03/28/23 MSCA 31006 IP02 Moushumi Pardesi

Assignment 1

2.1 Given $E(X) = 2$, $Var(X) = 9$, $E(Y) = 0$, $Var(Y) = 4$
$Cor(X,Y) = 0.25$

(a) $Cov(X,Y) = Cor(X,Y) \times \sigma_X \times \sigma_Y$
$= 0.25 \times \sqrt{9} \times \sqrt{4}$
$= 1.5$ ——— (1)

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$
$= 9 + 4 + 2(1.5)$ —— (from (1))
$= 16$

(b) $Cov(X, X+Y) = E[(X - E(X))(X+Y - E(X) - E(Y))]$
$= E[X^2 - 2(X+Y)E(X) - XE(Y) + E(X)^2 + XY + E(X)E(Y)]$
$= Var(X) + E(X)^2 - 2(4) + E(X)^2 + E(XY)$
$= 2.5$

(c) $Cov(X+Y, X-Y) = Cov(X,X) - Cov(Y,Y)$
$= Var(X) - Var(Y)$
$= 9 - 4$
$= 5$ ——— (1)

$Var(X-Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$
$= 9 + 4 + 2(1.5)$
$= 16$ ——— (2)

$$\mathrm{Cor}(X+Y, X-Y) = \frac{\mathrm{Cov}(X+Y, X-Y)}{\sigma_{X+Y} \times \sigma_{X-Y}}$$

$$= \frac{5}{\sqrt{16} \times \sqrt{16}} \quad \text{— (from ①, ②, (a))}$$

$$= 0.3125$$

2.2 X and Y are dependent but $Var(X) = Var(Y)$

$$\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X,X) - \mathrm{Cov}(Y,Y) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,X)$$

$$= Var(X) - Var(Y)$$

$$= 0$$

2.5 $Y_t = 5 + 2t + X_t$

$\{X_t\}$ is zero mean stationary series with autocovariance function γ_K

(a) $E(Y_t) = E[5 + 2t + X_t]$

$= 5 + 2t + E[X_t]$

$= 5 + 2t$

(b) $Cov(Y_t, Y_{t+n}) =$

$E[(Y_t - E(Y_t)) \times (Y_{t+n} - E(Y_{t+n}))]$

$= E[((5 + 2t + X_t - (5 + 2t)) \times (5 + 2(t+h) + X_{t+n} - (5 + 2(t+h)))]$

$= E[(X_t - X_{t+n})(2n + 2t)]$

Since X_t is a stationary series, γ_K only depends on time lag k.

$Cov(Y_t, Y_{t+n}) = 2h\gamma_n + 2\gamma_0$

where γ_n is the autocovariance function for X_t at lag h and γ_0 is the variance of X_t.

$\therefore$ autocovariance for Y_t will be

$Cov(Y_t, Y_{t+n}) = 2h\gamma_n + 2\gamma_0$

★ (c) From (a) we have mean of Y_t which is $5+2t$, which is not constant over time.

Therefore $\{Y_t\}$ is not stationary.

Autocovariance function for Y_t is $2h\gamma_h + 2\gamma_0$ where γ_h is autocovariance of X_t. Since X_t is a stationary series, we know that γ_h is stationary.

However, autocovariance function of Y_t is not constant over time with a factor of $2h$.

So $\{Y_t\}$ is not a stationary series.

2.6 $\{X_t\}$ is a stationary time series

$$Y_t = \begin{cases} X_t & \text{when } t \text{ is odd} \\ X_t + 3 & \text{when } t \text{ is even} \end{cases}$$

(a) $Cov(Y_t, Y_{t-k}) = E[(Y_t - E(Y_t)) \times (Y_{t-k} - E(Y_{t-k}))]$

$= E[(X_t - E(X_t)) \times (X_{t-k} - E(X_{t-k}))]$ if k is odd

$= E[(X_t + 3 - E(X_t+3)) \times (X_{t-k} + 3 - E(X_{t-k}+3))]$
if k is even

It is apparent from above that when k is odd, Y_t and Y_{t-k} are odd too and equal the X_t and X_{t-k}.

So for odd k,

$$\mathrm{cov}(Y_t, Y_{t-k}) = \mathrm{Cov}(X_t, X_{t-k})$$

When k is even, Y_t and Y_{t-k} are both even.

$$\mathrm{Cov}(Y_t, Y_{t-k}) = E(X_t \times X_{t-k}) - E(X_t) \times E(X_{t-k}) + 3(E(X_{t-k})) - 3(E(X_t)) + 9$$

Since X_t is stationary, $E(X_t)$ and $E(X_{t-k})$ are constant.

So the first 2 terms of the above equation only depend on lag k. Last 2 terms depend on $E(X_{t-k})$ and $E(X_t)$ which are only functions of lag k

So $\mathrm{cov}(Y_t, Y_{t-k})$ is independent of t and only depends on lag k

(b) $E(Y_t) = \begin{cases} E(X_t) & \text{for odd } t \\ E(X_t + 3) & \text{for even } t \end{cases}$

Since X_t is stationary, $E(X_t)$ is constant over t.

So $E(Y_t)$ alternates between $E(X_t)$ and $E(X_t) + 3$, so it is not constant over t.

Moreover,

$\gamma_{Y(k)} = Cov(X_t, X_{t-k})$ for odd k

$\gamma_{Y(k)} = Cov(X_t + 3, X_{t-k})$ for even k

So autocovariance function of Y_t for odd lags k is not dependant on t. But it differs for even lag k and thus overall it is not constant over t.

Thus $\{Y_t\}$ is not stationary.

2.7 $\{Y_t\}$ is stationary with autocovariance function γ_K

(a) $W_t = Y_t - Y_{t-1}$

$E(W_t) = E(Y_t) - E(Y_{t-1})$

$= 0$ (because both will be constant for stationary Y_t)

$\therefore$ Mean of W_t doesnot depend on t.

$\text{Cov}(W_t, W_{t-K}) = E(W_t W_{t-K}) - E(W_t) E(W_{t-K})$

$= E[(Y_t - Y_{t-1})(Y_{t-K} - Y_{t-K-1})]$

$= \gamma_K + \gamma_K - \gamma_{K+1} - \gamma_{K-1}$

$= 2\gamma_K - \gamma_{K+1} - \gamma_{K-1} \rightarrow$ function of K

$\therefore$ W_t is stationary

(b) $U_t = Y_t - 2Y_{t-1} + Y_{t-2}$

$= Y_t - Y_{t-1} - (Y_{t-1} - Y_{t-2}) - Y_{t-2}$

$= W_t - W_{t-1} - W_{t-2}$

Since we showed in (a) that W_t is stationary, U_t is also stationary because it is a linear combination of W_t.

2.8 $\{Y_t\}$ is stationary with autocovariance function γ_k.

$$W_t = c_1 Y_t + c_2 Y_{t-1} + \ldots\ldots + c_n Y_{t-n+1}$$

$$E(W_t) = c_1 E(Y_t) + c_2 E(Y_{t-1}) + \ldots + c_n E(Y_{t-n+1})$$

since Y_t is stationary, $E(Y_t)$, $E(Y_{t-1})\ldots$ are constant and do not depend on t.

So $E(W_t)$ is constant.

$$\text{Cov}(W_t, W_{t-n}) = E[(c_1 Y_t + c_2 Y_{t-1} + \ldots\ldots c_n Y_{t-n+1})(c_1 Y_{t-k} + \ldots\ldots + c_n Y_{t-k-n+1})]$$

$\therefore$ W_t is stationary because it has constant mean and its autocovariance depends on lag k.

2.11 $Cov(X_t, X_{t-k}) = \gamma_k$ is not dependent on t
$E(X_t) = 3t$ which is dependent on t.

(a) $\{X_t\}$ is not stationary because its mean is not constant.

(b) $Y_t = 7 - 3t + X_t$

$E(Y_t) = E(7 - 3t + X_t) = E(7 - 3t - 3t) = 0$

$Cov(Y_t\ Y_{t-k}) = E[(Y_t)(Y_{t-k})] - E(Y_t)\ E(Y_{t-k})$

$= E(7 - 3t + X_t)(7 - 3(t-k) + X_{t-k}) - 7 \times 7$

$= E[(X_t)(X_{t-k})] - E(X_t)E(X_{t-k})$

$\therefore \{Y_t\}$ is stationary

2.12 $Y_t = e_t - e_{t-12}$

$E(Y_t) = E(e_t) - E(e_{t-12}) = 0$

$Cov(Y_t, Y_{t-k}) = E(Y_t Y_{t-k}) - E(Y_t)\ E(Y_{t-k})$

$= E[(e_t - e_{t-12})(e_{t-k} - e_{t-k-12})]$

$= E[e_t e_{t-k}] + E(e_{t-12} e_{t-k-12})$
$- E(e_{t-12} e_{t-k}) - E(e_t e_{t-k-12})$

$= 0$ for $k > 0$ or $k < 0$

For $k = 12$, $Cov(Y_t, Y_{t-k}) = E[e_{t-12}^2]$
$= -\sigma^2$

$\therefore$ Autocovariance and the autocorrelation $\neq 0$ when $k = 12$
$= 0$ when $k \neq 12$

2.14 (a) $Y_t = \theta_0 + te_t$

$$E(Y_t) = E(\theta_0 + te_t)$$
$$= E(\theta_0) + t\,E(e_t)$$
$$= E(\theta_0) + t \times 0 \quad = \theta_0$$

$$\gamma_K = Cov(Y_t, Y_{t-K}) = E[Y_t Y_{t-K}] - E[Y_t]E[Y_{t-K}]$$
$$= E[(\theta_0 + te_t)(\theta_0 + (t-K)e_{t-K}$$
$$= E(\theta_0^2) = 0$$

$$Var(Y_t) = t^2\, Var(e_t)$$
$$= \sigma^2 t^2$$

so the process has constant variance and constant autocovariance

$\therefore Y_t$ is stationary

(b) $W_t = \nabla Y_t = Y_t - Y_{t-1}$
$= \theta_0 + te_t - (\theta_0 + (t-1)e_{t-1})$
$= te_t - (t-1)e_{t-1}$

$E(W_t) = E(te_t) - E[(t-1)e_{t-1}]$
$= 0$
since e_t is white noise.

$\gamma_k = Cov(W_t, W_{t-k})$
$= E(W_t W_{t-k}) - E(W_t)E(W_{t-k})$
$= E(W_t W_{t-k})$
$= E[te_t - (t-1)e_{t+1}][(t-k)e_{t-k} - (t-k-1)e_{t-k-1}]$

$= 0$

$Var(W_t) = Var(te_t - (t-1)e_{t-1})$

So the process has constant mean and autocovariance

$\therefore \{W_t\}$ is stationary.

(c) $Y_t = e_t e_{t-1}$

$E(Y_t) = 0$

$Cov(Y_t, Y_{t-k}) = E(Y_t, Y_{t-k}) - E(Y_t)$
$E(Y_{t-k})$

$= E(e_t e_{t-1}, e_{t-k} e_{t-1-k})$
$\quad - E(e_t e_{t-1}) E(e_{t-k} e_{t-1-k})$
$= 0$

The process has constant mean and constant autocovariance.

$\therefore \{Y_t\}$ is stationary.

2.15 $Y_t = (-1)^t X$ and $E(X) = 0$

(a) $E(Y_t) = E(X(-1)^t)$
$= 0$ since $E(X) = 0$

(b) $Cov(Y_t, Y_{t-k}) = Cov(X(-1)^t, X(-1)^{t-k})$
$= Cov(X, X) \times (-1)^k$
$= (-1)^k \sigma^2$ for $k \geq 0$

(c) Y_t is stationary because it is only dependent on lag k.