Syntax

Type Variables	X, Y, Z		
Term Variables	x, y, z		
Types	T, S, U	$::=$	X
		$\mid$	I
		$\mid$	$T \oplus T$
		$\mid$	$T \otimes T$
		$\mid$	$T \multimap T$
		$\mid$	$\mu X.T$
Terms	t, u, v	$::=$	x
		$\mid$	$()$
		$\mid$	$\mathrm{inl}\ t$
		$\mid$	$\mathrm{inr}\ t$
		$\mid$	$t \times t$
		$\mid$	$t \mapsto t$
		$\mid$	$\mathrm{fold}\ t$
		$\mid$	$t + t$
Type Contexts	Θ	$::=$	
Term Contexts		$\mid$	Θ, X
	Γ	$::=$	
		$\mid$	$\Gamma, t : T$
Type Judgements			$\Theta \vdash T$
Term Judgements			$\Gamma \vdash t : T$
Expressions	f, e	$::=$	$u\ t$

Type Formation rules

$$\frac{}{\Theta, X \vdash X} \qquad \frac{}{\Theta \vdash I} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \oplus T_2} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \otimes T_2} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \multimap T_2} \qquad \frac{\Theta, X \vdash T}{\Theta \vdash \mu X.T}$$

Typing rules for Terms

$$\text{Variable}\ \frac{}{x : T \vdash x : T} \qquad \frac{\Gamma_2, \Gamma_1 \vdash t : T}{\Gamma_1, \Gamma_2 \vdash t : T}\ \text{Exchange}$$

$$I_L\ \frac{\Gamma \vdash t : T}{\Gamma, () : I \vdash t : T} \qquad \frac{}{\vdash () : I}\ I_R$$

$$\oplus_{L_l}\ \frac{\Gamma, t_1 : T_1 \vdash t : T}{\Gamma, \mathrm{inl}\ t_1 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \mathrm{inl}\ t_1 : T_1 \oplus T_2}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \frac{\Gamma, t_2 : T_2 \vdash t : T}{\Gamma, \mathrm{inr}\ t_2 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_2 : T_2}{\Gamma \vdash \mathrm{inr}\ t_2 : T_1 \oplus T_2}\ \oplus_{R_r}$$

$$\otimes_L\ \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T} \qquad \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2 \vdash t_2 : T_2}{\Gamma_1, \Gamma_2 \vdash t_1 \times t_2 : T_1 \otimes T_2}\ \otimes_R$$

$$\multimap_L\ \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2, t_2 : T_2 \vdash t : T}{\Gamma_1, \Gamma_2, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T} \qquad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2}\ \multimap_R$$

$$\mu_L\ \frac{\Gamma, u : U[X/\mu X.U] \vdash t : T}{\Gamma, \mathrm{fold}\ u : \mu X.U \vdash t : T} \qquad \frac{\Gamma \vdash u : U[X/\mu X.U]}{\Gamma \vdash \mathrm{fold}\ u : \mu X.U}\ \mu_R$$

$$\text{Linearity}_L\ \frac{\Gamma, t_1 : T_{12} \vdash t : T \qquad \Gamma, t_2 : T_{12} \vdash t : T}{\Gamma, t_1 + t_2 : T_{12} \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T_{12} \qquad \Gamma \vdash t_2 : T_{12}}{\Gamma \vdash t_1 + t_2 : T_{12}}\ \text{Linearity}_R$$

Typing rules for Expressions

$$\frac{\vdash u : T \multimap U \qquad \vdash t : T}{\vdash u\ t : U}\ \text{Application}$$

Substitution

$$
\begin{aligned}
Y[X/S] &= \begin{cases} S & \text{if } Y = X \\ Y & \text{otherwise} \end{cases} \\
I[X/S] &= I \\
T_1 \oplus T_2[X/S] &= T_1[X/S] \oplus T_2[X/S] \\
T_1 \otimes T_2[X/S] &= T_1[X/S] \otimes T_2[X/S] \\
T_1 \multimap T_2[X/S] &= T_1[X/S] \multimap T_2[X/S] \\
\mu\, Y.T[X/S] &= \mu\, Y.(T[X/S])
\end{aligned}
$$

Operational Semantics

$$\text{Match–var}\ \frac{}{x = t \quad \hookrightarrow \quad [x \to t]} \qquad \frac{}{[x \to t]\ x \quad \rightsquigarrow \quad t}\ \text{Subst-var}$$

$$\text{Match–unit}\ \frac{}{() = () \quad \hookrightarrow \quad [\,]} \qquad \frac{}{[\,]\ () \quad \rightsquigarrow \quad ()}\ \text{Subst–unit}$$

$$\text{Match–inl}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{inl } t = \text{inl } u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{inl } t \quad \rightsquigarrow \quad \text{inl } t'}\ \text{Subst–inl}$$

$$\text{Match–inr}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{inr } t = \text{inr } u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{inr } t \quad \rightsquigarrow \quad \text{inr } t'}\ \text{Subst–inr}$$

$$\text{Match–tensor}\ \frac{t_1 = u_1 \quad \hookrightarrow \quad [\Xi_1] \qquad t_2 = u_2 \quad \hookrightarrow \quad [\Xi_2]}{t_1 \times t_2 = u_1 \times u_2 \quad \hookrightarrow \quad [\Xi_1, \Xi_2]} \qquad \frac{[\Xi_1]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi_2]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi_1, \Xi_2]\ t_1 \times t_2 \quad \rightsquigarrow \quad t_1' \times t_2'}\ \text{Subst–tensor}$$

$$\text{Match–arrow}\ \frac{t_1 = u_1 \quad \hookrightarrow \quad [\Xi_1] \qquad t_2 = u_2 \quad \hookrightarrow \quad [\Xi_2]}{t_1 \mapsto t_2 = u_1 \mapsto u_2 \quad \hookrightarrow \quad [\Xi_1, \Xi_2]} \qquad \frac{[\Xi_1]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi_2]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi_1, \Xi_2]\ t_1 \mapsto t_2 \quad \rightsquigarrow \quad t_1' \mapsto t_2'}\ \text{Subst–arrow}$$

$$\text{Match–fold}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{fold } t = \text{fold } u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{fold } t \quad \rightsquigarrow \quad \text{fold } t'}\ \text{Subst–fold}$$

$$\text{Match–lin}\ \frac{t_1 = u \quad \hookrightarrow \quad [\Xi] \qquad t_2 = u \quad \hookrightarrow \quad [\Xi]}{t_1 + t_2 = u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi]\ t_1 + t_2 \quad \rightsquigarrow \quad t_1' + t_2'}\ \text{Subst-lin}$$

$$\text{Match–lin–l}\ \frac{t = u_1 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_1\ t \quad \Downarrow \quad t_1'}{(t_1 + t_2)\ t \quad \Downarrow \quad t_1'}\ \text{App–lin–l}$$

$$\text{Match–lin–c}\ \frac{t = u_1 \quad \hookrightarrow \quad [\Xi] \qquad t = u_2 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_1\ t \quad \Downarrow \quad t_1' \qquad t_2\ t \quad \Downarrow \quad t_2'}{(t_1 + t_2)\ t \quad \Downarrow \quad t_1' + t_2'}\ \text{App–lin–c}$$

$$\text{Match–lin–r}\ \frac{t = u_2 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_2\ t \quad \Downarrow \quad t_2'}{(t_1 + t_2)\ t \quad \Downarrow \quad t_2'}\ \text{App–lin–r}$$

$$\frac{t_1 = t \quad \hookrightarrow \quad [\Xi] \qquad [\Xi]\ t \quad \rightsquigarrow \quad t'}{(t_1 \mapsto t_2)\ t \quad \Downarrow \quad t'}\ \text{App}$$

Type Interpretation

V is Symmetric Monoidal Closed Category with Finite Biproduct.

Π_i is Projection Functor.

K_I is Constant–I–Functor.

$[-,-]$ is internal Hom Functor.

$$
\begin{aligned}
[\![\Theta \vdash T]\!] &: \mathbf{V}^{|\Theta|} \to \mathbf{V} \\
[\![\Theta \vdash X_i]\!] &= \Pi_i \\
[\![\Theta \vdash I]\!] &= K_I \\
[\![\Theta \vdash T_1 \oplus T_2]\!] &= \oplus \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \otimes T_2]\!] &= \otimes \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \multimap T_2]\!] &= [[\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!]] \\
[\![\Theta \vdash \mu X.T]\!] &= [\![\Theta, X \vdash T]\!]^{\sharp}
\end{aligned}
$$

Denotational Semantics (No Variable)

id_T is identity morphism of Object T.

0 is zero morphism.

ι_i is injection morphism.

pi_i is projection morphism.

$\iota_i^{-1} = \pi_i$

$\pi_i \circ \iota_j = id$ if $i = j$

$\pi_i \circ \iota_j = 0$ otherwise

$fold^{-1} = unfold$

$f \circ (g \oplus h) = (f \circ g) \oplus (f \circ h)$

$(f \oplus g) \circ h = (f \circ h) \oplus (g \circ h)$

$id_A \otimes id_B = id_{A \otimes B}$

$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)$

$$
\begin{aligned}
[\![()]\!] &:= id_I \\
[\![\mathrm{inl}\ v]\!] &:= \iota_1 \circ [\![v]\!] \\
[\![\mathrm{inr}\ v]\!] &:= \iota_2 \circ [\![v]\!] \\
[\![v_1 \times v_2]\!] &:= [\![v_1]\!] \otimes [\![v_2]\!] \\
[\![v_1 \mapsto v_2]\!] &:= k_{[\![v_2]\!]} \circ [\![v_1]\!]^{-1} \\
[\![\mathrm{fold}\ v]\!] &:= fold \circ [\![v]\!] \\
[\![v_1 + v_2]\!] &:= [\![v_1]\!] \oplus [\![v_2]\!] \\
[\![v_1\ \ v_2]\!] &:= [\![v_1]\!]([\![v_2]\!](*))
\end{aligned}
$$