.

Therefore:

$$A_a = 2\pi r^2 (\cos(n\theta) - \cos((n+1)\theta))$$

3.2 Energy Absorbed

The energy absorbed by the planet comes from the sun and the radiative forcing caused by carbon dioxide in the atmosphere.

To derive the first portion of the equation for energy absorbed - the energy absorbed from the sun - the inverse square law is used to determine the intensity of the light received by the whole planet. The intensity is converted to energy by multiplying by the perceived area of the specific latitude belt. Furthermore, it is multiplied the reflectivity, or $1 -$ albedo, to account for reflected light. This gives:

$$Q_{sun} = \frac{LA_p(1-a)}{4\pi D^2}$$

where L is the luminosity and D is the distance between the planet and the sun.

For the second portion of the equation, the equation for radiative forcing is used:

$$\Delta F = \alpha \ln\left(\frac{C}{C_0}\right)$$

where α is the climate sensitivity parameter, C is the concentration of carbon dioxide in the atmosphere, and C_0 is the pre-industrial concentration of carbon dioxide in the atmosphere. C_0 has been estimated to be 270 ppm.

For carbon dioxide, the value of the climate sensitivity parameter is $5.35 W\ m^{-2}$ [2]. Radiative forcing is the increase in intensity caused by greenhouse gases. The above formula only gives the change when C_0 is increased to C, so we need to find the radiative forcing caused by C_0.

Firstly, the temperature of the planet without the greenhouse effect is calculated by saying that $Q_{in} = Q_{out}$:

$$\frac{LA_p(1-a)}{4\pi D^2} = A_a \sigma T^4 \tag{1}$$

solving for T gives a temperature of $255K$

However, the pre-industrial temperature of the earth has been estimated to be $288K$. This is due to the radiative forcing caused by C_0, ΔF_{C_0}

$$A_a \sigma T^4 = \frac{LA_p(1-a)}{4\pi D^2} + \Delta F_{C_0} \times A_a \tag{2}$$

By substituting $255K$ for T in (1), and subtracting that equation from equation (2) which has T as $288K$, we obtain:

$$A_a \sigma (288^4 - 255^4) = \Delta F_{C_0} \times A_a$$
$$\Delta F_{C_0} = 150$$

This value can be added onto the radiative forcing caused by an increase in carbon dioxide concentration. This gives us our final equation for change in radiative forcing:

$$\Delta F = A_a(150 + 5.35 \ln(\frac{C}{270}))$$

3.3 Energy Emitted

To calculate the energy emitted from the planet, due to the fact that the most of the radiation from the sun is of a high wavelength, it is appropriate to use the Stefan-Boltzmann law as an approximation of Planck's law.

$$Q_{out} = A\sigma e T^4$$

where σ is Stefan's constant, e is the emissivity of the planet, and T is the temperature of the latitude belt

3.4 Convectional Energy Transferred Between Latitude Belts

An equation for the convectional energy transferred between latitude belts can be derived by using the equation for convection:

$$Q = hA_a \Delta T$$

where h is the convection heat transfer coefficient.

For a specific latitude belt, the heat transferred to the belt above and below needs to be calculated. This is done by considering couples of neighbouring belts individually. Thus:

$$\Delta T = T_{n1} - T_c$$

where T_{n1} is the temperature of the neighbour above the current latitude belt and T_c is the temperature of the current latitude belt.

The overall area of each couple is the sum of their actual areas:

$$A = (A_{ac} + A_{an1})$$

where A_{ac} is the actual area of the current latitude belt and A_{an1} is the actual area of the neighbour above the current latitude belt.

Inputting this information into the equation for convection and considering the upper and lower neighbours of each latitude belt gives this:

$$Q_{transferred} = 2.3(A_{ac} + A_{an1})(T_{n1} - T_c) + 2.3(A_{an2} + A_{ac})(T_{n2} - T_c)$$

Note that $h = 2.3$

3.5 Albedo

Each latitude belt can have a different albedo (reflectivity) according to its snow, water, and land proportions. The equation below is an average of the albedos of land and water weighted based on the land-water proportion. It doesn't include snow as - in our model - each belt is either completely snow-covered or has a land-water proportion.

Therefore:

$$a = 2a_L(1 - W) + 2a_W(1 + W)$$

where a_L is the albedo of land, W is the land-water proportion, and a_W is the albedo of water

3.6 Temperature Change

To calculate the temperature change, we can use the specific heat capacity equation:

$$Q = mc\Delta T$$

This is calculated for the atmosphere, which has a mass of 5.148×10^{18} kg and a specific heat capacity of 700 J K^{-1} kg^{-1}. The mass of the atmosphere needs to be divided by the number of latitude belts, N, in order to get the mass of each latitude belt's atmosphere.
A constant that describes the amount of energy absorbed by the atmosphere compared to the surface is multiplied to this equation so that it takes in the correct amount of energy.

Therefore:

$$\Delta T = \frac{0.27N(Q_{in} - Q_{out})}{mc}$$

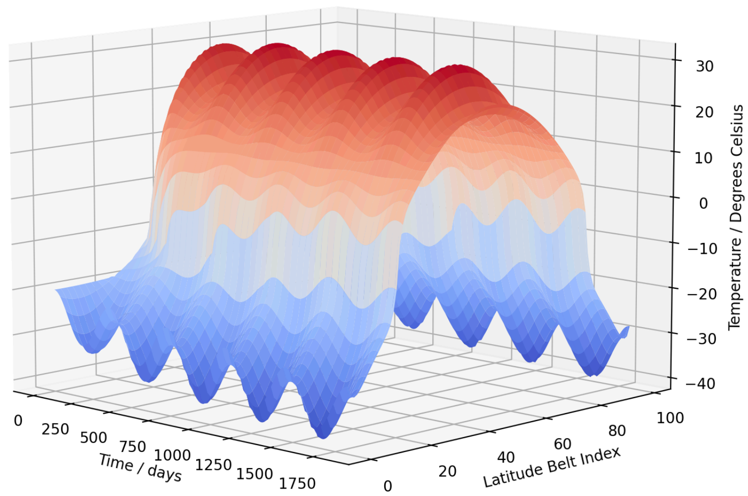
4 Investigating Different Conditions

4.1 Concentration of Carbon Dioxide

4.1.1 No Change in Carbon Dioxide

When there is no change in carbon dioxide, the temperature of the planet will balance at a certain temperature. The starting temperature will be -20°C as this ensures that snow is formed at the beginning.

This is the graph of the temperature of the planet at each latitude belt over time with no change in carbon dioxide:

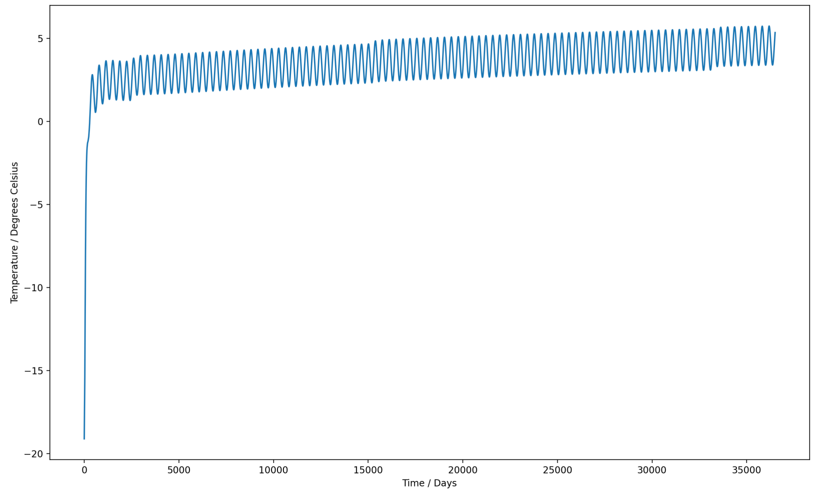


In this 3D surface graph, the oscillations are caused by the fact that the planet is at an angle of 23° perpendicular to its orbit (obliquity). This causes certain parts of the planet to be angled more directly to the sun at certain times of the year.

However, taking this fact into account in the perceived area equation has caused a slight uptick at the extreme latitude belts. This is due to the fact that when $\sin(n\theta)$ is closer to 0, the approximations in the formula for perceived area have a much higher percentage error which causes this effect.

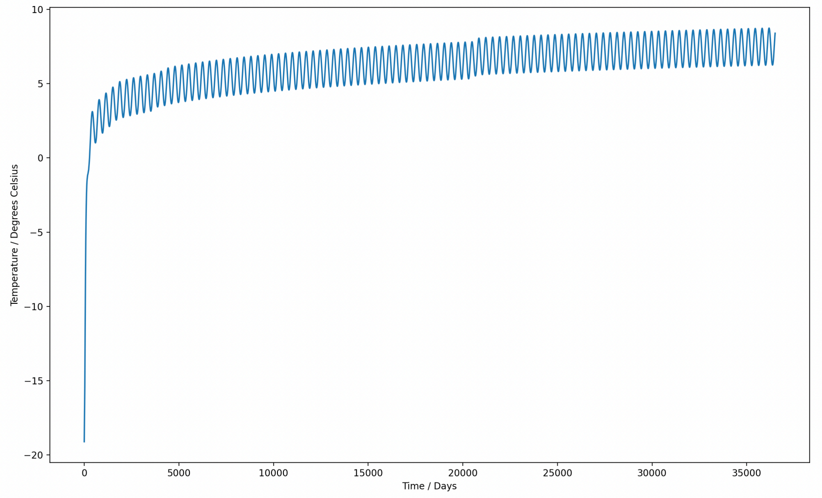
4.1.2 Arithmetically Progressing Carbon Dioxide

In this section, a constant value will be added to the carbon dioxide concentration of the atmosphere. A rise in temperature in all latitude belts - but especially at the poles (due to snow melting) - should be observed. For this, a 2D graph of the average surface temperature of the planet against time would make it easier to see temperature change:



As expected, there is a general rise in the average temperature of the surface. The sudden jumps in temperature are likely due to the snow on snow-covered latitude belts suddenly melting, thus reducing reflection.

Increasing the value that is added to the concentration of carbon dioxide should cause the change in temperature to increase.



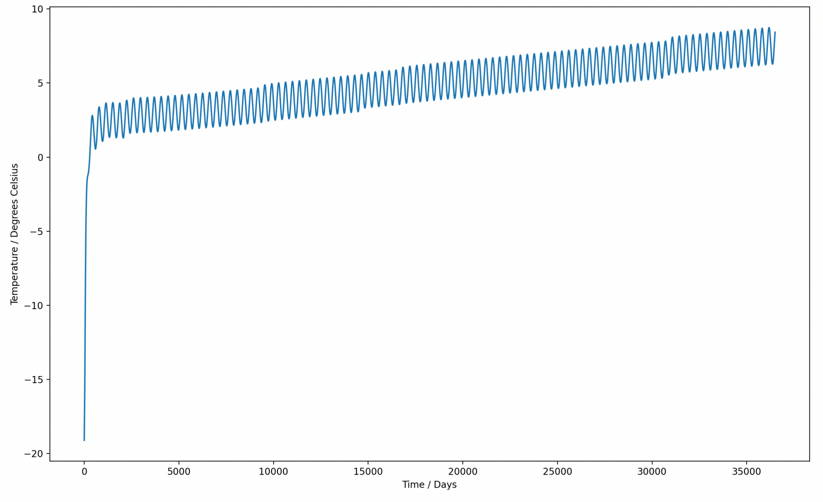
It does, as expected.

4.1.3 Geometrically Progressing Carbon Dioxide

Geometrically progressing the amount of carbon dioxide is a better approximation of the increase in carbon dioxide if humans continue to emit at our current rate.

The value that is multiplied to the carbon dioxide concentration is consistent with the value used for the first arithmetic progression.

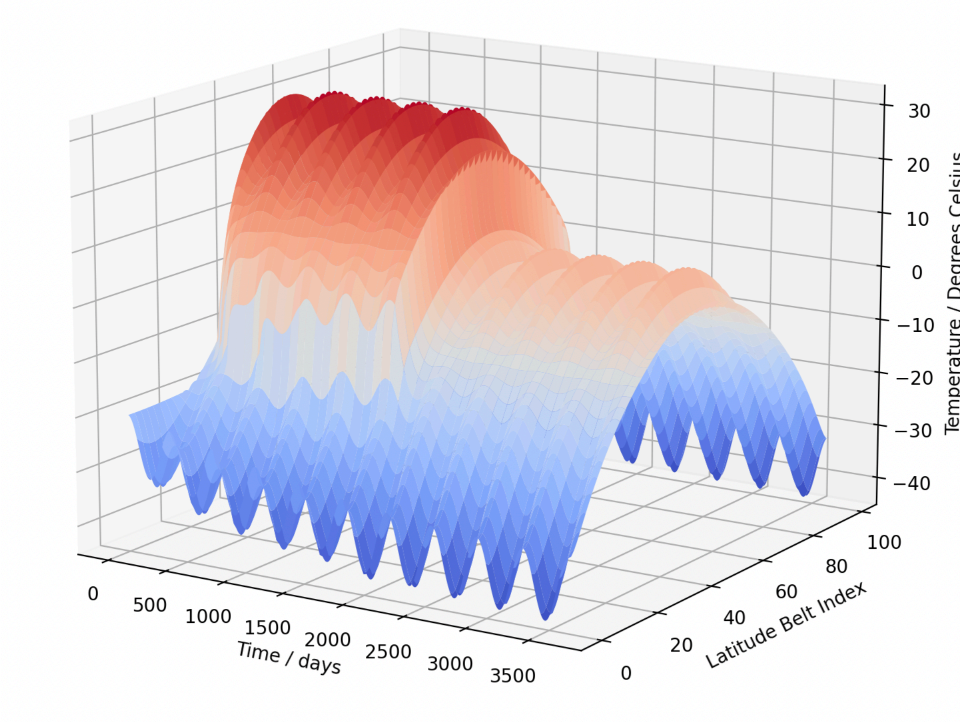
A greater rise in temperature is expected.



Here, the gradient of the geometric progression is constant. This is in contrast to the arithmetic progression which had a decreasing gradient. If the simulation is run for a longer time, the geometrically progressing planet will see higher temperatures in a shorter amount of time when compared to the arithmetically progressing planet.

4.2 Change in Albedo

An event such as a volcanic eruption or a nuclear explosion could cause aerosols to form in the air, reflecting sunlight. This would result in an increase in albedo. Using a higher value for albedo at a random time in the simulation to represent such a situation should show a decrease in temperature.



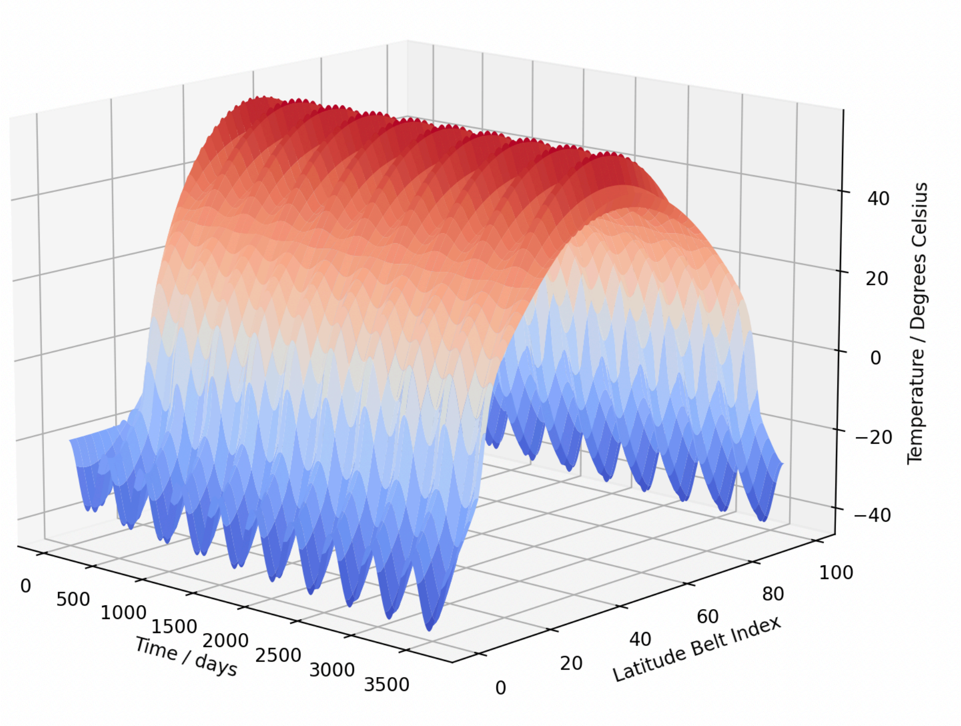
The value used here for albedo is that of if the volcano at Yellowstone Park were to erupt.
An interesting observation in this is graph is that the uptick effect at the poles is lessened. This is likely due to the albedo of the poles changing less than the albedo of the equator.

4.3 Initial Conditions

Here, we tried experimenting with changing some of the initial conditions of the model.

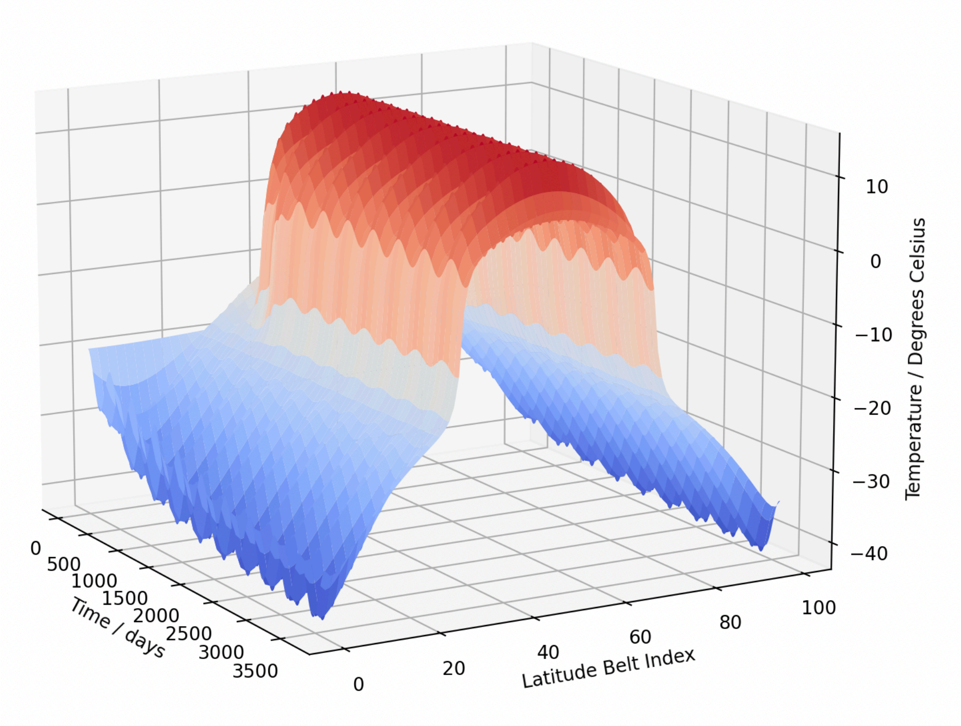
4.3.1 Distance Between Planet and Sun

Decreasing the distance between the planet and the sun should cause a rise in temperature and an increase in the distance should cause a decrease in temperature.
Smaller distance:



A higher maximum temperature is observed as expected.

Greater distance:

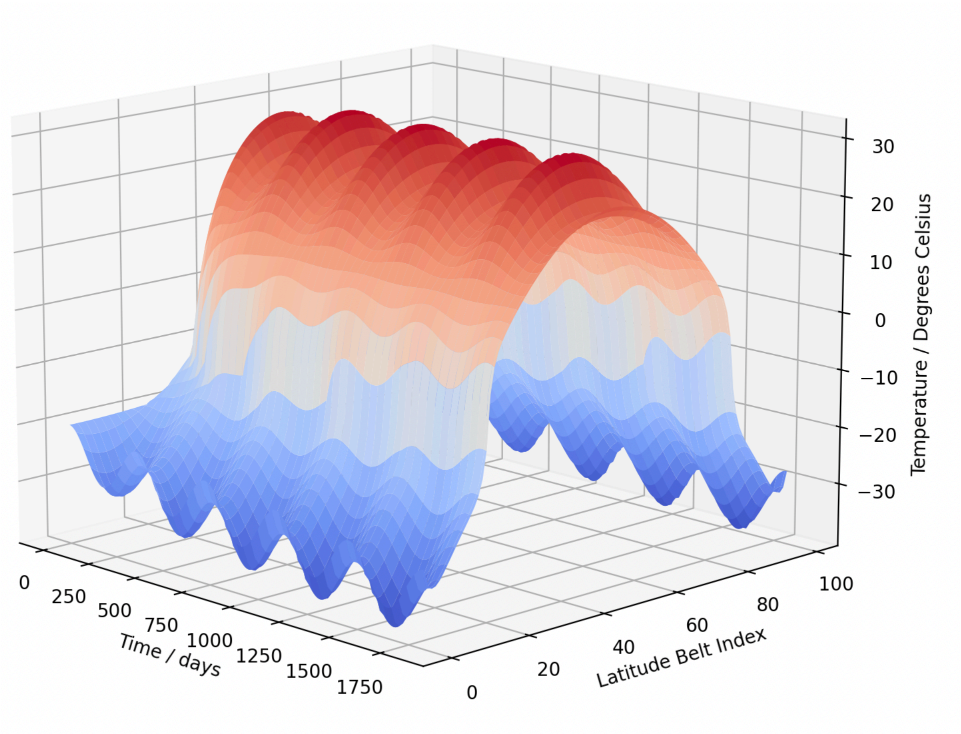


A lower maximum temperature is observed, as expected.

As well as the obvious temperature change, the change between the warm and the cold belts is less sudden in the planet that is closer to the sun than the one that is further away from the sun. This is due to the closer planet only having snow at the extreme latitudes.

4.3.2 Mass of Atmosphere

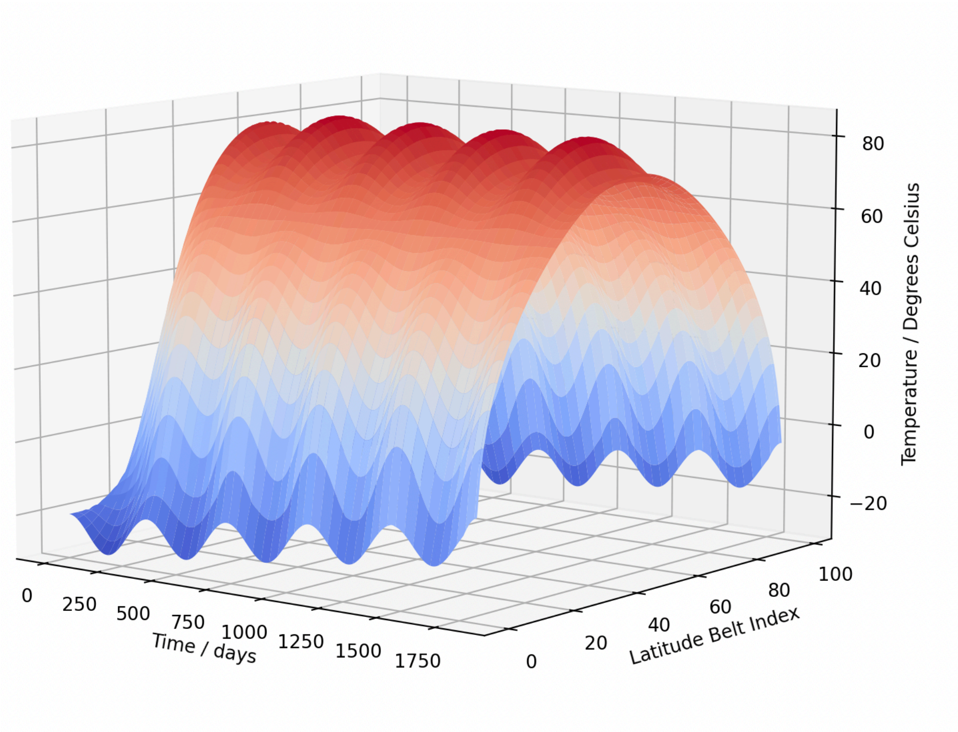
The value of ΔT should be lower with a heavier atmosphere according to the equation used.



The lower value of ΔT has resulted in a decreased the rate at which temperature increases.

4.3.3 Emissivity

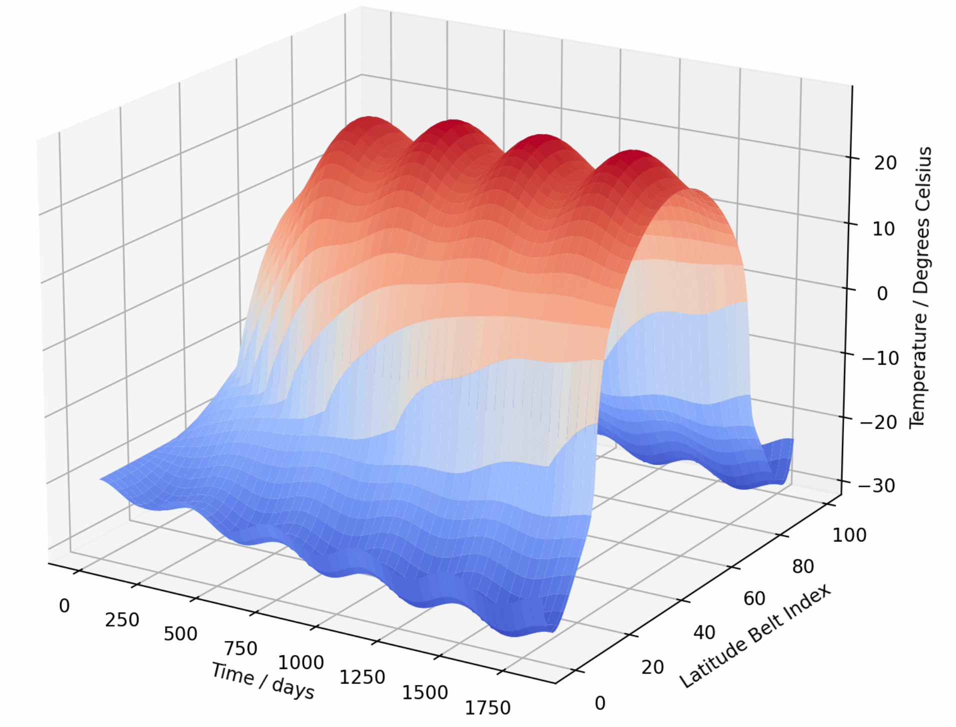
With a lower emissivity, the value of Q_{out} will decrease, thus increasing the maximum temperature of the planet.



The maximum temperature has increased, as expected.

4.3.4 Size of Planet

Due to how widely the value for the radius is used, it is difficult to predict how changing it would affect the temperature of the planet over time.

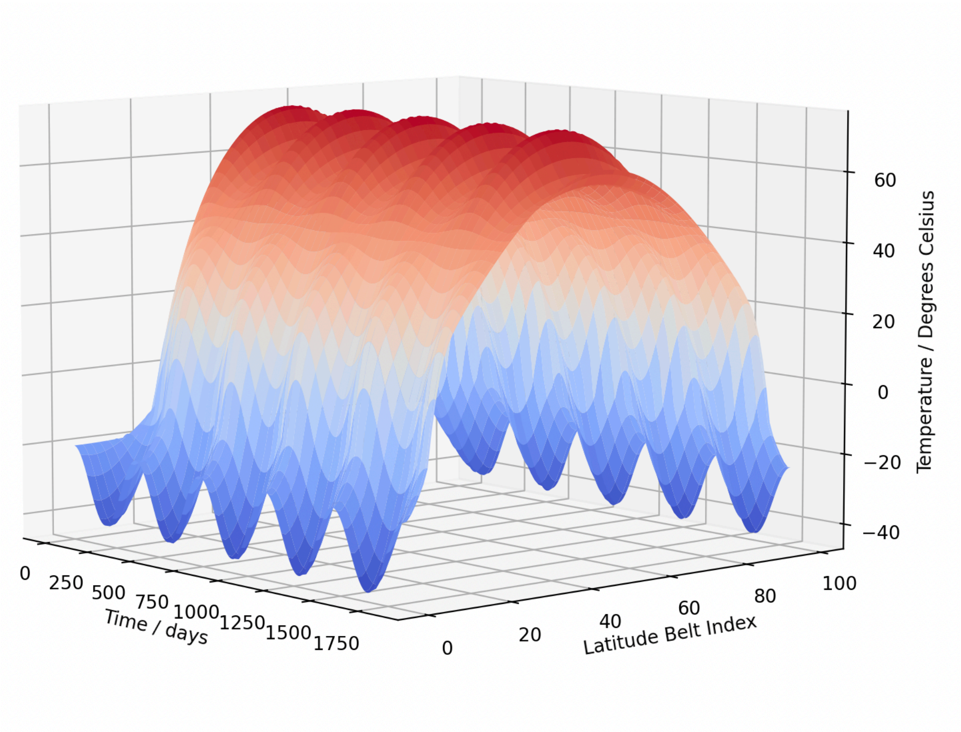


Decreasing the radius of the planet seems to slow the rate at which its temperature evolution progresses.

However, this result is likely to be inaccurate as some of the constants used are based off measurements from our planet and would not work for a planet of a smaller size.

4.3.5 Intensity

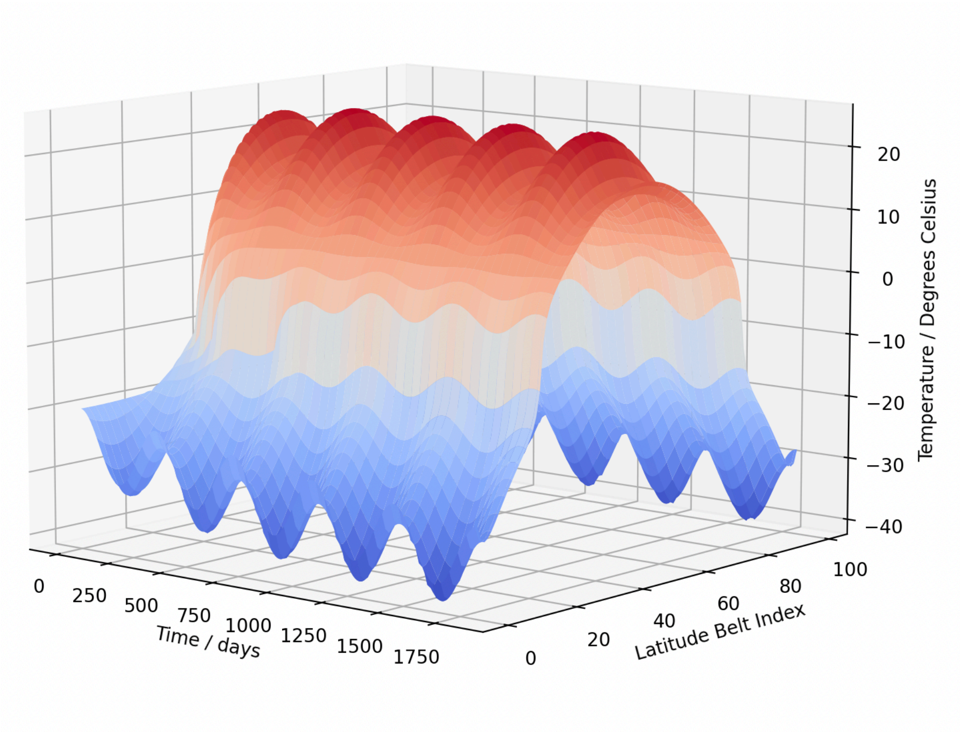
Increasing the intensity of the sun should increase the temperature of the planet.



As expected, increasing the intensity of the sun caused the temperature to increase.

4.3.6 Water Proportion

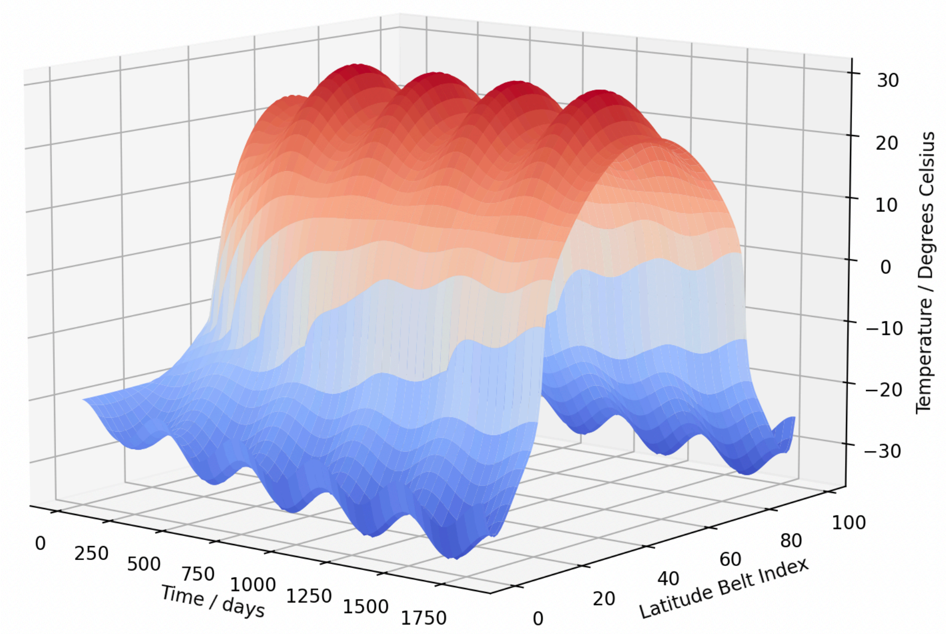
Water - on average - has a lower albedo than land, therefore, a decrease in water proportion should result in an decrease in maximum temperature.



As predicted, the maximum temperature of the planet has decreased.

4.3.7 Heat Capacity of the Atmosphere

Increasing the heat capacity of the atmosphere should yield similar results to increasing the mass of the atmosphere. Therefore, an increase in heat capacity should cause the value of ΔT to be smaller which will decrease the rate at which the temperature evolution of the planet progresses.



As expected, the rate at which the temperature increases is lower.

5 Conclusions

5.1 Limitations

Computational power is a limitation to our model. The more latitude belts used, the more computationally intensive it becomes. Therefore, we have been forced to use lower numbers due to the project being coded on a regular laptop which results in a less accurate model.

An issue with the latitude belt approach is that it assumes each latitude belt to have a uniform biome which is not the case in reality.

All climate models are limited to using estimates for certain values, such as the pre-industrial carbon dioxide levels, therefore our model can only be as accurate as these estimates.

Some of the constants used only apply to planet Earth, which makes it difficult for our model to predict the behaviour of the climate of a different planet.

5.2 Future Extensions

- Two-body problem
- Longitude belts
- Other greenhouse gases
- More in-depth atmosphere
- Use of heat equation (for conductive energy transfer)
- Energy used for evaporation/freezing
- Carbon cycle

6 References

1. https://blogs.bl.uk/science/2016/12/the-first-paper-on-carbon-dioxide-and-global-warming.html [British Library - Accessed 09/03/23]

2. https://www.ipcc.ch/site/assets/uploads/2018/03/TAR-06.pdf [IPCC - Accessed 04/05/23]