Time Series Analysis & Forecasting

Class 7

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ARCH Model

In 1982, Robert Engle developed the Autoregressive Conditional Heteroscedasticity (ARCH) model.

Assume we have a TS $\{\varepsilon_t\}$ with heteroscedasticity.

ARCH(q) model represented as

$$\varepsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + .. + \alpha_q \varepsilon_{t-q}^2$$

Where e_t is iid with mean = 0 and variance = 1

When to use an ARCH Model

ARMA (p,q) process

$$\phi(B)Y_t = \theta(B)\varepsilon_t$$

where ε_t is assumed to be error and $\{\varepsilon_t\}$ can be a TS with heteroscedasticity.

Need to verify and detect heteroscedasticity:

McLeod Li test for Box-Jenkins models that calculates Ljung Box statistic for the squared TS

$$H_0: no\ ARCH\ effect\ in\ data$$

Breusch-Pagan test for linear regression model with n parameters:

$$H_0: homoscedastic\ \chi^2\ n-1\ distribution$$

GARCH Model

In 1986, Robert Engle's doctoral student Tim Bollerslev develop the Generalized ARCH (GARCH) model.

GARCH(p, q) model is represented as

$$\varepsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + .. + \beta_p \sigma_{t-p}^2 +$$
$$\alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + .. + \alpha_q \varepsilon_{t-q}^2$$

ARMA – GARCH Model

Consider $\{y_t\}$ that has unconditional mean zero and is heteroscedastic (varying variance).

Model with $ARMA(p,q) - GARCH(p,q)$ process where the conditional mean value of the time series is modeled using ARMA and the conditional varying variance is modeled using GARCH.

$$\phi(B)y_t = \theta(B)\varepsilon_t$$

$$E(\varepsilon_t) = 0$$

$$\varepsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + \beta_2\sigma_{t-2}^2 + .. + \beta_p\sigma_{t-p}^2 +$$

$$\alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + .. + \alpha_q\varepsilon_{t-q}^2$$

The conditional mean process due to ARMA has essentially the same shape as the conditional variance process due to GARCH, just the lag orders may differ (allowing for a nonzero unconditional mean of yt should not change this result significantly).

GARCH to ARCH Model

GARCH(p, q) model is represented as

$$\varepsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + .. + \beta_p \sigma_{t-p}^2 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + .. + \alpha_q \varepsilon_{t-q}^2$$

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2$$

GARCH(1,1) model

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2$$

Equivalent to ARCH(k) model (k is very large)

$$\sigma_t^2 = \frac{\alpha_0}{1-\beta} + \alpha_1 \sum_{j=1}^{k} \beta^{j-1} \varepsilon_{t-j}^2$$

Different GARCH Models

IGARCH Integrated GARCH – to model persistent changes in volatility

APARCH Asymmetric Power ARCH – no symmetric square values effect

NGARCH Non-linear GARCH

EGARCH Exponential GARCH

Cross-correlation Function (CCF)

- Cross-covariance

$$\gamma_{xy}(k) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)]$$

- CCF

$$\rho_{xy}(k) = \frac{\gamma_{xy}(k)}{\sigma_x \sigma_y}$$

- Note:

$$\gamma_{xy}(k) = \gamma_{yx}(-k)$$

$$\rho_{xy}(k) = \rho_{yx}(-k)$$

VAR Model

- Vector AutoRegressive order 1 VAR(1) represented by

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

- If $\phi_{1,12} = \phi_{1,21} = 0$, then we z_{1t} and z_{2t} are not dynamically correlated
- z_{1t} and z_{2t} are said to have a transfer function relationship => z_{1t} can be adjusted to influence z_{2t} and vice-versa
- Specification done with Likelihood Ratio Tests or Information Criteria (more effective).
- Parameters estimated with
    - OLS
    - Maximum Likelihood
    - Bayesian estimation

VMA Model

- Vector Moving Average model of order 1 VMA(1) is represented by

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta_{1,11} & \theta_{1,12} \\ \theta_{1,21} & \theta_{1,22} \end{bmatrix} \begin{bmatrix} a_{1t-1} \\ a_{2t-1} \end{bmatrix}$$

- Specification done with cross-correlation matrices that satisfy $\rho_j = 0, j > q$ to determine VMA(q).

- Parameters estimated with
  - Conditional and Exact Likelihood

VARMA Model

- Vector AutoRegressive Moving Average model VARMA(1, 1) is represented by

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} \theta_{1,11} & \theta_{1,12} \\ \theta_{1,21} & \theta_{1,22} \end{bmatrix} \begin{bmatrix} a_{1t-1} \\ a_{2t-1} \end{bmatrix}$$

- Specification done using Kronecker Index (`MTS::Kronid()`) or Scalar Component Model (SCM).
- Parameters estimated with
  - Conditional and Exact Likelihood
- Transfer function models are a special case of VARMA models.

VARMA Identifiability

- Unlike VAR and VME models, VARMA models encounter problem of identifiability – i.e. non-uniqueness in model specification (get more than 1 VARMA model for the same set of AR and MA polynomials).

- There are cases for which a VMA(1) model can also be written as a VAR(1) model.

- This is harmless since either model can be used in real life application.

R code

```
library("MTS")

xt <- matrix(rnorm(1500), 500, 3)

MTSplot(xt)

p1<- matrix(c(0.2,-0.6,0.3,1),2,2)

sig <- matrix(c(4,0.8,0.8,1),2,2)

th1 <- matrix(c(-0.5,0,0,-0.6),2,2)

m1 <- VARMAsim(300, arlags=c(1), malags = c(1),phi=p1,theta=th1,
sigma=sig )

zt <- m1$series

MTSplot(zt)


Kronid(zt)
```

R code

```
library("astsa")
data(cmort)
data(tempr)
data(part)
zt <- cbind(cmort, tempr, part)

# CCF
acf(zt, 50)
ccf(cmort, tempr, 50)

mod1 <- MTS::VARMA(zt)
summary(mod1)
acf(resid(mod1))
VARMApred(mod1, h=5)

Kronid(zt)
mod2 <- MTS::VARMA(zt, p = 2, q= 1)
```

R code

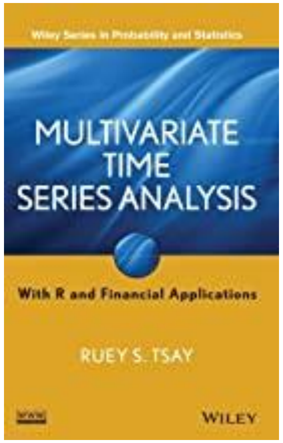
```
library("astsa")

data("mts-examples")

ibmspko.mts <- ts(ibmspko[,-1],start=c(1961,1),frequency=12)

mod3 <- VARMA(ibmspko.mts)

summary(mod3)

acf(resid(mod3))

VARMApred(mod, h=10)
```

Textbook Chapters

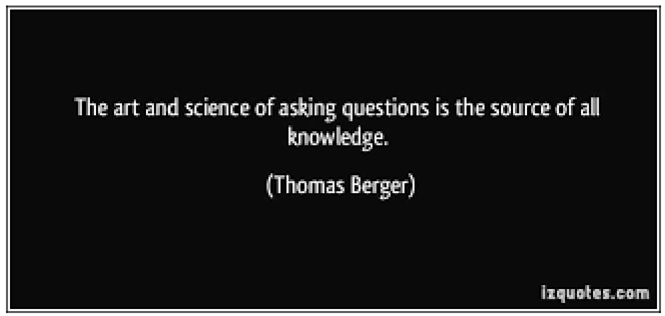
Materials covered available in book chapters:

PTS: 6

R.S. Tsay, *Multivariate Time Series Analysis*, Wiley, 2013.



Thank You


The art and science of asking questions is the source of all knowledge.
(Thomas Berger)
izquotes.com