

## Syntax

Type Variables	$X, Y, Z$
Term Variables	$x, y, z$
Types	$T, S, U ::= X$ $  I$ $  T \oplus T$ $  T \otimes T$ $  T \multimap T$ $  \mu X. T$
Terms	$t, u, v ::= x$ $  ()$ $  \text{inl } t$ $  \text{inr } t$ $  t \times t$ $  t \mapsto t$ $  \text{fold } t$ $  \text{trace } t$ $  t + t$ $  t \mathbin{\text{;}} t$ $  t^\dagger$ $  \emptyset$ $  \text{id}$
Type Contexts	$\Theta ::=$ $  \Theta, X$
Term Contexts	$\Gamma ::=$ $  \Gamma, t : T$
Type Judgements	$\Theta \vdash T$
Term Judgements	$\Gamma \vdash t : T$
Expressions	$e, f, g ::= t$ $  t @ t$
Expr judgement	$::= \vdash e : T$
Var Environment	$\Xi ::= \{ \}$ $\Xi, \{x \rightarrow t\}$

## Type Formation rules

$$\frac{}{\Theta, X \vdash X} \quad \frac{}{\Theta \vdash I} \quad \frac{\Theta \vdash T_1 \quad \Theta \vdash T_2}{\Theta \vdash T_1 \oplus T_2 :} \quad \frac{\Theta \vdash T_1 \quad \Theta \vdash T_2}{\Theta \vdash T_1 \otimes T_2 :} \quad \frac{\Theta \vdash T_1 \quad \Theta \vdash T_2}{\Theta \vdash T_1 \multimap T_2} \quad \frac{\Theta, X \vdash T}{\Theta \vdash \mu X. T}$$

## Type Substitution

$$\begin{aligned}
X[X \rightarrow S] &= S \\
Y[X \rightarrow S] &= Y \\
I[X \rightarrow S] &= I \\
T_1 \oplus T_2[X \rightarrow S] &= T_1[X \rightarrow S] \oplus T_2[X \rightarrow S] \\
T_1 \otimes T_2[X \rightarrow S] &= T_1[X \rightarrow S] \otimes T_2[X \rightarrow S] \\
T_1 \multimap T_2[X \rightarrow S] &= T_1[X \rightarrow S] \multimap T_2[X \rightarrow S] \\
\mu Y. T[X \rightarrow S] &= \mu Y. (T[X \rightarrow S])
\end{aligned}$$

$$\begin{array}{c}
\text{Variable} \frac{}{x : T \vdash x : T} \quad \frac{\Gamma_2, \Gamma_1 \vdash t : T}{\Gamma_1, \Gamma_2 \vdash t : T} \text{Exchange} \\
\\
I_L \frac{\Gamma \vdash t : T}{\Gamma, () : I \vdash t : T} \quad \frac{}{\vdash () : I} I_R \\
\\
\oplus_{L_l} \frac{\Gamma, t_1 : T_1 \vdash t : T}{\Gamma, \text{inl } t_1 : T_1 \oplus T_2 \vdash t : T} \quad \frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl } t_1 : T_1 \oplus T_2} \oplus_{R_l} \\
\\
\oplus_{L_r} \frac{\Gamma, t_2 : T_2 \vdash t : T}{\Gamma, \text{inr } t_2 : T_1 \oplus T_2 \vdash t : T} \quad \frac{\Gamma \vdash t_2 : T_2}{\Gamma \vdash \text{inr } t_2 : T_1 \oplus T_2} \oplus_{R_r} \\
\\
\otimes_L \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T} \quad \frac{\Gamma_1 \vdash t_1 : T_1 \quad \Gamma_2 \vdash t_2 : T_2}{\Gamma_1, \Gamma_2 \vdash t_1 \times t_2 : T_1 \otimes T_2} \otimes_R \\
\\
\multimap_L \frac{\Gamma_1 \vdash t_1 : T_1 \quad \Gamma_2, t_2 : T_2 \vdash t : T}{\Gamma_1, \Gamma_2, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T} \quad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2} \multimap_R \\
\\
\mu_L \frac{\Gamma, u : U[X \rightarrow \mu X.U] \vdash t : T}{\Gamma, \text{fold } u : \mu X.U \vdash t : T} \quad \frac{\Gamma \vdash t : T[X \rightarrow \mu X.T]}{\Gamma \vdash \text{fold } t : \mu X.T} \mu_R \\
\\
\text{Trace}_L \frac{\Gamma, u : S \oplus U_1 \multimap S \oplus U_2 \vdash t : T}{\Gamma, \text{trace } u : U_1 \multimap U_2 \vdash t : T} \quad \frac{\Gamma \vdash t : U \oplus T_1 \multimap U \oplus T_2}{\Gamma \vdash \text{trace } t : T_1 \multimap T_2} \text{Trace}_R \\
\\
\text{Linearity}_L \frac{\Gamma, t_1 : U \vdash t : T \quad \Gamma, t_2 : U \vdash t : T}{\Gamma, t_1 + t_2 : U \vdash t : T} \quad \frac{\Gamma \vdash t_1 : T \quad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1 + t_2 : T} \text{Linearity}_R \\
\\
\text{Composition}_L \frac{\Gamma, t_1 : T_1 \multimap T_2, t_2 : T_2 \multimap T_3 \vdash t : T}{\Gamma, t_1 \circ t_2 : T_1 \multimap T_3 \vdash t : T} \quad \frac{\Gamma_1 \vdash t_1 : T_1 \multimap T_2 \quad \Gamma_2 \vdash t_2 : T_2 \multimap T_3}{\Gamma_1, \Gamma_2 \vdash t_1 \circ t_2 : T_1 \multimap T_3} \text{Composition}_R \\
\\
\dagger_L \frac{\Gamma, u : T_2 \multimap T_1 \vdash t : T}{\Gamma, u^\dagger : T_1 \multimap T_2 \vdash t : T} \quad \frac{\Gamma \vdash t : T_2 \multimap T_1}{\Gamma \vdash t^\dagger : T_1 \multimap T_2} \dagger_R \\
\\
\text{id}_L \frac{\Gamma \vdash t : T}{\Gamma, \text{id} : U \multimap U \vdash t : T} \quad \frac{}{\vdash \text{id} : T \multimap T} \text{id}_R \\
\\
\text{Application} \frac{\vdash t : T_1 \multimap T_2 \quad \vdash t_1 : T_1}{\vdash t @ t_1 : T_2}
\end{array}$$

$$\vdash t : T \triangleright$$

$$\text{Variable} \frac{\Gamma(x) = T}{\Gamma \vdash x : T \triangleright \Gamma \setminus x}$$

$$I_L \frac{\Gamma \vdash t : T \triangleright \Gamma'}{\Gamma, () : I \vdash t : T \triangleright \Gamma'} \quad \frac{}{\vdash () : I \triangleright} I_R$$

$$\oplus_{L_l} \frac{\Gamma, t_1 : T_1 \vdash t : T \triangleright \Gamma'}{\Gamma, \text{inl } t_1 : T_1 \oplus T_2 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma'}{\Gamma \vdash \text{inl } t_1 : T_1 \oplus T_2 \triangleright \Gamma'} \oplus_{R_l}$$

$$\oplus_{L_r} \frac{\Gamma, t_2 : T_2 \vdash t : T \triangleright \Gamma'}{\Gamma, \text{inr } t_2 : T_1 \oplus T_2 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t_2 : T_2 \triangleright \Gamma'}{\Gamma \vdash \text{inr } t_2 : T_1 \oplus T_2 \triangleright \Gamma'} \oplus_{R_r}$$

$$\otimes_L \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T \triangleright \Gamma'}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma' \quad \Gamma' \vdash t_2 : T_2 \triangleright \Gamma''}{\Gamma \vdash t_1 \times t_2 : T_1 \otimes T_2 \triangleright \Gamma''} \otimes_R$$

$$\multimap_L \frac{\Gamma \vdash t_1 : T_1 \triangleright \Gamma' \quad \Gamma', t_2 : T_2 \vdash t : T \triangleright \Gamma''}{\Gamma, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T \triangleright \Gamma''} \quad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2 \triangleright \Gamma'}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2 \triangleright \Gamma'} \multimap_R$$

$$\mu_L \frac{\Gamma, u : U[X \rightarrow \mu X.U] \vdash t : T \triangleright \Gamma'}{\Gamma, \text{fold } u : \mu X.U \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t : T[X \rightarrow \mu X.T] \triangleright \Gamma'}{\Gamma \vdash \text{fold } t : \mu X.T \triangleright \Gamma'} \mu_R$$

$$\text{Trace}_L \frac{\Gamma, u : S \oplus U_1 \multimap S \oplus U_2 \vdash t : T \triangleright \Gamma'}{\Gamma, \text{trace } u : U_1 \multimap U_2 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t : U \oplus T_1 \multimap U \oplus T_2 \triangleright \Gamma'}{\Gamma \vdash \text{trace } t : T_1 \multimap T_2 \triangleright \Gamma'} \text{Trace}_R$$

$$\text{Linearity}_L \frac{\Gamma, t_1 : U \vdash t : T \triangleright \Gamma' \quad \Gamma, t_2 : U \vdash t : T \triangleright \Gamma'}{\Gamma, t_1 + t_2 : U \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t_1 : T \triangleright \Gamma' \quad \Gamma \vdash t_2 : T \triangleright \Gamma'}{\Gamma \vdash t_1 + t_2 : T \triangleright \Gamma'} \text{Linearity}_R$$

$$\text{Comp}_L \frac{\Gamma, t_1 : T_1 \multimap T_2, t_2 : T_2 \multimap T_3 \vdash t : T \triangleright \Gamma'}{\Gamma, t_1 \circ t_2 : T_1 \multimap T_3 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t_1 : T_1 \multimap T_2 \triangleright \Gamma' \quad \Gamma' \vdash t_2 : T_2 \multimap T_3 \triangleright \Gamma''}{\Gamma \vdash t_1 \circ t_2 : T_1 \multimap T_3 \triangleright \Gamma''} \text{Comp}_R$$

$$\dagger_L \frac{\Gamma, u : T_2 \multimap T_1 \vdash t : T \triangleright \Gamma'}{\Gamma, u^\dagger : T_1 \multimap T_2 \vdash t : T \triangleright \Gamma'} \quad \frac{\Gamma \vdash t : T_2 \multimap T_1 \triangleright \Gamma'}{\Gamma \vdash t^\dagger : T_1 \multimap T_2 \triangleright \Gamma'} \dagger_R$$

$$\text{id}_L \frac{\Gamma \vdash t : T \triangleright \Gamma'}{\Gamma, \text{id} : U \multimap U \vdash t : T \triangleright \Gamma'} \quad \frac{}{\vdash \text{id} : T \multimap T \triangleright} \text{id}_R$$

$$\text{Application} \frac{\vdash t : T_1 \multimap T_2 \triangleright \quad \vdash t_1 : T_1 \triangleright}{\vdash t @ t_1 : T_2 \triangleright}$$

$$\begin{array}{c}
 \vdash t : X \triangleright \mid C \\
 \\
 \text{Variable} \frac{\Gamma(x) = T}{\Gamma \vdash x : T \triangleright \Gamma \setminus (x : T) \mid \{}} \\
 \\
 I_L \frac{\Gamma \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, () : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = I\}} \quad \frac{}{\vdash () : V \triangleright \mid \{V = I\}} I_R \\
 \\
 \oplus_{L_l} \frac{\Gamma, t_1 : X_1 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, \text{inl } t_1 : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \oplus X_2\}} \quad \frac{\Gamma \vdash t_1 : X_1 \triangleright \Gamma' \mid C}{\Gamma \vdash \text{inl } t_1 : V \triangleright \Gamma' \mid C \cup \{V = X_1 \oplus X_2\}} \oplus_{R_l} \\
 \\
 \oplus_{L_r} \frac{\Gamma, t_2 : X_2 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, \text{inr } t_2 : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \oplus X_2\}} \quad \frac{\Gamma \vdash t_2 : X_2 \triangleright \Gamma' \mid C}{\Gamma \vdash \text{inr } t_2 : V \triangleright \Gamma' \mid C \cup \{X_1 \oplus X_2\}} \oplus_{R_r} \\
 \\
 \otimes_L \frac{\Gamma, t_1 : X_1, t_2 : X_2 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, t_1 \times t_2 : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \otimes X_2\}} \quad \frac{\Gamma \vdash t_1 : X_1 \triangleright \Gamma' \mid C_1 \quad \Gamma' \vdash t_2 : X_2 \triangleright \Gamma'' \mid C_2}{\Gamma \vdash t_1 \times t_2 : V \triangleright \Gamma'' \mid C_1 \cup C_2 \cup \{V = X_1 \otimes X_2\}} \otimes_R \\
 \\
 \multimap_L \frac{\Gamma \vdash t_1 : X_1 \triangleright \Gamma' \mid C_1 \quad \Gamma', t_2 : X_2 \vdash t : T \triangleright \Gamma'' \mid C_2}{\Gamma, t_1 \mapsto t_2 : V \vdash t : T \triangleright \Gamma'' \mid C_1 \cup C_2 \cup \{V = X_1 \multimap X_2\}} \quad \frac{\Gamma, t_1 : X_1 \vdash t_2 : X_2 \triangleright \Gamma' \mid C}{\Gamma \vdash t_1 \mapsto t_2 : V \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_2\}} \multimap_R \\
 \\
 \mu_L \frac{\Gamma, u : U[Y \rightarrow \mu Y.U] \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, \text{fold}_{\mu Y.U} u : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = \mu Y.U\}} \quad \frac{\Gamma \vdash t : T[X \rightarrow \mu X.T] \triangleright \Gamma' \mid C}{\Gamma \vdash \text{fold}_{\mu X.T} t : V \triangleright \Gamma' \mid C \cup \{V = \mu X.T\}} \mu_R \\
 \\
 \text{Trace}_L \frac{\Gamma, u : U \oplus X_1 \multimap U \oplus X_2 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, \text{trace}_U u : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_2\}} \quad \frac{\Gamma \vdash t : T \oplus X_1 \multimap T \oplus X_2 \triangleright \Gamma' \mid C}{\Gamma \vdash \text{trace}_T t : V \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_2\}} \text{Trace}_R \\
 \\
 \text{Lin}_L \frac{\Gamma, t_1 : X \vdash t : T \triangleright \Gamma' \mid C_1 \quad \Gamma, t_2 : X \vdash t : T \triangleright \Gamma' \mid C_2}{\Gamma, t_1 + t_2 : V \vdash t : T \triangleright \Gamma' \mid C_1 \cup C_2 \cup \{V = X\}} \quad \frac{\Gamma \vdash t_1 : X \triangleright \Gamma' \mid C_1 \quad \Gamma \vdash t_2 : X \triangleright \Gamma' \mid C_2}{\Gamma \vdash t_1 + t_2 : V \triangleright \Gamma' \mid C_1 \cup C_2 \cup \{V = X\}} \text{Lin}_R \\
 \\
 \text{Comp}_L \frac{\Gamma, t_1 : X_1 \multimap X_2, t_2 : X_2 \multimap X_3 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, t_1 \circ t_2 : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_3\}} \\
 \frac{\Gamma \vdash t_1 : X_1 \multimap X_2 \triangleright \Gamma' \mid C_1 \quad \Gamma' \vdash t_2 : X_2' \multimap X_3 \triangleright \Gamma'' \mid C_2}{\Gamma \vdash t_1 \circ t_2 : V \triangleright \Gamma'' \mid C_1 \cup C_2 \cup \{V = X_1 \multimap X_3\}} \text{Comp}_R \\
 \\
 \dagger_L \frac{\Gamma, u : X_2 \multimap X_1 \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, u^\dagger : \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_2\}} \quad \frac{\Gamma \vdash t : X_2 \multimap X_1 \triangleright \Gamma' \mid C}{\Gamma \vdash t^\dagger : V \triangleright \Gamma' \mid C \cup \{V = X_1 \multimap X_2\}} \dagger_R \\
 \\
 \text{id}_L \frac{\Gamma \vdash t : T \triangleright \Gamma' \mid C}{\Gamma, \text{id} : V \vdash t : T \triangleright \Gamma' \mid C \cup \{V = X \multimap X\}} \quad \frac{}{\vdash \text{id} : V \triangleright \mid \{V = X \multimap X\}} \text{id}_R \\
 \\
 \text{Application} \frac{\vdash t : X_1 \multimap X_2 \triangleright \mid C_1 \quad \vdash t_1 : X_1 \triangleright \mid C_2}{\vdash t @ t_1 : V \triangleright \mid C_1 \cup C_2 \cup \{V = X_2\}}
 \end{array}$$

## Unification

$$\begin{array}{l}
 \text{unify} := \text{Set of Constraint} \rightarrow \text{Substitution} \\
 \text{unify}(\{\}) = [] \\
 \text{unify}(\{X = T\} \cup C) = \text{unify}([X \rightarrow T]C) \circ [X \rightarrow T] \\
 \text{unify}(\{T = X\} \cup C) = \text{unify}([X \rightarrow T]C) \circ [X \rightarrow T] \\
 \text{unify}(\{S_1 \oplus S_2 = T_1 \oplus T_2\} \cup C) = \text{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
 \text{unify}(\{S_1 \otimes S_2 = T_1 \otimes T_2\} \cup C) = \text{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
 \text{unify}(\{S_1 \multimap S_2 = T_1 \multimap T_2\} \cup C) = \text{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
 \text{unify}(\{\mu X.S = \mu Y.T\} \cup C) = \text{unify}(C \cup \{X = Y, S = T\})
 \end{array}$$

$|\Theta|$  はコンテキスト  $\Theta$  に含まれる型変数の数

圏  $\mathbf{V}$  はトレース双積付きダガーコンパクト圏 (Dagger Compact Category with Traced Finite Biproduct)

- $F : \mathbf{V}^n \rightarrow \mathbf{V}$  は  $n$  多重関手
- $\Pi_i : \mathbf{V}^{|\Theta|} \rightarrow \mathbf{V}$  は射影関手
- $K_I : \mathbf{V}^{|\Theta|} \rightarrow \mathbf{V}$  は定数  $I$  関手
- $\otimes : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$  はテンソル積関手
- $\oplus : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$  は双積関手
- $(-)^{\star} : \mathbf{V}^{op} \rightarrow \mathbf{V}$  は充満忠実自己反変関手
- $[-, -] : \mathbf{V}^{op} \times \mathbf{V} \rightarrow \mathbf{V}$  は内部ホム関手
- $Id_{\mathbf{V}}$  は恒等関手
- 圏  $\mathbf{V}$  と関手  $F : \mathbf{V}^n \rightarrow \mathbf{V}$  ( $n \geq 1$ ) について, パラメトライズされた  $F$  の始代数は, 以下を満たす組  $(F^{\sharp}, \phi^F)$ 
  - $F^{\sharp} : \mathbf{V}^{n-1} \rightarrow \mathbf{V}$  は関手
  - $\phi^F : F \circ \langle Id, F^{\sharp} \rangle \Rightarrow F^{\sharp} : \mathbf{V}^{n-1} \rightarrow \mathbf{V}$  は自然同型
  - 全ての  $T \in |\mathbf{V}^{n-1}|$  について, 組  $(F^{\sharp}(T), \phi_T^F)$  は  $F(T, -)$ - 始代数

$$\begin{aligned}
 \llbracket \Theta \vdash T \rrbracket &: \mathbf{V}^{|\Theta|} \rightarrow \mathbf{V} \\
 \llbracket \Theta \vdash X_i \rrbracket &= \Pi_i \\
 \llbracket \Theta \vdash I \rrbracket &= K_I \\
 \llbracket \Theta \vdash T_1 \oplus T_2 \rrbracket &= \oplus \circ \langle \llbracket \Theta \vdash T_1 \rrbracket, \llbracket \Theta \vdash T_2 \rrbracket \rangle \\
 \llbracket \Theta \vdash T_1 \otimes T_2 \rrbracket &= \otimes \circ \langle \llbracket \Theta \vdash T_1 \rrbracket, \llbracket \Theta \vdash T_2 \rrbracket \rangle \\
 \llbracket \Theta \vdash T_1 \multimap T_2 \rrbracket &= \otimes \circ \langle \llbracket \Theta \vdash T_1 \rrbracket^{\star}, \llbracket \Theta \vdash T_2 \rrbracket \rangle \\
 \llbracket \Theta \vdash \mu X. T \rrbracket &= \llbracket \Theta, X \vdash T \rrbracket^{\sharp} \\
 \llbracket \Theta \vdash T[X \rightarrow U] \rrbracket &= \llbracket \Theta \vdash T \rrbracket \circ \langle Id, \llbracket \Theta \vdash U \rrbracket \rangle \\
 \llbracket T \rrbracket &:= \llbracket \vdash T \rrbracket(*) \in |\mathbf{V}|
 \end{aligned}$$

$*$  は, スモール圏の圏  $\mathbf{Cat}$  における終対象  $\mathbf{1}$  の唯一の対象

— Denotational Semantics —

$$\begin{aligned}
 \llbracket t : T \rrbracket &\in \llbracket T \rrbracket \\
 \llbracket id : T \rrbracket &= 1 \\
 \llbracket \emptyset : T \rrbracket &= 0 \\
 \llbracket x : T \rrbracket &= x \\
 \llbracket () : I \rrbracket &= * \\
 \llbracket inl\ t_1 : T_1 \oplus T_2 \rrbracket &= \iota_1(\llbracket t_1 : T_1 \rrbracket) \\
 \llbracket inr\ t_2 : T_1 \oplus T_2 \rrbracket &= \iota_2(\llbracket t_2 : T_2 \rrbracket) \\
 \llbracket t_1 \times t_2 : T_1 \otimes T_2 \rrbracket &= \llbracket t_1 : T_1 \rrbracket \otimes \llbracket t_2 : T_2 \rrbracket \\
 \llbracket t_1 \multimap t_2 : T_1 \multimap T_2 \rrbracket &= \llbracket t_1 : T_1 \rrbracket^{\star} \otimes \llbracket t_2 : T_2 \rrbracket \\
 \llbracket fold\ t : \mu X. T \rrbracket &= fold_{\mu X. T}(\llbracket t : T[X/\mu X. T] \rrbracket) \\
 \llbracket t_1 + t_2 : T \rrbracket &= \frac{1}{\sqrt{2}}(\llbracket t_1 : T \rrbracket + \llbracket t_2 : T \rrbracket) \\
 \llbracket t_1 \circ t_2 : T_1 \multimap T_3 \rrbracket &= \llbracket t_2 : T_2 \multimap T_3 \rrbracket \circ \llbracket t_1 : T_1 \multimap T_2 \rrbracket \\
 \llbracket trace_U\ t : T_1 \multimap T_2 \rrbracket &= Tr_{T_1, T_2}^U(\llbracket t : (U \oplus T_1) \multimap (U \oplus T_2) \rrbracket) \\
 \llbracket t^{\dagger} : T_1 \multimap T_2 \rrbracket &= \llbracket t : T_2 \multimap T_1 \rrbracket^{-1} \\
 \llbracket t @ t_1 : T_2 \rrbracket &= \llbracket t : T_1 \multimap T_2 \rrbracket(\llbracket t_1 : T_1 \rrbracket)
 \end{aligned}$$

— partial order —

$$\emptyset < x < () < inl\ t < inr\ t < t_1 \times t_2 < t_1 \multimap t_2 < fold\ t < trace\ t < t_1 + t_2$$

$$\begin{array}{c}
 \frac{}{x ; t \rightarrow \{x \rightarrow t\}} \quad \frac{t \in \Xi(x)}{\Xi \triangleright x \rightsquigarrow t \triangleright \Xi \setminus \{x \rightarrow t\}} \quad \frac{x \notin \text{Dom}(\Xi)}{\Xi \triangleright x \rightsquigarrow \emptyset \triangleright \Xi} \\
 \\
 \frac{}{() ; () \rightarrow \{\}} \quad \frac{}{\Xi \triangleright () \rightsquigarrow () \triangleright \Xi} \\
 \\
 \frac{t ; u \rightarrow \Xi}{\text{inl } t ; \text{inl } u \rightarrow \Xi} \quad \frac{}{\text{inl } t ; \text{inr } u \rightarrow \perp} \quad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \text{inl } t \rightsquigarrow \text{inl } t' \triangleright \Xi'} \\
 \\
 \frac{t ; u \rightarrow \Xi}{\text{inr } t ; \text{inr } u \rightarrow \Xi} \quad \frac{}{\text{inr } t ; \text{inl } u \rightarrow \perp} \quad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \text{inr } t \rightsquigarrow \text{inr } t' \triangleright \Xi'} \\
 \\
 \frac{t_1 ; u_1 \rightarrow \Xi_1 \quad t_2 ; u_2 \rightarrow \Xi_2}{t_1 \times t_2 ; u_1 \times u_2 \rightarrow \Xi_1 \cup_{\perp}^{\times} \Xi_2} \quad \frac{\Xi \triangleright t_1 \rightsquigarrow t'_1 \triangleright \Xi' \quad \Xi' \triangleright t_2 \rightsquigarrow t'_2 \triangleright \Xi''}{\Xi \triangleright t_1 \times t_2 \rightsquigarrow t'_1 \times t'_2 \triangleright \Xi''} \\
 \\
 \frac{t_1 ; u_1 \rightarrow \Xi_1 \quad t_2 ; u_2 \rightarrow \Xi_2}{t_1 \mapsto t_2 ; u_1 \mapsto u_2 \rightarrow \Xi_1 \cup_{\perp}^{\times} \Xi_2} \quad \frac{\Xi \triangleright t_1 \rightsquigarrow t'_1 \triangleright \Xi' \quad \Xi' \triangleright t_2 \rightsquigarrow t'_2 \triangleright \Xi''}{\Xi \triangleright t_1 \mapsto t_2 \rightsquigarrow t'_1 \mapsto t'_2 \triangleright \Xi''} \\
 \\
 \frac{t ; u \rightarrow \Xi}{\text{fold } t ; \text{fold } u \rightarrow \Xi} \quad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \text{fold } t \rightsquigarrow \text{fold } t' \triangleright \Xi'} \\
 \\
 \frac{t ; u \rightarrow \Xi}{\text{trace } t ; \text{trace } u \rightarrow \Xi} \quad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \text{trace } t \rightsquigarrow \text{trace } t' \triangleright \Xi'} \\
 \\
 \frac{t ; u_1 \rightarrow \Xi_1 \quad t ; u_2 \rightarrow \Xi_2}{t ; u_1 + u_2 \rightarrow \Xi_1 \cup_{\perp}^{+} \Xi_2} \quad \frac{t_1 ; u \rightarrow \Xi_1 \quad t_2 ; u \rightarrow \Xi_2}{t_1 + t_2 ; u \rightarrow \Xi_1 \cup_{\perp}^{+} \Xi_2} \\
 \frac{\Xi \triangleright t_1 \rightsquigarrow t'_1 \triangleright \Xi' \quad \Xi' \triangleright t_2 \rightsquigarrow t'_2 \triangleright \Xi''}{\Xi \triangleright t_1 + t_2 \rightsquigarrow t'_1 + t'_2 \triangleright \Xi''} \\
 \\
 \frac{t_1 ; u_1 \rightarrow \Xi_1 \quad t_2 ; u_2 \rightarrow \Xi_2}{t_1 \circ t_2 ; u_1 \circ u_2 \rightarrow \Xi_1 \cup_{\perp}^{\times} \Xi_2} \quad \frac{\Xi \triangleright t_1 \rightsquigarrow t'_1 \triangleright \Xi' \quad \Xi' \triangleright t_2 \rightsquigarrow t'_2 \triangleright \Xi''}{\Xi \triangleright t_1 \circ t_2 \rightsquigarrow t'_1 \circ t'_2 \triangleright \Xi''} \\
 \\
 \frac{}{\text{id} ; \text{id} \rightarrow \{\}} \quad \frac{}{\Xi \triangleright \text{id} \rightsquigarrow \text{id} \triangleright \Xi} \\
 \\
 \frac{}{\emptyset ; \emptyset \rightarrow \{\}} \quad \frac{}{\Xi \triangleright \emptyset \rightsquigarrow \emptyset \triangleright \Xi} \\
 \\
 \frac{t ; u \rightarrow \Xi}{t^{\dagger} ; u^{\dagger} \rightarrow \Xi} \quad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright t^{\dagger} \rightsquigarrow t'^{\dagger} \triangleright \Xi'}
 \end{array}$$

$$\begin{array}{c}
 \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{(t_1 + t_2) + t_3 \equiv t'_1 + (t'_2 + t'_3)} \quad \frac{t \equiv t'}{\emptyset + t \equiv t'} \quad \frac{}{t_1 + t_2 \equiv t_2 + t_1} \quad \frac{}{t + t \equiv t} \\
 \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{(t_1 \circ t_2) \circ t_3 \equiv t'_1 \circ (t'_2 \circ t'_3)} \quad \frac{t \equiv t'}{\text{id} \circ t \equiv t'} \quad \frac{t \equiv t'}{t \circ \text{id} \equiv t'} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 \circ t_2 \equiv t'_1 \circ t'_2} \\
 \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{t_1 \circ (t_2 + t_3) \equiv t'_1 \circ t'_2 + t'_1 \circ t'_3} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{(t_1 + t_2) \circ t_3 \equiv (t'_1 \circ t'_3) + (t'_2 \circ t'_3)} \quad \frac{}{\emptyset \circ t \equiv \emptyset} \quad \frac{}{t \circ \emptyset \equiv \emptyset}
 \end{array}$$

$$\frac{}{\text{inl } \emptyset \equiv \emptyset} \quad \frac{}{\text{inr } \emptyset \equiv \emptyset} \quad \frac{}{t_1 \times \emptyset \equiv \emptyset} \quad \frac{}{\emptyset \times t_2 \equiv \emptyset} \quad \frac{}{t_1 \mapsto \emptyset \equiv \emptyset} \quad \frac{}{\emptyset \mapsto t_2 \equiv \emptyset} \quad \frac{}{\text{fold } \emptyset \equiv \emptyset} \quad \frac{}{\text{trace } \emptyset \equiv \emptyset}$$

$$\frac{}{\emptyset \equiv \emptyset} \quad \frac{}{\text{id} \equiv \text{id}}$$

$$\frac{}{() \equiv ()} \quad \frac{}{x \equiv x}$$

$$\frac{t \equiv t'}{\text{inl } t \equiv \text{inl } t'} \quad \frac{t \equiv t'}{\text{inr } t \equiv \text{inr } t'} \\
 \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 \times t_2 \equiv t'_1 \times t'_2} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 \mapsto t_2 \equiv t'_1 \mapsto t'_2}$$

$$\frac{t \equiv t'}{\text{fold } t \equiv \text{fold } t'} \quad \frac{t \equiv t'}{\text{trace } t \equiv \text{trace } t'}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{\text{inl } (t_1 + t_2) \equiv \text{inl } t'_1 + \text{inl } t'_2} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{\text{inr } (t_1 + t_2) \equiv \text{inr } t'_1 + \text{inr } t'_2}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{(t_1 + t_2) \times t_3 \equiv (t'_1 \times t'_3) + (t'_2 \times t'_3)} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{t_1 \times (t_2 + t_3) \equiv (t'_1 \times t'_2) + (t'_1 \times t'_3)}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{(t_1 + t_2) \mapsto t_3 \equiv (t'_1 \mapsto t'_3) + (t'_2 \mapsto t'_3)} \quad \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t_3 \equiv t'_3}{t_1 \mapsto (t_2 + t_3) \equiv (t'_1 \mapsto t'_2) + (t'_1 \mapsto t'_3)}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{\text{fold } (t_1 + t_2) \equiv \text{fold } t'_1 + \text{fold } t'_2}$$

$$\frac{t_1^\dagger \equiv t'_1 \quad t_2^\dagger \equiv t'_2}{(t_1 + t_2)^\dagger \equiv t'_1 + t'_2} \quad \frac{}{\emptyset^\dagger \equiv \emptyset} \quad \frac{t_1^\dagger \equiv t'_1 \quad t_2^\dagger \equiv t'_2}{(t_1 \circ t_2)^\dagger \equiv t'_2 \circ t'_1} \quad \frac{}{\text{id}^\dagger \equiv \text{id}} \quad \frac{t \equiv t'}{(t^\dagger)^\dagger \equiv t'}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{(t_1 \mapsto t_2)^\dagger \equiv t'_2 \mapsto t'_1} \quad \frac{t^\dagger \equiv t'}{(\text{trace } t)^\dagger \equiv \text{trace } t'}$$

$$\frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2 \quad t \equiv t' \quad t'_1 ; t' \rightarrow \Xi \quad \Xi \triangleright t'_2 \rightsquigarrow t'' \triangleright}{(t_1 \mapsto t_2) @ t \equiv t''}$$

$$\frac{}{\emptyset @ t \equiv \emptyset} \quad \frac{t \equiv t'}{\text{id} @ t \equiv t'}$$

$$\frac{t_1 @ t \equiv t'_1 \quad t_2 @ t \equiv t'_2}{(t_1 + t_2) @ t \equiv t'_1 + t'_2} \quad \frac{t @ t_1 \equiv t'_1 \quad t @ t_2 \equiv t'_2}{t @ (t_1 + t_2) \equiv t'_1 + t'_2}$$

$$\frac{t_1 @ t \equiv t' \quad t_2 @ t' \equiv t''}{(t_1 \circ t_2) @ t \equiv t''} \quad \frac{t^\dagger \equiv t' \quad t' @ u \equiv u'}{t^\dagger @ u \equiv u'}$$

$$\frac{t @ (\text{inr } u) \equiv \text{inr } u'}{(\text{trace } t) @ u \equiv u'} \quad \frac{t @ (\text{inl } u) \equiv \text{inl } u' \quad (\text{trace } t) @ (\text{inl } u') \equiv u''}{(\text{trace } t) @ (\text{inl } u) \equiv u''}$$

$$\frac{t @ (\text{inr } u) \equiv \text{inl } u' \quad (\text{trace } t) @ (\text{inl } u') \equiv u''}{(\text{trace } t) @ u \equiv u''} \quad \frac{t @ (\text{inl } u) \equiv \text{inr } u'}{(\text{trace } t) @ (\text{inl } u) \equiv u'}$$