Time Series Analysis & Forecasting

Class 1

Arnab Bose, Ph.D.

MSc Analytics

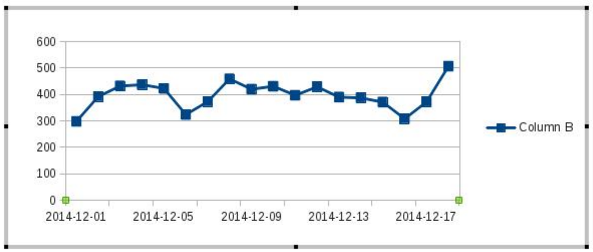
University of Chicago

Time Series (TS)

- Data is measured at different time points –

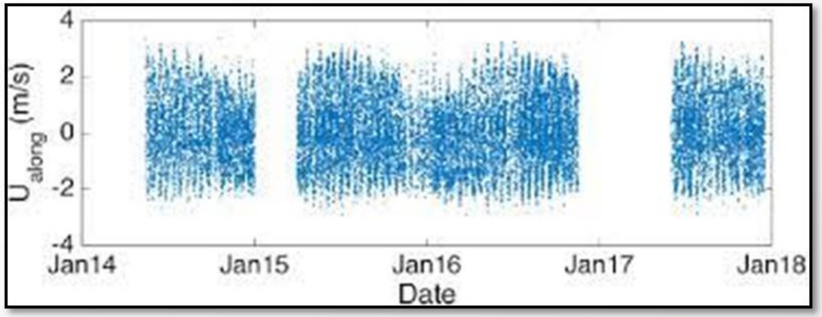
a. Uniform TS – uniform time intervals

2014-12-01	298
2014-12-02	392
2014-12-03	432
2014-12-04	437
2014-12-05	423
2014-12-06	324
2014-12-07	373
2014-12-08	459
2014-12-09	420
2014-12-10	431
2014-12-11	397
2014-12-12	429
2014-12-13	390
2014-12-14	387
2014-12-15	371
2014-12-16	308
2014-12-17	372
2014-12-18	507



https://stats.stackexchange.com/questions/129339/how-to-calculate-probabilities-based-on-cumulative-of-time-series

b. Non-uniform TS – non-uniform time intervals (uneven spacing)



https://www.researchgate.net/figure/Example-of-a-non-uniform-time-series-of-along-channel-velocity-from-ferry-based_fig4_333984372

- Data is represented as $\{y_t\} = \{y_1, y_2, y_3, \dots, y_t\}$
- Primary objective of TS analysis is to develop mathematical models that provide plausible description of the data
- Data collection -> separate field of study that outlines the challenges in data collection

Random Variable (RV)

A variable whose values are defined by a probability distribution. The distribution specifies the probability that the value of the variable is within any given interval.

RV can be

discrete – taking finite or countable list of values

continuous – taking any numerical value

If 2 RVs are independent then they are uncorrelated.

If 2 RVs have joint Gaussian distribution and are uncorrelated, then they are independent.

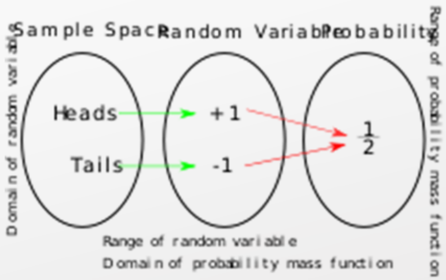
Random Variable (RV)

A variable whose values are defined by a probability distribution

The distribution specifies the probability that the value of the variable is within any given interval

RV can be:

- discrete : taking finite or countable list of values
- continuous : taking any numerical value

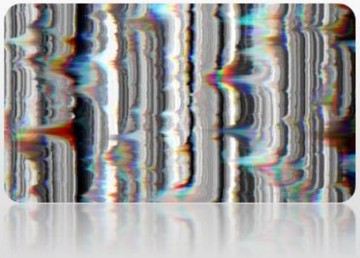


https://en.wikipedia.org/wiki/Random_variable

If 2 RVs

1. are independent then they are uncorrelated
2. have joint Gaussian distribution and are uncorrelated, then they are independent

White Noise



1. Collection of uncorrelated random variables w_t that are uncorrelated with mean 0 and variance σ_w^2
2. White – analogy with white light where all possible frequencies are present with equal strength
3. Gaussian white noise is independent and identically distributed (iid)

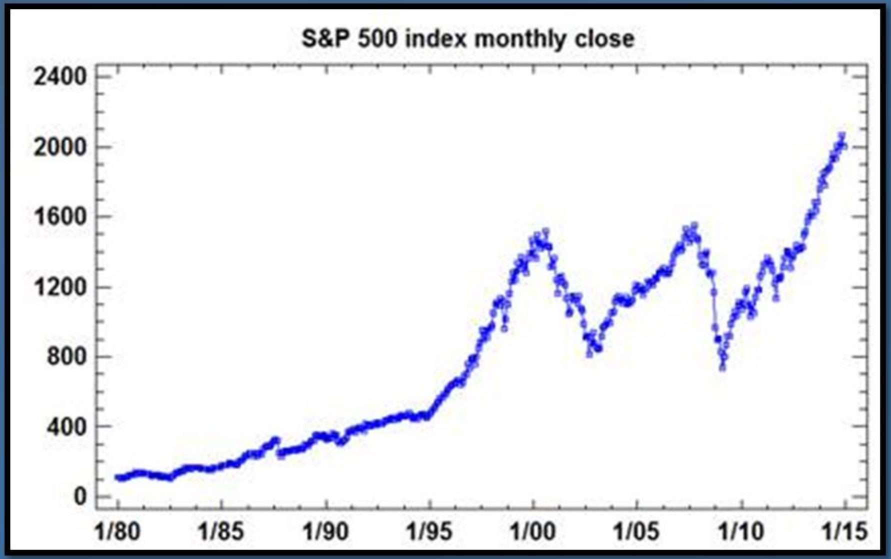
White Noise – Practical Example

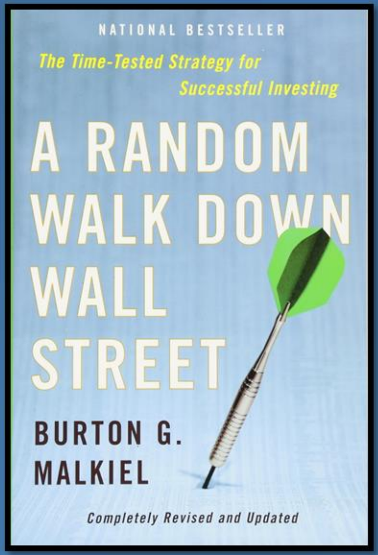
Random Walk

1. Mathematically represented as $y_t = y_{t-1} + w_t$

2. Current value y_t is previous value y_{t-1} plus completely random white noise w_t

Random Walk – Practical Example





TS Characteristics – Autocovariance and Autocorrelation

1. **Autocovariance** – measures the linear dependency between two points in the same TS observed at different times.

   Represented with 2 notations (note that the important concept of time lag or the time difference between the 2 TS)

$$\gamma_{s,t} = Cov(y_s, y_t)$$

$$\gamma_k = Cov(y_s, y_{s-k})$$

2. **Autocorrelation** – normalized autocovariance given by

$$\rho_{s,t} = \frac{Cov(y_s, y_t)}{\sqrt{Cov(y_s, y_s) * Cov(y_t, y_t)}} = \frac{Cov(y_s, y_t)}{\sqrt{Var(y_s) * Var(y_t)}}$$

Stationarity – Strict

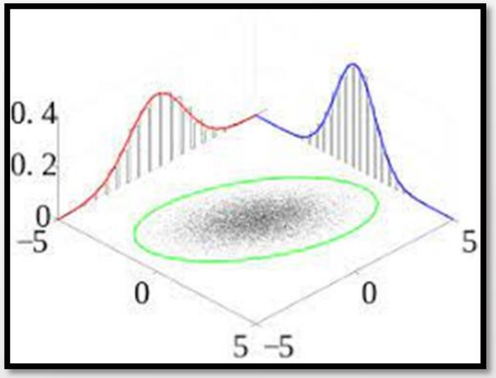
TS is said to be stationary – the process is in statistical equilibrium.

Basic idea is that the laws of probability that govern the process behavior do not change over time.

STRICT stationarity

1. When the probability behavior of every collection of TS data points $\{x_{t1}, x_{t2}, x_{t3}, \ldots . x_{tk}\}$ is identical to that of time shifted data points $\{x_{t1+h}, x_{t2+h}, x_{t3+h}, \ldots . x_{tk+h}\}$ for all h.

2. Another definition – the joint probability distribution is constant over time.



https://en.wikipedia.org/wiki/Joint_probability_distribution

Stationarity – Weak

WEAK stationarity aka Wide Sense Stationary (WSS) or Covariance Stationary

TS $\{y_t\}$ has a finite variance process such that

1. Mean is constant and does not depend on time
2. Auto covariance function $\gamma_{s,t}$ depends on s, t only through their difference |t-s|

NOTE:

1. IF TS is Gaussian joint distribution (meaning the the RVs are Gaussian) then Weak same as Strict stationary
2. Finite 2nd moments are not assumed for strict => strict does not imply weak

Non-Stationarity

Trend

Time Series with trend is non-stationary but can be made stationary by removing the trend → Trend Stationarity

Seasonality

Time Series with seasonality (repeating cycles) is non-stationary.

Examples of seasonality = hourly (repeating pattern every 24 hours)

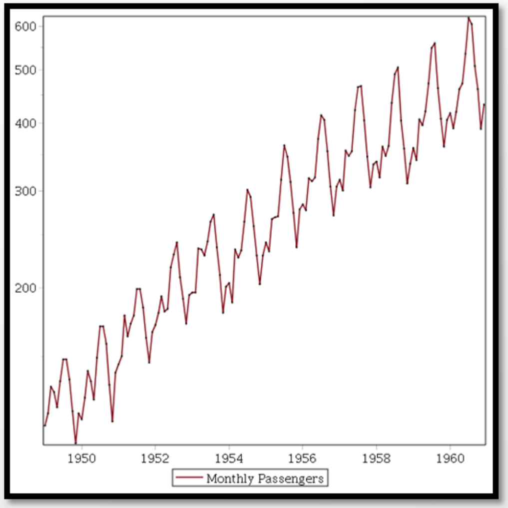
= weekly

= monthly

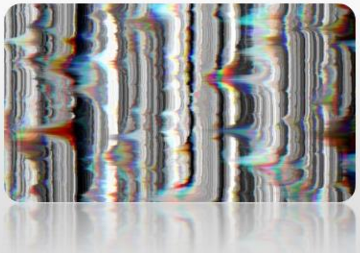
= quarterly

= half-yearly

= yearly



White Noise Stationarity – first and second order moments



1. Mean = μ = 0

2. Auto covariance $\gamma_{s,t} = \begin{cases} \sigma_w^2, & s = t \\ 0, & s \neq t \end{cases}$

3. So white noise is weak stationary since the auto-covariance is dependent on s - t

Random Walk Stationarity – first order moment

Process

$$y_t = y_{t-1} + w_t$$
$$= y_{t-2} + w_{t-1} + w_t$$
$$\vdots$$
$$= y_0 + w_t + w_{t-1} + w_{t-2} + \cdots + w_1$$

Assume $y_0 = 0$

Mean $\mu = E[y_t] = E[y_0 + \sum_{s=1}^{t} w_s] = E[y_0] + \sum_{s=1}^{t} E[w_s] = 0$

which is time-invariant

Random Walk Stationarity – second order moment

Autocovariance

$$\gamma_{t,s} = E[y_t\, , y_s]$$

$$= E\left[\sum_{k=1}^{t} w_k \sum_{m=1}^{s} w_m\right] = E\left[\sum_{k=1}^{\min(t,s)} w_k^2 + \sum_{m=1}^{t} \sum_{n=1,n\neq m}^{s} w_m w_n\right]$$

$$= E\left[\sum_{k=1}^{\min(t,s)} w_k^2\right] = \sum_{\min(t,s)} w^2$$

Assume $t \leq s$

$$= t\sigma^2$$

NOTE: autocovariance depends on the particular t, s and not on the distance between two time points => non-stationary.

Autocorrelation

$$\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} =$$

$$\frac{t\sigma^2}{\sqrt{t\sigma^2 s\sigma^2}} = \sqrt{\frac{t}{s}}, t \leq s$$

Stationarity ↔ Independent and Identically Distributed (iid)

iid => strict stationarity

Since the joint distribution is the product of common marginal distribution that does not depend on time.

Strict stationarity $\neq>$ iid

Strict stationarity implies RVs are identically distributed but nothing about independence.

Eg. TS {X, X, X, …. }

Strict Stationarity ↔ Weak Stationarity

Strict stationarity => weak stationarity

IFF mean and all the covariance exist and are finite.

Eg, iid Cauchy process is strictly stationary but not weakly since it has not finite 2^{nd} order moment.

In other words, a strict stationary process with finite 2^{nd} order moment is weak stationary*.

Strict stationarity => weak stationarity

For example, a white noise is weak stationary and also strict stationary if it is a Gaussian white noise.

NOTE: general white noise only implies uncorrelation while Gaussian white noise also implies independence. Because if a process is Gaussian, uncorrelation implies independence. Therefore, a Gaussian white noise is just iid N(0, σ2).

[* https://stats.stackexchange.com/questions/159899/strictly-stationary-time-series-with-infinite-moments](https://stats.stackexchange.com/questions/159899/strictly-stationary-time-series-with-infinite-moments)

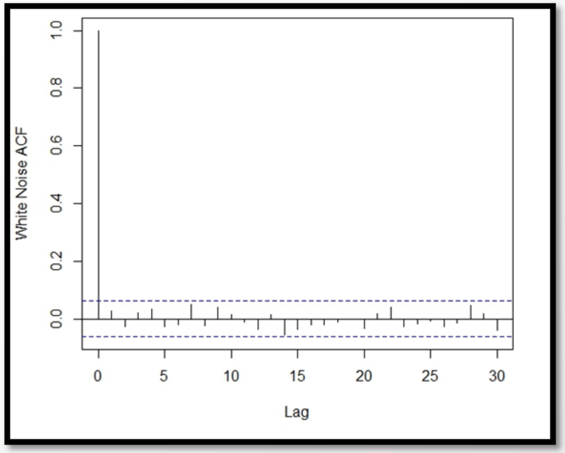
Exploratory Data Analysis

1. Idea is to understand Time Series datasets, both univariate and multivariate using
   a. Qualitative – analyze plots
   b. Quantitative – use tests and numbers
2. Univariate – test for stationarity for example with a unit root test such as Augmented Dickey Fuller (ADF)
3. Augmented in ADF – no assumption about the error term u_t being uncorrelated
4. To transform a non-stationary TS to stationary TS, iteratively take the difference of the non-stationary TS and verify if the differenced TS is stationary
5. If TS has a unit root, then the first difference is stationary

Stationarity Tests – Qualitative

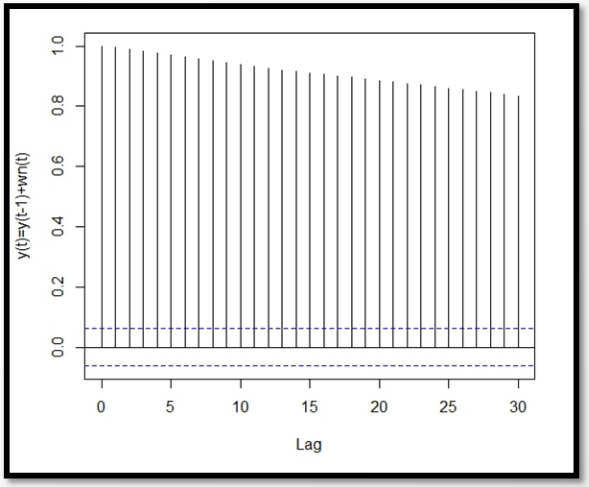
1. Stationary

Dying down quickly



2. Non-stationary

Dying down extremely slowly



Stationarity Tests – Quantitative

1. Augmented Dickey-Fuller (ADF) Test

The option alternative in adf.test() takes the value 'stationary' or 'explosive'. The value 'explosive' is used to test if the series is stationary about a linear time trend. This means that a constant and trend are to be included in the DF or ADF test regression.

R code

adf.test() – check the p-value

2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS) Test

The KPSS test will often select fewer differences than the ADF test or a PP test. A KPSS test has a null hypothesis of stationarity, whereas the ADF and PP (Phillips-Perron) tests assume that the data have I(1) non-stationarity.

R code

kpss.test()

H_0 – TS is stationary (level – default or trend)

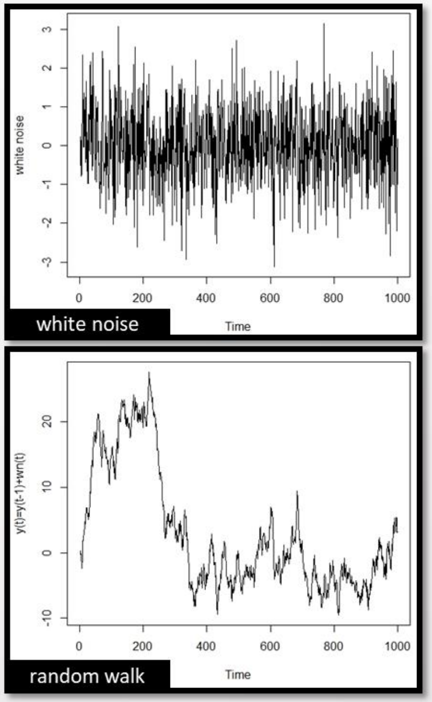
Statistical Significance of *p*-Value

1. P-values indicate how incompatible the data are with a specified statistical model
2. P-values do not measure the probability that the studied hypothesis is true
3. Scientific conclusions and business or policy decisions should not be based on specific threshold
4. Proper inference requires full reporting and transparency
5. P-values do not measure the size or importance of an effect
6. P-value by itself does not provide a good measure of evidence regarding a model or hypothesis

https://amstat.tandfonline.com/doi/pdf/10.1080/00031305.2016.1154108?needAccess=true

R code – white noise + random walk

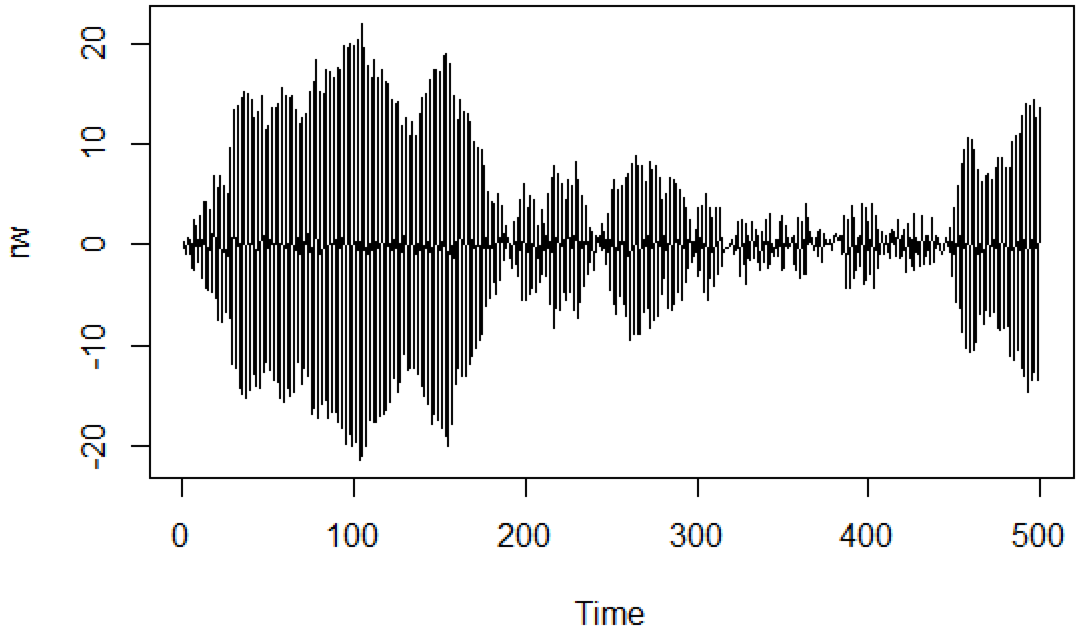
```
wn <- rnorm(500, 0, 1) # White Noise
rw <- cumsum(wn) # Random Walk

plot.ts(wn, main="white noise")
plot.ts(rw, main="random walk")

acf(wn)
acf(rw)

library("tseries")
adf.test(wn)
adf.test(rw)
```



R code – TS with $\boldsymbol{\rho = -1}$

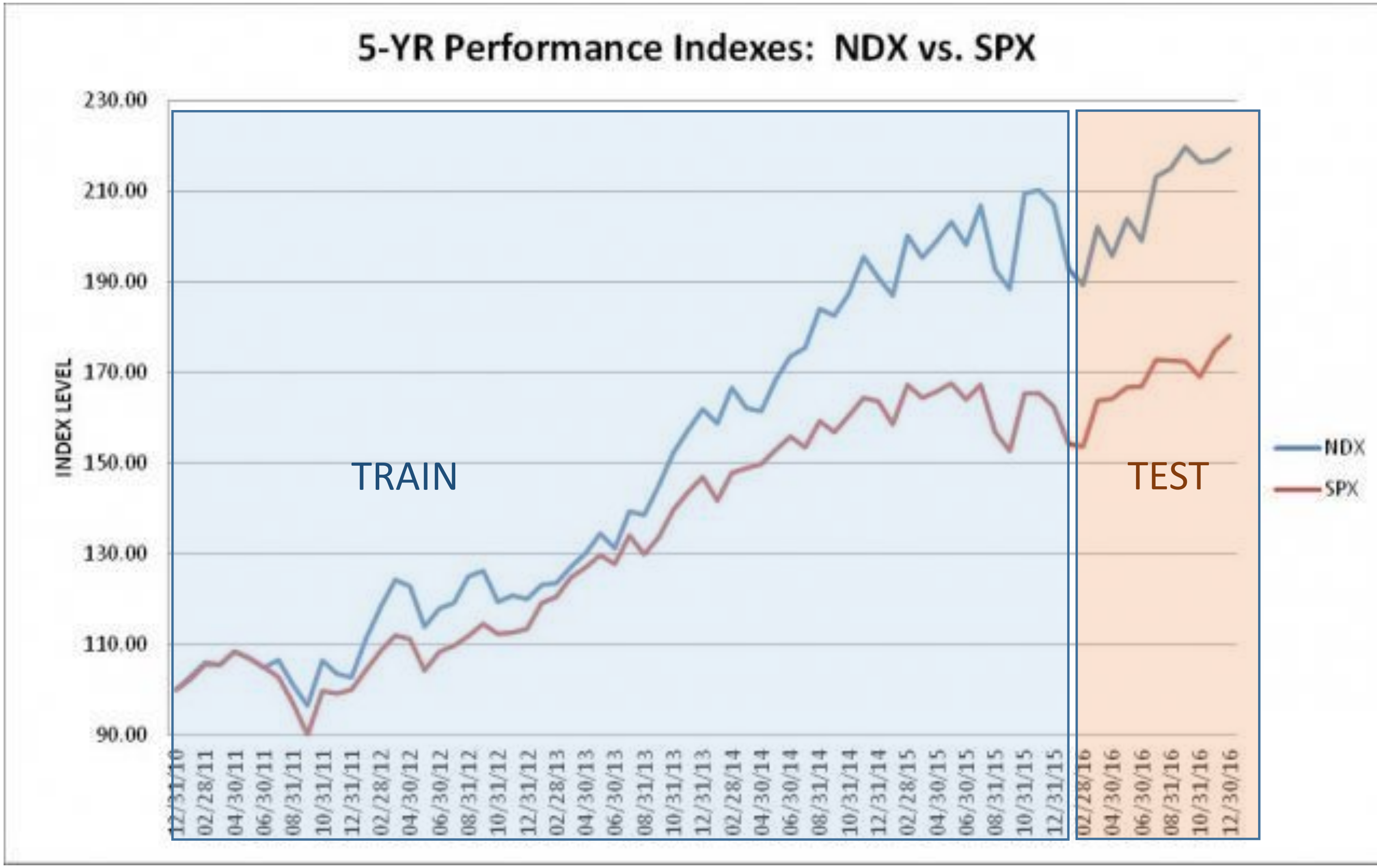
```
rw <- c(0)
wn <- rnorm(500, 0, 1)
rw[1] <- wn[1]

for (i in 2:length(wn))
  rw[i] <- -1 * rw[i-1] + wn[i]

ts.plot(rw)
adf.test(rw)
kpss.test(rw)
```

Random walk with $\boldsymbol{\rho = -1}$



Time Travel – What ?!?

Go “back in time” to evaluate a TS Model



Exponential Smoothing

- A method to smooth time series using weight $0 \leq \alpha \leq 1$ where the recent observations are weighted more than the less recent ones.

- Mathematically represent smooth TS as level

$$l_t = \alpha x_t + (1 - \alpha)\, l_{t-1}$$

- Forecast

$$\hat{y}_{t+\tau}(t) = l_t$$

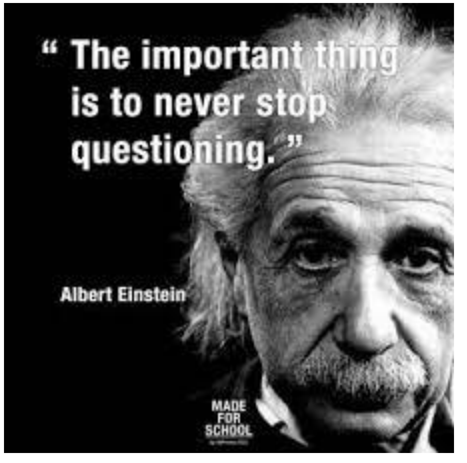
Textbook Chapters

Materials covered available in book chapters:

FPP: 1, 2, 4, 6, 8

PTS: 1 – 5

Thank You


" The important thing
is to never stop
questioning."
Albert Einstein
MADE
FOR
SCHOOL