Poisson Distribution :-

Defn : A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = p(x,\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} , & x=0,1,2,\cdots ; \lambda>0 \\ 0 , & \text{Otherwise} \end{cases}$$

Here, λ is known as the parameter of the distribution.
We shall use the notation $X \sim P(\lambda)$, to denote that X is a Poisson variate with parameter λ.

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Moments of the Poisson Distribution :-

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, p(x,\lambda)$$

add & sub 1

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^{x-1+1}}{x!}$$

$$= e^{-\lambda}\lambda \sum_{x=0}^{\infty} \frac{x \lambda^{x-1}}{x!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{x \lambda^{x-1}}{x(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left\{1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right\} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Hence, the mean of the Poisson distribution is λ

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x,\lambda)$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x(x-1)\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \quad \because \text{from } ④$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\mu_3' = E(X^3)$$

$$= \sum_{x=0}^{\infty} [x(x-1)(x-2) + 3x(x-1) + x] \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} + 3\sum_{x=0} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} x(x-1)(x-2)\frac{\lambda^{x-3+3}}{x!} + 3e^{-\lambda}\sum_{x=0}^{\infty} x(x-1)\frac{\lambda^{x-2+2}}{x!} + \cancel{e^{-\lambda}\sum_{x=0} x\lambda^x}$$

$$+ e^{-\lambda}\sum \frac{x\lambda^{x-1+1}}{x!}$$

$$= e^{-\lambda}\sum_{x=3}^{\infty} \frac{\cancel{x(x-1)(x-2)}\lambda^{x-3}\cdot\lambda^3}{\cancel{x(x-1)(x-2)}(x-3)!} + 3e^{-\lambda}\sum_{x=2}^{\infty} \frac{\cancel{x(x-1)}\lambda^{x-2}\cdot\lambda^2}{\cancel{x(x-1)}(x-2)!}$$

$$+ e^{-\lambda} \sum \frac{\cancel{x}\ \lambda^{x-1}}{\cancel{x}(x-1)!} \cdot \lambda$$

$$= e^{-\lambda} \lambda^3 \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} + 3e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$+ e^{-\lambda} \lambda \sum_{x=3} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda^3 \cdot e^{\lambda} + 3 e^{-\lambda} \lambda^2 e^{\lambda} + e^{-\lambda} \lambda \cdot e^{\lambda}$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} x^4 \cdot p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} \left[x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x \right] \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \lambda^4 \left\{ \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} \right\} + 6 e^{-\lambda} \lambda^3 \left\{ \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 7 e^{-\lambda} \lambda^2 \left\{ \sum \frac{\lambda^{x-2}}{(x-2)!} \right\}$$

$$+ \lambda$$

$$= e^{-\lambda} \lambda^4 e^{\lambda} + 6 e^{-\lambda} \lambda^3 e^{\lambda} + 7 e^{-\lambda} \lambda^2 e^{\lambda} + \lambda$$

$$\mu_4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four Central moments are now Obtained as follows.

$\mu_2 = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Thus mean & Variance of the Poisson distribution are ~~eq~~ each equal to λ.

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$.

$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3$.

$= \cancel{\lambda^3} + \cancel{3\lambda^2} + \lambda - \cancel{3\lambda^3} - \cancel{3\lambda^2} + \cancel{2\lambda^3}$.

$\mu_3 = \lambda$.

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$.

$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda)$
$+ 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4$

$\mu_4 = 3\lambda^2 + \lambda$.

Coefficent of Skewness : $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}$

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \dfrac{1}{\lambda}$.

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Moments Generating function of the Poisson Distribution:-

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \cdots\right\} = e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$