The Black-Scholes Option Pricing Model

European Call Options

The Black-Scholes model for the valuation of a European call option on an asset that pays a dividend is as follows:

$$C = S \cdot e^{-\delta \cdot T} \cdot N(d_1) - X \cdot e^{-r \cdot T} \cdot N(d_2)$$

where $$d_1 = \frac{ln\left(S/_X\right) + \left(r - \delta + \frac{\sigma^2}{2}\right) \cdot T}{\sigma \cdot \sqrt{T}}$$ and $$d_2 = d_1 - \sigma \cdot \sqrt{T}$$

- *S* is the current market value of the underlying asset
- σ is the standard deviation of (log) returns on the underlying asset (per year)
- δ is the annual continuously-compounded dividend yield on the underlying asset
- *r* is the annual continuously-compounded risk-free rate of return
- *X* is the exercise (or strike) price of the option
- *T* is the time until maturity of the option (in years)
- $N(d_1)$ and $N(d_2)$ represent the cumulative unit normal probability density function evaluated at d_1 and d_2, respectively. In other words, d_1 is a z-score, and $N(d_1)$ is the cumulative probability that a unit normal random variable < d_1. Evaluate this using Microsoft Excel's =Norm.S.Dist(d_1 , TRUE) function.

The call option's *delta* is the "hedge ratio". It measures the local sensitivity of the option's value to changes in the value of the underlying asset.

$$Call\ Delta = \frac{\partial C}{\partial S} = e^{-\delta \cdot T} \cdot N(d_1)$$

The call option's *gamma* measures the local rate of change in the delta as the value of the underlying changes, and it provides a metric for how "wrong" the delta-hedge can be.

$$Call\ Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta \cdot T}}{S \cdot \sigma \cdot \sqrt{T}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{d_1^2}{2}}$$

European Put Options

By application of Put-Call Parity, we can write the Black-Scholes model for a European put as follows:

$$P = C - S \cdot e^{-\delta \cdot T} + X \cdot e^{-r \cdot T}$$

$$Put\ Delta = \frac{\partial P}{\partial S} = e^{-\delta \cdot T} \cdot (Call\ Delta - 1)$$

$$Put\ Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{e^{-\delta \cdot T}}{S \cdot \sigma \cdot \sqrt{T}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{d_1^2}{2}}$$