Syntax

Type Variables	X, Y, Z		
Term Variables	x, y, z		
Trace Labels	l		
Types	T, S, U	$::=$	X
		$\mid$	I
		$\mid$	$T \oplus T$
		$\mid$	$T \otimes T$
		$\mid$	$T \multimap T$
		$\mid$	$\mu X.T$
Terms	t, u, v	$::=$	x
		$\mid$	$()$
		$\mid$	inl t
		$\mid$	inr t
		$\mid$	$t \times t$
		$\mid$	$t \mapsto t$
		$\mid$	fold t
		$\mid$	$t + t$
		$\mid$	$l\ t$
Type Contexts	Θ	$::=$	
Term Contexts		$\mid$	Θ, X
	Γ	$::=$	
		$\mid$	$\Gamma, t : T$
Type Judgements			$\Theta \vdash T$
Term Judgements			$\Gamma \vdash t : T$
Expressions	e, f, g	$::=$	t
		$\mid$	$f\ t$
		$\mid$	$f \mathbin{⨾} f$
		$\mid$	$f^{\dagger}$
		$\mid$	t where $l :: T$

Type Formation rules

$$\frac{}{\Theta, X \vdash X} \qquad \frac{}{\Theta \vdash I} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \oplus T_2} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \otimes T_2} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \multimap T_2} \qquad \frac{\Theta, X \vdash T}{\Theta \vdash \mu X.T}$$

Substitution

$$\begin{aligned}
X[X/S] &= S \\
Y[X/S] &= Y \\
I[X/S] &= I \\
T_1 \oplus T_2[X/S] &= T_1[X/S] \oplus T_2[X/S] \\
T_1 \otimes T_2[X/S] &= T_1[X/S] \otimes T_2[X/S] \\
T_1 \multimap T_2[X/S] &= T_1[X/S] \multimap T_2[X/S] \\
\mu Y.T[X/S] &= \mu Y.(T[X/S])
\end{aligned}$$

Term Typing rules

$$\text{Variable}\ \frac{}{x : T \vdash x : T} \qquad \frac{\Gamma_2, \Gamma_1 \vdash t : T}{\Gamma_1, \Gamma_2 \vdash t : T}\ \text{Exchange}$$

$$I_L\ \frac{\Gamma \vdash t : T}{\Gamma, () : I \vdash t : T} \qquad \frac{}{\vdash () : I}\ I_R$$

$$\oplus_{L_l}\ \frac{\Gamma, t_1 : T_1 \vdash t : T}{\Gamma, \text{inl}\ t_1 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl}\ t_1 : T_1 \oplus T_2}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \frac{\Gamma, t_2 : T_2 \vdash t : T}{\Gamma, \text{inr}\ t_2 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_2 : T_2}{\Gamma \vdash \text{inr}\ t_2 : T_1 \oplus T_2}\ \oplus_{R_r}$$

$$\otimes_L\ \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T} \qquad \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2 \vdash t_2 : T_2}{\Gamma_1, \Gamma_2 \vdash t_1 \times t_2 : T_1 \otimes T_2}\ \otimes_R$$

$$\multimap_L\ \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2, t_2 : T_2 \vdash t : T}{\Gamma_1, \Gamma_2, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T} \qquad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2}\ \multimap_R$$

$$\mu_L\ \frac{\Gamma, u : U[X / \mu X.U] \vdash t : T}{\Gamma, \text{fold}\ u : \mu X.U \vdash t : T} \qquad \frac{\Gamma \vdash t : T[X / \mu X.T]}{\Gamma \vdash \text{fold}\ t : \mu X.T}\ \mu_R$$

$$\text{Linearity}_L\ \frac{\Gamma, t_1 : U \vdash t : T \qquad \Gamma, t_2 : U \vdash t : T}{\Gamma, t_1 + t_2 : U \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T \qquad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1 + t_2 : T}\ \text{Linearity}_R$$

$$\text{Trace}_L\ \frac{[]\Gamma, u : S \vdash t : T[\Lambda]}{[l : S]\Gamma, l\ u : U \vdash t : T[\Lambda]} \qquad \frac{[\Lambda]\Gamma \vdash t : U[]}{[\Lambda]\Gamma \vdash l\ t : T[l : U]}\ \text{Trace}_R$$

Expression Typing rules

$$\text{Application}\ \frac{\vdash f : T_1 \multimap T_2 \qquad \vdash t : T_2}{\vdash f\ t : T_1} \qquad \frac{\vdash f : T_1 \multimap T_2 \qquad \vdash g : T_2 \multimap T_3}{\vdash f \mathbin{;} g : T_1 \multimap T_3}\ \text{Composition}$$

$$\text{Dagger}\ \frac{\vdash f : T \multimap U}{\vdash f^{\dagger} : U \multimap T} \qquad \frac{[l : U] \vdash t : T[l : U]}{\vdash t\ \text{where}\ l :: U : T}\ \text{Trace}$$

Operational Semantics

$$\text{Match–var}\ \frac{}{x = t \quad \hookrightarrow \quad [x \to t]} \qquad \frac{}{[x \to t]\ x \quad \rightsquigarrow \quad t}\ \text{Subst-var}$$

$$\text{Match–unit}\ \frac{}{() = () \quad \hookrightarrow \quad [\,]} \qquad \frac{}{[\,]\ () \quad \rightsquigarrow \quad ()}\ \text{Subst–unit}$$

$$\text{Match–inl}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{inl}\ t = \text{inl}\ u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{inl}\ t \quad \rightsquigarrow \quad \text{inl}\ t'}\ \text{Subst–inl}$$

$$\text{Match–inr}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{inr}\ t = \text{inr}\ u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{inr}\ t \quad \rightsquigarrow \quad \text{inr}\ t'}\ \text{Subst–inr}$$

$$\text{Match–tensor}\ \frac{t_1 = u_1 \quad \hookrightarrow \quad [\Xi_1] \qquad t_2 = u_2 \quad \hookrightarrow \quad [\Xi_2]}{t_1 \times t_2 = u_1 \times u_2 \quad \hookrightarrow \quad [\Xi_1, \Xi_2]} \qquad \frac{[\Xi_1]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi_2]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi_1, \Xi_2]\ t_1 \times t_2 \quad \rightsquigarrow \quad t_1' \times t_2'}\ \text{Subst–tensor}$$

$$\text{Match–arrow}\ \frac{t_1 = u_1 \quad \hookrightarrow \quad [\Xi_1] \qquad t_2 = u_2 \quad \hookrightarrow \quad [\Xi_2]}{t_1 \mapsto t_2 = u_1 \mapsto u_2 \quad \hookrightarrow \quad [\Xi_1, \Xi_2]} \qquad \frac{[\Xi_1]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi_2]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi_1, \Xi_2]\ t_1 \mapsto t_2 \quad \rightsquigarrow \quad t_1' \mapsto t_2'}\ \text{Subst–arrow}$$

$$\text{Match–fold}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{\text{fold}\ t = \text{fold}\ u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t'}{[\Xi]\ \text{fold}\ t \quad \rightsquigarrow \quad \text{fold}\ t'}\ \text{Subst–fold}$$

$$\text{Match–lin}\ \frac{t_1 = u \quad \hookrightarrow \quad [\Xi] \qquad t_2 = u \quad \hookrightarrow \quad [\Xi]}{t_1 + t_2 = u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t_1 \quad \rightsquigarrow \quad t_1' \qquad [\Xi]\ t_2 \quad \rightsquigarrow \quad t_2'}{[\Xi]\ t_1 + t_2 \quad \rightsquigarrow \quad t_1' + t_2'}\ \text{Subst-lin}$$

$$\text{Match–trace}\ \frac{t = u \quad \hookrightarrow \quad [\Xi]}{l\ t = l\ u \quad \hookrightarrow \quad [\Xi]} \qquad \frac{[\Xi]\ t \quad \rightsquigarrow \quad t' \qquad [l, u]u\ t' \quad \Rightarrow \quad t''}{[l, u][\Xi]\ l\ t \quad \rightsquigarrow \quad t''}\ \text{Subst–trace}$$

$$\text{Match–lin–l}\ \frac{t = u_1 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_1\ t \quad \Rightarrow \quad t_1'}{(t_1 + t_2)\ t \quad \Rightarrow \quad t_1'}\ \text{App–lin–l}$$

$$\text{Match–lin–c}\ \frac{t = u_1 \quad \hookrightarrow \quad [\Xi] \qquad t = u_2 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_1\ t \quad \Rightarrow \quad t_1' \qquad t_2\ t \quad \Rightarrow \quad t_2'}{(t_1 + t_2)\ t \quad \Rightarrow \quad t_1' + t_2'}\ \text{App–lin–c}$$

$$\text{Match–lin–r}\ \frac{t = u_2 \quad \hookrightarrow \quad [\Xi]}{t = u_1 + u_2 \quad \hookrightarrow \quad [\Xi]} \qquad \frac{t_2\ t \quad \Rightarrow \quad t_2'}{(t_1 + t_2)\ t \quad \Rightarrow \quad t_2'}\ \text{App–lin–r}$$

$$\text{App–arrow}\ \frac{t_1 = t \quad \hookrightarrow \quad [\Xi] \qquad [\Xi]\ t_2 \quad \rightsquigarrow \quad t'}{(t_1 \mapsto t_2)\ t \quad \Rightarrow \quad t'} \qquad \frac{f\ t \quad \Rightarrow \quad t' \qquad g\ t' \quad \Rightarrow \quad t''}{(f \fatsemi g)\ t \quad \Rightarrow \quad t''}\ \text{App–comp}$$

$$\text{App–dagger}\ \frac{f^\dagger \quad \looparrowright \quad f' \qquad f'\ t \quad \Rightarrow \quad t'}{f^\dagger\ t \quad \Rightarrow \quad t'} \qquad \frac{[l, u]u\ t \quad \Rightarrow \quad t'}{(u\ \text{where}\ l :: U)\ t \quad \Rightarrow \quad t'}\ \text{App–trace}$$

$$\text{Flip–arrow}\ \frac{}{(t_1 \mapsto t_2)^\dagger \quad \looparrowright \quad t_2 \mapsto t_1} \qquad \frac{t_1^\dagger \quad \looparrowright \quad t_1' \qquad t_2^\dagger \quad \looparrowright \quad t_2'}{(t_1 + t_2)^\dagger \quad \looparrowright \quad t_1' + t_2'}\ \text{Flip–lin}$$

$$\text{Flip–app}\ \frac{f\ t \quad \Rightarrow \quad t' \qquad t'^\dagger \quad \looparrowright \quad t''}{(f\ \ t)^\dagger \quad \looparrowright \quad t''} \qquad \frac{f^\dagger \quad \looparrowright \quad f' \qquad g^\dagger \quad \looparrowright \quad g'}{(f \fatsemi g)^\dagger \quad \looparrowright \quad g' \fatsemi f'}\ \text{Flip–comp}$$

$$\text{Flip–trace}\ \frac{t^\dagger \quad \looparrowright \quad t'}{(t\ \text{where}\ l :: T)^\dagger \quad \looparrowright \quad t'\ \text{where}\ l :: T}$$

Type Interpretation

$\mathbf{V}$ is Compact Closed Category with Finite Biproduct.

Π_i is projection functor.

K_I is constant I functor.

$[-,-]$ is internal hom functor.

$(-)^*$ is contravariant anafunctor. Using Axiom of Choice, we can define it on strict functor (see `https://ncatlab.org/nlab/show/rigid+monoidal+category#remarks`).

Id is identity functor.

$$
\begin{aligned}
[\![\Theta \vdash T]\!] &: \mathbf{V}^{|\Theta|} \to \mathbf{V} \\
[\![\Theta \vdash X_i]\!] &= \Pi_i \\
[\![\Theta \vdash I]\!] &= K_I \\
[\![\Theta \vdash T_1 \oplus T_2]\!] &= \oplus \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \otimes T_2]\!] &= \otimes \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \multimap T_2]\!] &= \otimes \circ [(-)^*, Id] \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash \mu X.T]\!] &= [\![\Theta, X \vdash T]\!]^\sharp
\end{aligned}
$$

Denotational Semantics

$\mathbf{V}(A, B)$ is morphism which Domain and Codomain are $A \in \mathrm{Obj}(\mathbf{V})$ and $B \in \mathrm{Obj}(\mathbf{V})$, respectively.

id_T is identity morphism of Object T.

$0_{A,B}$ is zero morphism of $\mathbf{V}(A, B)$.

ι_i is injection morphism.

π_i is projection morphism.

$\pi_i = \iota_i^{-1}$

$\pi_i \circ \iota_j = id$ if $i = j$

$\pi_i \circ \iota_j = 0$ otherwise

$unfold_{\mu X.T} = fold_{\mu X.T}^{-1}$

$f \circ (g \oplus h) = (f \circ g) \oplus (f \circ h)$

$(f \oplus g) \circ h = (f \circ h) \oplus (g \circ h)$

$id_A \otimes id_B = id_{A \otimes B}$

$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)$

$f^{**} = f$

$(f \otimes g)^* = f^* \otimes g^*$

$\Delta_T : T \to T \oplus T$

$\nabla_T : T \oplus T \to T$

$$
\begin{aligned}
[\![t : T]\!] &\in \mathbf{V}(I, T) \\
[\![x : T]\!] &:= x_T \\
[\![() : T]\!] &:= id_I \\
[\![\mathrm{inl}\ t_1 : T_1 \oplus T_2]\!] &:= \iota_1 \circ [\![t_1 : T_1]\!] \\
[\![\mathrm{inr}\ t_2 : T_1 \oplus T_2]\!] &:= \iota_2 \circ [\![t_2 : T_2]\!] \\
[\![t_1 \times t_2 : T_1 \otimes T_2]\!] &:= [\![t_1 : T_1]\!] \otimes [\![t_2 : T_2]\!] \\
[\![t_1 \mapsto t_2 : T_1 \multimap T_2]\!] &:= [\![t_1 : T_1]\!]^* \otimes [\![t_2 : T_2]\!] \\
[\![\mathrm{fold}\ t : \mu X.T]\!] &:= fold_{\mu X.T} \circ [\![t : T[X / \mu X.T]]\!] \\
[\![t_1 + t_2 : T]\!] &:= \nabla_T \circ [\![t_1 : T]\!] \oplus [\![t_2 : T]\!] \circ \Delta_T
\end{aligned}
$$

Term Typing rules

$$\varnothing \vdash t : T \triangleright \varnothing$$

$$\text{Variable}\ \frac{\Gamma(x) = T}{\Gamma \vdash x : T \triangleright \Gamma \setminus \{x\}}$$

$$I_L\ \frac{\Gamma \vdash t : T \triangleright \Gamma'}{\Gamma, () : I \vdash t : T \triangleright \Gamma'} \qquad \frac{}{\varnothing \vdash () : I \triangleright \varnothing}\ I_R$$

$$\oplus_{L_l}\ \frac{\Gamma, t_1 : T_1 \vdash t : T \triangleright \Gamma'}{\Gamma, \text{inl } t_1 : T_1 \oplus T_2 \vdash t : T \triangleright \Gamma'} \qquad \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma'}{\Gamma \vdash \text{inl } t_1 : T_1 \oplus T_2 \triangleright \Gamma'}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \frac{\Gamma, t_2 : T_2 \vdash t : T \triangleright \Gamma'}{\Gamma, \text{inr } t_2 : T_1 \oplus T_2 \vdash t : T \triangleright \Gamma'} \qquad \frac{\Gamma \vdash t_2 : T_2 \triangleright \Gamma'}{\Gamma \vdash \text{inr } t_2 : T_1 \oplus T_2 \triangleright \Gamma'}\ \oplus_{R_r}$$

$$\otimes_L\ \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T \triangleright \Gamma'}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T \triangleright \Gamma'} \qquad \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma' \qquad \Gamma' \vdash t_2 : T_2 \triangleright \Gamma''}{\Gamma \vdash t_1 \times t_2 : T_1 \otimes T_2 \triangleright \Gamma''}\ \otimes_R$$

$$\multimap_L\ \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma' \qquad \Gamma', t_2 : T_2 \vdash t : T \triangleright \Gamma''}{\Gamma, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T \triangleright \Gamma''} \qquad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2 \triangleright \Gamma'}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2 \triangleright \Gamma'}\ \multimap_R$$

$$\mu_L\ \frac{\Gamma, u : U[X / \mu X.U] \vdash t : T \triangleright \Gamma'}{\Gamma, \text{fold } u : \mu X.U \vdash t : T \triangleright \Gamma'} \qquad \frac{\Gamma \vdash t : T[X / \mu X.T] \triangleright \Gamma'}{\Gamma \vdash \text{fold } t : \mu X.T \triangleright \Gamma'}\ \mu_R$$

$$\text{Linearity}_L\ \frac{\Gamma, t_1 : U \vdash t : T \triangleright \Gamma' \qquad \Gamma, t_2 : U \vdash t : T \triangleright \Gamma'}{\Gamma, t_1 + t_2 : U \vdash t : T \triangleright \Gamma'} \qquad \frac{\Gamma \vdash t_1 : T \triangleright \Gamma' \qquad \Gamma \vdash t_2 : T \triangleright \Gamma'}{\Gamma \vdash t_1 + t_2 : T \triangleright \Gamma'}\ \text{Linearity}_R$$

$$\text{Trace}_L\ \frac{[l : S]\Gamma, u : S \vdash t : T \triangleright \Gamma'}{[l : S]\Gamma, l\ u : U \vdash t : T \triangleright \Gamma'} \qquad \frac{[l : U]\Gamma \vdash t : U \triangleright \Gamma'}{[l : U]\Gamma \vdash l\ t : T \triangleright \Gamma'}\ \text{Trace}_R$$