Computer Vision

Filtering in frequency domain

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Few other links

- Ref Book: Rafael C. Gonzalez and Richard E. Woods, *Digital Image Processing,* 3rd edition, Prentice Hall, 2009

- Fourier Transform of an Impulse Sampled Signal by Adam Panagos
  - https://www.adampanagos.org/or-otherftexamples
  - https://www.youtube.com/watch?v=mxdf_fSE2Gg
- Computer Vision by Instructor: S. Narasimhan at Carnegie Mellon School of Computer Science
  - www.cs.cmu.edu/~16385/s14/lec_slides/lec-4.ppt
- (For convolution)

  http://www.songho.ca/dsp/convolution/convolution2d_example.html
- Image Processing Learning Resources:

  https://homepages.inf.ed.ac.uk/rbf/HIPR2/hipr_top.htm

  http://aishack.in/

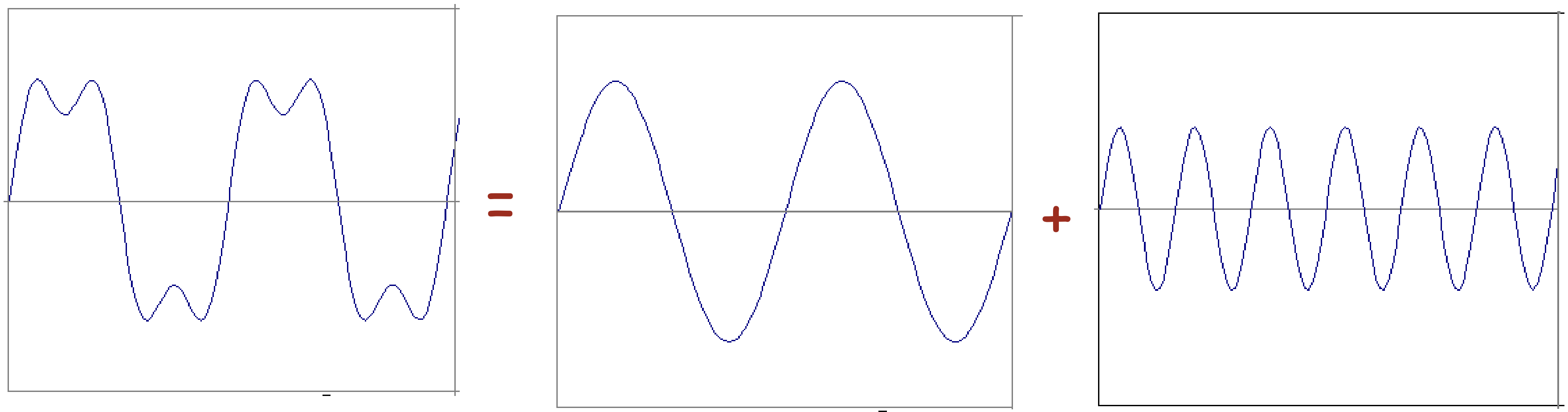
Jean Baptiste Joseph Fourier (1768-1830)

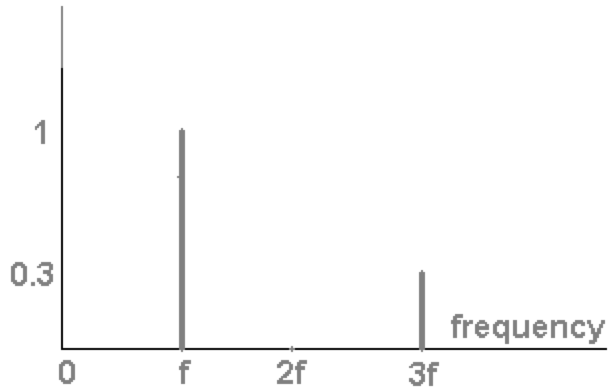
- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

www.cs.cmu.edu/~16385/s14/lec_slides/lec-4.ppt

Time and Frequency

- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi (3f)\, t)$





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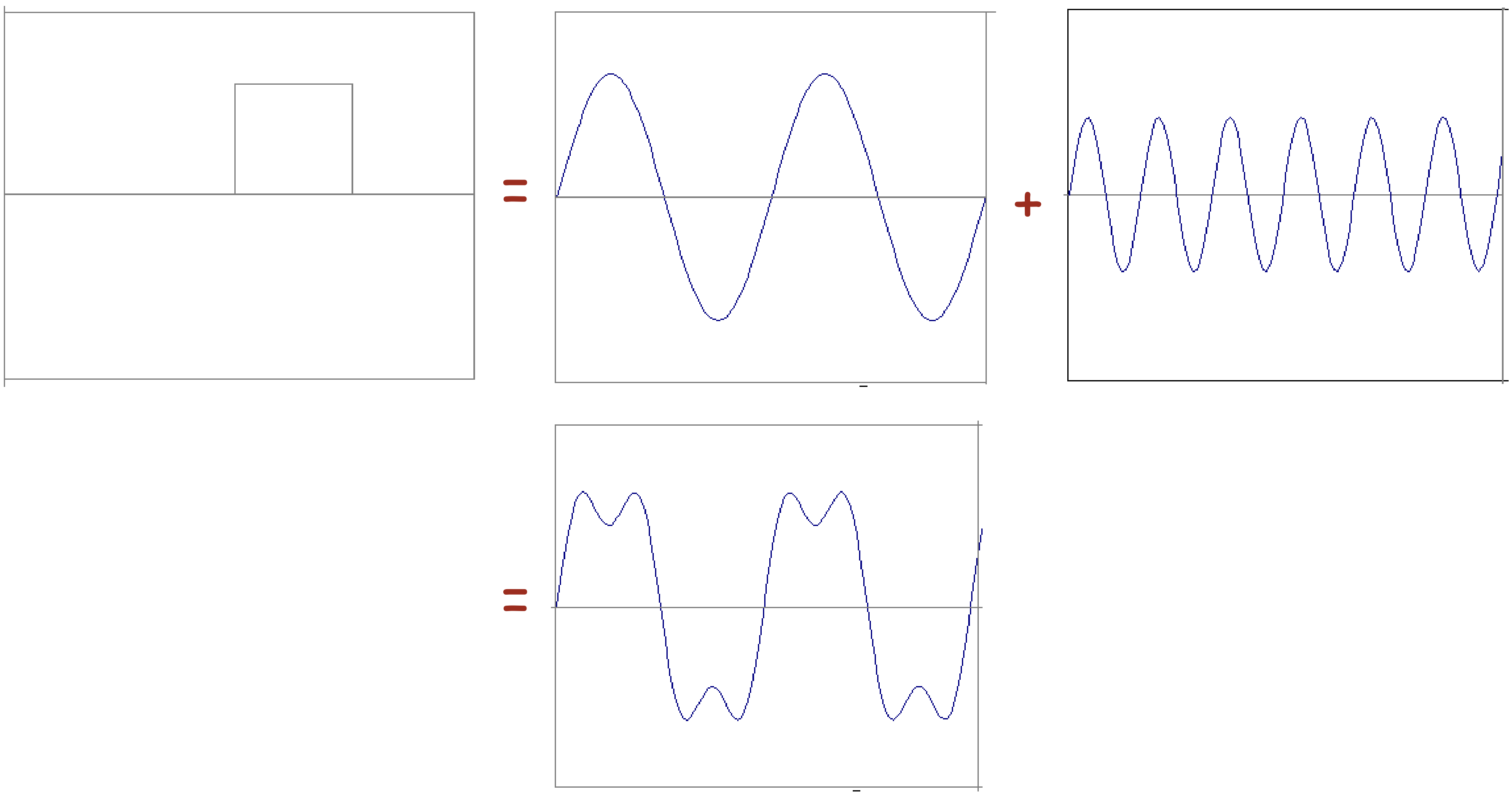
Frequency Spectra



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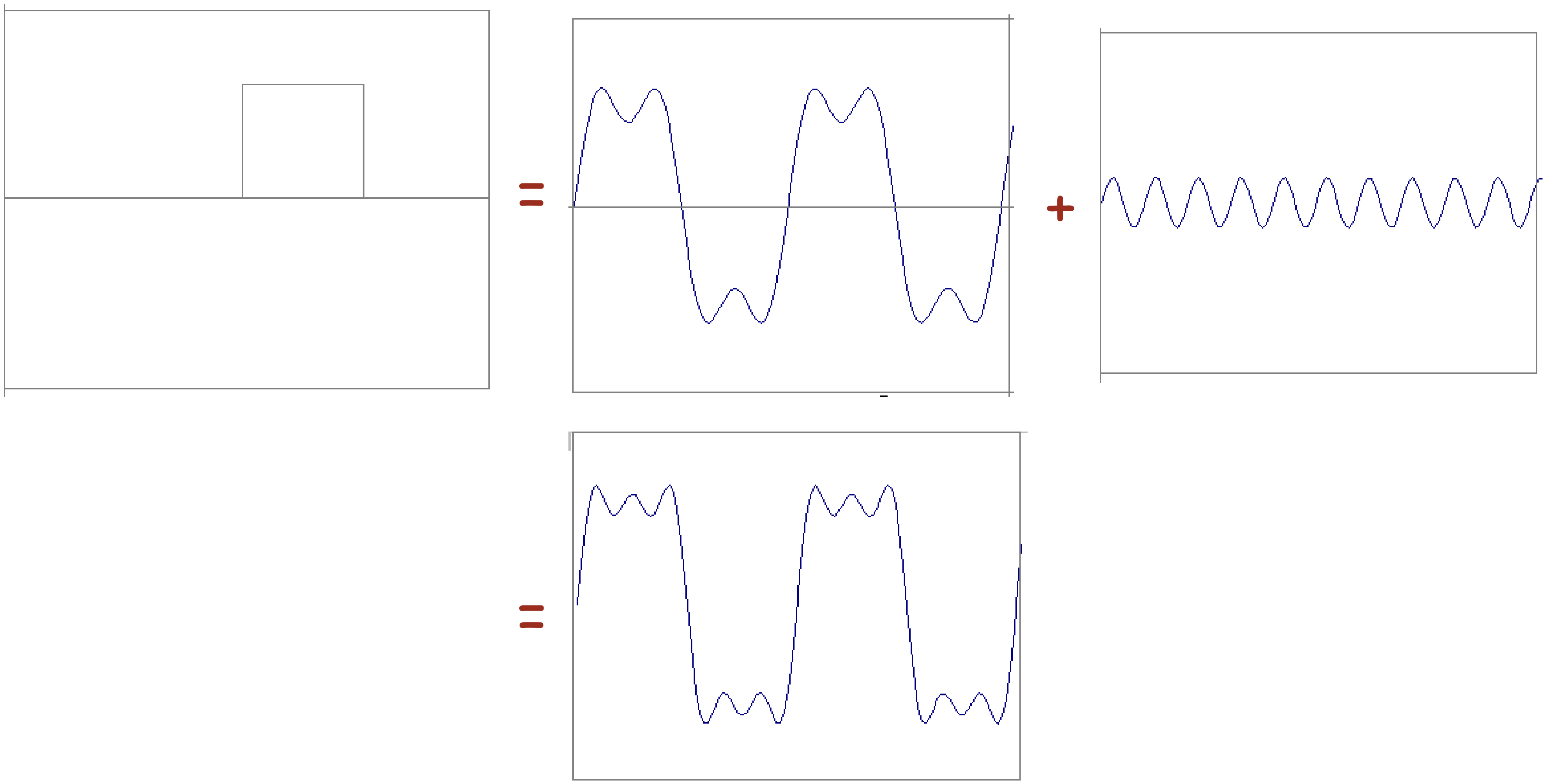
Frequency Spectra



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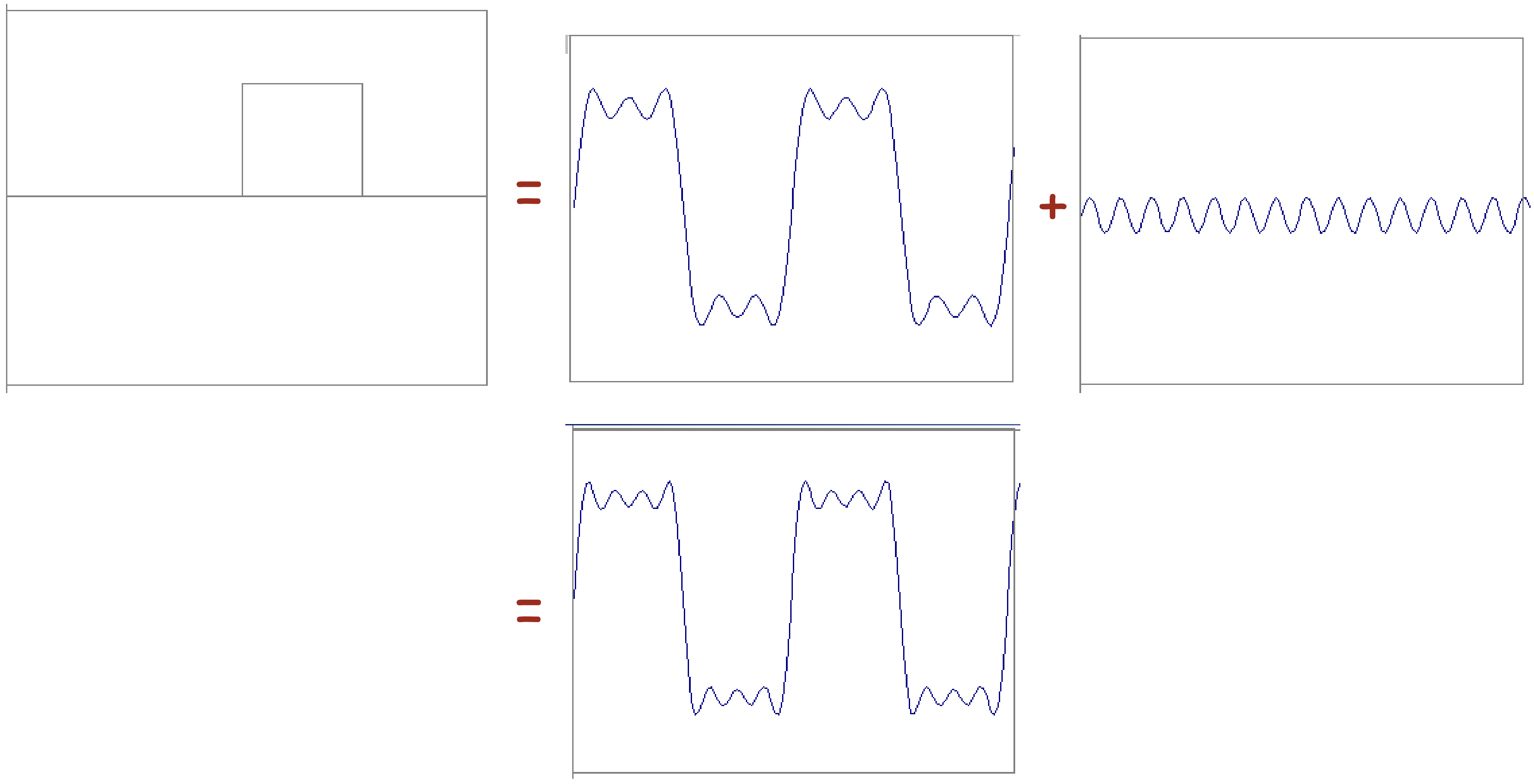
Frequency Spectra



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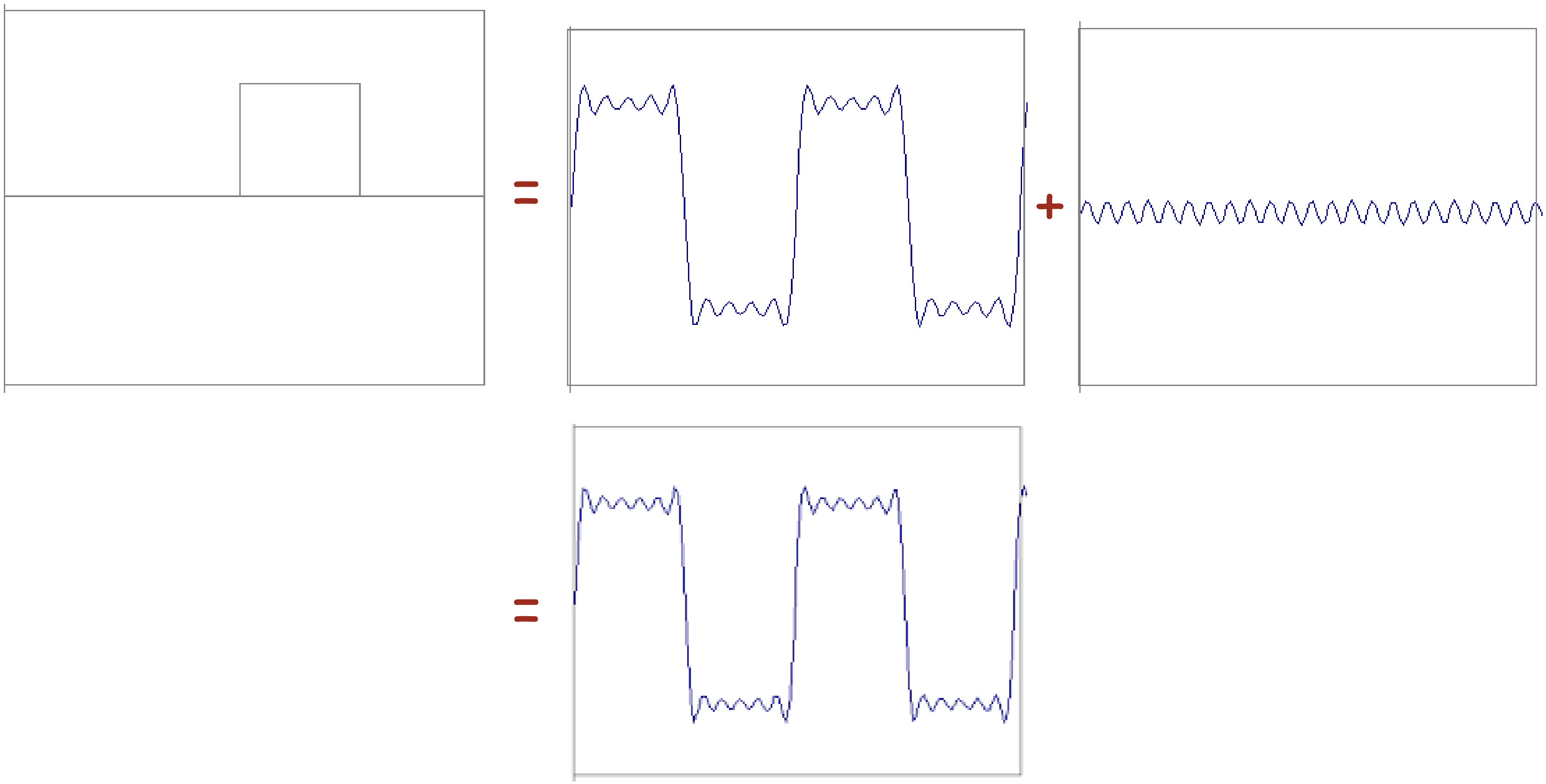
Frequency Spectra



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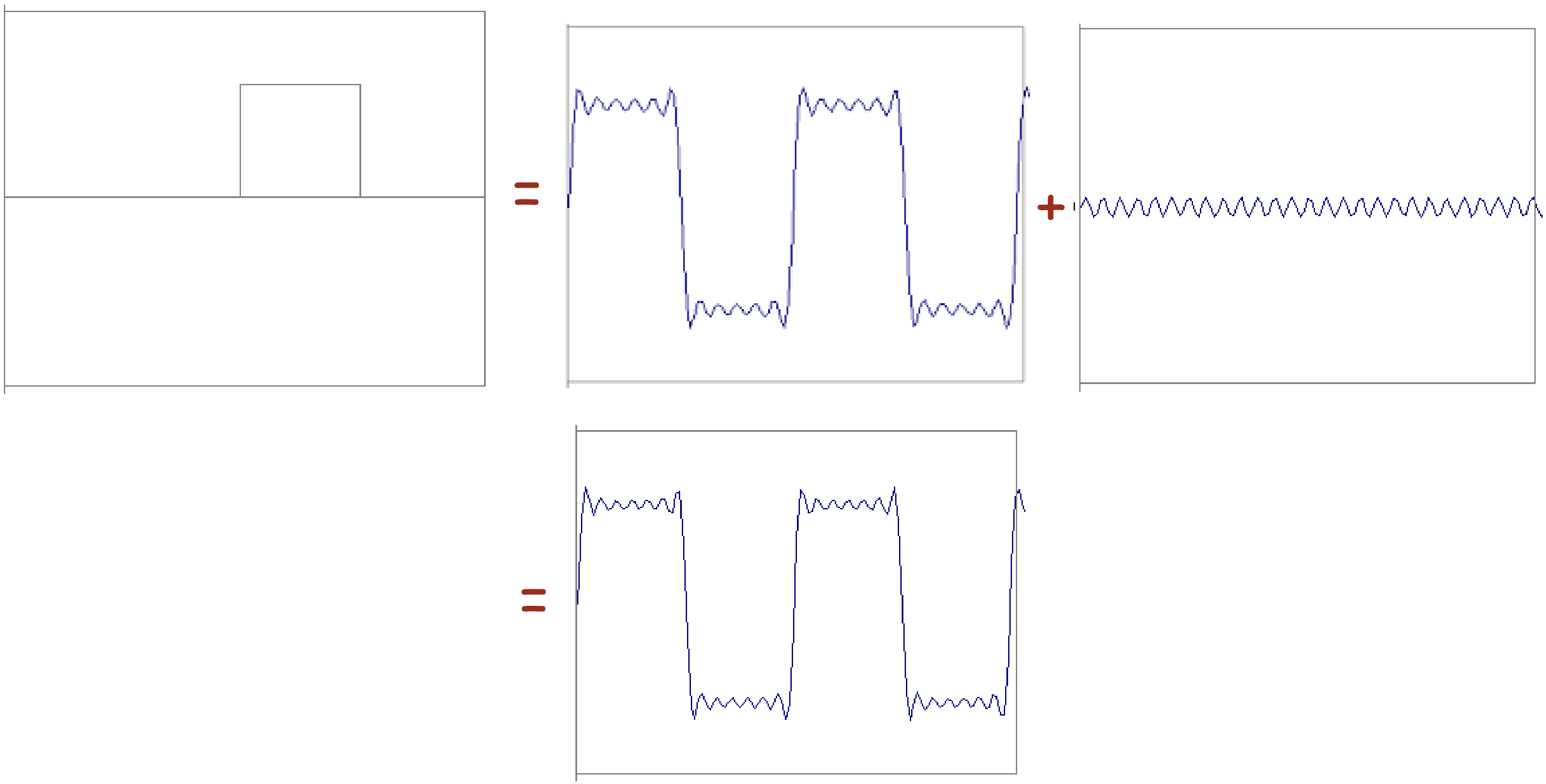
Frequency Spectra



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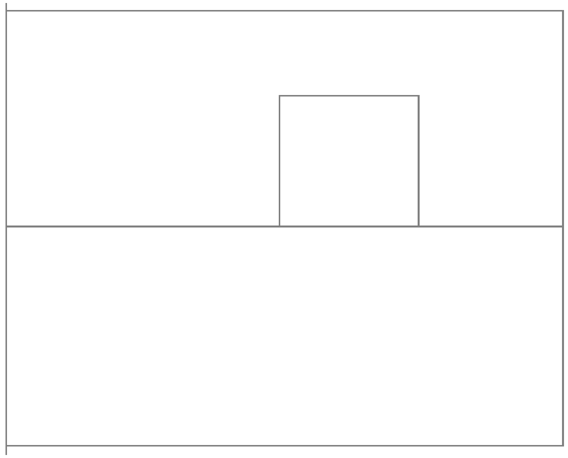
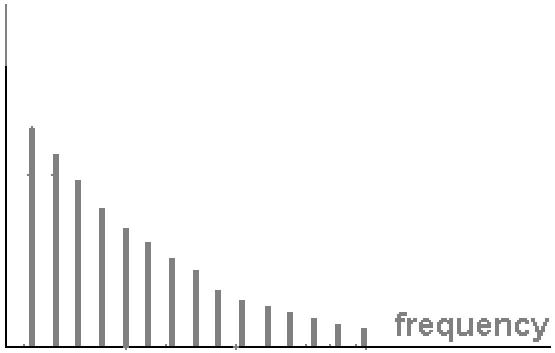
Frequency Spectra



=

$$\boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



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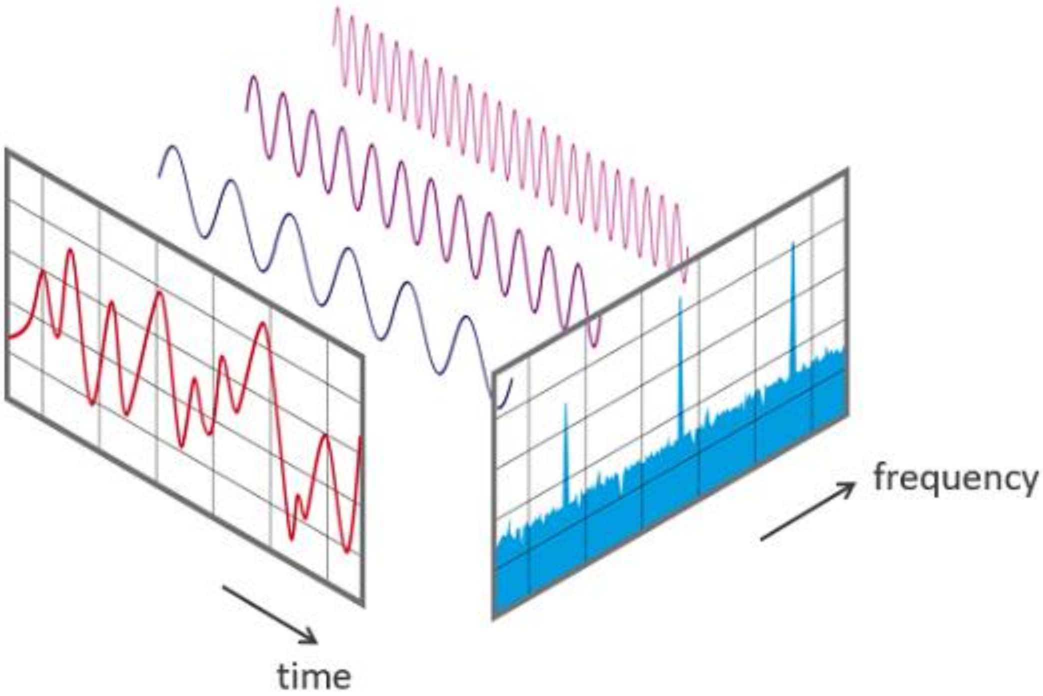


https://www.nti-audio.com/en/support/know-how/fast-fourier-transform-fft

Fourier transform

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Using Euler's formula

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)] dt$$

f(t) is a continuous variable t that is periodic with period T, Known as Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi n}{T}t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$$

Fourier transform

For that reason the Fourier transform F(u)

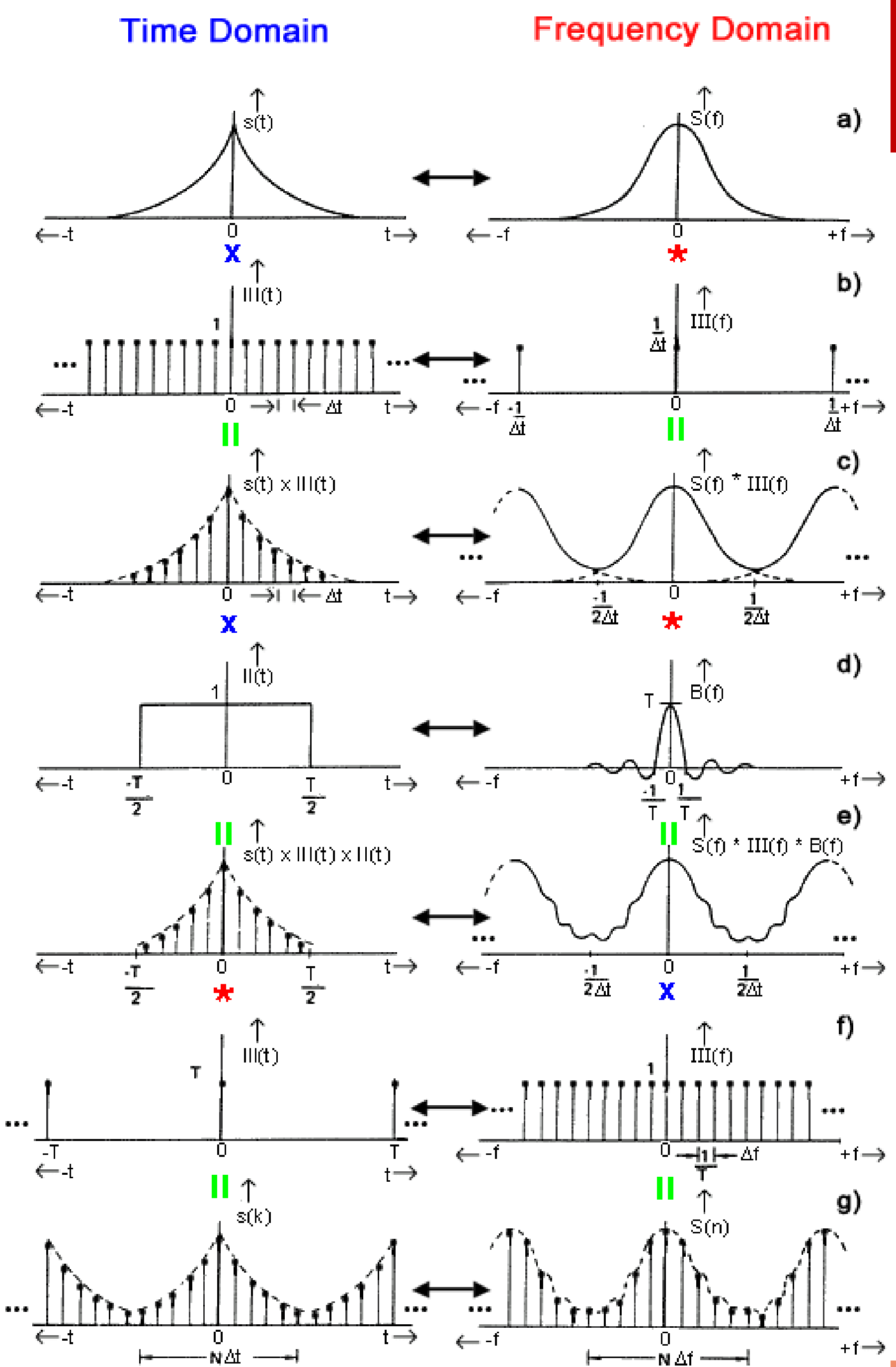
$$F(u) = R(u) + jI(u)$$

Which often appear as

$$F(u) = |F(u)|\, e^{j\phi(u)}$$

$$|F(u)| = \sqrt{R(u)^2 + I(u)^2}$$

$$\tan(\phi(u)) = \frac{I(u)}{R(u)}$$



https://math.stackexchange.com/questions/484553/fourier-transform-graph-what-are-the-negative-frequencies?rq=1

Fourier transform

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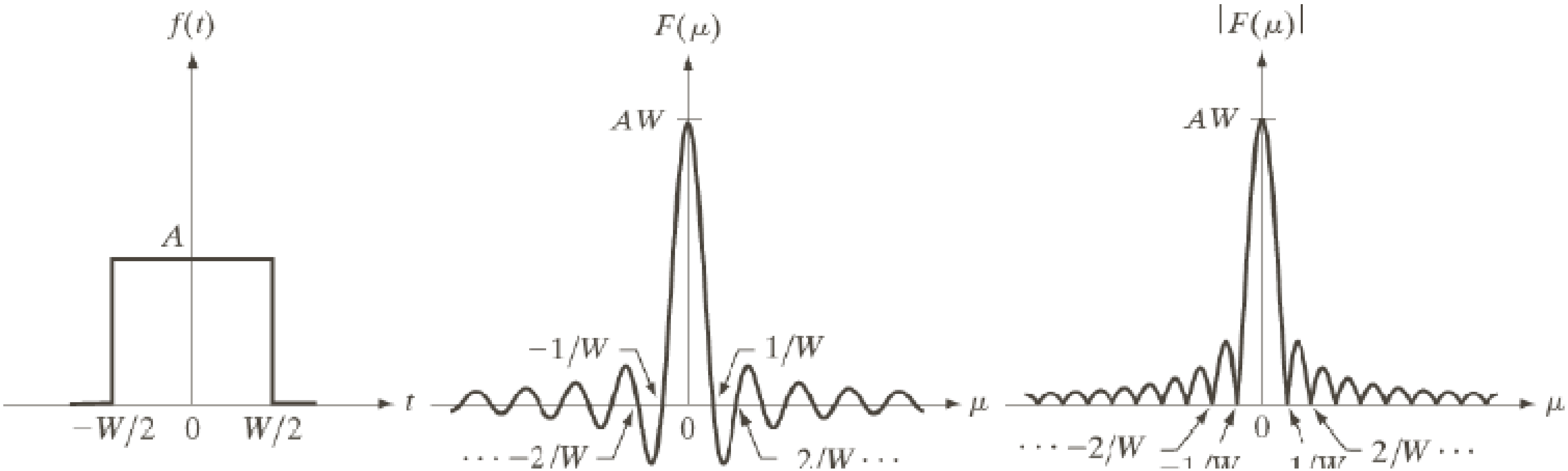


$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt = \int_{-W/2}^{W/2} A e^{-j2\pi ut} dt = \frac{-A}{j2\pi u}\left[e^{-j2\pi ut}\right]_{-W/2}^{W/2}$$

$$= AW \frac{\sin(\pi \mu W)}{(\pi \mu W)}$$

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi u x}$)

Discrete Fourier transform (DFT)

- Interested to identify discrete samples in frequency domain

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-j2\pi\mu t}\, dt$$

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

Suppose that we want to obtain M equally spaced samples of taken over the period μ=0 to μ=1/ΔT. This is accomplished by taking the samples at the following frequencies:

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \dots, M-1$$

DFT: $$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \dots, M-1$$

IDFT: $$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \dots, M-1$$

2D - DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

where f(x,y) is a digital image of size MxN
variables *u* and v ranges: u = 0, 1, 2, .. ,M- 1 and v = 0, 1, 2,.. , N - 1

Fourier transform

$$|F(u, v)| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$$

$$\phi(u, v) = \arctan\left[\frac{I(u, v)}{R(u, v)}\right]$$

$$F(u, v) = |F(u, v)|e^{j\phi(u,v)}$$

Frequency Domain Processing

- What does frequency mean in an image?

  - High frequency components – fast changing/sharp features

  - Low frequency components – slow changing/smooth features

Fourier Transform: Image

- In MatLab:

```
F2=fft2(im);
F2=fftshift(F2);
Fabs2=FH_abs(F2);
imshow(fabs2)
```

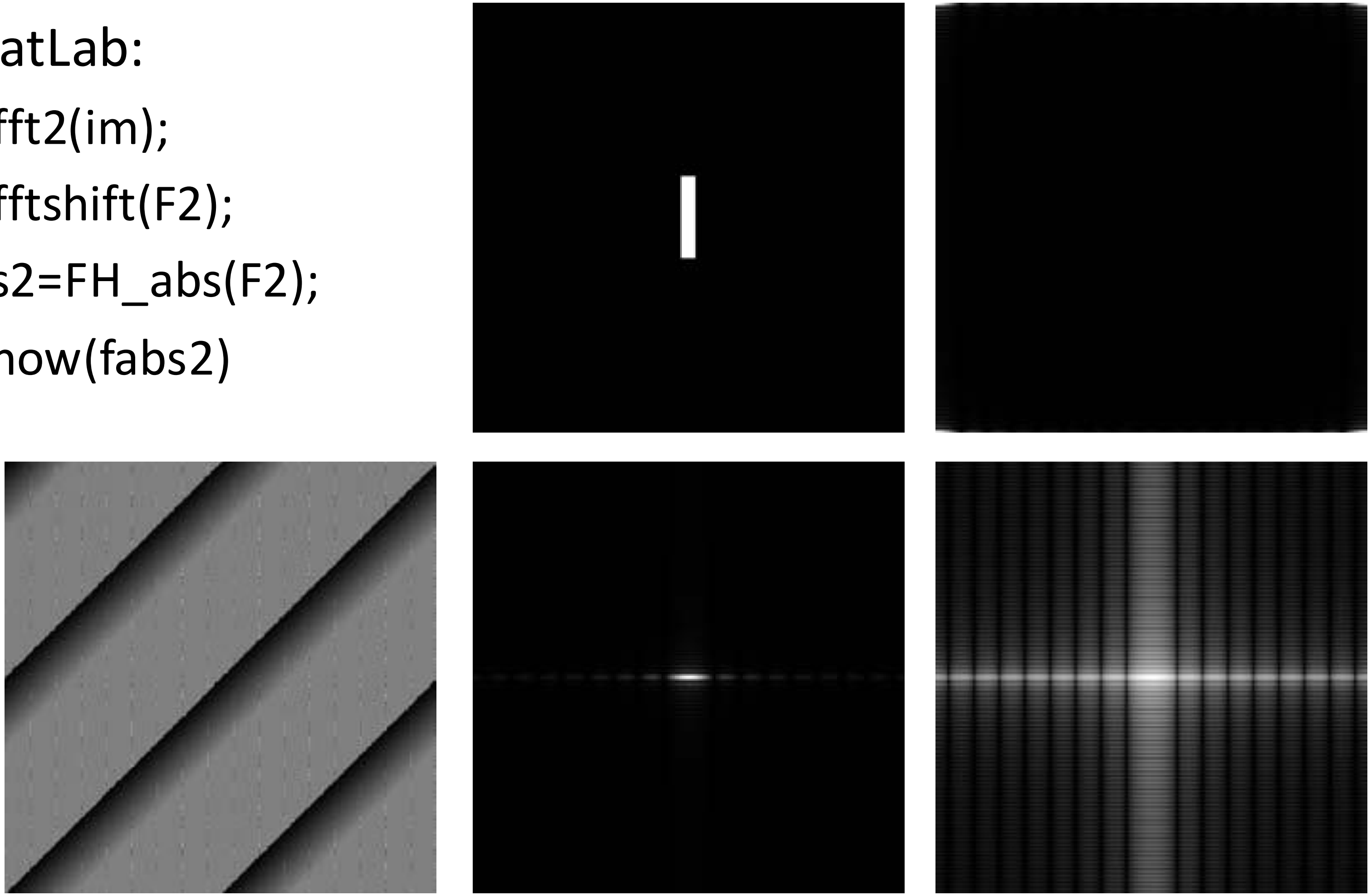


FIGURE: (a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The Coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

Fourier Transform: Image

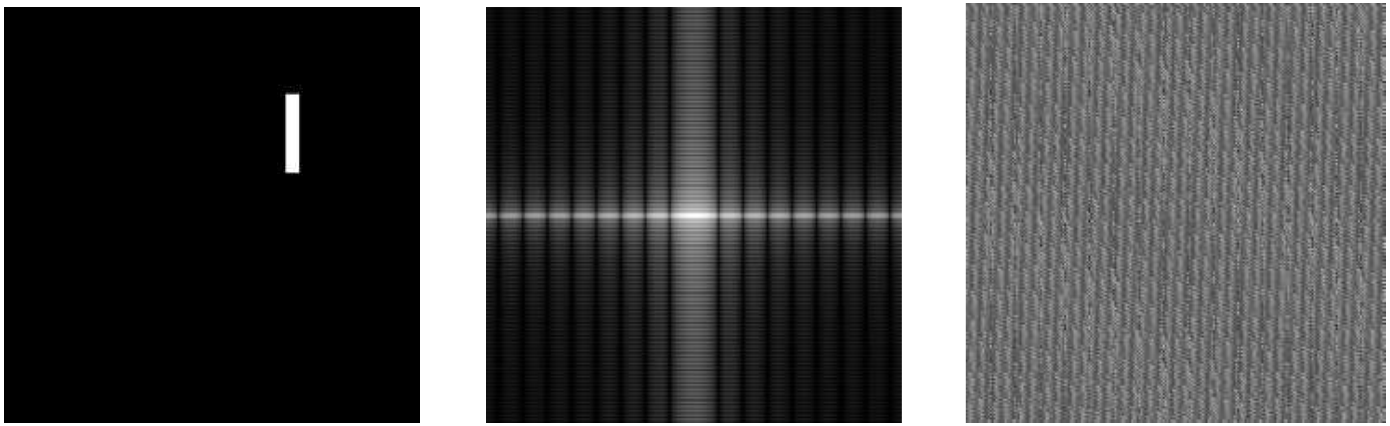


FIGURE: (a) The rectangle is translated, and (b) the corresponding spectrum.

The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image

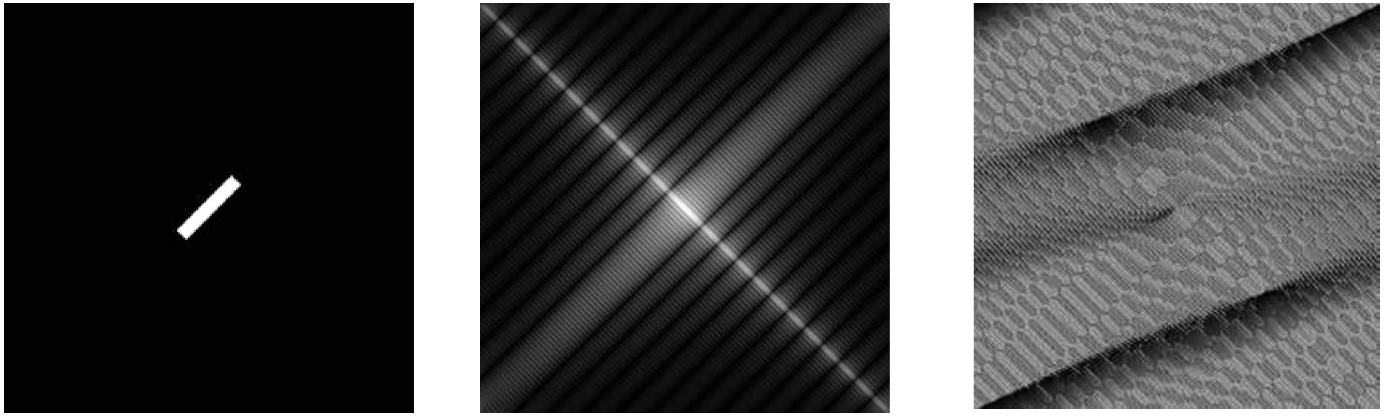


FIGURE:: (a) Rotated rectangle, and (b) the corresponding spectrum.

DC term

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

$$|F(0,0)| = MN|\overline{f}(x,y)|$$

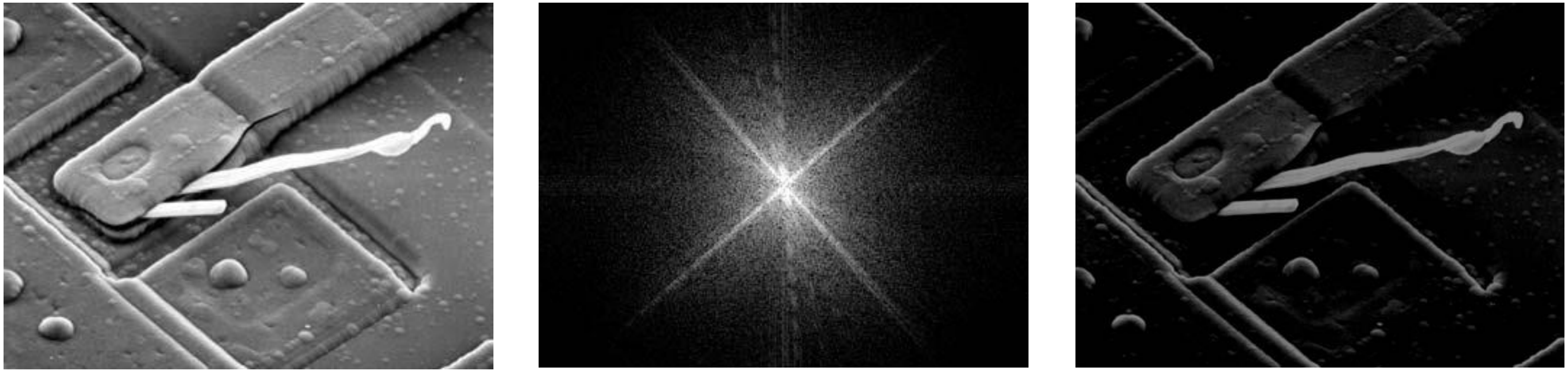
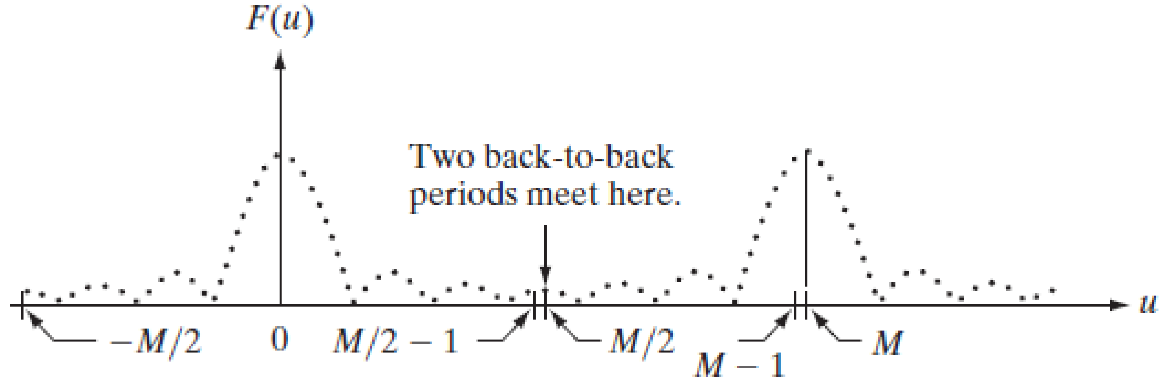


FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (c) Result of filtering the image in (a) by setting to 0 the term ($M/2$, $N/2$) in the Fourier transform.

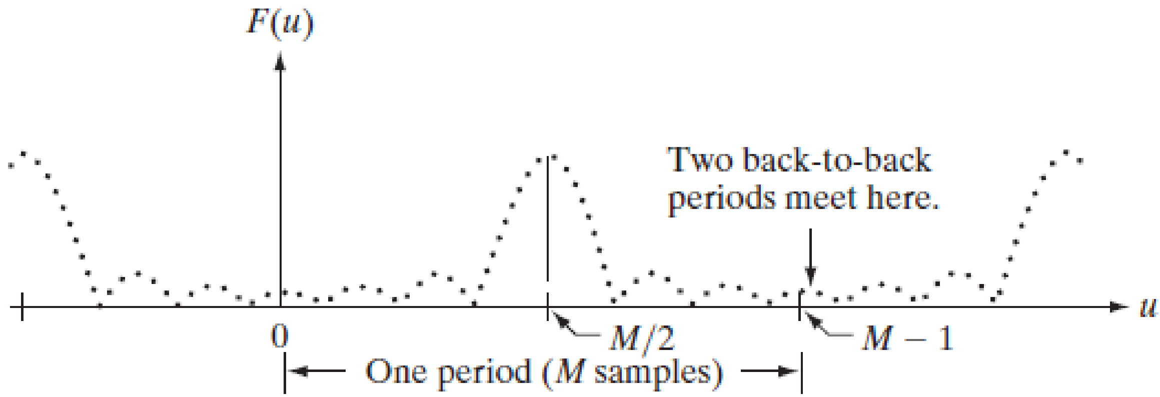
$$f(x)\,e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0)$$



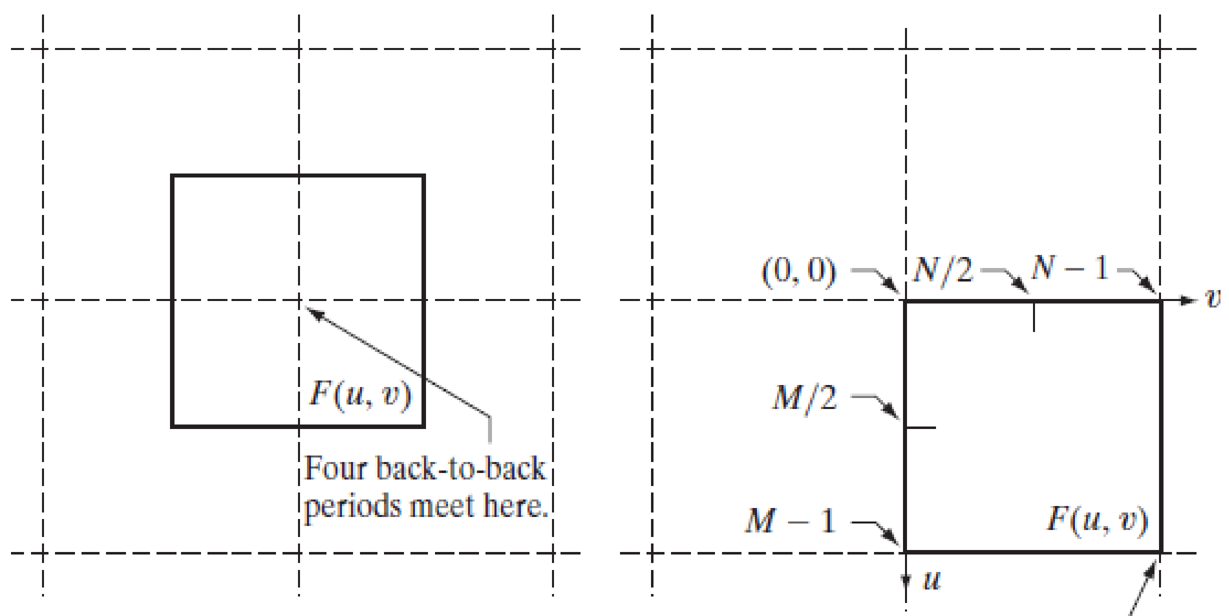
let $u_0 = M/2$

$e^{j\pi x}$ which is equal to $(-1)^x$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$



$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$



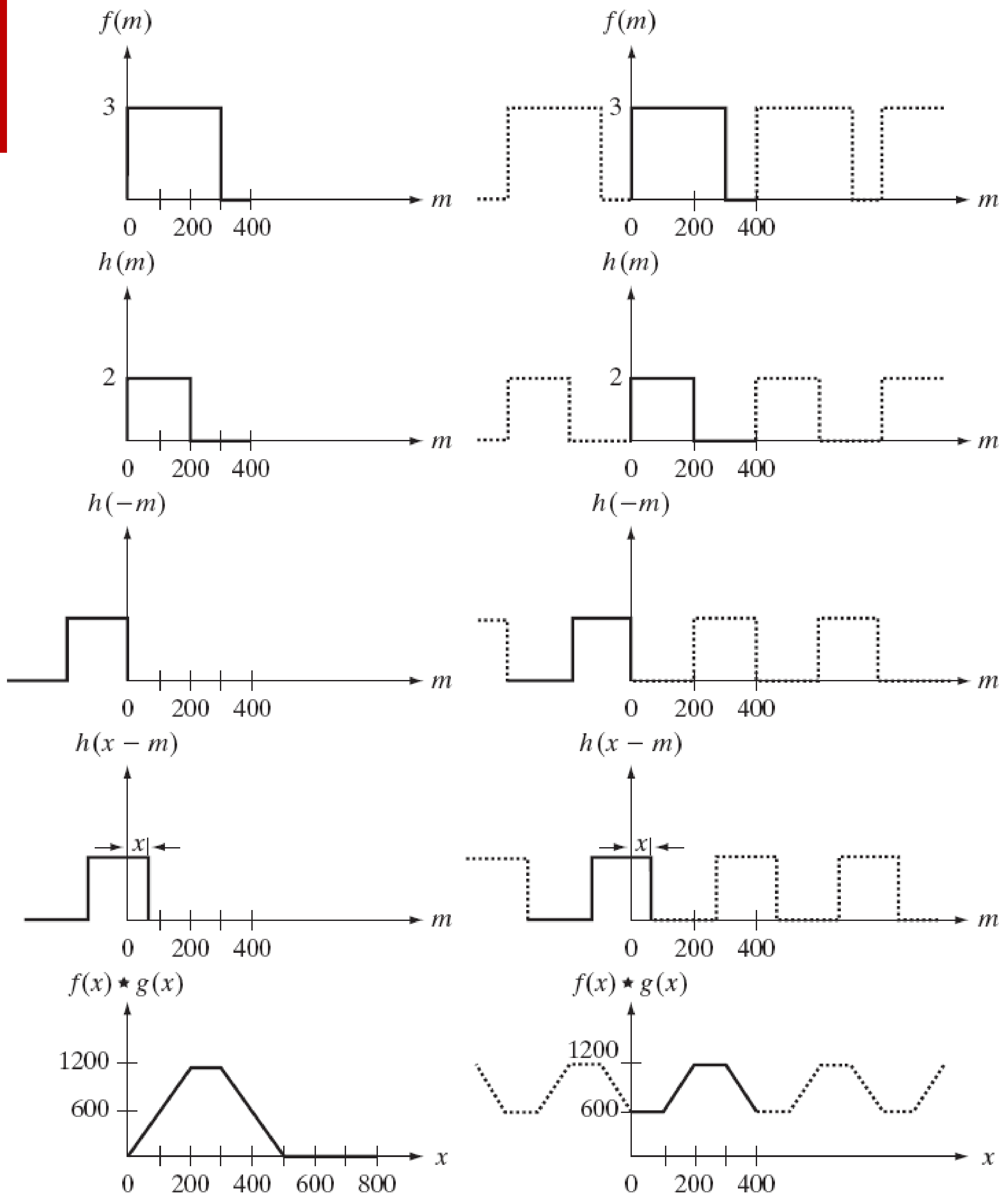
Wraparound

FIGURE: Left column: convolution of two discrete functions
The result in (e) is correct. Right column:

Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect Convolution result. To obtain the correct result, function padding must be used.

P ≥ A + B – 1

where A and B are length of f(x) and h(x) respectively



Frequency Domain Processing

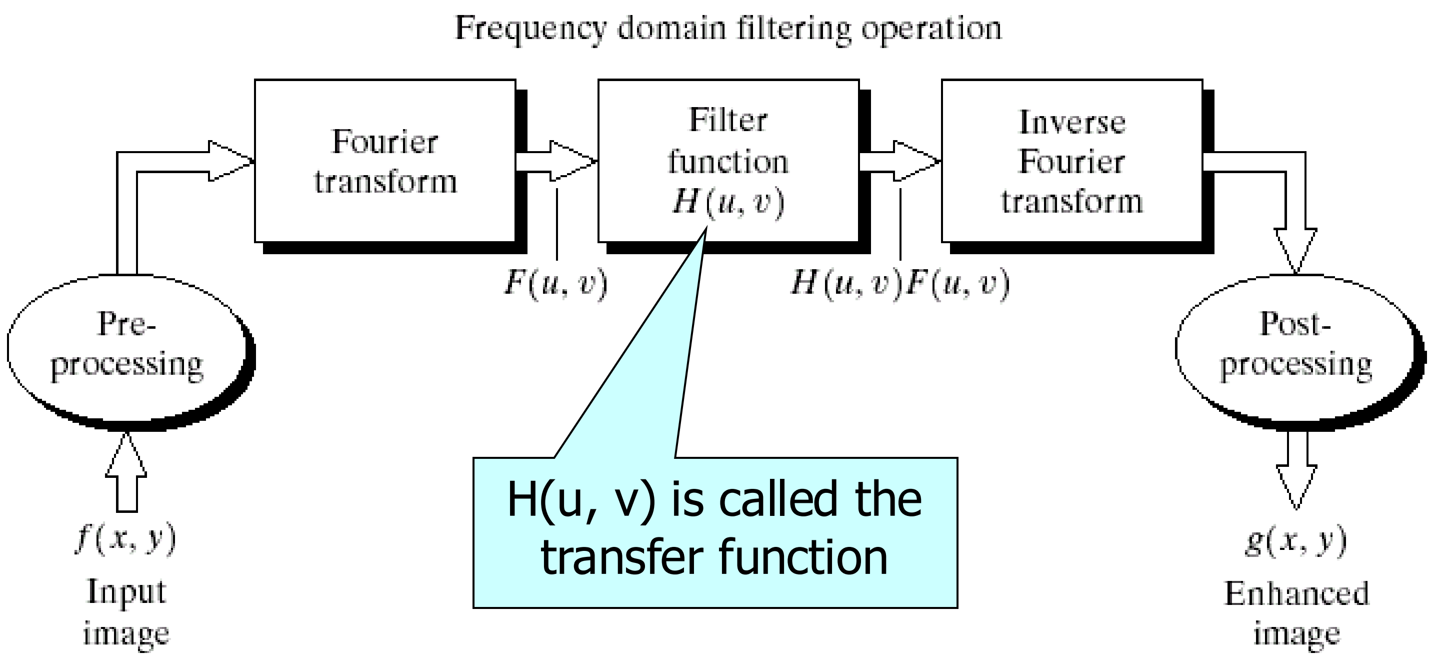


FIGURE 4.5 Basic steps for filtering in the frequency domain.

$$g(x, y) = \Im^{-1}[H(u, v)F(u, v)]$$

Filtering in the frequency domain

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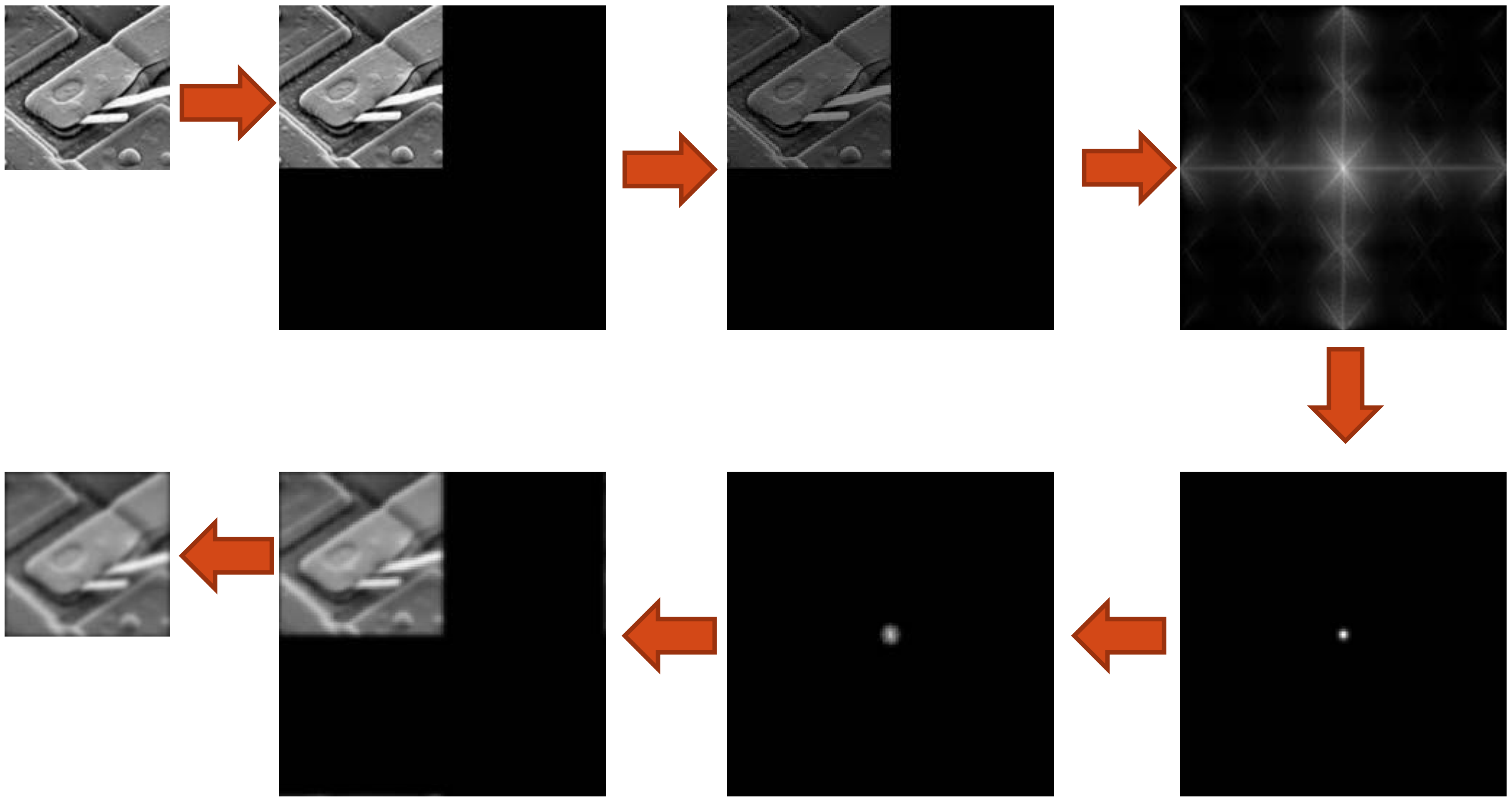
1. Given an input image $f(x, y)$ of size MxN obtain the padding parameters P $(P \geq 2M - 1)$ and $(Q \geq 2N - 1)$
2. Form a padded image $f_p(x,y)$, of size PxQ by appending the necessary number of zeros to f(x, y).
3. Multiply $f_p(x,y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size PxQ with center at coordinates $(P/2,Q/2)$.
6. Form the product

   $G(u,v) = H(u,v)F(u,v)$ using array multiplication
7. Obtain the processed image: $g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$

where the real part is selected in order to ignore parasitic complex components resulting from computational inaccuracies, and the subscript p indicates that we are dealing with padded arrays.

8. Obtain the final processed result, $g(x, y)$, by extracting the MxN region from the top, left quadrant of $g_p(x,y)$

Filtering in the frequency domain



Filtering in the frequency domain

- One way to take advantage of the properties of both domains is to specify a filter in the frequency domain, compute its IDFT, and then use the resulting, full-size spatial filter as a *guide* for constructing smaller spatial filter masks

- Filters based on Gaussian functions are of particular interest because, both the forward and inverse Fourier transforms of a Gaussian function are real Gaussian functions

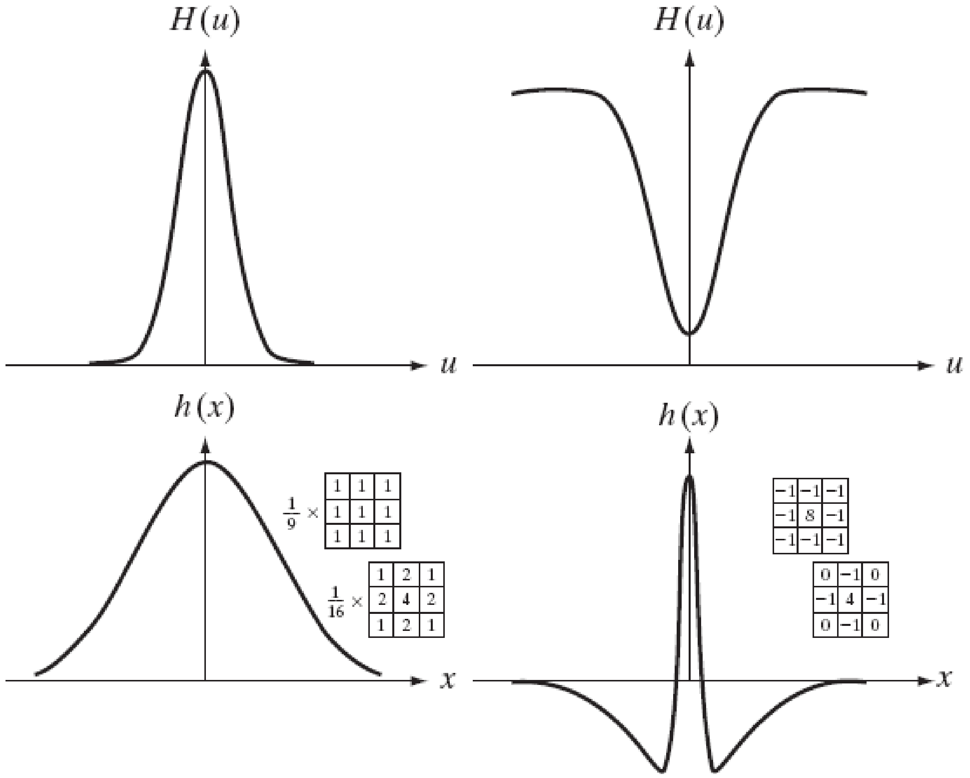
Gaussian filter

- 1-D frequency domain Gaussian filter

$$H(u) = Ae^{-u^2/2\sigma^2}$$

- where σ is the standard deviation of the Gaussian curve. The corresponding filter in the spatial domain is obtained by taking the inverse Fourier transform of *H*(*u*)

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

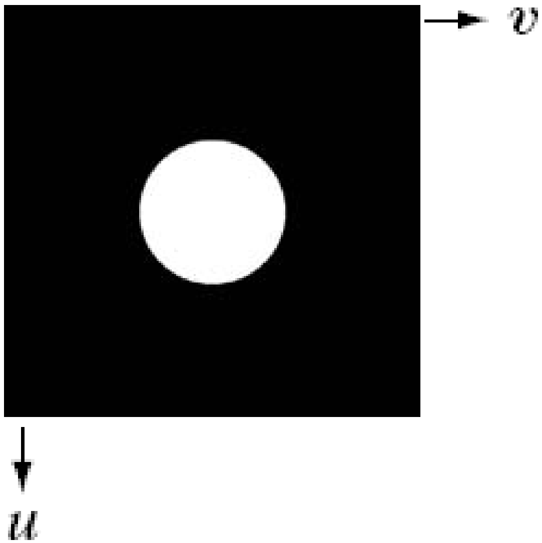


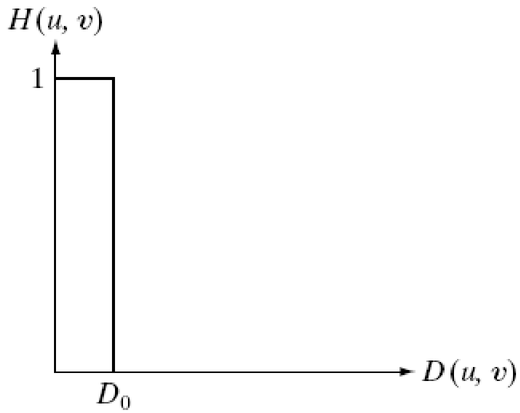
The functions behave reciprocally. When H(*u*) has a broad profile (large value of σ), *h*(*x*) has a narrow profile, and vice versa.

- **Ideal Lowpass Filters**

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

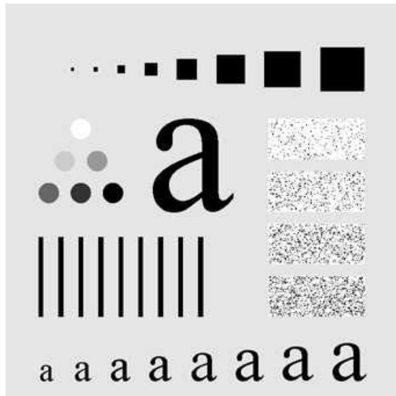
$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2}$$





total image power P_T $P_T = \sum_{u=0}^{P-1} \sum_{v=0}^{Q-1} P(u,v)$ $\alpha = 100\left[\sum_u \sum_v P(u,v)/P_T\right]$

Image smoothing using frequency domain



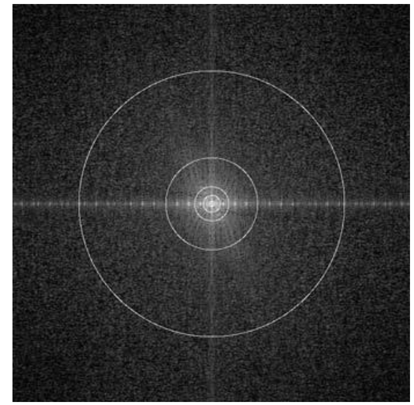


FIGURE: (a) Original image. (b) its Fourier spectrum
Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Image smoothing using frequency domain filters

FIGURE: (a) Representation in the spatial domain of an ILPF of radius 5 and size.
(b) Intensity profile of a horizontal line passing through the center of the image.

Image smoothing using frequency domain

- **Butterworth Lowpass Filters**

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$

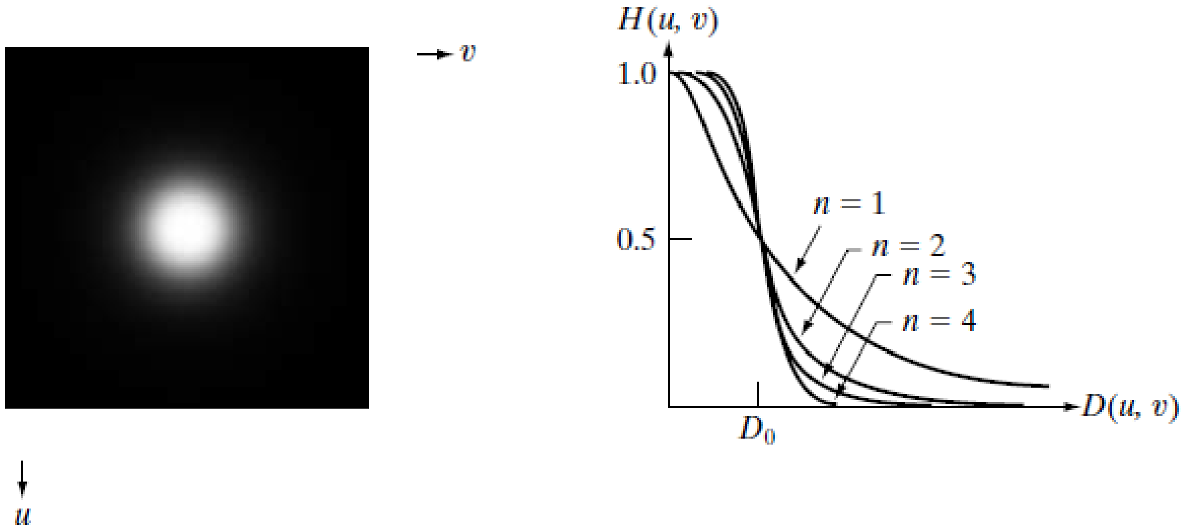


FIGURE: (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Lowpass Filters

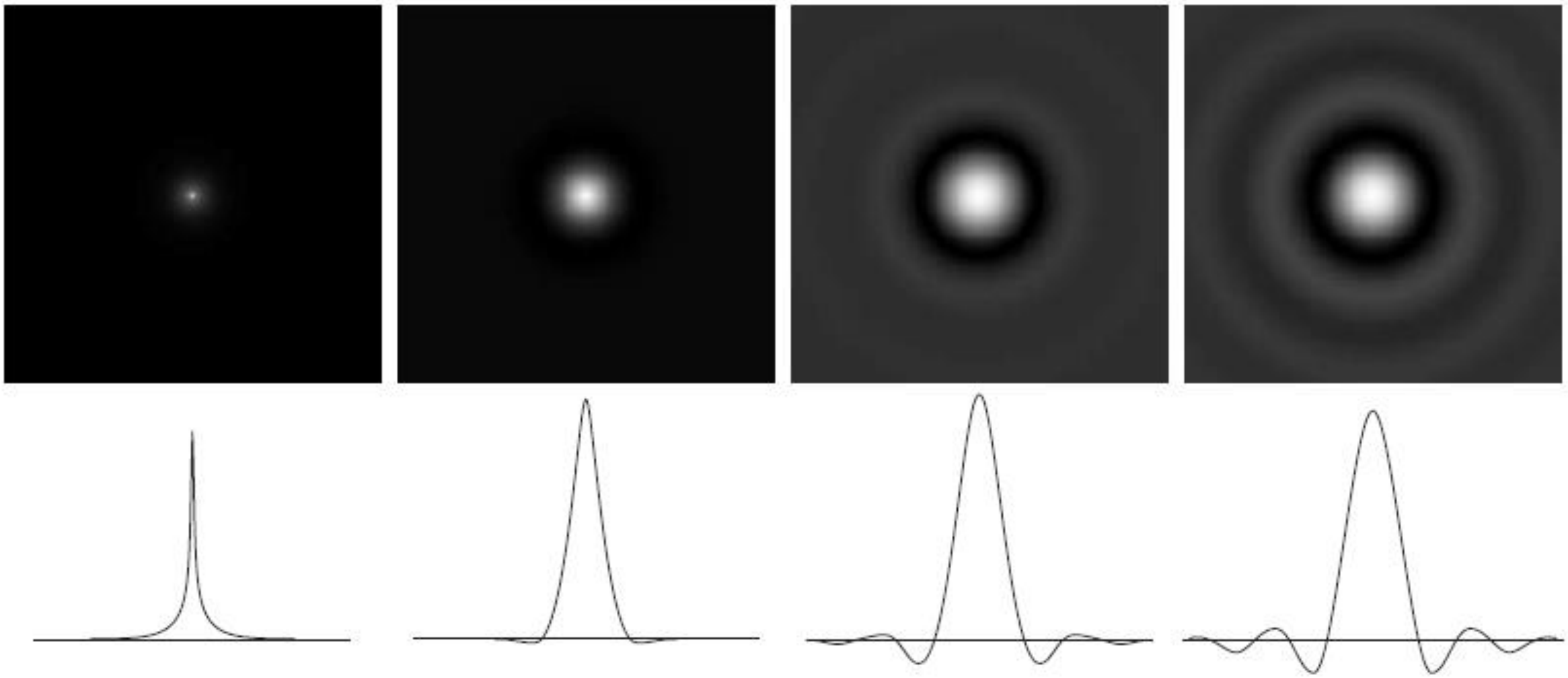


FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is and the cutoff frequency is 5).
Observe how ringing increases as a function of filter order.

Butterworth Lowpass Filters

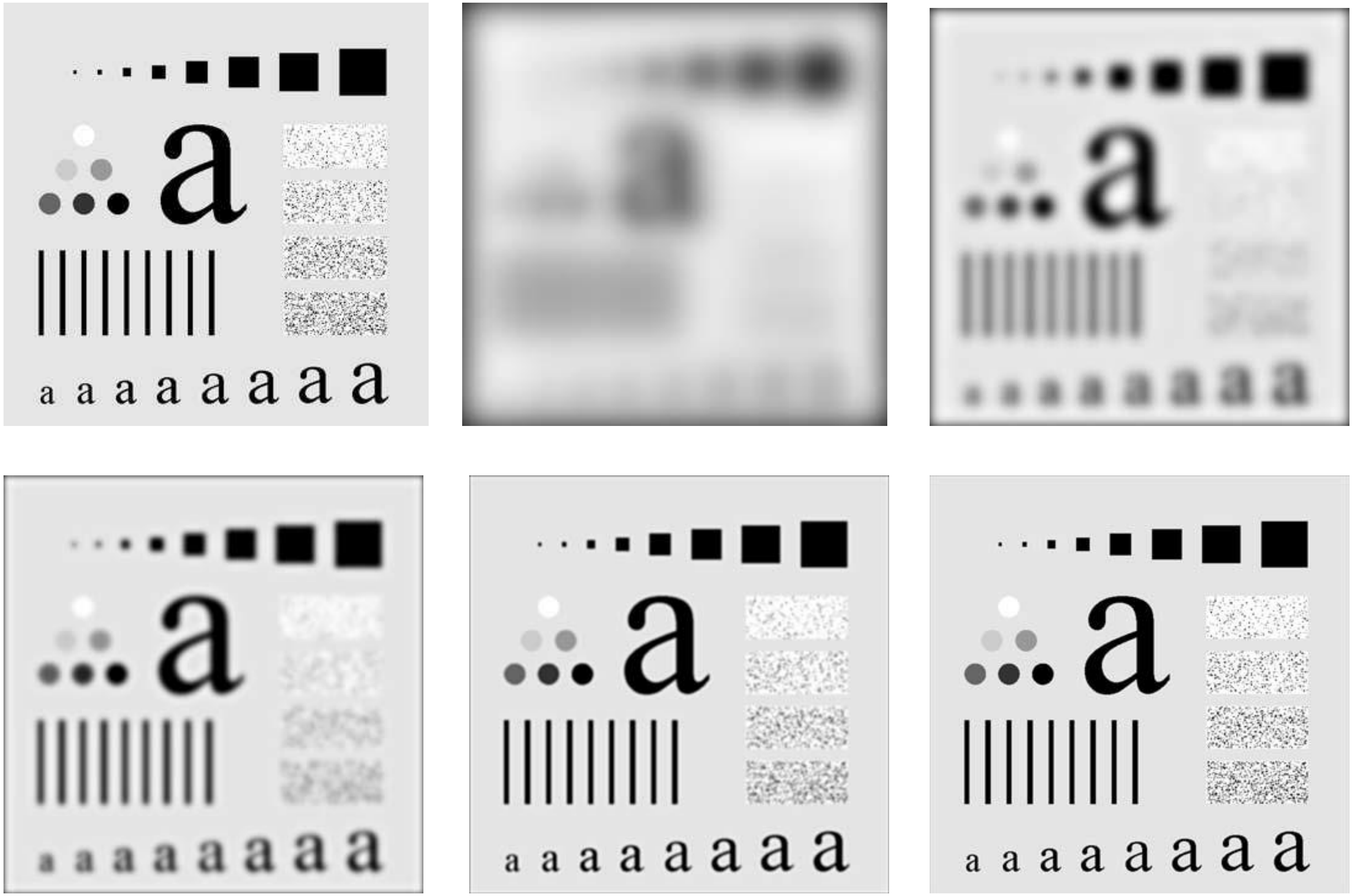
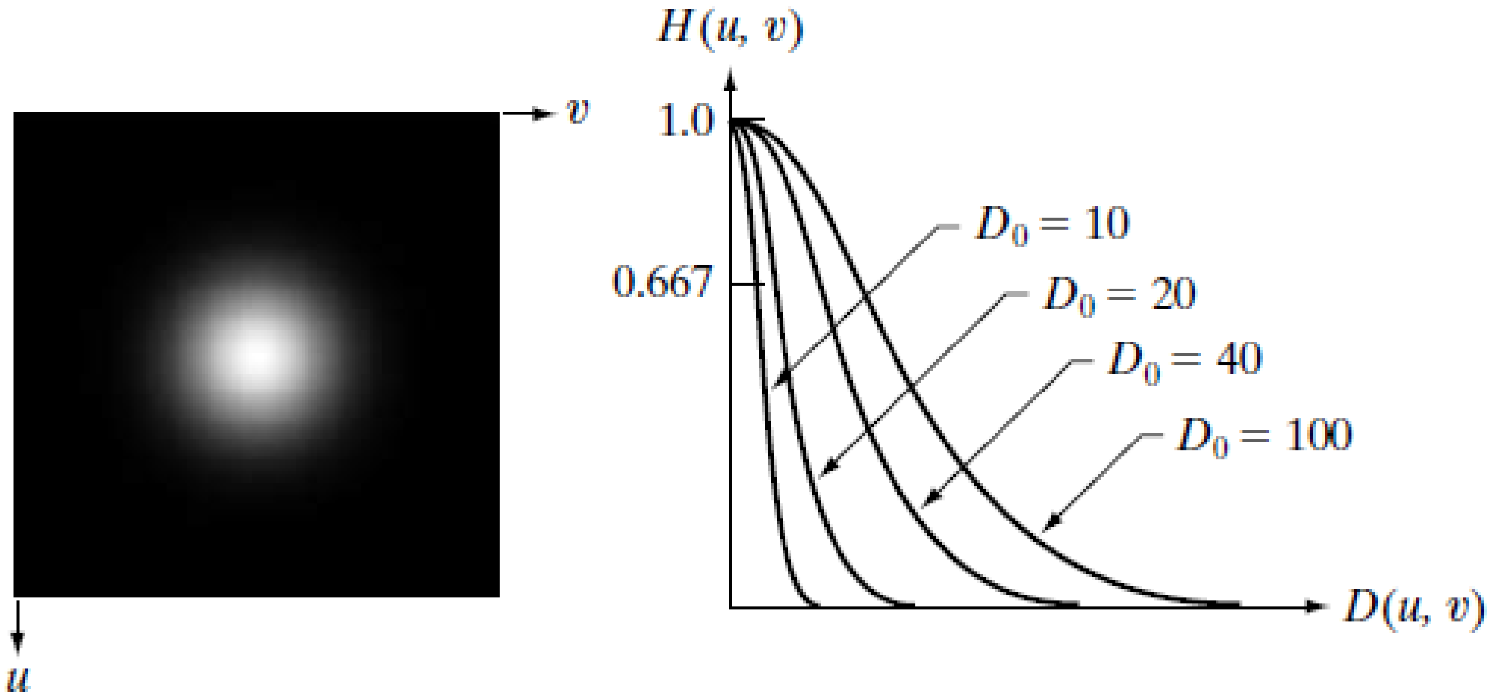


FIGURE: (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii values 10, 30, 60, 160, and 460

- **Gaussian Lowpass Filters** $H(u,v) = e^{-D^2(u,v)/2D_0^2}$



Gaussian Lowpass Filters

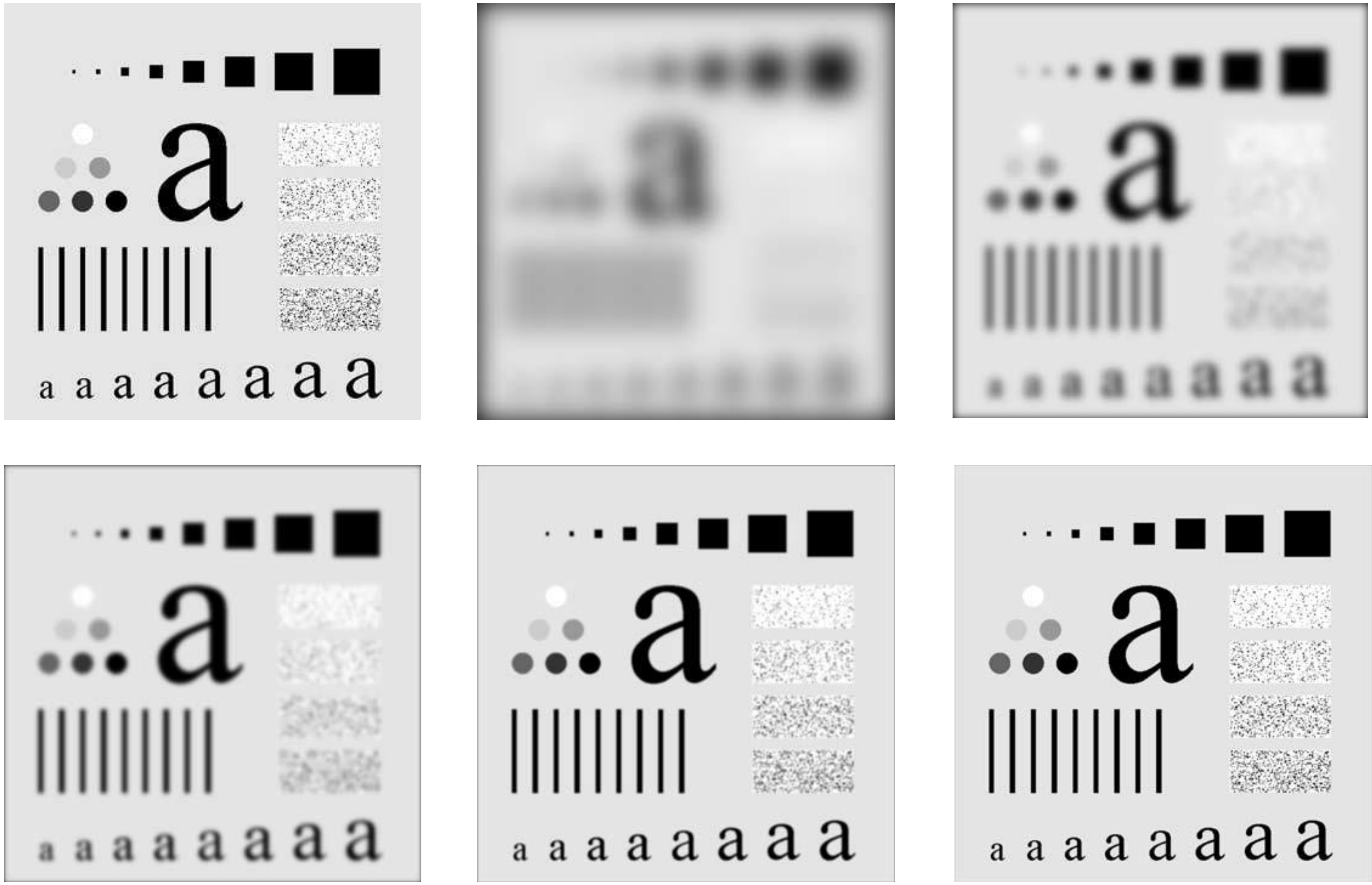


FIGURE: (a) Original image. (b)–(f) Results of filtering using GLPFs, with cutoff frequencies at the radii values 10, 30, 60, 160, and 460

Image Sharpening Using Frequency Domain Filters

- Because edges and other abrupt changes in intensities are associated with high-frequency components, image sharpening can be achieved in the frequency domain by highpass filtering, which attenuates the low-frequency components without disturbing high-frequency information in the Fourier transform

$$H_{\text{HP}}(u, v) = 1 - H_{\text{LP}}(u, v)$$

Image Sharpening Using Frequency Domain Filters

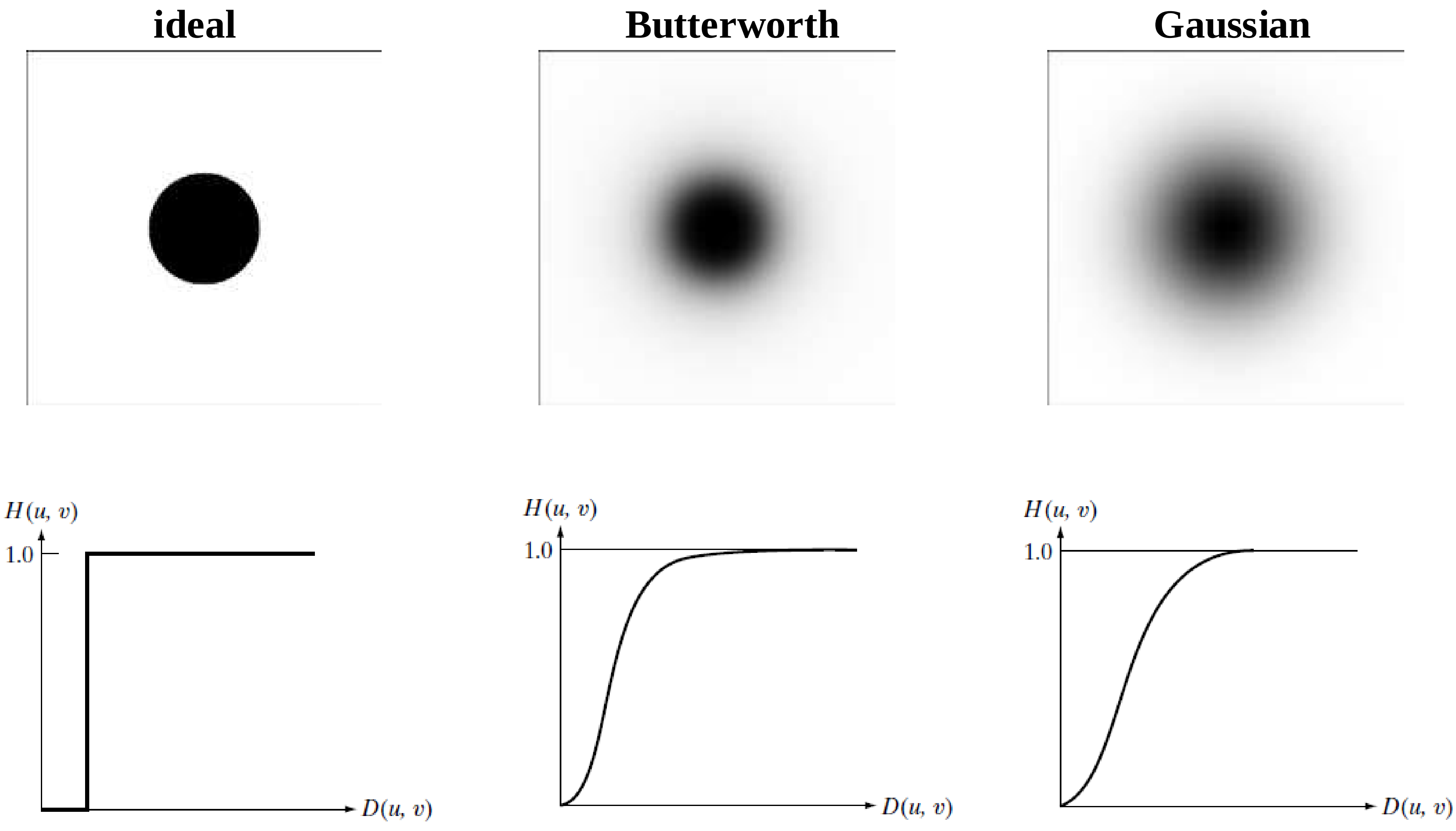


FIGURE: Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows:The same sequence for typical Butterworth and Gaussian highpass filters.

Image Sharpening Using Frequency Domain Filters

IHPF with $D_0 = 30$, 60, and 160.

BHPF of order 2 with $D_0 = 30$, 60, and 160, These results are much smoother than those obtained with an IHPF.

GHPF with $D_0 = 30$, 60, and 160

Image Sharpening Using Frequency Domain Filters

FIGURE 4.57 (a) Thumb print. (b) Result of Butterworth highpass filtering (a). (c) Result of thresholding (b).

Homomorphic Filtering

- The illumination-reflectance model

$$f(x, y) = i(x, y)r(x, y)$$

Illumination component *i(x,y)* : characterized by slow spatial variations

Reflectance component *r(x,y)*: tends to vary abruptly, particularly at the junctions of dissimilar objects.

These characteristics lead to associating the low frequencies of the Fourier transform of the logarithm of an image with illumination and the high frequencies with reflectance. Although these associations are rough approximations.

Homomorphic Filtering

$$\Im[f(x, y)] \neq \Im[i(x, y)]\Im[r(x, y)]$$

$$\begin{aligned} z(x, y) &= \ln f(x, y) \\ &= \ln i(x, y) + \ln r(x, y) \end{aligned}$$

$$\begin{aligned} \Im\{z(x, y)\} &= \Im\{\ln f(x, y)\} \\ &= \Im\{\ln i(x, y)\} + \Im\{\ln r(x, y)\} \end{aligned}$$

$$Z(u, v) = F_i(u, v) + F_r(u, v)$$

$$\begin{aligned} S(u, v) &= H(u, v)Z(u, v) \\ &= H(u, v)F_i(u, v) + H(u, v)F_r(u, v) \end{aligned}$$

$$\begin{aligned} s(x, y) &= \Im^{-1}\{S(u, v)\} \\ &= \Im^{-1}\{H(u, v)F_i(u, v)\} + \Im^{-1}\{H(u, v)F_r(u, v)\} \end{aligned}$$

$$g(x, y) = e^{s(x,y)}$$

Homomorphic Filtering

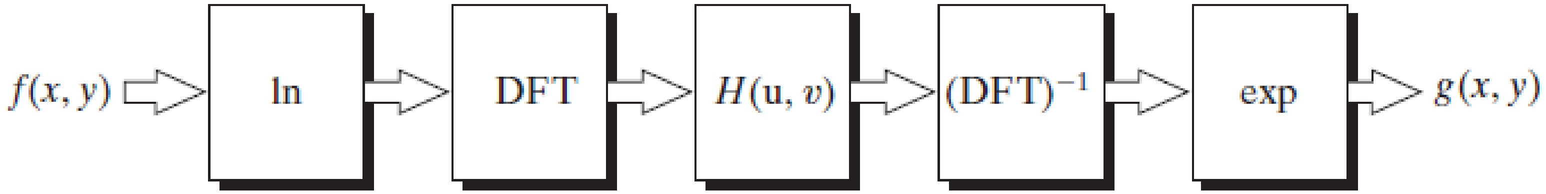


FIGURE: Summary of steps in homomorphic filtering.

The *homomorphic filter H(u,v) function* then can operate on these components separately

Homomorphic Filtering

$$H(u, v) = (\gamma_H - \gamma_L)\left[1 - e^{-c[D^2(u,v)/D_0^2]}\right] + \gamma_L$$

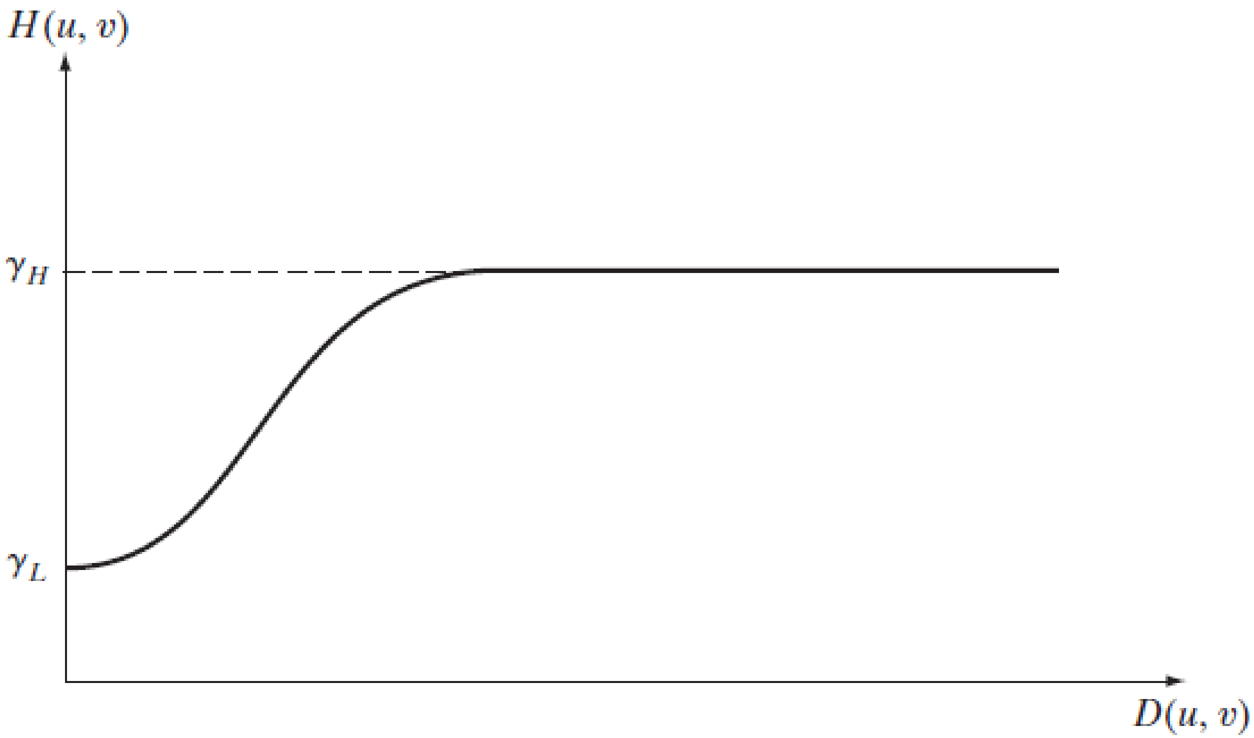


FIGURE: Radial cross section of a circularly symmetric homomorphic filter function. The vertical axis is at the center of the frequency rectangle and *D(u,v)* is the distance from the center.

Homomorphic filtering

Example for homomorphic filtering

Figure: (a) Input Image (b) Image enhanced by high-pass filtering (c) image enhanced by homomorphic filtering

Image Ref: https://en.wikipedia.org/wiki/Homomorphic_filtering

Selective Filtering

- **Bandreject and Bandpass Filters**

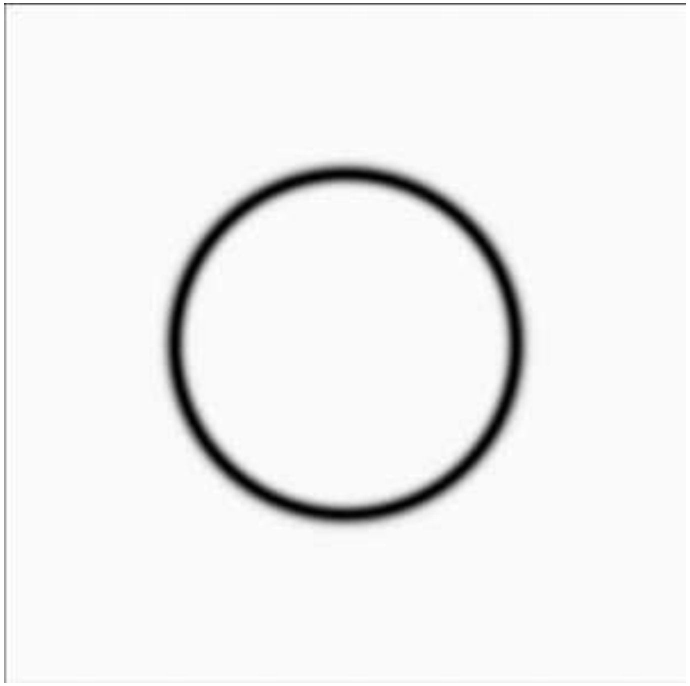


FIGURE: (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Notch Filters

- **Notch Filters**

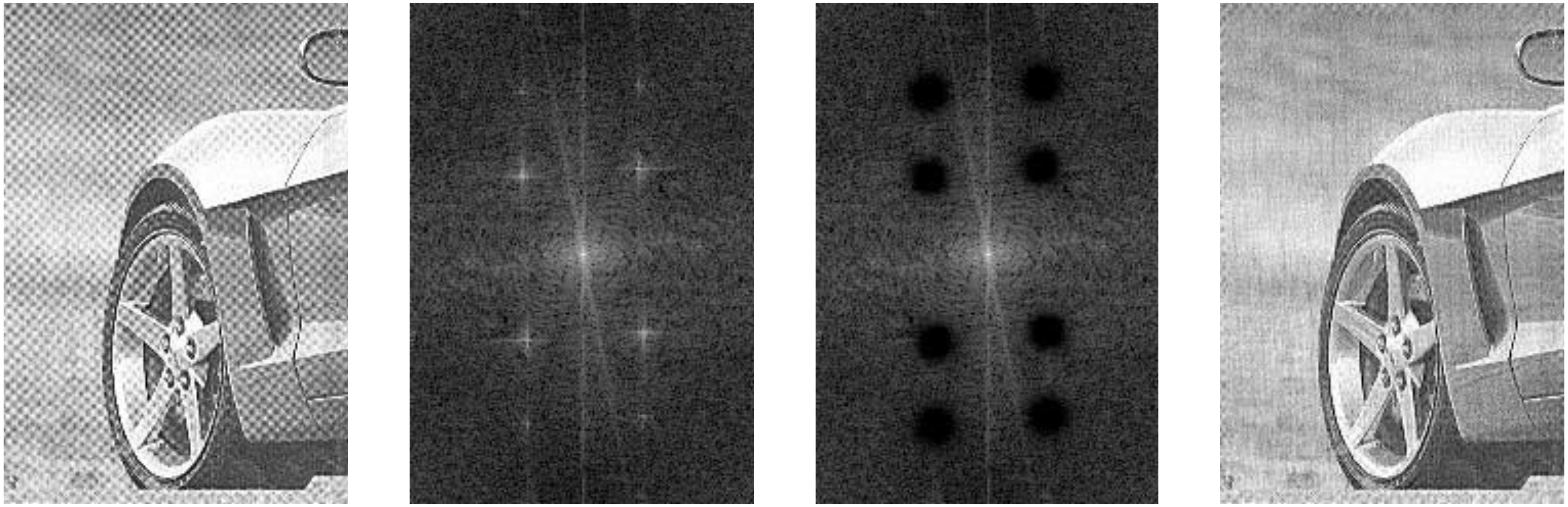
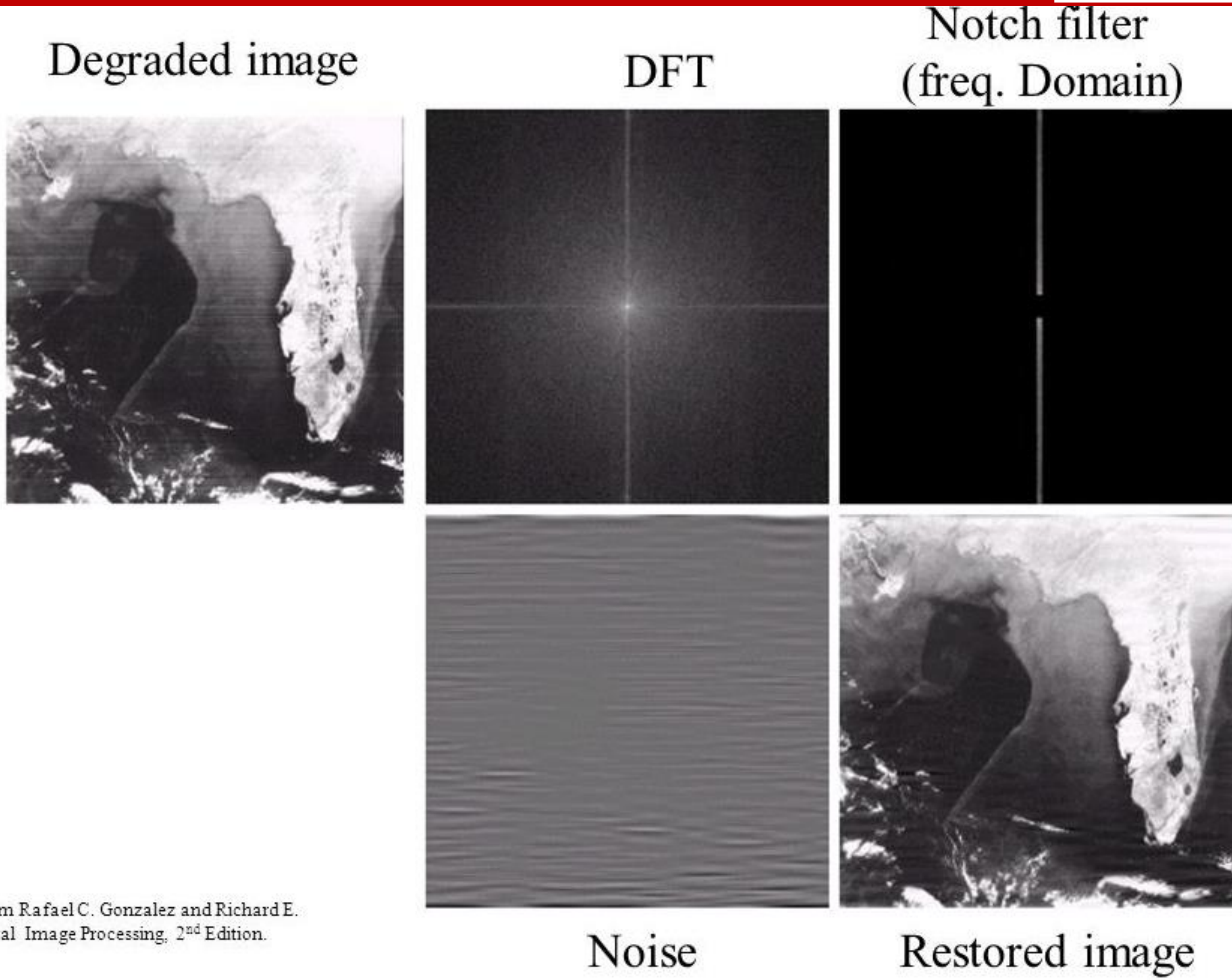


FIGURE: (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

Notch Filters



(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

Notch Filters

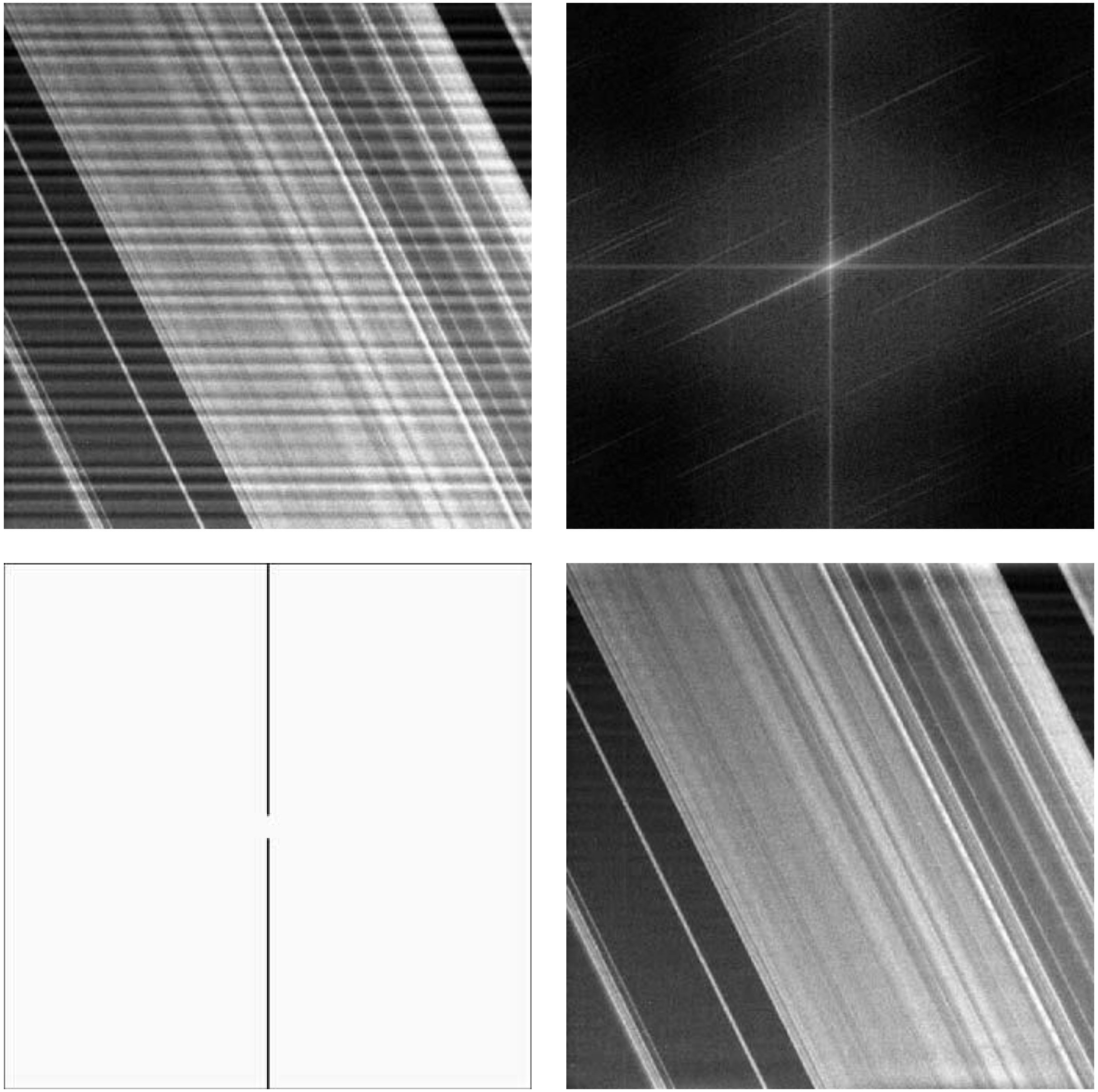


FIGURE: (a) image of the Saturn rings showing nearly Periodic interference. (b) Spectrum:The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering.The thin black border in (c) was added for clarity; it is not part of the data.

- End of Chapter

Thanks