Binomial Distribution

Def^n: A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = p(x) = \begin{cases} {}^nC_x \, p^x q^{n-x} & ; x = 0,1,2,\dots n. \\ & q = 1-p \\ 0 & ; \text{otherwise} \end{cases}$$

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Moments of Binomial Distribution:-

The first four moment about point of binomial distribution are obtained as follows.

$$\mu_1' = E(X) = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x \cdot {}^nC_x \, p^x q^{n-x} = \sum_{x=1}^{n} x \cdot \frac{n}{x} \, {}^{n-1}C_{x-1} \, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} n \, {}^{n-1}C_{x-1} \, p^{x-1+1} \cdot q^{n-x}$$

$$= \sum_{x=1}^{n} np \, {}^{n-1}C_{x-1} \, p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{n-x}$$

$$\mu_1' = np(p+q)^{n-1} = np(1)^{n-1} = np$$

$$\boxed{\mu_1' = np = \text{mean}}$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \, {}^nC_x \, p^x q^{n-x} + \underline{\sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}}$$

directly put from μ_1'

$$= \sum_{x=2}^{n} \cancel{x}(\cancel{x-1}) \cdot \frac{n}{\cancel{x}} \frac{(n-1)}{(\cancel{x-1})} \, {}^{n-2}C_{x-2} \, p^x q^{n-x} + \underline{\sum_{x=1}^{n} \cancel{x} \cdot \frac{n}{\cancel{x}} \, {}^{n-1}C_{x-1} \, p^x q^{n-x}}$$

add & sub 1.

$$= n(n-1) \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^x q^{n-x} + n \underline{\sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} \cdot p \cdot q^{n-x}}$$

add & sub 2

$$= n(n-1)\sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2}\cdot p^2\, q^{n-x} + np\sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x} + np\,(p+q)^{n-1}$$

$$= n(n-1)p^2 \,(p+q)^{n-2} + np\,(1)$$

$$= n(n-1)\,p^2\,(1) + np$$

$$= n(n-1)p^2 + np.$$

$$\therefore \boxed{\mu_2' = n(n-1)p^2 + np}$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 p(x)$$

$$= \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)\, {}^nC_x\, p^x q^{n-x} + \sum_{x=0}^{n} 3x(x-1)\, {}^nC_x\, p^x q^{n-x} + \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x}.$$

$$= \sum_{x=3}^{n} x(x-1)(x-2)\cdot \frac{n(n-1)(n-2)}{x(x-1)(x-2)}\, {}^{n-3}C_{x-3}\, p^x q^{n-x} + 3n(n-1)p^2 + np.$$

from (*) from (*)

$$= n(n-1)(n-2) \sum_{x=3}^{n} {}^{n-3}C_{x-3}\, p^{x-3}\cdot p^3\, q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} {}^{n-3}C_{x-3}\, p^{x-3} q^{n-x} + 3n(n-1)p^2 + np.$$

$$= n(n-1)(n-2)\,p^3\,(p+q)^{n-3} + 3n(n-1)p^2 + np.$$

$$= n(n-1)(n-2)\,p^3 + 3n(n-1)p^2 + np.$$

Similarly

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{n} x^4 \, {}^nC_x \, p^x q^{n-x}.$$

On simplification we get

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

$$\therefore \mu_1 = 0$$

$$= E(X^2) - [E(X)]^2$$

$$Var(x) = \mu_2 = \mu_2' - \mu_1'^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= (n^2 - n)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\boxed{\mu_2 = np(1-p) = npq} \qquad \because p+q=1.$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\}$$

$$-3\{n(n-1)p^2 + np\}np + 2(np)^3\}$$

$$= np\{(n-1)(n-2)p^2 + 3(n-1)p + 1 - 3n(n-1)p^2 - 3np + 2n^2p^2\}$$

$$= np\{(n^2-3n+2)p^2 + 3np - 3p + 1 - 3n^2p^2 + 3np^2 - 3np + 2n^2p^2\}$$

$$= np\{\cancel{n^2p^2} - \cancel{3np^2} + 2p^2 + \cancel{3np} - 3p + 1 - \cancel{3n^2p^2} + \cancel{3np^2} - \cancel{3np} + \cancel{2n^2p^2}\}$$

$$= np\{2p^2 - 3p + 1\}$$

$$= np\{2p^2 - 2p - p + 1\} \qquad q = 1-p$$

$$= np\{2p^2 - 2p + q\}$$

$$= np\{2p(p-1) + q\} \qquad 1 = p+q$$

$$= np\{-2pq + q\} \qquad 1-2p = p+q-2p$$

$$= npq\{1-2p\} \qquad = q-p$$

$$= npq\{p+q-2p\}$$

$$\boxed{\mu_3 = npq\{q-p\}}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$\boxed{\mu_4 = npq\{1+3(n-2)pq\}}$$

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \dfrac{(q-p)^2}{npq} = \dfrac{(1-2p)^2}{npq}$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[1+3(n-2)pq]}{n^2p^2q^2} = \frac{1+3(n-2)pq}{npq} = 3 + \frac{1-6pq}{npq}$$

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Moments Generating Function of Binomial Distribution :

$$\cancel{E\{e^{t(x-np)}\}} = \cancel{e^{-tnp}} \cdot \cancel{E(e^{tx})}$$

$$= \cancel{e^{-tnp} \cdot M_x(t)}$$

Let $X \sim B(n, p)$, then.

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x q^{n-x} = (q+pe^t)^n$$