Syntax

Type Variables	X, Y, Z	
Term Variables	x, y, z	
Types	T, S, U	$::= X$
		$\mid\ I$
		$\mid\ T \oplus T$
		$\mid\ T \otimes T$
		$\mid\ T \multimap T$
		$\mid\ \mu X.T$
Terms	t, u, v	$::= x$
		$\mid\ ()$
		$\mid\ \text{inl}\ t$
		$\mid\ \text{inr}\ t$
		$\mid\ t \times t$
		$\mid\ t \mapsto t$
		$\mid\ \text{fold}\ t$
		$\mid\ \text{trace}\ t$
		$\mid\ t + t$
		$\mid\ t \fatsemi t$
		$\mid\ t^{\dagger}$
		$\mid\ \varnothing$
		$\mid\ \text{id}$
Type Contexts	Θ	$::=$
		$\mid\ \Theta, X$
Term Contexts	Γ	$::=$
		$\mid\ \Gamma, t : T$
Type Judgements		$\Theta \vdash T$
Term Judgements		$\Gamma \vdash t : T$
Expressions	e, f, g	$::= t$
		$\mid\ t \ @\ t$
Expr judgement		$::= \vdash e : T$
Var Environment	Ξ	$::= \{\ \}$
		$\Xi, \{x \to t\}$

Type Formation rules

$$\frac{}{\Theta, X \vdash X} \qquad \frac{}{\Theta \vdash I} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \oplus T_2 :} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \otimes T_2 :} \qquad \frac{\Theta \vdash T_1 \qquad \Theta \vdash T_2}{\Theta \vdash T_1 \multimap T_2} \qquad \frac{\Theta, X \vdash T}{\Theta \vdash \mu X.T}$$

Type Substitution

$$\begin{aligned}
X[X \to S] &= S \\
Y[X \to S] &= Y \\
I[X \to S] &= I \\
T_1 \oplus T_2[X \to S] &= T_1[X \to S] \oplus T_2[X \to S] \\
T_1 \otimes T_2[X \to S] &= T_1[X \to S] \otimes T_2[X \to S] \\
T_1 \multimap T_2[X \to S] &= T_1[X \to S] \multimap T_2[X \to S] \\
\mu Y.T[X \to S] &= \mu Y.(T[X \to S])
\end{aligned}$$

Typing rules

$$\text{Variable}\ \frac{}{x : T \vdash x : T} \qquad \frac{\Gamma_2, \Gamma_1 \vdash t : T}{\Gamma_1, \Gamma_2 \vdash t : T}\ \text{Exchange}$$

$$I_L\ \frac{\Gamma \vdash t : T}{\Gamma, () : I \vdash t : T} \qquad \frac{}{\vdash () : I}\ I_R$$

$$\oplus_{L_l}\ \frac{\Gamma, t_1 : T_1 \vdash t : T}{\Gamma, \text{inl}\ t_1 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash \text{inl}\ t_1 : T_1 \oplus T_2}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \frac{\Gamma, t_2 : T_2 \vdash t : T}{\Gamma, \text{inr}\ t_2 : T_1 \oplus T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t_2 : T_2}{\Gamma \vdash \text{inr}\ t_2 : T_1 \oplus T_2}\ \oplus_{R_r}$$

$$\otimes_L\ \frac{\Gamma, t_1 : T_1, t_2 : T_2 \vdash t : T}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2 \vdash t : T} \qquad \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2 \vdash t_2 : T_2}{\Gamma_1, \Gamma_2 \vdash t_1 \times t_2 : T_1 \otimes T_2}\ \otimes_R$$

$$\multimap_L\ \frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2, t_2 : T_2 \vdash t : T}{\Gamma_1, \Gamma_2, t_1 \mapsto t_2 : T_1 \multimap T_2 \vdash t : T} \qquad \frac{\Gamma, t_1 : T_1 \vdash t_2 : T_2}{\Gamma \vdash t_1 \mapsto t_2 : T_1 \multimap T_2}\ \multimap_R$$

$$\mu_L\ \frac{\Gamma, u : U[X \to \mu X.U] \vdash t : T}{\Gamma, \text{fold}\ u : \mu X.U \vdash t : T} \qquad \frac{\Gamma \vdash t : T[X \to \mu X.T]}{\Gamma \vdash \text{fold}\ t : \mu X.T}\ \mu_R$$

$$\text{Trace}_L\ \frac{\Gamma, u : S \oplus U_1 \multimap S \oplus U_2 \vdash t : T}{\Gamma, \text{trace}\ u : U_1 \multimap U_2 \vdash t : T} \qquad \frac{\Gamma \vdash t : U \oplus T_1 \multimap U \oplus T_2}{\Gamma \vdash \text{trace}\ t : T_1 \multimap T_2}\ \text{Trace}_R$$

$$\text{Linearity}_L\ \frac{\Gamma, t_1 : U \vdash t : T \qquad \Gamma, t_2 : U \vdash t : T}{\Gamma, t_1 + t_2 : U \vdash t : T} \qquad \frac{\Gamma \vdash t_1 : T \qquad \Gamma \vdash t_2 : T}{\Gamma \vdash t_1 + t_2 : T}\ \text{Linearity}_R$$

$$\text{Composition}_L\ \frac{\Gamma, t_1 : T_1 \multimap T_2, t_2 : T_2 \multimap T_3 \vdash t : T}{\Gamma, t_1 \fatsemi t_2 : T_1 \multimap T_3 \vdash t : T} \qquad \frac{\Gamma_1 \vdash t_1 : T_1 \multimap T_2 \qquad \Gamma_2 \vdash t_2 : T_2 \multimap T_3}{\Gamma_1, \Gamma_2 \vdash t_1 \fatsemi t_2 : T_1 \multimap T_3}\ \text{Composition}_R$$

$$\dagger_L\ \frac{\Gamma, u : T_2 \multimap T_1 \vdash t : T}{\Gamma, u^\dagger : T_1 \multimap T_2 \vdash t : T} \qquad \frac{\Gamma \vdash t : T_2 \multimap T_1}{\Gamma \vdash t^\dagger : T_1 \multimap T_2}\ \dagger_R$$

$$\text{id}_L\ \frac{\Gamma \vdash t : T}{\Gamma, \text{id} : U \multimap U \vdash t : T} \qquad \frac{}{\vdash \text{id} : T \multimap T}\ \text{id}_R$$

$$\text{Application}\ \frac{\vdash t : T_1 \multimap T_2 \qquad \vdash t_1 : T_1}{\vdash t\ @\ t_1 : T_2}$$

Syntax Directed Typing rules

$$\vdash\ t : T \triangleright$$

$$\text{Variable}\ \frac{\Gamma(x) = T}{\Gamma\ \vdash\ x : T\ \triangleright\ \Gamma \setminus x}$$

$$I_L\ \frac{\Gamma\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, () : I\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{}{\Gamma\ \vdash\ () : I\ \triangleright\ \Gamma}\ I_R$$

$$\oplus_{L_l}\ \frac{\Gamma, t_1 : T_1\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, \text{inl}\ t_1 : T_1 \oplus T_2\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t_1 : T_1\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ \text{inl}\ t_1 : T_1 \oplus T_2\ \triangleright\ \Gamma'}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \frac{\Gamma, t_2 : T_2\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, \text{inr}\ t_2 : T_1 \oplus T_2\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t_2 : T_2\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ \text{inr}\ t_2 : T_1 \oplus T_2\ \triangleright\ \Gamma'}\ \oplus_{R_r}$$

$$\otimes_L\ \frac{\Gamma, t_1 : T_1, t_2 : T_2\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, t_1 \times t_2 : T_1 \otimes T_2\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t_1 : T_1\ \triangleright\ \Gamma' \qquad \Gamma'\ \vdash\ t_2 : T_2\ \triangleright\ \Gamma''}{\Gamma\ \vdash\ t_1 \times t_2 : T_1 \otimes T_2\ \triangleright\ \Gamma''}\ \otimes_R$$

$$\multimap_L\ \frac{\Gamma\ \vdash\ t_1 : T_1\ \triangleright\ \Gamma' \qquad \Gamma', t_2 : T_2\ \vdash\ t : T\ \triangleright\ \Gamma''}{\Gamma, t_1 \mapsto t_2 : T_1 \multimap T_2\ \vdash\ t : T\ \triangleright\ \Gamma''} \qquad \frac{\Gamma, t_1 : T_1\ \vdash\ t_2 : T_2\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ t_1 \mapsto t_2 : T_1 \multimap T_2\ \triangleright\ \Gamma'}\ \multimap_R$$

$$\mu_L\ \frac{\Gamma, u : U[X \to \mu X.U]\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, \text{fold}\ u : \mu X.U\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t : T[X \to \mu X.T]\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ \text{fold}\ t : \mu X.T\ \triangleright\ \Gamma'}\ \mu_R$$

$$\text{Trace}_L\ \frac{\Gamma, u : S \oplus U_1 \multimap S \oplus U_2\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, \text{trace}\ u : U_1 \multimap U_2\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t : U \oplus T_1 \multimap U \oplus T_2\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ \text{trace}\ t : T_1 \multimap T_2\ \triangleright\ \Gamma'}\ \text{Trace}_R$$

$$\text{Linearity}_L\ \frac{\Gamma, t_1 : U\ \vdash\ t : T\ \triangleright\ \Gamma' \qquad \Gamma, t_2 : U\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, t_1 + t_2 : U\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t_1 : T\ \triangleright\ \Gamma' \qquad \Gamma\ \vdash\ t_2 : T\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ t_1 + t_2 : T\ \triangleright\ \Gamma'}\ \text{Linearity}_R$$

$$\text{Comp}_L\ \frac{\Gamma, t_1 : T_1 \multimap T_2, t_2 : T_2 \multimap T_3\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, t_1 \fatsemi t_2 : T_1 \multimap T_3\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t_1 : T_1 \multimap T_2\ \triangleright\ \Gamma' \qquad \Gamma'\ \vdash\ t_2 : T_2 \multimap T_3\ \triangleright\ \Gamma''}{\Gamma\ \vdash\ t_1 \fatsemi t_2 : T_1 \multimap T_3\ \triangleright\ \Gamma''}\ \text{Comp}_R$$

$$\dagger_L\ \frac{\Gamma, u : T_2 \multimap T_1\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, u^\dagger : T_1 \multimap T_2\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{\Gamma\ \vdash\ t : T_2 \multimap T_1\ \triangleright\ \Gamma'}{\Gamma\ \vdash\ t^\dagger : T_1 \multimap T_2\ \triangleright\ \Gamma'}\ \dagger_R$$

$$\text{id}_L\ \frac{\Gamma\ \vdash\ t : T\ \triangleright\ \Gamma'}{\Gamma, \text{id} : U \multimap U\ \vdash\ t : T\ \triangleright\ \Gamma'} \qquad \frac{}{\vdash\ \text{id} : T \multimap T \triangleright}\ \text{id}_R$$

$$\text{Application}\ \frac{\vdash\ t : T_1 \multimap T_2 \triangleright \qquad \vdash\ t_1 : T_1 \triangleright}{\vdash\ t\ @\ t_1 : T_2 \triangleright}$$

Constraint–Based Type Inference rules

$$\vdash\ t : X \triangleright\ \mid\ C$$

$$\text{Variable}\ \dfrac{\Gamma(x) = V_2}{\Gamma\ \vdash\ x : V_1\ \triangleright\ \Gamma \setminus (x : T)\ \mid\ \{V_1 = V_2\}}$$

$$I_L\ \dfrac{\Gamma\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, () : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = I\}} \qquad \dfrac{}{\Gamma\ \vdash\ () : V\ \triangleright\ \Gamma\ \mid\ \{V = I\}}\ I_R$$

$$\oplus_{L_l}\ \dfrac{\Gamma, t_1 : X_1\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, \text{inl } t_1 : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \oplus X_2\}} \qquad \dfrac{\Gamma\ \vdash\ t_1 : X_1\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ \text{inl } t_1 : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \oplus X_2\}}\ \oplus_{R_l}$$

$$\oplus_{L_r}\ \dfrac{\Gamma, t_2 : X_2\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, \text{inr } t_2 : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \oplus X_2\}} \qquad \dfrac{\Gamma\ \vdash\ t_2 : X_2\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ \text{inr } t_2 : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{X_1 \oplus X_2\}}\ \oplus_{R_r}$$

$$\otimes_L\ \dfrac{\Gamma, t_1 : X_1, t_2 : X_2\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, t_1 \times t_2 : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \otimes X_2\}} \qquad \dfrac{\Gamma\ \vdash\ t_1 : X_1\ \triangleright\ \Gamma'\ \mid\ C_1 \qquad \Gamma'\ \vdash\ t_2 : X_2\ \triangleright\ \Gamma''\ \mid\ C_2}{\Gamma\ \vdash\ t_1 \times t_2 : V\ \triangleright\ \Gamma''\ \mid\ C_1 \cup C_2 \cup \{V = X_1 \otimes X_2\}}\ \otimes_R$$

$$\multimap_L\ \dfrac{\Gamma\ \vdash\ t_1 : X_1\ \triangleright\ \Gamma'\ \mid\ C_1 \qquad \Gamma', t_2 : X_2\ \vdash\ t : T\ \triangleright\ \Gamma''\ \mid\ C_2}{\Gamma, t_1 \mapsto t_2 : V\ \vdash\ t : T\ \triangleright\ \Gamma''\ \mid\ C_1 \cup C_2 \cup \{V = X_1 \multimap X_2\}} \qquad \dfrac{\Gamma, t_1 : X_1\ \vdash\ t_2 : X_2\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ t_1 \mapsto t_2 : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_2\}}\ \multimap_R$$

$$\mu_L\ \dfrac{\Gamma, u : U[Y \to \mu\, Y.U]\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, \text{fold}_{\mu\, Y.U}\ u : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = \mu\, Y.U\}} \qquad \dfrac{\Gamma\ \vdash\ t : T[X \to \mu\, X.T]\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ \text{fold}_{\mu\, X.T}\ t : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = \mu\, X.T\}}\ \mu_R$$

$$\text{Trace}_L\ \dfrac{\Gamma, u : U \oplus X_1 \multimap U \oplus X_2\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, \text{trace}_U\ u : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_2\}} \qquad \dfrac{\Gamma\ \vdash\ t : T \oplus X_1 \multimap T \oplus X_2\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ \text{trace}_T\ t : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_2\}}\ \text{Trace}_R$$

$$\text{Lin}_L\ \dfrac{\Gamma, t_1 : X\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C_1 \qquad \Gamma, t_2 : X\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C_2}{\Gamma, t_1 + t_2 : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C_1 \cup C_2 \cup \{V = X\}} \qquad \dfrac{\Gamma\ \vdash\ t_1 : X\ \triangleright\ \Gamma'\ \mid\ C_1 \qquad \Gamma\ \vdash\ t_2 : X\ \triangleright\ \Gamma'\ \mid\ C_2}{\Gamma\ \vdash\ t_1 + t_2 : V\ \triangleright\ \Gamma'\ \mid\ C_1 \cup C_2 \cup \{V = X\}}\ \text{Lin}_R$$

$$\text{Comp}_L\ \dfrac{\Gamma, t_1 : X_1 \multimap X_2, t_2 : X_2 \multimap X_3\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, t_1 \fatsemi t_2 : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_3\}}$$

$$\dfrac{\Gamma\ \vdash\ t_1 : X_1 \multimap X_2\ \triangleright\ \Gamma'\ \mid\ C_1 \qquad \Gamma'\ \vdash\ t_2 : X_2' \multimap X_3\ \triangleright\ \Gamma''\ \mid\ C_2}{\Gamma\ \vdash\ t_1 \fatsemi t_2 : V\ \triangleright\ \Gamma''\ \mid\ C_1 \cup C_2 \cup \{V = X_1 \multimap X_3\}}\ \text{Comp}_R$$

$$\dagger_L\ \dfrac{\Gamma, u : X_2 \multimap X_1\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, u^\dagger :\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_2\}} \qquad \dfrac{\Gamma\ \vdash\ t : X_2 \multimap X_1\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma\ \vdash\ t^\dagger : V\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X_1 \multimap X_2\}}\ \dagger_R$$

$$\text{id}_L\ \dfrac{\Gamma\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C}{\Gamma, \text{id} : V\ \vdash\ t : T\ \triangleright\ \Gamma'\ \mid\ C \cup \{V = X \multimap X\}} \qquad \dfrac{}{\vdash\ \text{id} : V \triangleright\ \mid\ \{V = X \multimap X\}}\ \text{id}_R$$

$$\text{Application}\ \dfrac{\vdash\ t : X_1 \multimap X_2 \triangleright\ \mid\ C_1 \qquad \vdash\ t_1 : X_1 \triangleright\ \mid\ C_2}{\vdash\ t\ @\ t_1 : V \triangleright\ \mid\ C_1 \cup C_2 \cup \{V = X_2\}}$$

Unification

$$\begin{aligned}
\mathit{unify} &:= \text{Set of Constraint} \to \text{Substitution} \\
\mathit{unify}(\{\}) &= [] \\
\mathit{unify}(\{X = T\} \cup C) &= \mathit{unify}([X \to T]C) \circ [X \to T] \\
\mathit{unify}(\{T = X\} \cup C) &= \mathit{unify}([X \to T]C) \circ [X \to T] \\
\mathit{unify}(\{S_1 \oplus S_2 = T_1 \oplus T_2\} \cup C) &= \mathit{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
\mathit{unify}(\{S_1 \otimes S_2 = T_1 \otimes T_2\} \cup C) &= \mathit{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
\mathit{unify}(\{S_1 \multimap S_2 = T_1 \multimap T_2\} \cup C) &= \mathit{unify}(C \cup \{S_1 = T_1, S_2 = T_2\}) \\
\mathit{unify}(\{\mu\, X.S = \mu\, Y.T\} \cup C) &= \mathit{unify}(C \cup \{X = Y, S = T\})
\end{aligned}$$

Type Interpretation

$|\Theta|$ はコンテキスト Θ に含まれる型変数の数

圏 $\mathbf{V}$ はトレース双積付きダガーコンパクト圏 (Dagger Compact Category with Traced Finite Biproduct)

- $F : \mathbf{V}^n \to \mathbf{V}$ は n 多重関手
- $\Pi_i : \mathbf{V}^{|\Theta|} \to \mathbf{V}$ は射影関手
- $K_I : \mathbf{V}^{|\Theta|} \to \mathbf{V}$ は定数 I 関手
- $\otimes : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ はテンソル積関手
- $\oplus : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ は双積関手
- $(-)^\star : \mathbf{V}^{op} \to \mathbf{V}$ は充満忠実自己反変関手
- $[-,-] : \mathbf{V}^{op} \times \mathbf{V} \to \mathbf{V}$ は内部ホム関手
- $Id_{\mathbf{V}}$ は恒等関手
- 圏 $\mathbf{V}$ と関手 $F : \mathbf{V}^n \to \mathbf{V} \quad (n \geq 1)$ について，パラメトライズされた F の始代数は，以下を満たす組 $(F^\sharp, \phi^F)$
  - $F^\sharp : \mathbf{V}^{n-1} \to \mathbf{V}$ は関手
  - $\phi^F : F \circ \langle Id, F^\sharp \rangle \Rightarrow F^\sharp : \mathbf{V}^{n-1} \to \mathbf{V}$ は自然同型
  - 全ての $T \in |\mathbf{V}^{n-1}|$ について，組 $(F^\sharp(T), \phi^F_T)$ は $F(T,-)$- 始代数

$$
\begin{aligned}
[\![\Theta \vdash T]\!] &: \mathbf{V}^{|\Theta|} \to \mathbf{V} \\
[\![\Theta \vdash X_i]\!] &= \Pi_i \\
[\![\Theta \vdash I]\!] &= K_I \\
[\![\Theta \vdash T_1 \oplus T_2]\!] &= \oplus \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \otimes T_2]\!] &= \otimes \circ \langle [\![\Theta \vdash T_1]\!], [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash T_1 \multimap T_2]\!] &= \otimes \circ \langle [\![\Theta \vdash T_1]\!]^\star, [\![\Theta \vdash T_2]\!] \rangle \\
[\![\Theta \vdash \mu X.T]\!] &= [\![\Theta, X \vdash T]\!]^\sharp \\
[\![\Theta \vdash T[X \to U]]\!] &= [\![\Theta \vdash T]\!] \circ \langle Id, [\![\Theta \vdash U]\!] \rangle \\
[\![T]\!] &:= [\![\vdash T]\!](*) \in |\mathbf{V}|
\end{aligned}
$$

$*$ は，スモール圏の圏 **Cat** における終対象 **1** の唯一の対象

Denotational Semantics

$$
\begin{aligned}
[\![t : T]\!] &\in [\![T]\!] \\
[\![\mathrm{id} : T]\!] &= 1 \\
[\![\varnothing : T]\!] &= 0 \\
[\![x : T]\!] &= x \\
[\![() : I]\!] &= * \\
[\![\mathrm{inl}\ t_1 : T_1 \oplus T_2]\!] &= \iota_1([\![t_1 : T_1]\!]) \\
[\![\mathrm{inr}\ t_2 : T_1 \oplus T_2]\!] &= \iota_2([\![t_2 : T_2]\!]) \\
[\![t_1 \times t_2 : T_1 \otimes T_2]\!] &= [\![t_1 : T_1]\!] \otimes [\![t_2 : T_2]\!] \\
[\![t_1 \mapsto t_2 : T_1 \multimap T_2]\!] &= [\![t_1 : T_1]\!]^\star \otimes [\![t_2 : T_2]\!] \\
[\![\mathrm{fold}\ t : \mu X.T]\!] &= fold_{\mu X.T}([\![t : T[X / \mu X.T]]\!]) \\
[\![t_1 + t_2 : T]\!] &= \frac{1}{\sqrt{2}}([\![t_1 : T]\!] + [\![t_2 : T]\!]) \\
[\![t_1 \fatsemi t_2 : T_1 \multimap T_3]\!] &= [\![t_2 : T_2 \multimap T_3]\!] \circ [\![t_1 : T_1 \multimap T_2]\!] \\
[\![\mathrm{trace}_U\ t : T_1 \multimap T_2]\!] &= Tr^U_{T_1, T_2}([\![t : (U \oplus T_1) \multimap (U \oplus T_2)]\!]) \\
[\![t^\dagger : T_1 \multimap T_2]\!] &= [\![t : T_2 \multimap T_1]\!]^{-1} \\
[\![t\ @\ t_1 : T_2]\!] &= [\![t : T_1 \multimap T_2]\!]([\![t_1 : T_1]\!])
\end{aligned}
$$

partial order

$$\varnothing \prec x \prec () \prec \mathrm{inl}\ t \prec \mathrm{inr}\ t \prec t_1 \times t_2 \prec t_1 \mapsto t_2 \prec \mathrm{fold}\ t \prec \mathrm{trace}\ t \prec t_1 + t_2$$

Term Matching and Substitution

$$\frac{}{x \;;\; t \rightharpoonup \{x \to t\}} \qquad \frac{t \in \Xi(x)}{\Xi \triangleright x \rightsquigarrow t \triangleright \Xi \setminus \{x \to t\}} \qquad \frac{x \notin Dom(\Xi)}{\Xi \triangleright x \rightsquigarrow \varnothing \triangleright \Xi}$$

$$\frac{}{() \;;\; () \rightharpoonup \{\}} \qquad \frac{}{\Xi \triangleright () \rightsquigarrow () \triangleright \Xi}$$

$$\frac{t \;;\; u \rightharpoonup \Xi}{\mathrm{inl}\ t \;;\; \mathrm{inl}\ u \rightharpoonup \Xi} \qquad \frac{}{\mathrm{inl}\ t \;;\; \mathrm{inr}\ u \rightharpoonup \bot} \qquad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \mathrm{inl}\ t \rightsquigarrow \mathrm{inl}\ t' \triangleright \Xi'}$$

$$\frac{t \;;\; u \rightharpoonup \Xi}{\mathrm{inr}\ t \;;\; \mathrm{inr}\ u \rightharpoonup \Xi} \qquad \frac{}{\mathrm{inr}\ t \;;\; \mathrm{inl}\ u \rightharpoonup \bot} \qquad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \mathrm{inr}\ t \rightsquigarrow \mathrm{inr}\ t' \triangleright \Xi'}$$

$$\frac{t_1 \;;\; u_1 \rightharpoonup \Xi_1 \qquad t_2 \;;\; u_2 \rightharpoonup \Xi_2}{t_1 \times t_2 \;;\; u_1 \times u_2 \rightharpoonup \Xi_1 \cup^{\times}_{\bot} \Xi_2} \qquad \frac{\Xi \triangleright t_1 \rightsquigarrow t_1' \triangleright \Xi' \qquad \Xi' \triangleright t_2 \rightsquigarrow t_2' \triangleright \Xi''}{\Xi \triangleright t_1 \times t_2 \rightsquigarrow t_1' \times t_2' \triangleright \Xi''}$$

$$\frac{t_1 \;;\; u_1 \rightharpoonup \Xi_1 \qquad t_2 \;;\; u_2 \rightharpoonup \Xi_2}{t_1 \mapsto t_2 \;;\; u_1 \mapsto u_2 \rightharpoonup \Xi_1 \cup^{\times}_{\bot} \Xi_2} \qquad \frac{\Xi \triangleright t_1 \rightsquigarrow t_1' \triangleright \Xi' \qquad \Xi' \triangleright t_2 \rightsquigarrow t_2' \triangleright \Xi''}{\Xi \triangleright t_1 \mapsto t_2 \rightsquigarrow t_1' \mapsto t_2' \triangleright \Xi''}$$

$$\frac{t \;;\; u \rightharpoonup \Xi}{\mathrm{fold}\ t \;;\; \mathrm{fold}\ u \rightharpoonup \Xi} \qquad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \mathrm{fold}\ t \rightsquigarrow \mathrm{fold}\ t' \triangleright \Xi'}$$

$$\frac{t \;;\; u \rightharpoonup \Xi}{\mathrm{trace}\ t \;;\; \mathrm{trace}\ u \rightharpoonup \Xi} \qquad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright \mathrm{trace}\ t \rightsquigarrow \mathrm{trace}\ t' \triangleright \Xi'}$$

$$\frac{t \;;\; u_1 \rightharpoonup \Xi_1 \qquad t \;;\; u_2 \rightharpoonup \Xi_2}{t \;;\; u_1 + u_2 \rightharpoonup \Xi_1 \cup^{+}_{\bot} \Xi_2} \qquad \frac{t_1 \;;\; u \rightharpoonup \Xi_1 \qquad t_2 \;;\; u \rightharpoonup \Xi_2}{t_1 + t_2 \;;\; u \rightharpoonup \Xi_1 \cup^{+}_{\bot} \Xi_2}$$

$$\frac{\Xi \triangleright t_1 \rightsquigarrow t_1' \triangleright \Xi' \qquad \Xi' \triangleright t_2 \rightsquigarrow t_2' \triangleright \Xi''}{\Xi \triangleright t_1 + t_2 \rightsquigarrow t_1' + t_2' \triangleright \Xi''}$$

$$\frac{t_1 \;;\; u_1 \rightharpoonup \Xi_1 \qquad t_2 \;;\; u_2 \rightharpoonup \Xi_2}{t_1 \fatsemi t_2 \;;\; u_1 \fatsemi u_2 \rightharpoonup \Xi_1 \cup^{\times}_{\bot} \Xi_2} \qquad \frac{\Xi \triangleright t_1 \rightsquigarrow t_1' \triangleright \Xi' \qquad \Xi' \triangleright t_2 \rightsquigarrow t_2' \triangleright \Xi''}{\Xi \triangleright t_1 \fatsemi t_2 \rightsquigarrow t_1' \fatsemi t_2' \triangleright \Xi''}$$

$$\frac{}{\mathrm{id} \;;\; \mathrm{id} \rightharpoonup \{\}} \qquad \frac{}{\Xi \triangleright \mathrm{id} \rightsquigarrow \mathrm{id} \triangleright \Xi}$$

$$\frac{}{\varnothing \;;\; \varnothing \rightharpoonup \{\}} \qquad \frac{}{\Xi \triangleright \varnothing \rightsquigarrow \varnothing \triangleright \Xi}$$

$$\frac{t \;;\; u \rightharpoonup \Xi}{t^{\dagger} \;;\; u^{\dagger} \rightharpoonup \Xi} \qquad \frac{\Xi \triangleright t \rightsquigarrow t' \triangleright \Xi'}{\Xi \triangleright t^{\dagger} \rightsquigarrow t'^{\dagger} \triangleright \Xi'}$$

Operational Sematics

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{(t_1 + t_2) + t_3 \equiv t_1' + (t_2' + t_3')} \qquad \frac{t \equiv t'}{\varnothing + t \equiv t'} \qquad \frac{}{t_1 + t_2 \equiv t_2 + t_1} \qquad \frac{}{t + t \equiv t}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{(t_1 \fatsemi t_2) \fatsemi t_3 \equiv t_1' \fatsemi (t_2' \fatsemi t_3')} \qquad \frac{t \equiv t'}{\mathrm{id} \fatsemi t \equiv t'} \qquad \frac{t \equiv t'}{t \fatsemi \mathrm{id} \equiv t'} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{t_1 \fatsemi t_2 \equiv t_1' \fatsemi t_2'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{t_1 \fatsemi (t_2 + t_3) \equiv t_1' \fatsemi t_2' + t_1' \fatsemi t_3'} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{(t_1 + t_2) \fatsemi t_3 \equiv (t_1' \fatsemi t_3') + (t_2' \fatsemi t_3')} \qquad \frac{}{\varnothing \fatsemi t \equiv \varnothing} \qquad \frac{}{t \fatsemi \varnothing \equiv \varnothing}$$

$$\frac{}{\mathrm{inl}\ \varnothing \equiv \varnothing} \qquad \frac{}{\mathrm{inr}\ \varnothing \equiv \varnothing} \qquad \frac{}{t_1 \times \varnothing \equiv \varnothing} \qquad \frac{}{\varnothing \times t_2 \equiv \varnothing} \qquad \frac{}{t_1 \mapsto \varnothing \equiv \varnothing} \qquad \frac{}{\varnothing \mapsto t_2 \equiv \varnothing} \qquad \frac{}{\mathrm{fold}\ \varnothing \equiv \varnothing} \qquad \frac{}{\mathrm{trace}\ \varnothing \equiv \varnothing}$$

$$\frac{}{\varnothing \equiv \varnothing} \qquad \frac{}{\mathrm{id} \equiv \mathrm{id}}$$

$$\frac{}{() \equiv ()} \qquad \frac{}{x \equiv x}$$

$$\frac{t \equiv t'}{\mathrm{inl}\ t \equiv \mathrm{inl}\ t'} \qquad \frac{t \equiv t'}{\mathrm{inr}\ t \equiv \mathrm{inr}\ t'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{t_1 \times t_2 \equiv t_1' \times t_2'} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{t_1 \mapsto t_2 \equiv t_1' \mapsto t_2'}$$

$$\frac{t \equiv t'}{\mathrm{fold}\ t \equiv \mathrm{fold}\ t'} \qquad \frac{t \equiv t'}{\mathrm{trace}\ t \equiv \mathrm{trace}\ t'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{\mathrm{inl}\ (t_1 + t_2) \equiv \mathrm{inl}\ t_1' + \mathrm{inl}\ t_2'} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{\mathrm{inr}\ (t_1 + t_2) \equiv \mathrm{inr}\ t_1' + \mathrm{inr}\ t_2'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{(t_1 + t_2) \times t_3 \equiv (t_1' \times t_3') + (t_2' \times t_3')} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{t_1 \times (t_2 + t_3) \equiv (t_1' \times t_2') + (t_1' \times t_3')}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{(t_1 + t_2) \mapsto t_3 \equiv (t_1' \mapsto t_3') + (t_2' \mapsto t_3')} \qquad \frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t_3 \equiv t_3'}{t_1 \mapsto (t_2 + t_3) \equiv (t_1' \mapsto t_2') + (t_1' \mapsto t_3')}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{\mathrm{fold}\ (t_1 + t_2) \equiv \mathrm{fold}\ t_1' + \mathrm{fold}\ t_2'}$$

$$\frac{t_1^\dagger \equiv t_1' \quad t_2^\dagger \equiv t_2'}{(t_1 + t_2)^\dagger \equiv t_1' + t_2'} \qquad \frac{}{\varnothing^\dagger \equiv \varnothing} \qquad \frac{t_1^\dagger \equiv t_1' \quad t_2^\dagger \equiv t_2'}{(t_1 \fatsemi t_2)^\dagger \equiv t_2' \fatsemi t_1'} \qquad \frac{}{\mathrm{id}^\dagger \equiv \mathrm{id}} \qquad \frac{t \equiv t'}{(t^\dagger)^\dagger \equiv t'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2'}{(t_1 \mapsto t_2)^\dagger \equiv t_2' \mapsto t_1'} \qquad \frac{t^\dagger \equiv t'}{(\mathrm{trace}\ t)^\dagger \equiv \mathrm{trace}\ t'}$$

$$\frac{t_1 \equiv t_1' \quad t_2 \equiv t_2' \quad t \equiv t' \quad t_1' \,;\, t' \rightharpoonup \Xi \quad \Xi \triangleright t_2' \rightsquigarrow t'' \triangleright}{(t_1 \mapsto t_2) \mathbin{@} t \equiv t''}$$

$$\frac{}{\varnothing \mathbin{@} t \equiv \varnothing} \qquad \frac{t \equiv t'}{\mathrm{id} \mathbin{@} t \equiv t'}$$

$$\frac{t_1 \mathbin{@} t \equiv t_1' \quad t_2 \mathbin{@} t \equiv t_2'}{(t_1 + t_2) \mathbin{@} t \equiv t_1' + t_2'} \qquad \frac{t \mathbin{@} t_1 \equiv t_1' \quad t \mathbin{@} t_2 \equiv t_2'}{t \mathbin{@} (t_1 + t_2) \equiv t_1' + t_2'}$$

$$\frac{t_1 \mathbin{@} t \equiv t' \quad t_2 \mathbin{@} t' \equiv t''}{(t_1 \fatsemi t_2) \mathbin{@} t \equiv t''} \qquad \frac{t^\dagger \equiv t' \quad t' \mathbin{@} u \equiv u'}{t^\dagger \mathbin{@} u \equiv u'}$$

$$\frac{t \mathbin{@} (\mathrm{inr}\ u) \equiv \mathrm{inr}\ u'}{(\mathrm{trace}\ t) \mathbin{@} u \equiv u'} \qquad \frac{t \mathbin{@} (\mathrm{inl}\ u) \equiv \mathrm{inl}\ u' \quad (\mathrm{trace}\ t) \mathbin{@} (\mathrm{inl}\ u') \equiv u''}{(\mathrm{trace}\ t) \mathbin{@} (\mathrm{inl}\ u) \equiv u''}$$

$$\frac{t \mathbin{@} (\mathrm{inr}\ u) \equiv \mathrm{inl}\ u' \quad (\mathrm{trace}\ t) \mathbin{@} (\mathrm{inl}\ u') \equiv u''}{(\mathrm{trace}\ t) \mathbin{@} u \equiv u''} \qquad \frac{t \mathbin{@} (\mathrm{inl}\ u) \equiv \mathrm{inr}\ u'}{(\mathrm{trace}\ t) \mathbin{@} (\mathrm{inl}\ u) \equiv u'}$$